Sufficiently Specialized Economies Have Nonempty Cores

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Abstract

An economy with a nonempty core may plausibly be regarded as socially stable since there exists allocations against which no group in the economy wishes to "recontract out." Aside from classical economies, it is not generally known what are the primitives of an economy that give rise to a nonempty core. This paper finds a class of perturbations that operate directly on economic primitives to generate a nonempty core. These perturbations are characterized by two properties which have economic content. The first is a notion of specialization — individuals hold goods and essential inputs to productive processes that are not readily available elsewhere in the economy. The second is a curvature condition. Each agent's preferences must display sufficient curvature so that another person's specialized holdings are valued by the agent. It is shown that for any economy of a general class that includes possibly local nonconvexities and a wide variety of property rights configurations, if the economy is sufficiently specialized and the curvature condition is satisfied, then the corresponding NTU game is balanced. Hence, the economy has a nonempty core.

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1 Introduction

An economy with a nonempty core may plausibly be regarded as socially stable as it yields outcomes from which no group of individuals could feasibly "recontract out." What kinds of economies have nonempty cores?

The answer to this question is well known for the class of Arrow-Debreu, convex economies. These economies are typically characterized by convex preferences, nonincreasing returns to scale production possibilities, and individually assigned endowments (i.e., private rather than collective ownership). We will refer to these economies as *classical* economies. The canonical economy of this form is the standard exchange economy with consumers having concave utility functions. A series of papers in the late sixties and early seventies (e.g., Scarf (1967), Shapley and Shubik (1969), Shapley (1973), Billera (1974), and Billera and Bixby(1973)) established that any economy that generates a *balanced* NTU (nontransferable utility) game has a nonempty core. The classical economies described above satisfy this criterion.[2] Also, many nonclassical economies such as the coalitional production economies found in Boehm (1974) and Border (1984) are balanced.

Because the balancedness criterion is defined directly on NTU games it is generally applicable to a wide variety of environments and social situations. However, though elegant, it remains unclear beyond classical economies how balancedness translates into economic fundamentals. The abstraction inherent in the NTU formulation makes balancedness difficult to translate into primitives of the underlying economy.

A much stronger hypothesis, *outcome balancedness*, <u>is</u> defined on economic fundamentals (the concept originated from Boehm (1974) though our terminology is Border's (1985)). Economies that are outcome balanced generate balanced NTU games in the usual sense, and many classical economies such as classical exchange economies do satisfy this stronger condition. Since outcome balancedness is a condition defined on physical outcomes of the economy instead of utility outcomes of the corresponding NTU game, it has the advantage of being "closer" to economic settings than the more general balancedness condition. However, since outcome balancedness mimics, in some sense, the balancedness condition for the physical environment, it suffers from the same interpretational problems. Moreover, many economies of interest including many classical economies with production are not outcome balanced.

One way to address this issue is through the following approach. Suppose that one starts with an economy that has an empty core. Is there then a systematic way of changing the primitives of the economy so that the resulting new economy is balanced? Does this systematic method have a reasonable interpretation in terms of these primitives?

This paper finds a class of "perturbations" which operates along a sequence of economies to generate balancedness for economies far enough out on the sequence. Two general conditions of interest interact with one another along the sequence to generate balancedness. The first is a notion of specialization. Specialization means that individuals hold goods and essential inputs to productive processes that are not readily available elsewhere in the economy. The second condition is a curvature condition. It is necessary that, say, agent A's preferences display sufficient curvature so that agent B's specialized holdings are valued by agent A. This condition should be regarded as a complementary assumption to specialization since this curvature is necessary to effectively

[2] In fact, they go a step further in showing that these economies typically generate *totally balanced* games, that is, balanced on all possible subgames. Since every totally balanced game comes from a certain classical economy of a general class described in Billera (1974), there is an isomorphism between a family of equivalence classes of classical economies and the class of NTU games.

translate specialization into utility space. The two conditions taken together guarantee that each agent possesses valuable goods for which there are no close substitutes. Note that without *some* type of curvature condition, it may be inevitable that certain "corner solutions" arise which allow coalitions to exclude an agent despite his exclusive holdings of some goods.

We give an elementary proof of the following result: given a sequence of economies which belong to a certain class (to be defined shortly), if the economies along the sequence are increasingly specialized, and if the curvature condition holds, then corresponding NTU game for economies far enough along the sequence is balanced. Hence these economies have a nonempty core.

The notion of an *economy* used here is permissive in the sense that we allow for an almost unlimited variety of property rights configurations,[3] including proprietary productive processes; it is restrictive in the sense that agents' utility functions are assumed component-wise, unboundedly and strictly increasingly concave, and there are assumed to be no income effects. The feasible sets for all coalitions are also assumed to be compact and the aggregate feasible set is convex (though coalitionally feasible sets need not be).

Since these latter conditions are more restrictive than many standard models of core existence, our economic interpretation of balancedness comes at the cost of some generality. Moreover, the definition of specialization considered here is to some extent nonparametric; we cannot say in absolute terms how much more specialized is one economy than another. Our definition *does* transform economies along a sequence from those that may have an empty core to those whose core is nonempty.

For this reason it is inaccurate, if tempting, to view the perturbations in the same way as the replica economies in the core convergence literature.[4] Both involve systematic changes along a sequence of economies, and both relate to the core. However, while that literature was interested in measuring precisely how close is the core to competitive allocations, the purpose here is to interpret balancedness by the economic content of the perturbations that asymptotically generate balancedness. Hence, it is the *nature* of the perturbations, not their *magnitude*, that is of primary importance here.

The analysis of the present paper relates to a model of the core with increasing returns by Ichiichi and Quinzii (IQ) (1983), and a model of coalitional property rights by Glomm and Lagunoff (GL) (1995). IQ prove the nonemptiness of the core of certain types of production economies with increasing returns. They do this by proving the existence of a hybrid equilibrium contained in the core. The solution utilizes both coalitional stability and a decentralized price system. As with the present analysis, joint restrictions on both the production and consumption sectors are employed. In particular, the degree of nonconvexity in the production set is restricted. Although the analogy is loose, the Increasing Specialization assumption in the present framework is roughly similar to an asymptotic restriction of the degree of nonconvexity. Moreover, the Curvature condition here is reminiscent of an assumption in IQ which similarly serves to exclude "corner solutions" of the type mentioned above.[5]

The GL model generalizes the standard, "private property" Arrow-Debreu exchange economy. They examine a class of nonclassical exchange economies in which ownership of goods and services

[3]Moulin and Peleg (1982) consider a concept of *coalition effectiveness* that may be used here to describe a *property rights configuration*.

[4]See Anderson (1986) for a survey.

[5]The precise relation between the two analyses are difficult to ascertain since IQ employ joint restrictions on both exogenous variables and derived variables such as prices. Meanwhile, the present analysis does not the consider decentralization issue at all.

is attributable to coalitions of various sizes. GL then prove a special case of the result given here for property rights regimes that exhibit certain exclusionary properties. The main differences between the present paper and GL is that the notion of specialization is generalized here, and the present analysis also allows for production and local (coalitional) nonconvexities.

The paper is organized as follows. Section 2 provides preliminary definitions. The class of economies and their corresponding NTU games are defined. The balancedness condition is defined for these games. Section 3 gives some examples of economies which exhibit "specialization" and "curvature." These concepts are developed formally in Section 4. Section 5 gives the main result: sufficiently specialized economies are balanced, hence have nonempty cores. Section 6 provides the proof.

2 Preliminaries

2.1 The Economy

We consider the following class of environments. There is a set $I = \{1, \ldots, n\}$ of agents who can consume ℓ goods or commodities. An arbitrary agent will be denoted by i, an arbitrary good by k. We assume that an agent's preferences are represented by a continuous, concave, and component-wise, unboundedly strictly increasing function $u^i : \Re^\ell_+ \to \Re$, $i \in I$, normalized so that $u^i(0) = 0$. We also assume that u^i displays no income effects. Formally, we assume that for any $x, y \in \Re^\ell_+$ with $u^i(x) \geq u^i(y)$, then $u^i(x+z) \geq u^i(y+z)$ for all $z \in \Re^\ell_+$.[6] Let $u = (u^1, \ldots, u^n)$

An economy will be described here in reduced form by a resource-feasibility correspondence $\omega : 2^I \backslash \{\emptyset\} \to\to \Re^\ell_+$ that satisfies

(i) For each $C \subseteq I$, $\omega(C)$ is a compact and comprehensive set in $\Re^\ell_+$.[7]

(ii) For any pair C_1, C_2 with $C_1 \cap C_2 = \emptyset$, $\omega(C_1) \cup \omega(C_2) \subseteq \omega(C_1 \cup C_2)$

(iii) $\omega(I)$ is convex.

Here, $\omega(C)$ is interpreted as the set of feasible aggregate resource vectors achievable by coalition C. Notationally, we will find it convenient to construct economy-wide allocations that are feasible for a particular coalition without using the projection mapping. For each coalition $C \subseteq I$, define a *C-feasible* allocation $x(C) \in \Re^{n\ell}_+$ to be one that satisfies $\sum_{i \in C} x^i(C) \in \omega(C)$ and $x^j(C) = 0$ if $j \notin C$.

This formulation of an economy is a generalization of the usual Arrow-Debreu formulation of an economy in which each individual i is privately endowed with some vector $e^i \in \Re^\ell_+$. In the case of the standard exchange economy, $\omega(C) = \{x \in \Re^\ell_+ |\ x \leq \sum_{i \in C} e^i\}$ for each coalition $C \subseteq I$. In the case of a "blue-print" production technology (equal access) with production set Y, $\omega(C) = \{x \leq y + \sum_{i \in C} e^i |\ y \in Y\}$.

The definition of the outcome correspondence ω, however, also includes the coalitional production economies of Boehm (1974) and Border (1984), and the coalitional property rights economies of Glomm and Lagunoff (1992). In the case of the former, coalition C has joint proprietary rights to the coalitional production set Y^C, while in the case of the latter, a quantity of good k is jointly held by C only if C includes all the members with joint legal rights to that quantity.

[6]This condition is also referred to as *quasi-linearity*.

[7]The set $\omega(C)$ is *comprehensive* if for any $x \in \omega(C)$, then any bundle $y \in \Re^\ell_+$ with $y \leq x$ satisfies $y \in \omega(C)$.

Our formulation excludes economies with global externalities, but does include economies with club goods and local public goods such as in Greenberg (1983), Scotchmer (1985), and Wooders (1978).

Formally, an *economy* $\mathcal{E}$ will be defined by the pair $\mathcal{E} = (\omega, u)$. The *core* of an economy $\mathcal{E}$ is an I-feasible allocation $x(I) \in \Re_+^{n\ell}$ for which there is no coalition C and no C-feasible allocation $x(C)$ such that $u^i(x^i(C)) > u^i(x^i(I))$ for all $i \in C$.

2.2 NTU Games

The interest of this paper is in a, by now standard, sufficient condition for the existence of a nonempty core.

The NTU game $\mathcal{U}$ derived from an economy $\mathcal{E} = (\omega, u)$ (usually called a "market game" when it is derived from a classical economy) is a correspondence $\mathcal{U} : 2^n \rightarrow\rightarrow \Re^n$ defined by

$$\mathcal{U}(C) \;\equiv\; \left\{ \bar{u} \in \Re^n \,\middle|\, \exists\, C\text{-feasible } x(C), \text{ s.t. } \bar{u}^i \leq u^i(x^i(C)),\ \forall i \in C \right\}, \tag{1}$$

for each coalition $C \subseteq I$. It will be sufficient for our purposes to consider those elements in $\mathcal{U}(C)$ that map from C-feasible points $x(C)$. We denote such elements by

$$u(x(C)) \equiv \left((u^i(x^i(C)))_{i \in C}, \overbrace{0, \ldots, 0}^{n-|C|} \right).$$

The balancedness criterion used here is standard. Let $\mathcal{B}$ denote a nonempty collection of subsets $B \subseteq I$. Let $\mathcal{B}(i) \equiv \{B \in \mathcal{B} |\ i \in B\}$. The collection $\mathcal{B}$ is *balanced* if there is a tuple $(\lambda_B)_{B \in \mathcal{B}}$ where $\lambda_B \geq 0$ satisfying for each agent i,

$$\sum_{B \in \mathcal{B}(i)} \lambda_B \;=\; 1.$$

The collection $\mathcal{B}$ is *minimally balanced* if it strictly contains no balanced collection $\mathcal{B}'$.

An NTU game $\mathcal{U}$ is *balanced*[8] if for any balanced collection[9] $\mathcal{B}$ of I with balancing weights $(\lambda_B)_{B \in \mathcal{B}}$, and for each B-feasible allocation $x(B)$ with $B \in \mathcal{B}$,

$$\sum_{B \in \mathcal{B}} \lambda_B u(x(B)) \;\in\; \mathcal{U}(I). \tag{2}$$

Well known theorems of Scarf and Billera have shown that economies that correspond to balanced NTU games have nonempty cores. We do not provide a proof of their results here. However, showing that balancedness holds under certain conditions is the crux of the paper.

3 Some Examples

What does balancedness mean in terms of economic primitives — preferences, technologies and endowment structure? Some examples are given here which display the two key concepts of specialization and curvature to be formalized in section 4.

Example 1 Consider the following example of exchange.[10] There are three individuals and four

[8] This paper uses the definition of Shapley and Shubik (1969) and applied by Billera (1974) to NTU games rather than the generally weaker definition used in Scarf (1967)).

[9] We note here that there is no loss in generality in considering balanced collections $\mathcal{B}$ in which there is no pair of disjoint coalitions (see Shapley (1973)).

[10] This example is similar to Example 8 in GL.

goods. The distribution of endowments and the "property rights" structure is given in the matrix in Table 1 below. In Table 1 there are $\beta + 2\alpha$ units of each of the first three goods distributed so that agent i ($i = 1, 2, 3$), holds β units of good i. Agent j, $j \neq i$, holds α units of good i where $\alpha < \beta$. The fourth good, of which there is γ units, may be regarded as "communally owned" among the three agents. This good is allocated via majority rule. That is, up to γ units of good 4 may be consumed by a coalition C if and only if $|C| \geq 2$. Otherwise, if $|C| < 2$ then none of the fourth good is held.

	good 1	good 2	good 3	good 4
agent 1	β	α	α	0
agent 2	α	β	α	0
agent 3	α	α	β	0
agents 1 and 3	0	0	0	γ
agents 1 and 2	0	0	0	γ
agents 2 and 3	0	0	0	γ

Table 1

Observe that if β grows relative to α then the holdings of the first three goods are *increasingly specialized* in the sense that individual 1 owns an increasing proportion of the first good, individual 2 owns an increasing proportion of the second good, etc. What is interesting about this example is that whether or not the NTU game corresponding to this economy is balanced depends crucially on the degree of this type of specializaton in endowments over the privately held goods. The majority rule over the fourth good creates "voting cycles" that are typically responsible for empty cores and violations of balancedness.[11] The specialized endowment holdings, if they are suffciently large and if agents' preferences have sufficient curvature so that there are enough gains from trade in the private holdings, may prevent such cycles.

In terms of this example, this means that if β is sufficiently large relative to both α and γ then the core of this economy is "more likely" to be nonempty. If β is large relative to α then each agent owns relatively large specialized holdings. If β is large relative to γ then these specialized holdings have a greater weight in an agent's consumption bundle than the communally held good.

Whether or not the core is actually nonempty depends on the curvature of the utility functions of the three agents. If, for example, the three agents have identical utility functions in which the first three goods are perfect substitutes, then no amount of specialization can rescue the economy — the core is empty. There has to be a reason *not to exclude an individual* in the majorities that form to allocate the fourth good. To prevent such exclusion requires that each agent prefer to smooth his consumption over all (the first three) goods. Some form of strict concavity is required.

[11]However, the majority rule alluded to here is not actually a majority *voting* mechanism. Rather, it refers to the property rights structure under which the majority coalitions have the ability to claim the entire quantity of the fourth good.

Using the symmetric utility representation $u^i(x^i) = \sum_{k=1}^{4}(x_k^i)^{\frac{1}{2}}$, it is not difficult to check that if, for example, $\beta = 3, \alpha = 0$ and $\gamma = 3$, then the core of this economy is nonempty.

Example 2 It is straightforward to extend this example to include production. Suppose $\gamma = 0$ and, instead, the fourth good is produced by inputs from the other three with production function $x_4 = \min\{x_1, x_2, x_3\}$. With this technology both specializaton and curvature are manifested through the productive process rather than only through endowments and utilities, resp. Nevertheless, the idea remains the same. Even if agents only value the fourth good, specialized holdings of productive inputs combined with extreme curvature in the production function allows the core to be nonempty despite the voting cycle in the only good that has (utility) value.[12]

In the example with production, curvature in the production function plays a crucial role. If, for example, all goods are perfect substitutes in the production process as with the function $x_4 = x_1 + x_2 + x_3$, then majority rule over the fourth good yields an empty core regardless of how specialized are the holdings of the first three goods.

Example 3 A faulty conclusion that one might draw from the first example is that private property is necessary for agents to be specialized, and hence, necessary for the existence of a nonempty core. Consider an example of exchange with (the same) three agents but only two goods. The endowment and property rights structure is exhibited in the matrix in Table 2.

	good 1	good 2
agents 1,2, and 3	β	0
agents 1 and 3	0	γ
agents 1 and 2	0	γ
agents 2 and 3	0	γ

Table 2

Each of the two goods are collectively held by all three agents. The analysis hinges crucially upon the compatibility of claims of the coalitions that can form. For good 2 consider the majority rule as in the previous examples. For good 1 consider a *unanimity* rule in which up to β units of good 1 can be consumed by C if and only if $C = I$.

It should be clear that with enough curvature in agents' utilities, the core is nonempty if β is large enough relative to γ. The reason is that the unanimity rule endows each agent with veto power over the use of good 1. In this sense, each agent is specialized in the holding of good 1 where specialization means that each agent holds on to good(s) that are not available elsewhere, i.e., *without the presence of that agent*.

[12]However, the main result in section 5 does not cover the case in which some goods are not valued as we assume that utilities are strictly monotonic.

4 A Formalization

4.1 Specialization

The intuition of the examples above is formalized. Consider a sequence of economies $\{\mathcal{E}_t\}_{t=0}^{\infty} = \{\omega_t, u\}_{t=0}^{\infty}$ in which there are only changes in the physical environment defined by ω_t. The sequence is assumed monotone in the sense that $\omega_{t+1}(C) \supseteq \omega_t(C)$ for each $C \subseteq I$. We formalize a notion of specialization for an agent i which looks at how the sequence $\{\omega_t\}$ progresses when the aggregate resource set $\omega_t(I)$ increases relative to the aggregate resource set without the presence of agent i, $\omega_t(I \setminus \{i\})$.

Definition 1 Given a monotone sequence $\{\mathcal{E}_t\}_{t=0}^{\infty} = \{(\omega_t, u)\}_{t=0}^{\infty}$, an agent i is said to be *increasingly specialized in good* k if $proj_k \omega_t(I)$ is unboundedly strictly increasing in t while $proj_k \omega_t(I \setminus \{i\})$ remains constant on t. (where "$proj_k$" is the projection mapping onto the k^{th} good coordinate).

Denote the set of commodities in which i is increasingly specialized by $\pi(i)$. Observe that $proj_k \omega_t(I)$ is a compact interval in $\Re_+$ of the form $[\underline{w}_k, \overline{w}_k]$. The assumption that $proj_k \omega_t(I)$ is unboundedly strictly increasing means that the number $\overline{w}_k$ is unboundedly strictly increasing.

Definition 2 A monotone sequence of economies, $\{\mathcal{E}_t\}_{t=0}^{\infty} = \{(\omega_t, u)\}_{t=0}^{\infty}$, is *increasingly specialized* if

1. Each agent i is increasingly specialized in at least one good (i.e., $\pi(i) \neq \emptyset, \ \forall i$)
2. For each t there is some $\nu_t \in \omega_t(I)$ such that for each i, $\omega_t(I \setminus \{i\})$ is bounded above by ν_t.
3. If for any $C \subseteq I$, no agent in C is increasingly specialized in good k (i.e., $k \notin \bigcup_{i \in C} \pi(i)$), then $proj_k \omega_t(C)$ is constant on t.

Part 1 of Definition 2 assumes that, along the sequence $\{\mathcal{E}_t\}$, every agent is increasingly specialized in at least one good. Part 2 assumes, further, that there is an aggregate resource vector for the economy $\mathcal{E}_t$ that uniformly bounds the aggregate resource vectors that are feasible if one agent is missing. Though somewhat restrictive, observe that all exchange economies, classical and non-classical, satisfy this condition. Examples 1 and 3 in Section 3 are two such nonclassical examples. Finally, part 3 of the Definition implies that $proj_k \omega_t(C)$ is constant if no one in C is specialized in good k. Hence, if no one is specialized in good k then coalition C's aggregate holdings of good k is bounded by $\max\{x_k : \ x_k \in proj_k \omega_0(C)\}$.

4.2 Curvature

The idea that increasing specialization must hold in utility space is formalized here.

Definition 3 We will say that a monotone sequence of economies, $\{\mathcal{E}_t\}_{t=0}^{\infty} = \{\omega_t, u\}_{t=0}^{\infty}$, satisfies the *Curvature Condition* if for each $i \in I$, for each minimally balanced collection $\mathcal{B}$ with no disjoint sets, for each collection of balancing weights (λ_B), and for each $x_0 \in \omega_0(I)$, the following implication holds.

For any collection of sequences, $\{z_t(B)\}_{t=1}^{\infty}$, $B \in \mathcal{B}(i)$, that satisfies

(a) $z_t(B) \in \omega_t(B)$ for each t and each $B \in \mathcal{B}(i)$; and

(b) there exists some pair $B, B' \in \mathcal{B}(i)$ and a pair k and k' so that $z_{tk}(B) = z_{tk'}(B') = 0$, while $z_{tk'}(B)$ and $z_{tk}(B')$ increase without bound,

then there exists some $\bar{t}$ and $\epsilon > 0$ so that for each $t \geq \bar{t}$,

$$\sum_{B \in \mathcal{B}(i)} \lambda_B u^i(x_0 + z_t(B)) \ < \ u^i(\sum_{B \in \mathcal{B}(i)} \lambda_B z_t(B)) \ - \ \epsilon. \tag{3}$$

The Curvature Condition is a joint restriction on utility functions and on aggregate feasible resource vectors along the sequence of economies. It states loosely that utility functions have enough curvature on the sequence of economies so as to eventually overwhelm the effect of any initial point that is feasible in the initial economy $\mathcal{E}_0$. To see what this means observe that (3) is almost the standard Jensen's Inequality (if $x_0 = 0$ and $\epsilon = 0$). Therefore, (3) states that the effect of any fixed initial point x_0 on the weighted utility is uniformly dominated by the utility of the weighted allocation of unboundedly increasing consumption allocations if t becomes sufficiently large.

The $z_t(B)$ sequences represent some transformation of the initial economy $\mathcal{E}_0$. The restrictions on these sequences given by (a) and (b) guarantee that the condition can be satisfied for a large class of utility functions. The reason is that such sequences grow farther apart as t increases. This makes it possible for utilities with enough curvature to satisfy Jensen's Inequality with a uniform bound if t is sufficiently large and/or x_0 is sufficiently small. Note that the more restrictive the requirements on the sequences, or the smaller is the initial feasible point, the more likely the Curvature Condition will be satisfied.

From a simplified version of the economy in Example 1 in Section 3, a sequence of economies can easily be constructed to be increasingly specialized and satisfy the Curvature Condition. Consider the economy in Table 1. Letting $\alpha = 0$, define the sequence $\{\omega_t\}$ so that for each t, $\omega_t(I) = \{x \in \Re^4_+ | \ x \leq (\beta_t, \beta_t, \beta_t, \gamma)\}$, where β_t is assumed to increase in t without bound. Assume the feasible endowments of the subcoalitions $\omega_t(C)$, $C \subset I$ conform to Table 1. Clearly, this sequence is monotone, and it is increasingly specialized since agent i, $i = 1, 2, 3$ is increasingly specialized in good i. No individual is increasingly specialized in good 4. Now fix some $x_0 \in \omega_0(I)$. Without loss of generality we can take x_0 to be the upper bound $(\beta_0, \beta_0, \beta_0, \gamma)$. The only relevent balanced collection is the collection $\mathcal{B} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$ with weights $1/2$ for each coalition. Finally, consider agent 1. For any pair

$$\begin{array}{rcccl} z_t(\{1,2\}) & = & (z_{1t}, z_{2t}, 0, z_4) & \in & \omega_t(\{1,2\}), \quad \text{and} \\ z_t(\{1,3\}) & = & (z_{1t}, 0, z_{3t}, z_4) & \in & \omega_t(\{1,3\}), \end{array}$$

where z_{2t} and z_{3t} increase without bound, it suffices to verify that (3) holds. That is, we must verify that there is some $\bar{t}$ and $\epsilon > 0$ such that for all $t \geq \bar{t}$,

$$\frac{1}{2}u^1(\beta_0{+}z_{1t}, \beta_0{+}z_{2t}, \beta_0, \gamma{+}z_4){+}\frac{1}{2}u^1(\beta_0{+}z_{1t}, \beta_0, \beta_0{+}z_{3t}, \gamma{+}z_4) \ < \ u^1(z_{1t}, \frac{1}{2}z_{2t}, \frac{1}{2}z_{3t}, \gamma{+}z_4) - \epsilon \tag{4}$$

Letting preferences be given by $u^i(x^i) = \sum_k (x^i_k)^{\frac{1}{2}}$, the inequality (4) reduces to

$$\epsilon \ < \ [z_{1t}^{\frac{1}{2}} - (\beta_0 + z_{1t})^{\frac{1}{2}}] \ + \ [(\frac{1}{2}z_{2t})^{\frac{1}{2}} - \frac{1}{2}(\beta_0 + z_{2t})^{\frac{1}{2}}] \ + \ [(\frac{1}{2}z_{3t})^{\frac{1}{2}} - \frac{1}{2}(\beta_0 + z_{3t})^{\frac{1}{2}}] - \beta_0^{\frac{1}{2}}. \tag{5}$$

Observe that while the first term in the brackets on the right-hand side of (5) is bounded, the second and third bracketed terms are unboundedly increasing in t. Hence, given the preferences above for all three agents, this sequence of economies satisfies the Curvature Condition. Our main result states that the sequence of cores corresponding to the sequence of these economies are nonempty sets for all but finitely many t.

5 The Result

Theorem *Let $\{\mathcal{E}_t\}_{t=0}^{\infty} = \{(\omega_t, u)\}_{t=0}^{\infty}$ denote a sequence of monotone economies that (1) is increasingly specialized, and (2) satisfies the Curvature Condition. Then there is a $\bar{t}$ such that for each $t \geq \bar{t}$, the NTU game derived from $\mathcal{E}_t$ is balanced. (Hence the core of $\mathcal{E}_t$ is nonempty.)*

This result demonstrates how changes made to the initial economy $\mathcal{E}_0$ create a transformation of the original economy which is eventually balanced. Systematic changes are made only to ω_t as t varies, and only in a way that increases specialization in $\mathcal{E}_0$ as specified by Definition 2. Utilities are held constant. For large enough t the economy $\mathcal{E}_t$ is balanced. Hence, sufficient specialization gives the balancedness condition.

Remark: An Alternative Approach:

The approach in the main result is to examine a sequence of economies, each of which vary in the structure of ω (which is determined by endowments and technology). Each successive economy has certain endowments which are larger than before. Preferences are held fixed. An alternative approach is to fix ω and vary, instead, the preferences. One might call this the *normalized approach* since it examines a fixed endowment structure which is already "specialized" and then varies preferences in a way that increases the effect of the specialization in utility space. As an example consider:[13]

There are two agents, $i = 1, 2$, two goods, $k = a, b$, and,

$$\begin{aligned} \omega(I) &= \{x \in \Re^2_+ \mid x_a \leq 1,\ x_b \leq 1\} \\ \omega(\{1\}) &= \{x \in \Re^2_+ \mid x_a = 0,\ x_b \leq 1\} \\ \omega(\{2\}) &= \{x \in \Re^2_+ \mid x_a \leq 1,\ x_b = 0\} \end{aligned}$$

Increased specialization is reflected through increased preference complementarities between goods. Define the sequence of economies $\{\mathcal{E}_t\} = \{\omega, u_t\}$ where ω is fixed as above and

$$u^i_t = \frac{1}{1-t}\left[(x^i_a)^{1-t} + (x^i_b)^{1-t}\right] \tag{6}$$

Here, u^i_t converges to the perfect complements case as $t \to \infty$. The effect of increasing the parameter t is not unlike that of increasing β in Examples 1-3 in Section 3. The advantage of this approach is that the sequence of economies has a well defined "limit" economy, the perturbations of which exhibit a nonempty core. The disadvantage is that it is difficult to express increased specialization (globally) in a more general class of preferences than, say, some parameterized class of utilities (such as the CES case above).

[13] I thank a referee for suggesting this approach and suggesting the coalitional structure in the example.

6 Proof of the Theorem

The proof is elementary though somewhat involved. For each economy $\mathcal{E}_t$ in the sequence $\{\mathcal{E}_t\}$ we denote the corresponding NTU game by $\mathcal{U}_t$ which is defned analogously to (1). Let $\mathcal{B}$ denote a balanced collection of I which we assume, without loss of generality, contains no disjoint sets (see Shapley (1973)). Let (λ_B) be a collection of balancing weights for $\mathcal{B}$. We must show that, under the hypothesis of the Theorem,

$$u_t \in \sum_{B \in \mathcal{B}} \lambda_B \mathcal{U}_t(B) \quad \text{implies} \quad u_t \in \mathcal{U}_t(I) \tag{7}$$

for t sufficiently large.

By the Specialization assumption there is a sequence $\{\nu_t\}_{t=0}^{\infty}$ such that for each t, $\nu_t \in \omega_t(I)$ and ν_t bounds from above each $\omega_t(B)$, $B \in \mathcal{B}$. Without loss of generality we may assume that $\nu_{t+1} \geq \nu_t$ for all t. In particular, we assume that ν_{tk} increases without bound whenever $proj_k \omega_t(I)$ increases without bound, and remains constant whenever $proj_k \omega_t(I)$ is. Furthermore, the convexity of $\omega_t(I)$ allows us to consider a ν_t with the following property: each $\nu_t \in \omega_t(I)$ is *efficient* if there is no other $\nu'_t \in \omega_t(I)$ satisfying $\nu'_{tk} \geq \nu_{tk}$ for all k and $\nu'_{tk} > \nu_{tk}$ for some k. We assume therefore that ν_t is efficient for all t.

Consider the utility allocation $u_t \in \sum_{B \in \mathcal{B}} \lambda_B \mathcal{U}_t(B)$. This utility vector u_t may be expressed, for each i, by

$$u_t^i \;=\; \sum_{B \in \mathcal{B}(i)} \lambda_B u^i(\psi^i + z_t^i(B)) \tag{8}$$

where, for each $B \in \mathcal{B}$ and each t, $(\psi^i + z_t^i(B))_{i \in B}$ is a B-feasible allocation for ω_t and satisfies $\psi^i \in \omega_0(I)$ and $z_t^i(B) \in \Re^\ell$ for each i, and $\sum_{i \in B} \psi^i \leq \nu_{0k}$.

For each coalition B in the collection $\mathcal{B}$, let $\bar{z}_t(B) \equiv \sum_{i \in B} z_t^i(B)$ denoting the increment from the amount ν_{0k} in the initial economy $\mathcal{E}_0$ to the "t^{th}" economy $\mathcal{E}_t$. Without loss of generality we suppose that $\bar{z}_t(B) + \sum_{i \in B} \psi^i$ is contained in the upper boundary of $\omega_t(B)$. In this construction, the aggregate coalitional bundles are decomposable into an initial upper bound ν_0, and a part $\bar{z}_t$ which increments ν_{0k}.

It therefore suffices to show, in utility vector form,[14]

$$\sum_{B} \lambda_B u(\psi + z_t(B)) \;\in\; \mathcal{U}_t(I) \tag{9}$$

for large enough t. We do so by making use of the following claims.

Fix k and define $C(k) = \{i \mid k \in \pi(i)\}$. By definition, each $i \in C(k)$ is increasingly specialized in good k.

[14]From here on, we adopt the convention that $u^i(\psi^i + z_t^i(B)) = 0$ whenever $i \notin B$.

Lemma 1 *For each k and each $B \in \mathcal{B}$, $\bar{z}_{tk}(B) > 0$ only if $C(k) \neq \emptyset$ and $C(k) \subseteq B$.*

proof of Lemma 1 The fact that $C(k) \neq \emptyset$ follows from part 3 of Specialization Assumption (Definition 2). Suppose that $C(k) \not\subseteq B$. Then there is some $i \in C(k)$ where $i \notin B$. By the Specialization Assumption, $proj_k \omega_t(B) \subseteq proj_k \omega_t(I \backslash \{i\}) \subseteq proj_k \omega_0(I)$, and so $\bar{z}_{tk}(B) + \sum_{i \in B} \psi^i_k \leq \nu_{tk} = \nu_{0k}$ by our earlier construction of $\{\nu_t\}$. Given the construction of $\bar{z}_t(B)$ it follows that $\bar{z}_{tk}(B) \leq 0$. □

Lemma 2 $\sum_{B \in \mathcal{B}} \lambda_B \bar{z}_t(B) \ \leq \nu_t, \ \ \forall t.$

proof of Lemma 2 Fix t. Suppose that $\bar{z}_{tk}(B) > 0$ for some k and some B. By Lemma 1, $C(k) \neq \emptyset$ and $C(k) \subseteq B$. Hence the balancing weights must satisfy

$$\sum_{\{B \in \mathcal{B} | \ C(k) \subseteq B\}} \lambda_B \ \leq \ 1.$$

From the definition of Specialization, $\bar{z}_t(B) \leq \nu_t$ for each $B \in \mathcal{B}$. Therefore, by Lemma 1,

$$\sum_{B \in \mathcal{B}} \lambda_B \bar{z}_{tk}(B) \ = \sum_{\{B \in \mathcal{B} | \ C(k) \subseteq B\}} \lambda_B \bar{z}_{tk}(B) \ \leq \ \nu_{tk},$$

concluding the proof of Lemma 2. □

By Lemma 2, we can find some $y_t \in \Re^\ell_+$ that satisfies

$$n y_t + \sum_{B \in \mathcal{B}} \lambda_B \bar{z}_t(B) = \nu_t, \ \forall t. \tag{10}$$

By (10), the vector $\left(y_t + \sum_{B \in \mathcal{B}(i)} \lambda_B z^i_t(B)\right)_{i \in I}$ is I-feasible for ω_t, therefore

$$\left(u^i(y_t + \sum_{B \in \mathcal{B}(i)} \lambda_B z^i_t(B))\right)_{i \in I} \in \mathcal{U}_t(I). \tag{11}$$

Notice that if $u^i(y_t + \sum_{B \in \mathcal{B}(i)} \lambda_B z^i_t(B)) > \sum_{B \in \mathcal{B}(i)} \lambda_B u^i(\psi^i + z^i_t(B))$ for each i, then we would be done. The remainder of the proof will verify (9) by establishing an approximation of this inequality.

Lemma 3 *For each $i \in I$ and each $k \in \pi(i)$, there is some $B \in \mathcal{B}(i)$ such that:*
if y_{tk} is bounded from above then $\bar{z}_{tk}(B)$ is unboundedly increasing.

proof of Lemma 3 Fix i and let $k \in \pi(i)$. Then $proj_k \omega_t(I)$ increases unboundedly. From the way in which ν_t was constructed, it follows that ν_{tk} also increases unboundedly. Suppose that y_{tk} is bounded. Then (10) implies that $\sum_B \lambda_B \bar{z}_{tk}(B)$ increases unboundedly which, in turn, implies that for some B', $\bar{z}_{tk}(B')$ increases unboundedly. By Lemma 1, $i \notin B'$ would imply $\bar{z}_{tk}(B') = 0$. It therefore must be the case that $i \in B'$. □

Now suppose that y_t is unbounded. Given that u^i is unboundedly increasing in each good, fix t sufficiently large so that for each i, $u^i(y_t) > u^i(\psi^i)$. Then, since $\sum_{B\in\mathcal{B}(i)} \lambda_B z_t^i(B) \geq 0$, by the quasi-linearity assumption on u^i we have

$$u^i(y_t + \sum_{B\in\mathcal{B}(i)} \lambda_B z_t^i(B)) \; > \; u^i(\psi^i + \sum_{B\in\mathcal{B}(i)} \lambda_B z_t^i(B)) \; \geq \; \sum_{B\in\mathcal{B}(i)} \lambda_B u^i(\psi^i + z_t^i(B))$$

where the last inequality follows from the concavity of u^i. But since this holds for all i this means that utility vector in (11) dominates the utility vector in (9), which would conclude the proof. Therefore, without loss of generality, in the sequel we assume that y_t is bounded above. Hence, we can apply Lemma 3 to assert that for each i and each $k \in \pi(i)$, $\bar{z}_{tk}(B)$ is unboundedly increasing. We may also assume that for any $i \in B$, $z_t^i(B)$ increases without bound whenever $\bar{z}_t(B)$ does so.

Lemma 4 *Suppose that y_t is bounded above. Then there is at least one agent $i \in I$ such that the sequences $\{z_t^i(B)\}$, $B \in \mathcal{B}(i)$ satisfy the hypothesis of the Curvature Condition. Specifically, the sequences satisfy (a) and (b) of Definition 3.*

proof of Lemma 4 Suppose that the Lemma is false. Specifically, the sequences violate (b) of Definition 3 (since (a) holds by construction). Then for each i, each k, and any pair $B_1, B_2 \in \mathcal{B}(i)$,

$$\bar{z}_{tk}(B_1) = 0 \;\text{ iff }\; \bar{z}_{tk}(B_2) = 0. \tag{12}$$

Fix one such i. Since y_t is assumed bounded above, Lemma 3 implies that for each $k \in \pi(i)$, $z_{tk}^i(B)$ increases unboundedly for some $B \in \mathcal{B}(i)$ Then by the hypothesis in (12), $z_{tk}^i(B_1)$ and $z_{tk}^i(B_2)$ also increase without bound and so, by Lemma 1, $C(k) \subseteq B_1$ and $C(k) \subseteq B_2$ for every $k \in \pi(i)$.

Since $\mathcal{B}$ is minimal, the coalitions B_1 and B_2 intersect without inclusion. Therefore, there is some agent j where $j \notin B_1$ and yet $j \in B_2$. Observe that $j \notin B_1$ implies $j \notin C(k)$ for every $k \in \pi(i)$. This, in turn, implies $\pi(i) \cap \pi(j) = \emptyset$. Nevertheless, by Specialization, $\pi(j) \neq \emptyset$. Let $k' \in \pi(j)$. Again, by Lemma 3 there is some $B_3 \in \mathcal{B}(j)$ for which $z_{tk'}^j(B_3)$ increases without bound, implying $j \in C(k') \subseteq B_3$. Also, $i \notin B_3$ since otherwise, the maintained hypothesis in (12) would be violated for B_1 and B_3. Since $j \notin B_1$, it follows that $z_{tk'}^j(B_1) = 0$ and so $z_{tk'}^j(B_2) = 0$ as well. We conclude that, given $k' \in \pi(j)$ and any $k \in \pi(i)$,

$$z_{tk}^j(B_3) = z_{tk'}^j(B_2) = 0$$

while $z_{tk'}^j(B_3)$ and $z_{tk}^j(B_2)$ increase without bound.

Hence, agent j has a nonempty subcollection of $\mathcal{B}(j)$, $\{B_2, B_3\}$, in which $z_t^j(B_2)$ and $z_t^j(B_3)$ satisfy hypothesis (b) of the Curvature Condition. □

Following the notation in Lemma 4, we assume that, for an agent j, $\{z_t^j(B)\}$, $B \in \mathcal{B}(j)$ satisfies the hypothesis of the Curvature Condition. Then there is some $\bar{t}$ and $\epsilon > 0$ such that for all $t \geq \bar{t}$,

$$\sum_B \lambda_B u^j(\psi^j + z_t^j(B)) \; < \; u^j(y_t + \sum_B \lambda_B z_t^j(B)) \; - \epsilon. \tag{13}$$

Given the uniform bound ϵ, a number $\gamma > 0$ can be found close enough to 0 so that

$$\sum_B \lambda_B u^j(\psi^j + z_t^j(B)) \; < \; u^j(y_t + (1-\gamma) \sum_B \lambda_B z_t^j(B)) \tag{14}$$

for each $t \geq \bar{t}$.

Let $(\gamma^i)_{i \neq j}$ satisfy $\gamma^i > 0$ for each $i \neq j$, and $\sum_{i \neq j} \gamma_i = \gamma$. Recall that $\sum_B \lambda_B z_t^j(B)$ has unboundedly increasing components, and each u^i, $i \neq j$, is unboundedly increasing in each component. Therefore, for each i there is some t^i such that for each $t \geq t^i$,

$$\sum_B \lambda_B u^i(\psi^i + z_t^i(B)) \; < \; u^i(y_t + \sum_B \lambda_B z_t^i(B) + \gamma^i \sum_B \lambda_B z_t^j(B)). \tag{15}$$

Finally, let $\hat{t} = \max\{\bar{t}, (t^i)_{i \neq j}\}$. We conclude that (9) holds for all $t \geq \hat{t}$, which concludes the proof. □ □

References

[1] Anderson, R. (1986), "Notions of Core Convergence," Chapter 2, *Handbook of Mathematical Economics*, Amsterdam: North Holland.

[2] Billera, L. (1974), "On Games without Side Payments Arising from a General Class of Markets," *Journal of Mathematical Economics*, 1:129-39.

[3] Billera, L. and R. Bixby (1973), "A Characterization of Polyhedral Market Games," *International Journal of Game Theory*, 2:253-61.

[4] Boehm, V. (1974), "The Core of an Economy with Production," *Review of Economic Studies*, 41:429-36.

[5] Border, K. (1984), "A Core Existence Theorem for Games without Ordered Preferences," *Econometrica*, 52: 1537-42.

[6] Border, K. (1985), *Fixed Point Theorems with Applications to Economics and Game Theory*, Cambridge University Press, Cambridge.

[7] Glomm, G. and R. Lagunoff (1995), "On the Social Stability of Coalitional Property Rights Regimes," mimeo, University of Pennsylvania.

[8] Greenberg, J. (1983), "Local Public Goods with Mobility: Existence and Optimality of a General Equilibrium," *Journal of Economic Theory*, 30:17-33.

[9] Ichiichi, T. and M. Quinzii (1983), "Decentralization of the Core of a Production Economy with Increasing Returns," *International Economic Review*, 24, 397-412.

[10] Lagunoff, R. (1994) "A Simple Noncooperative Core Story," *Games and Economic Behavior*, 7: 54-61.

[11] Moldovanu, B. and E. Winter (1991), "Order Independent Equilibria," manuscript, University of Bonn.

[12] Moulin, H. and B. Peleg (1982), "The Core of Effectivity Functions and Implementation Theory," *Journal of Mathematical Economics*, 10: 115-45.

[13] Okada, A. (1991), "Noncooperative Bargaining and the Core of an N-Person Characteristic Function Game," Kyoto Institute for Economic Research, Discussion Paper #336, October.

[14] Perry, M. and P. Reny (1994), "A Noncooperative View of Coalition Formation and the Core," *Econometrica*, 62: 795-818.

[15] Scarf, H. (1967), "The Core of an N Person Game," *Econometrica*, 35:50-69.

[16] Scotchmer, S. (1985), "Profit Maximizing Clubs," *Journal of Public Economics*," 27:25-45.

[17] Shapley, L. (1973), "On Balanced Games without Side Payments," in *Mathematical Programming*, T.C. Hu and S.M. Robinson, eds., Academic Press: New York.

[18] Shapley, L. and M. Shubik (1969), "On Market Games," *Journal of Economic Theory*, 1:9-25.

[19] Wooders, M. (1978), "Equilibria, the Core, and Jurisdiction Structures in Economies with a Local Public Good," *Journal of Economic Theory*, 18:328-48.