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MISSPECIFICATION AVERSE PREFERENCES

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Abstract. We study a decision maker who approaches an uncertain decision problem by formulating a set of plausible probabilistic models of the environment but is aware that these models are only stylized and incomplete approximations. The agent is effectively facing two layers of uncertainty. Not only is the decision maker uncertain regarding what model in this set has the best fit (model ambiguity), but she is also concerned that the best-fit model itself might be a poor description of the environment (model misspecification). We develop an axiomatic foundation for a general class of preferences that capture concern toward these two layers of uncertainty and allow us to compare individuals' degrees of aversion to model misspecification and model ambiguity independently of each other.

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1. Introduction

Economic agents often employ simplified and stylized descriptions of the complex environment they face to help guide their decisions. This implies that model misspecification is a pervasive phenomenon affecting many decision problems. For example, a policymaker might have an incorrect description of how the economy would respond to a fiscal or monetary stimulus, or a company's marketing department might have a wrong assessment of how demand would react to changes in the price of a product. As a result, a growing literature studies the implications of using misspecified models in the context of decision making and strategic interaction (see Section 1.1 for a comprehensive literature review). A common assumption in this literature is that once agents have settled on using a specific statistical model of the environment, they disregard the possibility of it being misspecified and act in a fully Bayesian fashion, evaluating alternatives by their expected utility with respect to that model. However, sophisticated enough agents should realize that their model is only a simplified approximation of reality. As suggested by Hansen and Sargent (2001), an economic agent who is concerned with acting on the basis of an incorrectly formulated model should make decisions that are *robust*; that is, policies that work reasonably well across all models that are close enough to the reference model. Following this idea, the first axiomatic treatments of decision criteria featuring misspecification aversion have been proposed by Cerreia-Vioglio et al. (2020) and Lanzani (2022).

In this paper, we provide an axiomatic foundation of a general class of preferences that are averse to the possibility of misspecification. The main contribution of our model is a way of meaningfully disentangling misspecification aversion from the more commonly studied aversion to model ambiguity. We adopt a version of the Anscombe-Aumann framework where uncertainty is captured by a set of states of the world Ω, and the decision maker (henceforth, DM) needs to choose an act f that maps states of the world to outcomes. The DM does not know the true *data-generating process (DGP)* governing the environment, but she has statistical information in her possession. This is given by a set $\mathcal{M}$ of distributions over states of the world. Each *model* $m \in \mathcal{M}$ can be interpreted as an alternative plausible hypothesis regarding the DGP. Being aware that models are only imperfect and stylized descriptions of the real environment, the DM might become concerned that, in fact, no hypothesized model in $\mathcal{M}$ is an accurate approximation of the DGP; or, in other words, that the true DGP is not contained in $\mathcal{M}$. Moreover, the DM also has at her disposal a best-fit map, identifying the model that is the best approximation of the true DGP based on different state realizations.

In this framework, the DM evaluates each uncertain alternative according to the following two-step procedure. First, if the DM were told sufficient information to determine that a distribution $m \in \mathcal{M}$ is the best-fit model, she would evaluate an act f according to the following misspecification-robust criterion

$$V^m(f) = \min_{p \in \Delta(\Omega)} \{\mathbb{E}_p[u(f)] + c(p, m)\} \tag{1}$$

where u is a utility over outcomes and $c(\cdot, m)$ is an index of misspecification aversion. That is, even conditional on observing sufficient information to determine that m is the best model, the DM would not completely trust it out of misspecification concerns. Therefore, in evaluating an act f, she would also take into account other distributions p outside of $\mathcal{M}$ that are not too far apart in a probabilistic sense from m. The index $c(\cdot, m)$ captures the DM's confidence in the model m. An important special case is given by $c(\cdot, m) = \lambda R(\cdot || m)$, where R is the relative entropy and $\lambda > 0$ is a parameter of misspecification aversion. When the DM's concern for misspecification is high, λ is low, and therefore, she would give preference to acts that perform robustly well across a larger set of models around m. In the extreme case of λ approaching infinity, the

DM becomes misspecification neutral and evaluates acts according to their expected utility given m.

Second, being also uncertain about the identity of the best-fit model, the DM aggregates together the misspecification-robust evaluations:

$$V(f) = \hat{I}\left((V^m(f))_{m\in\mathcal{M}}\right) = \hat{I}\left(\min_{p\in\Delta(\Omega)} \{\mathbb{E}_p[u(f)] + c(p,\cdot)\}\right) \tag{2}$$

where $\hat{I} : \mathbb{R}^{\mathcal{M}} \to \mathbb{R}$ is a normalized, monotone, and quasiconcave aggregator capturing the DM's attitudes toward the ambiguity regarding what model is the best-fit one. Normalization allows us to interpret $\hat{I}$ as a utility certainty equivalent of the uncertain (because of model ambiguity) profile of misspecification-robust evaluations.

We illustrate how our framework distinguishes concern toward misspecification from attitudes toward model ambiguity. We show that we can rank two agents in terms of their degree of misspecification aversion by only comparing their misspecification index c (without imposing any mutual restrictions on their aggregators $\hat{I}$) and, similarly, we can rank agents in terms of their attitudes toward model ambiguity by only comparing their aggregator $\hat{I}$ (without imposing any mutual restrictions on their misspecification aversion indexes). Specifically, DM1 is more misspecification averse than DM2 if and only if $c_1(\cdot, m) \leq c_2(\cdot m)$ for all hypothesized models $m \in \mathcal{M}$. On the other hand, DM1 is more averse to model ambiguity than DM2 if and only if $\hat{I}_1 \leq \hat{I}_2$. That is, the first individual is more model ambiguity averse if she is willing to accept lower certainty equivalents than the second to eliminate the ambiguity regarding the identity of the best-fit model.

We provide an axiomatization of two important special cases of the aggregator $\hat{I}$. First, we show that if the DM confronts the uncertainty regarding the identity of the best-fit model according to the expected utility tenets, she aggregates the misspecification-robust evaluations in a Bayesian fashion:

$$V_{\phi,\mu}(f) = \int_{\mathcal{M}} \phi \left(\min_{p\in\Delta(\Omega)} \{\mathbb{E}_p[u(f)] + c(p,m)\} \right) d\mu(m) \tag{3}$$

where μ is the DM's subjective prior over the set of models $\mathcal{M}$ and ϕ captures the DM's attitudes toward model ambiguity. If the DM is neutral toward model ambiguity and shows a uniform concern for misspecification, this criterion becomes the average robust control representation axiomatized by Lanzani (2022).[1] If, instead, the DM is misspecification neutral, that is, when $c(\cdot,m)$ assigns an infinite penalization to any probability model different from m itself, this criterion becomes the well-known smooth ambiguity model of Klibanoff et al. (2005). Second, we show that if the DM is cautious and evaluates the uncertainty about the best-fit model according to a worst-case scenario approach, then the aggregator takes on a maxmin form and we obtain the criterion proposed by Cerreia-Vioglio et al. (2020):

$$V_{min}(f) = \min_{p\in\Delta(\Omega)} \left\{ \mathbb{E}_p[u(f)] + \min_{m\in\mathcal{M}} c(p,m) \right\}. \tag{4}$$

Before moving on to discussing the related literature, we provide a brief discussion of the two main axioms underpinning our decision criterion. The first, which we call *misspecification aversion*, requires that even after conditioning on the event that a given $m \in \mathcal{M}$ is the best-fit model, the DM's preferences still do not satisfy full-fledged independence but only a weaker version of it (in our case, weak c-independence). Intuitively, suppose the DM observed the information sufficient to determine that m is the best approximation in $\mathcal{M}$. If she were completely certain that the true DGP is included in $\mathcal{M}$, she would conclude as a matter of fact that m *is* the correct description of the environment and evaluate uncertain alternatives according to their expected

[1] To be precise, we would also need to impose that the conditional misspecification-robust evaluations are the multiplier preferences proposed by Hansen and Sargent (2001) and axiomatized by Strzalecki (2011).

utility given m. The fact that even after m is revealed to be the best-fit model, the DM's preferences might still feature violations of independence reflects her mistrust of the best-fit model m and, thereby, a concern for the set of hypothesized models being misspecified. The second axiom is *consistency*. It requires that the DM prefers an act f to g whenever f is preferred to g conditional on each $m \in \mathcal{M}$. This axiom connects the subjective preferences of the DM to the statistical information encoded in the set of models $\mathcal{M}$. It captures the idea that even if aware of the possibility of misspecification, the DM still puts substantive trust in the set of models. If they provide a unanimous ranking of two alternatives, after taking into account misspecification concerns, then the DM's preferences comply with that ranking.

The rest of the paper is structured as follows. Section 1.1 discusses the relevant literature. Section 2 lays out the decision framework and the notions of the hypothesized set of models and best-fit map. Section 3 introduces and discusses the axioms characterizing the misspecification averse preferences. Section 4 states and discusses the representation results. Section 5 concludes. All proofs can be found in the Appendix.

1.1. **Related Literature.**

Preferences and Sufficient Statistics.

This paper is related to the literature connecting statistical information to choice behavior (see, for example, Amarante (2009), Al-Najjar and De Castro (2014), Epstein and Seo (2010), and Klibanoff et al. (2014)). Within this class, the closest paper to ours is Cerreia-Vioglio et al. (2013). Building on their setup, we incorporate misspecification aversion in the preferences of a DM using exogenous, statistical information to inform her choices. In their case, since the DM does not care about model misspecification, preferences conditional on a model m are expected utility. Therefore, their consistency axiom requires the DM's preferences to comply whenever two acts are unanimously ranked according to their expected utility certainty equivalents:

$$\forall m \in \mathcal{M},\ \int_S f dm \succsim \int_S g dm \implies f \succsim g.$$

They show that this implies their representation only depends on the profile of expected utility evaluations $(\mathbb{E}_P[u(f)])_{P\in\mathcal{P}}$, so that preferences are represented via an aggregator of the map $m \mapsto \mathbb{E}_m[u(f)]$. In our case, however, even after observing the missing information sufficient to pin down a unique best-fit model $m \in \mathcal{M}$, the DM, out of misspecification concerns, would only trust m to be the best approximation to the DGP, but not necessarily the correct one. Therefore, her preferences conditional on m are not necessarily expected utility, but can still display a preference for robustness across models that are in a vicinity of m. In Theorem 1 we show that the representation of our class of misspecification averse preferences only depends on the profile of misspecification robust conditional evaluations $(\min_p \mathbb{E}_p[u(f)] + c(p,m))_{m\in\mathcal{M}}$, so that the representation can be expressed as a certainty equivalent $\hat{I}$ of the map $m \mapsto \min_p \mathbb{E}_p u(f) + c(p,m)$. The fact that this map is no longer linear in the models $m \in \mathcal{M}$ is the main technical difficulty that we deal with in this paper.[2] Moreover, we show that also in our case, axioms on preferences over acts can be translated into properties of the certainty equivalent $\hat{I}$ without having to resort to second-order acts.

This paper is also related to the recent axiomatization by Denti and Pomatto (2022) of identifiable smooth ambiguity preferences. Without positing an exogenous set of probabilistic models, and abstracting from misspecification concerns, they find conditions under which preferences are represented by the smooth ambiguity criterion, where the beliefs involved in the representation are identifiable; that is, they are completely orthogonal for some kernel κ. In this paper, we start with the DM having an exogenously given set of models and a best-fit map, connect the DM's preferences

[2]In this respect, this paper is also related to Mu et al. (2021). In a different context, they show that monotone additive statistics can be represented as averages of CARA certainty equivalents.

to this statistical models by consistency, but allow the DM to display aversion toward misspecification.

Decision Criteria incorporating Misspecification Concerns.

There are a few papers axiomatizing preferences that display aversion to model misspecification. Cerreia-Vioglio et al. (2013) axiomatize the criterion (4) in a two-preference setup à la Gilboa et al. (2010). The DM has an objectively rational preference that satisfies weak c-independence but is incomplete and a subjectively rational preference that is complete, but satisfies independence only on constant acts. These two preferences are linked via two axioms originated in Gilboa et al. (2010). The first is consistency. It requires that the subjectively rational preference is a completion of the objectively rational. The second is that the DM exercises caution; that is, if the objectively rational preference is not confident enough to rank an uncertain act over a deterministic one, then the deterministic act is chosen by the subjectively rational preference (when in doubt, go with the certain alternative). Moreover, the two preferences are informed by the set of probabilistic models via coherence requirements analogous to those given in this paper. They also propose a foundation of a more general aggregator of the misspecification-robust evaluations in a setup involving a two-preference family indexed by varying sets of posited models. Lanzani (2022) also adopts the view of Cerreia-Vioglio et al. (2013) by considering states of the world that describe both the realization of the payoff relevant state and the distribution over such payoff states. They assume that the DM has variational preferences and obtain the average robust criterion (3) by imposing that preferences on bets over models satisfy the sure thing principle and uncertainty neutrality (thus obtaining an affine ϕ). Moreover, they propose axioms that characterize the asymptotic behavior of the index of misspecification concern when the DM's preferences evolve in reaction to the arrival of new information. We show (Theorems 2 and 3) that the criteria introduced by Lanzani (2022) and Cerreia-Vioglio et al. (2020) both fall within the general class of misspecification averse preferences studied in this paper and represent two opposite ends of the spectrum; the average robust criterion is neutral toward model ambiguity, while the maxmin criterion displays an extreme form of model ambiguity aversion. One contribution of our paper is to allow more flexible attitudes toward model ambiguity while proposing a way to disentangle those from the degree of misspecification aversion. This is reflected in the fact that the representation parameters capturing model misspecification aversion (the index c) and model ambiguity aversion (the aggregator $\hat{I}$) are independent of each other.

Learning with Misspecified Models.

Starting with Esponda and Pouzo (2016), many papers have examined the asymptotic behavior of actions and beliefs when agents take repeated decisions in a stochastic environment of which they have a possibly incorrect or only partial understanding (see, for instance, Frick, Iijima, and Ishii, 2022; Fudenberg, Lanzani, and Strack, 2021). In all these models, agents are not concerned about misspecification and are expected utility maximizers. A key result is that misspecification is asymptotically persistent and thus matters in shaping agents' behavior and beliefs, even when agents collect many observations generated by the true data-generating process. A different strand of the literature allows agents to realize that their model is misspecified and switch to a competing alternative (see, for example, Ba (2021), Fudenberg and Lanzani (2023), and He and Libgober (2021)). The main difference to our misspecification averse preferences axiomatized is that agents act in a fully Bayesian fashion once they have selected one of the competing models on the basis of a statistical fitness test.

2. Decision Framework

We begin by describing the decision environment faced by the DM. Uncertainty is described by a state space Ω endowed with a countably generated σ-algebra $\mathcal{G}$.

Let X be the space of *consequences*, a convex subset of a linear space. The DM needs to choose *simple acts*, that is, simple functions $f : \Omega \to X$ mapping states to consequences that are measurable with respect to $\mathcal{G}$. Denote by $\mathcal{F}$ the set of all simple acts. Abusing notation, we denote by $x \in X$ also the constant act yielding consequence x in each state $\omega \in \Omega$. For each $f, f' \in \mathcal{F}$ and $\alpha \in [0,1]$, the convex combination $\alpha f + (1-\alpha)f'$ is the simple act given by:

$$(\alpha f + (1-\alpha)f')(\omega) \coloneqq \alpha f(\omega) + (1-\alpha)f'(\omega)$$

for all $\omega \in \Omega$. Given any $E \in \mathcal{G}$ and simple acts $f, g \in \mathcal{F}$, let fEg be the act taking value $f(\omega)$ if $\omega \in E$ and value $g(\omega)$ if $\omega \in \Omega \setminus E$. If $\mathcal{E}$ is a sub-σ-algebra of $\mathcal{G}$, denote by $\mathcal{F}(\mathcal{E})$ the subset of simple acts in $\mathcal{F}$ that are measurable with respect to $\mathcal{E}$.

Let $\succsim$ be a preference relation over $\mathcal{F}$. Denote by $\succ$ and $\sim$ respectively the asymmetric and symmetric part of $\succsim$. An event $E \in \mathcal{G}$ is *null* if for all acts $f, f' \in \mathcal{F}$, $f|_{\Omega\setminus E} = \tilde{f}|_{\Omega\setminus E}$ implies that $f \sim f'$.

2.1. **Probabilistic Models and Best-Fit Map.** Let $\Delta \coloneqq \Delta(\Omega, \mathcal{G})$ denote the space of countably additive probability measures on $(\Omega, \mathcal{G})$. Endow Δ with the natural σ-algebra $\mathcal{D}$ generated by the family of evaluations maps and any subset of Δ, with its relative σ-algebra.[3]

We assume that the DM is equipped with a set $\mathcal{M} \subseteq \Delta$ of probability distributions over states of the world that, given some external information, she believes are plausible descriptions of the uncertain environment she is facing. In keeping with the classical setup of Wald (1950), each model $m \in \mathcal{M}$ can be interpreted as an alternative hypothesis regarding the DGP, based on substantive motivations, like scientific theories and empirical evidence. The models in $\mathcal{M}$ are sometimes referred to in the literature (see, for example, Hansen and Sargent (2022) and Cerreia-Vioglio et al.

[3]Appendix A provides rigorous definitions of the mathematical concepts and details regarding the notation.

(2020)) as *structured*, to remark their special status in the eye of the DM as opposed to other distributions outside $\mathcal{M}$. Following Box (1976, 1979) and Cox (1995)'s idea that models are only approximations, we do not assume that the set of models includes the DGP, that is, the true probability law governing state uncertainty. Moreover, we allow for the possibility that the DM is aware of this fact, and perceives the possibility that her set of models might be *misspecified*.

Our aim in this paper is to discern between ambiguity about which probabilistic model is the best approximation to the DGP and concern about misspecification, that is, the fact that no hypothesized model is an accurate approximation of the DGP. Uncertainty about models is usually motivated in terms of "lack of information" preventing the DM from selecting the best one. Following Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2013),[4] we formalize this missing information via the idea of sufficient statistics and information (Dynkin, 1978). Specifically, we assume that the measurable space of states of the world $(\Omega, \mathcal{G})$ and the posited set of probabilistic models $\mathcal{M}$ admit a *best-fit map* $\mathfrak{q} : \Omega \to \mathcal{M}$ and a *sufficient* σ-algebra $\mathcal{A} \subseteq \mathcal{G}$ such that

(i) $\mathcal{A}$ is the σ-algebra generated by $\mathfrak{q}$,

(ii) $m(\{\omega \in \Omega : \mathfrak{q}(\omega) = m\}) = 1$ for all $m \in \mathcal{M}$.[5]

We interpret this framework as follows. Suppose that the well-specified description of the environment is given by the set of models $\mathcal{P}$ and a map $p : \omega \mapsto p^\omega \in \Delta$ such that $P(\{\omega \in \Omega : p^\omega = P\}) = 1$ for all $P \in \mathcal{P}$. The interpretation is that the realization of ω also pins down what is the true DGP p^ω, so that the σ-algebra of events $\mathcal{A}$ that makes p^ω measurable captures the sufficient information to determine

[4]See also Amarante (2009), Al-Najjar and De Castro (2014), Epstein and Seo (2010), and Klibanoff et al. (2014) for related approaches.

[5]The requirement that each model $m \in \mathcal{M}$ is selected by the best-fit map with probability one according to m is equivalent to the notion of sufficient statistics introduced by Dynkin (1978) and is related to the strong law of large numbers. In mathematics and probability, this property is what is known as complete orthogonality of the set of probability models $\mathcal{M}$. See, for example, Mauldin et al. (1983) and Weis (1984).

what is the true probability law over states. Moreover, the statement that $p^\omega = P$ with probability one according to P is a requirement that the correct description of the environment is not contradictory; that is, whenever P is the true DGP, then it is selected with probability one by the map p^ω.[6] However, the DM posits a misspecified set of models $\mathcal{P}_0$ that does not necessarily include all models in $\mathcal{P}$[7]. Now, suppose the DM observed the missing information in $\mathcal{A}$ that would be sufficient to infer $P \in \mathcal{P}$. However, since the DM has posited a misspecified set of models, the insight from Berk (1966) suggests she would select from $\mathcal{P}_0$ the closest model to P; that is, the unique minimizer $q^*(P) \in \mathcal{P}_0$ solving $\min_{q \in \mathcal{P}_0} R(q||P)$, where $R(\cdot||\cdot)$ is the relative entropy.[8] Define the function $\mathfrak{q}(\omega) \coloneqq q^*(p^\omega)$ and the set $\mathcal{M} \coloneqq \{m \in \Delta : \exists P \in \mathcal{P},\ m = q^*(P)\}$. By the measurable maximum theorem, $p \mapsto \arg\max_{q \in \mathcal{P}_0 : q \ll p} R(q||p)$ is measurable, so that $\mathfrak{q}$ is also measurable with respect to $\mathcal{A}$. Moreover, for all $m \in \mathcal{M}$ we would, indeed, have that $m\left(\{\omega : \mathfrak{q}(\omega) = m\}\right) = 1$.[9] It is in this sense that we interpret $\mathfrak{q}$ as a *best-fit map* and the information in $\mathcal{A}$ as the sufficient information to determine the best approximation of the DGP among those in $\mathcal{M}$. That is, if the DM were able to observe ω, she would infer that the model $m_\omega = \mathfrak{q}(\omega)$ is the model that closest resembles the true DGP.

Example 1 (Exchangeability): Suppose S is an underlying finite set of *contemporaneous states* and assume that at each time period $t \in \mathbb{N}$, the uncertainty is described by the realization of a state $s_t \in S$. Then, a state of the world is an infinite sequence of realizations from S, and the state space is given by the sequence space $\Omega = S^{\mathbb{N}}$. In this case, the relevant σ-algebra $\mathcal{G}$ is the one generated by all cylinders. Let Π be

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[6] We can see the analogy to the strong law of large numbers if we interpret each ω as the realization of an infinite sequence of random variables and p^ω as the limit of a consistent estimator.

[7] Assume that $\mathcal{P}_0$ is compact and convex and that for each $P \in \mathcal{P}$, there exists a model $q \in \mathcal{P}_0$ such that q is absolutely continuous with respect to P (written $q \ll P$).

[8] That is, for every $q, p \in \Delta$, $R(q||p) = \int_\Omega \ln \frac{dq}{dp} dq$ if $q \ll p$ and equal to ∞ otherwise.

[9] For each $m \in \mathcal{M}$, there exists $P \in \mathcal{P}$ such that $m = q^*(P)$, so that $\{\omega : p^\omega = P\} \subseteq \{\omega : \mathfrak{q}(\omega) = q^*(p^\omega)\}$ and, therefore, $P(\Omega \setminus \{\omega : \mathfrak{q}(\omega) = m\}) \leq P(\Omega \setminus \{\omega : p^\omega = P\}) = 0$. Since $m \ll P$, it then must be the case that $m(\Omega \setminus \{\omega : \mathfrak{q}(\omega) = m\}) = 0$.

the group of all finite permutations of $\mathbb{N}$ and consider the set $\mathcal{A}$ of all exchangeable events; that is, events $E \in \mathcal{G}$ for which $\pi^{-1}E = E$ for all permutations $\pi \in \Pi$. If we let $\mathcal{P} = \{P : \forall \pi \in \Pi,\ \forall E \in \mathcal{G},\ P(\pi^{-1}E) = P(E)\}$ be the set of all exchangeable probability measures, we know that $(\Omega, \mathcal{G}, \mathcal{P})$ is a Dynkin space with sufficient σ-algebra $\mathcal{A}$ and the set of extreme points of $\mathcal{P}$ is given by the models $P \in \mathcal{P}$ that take on 0-1 values on the sufficient σ-algebra $\mathcal{A}$. For example, if $S = \{0, 1\}$, the set of models could be given by the iid Bernoulli distributions with success parameter in between $\underline{p} < \bar{p}$; that is, $\mathcal{P}_0 = \{q = \times_{n \in \mathbb{N}} \mathbf{p} : \mathbf{p} \sim Ber(p)$ for some $p \in (\underline{p}, \bar{p}).\}$

3. Uncertainty Averse Preferences, Coherence, and Consistency

In the sequel, we fix a measurable state space $(\Omega, \mathcal{G})$ and a set of probabilistic models $\mathcal{M}$, admitting a best-fit map $\mathfrak{q}$ and sufficient σ-algebra $\mathcal{A}$ satisfying the properties outlined in the previous section.

3.1. Uncertainty Averse Preferences.

We assume that the DM's preferences of the DM satisfy the axioms of Cerreia-Vioglio et al. (2011).

Axiom 1 (Uncertainty Averse Preferences):

(i) Weak Order. *$\succsim$ is complete and transitive.*

(ii) Monotonicity. *For all $f, f' \in \mathcal{F}$, if $f(\omega) \succsim f'(\omega)$ for all $\omega \in \Omega$, then $f \succsim f'$.*

(iii) Mixture Continuity. *If $f, f', f'' \in \mathcal{F}$, the sets $\{\alpha \in [0,1] : \alpha f' + (1-\alpha) f'' \succsim f\}$ and $\{\alpha \in [0,1] : f \succsim \alpha f' + (1-\alpha) f''\}$ are both closed.*

(iv) Risk Independence. *For all $x, y, z \in X$ and $\alpha \in [0,1]$,*

$$x \succsim y \iff \alpha x + (1-\alpha) z \succsim \alpha y + (1-\alpha) z \ .$$

(v) Uncertainty Aversion. *For all $f, f' \in \mathcal{F}$ and $\alpha \in (0,1)$,*

$$f \sim f' \implies \alpha f' + (1-\alpha) f \succsim f \ .$$

(vi) Unboundedness. *There exist $x, y \in X$ such that $x \succ y$ and for all $\alpha \in (0,1)$, there are $z, z' \in X$ such that*

$$\alpha z + (1-\alpha)y \succ x \succ y \succ \alpha x + (1-\alpha)z' \ .$$

The first four requirements guarantee that the preferences are a continuous and monotone weak order satisfying independence when restricted to constant acts. Then, the theorem of Herstein and Milnor (1953) implies that the preferences are represented on X by an affine utility u. If we interpret the mixture space X as the set of simple lotteries over outcomes, these axioms imply that the DM evaluates lotteries - i.e., constant acts not involving ambiguity - according to their objective expected utility. Requirement (v) is due to Schmeidler (1989) and captures a preference for hedging against uncertainty. The last requirement is mostly for technical convenience, and it guarantees that the utility over consequences u will be unbounded above and below.

Finally, the next axiom guarantees that preferences are robust to small perturbations and guarantees the countable additivity of the subjective probabilities.

Axiom 2 (Monotone Continuity): *For all $f, f' \in \mathcal{F}$ and $x \in X$, for all $(E_n)_{n\in\mathbb{N}} \subseteq \mathcal{G}$ such that $E_1 \supseteq E_2 \supseteq \cdots$ and $\bigcap_{n\in\mathbb{N}} E_n = \emptyset$, if $f \succ f'$, then, there exists $n_0 \in \mathbb{N}$ such that $xE_{n_0}f \succ f'$.*

3.2. **Coherence and Consistency.** For each model $m \in \mathcal{M}$, denote by $E^m := \mathfrak{q}^{-1}(m) \in \mathcal{A}$ the set of states of the world for which the best-fit map would imply that m is the best approximation of the DGP. Moreover, given $p \in \Delta$, for each simple act f, define

$$\mathbb{E}_p[f] := \sum_{x\in X} xp\left(f^{-1}(x)\right)$$

be the "average" of f according to the probability model p. Notice that since f has finite image and X is convex, $\mathbb{E}_p[f] \in X$ and it is the certainty equivalent of f for an Anscombe-Aumann EU maximizer who holds belief p over the state space Ω. The following axiom captures the idea that the preferences of the DM are coherent with the statistical framework embodied by the set of models and the best-fit map.

Axiom 3 (Coherence):

(i) *For all models* $m \in \mathcal{M}$, E^m *is nonnull and* $fE^m h \succsim gE^m h$ *if and only if* $fE^m h' \succsim gE^m h'$ *for all* $f, g, h, h' \in \mathcal{F}$.

(ii) *For all* $m \in \mathcal{M}$ *and* $f, g, h \in \mathcal{F}$,

$$f = g \quad a.e.\ [m] \implies fE^m h \sim gE^m h\ .$$

(iii) *For all* $m \in \mathcal{M}$, *if* $p(E^m) = 1$ *but* $p \neq m$, *then there exist* $f \in \mathcal{F}$ *and* $x \in X$ *such that* $fE^m x \succsim x$ *but* $x \succ \mathbb{E}_p[fE^m x]$.

(iv) *For all* $x \in X$ *and* $f \in \mathcal{F}$, *the set* $\{m \in \mathcal{M} : fE^m x \succsim x\}$ *is measurable.*

Coherence requires that the DM's preferences are adapted to the statistical information implied by the hypothesized models and the best-fit map. Point *(i)* requires that for each model m, the preferences of the DM deem possible the event that m is indeed the best approximation of the true DGP.[10] Moreover, the second part of the first point requires that the DM is able to identify the event that each model is the best approximation of the DGP and make conditional assessments of the acts based on this event. In particular, this guarantees that we can define nontrivial preferences $\succsim^m$ conditional on a model $m \in \mathcal{M}$ being the best approximation to the true DGP in an unambiguous way: for all $f, g \in \mathcal{F}$,

$$f \succsim^m g \iff (\exists h \in \mathcal{F},\ fE^m h \succsim gE^m h)\ .$$

The second requirement ensures that the preferences of the DM incorporate the information provided by the best-fit map and are coherent with the selected best-fit model. If two acts are equal with probability one according to a model $m \in \mathcal{M}$,

[10] We can think of this as a parsimony requirement: if the DM thought that a model could never be the best-fit one, then she might just as well drop it altogether.

the fact that the DM pays special attention to the model when it is the best-fit one suggests that she will be indifferent between them conditional on the event that m is, indeed, the best approximation. The third point clarifies the interpretation of m as having a special status in the eyes of the DM compared to other models not in $\mathcal{M}$. In principle, any p assigning probability one to E^m is not contradicted by observing evidence E^m. However, since p is not selected by $\mathfrak{q}$, the DM's preferences display instances of "incoherence" with it. That is, the DM's preferences prefer a (possibly) uncertain alternative f to the constant act x conditional on E^m even if x is strictly preferred to the p-certainty equivalent of f.[11] Finally, the last point is a measurability requirement of preferences with respect to the sufficient σ-algebra. To summarize, coherence implies that each model $m \in \mathcal{M}$ induces a well-defined and nontrivial conditional preference $\succsim^m$ that ranks as indifferent acts that are equal with probability one according to m and changes in a measurable fashion with respect to the models.

The next axiom is key in tying together the DM's subjective preferences with the set of models and the conditional preferences they induce.

Axiom 4 (Consistency): *For all $f, f', g \in \mathcal{F}$,*

$$(\forall m \in \mathcal{M}, \quad fE^m g \succsim f'E^m g) \implies f \succsim f' \ .$$

This assumption is analogous to the consistency axiom introduced in Gilboa et al. (2010) and Cerreia-Vioglio et al. (2013). We can think of the set of models $\mathcal{M}$ as identifying an objective preference over acts. If an act f dominates act f' conditional on each model $m \in \mathcal{M}$, then f is objectively preferred by the DM to f'. Consistency requires that the subjective preferences of the DM are informed by the objective preferences.

[11] This axiom is not strictly needed to obtain the representation and only clarifies the interpretation of each m being the unique reference model for the DM after observing the event E^m. In particular, this implies that the misspecification index $c(\cdot, m)$ is uniquely minimized at m.

3.3. **Misspecification Aversion.** We next state a conditional version of the axiom characterizing the variational preferences of Maccheroni et al. (2006). That is, the preferences after conditioning on the event that the model $m \in \mathcal{M}$ is the best-fit one satisfy a weaker form of independence, weak certainty independence, but they still do not need to satisfy full-fledged independence because of misspecification concerns.

Axiom 5 (Misspecification Aversion): *For all models $m \in \mathcal{M}$, $f, f' \in \mathcal{F}$, $x, y \in X$, and $\alpha \in (0,1)$,*

$$\alpha f + (1-\alpha)x \succsim \alpha f' E^m f + (1-\alpha)x \implies \alpha f + (1-\alpha)y \succsim \alpha f' E^m f + (1-\alpha)y \ .$$

We interpret Axiom 5 as capturing the idea that the DM is aware that the set of models is possibly misspecified and is concerned about it. Recall that we interpret ambiguity as the lack of information needed to pin down a unique probability distribution over states of the world. Now, suppose the DM was able to observe sufficient information to determine that a model m is the best-fit among all those in $\mathcal{M}$. If she was completely certain that the true DGP is included in $\mathcal{M}$, she should conclude as a matter of fact that m *is* the correct description of the uncertainty about the states. In this case, there is no reason why the DM's preferences should exhibit any ambiguity aversion but should instead behave according to the subjective expected utility tenets. That violations of independence are still allowed, even after being told that m is the best-fit model, reflects the idea that the does not trust that the best-fit model m is, in fact, the true DGP, reflecting a concern for the set of models being misspecified.

To summarize, we define the preferences under analysis as a binary relations satisfying all the axioms discussed so far.

Definition 1 (Misspecification Averse Preferences): A preference relation $\succsim$ on $\mathcal{F}$ is said to be *Misspecification Averse* if it satisfies Axioms 1, 2, 3, 4, and 5.

4. Representation of Misspecification Averse Preferences

In this section we present our main representation results. As a first step, we provide a representation for the preferences $\succsim^m$ conditional on the event that the model $m \in \mathcal{M}$ is the best-fit one. The result is that each $\succsim^m$ is a variational preference (Maccheroni et al., 2006).

Proposition 1: *Suppose $(\Omega, \mathcal{G}, \mathcal{M})$ admits a best-fit map $\mathfrak{q}$ and $\succsim$ is a misspecification averse preference relation. Then, there exist an affine and surjective utility function $u : X \to \mathbb{R}$ and a convex statistical distance $c : \Delta \times \mathcal{M} \to [0, \infty]$,*[12] *such that for each $m \in \mathcal{M}$, $f, g \in \mathcal{F}$,*

$$f \succsim^m g \iff I^m(u(f)) \geq I^m(u(g))$$

where $I^m : B(\mathcal{G}) \to \mathbb{R}$ is defined as

$$I^m(\varphi) = \min_{p \in \Delta} \left\{ \int_\Omega \varphi dp + c(p, m) \right\} \tag{5}$$

for all $\varphi \in B(\mathcal{G})$ it satisfies the following properties

(i) for all $\varphi, \varphi' \in B(\mathcal{G})$, $\varphi = \varphi'$ a.e. $[m] \implies I^m(\varphi) = I^m(\varphi')$.

(ii) for all $\xi \in B_0(\mathcal{M})$ such that $0 \leq \xi \leq 1$ there exists $\varphi \in B_0(\mathcal{G})$ such that $0 \leq \varphi \leq 1$ and $\xi(m) = I^m(\varphi)$ for all $m \in \mathcal{M}$.

Moreover, u is unique up to positive affine transformations, and c is unique given u.

We can interpret Proposition 1 in terms of a robust approach to the possibility of misspecification. Suppose that the DM has observed sufficient information to determine that m is the best-fit model she has available. Because of the possibility of misspecification, in evaluating an act f conditional on this information, the DM

[12]See Appendix A for a rigorous definition of the notion of statistical distance.

forms a robust evaluation of the act f

$$V^m(f) \coloneqq I^m(u(f)) = \min_{p \in \Delta} \left\{ \int_\Omega u(f)dp + c(p,m) \right\} . \tag{6}$$

The statistical distance $c(\cdot, m)$ captures how distant in a statistical sense a distribution p is from the best-fit model m. In particular, since the DM is concerned that m might not be an accurate approximation of the true DGP, she also takes into account other models p that are not too far apart from m. The index $c(\cdot, m)$ captures exactly the DM's confidence in the best-fit model m. When $c(\cdot, m)$ is (uniformly) lower, the DM potentially takes into account a larger set of models around m in evaluating an act; this reflects a lower trust in m or, conversely, a higher aversion to misspecification. An important and tractable case is when the misspecification index takes the form $c(\cdot, m) = \lambda R(\cdot||m)$ for all hypothesized models $m \in \mathcal{M}$, where $\lambda > 0$ is a parameter of misspecification aversion. In this case, the misspecification concern is proportional to the relative divergence with respect to the best-fit model, and it is uniform across models in $\mathcal{M}$ (see Lanzani (2022)), where a higher aversion toward misspecification is captured by a lower parameter λ. We now make this intuition about the statistical distance $c(\cdot, m)$ precise by adapting to the present context the approach to comparative uncertainty aversion due to Ghirardato and Marinacci (2002). Given two preferences $\succsim_1$ and $\succsim_2$, we say that $\succsim_1$ is *more misspecification averse* than $\succsim_2$ if for all $m \in \mathcal{M}$, $f \in \mathcal{F}$ and $x \in X$,

$$fE^m x \succsim_1 x \implies fE^m x \succsim_2 x \tag{7}$$

The idea behind this notion is that also in this case, constant acts are unaffected by the possibility that the set of hypothesized models is misspecified, since they are non-stochastic and, therefore, their evaluation does not depend on the probabilistic assessment of state uncertainty. Therefore, if it is true that after conditioning on any given model $m \in \mathcal{M}$, a DM is not concerned enough about misspecification to choose a constant act over an uncertain one, a fortiori, that should also be true for a less misspecification averse DM. The following result shows that this definition agrees with the notion that the statistical distance in the representation is an index of misspecification aversion.

Proposition 2: *Suppose* $(\Omega, \mathcal{G}, \mathcal{M})$ *admits a best-fit map* $\mathfrak{q}$ *and* $\succsim_1$ *and* $\succsim_2$ *are two misspecification averse preference relations. Then,* $\succsim_1$ *is more misspecification averse than* $\succsim_2$ *if and only if* u_2 *is a positive affine transformation of* u_1 *and, after normalizing* $u_1 = u_2$*,* $c_1(\cdot, m) \leq c_2(\cdot, m)$ *for all models* $m \in \mathcal{M}$*.*

Note that each function I^m can be seen as a non-linear expectation with respect to the best-fit model $m \in \mathcal{M}$. Indeed, while it fails to be linear, it satisfies many other characteristic properties of expectations, like monotonicity, normalization and same evaluation of functions that are almost surely equal. In particular, the map $\omega \mapsto I^{\mathfrak{q}(\cdot)}$ can be understood as a non-linear common conditional expectation of I^m.

Given the representation of the conditional preferences given in Proposition 1, we are able to associate to each act $f \in \mathcal{F}$ a function $m \mapsto I(f, m) := I^m(f)$ that maps each hypothesized model m to the misspecification-robust evaluation of act f conditional on m being the best-fit model. The axiom of Consistency then implies that if $I(f, m) \geq I(g, m)$ for all $m \in \mathcal{M}$, the DM is confident that f is better than g and thus $f \succsim g$. As remarked in Cerreia-Vioglio et al. (2020), this exemplifies the special status of the hypothesized models over distributions that are not in $\mathcal{M}$. If the misspecification-robust evaluations according to each model m rank unanimously an act over another, this is sufficient for the DM to decide to pick the first one. However, in general, the set of models will not provide a unanimous robust ranking of every pair of acts.

The first main result is that the representation of misspecification averse preferences only depends on $I(f,\cdot)$; that is, there exists a monotone, continuous, and quasiconcave aggregator of these misspecification-robust evaluations that represents the preferences of the DM.

THEOREM 1: *Suppose $(\Omega, \mathcal{G}, \mathcal{M})$ admits a best-fit map $\mathfrak{q}$. The following are equivalent:*

(i) $\succsim$ is a misspecification averse preference relation,

(ii) there exist a surjective utility function $u : X \to \mathbb{R}$, a convex statistical distance $c : \Delta \times \mathcal{M} \to [0, \infty]$, a monotone, normalized, quasiconcave, and lower semicontinuous functional $\hat{I} : B(\mathcal{M}, \mathcal{D}_{\mathcal{M}}) \to \mathbb{R}$, which is continuous on $B_0(\mathcal{M}, \mathcal{D}_{\mathcal{M}})$ such that for all $f, g \in \mathcal{F}$,

$$f \succsim g \iff \hat{I}\left(I(u(f), \cdot)\right) \geq \hat{I}\left(I((u(g), \cdot)\right) \tag{8}$$

where for all $m \in \mathcal{M}$, $I(\varphi, m) = I^m(\varphi)$ defined as

$$\forall \varphi \in B(\mathcal{G}),\ I(\varphi, m) = I^m(\varphi) = \min_{p \in \Delta} \left\{ \int_\Omega \varphi dp + c(p, m) \right\}.$$

and satisfies properties (i)-(ii) stated in Proposition 1.

Moreover, u is unique up to positive affine transformations, and c and $\hat{I}$ are unique given u.

We already discussed how $c(\cdot, m)$ can be interpreted as an index of the DM's uncertainty aversion. As we now make precise, $\hat{I}$ is an ambiguity certainty equivalent capturing attitudes toward the uncertainty regarding the best-fit model. To this end, say that $\succsim_1$ is *more averse to model ambiguity* than $\succsim_2$ if for all $f \in \mathcal{F}(\mathcal{A})$ and $x \in X$,

$$f \succsim_1 x \implies f \succsim_2 x. \tag{9}$$

The intuition for this definition is that acts that are measurable with respect to the sufficient information $\mathcal{A}$ are exactly those acts that are only affected by the uncertainty regarding what is the best approximation among the set of hypothesized models but not by misspecification concerns regarding the accuracy of the models in approximating the true DGP (notice that they need to be constant on each event E^m). Therefore, the definition above states that if $\succsim_1$ is more averse to model ambiguity than $\succsim_2$ then, whenever model ambiguity considerations are not enough for DM1 to prefer the certain outcome x to the act f that is affected by ambiguity about the best-fit model, then definitely they should not be enough for the less averse DM2.

PROPOSITION 3: *Suppose* $(\Omega, \mathcal{G}, \mathcal{M})$ *admits a best-fit map* $\mathfrak{q}$ *and* $\succsim_1$ *and* $\succsim_2$ *are two misspecification averse preference relations. Then,* $\succsim_1$ *is more model ambiguity averse than* $\succsim_2$ *if and only if* u_2 *is a positive affine transformation of* u_1 *and, after normalizing* $u_1 = u_2$, $\hat{I}_1 \leq \hat{I}_2$.

Since $\hat{I}_1$ and $\hat{I}_2$ are normalized, they can be interpreted as certainty equivalents of uncertain bets on the likelihood of which model is the best-fit one. The result can then be taken as stating that $\succsim_1$ is more averse to model ambiguity than $\succsim_2$ if DM1 is willing to accept lower certainty equivalents than DM2 as compensation for uncertain bets over the likelihood of the best approximation in $\mathcal{M}$. In this sense, Proposition 3 allows us to interpret the aggregator $\hat{I}$ as incorporating the DM's attitudes toward uncertainty about the identity of the best-fit model. This result, together with Proposition 2, clarifies how representation (8) achieves a separation of attitudes regarding the ambiguity about the identity of the best-fit model and misspecification concerns. Indeed, aversion to model ambiguity is captured by the aggregator $\hat{I}$, while the statistical distance $c(\cdot, m)$ is an index of the degree of aversion to the possibility that the set of hypothesized models is misspecified.

Before proceeding to study special cases of the aggregator $\hat{I}$, we provide a partial converse showing that if preferences are represented by the criterion (8) in Theorem 1, then there exists a best-fit map $\hat{\mathfrak{q}}$ with respect to which the preferences satisfy the coherence, consistency, and misspecification aversion axioms. To state the result, we introduce the following definition. The aggregator $\hat{I}$ is *strongly monotone* if for all $\xi_1, \xi_2 \in B(\mathcal{M})$ such that $\xi_1 > \xi_2$[13], then $\hat{I}(\xi_1) > \hat{I}(\xi_2)$.

PROPOSITION 4: *Assume that* $(\Omega, \mathcal{G})$ *is a standard Borel space and that* $\mathcal{M}$ *is a measurable subset of* $\Delta(\Omega)$. *Suppose that a preference relation* $\hat{\succsim}$ *is represented by the criterion* (8) *given in Theorem 1, with* I^m *satisfying properties* (i)-(ii) *of Proposition 1 for all* $m \in \mathcal{M}$. *Then, there exists a best-fit map* $\hat{\mathfrak{q}} : \Omega \to \mathcal{M}$ *such that* $m\left(\hat{\mathfrak{q}}^{-1}(m)\right) = 1$ *for all* $m \in \mathcal{M}$ *and for all* $m \in \mathcal{M}$, *if we define* $\hat{E}^m = \hat{\mathfrak{q}}^{-1}(m)$, *then*

$$f = g \quad a.e.\ [m] \implies f\hat{E}^m h \sim g\hat{E}^m h\ .$$

for all $f, g, h \in \mathcal{F}$. *If, furthermore,* $\hat{I}$ *is strongly monotone, then* $\hat{\succsim}$ *satisfies axioms 3, 4, and 5 given the best-fit map* $\hat{\mathfrak{q}}$.

The abstract form of $\hat{I}$ in the general representation of Theorem 1 is because no behavioral assumptions regarding the independence properties of the preference relation $\succsim$ have been made other than risk independence. The next two results characterize two specific shapes of the monotone aggregator of the misspecification-robust evaluations. The first result provides a foundation for a Bayesian version of the misspecification averse preferences, where the DM forms a subjective belief capturing her uncertainty regarding the identity of the best-fit model in $\mathcal{M}$.

THEOREM 2: *Suppose* $(\Omega, \mathcal{G}, \mathcal{M})$ *admits a best-fit map* $\mathfrak{q}$. *The following are equivalent:*

[13] Recall that $\xi_1 > \xi_2$ if $\xi_1 \geq \xi_2$ and $\xi_1(m) > \xi_2(m)$ for some $m \in \mathcal{M}$.

(i) $\succsim$ *is a misspecification averse preference relation whose restriction to* $\mathcal{F}(\mathcal{A})$ *admits an expected utility representation,*[14]

(ii) there exist a surjective and affine utility function $u : X \to \mathbb{R}$*, a convex statistical distance* $c : \Delta \times \mathcal{M} \to [0, \infty]$*, a strictly increasing, continuous, and concave function* $\phi : \mathbb{R} \to \mathbb{R}$ *and a nonatomic prior* $\mu \in \Delta^\sigma(\mathcal{M}, \mathcal{D}_{\mathcal{M}})$ *such that* $\succsim$ *is represented on* $\mathcal{F}$ *by:*

$$V(f) := \int_{\mathcal{M}} \phi\left(\min_{p\in\Delta} \int_{\Omega} u(f)dp + c(p,m)\right) d\mu(m) \ . \tag{10}$$

Moreover, u *is unique up to positive affine transformations,* c *is unique given* u*,* ϕ *is unique up to positive affine transformations given* u*, and* μ *is unique.*

As before, the DM's concern for misspecification is captured by the fact that even conditioning on the information revealing that m is the best-fit model, she still takes into account other distributions that are close enough to m. In this case, the perception of uncertainty regarding the identity of the best-fit model in the absence of the information in the sufficient sigma-algebra $\mathcal{A}$ and the attitudes toward this uncertainty are captured, respectively, by the Bayesian prior μ over the set of hypothesized models and the index of uncertainty aversion ϕ. The subjective belief μ quantifies what models the DM considers more likely to be good approximations of the true DGP. The concavity of ϕ captures the negative attitude exhibited by the DM toward this ambiguity about the best-fit model. The Bayesian criterion (10) can be seen as an extension of the smooth ambiguity model of Klibanoff et al. (2005) to incorporate misspecification concerns. We can recover the smooth ambiguity model by letting the misspecification aversion index c go to infinity (except on the diagonal, where it is always 0). This is equivalent to taking a limit case where the DM is neutral to misspecification. As already remarked in the introduction, this criterion becomes

[14]That is, $\succsim$ satisfies Savage (1954)'s Axioms P2-P6 when restricted to acts measurable with respect to $\mathcal{A}$.

the average robust control criterion axiomatized by Lanzani (2022) when the DM is neutral toward the ambiguity regarding the identity of the best-fit model. This would, indeed, imply that the index ϕ is affine. The relative entropy formulation of the misspecification aversion index $c(\cdot, m) = \lambda R(\cdot||m)$ could be obtained by imposing suitable versions of the multiplier preferences axioms discussed by Strzalecki (2011).

Finally, the next theorem shows that the criterion axiomatized in Cerreia-Vioglio et al. (2020) can arise as a special case of the representation in Theorem 1 when we assume that preferences exhibit a cautious attitude with respect to the uncertainty about the best-fit model.

Axiom 6 ($\mathcal{M}$-Caution): *For all $f \in \mathcal{F}$ and $x \in X$,*

$$\exists m \in \mathcal{M},\ x \succ f E^m x \implies x \succsim f.$$

This axiom is the conceptual analogue in our framework to the caution axiom in Gilboa et al. (2010). Indeed, the set of hypothesized models induces a typically incomplete dominance relation $\succsim_{\mathcal{M}}$, where for all $f, g \in \mathcal{F}$,

$$f \succsim_{\mathcal{M}} g \iff \forall m \in \mathcal{M},\ f \succsim^m g.$$

If $f \succsim_{\mathcal{M}} g$, this means that f is better than g according to each model $m \in \mathcal{M}$ after taking into account misspecification concerns. Because the DM trusts the set of models, when $f \succsim_{\mathcal{M}} g$, the DM is confident that f is better than g. Then, Axiom 6 can be rewritten as the requirement that if $f \not\succsim_{\mathcal{M}} x$, then $x \succsim f$. The interpretation is that if the DM is not sure that the uncertain act f is better than the constant (and therefore unaffected by uncertainty considerations) act x, then she should behave cautiously and prefer the certain act over the uncertain one. We also impose the following technical axiom.

Axiom 7 ($\mathcal{M}$-Lower Semicontinuity): *For all* $x \in X$ *and* $f \in \mathcal{F}$*, the set* $\{m \in \mathcal{M} : x \succsim fE^m x\}$ *is closed.*

This axiom is a strengthening of requirement (*iii*) in the axiom of Coherence (it requires closedness and not only measurability) and it is only needed to ensure that minima are achieved in the criterion. The following result shows that $\mathcal{M}$-Caution delivers the criterion of Cerreia-Vioglio et al. (2020).

Theorem 3: *Suppose* $(\Omega, \mathcal{G}, \mathcal{M})$ *is admits a best-fit map* $\mathfrak{q}$ *and* $\mathcal{M}$ *is compact.*[15] *The following are equivalent:*

(i) $\succsim$ *is a misspecification averse preference relation satisfying Axioms 6 and 7,*

(ii) there exists a surjective utility function $u : X \to \mathbb{R}$*, a convex statistical distance* $c : \Delta \times \mathcal{M} \to [0, \infty]$ *such that* $\succsim$ *is represented on* $\mathcal{F}$ *by:*

$$V(f) = \min_{p \in \Delta} \int_{\Omega} u(f)dp + \min_{m \in \mathcal{M}} c(p, m). \tag{11}$$

Moreover, u *is unique up to positive affine transformations, and* c *is unique given* u*.*

Notice that $\min_{m \in \mathcal{M}} c(m', m) = 0$ for all models $m' \in \mathcal{M}$. Therefore, $C_{\mathcal{M}}(\cdot) := \min_{m \in \mathcal{M}} c(\cdot, m)$ can be seen as a statistical distance between probability distributions and the set of hypothesized models $\mathcal{M}$ capturing the degree of misspecification concern of the DM, when she takes a worst-case scenario approach to the uncertainty regarding what is the best-fit model.

5. Conclusion

This paper provides an axiomatic foundation of general preferences that are misspecification averse. We study a framework where the DM formulates a possibly misspecified set of models that she considers plausible descriptions of the environment. We introduce the notion of a best-fit map that identifies the most suitable

[15]As for Axiom 7, closedness of $\mathcal{M}$ is only needed to ensure that minima are achieved.

approximation of the true DGP based on (in principle) observable states. This allows us to discern between the DM's concern about the set of models being misspecified and negative attitudes toward the uncertainty about what hypothesized models are more likely to be the best description of the environment. The main result is that the DM's preferences are a monotone and quasiconcave aggregation of misspecification-robust evaluations based on each model. As we saw in the paper, this representation achieves a separation of attitudes toward model ambiguity, captured by the aggregator, and misspecification concerns, captured by the misspecification-robust conditional evaluations. Specific shapes of the aggregator can be obtained by imposing additional suitable behavioral axioms on the DM's preferences. We show that two important decision criteria recently introduced in the literature by Lanzani (2022) and Cerreia-Vioglio et al. (2020) fall within the general class of misspecification averse preferences we studied. In particular, we provide specific axioms to obtain the Bayesian aggregator and the cautious criterion from the general case.

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Appendix

Appendix A. Mathematical Preliminaries

A.1. **Basic Notions.** Given an arbitrary measurable space $(Y, \mathcal{Y})$, we denote by $\Delta(Y, \mathcal{Y})$ and $\Delta^\sigma(Y, \mathcal{Y})$ respectively the space of finitely and countably additive probability measures on $(Y, \mathcal{Y})$. Sometimes, we will omit making explicit reference to the σ-algebra whenever no ambiguities can arise. Since both these spaces can be identified with subsets of the dual space of $B_0(Y, \mathcal{Y})$, the space of $\mathcal{Y}$-measurable simple functionals mapping Y to the real line, endowed with the supnorm $||\cdot||_\infty$, we endow them with the weak* topology. We endow $\Delta^\sigma(Y, \mathcal{Y})$ with the Borel σ-algebra generated by this topology; which is the same as the natural σ-algebra $\mathcal{D}^{Y,\mathcal{Y}}$ generated by the family of evaluations maps:

$$\forall E \in \mathcal{Y}, \quad E^* : \Delta^\sigma(Y, \mathcal{Y}) \to \mathbb{R},\ p \mapsto p(E)\ .$$

and any subset $\mathcal{Q}$ of Δ^σ, with the relative σ-algebra $\mathcal{D}^{Y,\mathcal{Y}}_{\mathcal{M}} := \mathcal{D}^{Y,\mathcal{Y}} \cap \mathcal{M}$. Moreover, denote by $B(Y, \mathcal{Y})$ the set of bounded $\mathcal{Y}$-measurable functionals from Y to $\mathbb{R}$. We know that $B(Y, \mathcal{Y})$ is the supnorm closure of $B_0(Y, \mathcal{Y})$.

Given a nonempty subset $\tilde{B}$ of $B(Y, \mathcal{Y})$, a functional $\Psi : \tilde{B} \to \mathbb{R}$ is said to be a *niveloid* if for all $\varphi, \varphi', \in \tilde{B}$,

$$\Psi(\varphi) - \Psi(\varphi') \leq \sup(\varphi - \varphi')$$

A niveloid is Lipschitz continuous with respect to the supnorm. Indeed:

$$\Psi(\varphi) - \Psi(\varphi') \leq \sup(\varphi - \varphi') \leq |\sup(\varphi - \varphi')| \leq \sup|\varphi - \varphi'| = ||\varphi - \varphi'||_\infty$$

$$\Psi(\varphi') - \Psi(\varphi) \leq \sup(\varphi' - \varphi) \leq |\sup(\varphi' - \varphi)| \leq \sup|\varphi' - \varphi'| = ||\varphi - \varphi'||_\infty$$

so that $|\Psi(\varphi) - \Psi(\varphi')| \leq ||\varphi - \varphi'||_\infty$ for all $\varphi, \varphi' \in \tilde{B}$. Moreover, the functional Ψ is said to be *normalized* if $\Psi(k) = k$ for all $k \in \mathbb{R}$ such that $k \in \tilde{B}$, where we identify each real number with the constant function yielding it everywhere. Finally, the functional Ψ is said to be *monotone* if whenever $\varphi, \varphi' \in \tilde{B}$ and $\varphi \geq \varphi'$, then $\Psi(\varphi) \geq \Psi(\varphi')$.[16] We say that Ψ is *monotone continuous* if for all $\varphi, \varphi' \in \tilde{B}$ and $k \in \tilde{B}$, for all monotone sequences $(E_n)_n \in \mathcal{Y}$ such that $E_n \downarrow \emptyset$, if $\Psi(\varphi) > \Psi(\varphi')$, then there exists $n_0 \in \mathbb{N}$ such that $\Psi(k\chi_{E_{n_0}} + \varphi\chi_{E^c_{n_0}})\varphi > \Psi(\varphi')$.

We define on $B(Y, \mathcal{Y})$ the *lattice operations* $\vee$ and $\wedge$ as follows: for all $\varphi, \varphi' \in B(Y, \mathcal{Y})$, $(\varphi \vee \varphi')(\omega) = \max\{\varphi(y), \varphi'(y)\}$ and $(\varphi \wedge \varphi')(\omega) = \min\{\varphi(y), \varphi'(y)\}$ for all $y \in Y$. We say that a nonempty subset L of $\subseteq B(Y, \mathcal{Y})$ is a *lattice* if for all $\varphi, \varphi' \in L$, $\varphi \vee \varphi', \varphi \wedge \varphi' \in L$. If $(\varphi_n)_N$ is a sequence of functions in $\subseteq B(Y, \mathcal{Y})$ and $\varphi \in B(Y, \mathcal{Y})$, we write $\varphi_n \to \varphi$ to mean that $(\varphi_n)_n$ converges uniformly to φ. If we want to stress that the uniformly convergent sequence is monotone, we write $\varphi_n \nearrow \varphi$ if $\varphi_n \leq \varphi_{n+1}$ for all $n \in \mathbb{N}$ and $\varphi_n \searrow \varphi$ if $\varphi_n \geq \varphi_{n+1}$ for all $n \in \mathbb{N}$. Finally, we write $\varphi_n \uparrow \varphi$ if $\varphi_n \leq \varphi_{n+1}$ for all $n \in \mathbb{N}$ and $(\varphi_n)_n$ converges pointwise to φ and, similarly, $\varphi_n \downarrow \varphi$ if $\varphi_n \geq \varphi_{n+1}$ for all $n \in \mathbb{N}$ and $(\varphi_n)_n$ converges pointwise to φ.

A.2. **Probabilities and Statistical Distances.** We now discuss some basic mathematical notions about probabilities and statistical distances. Fix an arbitrary measurable space $(Y, \mathcal{Y})$. For any $p, q \in \Delta(Y, \mathcal{Y})$, we write $p \ll q$ to denote that p is *absolutely continuous* with respect to q. Moreover, if $q \in \Delta(Y, \mathcal{Y})$ and f and g are $\mathcal{Y}$-measurable functions mapping Y to some arbitrary set, we write $f = g$ *a.e.* $[q]$ whenever $q(\{y \in Y : f(y) = g(y)\}) = 1$. As it is standard in measure-theoretic contexts, we assume throughout the convention $0 \cdot \infty = 0$. If f is a function mapping Y to some measurable space, we denote by $\sigma(f)$ the σ-algebra generated by f.

[16]See Maccheroni et al. (2006) and Cerreia-Vioglio et al. (2014) for an in-depth discussion of niveloids and their properties.

Given a convex subset C of $\Delta(Y,\mathcal{Y})$ and an extended real valued function $\varphi : C \to \bar{\mathbb{R}}$, we denote by $\operatorname{dom}\varphi$ the effective domain of φ, that is the subset of its domain on which φ takes on finite values; that is, $\operatorname{dom}\varphi \coloneqq \{p \in C : |\varphi(p)| < \infty\}$. Moreover, we say such function φ to be *grounded* if $\inf_{p\in C}\varphi(p) = 0$. Fix a subset $\mathcal{Q} \subseteq \Delta^\sigma(Y,\mathcal{Y})$ of countably additive probability measures. A function $c : \Delta(Y,\mathcal{Y}) \times \mathcal{Q} \to [0,\infty]$ is said to be a *statistical distance* if it satisfies the following two properties:

(i) for each $q \in \mathcal{M}$, $p = q$ implies $c(p,q) = 0$,

(ii) $c(\cdot,q)$ is lower semicontinuous for all $q \in \mathcal{Q}$.

Furthermore, a statistical distance c is convex if the section $c(\cdot,q)$ is a convex function for each $q \in \mathcal{Q}$ and is said to be a *divergence* if for all $q \in \mathcal{Q}$, $p \in \operatorname{dom} c(\cdot,q)$ implies that $p \ll q$.

A.3. **Structured Spaces.** Say that the triple $(\Omega,\mathcal{G},\mathcal{M})$ is a structured space if $(\Omega,\mathcal{G})$ is a measure space, where $\mathcal{G}$ is a countably generated σ-algebra and $\mathcal{M} \subseteq \Delta(\mathcal{G}) \coloneqq \Delta(\Omega,\mathcal{G})$ is a set of models admitting a best-fit map $\mathfrak{q}$ with sufficient σ-algebra $\mathcal{A}$. Given a structured space, let $\mathcal{D} \coloneqq \mathcal{D}^{\Omega,\mathcal{G}}$ and $\mathcal{D}_\mathcal{M} \coloneqq \mathcal{D}^{\Omega,\mathcal{G}}_\mathcal{M}$ respectively the natural σ-algebra on $\Delta^\sigma(\mathcal{G})$ and the relative σ-algebra on $\mathcal{M}$. Throughout the section, fix a structured space. In particular, recall that $E^m \coloneqq \mathfrak{q}^{-1}(m)$ and $m(E^m) = 1$ for all $m \in \mathcal{M}$. Denote by Λ the set of all the events in $\mathcal{G}$ that have probability either 0 or 1 according to all models $m \in \mathcal{M}$:

$$\Lambda \coloneqq \{E \in \mathcal{G} : \forall m \in \mathcal{M}, m(E) = 1 \text{ or } m(E) = 0\}.$$

LEMMA B.1: *The σ-algebra generated by $\mathfrak{q}$ is in Λ: $\mathcal{A} = \sigma(\mathfrak{q}) \subseteq \Lambda$. In particular, $m(E) \in \{0,1\}$ for all $E \in \mathcal{A}$ and model $m \in \mathcal{M}$.*

Proof of Lemma B.1: By definition of the σ-algebra $\mathcal{D}$, $\sigma(\mathfrak{q})$ is generated by the class:

$$\mathcal{C} := \left\{\mathfrak{q}^{-1}\left(\{p \in \Delta^\sigma(\mathcal{G}) : p(E) \leq x\}\right) : x \in [0,1],\ E \in \mathcal{G}\right\} .$$

Then, take any $x \in [0,1]$ and $E \in \mathcal{G}$. We have that for any $m \in \mathcal{M}$,

$$\begin{aligned} m\left(\mathfrak{q}^{-1}\left(\{p \in \Delta^\sigma(\mathcal{G}) : p(E) \leq x\}\right)\right) &= m\left(\{\omega \in \Omega : \mathfrak{q}^\omega(E) \leq x\}\right) \\ &= m\left(\{\omega \in \Omega : \mathfrak{q}^\omega(E) \leq x\} \cap E^m\right) \\ &= \begin{cases} 1 & \text{if } m(E) \leq x \\ 0 & \text{if } m(E) > 0 \end{cases}, \end{aligned}$$

and, therefore, $\mathfrak{q}^{-1}\left(\{p \in \Delta^\sigma(\mathcal{G}) : p(E) \leq x\}\right) \in \Lambda$, showing that $\mathcal{C} \subseteq \Lambda$.

It is clear that $\Omega, \emptyset \in \Lambda$ and that if $E \in \Lambda$, then $\Omega \setminus E \in \Lambda$. Moreover, if we take $(E)_{n\in\mathbb{N}} \subseteq \Lambda$, for each $m \in \mathcal{M}$, we have either of two cases. If $m(E_n) = 0$ for all $n \in \mathbb{N}$, then:

$$m(\cup_{n\in\mathbb{N}} E_n) \leq \sum_{n\in\mathbb{N}} m(E_n) = 0 \implies m(\cup_{n\in\mathbb{N}} E_n) = 0.$$

If, instead, there exists $k \in \mathbb{N}$ such that $m(E_k) = 1$, then:

$$m(\cup_{n\in\mathbb{N}} E_n) \geq m(E_k) = 1 \implies m(\cup_{n\in\mathbb{N}} E_n) = 1.$$

It follows that $\cup_{n\in\mathbb{N}} E_n \in \Lambda$. We can, thus, conclude that Λ is a σ-algebra containing $\mathcal{M}$ and, therefore, $\sigma(\mathfrak{q}) = \sigma(\mathcal{C}) \subseteq \Lambda$. ■

Suppose that $u : X \to \mathbb{R}$ is an affine and surjective function. If $\mathcal{E}$ is a sub-σ-algebra of $\mathcal{G}$, we can define the operator $u : \mathcal{F}(\mathcal{E}) \to B_0(\mathcal{E})$ as follows: for each $f \in \mathcal{F}(\mathcal{E})$,

$$u(f)(\omega) = u(f(\omega))$$

for all $\omega \in \Omega$.

Lemma B.2: *Suppose u is affine and surjective. Then, $u : \mathcal{F}(\mathcal{E}) \to B_0(\mathcal{E})$ is an affine operator and $\{u(f) : f \in \mathcal{F}(\mathcal{E})\} = B_0(\mathcal{E})$.*

Proof: Take any $f \in \mathcal{F}(\mathcal{E})$. Then, there exists a finite, measurable partition of Ω, $(E_i)_{i=1}^k \subseteq \mathcal{E}$, and consequences $(x_i)_{i=1}^k \subseteq X$ such that $f = \sum_{i=1}^k \chi_{E_i} x_i$. Then, for all E_i and for all $\omega \in E_i$,

$$u(f)(\omega) = u(f(\omega)) = u(x_i)$$

and therefore, $u(f) = \sum_{i=1}^k \chi_{E_i} u(x_i)$. Therefore, $u(f) \in B_0(\mathcal{E})$ for all $f \in \mathcal{F}(\mathcal{E})$ so that the operator is well-defined and $\{u(f) : f \in \mathcal{F}(\mathcal{E})\} \subseteq B_0(\mathcal{E})$. Moreover, take $\alpha \in (0,1)$ and $f, f' \in \mathcal{F}(\mathcal{E})$. We have that for all $\omega \in \Omega$,

$$\begin{aligned} u(\alpha f + (1-\alpha) f')(\omega) &= u((\alpha f(\omega) + (1-\alpha) f'(\omega)) \\ &= \alpha u(f(\omega)) + (1-\alpha) u(f'(\omega)) \\ &= \alpha u(f)(\omega) + (1-\alpha) u(f')(\omega) \end{aligned}$$

proving affinity. Finally, take any $\varphi \in B_0(\mathcal{E})$. Then, there exist a finite, measurable partition of Ω, $(E_i)_{i=1}^k \subseteq \mathcal{E}$, and reals $(r_i)_{i=1}^k \subseteq \mathbb{R}$ such that $\varphi = \sum_{i=1}^k \chi_{E_i} r_i$. Since $\operatorname{Im} u = \mathbb{R}$, for each r_i we can pick $x_i \in X$ such that $r_i = u(x_i)$. Setting $f = \sum_{i=1}^k \chi_{E_i} x_i$ we can see that $\varphi = u(f)$ and $\varphi \in \mathcal{F}(\mathcal{E})$. This shows that $B_0(\mathcal{E}) \subseteq \{u(f) : f \in \mathcal{F}(\mathcal{E})\}$. ■

Appendix B. Proof of Proposition 1

We say that a binary relation $\succsim$ over $\mathcal{F}$ is *solvable* if, for each act $f \in \mathcal{F}$, there exists a constant act $x_f \in X$ such that $x_f \sim f$. We call such (possibly non-unique) act the *certainty equivalent* of f. Next, we show that a preference relation that satisfies Axiom 1 is solvable.

Lemma B.3: *Suppose that $\succsim$ is a preference relation on $\mathcal{F}$ satisfying Axiom 1. Then, $\succsim$ is solvable.*

Proof of Lemma B.3: Fix any $f \in \mathcal{F}$. Since f takes on only finitely many values, we can pick x^* and x_* in X such that for all $\omega \in \Omega$, $x^* \succsim f(\omega) \succsim x_*$. By Axiom 1.ii, this implies that $x^* \succsim f \succsim x_*$. Now, $\{\alpha \in [0,1] : \alpha x^* + (1-\alpha)x_* \succsim f\}$ and $\{\alpha \in [0,1] : f \succsim \alpha x^* + (1-\alpha)x_*\}$ are closed by mixture continuity and are non-empty, since the first one contains 1 and the second one contains 0. Moreover, by completeness of $\succsim$, their union is the whole $[0,1]$. Since the closed, unit interval is connected, such sets must have a non-empty intersection. This shows the existence of $x_f \in X$ such that $x_f \sim f$. ■

We proceed by defining the preferences conditional on a given model $m \in \mathcal{M}$ being the best-fit model and show that they inherit some properties from the unconditional preferences. Let us first recall the following axioms characterizing the variational preferences axiomatized by Maccheroni et al. (2006).

Axiom B.1 (Variational):

- Weak Certainty Independence. *For all $f, f' \in \mathcal{F}$, $x, y \in X$, and $\alpha \in (0,1)$,*

$$\alpha f + (1-\alpha)x \succsim \alpha f' + (1-\alpha)x \implies \alpha f + (1-\alpha)y \succsim \alpha f' + (1-\alpha)y \ .$$

- Uncertainty Aversion. *For all $f, f' \in \mathcal{F}$ and $\alpha \in (0,1)$,*

$$f \sim f' \implies \alpha f' + (1-\alpha)f \succsim f \ .$$

Lemma B.4: *Suppose that $(\Omega, \mathcal{G}, \mathcal{M})$ is a structured space and that the preference relation $\succsim$ satisfies Axioms 1, 2, 3, 4, and 5. For all $m \in \mathcal{M}$, define $\succsim^m$ as follows: for all $f, f' \in \mathcal{F}$,*

$$f \succsim^m f' \iff \exists g \in \mathcal{F},\ fE^m g \succsim f'E^m g.$$

Then, $\succsim^m$ is well-defined, satisfies Axiom 1, 2, and B.1 and coincides with $\succsim$ when restricted to constant acts in X.

Proof of Lemma B.4: Fix any $m \in \mathcal{M}$ and consider $\succsim^m$ as defined in Equation B.4. We show that this is a well-defined binary relation over $\mathcal{F}$. Indeed, suppose that for $f, f' \in \mathcal{F}$, there exists some $g \in \mathcal{F}$ such that $fE^m g \succsim f'E^m g$. Then, Axiom 3 implies that $fE^m h \succsim f'E^m h$ for all $h \in \mathcal{F}$. Therefore, in the following, we just fix a $g \in \mathcal{F}$ and notice that $f \succsim^m f' \iff fE^m g \succsim f'E^m g$. Moreover, note that for any $f, f', g \in \mathcal{F}$ and $\alpha \in [0,1]$, $(\alpha f + (1-\alpha)f')E^m g = \alpha(fE^m g) + (1-\alpha)(f'E^m g)$. Indeed, if $\omega \in E^m$:

$$\begin{aligned}
((\alpha f + (1-\alpha)f')E^m g)(\omega) &= (\alpha f + (1-\alpha)f')(\omega) \\
&= \alpha f(\omega) + (1-\alpha)f'(\omega) \\
&= \alpha(fE^m g)(\omega) + (1-\alpha)(f'E^m g)(\omega) \\
&= (\alpha(fE^m g) + (1-\alpha)(f'E^m g))(\omega)
\end{aligned}$$

and, if $\omega \in \Omega \setminus E^m$:

$$\begin{aligned}
((\alpha f + (1-\alpha)f')E^m g)(\omega) &= g(\omega) \\
&= \alpha g(\omega) + (1-\alpha)g(\omega) \\
&= \alpha(fE^m g)(\omega) + (1-\alpha)(f'E^m g)(\omega) \\
&= (\alpha(fE^m g) + (1-\alpha)(f'E^m g))(\omega)\ .
\end{aligned}$$

Step 1: Weak Order. Take any $f, f' \in \mathcal{F}$. Then, since $\succsim$ is complete, it follows that either $fE^m g \succsim f'E^m g$ or $f'E^m g \succsim fE^m g$. That is, either $f \succsim^m f'$ or $f' \succsim^m f$, showing that $\succsim^m$ is complete. Moreover, suppose that there are $f, f', f'' \in \mathcal{F}$ such that $f \succsim^m f'$ and $f' \succsim^m f''$. Then, $fE^m g \succsim f'E^m g$ and $f'E^m g \succsim f''E^m g$. Since $\succsim$ is transitive, it follows that $fE^m g \succsim f''E^m g$ and, therefore, that $f \succsim^m f''$. This shows that $\succsim^m$ is also transitive.

Step 2: Mixture Continuity. Take any $f, f', f'' \in \mathcal{F}$. We show that $\{\alpha \in [0,1] : \alpha f' + (1-\alpha)f'' \succsim^m f\}$ is closed. Indeed, take any $\alpha_0 \in [0,1]$ and let $g = f$:

$$\begin{aligned} &\alpha_0 \in \{\alpha \in [0,1] : \alpha f' + (1-\alpha)f'' \succsim^m f\} \\ \iff &\alpha_0 f' + (1-\alpha_0)f'' \succsim^m f \\ \iff &(\alpha_0 f' + (1-\alpha_0)f'')E^m g \succsim fE^m g \\ \iff &\alpha_0(f'E^m f) + (1-\alpha_0)(f''E^m f) \succsim f \\ \iff &\alpha_0 \in \{\alpha \in [0,1] : \alpha(f'E^m f) + (1-\alpha)(f''E^m f) \succsim f\} \end{aligned}$$

so that $\{\alpha \in [0,1] : \alpha f' + (1-\alpha)f'' \succsim^m f\} = \{\alpha \in [0,1] : \alpha(f'E^m f) + (1-\alpha)(f''E^m f) \succsim f\}$ and the latter is closed by Axiom 1. By an analogous argument, it follows that also $\{\alpha \in [0,1] : f \succsim^m \alpha f' + (1-\alpha)f''\}$ is closed. Hence, $\succsim^m$ satisfies mixture continuity.

Step 3: Weak Certainty Independence. Take any $f, f' \in \mathcal{F}$, $x, y \in X$ and $\alpha \in (0,1)$. Then,

$$\alpha f + (1-\alpha)x \succsim^m \alpha f' + (1-\alpha)x \implies [\alpha f + (1-\alpha)x]E^m g \succsim [\alpha f' + (1-\alpha)x]E^m g$$

and letting $g = \alpha f + (1-\alpha)x$ this implies that:

$$\begin{aligned} \alpha f + (1-\alpha)x &\succsim [\alpha f' + (1-\alpha)x]E^m[\alpha f + (1-\alpha)x] \\ &= \alpha f'E^m f + (1-\alpha)x. \end{aligned}$$

But, then, by Axiom 5,

$$\begin{aligned}\alpha f + (1-\alpha)y &\succsim \alpha f' E^m f + (1-\alpha)y \\ &= [\alpha f' + (1-\alpha)y] E^m [\alpha f + (1-\alpha)y],\end{aligned}$$

which, then, implies that $\alpha f + (1-\alpha)y \succsim^m \alpha f' + (1-\alpha)y$. If follows that $\succsim^m$ satisfies Weak Certainty Independence. A fortiori, it satisfies Risk Independence.

Step 4: Non-triviality. Since E^m is nonnull, there must exist $f, f', g \in \mathcal{F}$ such that $fE^m g \succ f'E^m g$. Since f and f' are finite-valued, we can pick $x, y \in X$ so that $x \succsim f(\omega)$ and $f'(\omega) \succsim y$ for all $\omega \in E^m$. But then, monotonicity implies that

$$xE^m g \succsim fE^m g \succ f'E^m g \succsim yE^m g$$

and, by transitivity, $xE^m g \succ yE^m g$ so that $x \succ^m y$. It follows that $\succsim^m$ is non-trivial.

Step 5. $\succsim^m|_X = \succsim_X$. By Axiom 1, $\succsim$ is a non-trivial weak order satisfying mixture continuity and independence when restricted to X. By Steps 1-4, the same is true for $\succsim^m$. Then, by Herstein and Milnor (1953), there exist affine functions $u, u_m : X \to \mathbb{R}$ such that u represents $\succsim|_X$ and u_m represents $\succsim^m|_X$. Moreover, since both $\succsim$ and $\succsim^m$ are non-trivial, u and u_m are non-constant. Now, take any $x, y \in X$ such that $x \succsim y$. Then, for all $\omega \in \Omega$,

$$\begin{aligned}\omega \in E^m &\implies (xE^m g)(\omega) = x \succsim y = (yE^m g)(\omega) \\ \omega \in \Omega \setminus E^m &\implies (xE^m g)(\omega) = g(\omega) = (yE^m g)(\omega)\end{aligned}$$

so that, since $\succsim$ satisfies reflexivity and monotonicity by Axiom 1, $xE^m g \succsim yE^m g$ and, therefore, $x \succsim^m y$. Thus, for all $x, y \in X$:

$$\begin{aligned} u(x) \geq u(y) &\implies x \succsim|_X y \\ &\implies x \succsim y \\ &\implies x \succsim^m y \\ &\implies x \succsim^m|_X y \\ &\implies u_m(x) \geq u_m(y) \ . \end{aligned}$$

By Corollary B.3 in Ghirardato et al. (2004), there exists $a \in \mathbb{R}_{++}$ abd $b \in \mathbb{R}$ such that $u = au_m + b$. This implies the claim.

Step 6: Monotonicity. Take $f, f' \in \mathcal{F}$ and assume that $f(\omega) \succsim^m f'(\omega)$ for all $\omega \in \Omega$. Since by Step 4, $\succsim^m|_X = \succsim_X$, it is also the case that $f(\omega) \succsim f'(\omega)$ for all $\omega \in \Omega$. Then, since $\succsim$ satisfies Axiom 1, reflexivity and monotonicity imply that $fE^m g \succsim f'E^m g$ and, therefore, $f \succsim^m f'$, proving the statement.

Step 7: Unboundedness. This follows immediately by Step 5.

Step 8. Uncertainty Aversion

Take any $f, f' \in \mathcal{F}$ and $\alpha \in (0,1)$ and suppose that $f \sim^m f'$. Then, taking $g = f$ in the definition of $\succsim^m$ and since $\succsim$ satisfies Axiom 5, we have

$$\begin{aligned} f \sim^m f' &\implies f \sim f'E^m f \\ &\implies \alpha f + (1-\alpha) f'E^m f \succsim f \\ &\implies [\alpha f + (1-\alpha) f']E^m f \succsim fE^m f \\ &\implies \alpha f + (1-\alpha) f' \succsim^m f \end{aligned}$$

showing that $\succsim^m$ satisfies Uncertainty Aversion.

Step 9: Monotone Continuity.

Take any $f, f' \in \mathcal{F}$ such that $f \succ^m f'$, $x \in X$, and $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{G}$ such that $A_1 \supseteq A_2 \supseteq \cdots$ and $\bigcap_{n\in\mathbb{N}} A_n = \emptyset$. Taking $g = f$ in the definition of $\succsim^m$, we have that $f \succ f'E^m f$. Moreover, for each $n \in \mathbb{N}$, let $E_n := A_n \cap E^m$ and observe that $E_n = A_n \cap E^m \supseteq A_{n+1} \cap E^m = E_{n+1}$ and

$$\bigcap_{n\in\mathbb{N}} E_n = \bigcap_{n\in\mathbb{N}} (A_n \cap E^m) = (\bigcap_{n\in\mathbb{N}} A_n) \cap E^m = \emptyset \cap E^m = \emptyset.$$

Since $\succsim$ satisfies Axiom 2, we can find $n_0 \in \mathbb{N}$ such that $xE_{n_0}f \succ f'E^m f$. Moreover,

$$\omega \in E_{n_0} = A_{n_0} \cap E^m \implies (xE_{n_0}f)(\omega) = x = ((xA_{n_0}f)E^m f)(\omega),$$
$$\omega \in E^m \setminus A_{n_0} \implies (xE_{n_0}f)(\omega) = f(\omega) = ((xA_{n_0}f)E^m f)(\omega),$$
$$\omega \notin E^m \implies (xE_{n_0}f)(\omega) = f(\omega) = ((xA_{n_0}f)E^m f)(\omega).$$

Therefore, $(xA_{n_0}f)E^m f = xE_{n_0}f \succ f'E^m f$ which implies that $xA_{n_0}f \succ^m f'$ as we wanted to show. ■

We are now ready to prove Proposition 1.

Proof of Proposition 1: *(i) implies (ii)* Suppose that $(\Omega, \mathcal{G}, \mathcal{M})$ is a structured space and the preference relation $\succsim$ satisfies Axioms 1, 2, 3, 4, and 5. Since $\succsim$ is a non-trivial, continuous weak order satisfying independence when restricted to constant acts, we know by Herstein and Milnor (1953) that there exists an affine and non-constant function $u : X \to \mathbb{R}$ representing $\succsim$ over X. Moreover, such u is cardinally unique. Next, we show that $\operatorname{Im} u = \mathbb{R}$. Clearly, being u affine and X convex, $\operatorname{Im} u$ must be an interval. Pick $x, y \in X$ such that $x \succsim y$ and a monotonically decreasing sequence $(\alpha_n)_n \subseteq [0, 1]$ such that $\alpha_n \to 0$. Then, by unboundedness, for each $n \in \mathbb{N}$, there exists $z_n, z'_n \in X$ such that:

$$\alpha_n z_n + (1 - \alpha_n)y \succ x \succ y \succ \alpha_n z'_n + (1 - \alpha_n)x$$

Since u represents $\succsim$ on X and is affine, this implies:

$$\alpha_n u(z_n) + (1-\alpha_n)u(y) > u(x) > u(y) > \alpha_n u(z_n') + (1-\alpha_n)u(x)$$

and, rearranging:

$$u(z_n) > \frac{u(x)-u(y)}{\alpha_n} + u(y) \quad \text{and} \quad u(z_n') < -\frac{u(x)-u(y)}{\alpha_n} + u(x)$$

for all $n \in \mathbb{N}$. Therefore, $(u(z_n))_n$ and $(u(z_n'))_n$ are sequences in $\operatorname{Im} u$, the first monotonically increasing and diverging to $+\infty$, the second monotonically decreasing and diverging to $-\infty$. This implies that $\operatorname{Im} u = \mathbb{R}$.

Now, fix $m \in \mathcal{M}$. By Lemma B.4, $\succsim^m|_X = \succsim|_X$. Therefore, $\succsim^m$ is represented by u when restricted to constant acts in X. Define the functional $I_0^m : B_0(\mathcal{G}) \to \mathbb{R}$ as follows: for each $\varphi \in B_0(\mathcal{G})$, $I^m(\varphi) = u(x_{f_\varphi})$ where $f_\varphi \in \mathcal{F}$ is chosen such that $\varphi = u(f_\varphi)$ and $x_{f_\varphi} \sim^m f_\varphi$. This functional is well-defined by Lemmas B.4 and B.2. Moreover, define $V^m(f) \coloneqq I_0^m \circ u : \mathcal{F} \to \mathbb{R}$. Again, by Lemma B.2, V^m is a well-defined functional over $\mathcal{F}$. Moreover, it represents $\succsim^m$. Indeed, for any $f, f' \in \mathcal{F}$:

$$\begin{aligned} f \succsim^m f' &\iff x_f \succsim^m x_{f'} \\ &\iff u(x_f) \geq u(x_{f'}) \\ &\iff I_0^m(u(f)) \geq I_0^m(u(f')) \\ &\iff V^m(f) \geq V^m(f')\ . \end{aligned}$$

Lemma B.5: *I_0^m is a normalized and concave niveloid.*

Proof: *Step 1: Monotonicity.* Take $\varphi, \psi \in B_0(\mathcal{G})$ and assume that $\varphi \geq \psi$. By Lemma B.2, we can find $f_\varphi, f_\psi \in \mathcal{F}$ such that $u(f_\varphi) = \varphi$ and $u(f_\psi) = \psi$. Then, for all $\omega' \in \Omega$,

$$u(f_\varphi(\omega')) = u(f_\varphi)(\omega') = \varphi(\omega') \geq \psi(\omega') = u(f_\psi)(\omega') = u(f_\psi(\omega'))$$

and, therefore, $f_\varphi(\omega) \succsim^m f_\psi(\omega)$. Then, since by Lemma B.4, $\succsim^m$ satisfies monotonicity and transitivity, $f_\varphi \succsim^m f_\psi$ and, therefore, $x_{f_\varphi} \succsim^m x_{f_\psi}$. We can, thus, conclude that

$$I_0^m(\varphi) = u(x_{f_\varphi}) \geq u(x_{f_\psi}) = I_0^m(\psi)$$

which proves the claim.

Step 2: Normalization. Take $k \in \mathbb{R}$. Since $\operatorname{Im} u = \mathbb{R}$, we can find $x^k \in X$ such that $u(x^k) = k$. Then:

$$I_0^m(k) = u(x^k) = k$$

showing that I_0^m is normalized.

Step 3: Translation Invariance. Take any $\varphi, \psi \in B_0(\mathcal{G})$ and $k, r \in \mathbb{R}$. By Lemma B.2 and surjectivity, we can find $f_\varphi, f_\psi \in \mathcal{F}$ and $x^k, x^r \in X$ such that $u(f_\varphi) = \varphi$, $u(f_\psi) = \psi$, $u(x^k) = k$, and $u(x^r) = r$. Now, for any $\alpha \in (0,1)$, since u is an affine operator, we have for each $\xi \in \{\varphi, \psi\}$, $l \in \{k, r\}$,

$$u(\alpha f_\xi + (1-\alpha)x^l) = \alpha u(f_\xi) + (1-\alpha)u(x^l) = \alpha\xi + (1-\alpha)l \ .$$

Moreover, by Lemma B.4 and the fact that $I_0^m \circ u$ represents $\succsim^m$:

$$\begin{aligned}
&I_0^m\left(\alpha\varphi + (1-\alpha)k\right) = I_0^m\left(\alpha\psi + (1-\alpha)k\right) \\
\Longrightarrow\ &I_0^m\left(u(\alpha f_\varphi + (1-\alpha)x^k)\right) = I_0^m\left(u(\alpha f_\psi + (1-\alpha)x^k)\right) \\
\Longrightarrow\ &\alpha f_\varphi + (1-\alpha)x^k \sim^m \alpha f_\psi + (1-\alpha)x^k \\
\Longrightarrow\ &\alpha f_\varphi + (1-\alpha)x^r \sim^m \alpha f_\psi + (1-\alpha)x^r \\
\Longrightarrow\ &I_0^m\left(u(\alpha f_\varphi + (1-\alpha)x^r)\right) = I_0^m\left(u(\alpha f_\psi + (1-\alpha)x^r)\right) \\
\Longrightarrow\ &I_0^m\left(\alpha\varphi + (1-\alpha)r\right) = I_0^m\left(\alpha\psi + (1-\alpha)r\right) \ .
\end{aligned}$$

Then, for any $\varphi', \psi' \in B_0(\mathcal{G})$ and $k', r' \in \mathbb{R}$, by letting $\varphi = \varphi'/\alpha$, $\psi = \psi'/\alpha$, $k = k'/(1-\alpha)$, and $r = r'/(1-\alpha)$ in the previous implication:

$$I_0^m(\varphi' + k') = I_0^m(\psi' + k') \implies I_0^m(\varphi' + r') = I_0^m(\psi' + r') \ .$$

Then, take any $\xi \in B_0(\mathcal{G})$ and $l \in \mathbb{R}$. By Step 2, I_0^m is normalized and, therefore, $I_0^m(\xi) = I_0^m(I_0^m(\xi))$. By what is shown above, this implies:

$$I_0^m(\xi + l) = I_0^m(I_0^m(\xi) + l) = I_0^m(\xi) + l$$

proving the claim.

Step 4: Quasi-concavity. Take any $\varphi, \psi \in B_0(\mathcal{G})$ such that $I_0^m(\varphi) = I_0^m(\psi)$ and $\alpha \in (0,1)$. By Lemma B.2, we can find $f_\varphi, f_\psi \in \mathcal{F}$ such that $\varphi = u(f_\varphi)$ and $\psi = u(f_\psi)$. Then:

$$V^m(f_\varphi) = I_0^m(u(f_\varphi)) = I_0^m(\varphi) = I_0^m(\psi) = I_0^m(u(f_\psi)) = V^m(f_\psi)$$

so that $f_\varphi \sim^m f_\psi$. Since $\succsim^m$ satisfies Axiom B.1, uncertainty aversion implies that

$$\alpha f_\varphi + (1-\alpha) f_\psi \succsim^m f_\psi$$

and, therefore:

$$\begin{aligned} I_0^m(\alpha\varphi + (1-\alpha)\psi) &= I_0^m(\alpha u(f_\varphi) + (1-\alpha)u(f_\psi)) \\ &= I_0^m(u(\alpha f_\varphi + (1-\alpha) f_\psi)) \\ &= V^m(\alpha f_\varphi + (1-\alpha) f_\psi) \\ &\geq V^m(f_\psi) \\ &= I_0^m(u(f_\psi)) = I_0^m(\psi) \end{aligned}$$

proving the claim.

By Steps 1-4 and Theorem 4 in Cerreia-Vioglio et al. (2014), it follows that I_0^m is a normalized and concave niveloid. □

Denote by $I^m : B(\mathcal{G}) \to \mathbb{R}$ the unique normalized and concave niveloid extending I_0^m (see Lemma 25 in Maccheroni et al. (2006)). It is clear that $V^m = I^m \circ u$ on $\mathcal{F}$. Then, by Lemma 26 in Maccheroni et al. (2006), there exists a grounded, lower semicontinuous and convex function $c^m : \Delta \to [0,1]$ such that:

$$\begin{aligned} I^m(\varphi) &= \min_{p' \in \Delta(\mathcal{G})} \left\{ \int_\Omega \varphi dp' + c^m(p') \right\} \\ c^m(p) &= \sup_{\varphi' \in B(\mathcal{G})} \left\{ I^m(\varphi') - \int_\Omega \varphi' dp \right\} \end{aligned} \tag{12}$$

for all $\varphi \in B(\mathcal{G})$ and $p \in \Delta(\mathcal{G})$. Then, define $c(\cdot, m) := c^m(\cdot)$ for all $m \in \mathcal{M}$. We have that for each $m \in \mathcal{M}$ and for each $f, f' \in \mathcal{F}$,

$$\begin{aligned} f \succsim^m f' &\iff V^m(f) \geq V^m(f') \\ &\iff I^m(u(f)) \geq I^m(u(f')) \\ &\iff \min_{p \in \Delta(\mathcal{G})} \left\{ \int_\Omega u(f) dp + c(p,m) \right\} \geq \min_{p \in \Delta(\mathcal{G})} \left\{ \int_\Omega u(f') dp + c(p,m) \right\}, \end{aligned}$$

proving the representation in (5). We only need to check that $c(\cdot, m)$ is finite only on probabilities that are absolutely continuous with respect to m. This is the content of the next lemma.

Lemma B.6: *For all $m \in \mathcal{M}$, if $p \in \operatorname{dom} c(\cdot, m)$, then $p \ll m$ and $c(p,m) = 0$ if and only if $p = m$. In particular, c is a convex divergence.*

Proof of Lemma B.6: Fix any $m \in \mathcal{M}$.

We first show that if $p \in \operatorname{dom} c(\cdot, m)$, then p is absolutely continuous with respect to m. Suppose there exists a model $m \in \mathcal{M}$ and a $\hat{p} \in \operatorname{dom} c(\cdot, m)$ that is not absolutely continuous with respect to m. We show that $\succsim$ would violate Coherence. Indeed, we can find a measurable set $E \in \mathcal{G}$ such that $m(E) = 0$ but $\hat{p}(E) > 0$. Consider the sequence of acts $(f_n)_{n\in\mathbb{N}} \subseteq \mathcal{F}$ such that for each $n \in \mathbb{N}$, $f_n = x_n E x_0$ where, since u is surjective, we can pick $x_0 \in u^{-1}(0)$ and $x_n \in u^{-1}(-n)$. Since $m(E) = 0$, $f_n = x_0$ $a.e.[m]$ for any $n \in \mathbb{N}$. Since $\hat{p} \in \operatorname{dom} c(\cdot, m)$, $c(\hat{p}, m) < \infty$, so that there exists $N \in \mathbb{N}$ large enough such that $c(\hat{p}, m) < N \cdot \hat{p}(E)$. Therefore,

$$\begin{aligned} V^m(f_N) = I^m(u(f_N)) &= \min_{p\in\Delta} \left\{ \int_\Omega u(f_N) dp + c(p, m) \right\} \\ &= \min_{p\in\Delta} \left\{ \int_E -N \ dp + c(p, m) \right\} \\ &= \min_{p\in\Delta} \{ -N \ p(E) + c(p, m) \} \\ &\leq -N \ \hat{p}(E) + c(\hat{p}, m) \\ &< 0 = u(x_0) \end{aligned}$$

showing that $x_0 \succ^m f_N$ and, therefore, $x_0 E^m x_0 \not\precsim f_N E^m x_0$. But since $x_0 = f_N$ with probability 1 according to m, this violates Coherence.

We now show that $c(p, m) = 0$ if and only if $p = m$. Let $P_0 \coloneqq \{p_0 \in \Delta(\Omega) : c(p_0, m) = 0\}$. First of all, P_0 is non-empty because $c(\cdot, m)$ is grounded. Moreover, $P_0 \subseteq \{p_0 \in \Delta(\Omega) : p_0 \ll m\}$ by what just shown above. Take $p_0 \ll m$ such that $p_0 \neq m$. Then, by Coherence there must exist $f \in \mathcal{F}$ such that $f E^m x \succsim x$, but $x \succ \int_\Omega f dp_0$. But, then,

$$\int_\Omega u(f) dp_0 + c(p_0, m) \geq \min_{p\in\Delta(\Omega)} \left\{ \int_\Omega u(f) + c(p, m) \right\} \geq u(x) > u\left(\int_\Omega u(f) dp_0 \right) = \int_\Omega u(f) dp_0$$

which implies that $c(p_0, m) > 0$. Since this holds for all $p_0 \ll m$ sich that $p_0 \neq m$, it must be the case that $\emptyset \neq P_0 \subseteq \{m\}$. That is, $c(p, m) = 0$ if and only if $p = m$.

□

As an almost immediate consequence of Lemma B.6, we show that for any $m \in \mathcal{M}$, if $\varphi, \psi \in B(\mathcal{G})$ and $\varphi = \psi$ a.e. $[m]$, then $I^m(\varphi) = I^m(\psi)$. Indeed:

$$m(\{\omega : \varphi(\omega) \neq \psi(\omega)\}) = 0 \implies \forall p \ll m,\ p(\{\omega : \varphi(\omega) \neq \psi(\omega)\}) = 0$$

and, therefore,

$$I^m(\varphi) = \min_{p \ll m} \left\{ \int_\Omega \varphi \ dp + c(p,m) \right\} = \min_{p \ll m} \left\{ \int_\Omega \psi \ dp + c(p,m) \right\} = I^m(\psi).$$

Finally, as far as uniqueness, that u is cardinally unique follows from Herstein and Milnor (1953). Moreover, the uniqueness of c given u is guaranteed by the fact that $\succsim^m$ is an unbounded variational preference and Proposition 6 in Maccheroni et al. (2006). ■

Next, we show the characterization of the comparative notion of misspecification aversion.

Proof of Proposition 2: Suppose that $\succsim_1$ and $\succsim_2$ are two misspecification averse preferences. Let (u_1, c_1) and (u_2, c_2) represent respectively $(\succsim_2^m)_{m \in \mathcal{M}}$ and $(\succsim_2^m)_{m \in \mathcal{M}}$ as in Proposition 1 and define I_1^m and I_2^m accordingly for all $m \in \mathcal{M}$. Suppose that u_2 is a positive affine transformation of u_1 and $c_1 \leq c_2$. Without loss of generality, assume that $u_1 = u_2 = u$. Fix any $m \in \mathcal{M}$ and take any $f \in \mathcal{F}$ and $x \in X$ such that $fE^m x \succsim_1^m x$. Then, $f \succsim_1^m x$ and, therefore, $I_1^m(u(f)) \geq u(x)$. Then:

$$\begin{aligned} I_2^m(u(f)) &= \min_{p \in \Delta} \left\{ \int_\Omega u(f) dp + c_2(p,m) \right\} \\ &\geq \min_{p \in \Delta} \left\{ \int_\Omega u(f) dp + c_1(p,m) \right\} \\ &= u(x) \end{aligned}$$

so that $f \succsim_2^m x$, and, therefore, $fE^m x \succsim_2 x$.

As for the other direction, note that Equation 7 and nontriviality imply that u_2 is a positive affine transformation of u_1. Without loss of generality, set $u_1 = u_2 = u$. Fix any $m \in \mathcal{M}$ and take $\varphi \in B_0(\mathcal{G})$. Let $f \in \mathcal{F}$ be such that $u(f) = \varphi$ and $x \in X$ such that $f \sim_1^m x$. Then, condition 7 implies that $f \succsim_2^m x$, so that

$$I_1^m(\varphi) = I_1^m(u(f)) = u(x) \le I_2^m(u(f)) = I_2^m(\varphi).$$

Therefore, $I_1^m(\varphi) \le I_2^m(\varphi)$ for all $\varphi \in B_0(\mathcal{G})$. Since the latter is dense in the space $B(\mathcal{G})$, we conclude that $I_1 \le I_2$. Then, using Equation (12):

$$\begin{aligned} c_1(p,m) &= \sup_{\varphi' \in B(\mathcal{G})} \left\{ I_1^m(\varphi') - \int_\Omega \varphi' dp \right\} \\ &\le \sup_{\varphi' \in B(\mathcal{G})} \left\{ I_2^m(\varphi') - \int_\Omega \varphi' dp \right\} = c_2(p,m) \end{aligned}$$

for all $p \in \Delta$. ■

We conclude this section by proving the existence of a generalized conditional expectation. The next corollary shows that we are able to find a non-linear conditional expectation given $\mathcal{A}$ that is common to all hypothesized models $m \in \mathcal{M}$.

Corollary 1: *Suppose $(\Omega, \mathcal{G}, \mathcal{M})$ admits a best-fit map and $(I^m)_{m \in \mathcal{M}}$ are defined as in (5). Then, there exists a non-linear common conditional expectation of $(I^m)_{m\in\mathcal{M}}$ given $\mathcal{A}$. This is a map $I_\mathcal{A} : B(\mathcal{G}) \to \mathbb{R}^\Omega$ such that for all $\varphi \in B(\mathcal{G})$, $I_\mathcal{A}(\varphi)$ is in $B(\mathcal{A})$, $I_\mathcal{A}(\varphi)(\omega) = I^{\mathfrak{q}(\omega)}(\varphi)$ for all $\omega \in \Omega$ and for all $A \in \mathcal{A}$ and $m \in \mathcal{M}$,*

$$I^m\left(I_\mathcal{A}(\varphi)\chi_A\right) = I^m(\varphi\chi_A).$$

Proof of Corollary 1: First, we show that for any given $\varphi \in B(\mathcal{G})$, $I^m(\varphi)$ is measurable as a function of m.

Lemma B.7: *The map $m \mapsto I^m(\varphi)$ is a $\mathcal{D}_\mathcal{M}$-measurable and bounded functional for all $\varphi \in B(\Omega, \mathcal{G})$.*

Proof of Lemma B.7: Fix $\varphi \in B(\Omega, \mathcal{G})$ arbitrarily. We first show that $m \mapsto I^m(\varphi)$ is bounded. Indeed, since φ is bounded, there exist $k, K \in \mathbb{R}$ such that $k \leq \varphi \leq K$. By Lemma B.5, for each $m \in \mathcal{M}$, I^m is normalized and monotone and, therefore,

$$k = I^m(k) \leq I^m(\varphi) \leq I^m(K) = K$$

proving boundedness. We now show that $m \mapsto I^m(\varphi)$ is also measurable. Take any real number $r \in \mathbb{R}$. We want to show that $\{m \in \mathcal{M} : I^m(\varphi) > r\}$ is a measurable set in $\mathcal{D}_{\mathcal{M}}$. Since u is surjective, take x_r such that $u(x_r) = r$. Moreover, by Lemma B.2, we can pick f_φ such that $u(f_\varphi) = \varphi$. Then, we have:

$$\begin{aligned} \{m \in \mathcal{M} : I^m(\varphi) > r\} &= \{m \in \mathcal{M} : I^m(u(f_\varphi)) > u(x_r)\} \\ &= \{m \in \mathcal{M} : f_\varphi E^m x_r \succsim x_r\} \end{aligned}$$

and the latter is measurable since $\succsim$ satisfies Coherence. This proves that $m \mapsto I^m(\varphi)$ is bounded and measurable for any $\varphi \in B(\Omega, \mathcal{G})$. ■

Denote by $\mathfrak{q}_0$ the restriction of $\mathfrak{q}$ to Ω_0. Clearly, $\mathfrak{q}_0$ is $\mathcal{A}_{\Omega_0}/\mathcal{D}_{\mathcal{M}}$, where $\mathcal{A}_{\Omega_0}$ is the relative σ-algebra $\mathcal{A} \cap \Omega_0$. Fix any $\varphi \in B(\Omega, \mathcal{G})$. Since $m \mapsto I^m(\varphi)$ is bounded and $\mathcal{D}_{\mathcal{M}}$-measurable by Lemma B.7, it follows that the composition

$$\begin{array}{rccccc} I^{\mathfrak{q}(\cdot)}(\varphi): & (\Omega_0, \mathcal{A}_{\Omega_0}) & \to & (\mathcal{M}, \mathcal{D}_{\mathcal{M}}) & \to & (\mathbb{R}, \mathcal{B}(\mathbb{R})) \\ & \omega & \mapsto & \mathfrak{q}(\omega) & \mapsto & I^{\mathfrak{q}(\omega)}(\varphi) \end{array}$$

is a $\mathcal{A}_{\Omega_0}$-measurable and bounded functional. Obtain $I_{\mathcal{A}}(\varphi)$ by extending $I^{\mathfrak{q}(\cdot)}(\varphi)$ to the whole Ω in the following way: $I_{\mathcal{A}}(\varphi)(\omega) = I^{\mathfrak{q}(\omega)}\varphi$ if $\omega \in \Omega_0$ and $I_{\mathcal{A}}(\varphi)(\omega) = 0$ if $\omega \in \Omega \setminus \Omega_0$. It is easy to see that $I_{\mathcal{A}}(\varphi) \in B(\mathcal{A})$. Moreover, take any $A \in \mathcal{A}$ and fix $m \in \mathcal{M}$ arbitrarily. We know that $m(E^m) = 1$, so that $m(\Omega \setminus E^m) = 0$, where we recall that $E^m = \{\omega \in \Omega : \mathfrak{q}(\omega) = m\}$. By Lemma B.6, we also have that if $p \in \operatorname{dom} c(\cdot, m)$, it must be the case that p is absolutely continuous with respect to

m. Then, $p(\Omega \setminus E^m) = 0$ and $p(E^m) = 1$ for all $p \in \operatorname{dom} c(\cdot, m)$. Moreover, since $A \in \Lambda$ by Lemma B.1, we have that either $m(A) = 1$ or $m(A) = 0$. In any case, this implies that for any $p \in \operatorname{dom} c(\cdot, m)$,

$$p(A \cap E^m) = p(A)p(E^m) = p(A) = m(A).$$

Then:

$$\begin{aligned}
I^m(I_{\mathcal{A}}(\varphi)\chi_A) &= \min_{p\in\Delta}\left\{\int_\Omega I_{\mathcal{A}}(\varphi)(\omega)\chi_A(\omega)\ dp(\omega) + c(p,m)\right\}\\
&= \min_{p\in\operatorname{dom} c(\cdot,m)}\left\{\int_{A\cap E^m} I^{\mathfrak{q}(\omega)}(\varphi)\ dp(\omega) + c(p,m)\right\}\\
&= \min_{p\in\operatorname{dom} c(\cdot,m)}\left\{\int_{A\cap E^m} I^{m}(\varphi)\ dp(\omega) + c(p,m)\right\}\\
&= \min_{p\in\operatorname{dom} c(\cdot,m)}\left\{I^q(\varphi)\ q(A) + c(p,m)\right\}\\
&= I^m(\varphi)\ m(A)\\
&= I^m(\varphi\chi_A).
\end{aligned}$$

The last equality follows from the fact that $m(A) \in \{0, 1\}$. Indeed, if $m(A) = 0$,

$$I^m(\varphi\chi_A) = \min_{p\in\operatorname{dom} c(\cdot,m)}\left\{\int_A \varphi dp + c(p,m)\right\} = 0 = I^m(\varphi)m(A)$$

and if $m(A) = 1$,

$$\begin{aligned}
I^m(\varphi\chi_A) &= \min_{p\in\operatorname{dom} c(\cdot,m)}\left\{\int_A \varphi dp + c(p,m)\right\}\\
&= \min_{p\in\operatorname{dom} c(\cdot,m)}\left\{\int_\Omega \varphi dp + c(p,m)\right\} = I^m(\varphi)m(A)\ .
\end{aligned}$$

■

Appendix C. Structured Functionals

Throughout the section, assume that $(\Omega, \mathcal{G})$ is a measurable space $\mathcal{M}$ is a set of models admitting a best fit map $\mathfrak{q}$ with sufficient σ-algebra $\mathcal{A}$, that I^m is given as in the representation of Proposition 1 for all models $m \in \mathcal{M}$, and that $I_{\mathcal{A}}$ is the common generalized conditional expectation of $\mathcal{M}$ given $\mathcal{A}$, which exists by Corollary 1. Notice that for each $\varphi \in B(\mathcal{G})$, we can see $I^m(\varphi)$ as a function from models to $\mathbb{R}$:

$$I(\varphi, \cdot) : \mathcal{M} \to \mathbb{R}, \quad m \mapsto I(\varphi, m) := I^m(\varphi).$$

Define the operator $T : B(\Omega, \mathcal{G}) \to \mathbb{R}^{\mathcal{M}}$ such that for all $\varphi \in B(\Omega, \mathcal{G})$,

$$T(\varphi)(m) = I(\varphi, m)$$

for all $m \in \mathcal{M}$. By Lemma B.7, we have that $\operatorname{Im} T \subseteq B(\mathcal{M}, \mathcal{D}_{\mathcal{M}})$.

Lemma B.8: *$T : B(\Omega, \mathcal{G}) \to B(\mathcal{M}, \mathcal{D}_{\mathcal{M}})$ is Lipschitz continuous of order 1 with respect to supnorm convergence and is additive and homogeneous on $B(\mathcal{A})$.*

Proof of Lemma B.8: We prove the result in three steps.

Step 1: T is Lipschitz continuous. Take a $\varphi \in B(\mathcal{G})$ and take any sequence $(\varphi_n) \subseteq B(\mathcal{G})$ such that $||\varphi - \varphi_n||_\infty \to 0$. For each $m \in \mathcal{M}$, since I^m is a niveloid and, therefore, Lipschitz continuous of order 1, we have that:

$$|T(\varphi)(m) - T(\varphi_n)(m)| = |I^m(\varphi) - I^m(\varphi_n)| \leq ||\varphi - \varphi_n||_\infty$$

and, therefore,

$$||T(\varphi) - T(\varphi_n)||_\infty = \sup_{m \in \mathcal{M}} |T(\varphi)(m) - T(\varphi_n)(m)| \leq ||\varphi - \varphi_n||_\infty \to 0.$$

Step 2: T is additive on $B(\mathcal{A})$. Take $\varphi, \psi \in B(\mathcal{A})$. Since $B_0(\mathcal{A})$ is dense in $B(\mathcal{A})$, we can find sequences $(\varphi_n)_{n\in\mathbb{N}}$, $(\psi_n)_{n\in\mathbb{N}} \subseteq B_0(\mathcal{A})$ such that $\varphi_n \to \varphi$ and $\psi_n \to \psi$ in the supnorm. Fix any $n \in \mathbb{N}$. Then, there exists a partition $(E_i^n)_{i=1}^{k_n} \subseteq \mathcal{A}$ such that $\varphi_n = \sum_{i=1}^{k_n} r_i^n \chi_{E_i^n}$ and $\psi_n = \sum_{i=1}^{k_n} \tilde{r}_i^n \chi_{E_i^n}$ for reals $(r_i^n)_{i=1}^{k_n}, (\tilde{r}_i^n)_{i=1}^{k_n} \subseteq \mathbb{R}$. By Lemma B.1, for each $m \in \mathcal{M}$, there exists a unique $j_n(m) \in \{1, \ldots, k_n\}$ such that $m(E_{j_n(m)}^n) = 1$ and $m(E_i^n) = 0$ for all $i \neq j_n(m)$. Therefore, for all $m \in \mathcal{M}$, $\varphi_n = r_{j_n(m)}^n$ and $\psi_n = \tilde{r}_{j(m)}^n$ a.e.$[m]$ and, similarly $\varphi_n + \psi_n = r_{j_n(m)}^n + \tilde{r}_{j_n(m)}^n$, so that

$$
\begin{aligned}
T(\phi_n + \psi_n)(m) &= I^m(\varphi_n + \psi_n) = I^m(r_{j_n(m)}^n + \tilde{r}_{j_n(m)}^n) \\
&= r_{j_n(m)}^n + \tilde{r}_{j_n(m)}^n \\
&= I^m(r_{j_n(m)}^n) + I^m(\tilde{r}_{j_n(m)}^n) = I^m(\varphi_n) + I^m(\psi^n) = T(\varphi_n)(m) + T(\psi_n)(m).
\end{aligned}
$$

We conclude that $T(\varphi_n + \psi_n) = T(\varphi_n) + T(\psi_n)$ for all $n \in \mathbb{N}$. Since T is continuous with respect to supnorm convergence, taking limits we conclude that $T(\varphi + \psi) = T(\varphi) + T(\psi)$.

Step 3: T is homogeneous on $B(\mathcal{A})$. Take $\varphi \in B(\mathcal{A})$ and $\kappa \in \mathbb{R}$. As before, we can find a sequence $(\varphi_n)_{n\in\mathbb{N}} \subseteq B_0(\mathcal{A})$ such that $\varphi_n \to \varphi$ in the supnorm. Notice also that $||\kappa\varphi_n - \kappa\varphi||_\infty = |\kappa|||\varphi_n - \varphi||_\infty \to 0$. Fix any n and pick a partition $(E_i^n)_{i=1}^{k_n} \subseteq A$ such that $\varphi_n = \sum_{i=1}^{k_n} r_i^n \chi_{E_i^n}$ for reals $(r_i^n)_{i=1}^{k_n} \subseteq \mathbb{R}$. For each $m \in \mathcal{M}$, Lemma B.1 implies that there exists a unique $j_n(m) \in \{1, \ldots, k_n\}$ such that $m(E_{j_n(m)}^n) = 1$ and $m(E_i^n) = 0$ for all $i \neq j_n(m)$. Therefore, for all $m \in \mathcal{M}$, $\varphi_n = r_{j_n(m)}^n$ a.e. $[m]$ and, similarly $\kappa\varphi_n = \kappa r_{j_n(m)}^n$ a.e. $[m]$, so that

$$
T(\kappa\varphi_n)(m) = I^m(\kappa\varphi_n) = I^m(\kappa r_{j_n(m)}^n) = \kappa r_{j_n(m)}^n = \kappa I^m(r_{j_n(m)}^n) = \kappa I^m(\varphi_n) = \kappa T(\varphi_n)(m)
$$

Therefore, $T(\kappa\varphi_n) = \kappa T(\varphi_n)$ for all $n \in \mathbb{N}$ and taking limits and by continuity of T we conclude that $T(\kappa\varphi) = \kappa T(\varphi)$. ■

LEMMA B.9: *Let $T(B(\mathcal{A}))$ and $T(B_0(\mathcal{A}))$ be the images through T of $B(\mathcal{A})$ and $B_0(\mathcal{A})$ respectively. Then, $T(B(\mathcal{A})) = \operatorname{Im} T$ and $T(B_0(\mathcal{A}))$ is supnorm dense in $\operatorname{Im} T$.*

Moreover, T preserves lattice operations when restricted to $B(\mathcal{A})$. In particular, $\operatorname{Im} T$ *is a lattice.*

Proof of Lemma B.9: It is clear that

$$T(B(\mathcal{A})) = \{I(\varphi, \cdot) : \varphi \in B(\Omega, \mathcal{A})\}$$
$$\subseteq \{I(\varphi, \cdot) : \varphi \in B(\Omega, \mathcal{G})\} = \operatorname{Im} T$$

since $\mathcal{A}$ is a a sub-σ-algebra of $\mathcal{G}$. As for the reverse inclusions, take any $\xi \in \operatorname{Im} T$ and let $\varphi_\xi \in B(\mathcal{G})$ be such that $\xi = T(\varphi_\xi)$. Then, by Corollary 1, $I_{\mathcal{A}}(\varphi_\xi) \in B(\mathcal{A})$ and for all $m \in \mathcal{M}$,

$$T(I_{\mathcal{A}}(\varphi_\xi))(m) = I^m\left(I_{\mathcal{A}}(\varphi_\xi)\right) = I^m(\varphi_\xi) = T(\varphi_\xi)(m) = \xi(m),$$

so that $\xi \in T(B(\mathcal{A}))$, showing that $\operatorname{Im} T \subseteq T(B(\mathcal{A}))$. Next, we show that $T(B_0(\mathcal{A}))$ is supnorm dense in $\operatorname{Im} T$. Take $\xi \in \operatorname{Im} T$ and a corresponding $\varphi_\xi \in B(\mathcal{A})$ such that $\xi = T(\varphi_\xi)$ (which exists given what shown above). Since $B_0(\mathcal{A})$ is supnorm dense in $B(\mathcal{A})$, we can find a sequence $(\varphi_n)_n \subseteq B_0(\mathcal{A})$ such that $||\varphi_n - \varphi_\xi||_\infty \to 0$. Define $\xi_n = T(\varphi_n)$ for each $n \in \mathbb{N}$ and note that $(\xi_n)_n \subseteq T(B_0(\mathcal{A}))$. We show that ξ_n converges to ξ in the supnorm. Indeed, by Lemma B.8, T is Lipschitz of order 1 and, therefore

$$||\xi - \xi_n||_\infty = ||T(\varphi) - T(\varphi_n)||_\infty \leq ||\varphi - \varphi_n||_\infty \to 0.$$

Finally, we show that T preserves lattice operations on $B(\mathcal{A})$. Indeed, pick $\varphi, \tilde{\varphi} \in B(\mathcal{A})$ arbitrarily. Since $B_0(\mathcal{A})$ is supnorm dense in $B(\mathcal{A})$, we can take sequences $(\varphi)_n, (\tilde{\varphi}_n)_n \subseteq B_0(\mathcal{A})$ such that $||\varphi - \varphi_n||_\infty, ||\tilde{\varphi} - \tilde{\varphi}_n||_\infty \to 0$. For each $n \in \mathbb{N}$, we can find a finite partition $(E_n^i)_{i=1}^k$ and reals $(r_n^i)_{i=1}^k$, $(\tilde{r}_n^i)_{i=1}^k$ such that:

$$\varphi_n = \sum_{i=1}^k \chi_{E_n^i} r_n^i, \quad \tilde{\varphi}_n = \sum_{i=1}^k \chi_{E_n^i} \tilde{r}_n^i.$$

Fix any $m \in \mathcal{M}$. By Lemma B.1, for each $n \in \mathbb{N}$, there is a unique E_n^l in the partition such that $m(E_n^l) = 1$. Therefore, $\varphi_n = r_n^l$ and $\tilde{\varphi}_n = \tilde{r}_n^l$ a.e. $[m]$, so that by Proposition 1 and normalization, $I^m(\varphi_n) = I^m(r_n^l) = r_n^l$ and $I^m(\tilde{\varphi}_n) = I^m(\tilde{r}_n^l) = \tilde{r}_n^l$ for all $n \in \mathbb{N}$. Clearly, it is also the case that $\varphi_n \vee \tilde{\varphi}_n = r_n^l \vee \tilde{r}_n^l$ a.e. $[m]$ so that $I^m(\varphi_n \vee \tilde{\varphi}_n) = I^m(r_n^l \vee \tilde{r}_n^l) = r_n^l \vee \tilde{r}_n^l$ for all $n \in \mathbb{N}$. Therefore:

$$I^m(\varphi_n \vee \tilde{\varphi}_n) = r_n^l \vee \tilde{r}_n^l = I^m(\varphi_n) \vee I^m(\tilde{\varphi}_n)$$

for all $n \in \mathbb{N}$. Since lattice operations are continuous and I^m is Lipschitz, taking limits, it follows that

$$T(\varphi \vee \tilde{\varphi})(m) = I^m(\varphi \vee \tilde{\varphi}) = I^m(\varphi) \vee I^m(\tilde{\varphi}) = T(\varphi)(m) \vee T(\tilde{\varphi})(m).$$

Since m was chosen arbitrarily, we can conclude that $T(\varphi \vee \tilde{\varphi}) = T(\varphi) \vee T(\tilde{\varphi})$. That $\operatorname{Im} T$ is a lattice follows from the fact that $\operatorname{Im} T = T(B(\mathcal{A}))$ and $T|_{B(\mathcal{A})}$ preserves lattice operations. ■

Recall that $B_0(\mathcal{D}_\mathcal{M}) \coloneqq B_0(\mathcal{M}, \mathcal{D}_\mathcal{M})$ and $B(\mathcal{D}_\mathcal{M}) \coloneqq B(\mathcal{M}, \mathcal{D}_\mathcal{M})$ are, respectively, the spaces of simple and bounded functions on the set of models $\mathcal{M}$ measurable with respect to $\mathcal{D}_\mathcal{M}$. The following result shows that these spaces can be covered by applying the operator T respectively to $B_0(\mathcal{A})$ and $B(\mathcal{A})$. Further, characteristic functions of sets in $\mathcal{D}_\mathcal{M}$ can be recovered by applying the operator T to characteristic functions of sets in $\mathcal{A}$.

Lemma B.10: $\operatorname{Im} T = B(\mathcal{M}, \mathcal{D}_\mathcal{M})$. *Moreover,* $T(B_0(\mathcal{A})) = B_0(\mathcal{M}, \mathcal{D}_\mathcal{M})$ *and* $T(\{\chi_E : E \in \mathcal{A}\}) = \{\chi_D : D \in \mathcal{D}_\mathcal{M}\}$. *Moreover, for each* $\xi \in B_0(\mathcal{M}, \mathcal{D}_\mathcal{M})$ *such that* $0 \leq \xi \leq 1$, *there is* $\varphi \in B(\mathcal{A})$ *with* $0 \leq \varphi \leq 1$ *such that* $\xi = T(\varphi)$.

Proof of Lemma B.10: We prove the results via a series of steps.

Step (i). For all $E \in \mathcal{A}$, *there exists* $D_E \in \mathcal{D}_\mathcal{M}$ *such that* $T(\chi_E) = \chi_{D_E}$.

Proof: Take any $E \in \mathcal{A}$. By Lemma B.1, $E \in \Lambda$ and, thererfore, for all $m \in \mathcal{M}$, either $m(E) = 1$ or $m(E) = 0$. But then for all $m \in \mathcal{M}$:

$$m(E) = 1 \implies \chi_E = 1 \text{ a.e. } [m] \implies T(\chi_E)(m) = I^m(\chi_E) = I^m(1) = 1,$$

$$m(E) = 0 \implies \chi_E = 0 \text{ a.e. } [m] \implies T(\chi_E)(m) = I^m(\chi_E) = I^m(0) = 0.$$

Therefore, $\operatorname{Im} T(\chi_E) \in \{0, 1\}$. Moreover, by Lemma B.7, $D_E := [T(\chi_E)]^{-1}(\{1\}) \in \mathcal{D}_\mathcal{M}$ and $T(\chi_E) = \chi_{D_E}$ as we wanted to show. □

Step (ii). For all $D \in \mathcal{D}_\mathcal{M}$, there exists $E^D \in \mathcal{A}$ such that $T(\chi_{E^D}) = \chi_D$.

Proof: Take any $D \in \mathcal{D}_\mathcal{M}$ and let $E^D = \mathfrak{q}^{-1}(D)$. Since the space is structured, $E^D \in \mathcal{A}$ and $m(E^D) = 1$ if $m \in D$ and $m(E^D) = 0$ if $m \in \mathcal{M} \setminus D$. But then for all $m \in \mathcal{M}$:

$$m \in D \implies \chi_{E^D} = 1 \text{ a.e. } [m] \implies T(\chi_{E^D})(m) = I^m(\chi_{E^D}) = I^m(1) = 1,$$

$$m(E) \in \mathcal{M} \setminus D \implies \chi_{E^D} = 0 \text{ a.e. } [m] \implies T(\chi_{E^D})(m) = I^m(\chi_{E^D}) = I^m(0) = 0,$$

and we can, thus, conclude that $T(\chi_{E^D}) = \chi_D$. □

Steps (i) and (ii) together imply that $T(\{\chi_E : E \in \mathcal{A}\}) = \{\chi_D : D \in \mathcal{D}_\mathcal{M}\}$.

Step (iii). $T(B_0(\mathcal{A})) \subseteq B_0(\mathcal{M}, \mathcal{D}_\mathcal{M})$.

Proof: Take $\xi \in T(B_0(\mathcal{A}))$. By definition, there exists $\varphi_\xi \in B_0(\mathcal{A})$ such that $\xi = T(\varphi_\xi)$. Then, there exists a partition $(E_i)_{i=1}^k \subseteq \mathcal{A}$ and reals $(r_i)_{i=1}^k$ such that $\varphi_\xi = \sum_{i=1}^k \chi_{E_i} r_i$. By Step ($i$), we have that for each $i = 1, \ldots, k$, we can find $D_{E_i} \in \mathcal{D}_\mathcal{M}$ such that $T(\chi_{E_i}) = \chi_{D_{E_i}}$. Moreover, since for all $i = 1, \ldots, k$, $E_i \in \mathcal{A} \subseteq \Lambda$ by Lemma B.1, either $m(E_i) = 1$ or $m(E_i) = 0$ for each $m \in \mathcal{M}$. It follows that for each m, there is a unique element in the partition E_{j_m} such that $m(E_{j_m}) = 1$ and

$m(E_i) = 0$ if $i \neq j_m$. Then, for each $m \in \mathcal{M}$,

$$\varphi_\xi = r_{j_m} \text{ a.e. } [m] \implies T(\varphi_\xi)(m) = I^m(\varphi_\xi) = I^m(r_{j_m}) = r_{j_m}$$

and, since $\chi_{E_{j_m}} = 1$ a.e. $[m]$ and $\chi_{E_i} = 0$ a.e. $[m]$ for $i \neq j_m$,

$$\chi_{D_{E_{j_m}}}(m) = T(\chi_{E_{j_m}})(m) = I^m(\chi_{E_{j_m}}) = I^m(1) = 1 \implies m \in D_{E_{j_m}}$$
$$\forall i \neq j_m, \quad \chi_{D_{E_i}}(m) = T(\chi_{E_i})(m) = I^m(\chi_{E_i}) = I^m(0) = 0 \implies m \notin D_{E_i}.$$

It follows that $\varphi_\xi = \sum_{i=1}^k \chi_{D_{E_i}} r_i \in B_0(\mathcal{M}, \mathcal{D}_\mathcal{M})$. □

Step (iv). $B_0(\mathcal{M}, \mathcal{D}_\mathcal{M}) \subseteq T(B_0(\mathcal{A}))$, In particular, for all $D \in \mathcal{D}_\mathcal{M}$, there exists $E^D \in \mathcal{A}$ such that $\chi_D = T(\chi_{E^D})$.

Proof: Take any $\xi \in B_0(\mathcal{M}, \mathcal{D}_\mathcal{M})$. By definition, there exists a partition $(D_i)_{i=1}^k \subseteq \mathcal{D}_\mathcal{M}$ of $\mathcal{M}$ and reals $(r_i)_{i=1}^k$ such that $\xi = \sum_{i=1}^k \chi_{D_i} r_i$. By Step (ii), for each $i = 1, \ldots, k$, we can find $E^{D_i} \in \mathcal{A}$ such that $\chi_{D_i} = T(\chi_{E^{D_i}})$. Define $\varphi_\xi := \sum_{i=1}^k \chi_{E^{D_i}} r_i$. Clearly, $\varphi_\xi \in B_0(\mathcal{A})$. Moreover, for each $m \in \mathcal{M}$, let D_{j_m} be the unique element of the partition such that $m \in D_{j_m}$. We know by Lemma B.1 that since $E^{D_{j_m}} \in \mathcal{A}$, $m(E^{D_{j_m}}) \in \{0, 1\}$. If $m(E^{D_{j_m}}) = 0$, then $\chi_{E^{D_{j_m}}} = 0$ a.e. $[m]$ and, therefore, $T(E^{D_{j_m}})(m) = I^m(E^{D_{j_m}}) = I^m(0) = 0 \neq \chi_{D_{j_m}}(m) = 1$, a contradiction. We conclude that $m(E^{D_{j_m}}) = 1$ so that $\varphi_\xi = r_{j_m}$ a.e. $[m]$. Therefore,

$$T(\varphi_\xi)(m) = I^m(\varphi_\xi) = I^m(r_{j_m}) = r_{j_m} = r_{j_m}\chi_{D_{j_m}}(m) = \xi(m).$$

for all $m \in \mathcal{M}$. It follows that $T(\varphi_\xi) = \xi$, showing that $B_0(\mathcal{M}, \mathcal{D}_\mathcal{M}) \subseteq T(B_0(\mathcal{A}))$. □

Step (iii) and (iv) imply that $B_0(\mathcal{M}, \mathcal{D}_\mathcal{M}) = T(B_0(\mathcal{A}))$. Then, we have the following chain of inclusions:

$$B_0(\mathcal{M}, \mathcal{D}_\mathcal{M}) \subseteq T(B_0(\mathcal{A})) \subseteq B(\mathcal{M}, \mathcal{D}_\mathcal{M}).$$

Moreover, $B_0(\mathcal{M}, \mathcal{D}_{\mathcal{M}})$ is supnorm dense in $B(\mathcal{M}, \mathcal{D}_{\mathcal{M}})$ and by Lemma B.9, $T(B_0(\mathcal{A}))$ is supnorm dense in $\operatorname{Im} T$. Taking the supnorm closure of the previous chain of inclusions, we obtain that:

$$B(\mathcal{M}, \mathcal{D}_{\mathcal{M}}) = \operatorname{cl} B_0(\mathcal{M}, \mathcal{D}_{\mathcal{M}}) \subseteq \operatorname{cl} T(B_0(\mathcal{A})) = \operatorname{Im} T \subseteq \operatorname{cl} B(\mathcal{M}, \mathcal{D}_{\mathcal{M}}) = B(\mathcal{M}, \mathcal{D}_{\mathcal{M}})$$

and, therefore, we can conclude that $\operatorname{Im} T = B(\mathcal{M}, \mathcal{D}_{\mathcal{M}})$.

The last part of the result follows by steps *iii* and *iv* and by Lemma B.8. ■

LEMMA B.11:

(i) *If* $\xi, \xi' \in B_0(\mathcal{M}, \mathcal{D}_{\mathcal{M}})$ *are such that* $\xi \geq \xi'$, *then there exist* $\varphi_\xi, \varphi_{\xi'} \in B_0(\mathcal{A})$ *such that* $\varphi_\xi \geq \varphi_{\xi'}$ *and* $\xi = T(\varphi_\xi)$, $\xi' = T(\varphi_{\xi'})$.

(ii) *If* $(\xi_n)_n \subseteq T(B_0(\mathcal{A}))$ *is an increasing (decreasing) sequence uniformly bounded above (below) by a constant* K, *there exists an increasing (decreasing) sequence* $(\varphi_n)_n \subseteq B_0(\mathcal{A})$ *such that* $\xi_n = T(\varphi_n)$ *and* $\varphi_n \leq K$ $(\varphi_n \geq K)$ *for all* $n \in \mathbb{N}$.

(iii) *If* $\xi \in \operatorname{Im} T$ *and* $(\xi_n)_n \subseteq T(B_0(\mathcal{A}))$ *such that* $\xi_n \uparrow \xi$ $(\xi_n \downarrow \xi)$, *then we can find an increasing (decreasing) sequence* $(\varphi_n)_n \subseteq B_0(\mathcal{A})$ *and* $\varphi \in B(\mathcal{A})$ *such that* $\varphi_n \uparrow \varphi$ $(\varphi_n \downarrow \varphi)$, $\xi = T(\varphi)$, *and* $\xi_n = T(\varphi_n)$ *for all* $n \in \mathbb{N}$. *Moreover, if* $K \in \mathbb{R}$ *and* $\xi \leq K$ $(\xi \geq K)$, *then* $\varphi \leq K$ $(\varphi \geq K)$.

(iv) *If* $\xi, \xi' \in \operatorname{Im} T$ *are such that* $\xi \geq \xi'$, *then there exist* $\varphi_\xi, \varphi_{\xi'} \in B(\mathcal{A})$ *such that* $\varphi_\xi \geq \varphi_{\xi'}$ *and* $\xi = T(\varphi_\xi)$, $\xi' = T(\varphi_{\xi'})$.

PROOF OF LEMMA B.11: We prove the lemma in a number of steps.

PROOF OF (i): Take $\xi, \xi' \in T(B_0(\mathcal{A}))$ such that $\xi \geq \xi'$. By definition, we can pick $\varphi_\xi, \varphi_{\xi'} \in B_0(\mathcal{A})$ such that $\xi = T(\varphi_\xi)$ and $\xi' = T(\varphi_{\xi'})$. Moreover, we can find a partition $(E_i)_{i=1}^n \subseteq \mathcal{A}$ of Ω and reals $(r_i)_{i=1}^n, (r_i')_{i=1}^n$ such that

$$\varphi_\xi = \sum_{i=1}^n \chi_{E_i} r_i, \quad \varphi_{\xi'} = \sum_{i=1}^n \chi_{E_i} r_i'.$$

Take an element E_k in the partition. If $m(E_k) = 0$ for all $m \in \mathcal{M}$, we can assume wlog that $r_k = r'_k$. Indeed, for all $m \in \mathcal{M}$, $\varphi_{\xi'} = \sum_{i \neq k} \chi_{E_i} r'_i + \chi_{E_k} r_k$ a.e. $[m]$ and by Proposition 1, this implies

$$\xi' = T(\varphi_{\xi'})(m) = I^m(\varphi_{\xi'}) = I^m(\sum_{i \neq k} \chi_{E_i} r'_i + \chi_{E_k} r_k) = T(\sum_{i \neq k} \chi_{E_i} r'_i + \chi_{E_k} r_k)(m).$$

If there exists $m \in \mathcal{M}$ such that $m(E_k) \neq 0$, then $m(E_k) = 1$ since $E_k \in \mathcal{A} \subseteq \Lambda$ by Lemma B.1. Therefore, $\varphi_\xi = r_k$ and $\varphi_{\xi'} = r'_k$ a.e. $[m]$ and, therefore:

$$r_k = I^m(r_k) = I^m(\varphi_\xi) = T(\varphi_\xi)(m) = \xi(m),$$

$$r'_k = I^m(r'_k) = I^m(\varphi_{\xi'}) = T(\varphi_{\xi'})(m) = \xi'(m),$$

and, we conclude that $r_k = \xi(m) \geq \xi'(m) = r'_k$. We have thus shown that $r_i \geq r'_i$ for all $i = 1, \ldots, n$. Hence, it follows that $\varphi_\xi \geq \varphi_{\xi'}$. It is then immediate to see that since each I^m is normalized, if $\xi \leq K$ for some K in $\mathbb{R}$, we can find $\varphi_\xi \in B_0(\mathcal{A})$ such that $\varphi_\xi \leq K$ and $\xi = T(\varphi_\xi)$. □

Proof of (ii): Take a sequence $(\xi_n)_n \subseteq T(B_0(\mathcal{A}))$ and $K \in \mathbb{R}$ such that $\xi_n \leq \xi_{n+1} \leq K$ for all $n \in \mathbb{N}$. By Step (i), we can find a sequence $\varphi_{\xi_n} \in B_0(\mathcal{A})$ such that $\xi_n = T(\varphi_{\xi_n})$ and $\varphi_{\xi_n} \leq K$ for all $n \in \mathbb{N}$. However, this sequence is not necessarily increasing. Then, define for each $n \in \mathbb{N}$, $\varphi_n(\omega) = \sup_{k \leq n} \varphi_{\xi_k}(\omega)$ for all ω. Notice that $\varphi_n : \Omega \to \mathbb{R}$ is well-defined and in $B_0(\mathcal{A})$. Moreover, the sequence $(\varphi_n)_n$ so constructed is increasing and uniformly bounded above by K. Moreover, since T preserves lattice operations by Lemma B.9, we have that for each $n \in \mathbb{N}$,

$$T(\varphi_n) = T\left(\sup_{k \leq n} \varphi_{\xi_k}\right) = \sup_{k \leq n} T\left(\varphi_{\xi_k}\right) = \sup_{k \leq n} \xi_k = \xi_n,$$

where the last equality follows from the fact that $(\xi_n)_n$ is a monotonically increasing sequence. □

Proof of (iii): Take a sequence $(\xi_n)_n \subseteq T(B_0(\mathcal{A}))$ and $\xi \in \operatorname{Im} T$ such that $\xi_n \uparrow \xi$. Since ξ is bounded, $K_0 = \sup_{m \in \Omega} \xi$ is finite. Moreover, we have that $\xi_n \leq \xi \leq K_0$ for all $n \in \mathbb{N}$. By by point (ii), we can find an increasing sequence $(\varphi_n)_n \subseteq B_0(\mathcal{A})$ such that $\xi_n = T(\varphi_n)$ and $\varphi_n \leq K_0$ for all $n \in \mathbb{N}$. Since for each $\omega \in \Omega$, $(\varphi_n(\omega))_n$ is a monotonically increasing sequence of numbers bounded above by K_0, it converges to some $\lim_n \varphi_n(\omega) \leq K_0$. Therefore, the pointwise limit $\varphi \coloneqq \lim_n \varphi_n$ is well-defined, it is in $B(\mathcal{A})$, and it is uniformly bounded above by K_0. Moreover, we have that for all $n \in \mathbb{N}$,

$$k = \min_{\omega \in \Omega} \varphi_1(\omega) \leq \varphi_1 \leq \varphi_n \leq K_0 \implies ||\varphi_n||_\infty \leq \max\{|k|, |K_0|\}.$$

Therefore, $(\varphi_n)_n$ is uniformly bounded in the norm. Moreover, for each $m \in \mathcal{M}$, Thereom 13 in Maccheroni et al. (2006) and Proposition 5 in Cerreia-Vioglio et al. (2014), imply that I^m has the Lebesgue property. Therefore:

$$T(\varphi)(m) = I^m(\varphi) = I^m(\lim_n \varphi_n) = \lim_n I^m(\varphi_n) = \lim_n \xi_n(m) = \xi(m).$$

It is immediate to see that for all $k \in \mathbb{R}$ such that $\xi \leq K$, $K \geq K_0$ and, therefore, $\varphi \leq K$. □

Proof of (iv): Take $\xi_1, \xi_2 \in B(\mathcal{M}, \mathcal{D}_{\mathcal{M}})$ such that $\xi_1 \geq \xi_2$. Define $\tilde{\xi} = \xi_1 - \xi_2$ and notice that $\tilde{\xi} \geq 0$. By point (iii), we can find $\tilde{\varphi} \in B(\mathcal{A})$ such that $\tilde{\varphi} \geq 0$ and $\tilde{\xi} = T(\tilde{\varphi})$. Moreover, we can take $\phi_2 \in B(\mathcal{A})$ such that $\xi_2 = T(\varphi_2)$. Then, define $\varphi_1 \coloneqq \varphi_2 + \tilde{\varphi} \in B(\mathcal{A})$. Clearly, $\varphi_1 \geq \varphi_2$ and since T is linear on $B(\mathcal{A})$, we have that

$$T(\varphi_1) = T(\varphi_2 - \tilde{\varphi}) = T(\varphi_2) - T(\tilde{\varphi}) = \xi_2 - \tilde{\xi} = \xi_1$$

as we wanted to show. □

This concludes the proof of the lemma. ■

Proposition B.1: *The following are equivalent:*

(i) $I : B(\mathcal{A}) \to \mathbb{R}$ *is normalized, monotone, and such that for all* $\varphi, \varphi' \in B_0(\mathcal{A})$,

$$(\forall m \in \mathcal{M},\ I^m(\varphi) \geq I^m(\psi)) \implies I(\varphi) \geq I(\psi).$$

(ii) there exists a normalized and monotone functional $\hat{I} : B_0(\mathcal{M}, \mathcal{D}_{\mathcal{M}}) \to \mathbb{R}$ *such that for all* $\varphi \in B_0(\mathcal{A})$,

$$I(\varphi) = \hat{I}(T(\varphi)).$$

Moreover, $\hat{I}$ *is unique and*

- $\hat{I}$ *is continuous if and only if* I *is continuous.*
- $\hat{I}$ *is quasiconcave if and only if* I *is quasiconcave.*
- $\hat{I}$ *is monotone continuous if and only if* I *is monotone continuous.*

PROOF OF PROPOSITION B.1:

(i) implies (ii). Define $\hat{I} : B_0(\mathcal{D}_{\mathcal{M}}) \to \mathbb{R}$ as follows: for all $\xi \in B_0(\mathcal{D}_{\mathcal{M}})$,

$$\hat{I}(\xi) = I(\varphi_\xi),$$

where $\varphi_\xi \in B_0(\mathcal{A})$ is chosen so that $\xi = T(\varphi_\xi)$.

Step 1: $\hat{I}$ *is well-defined.* Pick $\xi \in B_0(\mathcal{D}_{\mathcal{M}})$ arbitrarily. That a $\varphi_\xi \in B_0(\mathcal{A})$ such that $\xi = T(\varphi_\xi)$ exists follows from Lemma B.10. Moreover, suppose there are two $\varphi, \psi \in B_0(\mathcal{A})$ such that $T(\varphi)(m) = I^m(\varphi) = \xi(m) = I^m(\psi) = T(\psi)(m)$ for all $m \in \mathcal{M}$. Then, by assumption, it must be the case that $I(\varphi) = I(\psi)$, showing that $\hat{I}$ is well-defined.

Step 2: $\hat{I}$ *is normalized.* Take any $k \in \mathbb{R}$. Then, since each I^m is normalized, it follows that $k = I^m(k) = T(k)(m)$ for all $m \in \mathcal{M}$. By definition, it follows that $\hat{I}(k) = I(k) = k$, where the last equality follows from the assumption that I is normalized. This proves the step.

Step 3: $\hat{I}$ is monotone. Take $\xi, \xi' \in \operatorname{Im} T$ such that $\xi \geq \xi'$. By Lemma B.10, $\xi, \xi' \in T(B_0(\mathcal{A}))$ and, therefore, Lemma B.11 implies that we can find $\varphi_\xi, \varphi_{\xi'} \in B_0(\mathcal{A})$ such that $\varphi_\xi \geq \varphi_{\xi'}$ and $\xi = T(\varphi_\xi)$, $\xi' = T(\varphi_{\xi'})$. Since I is monotone

$$\hat{I}(\xi) = \hat{I}(T(\varphi_\xi)) = I(\varphi_\xi) \geq I(\varphi_{\xi'}) = \hat{I}(T(\varphi_{\xi'})) = \hat{I}(\xi')$$

showing that also $\hat{I}$ is monotone.

Step 4: $\hat{I}$ is unique. Suppose there is another $\tilde{I} : B_0(\mathcal{M}, \mathcal{D}_\mathcal{M}) \to \mathbb{R}$ such that $I(\varphi) = \tilde{I}(T(\varphi))$ for all $\varphi \in B_0(\mathcal{A})$. Then, take any $\xi \in B_0(\mathcal{M}, \mathcal{D}_\mathcal{M})$. By Lemma B.10, there exists $\varphi_\xi \in B_0(\mathcal{A})$ and such that $\xi = T(\varphi_\xi)$. Then,

$$\tilde{I}(\xi) = \tilde{I}(T(\varphi_\xi)) = I(\varphi_\xi) = \hat{I}(T(\varphi_\xi)) = \hat{I}(\xi).$$

It follows that $\tilde{I} = \hat{I}$.

Step 5: $\hat{I}$ is continuous. Suppose that I is continuous. Fix any $\xi, \xi' \in B_0(\mathcal{D}_\mathcal{M})$ and $c \in \mathbb{R}$. First we show that the set $\{\alpha \in [0,1] : \hat{I}(\alpha\xi + (1-\alpha)\xi') \leq c\}$ is closed. If it is empty, it is closed. If it is nonempty, take any sequence $(\alpha_n)_n \subseteq L$ such that $\alpha_n \to \alpha_0$. By Lemma B.10, we can pick $\varphi, \varphi' \in B_0(\mathcal{A})$ such that $\xi = T(\varphi)$ and $\xi' = T(\varphi')$. Moreover, we can pick we a finite partition $(E_i)_{i=1}^k$ and reals $(r_i)_{i=1}^k$, $(r_i')_{i=1}^k$ such that:

$$\varphi = \sum_{i=1}^k \chi_{E_i} r_i, \quad \varphi' = \sum_{i=1}^k \chi_{E_i} r_i'.$$

Fix any $m \in \mathcal{M}$. Then, there is a unique E_{j_m} such that $m(E_{j_m}) = 1$ and $m(E_i) = 0$ if $i \neq j_m$. Therefore, it follows that for all $n \in \mathbb{N}$,

$$I^m(\alpha_n\varphi + (1-\alpha_n)\varphi') = \alpha_n r_{j_m} + (1-\alpha_n) r'_{j_m} = \alpha_n I^m(\varphi) + (1-\alpha_n) I^m(\varphi'),$$

$$I^m(\alpha_0\varphi + (1-\alpha_0)\varphi') = \alpha_0 r_{j_m} + (1-\alpha_0) r'_{j_m} = \alpha_0 I^m(\varphi) + (1-\alpha_0) I^m(\varphi').$$

Since $m \in \mathcal{M}$ was arbitrarily chosen, it follows that:

$$\forall n \in \mathbb{N}, \quad \alpha_n \xi + (1-\alpha_n)\xi = \alpha_n T(\varphi) + (1-\alpha_n)T(\varphi') = T(\alpha_n \varphi + (1-\alpha_n)\varphi')$$

$$\alpha_0 \xi + (1-\alpha_0)\xi = \alpha_0 T(\varphi) + (1-\alpha_0)T(\varphi') = T(\alpha_0 \varphi + (1-\alpha_0)\varphi')$$

Therefore, by definition of $\hat{I}$ and continuity of I:

$$\begin{aligned} c &\geq \liminf_n \hat{I}(\alpha_n \xi + (1-\alpha_n)\xi') \\ &= \liminf_n I(\alpha_n \varphi + (1-\alpha_n)\varphi') \\ &= I(\alpha_0 \varphi + (1-\alpha_0)\varphi') \\ &= \hat{I}(\alpha_0 \xi + (1-\alpha_0)\xi') \end{aligned}$$

and, therefore, $\alpha_0 \in \{\alpha \in [0,1] : \hat{I}(\alpha\xi + (1-\alpha)\xi') \leq c\}$, showing that this set is closed. By a symmetric argument, we can show that $\{\alpha \in [0,1] : \hat{I}(\alpha\xi + (1-\alpha)\xi') \geq c\}$ is also closed. Siince this holds for all $\xi, \xi' \in B_0(\mathcal{D}_{\mathcal{M}})$ and $c \in \mathbb{R}$, and $\hat{I}$ is monotone by Step 3, Proposition 43 in Cerreia-Vioglio et al. (2011) implies that $\hat{I}$ is continuous.

Step 6: $\hat{I}$ is quasiconcave. Fix any $\alpha \in \mathbb{R}$. We show that the set $U_c = \{\xi \in B_0(\mathcal{D}_{\mathcal{M}}) : \xi \geq c\}$ is convex. If it is empty, this holds vacuously true. Suppose it is nonempty. Take $\xi_1, \xi_2 \in U_c$ and $\alpha \in [0,1]$. By Lemma B.10, we can pick $\varphi_1, \varphi_2 \in B_0(\mathcal{A})$ such that $\xi_1 = T(\varphi_1$ and $\xi_2 = T = \varphi_2$. Notice that $I(\varphi_1) = \hat{I}(\xi_1) \geq c$ and $I(\varphi_1) = \hat{I}(\xi_1) \geq c$. Since I is quasiconcave, it follows that $I(\alpha\varphi_1 + (1-\alpha)\varphi_2) \geq c$. Now, pick a partition $\{E_i\}_{i=1}^k \subseteq \mathcal{F}$ and profiles of scalars $(r_i^1)_{i=1}^k, (r_i^2)_{i=1}^k \subseteq \mathbb{R}$ such that $\varphi_1 = \sum_{i=1}^k \chi_{E_i} r_i^1$ and $\varphi_2 = \sum_{i=2}^k \chi_{E_i} r_i^2$. Fix $m \in \mathcal{M}$. Since the partition is in $\mathcal{A}$, there is a unique j_m such that $m(E_{j_m}) = 1$ and $m(E_i) = 0$ if $i \neq j_m$. Therefore,

$$I^m(\alpha\varphi_1 + (1-\alpha)\varphi_2) = \alpha r_{j_m}^1 + (1-\alpha) r_{j_m}^2 = \alpha I^m(\varphi_1) + (1-\alpha) I^m(\varphi_2) = \alpha \xi_1(m) + (1-\alpha)\xi_2(m)$$

Therefore, we can conclude that $T(\alpha\varphi_1 + (1-\alpha)\varphi_2) = \alpha\xi_1 + (1-\alpha)\xi_2$. Then:

$$\hat{I}(\alpha\xi_1 + (1-\alpha)\xi_2) = I(\alpha\xi_1 + (1-\alpha)\xi_2) \geq c$$

and, therefore, $\alpha\xi_1 + (1-\alpha)\xi_2 \in U_c$, showing convexity. Since c was arbitrarily chosen, we conclude that $\hat{I}$ is quasiconcave.

Step 7: $\hat{I}$ is monotone continuous Take $\xi, \xi' \in B_0(\mathcal{D}_{\mathcal{M}})$ and $k \in \mathbb{R}$, a monotone sequence $(D_n)_n \in D_{\mathcal{M}}$ such that $D_n \downarrow \emptyset$, and assume that $\hat{I}(\xi) > \hat{I}(\xi')$. Then, we can find $\varphi, \varphi' \in B_0(\mathcal{A})$ such that $\xi = T(\varphi)$ and $T(\varphi') = \xi'$. It follows that $I(\varphi) = \hat{I}(\xi) > \hat{I}(\xi') = I(\varphi')$. Let $E_n := \mathfrak{q}^{-1}(D_n) \in \mathcal{A}$ and notice that $E_n \downarrow \emptyset$. Therefore, there exists n_0 such that $I(kE_{n_0}\varphi) > I(\varphi')$. Since $E_{n_0} \in \mathcal{A}$, for all $m \in \mathcal{M}$, $m(E_{n_0}) \in \{0, 1\}$ and

$$m(E_{n_0}) = 1 \implies kE_{n_0}\varphi = k \text{ a.e. } [m] \implies I^m(kE_{n_0}\varphi) = I^m(k) = k$$

$$m(E_{n_0}) = 0 \implies kE_{n_0}\varphi = \varphi \text{ a.e. } [m] \implies I^m(kE_{n_0}\varphi) = I^m(\varphi) = \xi(m)$$

Moreover, notice that $m(E_{n_0}) = 1$ if and only if $m \in D_{n_0}$ and $m(E_{n_0}) = 0$ if and only if $m \notin D_{n_0}$. Therefore, $kD_{n_0}\xi = T(kE_{n_0}\varphi)$ and we can conclude that $\hat{I}(kD_{n_0}\xi) = I(kE_{n_0}\varphi) > I(\varphi') = \hat{I}(\xi')$ as we wanted to show.

(ii) implies (i).

Suppose there exists a normalized, monotone, and continuous functional $\hat{I} : B_0(\mathcal{M}, \mathcal{D}_{\mathcal{M}}) \to \mathbb{R}$ such that for all $\varphi \in B_0(\mathcal{A})$, $I(\varphi) = \hat{I}(T(\varphi))$.

Step 1: I is normalized.

Take $k \in \mathbb{R}$. Since $\hat{I}$ is normalized, we have that $\hat{I}(k) = k$. Moreover, $T(k)(m) = I^m(k) = k$ for all $m \in \mathcal{M}$. Therefore, $I(k) = \hat{I}(T(k)) = \hat{I}(k) = k$, showing that I is normalized.

Step 2: I is monotone.

Take $\varphi, \varphi' \in B_0(\mathcal{A})$ such that $\varphi \geq \varphi'$. For all $m \in \mathcal{M}$, I^m is monotone and, therefore, $T(\varphi)(m) = I^m(\varphi) \geq I^m(\varphi') = T(\varphi')(m)$. But, then, since $\hat{I}$ is monotone

$$I(\varphi) = \hat{I}(T(\varphi)) \geq \hat{I}(T(\varphi')) = I(\varphi'),$$

showing that I is monotone.

Step 3: If $\varphi, \varphi' \in B_0(\mathcal{A})$ and $I^m(\varphi) \geq I^m(\varphi')$ for all $m \in \mathcal{M}$, then $I(\varphi) \geq I(\varphi')$.

Take any two $\varphi, \varphi' \in B_0(\mathcal{A})$ and assume that $I^m(\varphi) \geq I^m(\varphi')$ for all $m \in \mathcal{M}$. Then, $T(\varphi) \geq T(\varphi')$ and, therefore, since $\hat{I}$ is monotone:

$$I(\varphi) = \hat{I}(T(\varphi)) \geq \hat{I}(T(\varphi')) = I(\varphi').$$

Step 4: I is continuous. Take a sequence $(\varphi_n)_n \subseteq B_0(\mathcal{A})$ such that $\varphi_n \to \varphi \in B_0(\mathcal{A})$ uniformly. Since for each $m \in \mathcal{M}$, I^m is Lipschitz continuous, it follows that for all m, $|I^m(\varphi_n) - I^m(\varphi)| \leq ||\varphi - \varphi_n||_\infty$ so that:

$$||T(\varphi_n) - T(\varphi)||_\infty \leq ||\varphi - \varphi_n||_\infty \to 0.$$

Thus, $T(\varphi_n)$ converges uniformly to $T(\varphi)$ and by Lemma B.10, $T(\varphi_n), T(\varphi) \in B_0(\mathcal{D}_\mathcal{M})$. Therefore, by continuity of $\hat{I}$, we have that:

$$I(\varphi_n) = \hat{I}(T(\varphi_n)) \to \hat{I}(T(\varphi)) = I(\varphi)$$

showing that I is continuous.

Step 5: I is quasiconcave. Suppose $\hat{I}$ is quasiconcave. Take $\varphi_1, \varphi_2 \in B_0(\mathcal{A})$ and $\alpha \in [0, 1]$. Since I^m is concave, it follows that

$$I^m(\alpha\varphi_1 + (1-\alpha)\varphi_2) \geq \alpha I^m(\varphi_1) + (1-\alpha)I^m(\varphi_2)$$

for all $m \in \mathcal{M}$. Therefore, since $\hat{I}$ is monotone and quasiconcave,

$$\begin{aligned} I(\alpha\varphi_2 + (1-\alpha)\varphi_2) &= \hat{I}(T(\alpha\varphi_1 + (1-\alpha)\varphi_2)) \\ &\geq \hat{I}(\alpha T(\varphi_1) + (1-\alpha)T(\varphi_2)) \\ &\geq \min\{\hat{I}(T(\varphi_1)), \hat{I}(T(\varphi_2))\} = \min\{I(\varphi_1), I(\varphi_2)\} \end{aligned}$$

showing that I is quasiconcave.

Step 6: I is monotone continous. Take $\varphi, \varphi' \in B_0(\mathcal{A})$ and $k \in \mathbb{R}$, a monotone sequence $(E_n)_n \in \mathcal{A}$ such that $E_n \downarrow \emptyset$, and assume that $I(\varphi) > I(\varphi')$. Then, $\hat{I}(T(\varphi)) = I(\varphi) > I(\varphi') = \hat{I}(T(\varphi'))$. Notice that for each $n \in \mathbb{N}$, $E_n \in \mathcal{A}$ and, therefore, $m(E_n) \in \{0, 1\}$ for all $m \in \mathcal{M}$. Then, let $D_n = \{m \in \mathcal{M} : m(E_n) > \frac{1}{2}\}$ and notice that $m \in D_n$ if and only if $m(E_n) = 1$ and $m \notin D_n$ if and only if $m(E_n) = 0$. Clearly, D_n is a decreasing sequence of sets. We show that $\cap_n D_n = \emptyset$. Take any $m \in \mathcal{M}$. Since m is countably additive, by continuity of finite measures, it must be the case that $m(E_n) \to 0$. However, since $m(E_n) \in \{0, 1\}$ for all $n \in \mathbb{N}$, this implies that there is a N such that $m(E_n) = 0$ for all $n > N$. This implies that $m \notin E_n$ for $n > N$ and, therefore, $m \notin \cap_n D_n$. It follows that $D_n \downarrow \emptyset$. Since $\hat{I}$ is monotone continuous, there exists a n_0 such that $\hat{I}(\chi_{D_{n_0}} k + \chi_{D^c_{n_0}} T(\varphi)) > \hat{I}(T(\varphi'))$. Finally note that for all $m \in \mathcal{M}$,

$$\begin{aligned} m \in D_{n_0} &\implies m(D_{n_0}) = 1 \implies I^m(\chi_{D_{n_0}} k + \chi_{D^c_{n_0}} \varphi) = I^m(k) = k \\ m \in D^c_{n_0} &\implies m(D_{n_0}) = 0 \implies I^m(\chi_{D_{n_0}} k + \chi_{D^c_{n_0}} \varphi) = I^m(\varphi) = T(\varphi)(m). \end{aligned}$$

Hence, $T(\chi_{D_{n_0}} k + \chi_{D^c_{n_0}} \varphi) = \chi_{D_{n_0}} k + \chi_{D^c_{n_0}} T(\varphi)$ and, therefore,

$$\begin{aligned} I(\chi_{D_{n_0}} k + \chi_{D^c_{n_0}} \varphi) &= \hat{I}(T((\chi_{D_{n_0}} k + \chi_{D^c_{n_0}} \varphi)) \\ &= \hat{I}(\chi_{D_{n_0}} k + \chi_{D^c_{n_0}} T(\varphi)) \\ &> \hat{I}(T(\varphi')) = I(\varphi') \end{aligned}$$

as we wanted to show. ■

Appendix D. Proof of Theorem 1

Proof of Theorem 1: We know that $\succsim$ is represented by u when restricted to constant acts. Define the functional $I : B_0(\mathcal{G}) \to \mathbb{R}$ such that for each $\varphi \in B_0(\mathcal{G})$, $I(\varphi) := u(x_{f_\varphi})$, where $f_\varphi \in \mathcal{F}$ is chosen so that $\varphi = u(f_\varphi)$. By Lemma B.2, such act f_φ exists for all $\varphi \in B_0(\mathcal{G})$, while the certainty equivalent $x_{f_\varphi} \sim f_\varphi$ exists by Lemma B.3. Moreover, for any $\varphi \in B_0(\mathcal{G})$, if there are two $f_\varphi, f'_\varphi \in \mathcal{F}$ such that $u(f_\varphi) = \varphi = u(f'_\varphi)$, we then have that since u represents $\succsim$ over X,

$$\begin{aligned} u(f_\varphi)(\omega) = u(f'_\varphi)(\omega) &\implies u(f_\varphi(\omega)) = u(f'_\varphi(\omega)) \\ &\implies f_\varphi(\omega) \sim f'_\varphi(\omega) \end{aligned}$$

for all $\omega \in \Omega$. By Axiom 1.(ii) of monotonicity, it follows that $f_\varphi \sim f'_\varphi$ and, by transitivity, that $x_{f_\varphi} \sim x_{f'_\varphi}$. Therefore, we can conclude that $u(x_{f_\varphi}) = u(x_{f'_\varphi})$, showing that I is a well-defined functional on $B_0(\mathcal{G})$. It is easily seen that such functional is also normalized, monotone, and continuous.[17] Moreover, it is monotone continuous and its restriction to $B_0(\mathcal{A})$ is quasiconcave.

[17]See for example the proof of Theorem 1 (Omnibus) in the working paper version of Cerreia-Vioglio et al. (2022).

Define the function $V \coloneqq I \circ u : \mathcal{F} \to \mathbb{R}$. For all $f, f' \in \mathcal{F}$,

$$\begin{aligned} f \succsim f' &\iff x_{f'} \succsim x_{f'} \\ &\iff V(f) = I(u(f)) = u(x_f) \geq u(x_{f'}) = I(u(f)) = V(f') \ . \end{aligned}$$

This shows that V represents $\succsim$ on $\mathcal{F}$. Moreover, by Proposition 1, for each $m \in \mathcal{M}$, $\succsim^m$ is represented by $I^m \circ u$, where $I^m : B(\mathcal{G}) \to \mathbb{R}$ is as defined in (12). Moreover, let $I_{\mathcal{A}}$ be the generalized conditional expectation as in Corollary 1. Take now $\varphi, \psi \in B_0(\mathcal{G})$ such that $I^m(\varphi) \geq I^m(\psi)$ for all $m \in \mathcal{M}$. By Lemma B.2, we can find $f_\varphi, f_\psi \in \mathcal{F}$ such that $\varphi = u(f_\varphi)$ and $\psi = u(f_\psi)$. Then, $I^m(u(f_\varphi)) \geq I^m(u(f_\psi))$ for all $m \in \mathcal{M}$ so that $f_\varphi \succsim^m f_\psi$ for all $m \in \mathcal{M}$. Consistency implies that $f_\varphi \succsim f_\psi$. Therefore:

$$I(\varphi) = I(u(f_\varphi)) = V(f_\varphi) \geq V(f_\psi) = I(u(f_\psi)) \geq I(\psi) \ .$$

By this fact and since I is monotone, normalized, continuous, and quasiconcave, by Lemma B.11, there exists a unique monotone, normalized, continuous, and quasiconcave functional $\hat{I} : B_0(\mathcal{M}, \mathcal{D}_{\mathcal{M}}) \to \mathbb{R}$ such that $I(\varphi) = \hat{I}(T(\varphi))$ for all $\varphi \in B_0(\mathcal{A})$. Moreover, since I is monotone continuous, so is $\hat{I}$. By Theorem 21 in Cerreia-Vioglio et al. (2013), $\hat{I}$ admits a unique monotone, normalized, lower semicontinuous, and quasiconcave extension to $B(\mathcal{D}_{\mathcal{M}})$, which, abusing notation, we also denote by $\hat{I}$. Moreover, since $\hat{I}$ is monotone continuous when restricted to $B_0(\mathcal{D}_{\mathcal{M}})$, it is inner/outer continuous on $B(\mathcal{D}_{\mathcal{M}})$. Take now any $\varphi \in B_0(\mathcal{G})$. Since $I_{\mathcal{A}}(\varphi) \in B(\mathcal{A})$ and $B_0(\mathcal{A})$ is dense in $B(\mathcal{A})$, we can pick sequences $(\psi^l_n)_{n\in\mathbb{N}}, (\psi^u_n)_{n\in\mathbb{N}} \in B_0(\mathcal{A})$ such that $\psi^l_n \nearrow I_{\mathcal{A}}(\varphi)$ and $\psi^u_n \searrow I_{\mathcal{A}}(\varphi)$ uniformly. Fix any $m \in \mathcal{M}$. Since I^m is monotone, we have that for all $n \in \mathbb{N}$:

$$I^m(\psi^l_n) \leq I^m(I_{\mathcal{A}}(\varphi)) \leq I^m(\psi^u_n) \ .$$

By Proposition 1, we also have that $I^m(I_{\mathcal{A}}(\varphi)) = I^m(\varphi)$ and, therefore, we have that for all $n \in \mathbb{N}$,

$$I^m(\psi_n^l) \leq I^m(\varphi) \leq I^m(\psi_n^u)$$

for all $m \in \mathcal{M}$. By what shown above, we have that for all $n \in \mathbb{N}$:

$$\hat{I}(T(\psi_n^l)) = I(\psi_n^l) \leq I(\varphi) \leq I(\psi_n^u) \leq I(T(\psi_u^l)).$$

Since I^m is monotone and Lipschitz, we have that $T(\psi_n^l) \uparrow T(I_{\mathcal{A}}(\varphi)) = T(\varphi)$ and $T(\psi_n^u) \downarrow T(I_{\mathcal{A}}(\varphi)) = T(\varphi)$. Then, since $\hat{I}$ is inner/outer continuous, passing to the limit in the above sequence of inequality, we obtain:

$$\hat{I}(T(\varphi)) = \lim_n \hat{I}(T(\psi_n^l)) \leq I(\varphi) \leq \lim \hat{I}(T(\psi_u^l)) = \hat{I}(T(\varphi)).$$

This shows that $I(\varphi) = \hat{I}(T(\varphi))$ for all $\varphi \in B_0(\mathcal{G})$. It follows that for all $f, g \in \mathcal{F}$,

$$f \succsim g \iff I(u(f)) \geq I(u(g)) \iff \hat{I}(T(u(f))) \geq \hat{I}(T(u(g))).$$

■

Proof of Proposition 3: Let $\succsim_1$ and $\succsim_2$ Suppose that $\succsim_1$ and $\succsim_2$ are two misspecification averse preferences represented respectively by $(\hat{I}_1, u_1, c_1)$ and $(\hat{I}_2, u_2, c_2)$ as in Theorem 1. Suppose that $u_1 = u_2 = u$ and that $\hat{I}_1 \leq \hat{I}_2$. Take any $f \in \mathcal{F}(\mathcal{A})$ and $x \in X$ and assume that $f \succsim_1 x$. Since f is measurable with respect to $\mathcal{A}$, for each $m \in \mathcal{M}$, f must be constant on E^m and, therefore coherence and normalziation imply:

$$I_1(u(f), m) = I_1\left(u(f)\chi_{E^m}, m\right) = u(f|_{E^m}) = I_2\left(u(f)\chi_{E^m}, m\right) = I_2(u(f), m).$$

Then, we have that:

$$u(x) \leq \hat{I}_1 \left(I_1(u(f), \cdot) \right) \leq \hat{I}_2 \left(I_1(u(f), \cdot) \right) = \hat{I}_2 \left(I_2(u(f), \cdot) \right)$$

so that $f \succsim_2 x$.

As for the other direction, equation (9) and nontriviality automatically imply that u_2 is a positive affine transformation of u_1. Assume that $u_1 = u_2 = u$ and take $\xi \in B_0(\mathcal{D}_{\mathcal{M}})$. Then, by Lemmas B.10 and B.2, there exists $f \in \mathcal{F}(\mathcal{A})$ such that $\xi = I_1(u(f), \cdot)$. By the same argument given above, it is also the case that $\xi = I_2(u(f), \cdot)$. Take $x \in X$ such that $f \sim_1 x$. Then, condition (9) implies that $f \succsim_2 x$. Therefore:

$$\hat{I}_1(\xi) = \hat{I}_1(I_1(u(f), \cdot)) = u(x) \leq \hat{I}_2(I_2(u(f), \cdot)) = \hat{I}_2(\xi).$$

Thus, $\hat{I}_1 \leq \hat{I}_2$ on $B_0(\mathcal{D}_{\mathcal{M}})$. Since this set is dense in $B(\mathcal{D}_{\mathcal{M}})$, we can find a monotonically decreasing sequence $(\xi_n)_n \subseteq B_0(\mathcal{D}_{\mathcal{M}})$ such that $\xi_n \searrow \xi$. Then, $\hat{I}_1(\xi_n) \leq \hat{I}_2(\xi_n)$ for all $n \in \mathbb{N}$. Since $\hat{I}$ is inner/outer continuous, passing to the limit we can conclude that $\hat{I}_1(\xi) \leq \hat{I}_2(\xi)$.

Uniqueness follows by Lemma B.1 and routine arguments.

(ii) implies (i). It follows by routine arguments.

■

Proof of Theorem 4: Suppose the assumptions of the theorem are satisfied. Pick any $D \in \mathcal{D}_{\mathcal{M}}$. We want to show that there exists a $E^D \in \mathcal{G}$ such that $I^m(\chi_{E^D}) = \chi_D(m)$ for all $m \in \mathcal{M}$. By assumption, there exists $\varphi \in B_0(\mathcal{G})$ such that $0 \leq \varphi \leq 1$ and $I^m(\varphi) = \chi_D(m)$ for all $m \in \mathcal{M}$. Let $E^D = \{\omega \in \Omega : \varphi(\omega) > 0\}$ which clearly is in $\mathcal{G}$. Also notice that $\chi_D \geq \varphi$. Then,using monotonicity, if $m \in D$,

$$1 = \chi_D(m) = I^m(\varphi) \leq I^m(\chi_{E^D}) = \min_p p(E) + c(p, m) \leq m(E^D) \leq 1$$

showing that $\chi_D(m) = 1 = I^m(\chi_{E^D})$. On the other hand, suppose that $m \notin D$. Then

$$0 = \chi_D(m) = I^m(\varphi) = \min_{p \ll m} \int \varphi dp + c(p, m) = \int \varphi d\hat{p} + c(\hat{p}, m)$$

where $\hat{p}$ it the probability where the minimum in the above equation is attained. Then, since $\varphi \geq 0$ and $c(\hat{p}, m) \geq 0$, it must be the case that $\int \varphi d\hat{p} = 0$ which in turn implies that $\hat{p}(E^D) = 0$ and $c(\hat{p}, m) = 0$. Since the latter is uniquely minimized at m, it follows that $m = \hat{p}$ and, therefore, $m(E^D) = 0$. Thus,

$$I^m(\chi_{E^D}) = \min_{p \ll m} p(E^D) + c(p, m) = \min_{p \ll m} 0 + c(p, m) = 0 = \chi_D(M)$$

Since $(\Omega, \mathcal{G})$ is a standard Borel space and $\mathcal{M}$ is a measurable subset of $\mathcal{D}$, then also $(\mathcal{M}, \mathcal{D}_{\mathcal{M}})$ is a standard Borel space[18]. Thus, we can find a sequence $(D_n)_n \subseteq D_{\mathcal{M}}$ that separates points in $\mathcal{M}$. That is, if $m \neq m'$ for some $m, m' \in \mathcal{M}$, there exists D_n such that $m \in D_n$ and $m' \notin D_n$. By defining $\alpha_{\mathcal{M}} : \mathcal{M} \to \{0, 1\}^{\mathbb{N}}$, as $\alpha_{\mathcal{M}}(m) = (\chi_{D_n}(m))_n$, the fact that $(D_n)_n$ separates points in $\mathcal{M}$ implies that $\alpha_{\mathcal{M}}$ is injective. It is easy to see that it is also measurable. Since $(\mathcal{M}, \mathcal{D}_{\mathcal{M}})$ is a standard Borel space, the inverse $\alpha_{\mathcal{M}}^{-1} : \operatorname{Im} \alpha_{\mathcal{M}} \to \mathcal{M}$ exists and is also measurable. Furthermore, by what is shown above, for each $n \in \mathbb{N}$, we can find $E_n \in \mathcal{G}$ such that $I^m(E_n) = \chi_{D_n}(m)$ for all $m \in \mathcal{M}$. Define similarly $\alpha_\Omega : \Omega \to \{0, 1\}^{\mathbb{N}}$ as $\alpha_\Omega(\omega) = (\chi_{E_n}(\omega))_n$, which is also a measurable function. Fix arbitrarily $m_0 \in \mathcal{M}$ and define $\mathfrak{q} : \Omega \to \mathcal{M}$ as

$$\mathfrak{q}(\omega) = \begin{cases} \alpha_{\mathcal{M}}^{-1} \circ \alpha_\Omega(\omega) & \text{if } \alpha_\Omega(\omega) \in \operatorname{Im} \alpha_{\mathcal{M}} \\ m_0 & \text{otherwise.} \end{cases}$$

[18] See Theorems 17.23-17.24 and Corollary 13.4 in Kechris (2012)

Clearly, $\mathfrak{q}$ is measurable. We only need to show that $m(\mathfrak{q}^{-1}(m)) = 1$ for all $m \in \mathcal{M}$. Then, fix $m \in \mathcal{M}$. Take $n \in \mathbb{N}$ such that $m \in D_n$. Then, $\chi_{D_n}(m) = 1$ and, therefore,

$$1 \geq m(E_n) = m(E_n) + c(m, m) \geq \min_{p \ll m} p(E_n) + c(p, m) = I^m(\chi_{E_n}) = \chi_{D_n}(m) = 1$$

which implies that $m(E_n) = 1$. On the other hand, take $n \in \mathbb{N}$ such that $m \notin D_n$. Then $\chi_{D_n}(m) = 0$ and, therefore,

$$0 = \chi_{D_n}(m) = I^m(\chi_{E_n}) = \min_{p \ll m} p(E_n) + c(p, m) = \underbrace{\hat{p}(E_n)}_{\geq 0} + \underbrace{c(\hat{p}(E_n), m)}_{\geq 0}$$

where $\hat{p}$ attains the minimum in the problem $min_{p \ll m} p(E_n) + c(p, m)$. Then, $\hat{p}(E_n) = 0$ and $c(\hat{p}, m) = 0$. But since $c(\cdot, m) \geq 0$ is uniquely minimized at m, it follows that $m = \hat{p}$ and, therefore, $m(E_n) = 0$ and, thereby, $m(\Omega \setminus E_n) = 1$. With this, notice that

$$\begin{aligned}
\mathfrak{q}^{-1}(m) &\supseteq \{\omega \in \Omega : \mathfrak{q}(\omega) = m\} \\
&= \{\omega \in \Omega : \alpha_\Omega(\omega) = \alpha_{\mathcal{M}}(m)\} \\
&= \{\omega \in \Omega : \chi_{E_n}(\omega) = \chi_{D_n}(m) \text{ for all } n \in \mathbb{N}\} \\
&= \bigcap_{n \in \mathbb{N}} \{\omega \in \Omega : \chi_{E_n}(\omega) = \chi_{D_n}(m)\} \\
&= \left(\bigcap_{n : m \in D_n} \{\omega \in \Omega : \chi_{E_n}(\omega) = \chi_{D_n}(m)\} \right) \bigcap \left(\bigcap_{n : m \notin D_n} \{\omega \in \Omega : \chi_{E_n}(\omega) = \chi_{D_n}(m)\} \right) \\
&= (\cap_{n : m \in D_n} E_n) \bigcap (\cap_{n : m \notin D_n} \Omega \setminus E_n)
\end{aligned}$$

and, therefore,

$$1 \geq m\left(\mathfrak{q}^{-1}(m)\right) \geq m\left((\cap_{n : m \in D_n} E_n) \cap (\cap_{n : m \notin D_n} \Omega \setminus E_n)\right) = 1$$

implying the result.

Define $\hat{E}^m = \hat{\mathfrak{q}}^{-1}(m)$ for all m. Fix $m_0 \in \mathcal{M}$ and take $f_1, f_2 \in \mathcal{F}$ and fix any $g \in \mathcal{M}$. Note that since $m_0(\hat{E}^{m_0}) = 1$ and $m(\hat{E}^{m_0}) = 0$ whenever $m \neq m_0$, then $u(f_i\hat{E}^{m_0}g) = u(f_i)$ a.e.$[m_0]$ and $u(f_i\hat{E}^{m_0}g) = u(g)$ a.e. $[m]$ whenever $m \neq m_0$ for $i = 1, 2$. This implies that $I^{m_0}(u(f_i\hat{E}^{m_0}g)) = I^{m_0}(u(f_i))$ for $i = 1, 2$ and $I^m(u(f_1\hat{E}^{m_0}g)) = I^m(u(f_2\hat{E}^{m_0}g))$ for $m \neq m_0$. Then

$$\begin{aligned} I^{m_0}(u(f_1)) = I^{m_0}(u(f_2)) &\implies I(u(f_1\hat{E}^{m_0}g), \cdot) = I(u(f_2\hat{E}^{m_0}g), \cdot) \\ &\implies \hat{I}\left(I(u(f_1\hat{E}^{m_0}g)\right) = \hat{I}\left(I(u(f_2\hat{E}^{m_0}g)\right) \\ &\implies f_1\hat{E}^{m_0}g \sim f_2\hat{E}^{m_0}g \end{aligned}$$

and

$$\begin{aligned} I^{m_0}(u(f_1)) > I^{m_0}(u(f_2)) &\implies I(u(f_1\hat{E}^{m_0}g), \cdot) > I(u(f_2\hat{E}^{m_0}g), \cdot) \\ &\implies \hat{I}\left(I(u(f_1\hat{E}^{m_0}g)\right) > \hat{I}\left(I(u(f_2\hat{E}^{m_0}g)\right) \\ &\implies f_1\hat{E}^{m_0}g \succ f_2\hat{E}^{m_0}g \end{aligned}$$

This implies that $\forall f_1, f_2, g \in \mathcal{F}$, $f_1\hat{E}^{m_0}g \succsim f_1\hat{E}^m g$ if and only if $I^{m_0}(u(f_1)) \geq I^{m_0}(u(f_2))$ as we wanted to show. Checking the axioms is now routine. ■

D.1. **Proof of Theorems 2 and 3.** In this section we prove the general representation in Theorem 2. We start with the following lemma.

LEMMA B.12: *Suppose $\succsim$ is a misspecification averse preference whose restriction to $\mathcal{F}(\mathcal{A})$ satisfies Savage's P2-P6. There exist a non-constant, affine, and surjective $\tilde{u} : X \to \mathbb{R}$, a strictly increasing $\phi : \mathbb{R} \to \mathbb{R}$, and a non-atomic $\nu \in \Delta^\sigma(\Omega, \mathcal{A})$ such that for all $f, g \in \mathcal{F}(\mathcal{A})$,*

$$f \succsim g \iff \phi^{-1}\left(\int_\Omega \phi(\tilde{u}(f))d\nu\right) \geq \phi^{-1}\left(\int_\Omega \phi(\tilde{u}(g))d\nu\right) .$$

Moreover, ν is unique, $\tilde{u}$ is unique up to positive affine transformations, and ϕ is unique up to positive affine transformations given $\tilde{u}$.

Proof of Lemma B.12: Since when restricted to acts measurable with respect to $\mathcal{A}$, $\succsim$ satisfies the Axioms of Savage (1954) and monotone continuity, there exist a non-constant function $v : X \to \mathbb{R}$ and a non-atomic probability measure $\nu \in \Delta^\sigma(\Omega, \mathcal{A})$ such that for all $f, f' \in \mathcal{F}(\mathcal{A})$:

$$f \succsim f' \iff \int_\Omega v(f)d\nu \geq \int_\Omega v(f')d\nu.$$

Clearly, v represents $\succsim$ on X. By Herstein and Milnor (1953), there exists an affine $\tilde{u} : X \to \mathbb{R}$ representing $\succsim$ on X. Since $\succsim$ is unbounded, the argument in the proof of Proposition 1 shows that u must be surjective. Then, there exists a strictly increasing trasformation $\phi : \mathbb{R} \to \mathbb{R}$ such that $v = \phi \circ u$. Now, Take any $k, k' \in \operatorname{Im}\phi$ and $\lambda \in (0,1)$. Then, we can find $x_k, x_{k'} \in X$ such that $\phi(k) = \phi(u(x_k))$ and $\phi(k') = \phi(u(x_{k'}))$. Since ν is non-atomic, we can pick $E_\lambda \in \mathcal{A}$ such that $\nu(E_\lambda) = \lambda$. Then, $f_\lambda \coloneqq x_k E_\lambda x_{k'} \in \mathcal{F}(\mathcal{A})$ and, by Lemma B.3, we can find $x_{f_\lambda} \in X$ such that $x_{f_\lambda} \sim f_\lambda$. Clearly, both f_λ and x_{f_λ} are measurable with respect to $\mathcal{A}$. Therefore:

$$\begin{aligned}
\lambda\phi(k) + (1-\lambda)\phi(k'), &= \nu(E_\lambda)\phi(u(x_k)) + \nu(\Omega \setminus E_\lambda)\phi(u(x_{k'})) \\
&= \int_\Omega \phi(u(f_\lambda))d\nu \\
&= \int_\Omega \phi(u(x_{f_\lambda}))d\nu \\
&= \phi(u(x_{f_\lambda})) \in \operatorname{Im}\phi \ .
\end{aligned}$$

Thus, $\phi : \mathbb{R} \to \mathbb{R}$ is strictly increasing and has a convex image. It follows that ϕ is continuous.

The uniqueness of the representation follows by standard arguments. ■

Lemma B.13: *Suppose* $(\Omega, \mathcal{G}, \mathcal{M})$ *is a structured space and there exist a utility function* $u : X \to \mathbb{R}$*, a convex statistical distance* $c : \Delta \times \mathcal{M} \to [0, \infty]$*, a strictly increasing and continuous function* $\phi : \operatorname{Im} u \to \mathbb{R}$ *and a prior* $\mu \in \Delta^\sigma(\mathcal{M}, \mathcal{D}_\mathcal{M})$ *such that* $\succsim$ *is represented on* $\mathcal{F}$ *by*

$$\int_\mathcal{M} \phi\left(I^m(u(f))\right) d\mu(m)$$

where I^m *is defined as in* (5)*. Then, there exists a probability measure* $\nu \in \Delta^\sigma(\Omega, \mathcal{A})$ *such that the restriction of* $\succsim$ *to* $\mathcal{F}(\mathcal{A})$ *is represented by*

$$\int_\Omega \phi(u(f)) d\nu.$$

Moreover, ν *is nonatomic if* μ *is nonatomic.*

Proof of Lemma B.13: Suppose the premise holds and define the following measure: for all $A \in \mathcal{A}$,

$$\nu(A) = \int_\mathcal{M} m(A) d\mu(m)$$

and notice that $\nu \in \Delta^\sigma(\Omega, \mathcal{A})$ and $\nu(\Omega_0) = 1$. Moreover, for all $D \in \mathcal{D}_\mathcal{M}$, since

$$m \in D \implies m\left(\{\omega \in \Omega : \mathfrak{q}(\omega) \in D\}\right) \geq m\left(\{\omega \in \Omega : \mathfrak{q}(\omega) = m\}\right) = 1,$$

$$m \notin D \implies m\left(\{\omega \in \Omega : \mathfrak{q}(\omega) \in D\}\right) \leq 1 - m\left(\{\omega \in \Omega : \mathfrak{q}(\omega) = m\}\right) = 0$$

then,

$$\begin{aligned}
\nu \circ \mathfrak{q}^{-1}(D) &= \nu\left(\{\omega \in \Omega : \mathfrak{q}(\omega) \in D\}\right) \\
&= \int_\mathcal{M} m(\{\omega \in \Omega : \mathfrak{q}(\omega) \in D\})\, d\mu(m) \\
&= \int_D m(\{\omega \in \Omega : \mathfrak{q}(\omega) \in D\})\, d\mu(m) + \int_{\mathcal{M}\setminus D} m(\{\omega \in \Omega : \mathfrak{q}(\omega) \in D\})\, d\mu(m) \\
&= \int_D 1 d\mu(m) = \mu(D).
\end{aligned}$$

Therefore, for any $\psi \in B_0(\Omega, \mathcal{A})$, we have that

$$\begin{aligned}\int_{\mathcal{M}} \phi\left(I^m(\psi)\right) d\mu(m) &= \int_{\mathcal{M}} \phi\left(I^m(\psi)\right) d(\nu \circ \mathfrak{q}^{-1})(m) \\ &= \int_{\Omega_0} \phi\left(I^{\mathfrak{q}(\omega)}(\psi)\right) d\nu(\omega) \\ &= \int_{\Omega} \phi\left(I_{\mathcal{A}}(\psi)(\omega)\right) d\nu(\omega) \\ &= \int_{\Omega} \phi\left(\psi\right) d\nu.\end{aligned}$$

where we apply the change of variable formula and $I_{\mathcal{A}}$ is the generalized common conditional expectation of $(I^m)_{m\in\mathcal{M}}$ given $\mathcal{A}$ of Corollary 1. It follows that for all $f, g \in \mathcal{A}$, $f \succsim f$ if and only if

$$\int_{\Omega} \phi(u(f))d\nu \geq \int_{\Omega} \phi(u(g))d\nu.$$

as we wanted to show.

Furthermore, assume that μ is notatomic. We show that also ν is non-atomic. To this end, take $E \in \mathcal{A}$ such that $\nu(A) > 0$. Then, there exists by Lemma B.10 a set $D_E \in \mathcal{D}_{\mathcal{M}}$ such that $I^m(\chi_E) = \chi_{D_E}(m)$ for all $m \in \mathcal{M}$. Then,

$$\mu(D_E) = \int_{\mathcal{M}} \chi_{D_E}(m)d\mu(m) = \int_{\mathcal{M}} I^m(\chi_E)d\mu(m) = \int_{\mathcal{M}} m(E)d\mu(m) = \nu(E) > 0$$

where we use the fact that $m(E) \in \{0, 1\}$ for all $m \in \mathcal{M}$. Since μ is nonatomic, there exists a subset $D_0 \subseteq D_E$ in $\mathcal{D}_{\mathcal{M}}$ such that $0 < \mu(D_0) < \mu(D_E)$. Again by Lemma B.10, we can find $E^{D_0} \in \mathcal{A}$ such that $\chi_{D_0}(m) = I^m(\chi_{E^{D_0}})$ for all $m \in \mathcal{M}$. Then, let $E_0 \coloneqq E \cap E^{D_0} \subseteq E$. We have that for all $m \in \mathcal{M}$,

$$\chi_{D_0} = \chi_{D_0}\ \chi_{D_E} = I^m(\chi_{E^{D_0}})\ I^m(\chi_E) = I^m(\chi_{E_0})$$

where again we use Lemma B.1. Therefore,

$$\nu(E_0) = \int_{\mathcal{M}} m(E_0) d\mu(m) = \int_{\mathcal{M}} I^m(\chi_{E_0}) d\mu(m) = \int_{\mathcal{M}} \chi_{D_0} d\mu(m) = \mu(D_0)$$

so that $0 < \nu(E_0) < \nu(E)$, proving that ν is nonatomic. ■

Proof of Theorem 2: *(i) implies (ii).*

We know that $\succsim$ is represented by u when restricted to constant acts. Define the functional $I : B_0(\mathcal{G}) \to \mathbb{R}$ such that for each $\varphi \in B_0(\mathcal{G})$, $I(\varphi) \coloneqq u(x_{f_\varphi})$, where $f_\varphi \in \mathcal{F}$ is chosen so that $\varphi = u(f_\varphi)$. By Lemma B.2, such act f_φ exists for all $\varphi \in B_0(\mathcal{G})$, while the certainty equivalent $x_{f_\varphi} \sim f_\varphi$ exists by Lemma B.3. Moreover, for any $\varphi \in B_0(\mathcal{G})$, if there are two $f_\varphi, f'_\varphi \in \mathcal{F}$ such that $u(f_\varphi) = \varphi = u(f'_\varphi)$, we then have that since u represents $\succsim$ over X,

$$\begin{aligned} u(f_\varphi)(\omega) = u(f'_\varphi)(\omega) &\implies u(f_\varphi(\omega)) = u(f'_\varphi(\omega)) \\ &\implies f_\varphi(\omega) \sim f'_\varphi(\omega) \end{aligned}$$

for all $\omega \in \Omega$. By Axiom 1.(ii) of monotonicity, it follows that $f_\varphi \sim f'_\varphi$ and, by transitivity, that $x_{f_\varphi} \sim x_{f'_\varphi}$. Therefore, we can conclude that

$$I(\varphi) = u(x_{f_\varphi}) = u(x_{f'_\varphi}) = I(f'_\varphi)$$

showing that I is a well-defined functional on $B_0(\mathcal{G})$. It is easily seen that such functional is also normalized, monotone, and continuous.[19]

Define the function $V \coloneqq I \circ u : \mathcal{F} \to \mathbb{R}$. For all $f, f' \in \mathcal{F}$,

$$\begin{aligned} f \succsim f' &\iff x_{f'} \succsim x_{f'} \\ &\iff V(f) = I(u(f)) = u(x_f) \geq u(x_{f'}) = I(u(f)) = V(f') \ . \end{aligned}$$

[19]See for example the proof of Theorem 1 (Omnibus) in the working paper version of Cerreia-Vioglio et al. (2022).

This shows that V represents $\succsim$ on $\mathcal{F}$. Moreover, by Proposition 1, for each $m \in \mathcal{M}$, $\succsim^m$ is represented by $I^m \circ u$, where $I^m : B(\mathcal{G}) \to \mathbb{R}$ is as defined in (12). Moreover, let $I_{\mathcal{A}}$ be the generalized conditional expectation from Corollary 1. Take now $\varphi, \psi \in B_0(\mathcal{G})$ such that $I^m(\varphi) \geq I^m(\psi)$ for all $m \in \mathcal{M}$. By Lemma B.2, we can find $f_\varphi, f_\psi \in \mathcal{F}$ such that $\varphi = u(f_\varphi)$ and $\psi = u(f_\psi)$. Then, $I^m(u(f_\varphi)) \geq I^m(u(f_\psi))$ for all $m \in \mathcal{M}$ so that $f_\varphi \succsim^m f_\psi$ for all $m \in \mathcal{M}$. Consistency implies that $f_\varphi \succsim f_\psi$. Therefore:

$$I(\varphi) = I(u(f_\varphi)) = V(f_\varphi) \geq V(f_\psi) = I(u(f_\psi)) \geq I(\psi) \ .$$

Moreover, by Lemma B.12, there exist an unbounded and affine $\tilde{u} : X \to \mathbb{R}$, a strictly increasing $\phi : \mathbb{R} \to \mathbb{R}$, and a non-atomic $\nu \in \Delta^\sigma(\Omega, \mathcal{A})$ such that the restriction of $\succsim$ to $\mathcal{F}(\mathcal{A})$ is represented by the functional:

$$f \mapsto \phi^{-1}\left(\int_\Omega \phi(\tilde{u}(f))d\nu\right) \ .$$

Moreover, since $\Omega \setminus \Omega_0$ is null, $\nu(\Omega \setminus \Omega_0) = 0$. Without loss of generality, we can assume that $\tilde{u} = u$ and normalize $\phi(0) = 0$ and $\phi(1) = 1$. Now, define the map $J : B(\mathcal{A}) \to \mathbb{R}$ such that

$$J(\varphi) = \phi^{-1}\left(\int_\Omega \phi(\varphi)d\nu\right)$$

for all $\varphi \in B(\mathcal{A})$. Since ϕ is continuous and strictly increasing, J is well-defined, normalized, and continuous. Moreover, for all $f, g \in \mathcal{F}(\mathcal{A})$,

$$f \succsim g \iff J(u(f)) \geq J(u(g)) \ .$$

Moreover, take any $\varphi \in B_0(\mathcal{A})$. By Lemma B.2, we can choose $f_\varphi \in \mathcal{F}(\mathcal{A})$ such that $\varphi = u(f_\varphi) = f_\varphi$. Then, since both V and $J \circ u$ represent $\succsim$ on $\mathcal{F}(\mathcal{A})$,

$$I(\varphi) = I(u(f_\varphi)) = V(f_\varphi) = u(x_{f_\varphi}) = J\left(u(f_\varphi)\right) = J(\varphi) \ .$$

We conclude that $I(\varphi) = J(\varphi)$ for all $\varphi \in B_0(\mathcal{A})$. Take now any $\varphi \in B_0(\mathcal{G})$. Since $I_{\mathcal{A}}(\varphi) \in B(\mathcal{A})$ and $B_0(\mathcal{A})$ is dense in $B(\mathcal{A})$, we can pick sequences $(\psi^l_n)_{n\in\mathbb{N}}, (\psi^u_n)_{n\in\mathbb{N}} \in B_0(\mathcal{A})$ such that $\psi^l_n \nearrow I_{\mathcal{A}}(\varphi)$ and $\psi^u_n \searrow I_{\mathcal{A}}(\varphi)$ uniformly. Fix any $m \in \mathcal{M}$. Since I^m is monotone, we have that for all $n \in \mathbb{N}$:

$$I^m(\psi^l_n) \leq I^m(I_{\mathcal{A}}(\varphi)) \leq I^m(\psi^u_n) \ .$$

By Proposition 1, we also have that $I^m(I_{\mathcal{A}}(\varphi)) = I^m(\varphi)$ and, therefore, we have that for all $n \in \mathbb{N}$,

$$I^m(\psi^l_n) \leq I^m(\varphi) \leq I^m(\psi^u_n) \ .$$

Since m was chosen arbitrarily, this holds for all $m \in \mathcal{M}$. This and the fact that I and J coincide on $B_0(\mathcal{A})$ imply that for all $n \in \mathbb{N}$:

$$J(\psi^l_n) = I(\psi^l_n) \leq I(\varphi) \leq I(\psi^u_n) = J(\psi^u_n)$$

Passing to the limit and using the fact that J is continuous, we obtain that:

$$J(I_{\mathcal{A}}(\varphi)) \leq I(\varphi) \leq J(I_{\mathcal{A}}(\varphi)) \ .$$

That is:

$$\begin{aligned} I(\varphi) &= J(I_{\mathcal{A}}(\varphi)) \\ &= \phi^{-1}\left(\int_{\Omega} \phi\left(I_{\mathcal{A}}(\varphi)\right)\nu(d\tilde{\omega})\right) \\ &= \phi^{-1}\left(\int_{\Omega_0} \phi\left(\min_{p\in\Delta}\left\{\int_{\Omega} \varphi \ dp + c(p, \mathfrak{q}(\tilde{\omega}))\right\}\right)\nu(d\tilde{\omega})\right) \ . \end{aligned}$$

Finally, since $\mathfrak{q}_0 = \mathfrak{q}|_{\Omega_0}$ is a measurable transformation from $(\Omega_0, \mathcal{A}_0)$ to $(\mathcal{M}, \mathcal{D}_{\mathcal{M}})$, define the image measure $\mu \coloneqq \nu \circ \mathfrak{q}_0^{-1} \in \Delta^\sigma(\mathcal{M}, \mathcal{D}_{\mathcal{M}})$. Then, by Theorem 16.23 in

Billingsley (1995):

$$\begin{aligned} I(\varphi) &= \phi^{-1}\left(\int_{\Omega_0} \phi\left(\min_{p\in\Delta}\left\{\int_\Omega \varphi\ dp + c(p,\mathfrak{q}(\tilde{\omega}))\right\}\right) d\nu(\tilde{\omega})\right) \\ &= \phi^{-1}\left(\int_{\mathcal{M}} \phi\left(\min_{p\in\Delta}\left\{\int_\Omega \varphi\ dp + c(p,\mathfrak{q}(\tilde{\omega}))\right\}\right) d(\nu\circ\mathfrak{q}^{-1})(m)\right) \\ &= \phi^{-1}\left(\int_{\mathcal{M}} \phi\left(\min_{p\in\Delta}\left\{\int_\Omega \varphi\ dp + c(p,m)\right\}\right) d\mu(m)\right) \ . \end{aligned}$$

But, then, $\succsim$ is represented on $\mathcal{F}$ by

$$V(f) = I(u(f)) = \phi^{-1}\left(\int_{\mathcal{M}} \phi\left(\min_{p\in\Delta}\left\{\int_\Omega u(f)\ dp + c(p,m)\right\}\right) d\mu(m)\right)$$

as we wanted to show. Next, we show that if $\chi_D = T(\chi_E)$ for $E \in \mathcal{A}$ and $D \in \mathcal{D}_{\mathcal{M}}$, then $\nu(E) = \mu(D)$. Indeed,

$$\begin{aligned} \phi^{-1}(\nu(E)) &= \phi^{-1}\left(\int_{\Omega_0} \phi(\chi_E) d\nu\right) \\ &= J(\chi_{E^D}) = J(I_{\mathcal{A}}(\chi_E)) = I(\chi_E) \\ &= \phi^{-1}\left(\int_{\mathcal{M}} \phi\left(I(\chi_E,m)\right) d\mu(m)\right) \\ &= \phi^{-1}\left(\int_{\mathcal{M}} \phi\left(\chi_D\right) d\mu(m)\right) \\ &= \phi^{-1}(\mu(D)). \end{aligned}$$

and since ϕ^{-1} is strictly increasing, this implies that $\nu(E) = \mu(D)$. We now show that μ is nonatomic. Take $D \in \mathcal{D}_{\mathcal{M}}$ such that $\mu(D) > 0$. By Lemma B.10, there exists $E^D \in \mathcal{A}$ such that $\chi_D = T(\chi_{E^D})$. Therefore, by what shown above, $\nu(E^D) = \mu(D) > 0$. Since ν is nonatomic, we can find $E_0 \in \mathcal{A}$ such that $E_0 \subseteq E^D$ and $\nu E_0 > 0$. By Lemma B.10, we can find $D_{E_0} \in \mathcal{D}_{\mathcal{M}}$ such that $\chi_{D_{E_0}} = T(\chi_{E_0})$. Suppose $m \in D_{E_0}$. Then, $I(\chi_{E_0}, m) = 1$ and since $E_0 \in \mathcal{A}$, it must be the case that $m(E_0) = 1$. Since $E_0 \subseteq E^D$, $m(E^D) = 1$ and, therefore, $\chi_D(m) = I(\chi_{E^D}, m) = 1$, so that $m \in D$. This shows that $D_{E_0} \subseteq D$. Moreover, by what shows above $\mu(D_{E_0}) = \nu(E_0) > 0$.

This shows that μ is non-atomic. It only remains to show that ϕ is concave. Take $r_1, r_2 \in \mathbb{R}$ and $\alpha = 1/2$. Since ν is nonatomic, we can find E such that $\nu(E) = 1/2$. Moreover, we can pick $x_1, x_2 \in X$ such that $r_1 = u(x_1)$ and $r_2 = u(x_2)$. Then:

$$\begin{aligned} J(u(x_1 E x_2)) &= \phi^{-1}\left(\int_\Omega \phi\left(u(x_1 E x_2)\right) d\nu\right) \\ &= \phi^{-1}\left(\frac{1}{2}\phi(r_1) + \frac{1}{2}\phi(r_2)\right) \\ &= \phi^{-1}\left(\int_\Omega \phi\left(u(x_2 E x_1)\right) d\nu\right) = J(u(x_2 E x_1)). \end{aligned}$$

Thus, $x_1 E x_2 \sim x_2 E x_1$. Since $\succsim$ satisfies uncertainty aversion, it follows that

$$\frac{1}{2}x_1 + \frac{1}{2}x_2 = \frac{1}{2}x_1 E x_2 + \frac{1}{2}x_2 E x_1 \succsim x_1 E x_2$$

and, therefore, since ϕ is increasing:

$$\begin{aligned} \phi\left(\frac{1}{2}r_1 + \frac{1}{2}r_2\right) &= \phi\left(J\left(u\left(\frac{1}{2}x_1 + \frac{1}{2}x_2\right)\right)\right) \\ &\geq \phi\left(J\left(u\left(x_1 E x_2\right)\right)\right) \\ &= \frac{1}{2}\phi(r_1) + \frac{1}{2}\phi(r_2). \end{aligned}$$

This shows that ϕ is midpoint concave. Since it is also strictly increasing on the interval $\mathbb{R}$, we conclude it is concave.

Uniqueness follows by standard arguments.

(ii) implies (i). It is clear that $\succsim$ satisfies Axioms 1-5 are satisfied. Moreover, by Lemma B.13, there exists a nonatomic probability measure $\nu \in \Delta^\sigma(\Omega, \mathcal{A}$ such that the restriction of $\succsim$ to $\mathcal{F}(\mathcal{A})$ is represented by the functional

$$\int_\Omega \phi(u(f))d\nu.$$

This implies that $\succsim$ satisfies Savage (1954)'s P2-P6 when restricted to $\mathcal{F}(\mathcal{A})$. ■

Proof of Theorem 3: *(i) implies (ii).* We know that there exists an affine $u : X \to \mathbb{R}$ and a normalized, monotone, continuous, and quasiconcave functional $I : B_0(\mathcal{G}) \to \mathbb{R}$ such that $\succsim$ is represented by $I \circ u$ on $\mathcal{F}$. By Proposition 1, we know that for each $m \in \mathcal{M}$, there exists I^m given as in (5) such that $I^m \circ u$ represents $\succsim^m$ on $\mathcal{F}$. By consistency, we also know that for all $\varphi, \psi \in B_0(\mathcal{G})$, $I^m(\varphi) \geq I^m(\psi)$ for all $m \in \mathcal{M}$ implies that $I(\varphi) \geq I(\psi)$. Therefore, by Proposition B.1, there exists a unique normalized, monotone, and continuous $\hat{I} : B_0(\mathcal{D}_{\mathcal{M}})$ such that $\hat{I}(I(\varphi, \cdot)) = I(\varphi)$ for all $\varphi \in B_0(\mathcal{G})$. Moreover, $\hat{I}$ is quasiconcave and monotone continuous. Take $\xi \in B_0(\mathcal{D}_{\mathcal{M}})$. By Lemma B.10, we can find a $\varphi \in B_0(\mathcal{A})$ such that $\xi = T(\varphi)$ and $f \in B_0(\mathcal{G})$ such that $\varphi = u(f)$. Notice that since there exists a K such that $\xi(m) \geq K$ for all $m \in \mathcal{M}$, $r_0 \coloneqq \inf_{m \in \mathcal{M}} \xi(m) \geq K$ and, therefore, $r_0 \in \mathbb{R}$. Pick $r > r_0$. Then, we can find $x_0, x \in X$ such that $r_0 = u(x_0)$ and $r = u(x)$. Take a sequence $(\alpha_n) \in (0,1)$ such that $\alpha_n \downarrow 0$ and let $x_n = \alpha_n x + (1-\alpha_n)x_0$. Fix any $n \in \mathbb{N}$. By affinity of u,

$$u(x_n) = \alpha_n u(x) + (1-\alpha_n)u(x_0) = \alpha_n r + (1-\alpha_n) r_0 > r_0 = \inf_{m \in \mathcal{M}} \xi(m) = \inf_{m \in \mathcal{M}} I(u(f), m).$$

Therefore, there exists $m_n \in \mathcal{M}$ such that $u(x_n) > I(u(f), m_n)$. This implies that $x_n \succ_m f$ and, therefore, Caution implies that $x_n \succsim f$. That is,

$$\alpha_n r + (1-\alpha_n) r_0 = u(x_n) \geq I(u(f)) = I(\varphi) = \hat{I}(\xi).$$

This holds for all $n \in \mathbb{N}$ and passing to the limit, we obtain $r_0 \geq \hat{I}(\xi)$. On the other hand, we have that for all $m \in \mathcal{M}$, $r_0 = \inf_{m' \in \mathcal{M}} \xi(m') \leq \xi(m)$ and, therefore, since $\hat{I}$ is normalized and monotone, $r_0 = \hat{I}(r_0) \leq \hat{I}(\xi)$. It follows that $\hat{I}(\xi) = r_0 = \inf_{m \in \mathcal{M}} \xi(m)$. Therefore, $\hat{I}$. Now, for all $\xi, \xi' \in B(\mathcal{D}_{\mathcal{M}}$,

$$\hat{I}(\xi) - \hat{I}(\xi') = \inf_{m \in \mathcal{M}} \xi(m) - \inf_{m \in \mathcal{M}} \xi'(m) \leq \inf_{m \in \mathcal{M}} (\xi(m) - \xi(m)).$$

Thus, $\hat{I}$ is a niveloid, and it is, therefore, Lipschitz continuous. It follows that it admits a unique, monotone, and continuous extension to $B(\mathcal{D}_{\mathcal{M}})$, wich, abusing notation, we also denote $\hat{I}$. Then, pick any $\xi \in B(\mathcal{D}_{\mathcal{M}})$. Since $B_0(\mathcal{D}_{\mathcal{M}})$ is dense in $B(\mathcal{D}_{\mathcal{M}})$, we can find two sequences $(\xi_n^u)_n, (\xi_n^l)_n$ such that $\xi_n^u \searrow \xi$ and $\xi_n^l \nearrow \xi$. Since $\hat{I}$ is monotone, we have that for all $n \in \mathbb{N}$, $\xi_n^l \leq \xi \leq \xi_n^u$ and, therefore,

$$\hat{I}(\xi_n^l) = \inf_{m\in\mathcal{M}} \xi_n^l(m) \leq \inf_{m\in\mathcal{M}} \xi(m) \leq \inf_{m\in\mathcal{M}} \xi_n^u(m) = \hat{I}(\xi_n^u).$$

Since $\hat{I}$ is continuous, passing to the limit, we obtain that $\hat{I}(\xi) = \inf_{m\in\mathcal{M}} \xi$. Therefore, we have that for all $\varphi \in B_0(\mathcal{G})$,

$$\begin{aligned}\hat{I}(I(\varphi,\cdot)) &= \inf_{m\in\mathcal{M}} \min_{p\in\Delta(\mathcal{G})} \int_\Omega \varphi dp + c(p,m) \\ &= \inf_{p\in\Delta(\mathcal{G})} \inf_{m\in\mathcal{M}} \int_\Omega \varphi dp + c(p,m) \\ &= \inf_{p\in\Delta(\mathcal{G})} \int_\Omega \varphi dp + \inf_{m\in\mathcal{M}} c(p,m).\end{aligned}$$

Suppose that in addition $\mathcal{M}$ is closed and $\succsim$ satisfies. Fix any $\varphi \in B(\mathcal{G})$. Take any $r \in \mathbb{R}$ and pick $f \in \mathcal{F}$ and $x_r \in X$ such that $\varphi = u(f)$ and $r = u(x_r)$. Then:

$$\begin{aligned}\{m \in \mathcal{M} : I(\varphi,m) \leq r\} &= \{m \in \mathcal{M} : I^m(u(f)) \leq u(x_r)\} \\ &= \{m \in \mathcal{M} : x_r \succsim^m f\}\end{aligned}$$

and the latter is closed by axiom. Therefore, $m \mapsto I(\varphi,m)$ is lower semicontinuous. Therefore, the functional $\tilde{I}_\varphi : \Delta(\mathcal{G} \times \mathcal{M} \to \mathbb{R}$ defined as $\tilde{I}_\varphi(p,m) := I(\varphi,m) - \int_\Omega \varphi dp$ is lower semicontinuous in (p,m). Then, since

$$c(p,m) = \sup_{\varphi\in B_0(\mathcal{G})} \left\{ I(\varphi,m) - \int_\Omega \varphi dp \right\} = \sup_{\varphi\in B_0(\mathcal{G})} \tilde{I}_\varphi(p,m)$$

for all $(p,m) \in \Delta \times \mathcal{M}$ and by the theorem of the maximum (see Aliprantis and Border (2007), Lemma 17.29), we can conclude that c is lower semicontinuous in

(p, m). Then applyin Aliprantis and Border (2007), Lemma 17.30 twice, we obtain that $\inf_{m\in\mathcal{M}} c(\cdot, m) = \min_{m\in\mathcal{M}} c(\cdot, m)$ is lower semicontinuous and, therefore,

$$\begin{aligned}\hat{I}(I(\varphi, \cdot)) &= \inf_{p\in\Delta(\mathcal{G})} \int_\Omega \varphi dp + \inf_{m\in\mathcal{M}} c(p, m) \\ &= \min_{p\in\Delta(\mathcal{G})} \int_\Omega \varphi dp + \min_{m\in\mathcal{M}} c(p, m).\end{aligned}$$

Since $\varphi \in B_0(\mathcal{G})$ was arbitrarily chosen, we conclude that this holds everywhere on $B_0(\mathcal{G})$. Therefore, for all $f, g \in \mathcal{F}$,

$$\begin{aligned}f \succsim g &\iff I(u(f)) \geq I(u(g)) \\ &\iff \hat{I}(I(u(f), \cdot)) \geq \hat{I}(I(u(g), \cdot)) \\ &\iff \min_{p\in\Delta(\mathcal{G})} \int_\Omega u(f)dp + \min_{m\in\mathcal{M}} c(p, m) \geq \min_{p\in\Delta(\mathcal{G})} \int_\Omega u(g)dp + \min_{m\in\mathcal{M}} c(p, m)\end{aligned}$$

as we wanted to show. ■