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Abstract

We consider a principal-agent model with moral hazard, bilateral risk-neutrality, and limited liability. The principal knows only some of the actions the agent can take and evaluates contracts by their guaranteed payoff over possible unknown actions. We show that linear contracts are a robustly optimal way to incentivize the agent: any randomization over contracts can be improved by making each contract in its support linear. We then identify an optimal random linear contract characterized by a single parameter that bounds its continuous support. Several corollaries arise: the gain from randomization can be arbitrarily large; optimal randomization does not require commitment; and screening cannot improve the principal's guarantee.

1 Introduction

Recent years have witnessed a surge in contract-theoretic research on the optimal design of incentives when the designer does not know all the details of the contracting environment, or is otherwise reluctant to commit to a single probabilistic model of the situation. For moral hazard problems, the predominant approach is to assume that the designer evaluates each contract according to its payoff guarantee, i.e., its worst-case payoff across a class of

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environments. This approach has been successful in providing a possible foundation for linear contracts, a commonly observed contract form—see Carroll (2015) and subsequent work by, e.g., Dai and Toikka (2022), Walton and Carroll (2022), Carroll and Bolte (2023), Marku, Ocampo Diaz, and Tondji (2024), and Vairo (2025). Moreover, it does so without the need to resort to specific assumptions about how actions map to outputs and costs, a challenge for the Bayesian contracting literature.

The robust analysis of moral hazard problems, however, typically restricts attention to deterministic contracts. This restriction is substantive: Kambhampati (2023) shows in the framework of Carroll (2015) that randomizing over two linear contracts strictly improves the principal's guarantee compared to any optimal deterministic contract, one of which is linear. This raises the concern that the intuition about robust optimality of interest alignment the deterministic result formalizes is driven by restrictions on the contract space, rather than an inherent coherence between robustness and linearity. Moreover, because Kambhampati's (2023) result leaves open the form of optimal (necessarily random) contracts, it is no longer clear that robustness leads to contracts that are linear or, in any sense, simple.

Motivated by these concerns, we re-visit the canonical robust moral hazard problem of Carroll (2015) allowing for random contracts. In the model, the principal knows only some actions the agent can take and evaluates contracts based on their guaranteed payoff over all possible action sets containing the known actions. The agent is protected by limited liability and both parties are risk-neutral. The principal commits to a randomization over contracts and offers the realized (deterministic) contract to the agent. This allows the principal to hedge against the Knightian uncertainty regarding the agent's action set.

We first show that the alignment of the principal's and the agent's interests achieved by linear contracts provides a robustly optimal way to incentivize the agent, whether the contract is chosen randomly or not. Specifically, any mixture over contracts can be improved by making each contract in its support linear. The model thus retains the prediction from the deterministic case that the agent is offered a linear contract.

Our proof is based on explicitly formulating the guarantee of a (finitely supported) random contract as a linear program for adversarial Nature. This problem is in general a multidimensional mechanism design problem, where the agent's type is the realized contract. Nature designs for each type an action subject to usual incentive compatibility constraints and a type-dependent participation constraint stemming from the presence of actions known to the principal. Our proof proceeds by using the dual to Nature's problem to identify a random linear improvement contract. Applied to the deterministic case, the approach yields a short proof of Carroll's (2015) result, with linear programming duality used in place of his separating hyperplane argument.

Having established that it suffices to randomize over linear contracts, we consider the problem of choosing an optimal mixture over a grid of such contracts. This problem is made tractable by turning the principal's max-min problem into a max-max problem. This can be done by taking the dual of Nature's minimization problem. The max-max problem is, of course, simply a maximization problem and can be solved for analytically. This leads us to identify a closed-form solution for the optimal randomization over a fixed grid. As the size of the grid grows large, we obtain a guess of the form of an optimal random contract—a random linear contract characterized by a single parameter that bounds its continuous support.

To verify that the limit contract is optimal, we construct a saddle point in the zero-sum game played between the principal and Nature. This construction yields two important corollaries. First, given the equilibrium strategy of Nature, commitment to randomization is unecessary. Second, any randomization over menus of contracts cannot outperform our optimal random contract. That is, screening would not help despite only the agent knowing the true production technology.

Regarding the issue of the simplicity of robustly optimal contracts, our results admit two interpretations. On one hand, the optimal random contract we identify is simple because the agent is only ever offered a linear contract. Moreover, the randomization belongs to a one-parameter family of distributions. On the other hand, a new layer of complexity is introduced because it is optimal to randomize over a continuum of contracts.

That randomization may provide a hedge against Knightian uncertainty, or ambiguity, was suggested by Raiffa (1961) in response to Ellsberg (1961). If we adopt the interpretation that the environment is chosen by adversarial Nature, then the issue can be phrased as a question of timing or observability. Randomization mitigates uncertainty if Nature chooses an action set prior to the realization of a random contract. In fact, in our setting, we show that gain from randomization may be arbitrarily large. On the other hand, randomization has no value if Nature chooses an action set after the realization in which case the analysis reduces to that of Carroll (2015). Decision theory provides axiomatizations for both attitudes and even uncertainty regarding the timing of Nature's move (see, e.g., Saito, 2015; Ke and Zhang, 2020).

In practice, if the uncertainty regards the realization of some physical process such as the choice of agent or the determination of feasible production technologies, then it seems reasonable to assume that the resolution of a lottery over contracts just prior to one is presented to the agent should have no effect on these phenomena, and randomization should thus have value. Interestingly, this is the implicit position adopted in much of the recent literature on robust mechanism design (see, e.g., Carrasco, Luz, Kos, Messner, Monteiro, and Moreira (2018) and Che and Zhong (2024), who find random prices to be robustly optimal in single- and multi-dimensional screening environments). Allowing for randomization in max-min problems is also standard in computer science.

This work is related to the literature seeking foundations for linear and other commonly observed contracts following Holmström and Milgrom (1987), who concluded their fully Bayesian analysis of the optimality of linear intertemporal incentives by suggesting that the reason for the popularity of linear schemes might be their great robustness, and that the case for it could perhaps be made more effectively with a non-Bayesian model. We refer the reader to Carroll (2019) for a survey of this literature. Subsequent work has extended and generalized Carroll's (2015) approach to other environments with moral hazard and established the robust optimality of linear contracts therein (see the works cited in the opening paragraph). Related models are studied also by Antic (2021), Antic and Georgiadis (2024), Rosenthal (2023), Burkett and Rosenthal (2024), and Kambhampati (2024), who find worst-case optimal deterministic contracts different from linear contracts.

The rest of this paper is organized as follows. Section 2 sets up the model. Section 3 establishes that any finitely supported contract can be weakly improved by a randomization over linear contracts with a weakly smaller support. Section 4 sketches how to identify optimal random contracts defined on a grid and supplies a guess for the form of the optimal random contract by taking the size of the grid large. Section 5 verifies the guess by constructing a saddle point in the zero-sum game between the principal and Nature. Section 6 concludes with a discussion of corollaries and extensions of the main results. The Appendix contains proofs omitted from the main text.

2 Model

We consider the problem of a principal designing a contract to motivate an agent subject to moral hazard, with the principal facing non-quantifiable uncertainty over the agent's set of feasible actions.

Throughout, we write $uv := \sum_{i=1}^{d} u_i v_i$ for the usual inner product of any two vectors v and u in $\mathbb{R}^d$. Moreover, for any topological space B, we use $\Delta(B)$ to denote the set of all finitely supported probability measures on B and $\bar{\Delta}(B)$ to denote the set of all Borel probability measures on B.

Let $Y := \{y_1, \dots, y_n\} \subset \mathbb{R}_+$ be the finite set of possible output levels, labeled in increasing order so that $0 \equiv y_1 < \cdots < y_n$. It will be convenient to view the possible output levels as a vector $y = (y_1, \dots, y_n) \in \mathbb{R}^n_+$ and introduce the index set $I := \{1, \dots, n\}$.

An (unobservable) action for the agent is a pair $(\pi, c) \in \Delta(Y) \times \mathbb{R}_+$, where $\pi = (\pi_1, \dots, \pi_n)$ is the output distribution and c is the associated cost to the agent. A *tech-*

nology is a nonempty compact set of actions $A \subset \Delta(Y) \times \mathbb{R}_+$.

The agent is protected by limited liability, requiring payments to him to be non-negative. A *(deterministic) contract* is thus a non-negative n-vector $w = (w_1, \ldots, w_n) \in \mathbb{R}^n_+$, where w_i is the payment from the principal to the agent if output y_i is realized.

Both parties are assumed risk-neutral. Thus, if the agent plays the action (π, c) given contract w, then the principal's expected payoff is $\pi(y - w) = \sum_{i \in I} \pi_i(y_i - w_i)$, whereas the agent gets $\pi w - c = \sum_{i \in I} \pi_i w_i - c$.

The principal does not know the actual technology available to the agent. She is only aware of some technology A^0, referred to as the *known technology*, and views any technology $A \supseteq A^0$ as possible. We assume throughout that the known technology A^0 contains a surplus-generating action, i.e., there exists $(\pi, c) \in A^0$ such that $\pi y - c > 0$.

Faced with this uncertainty, the principal evaluates contracts based on their guaranteed performance over all technologies that contain the known one. To state this formally, given a contract w and a technology A, denote the agent's maximum payoff by

$$U(w|A) := \max_{(\pi,c) \in A} \pi w - c. \tag{2.1}$$

In the special case of the known technology A^0, we write

$$U^0(w) := U(w|A^0). \tag{2.2}$$

We note for future reference that $U^0(w)$ is continuous in the contract w by the theorem of the maximum. The corresponding payoff to the principal is

$$V(w|A) := \min_{(\pi,c) \in A} \pi(y - w) \quad \text{s.t.} \quad \pi w - c = U(w|A). \tag{2.3}$$

Note, in particular, that if the agent has multiple maximizers, then ties are broken against the principal. The principal's *guarantee* from the contract w is then

$$V(w) := \inf_{A \supseteq A^0} V(w|A). \tag{2.4}$$

We assume that the principal can hedge against the uncertainty over the agent's technology by randomizing over contracts. Formally, a *random contract* is a probability measure p on $\mathbb{R}^n_+$. Then, $\Delta(\mathbb{R}^n_+)$ is the space of finitely supported random contracts and $\bar{\Delta}(\mathbb{R}^n_+)$ is the space of all random contracts.

Given any random contract p, the agent observes the realized contract w before choosing an action. We can thus extend the principal's payoff in (2.3) to random contracts by setting

$$V(p|A) := \mathbb{E}_p[V(w|A)]. \tag{2.5}$$

Similarly, we extend the guarantee (2.4) to random contracts by setting

$$V(p) := \inf_{A \supseteq A^0} V(p|A). \tag{2.6}$$

The principal's *optimal guarantee* is the supremum of the guarantee (2.6) over all random contracts, or $\sup_{p \in \bar{\Delta}(\mathbb{R}^n_+)} V(p)$. A random contract is *optimal* if it attains this optimal guarantee. In contrast, the *optimal deterministic guarantee* is the supremum of (2.4) over all contracts, or equivalently, the supremum of (2.6) over all degenerate random contracts (i.e., contracts whose support is a singleton). A contract w is an *optimal deterministic contract* if it attains the optimal deterministic guarantee. We note that the existence of a known surplus-generating action ensures that the optimal deterministic guarantee, and hence the optimal guarantee, is positive.

Linear contracts play a central role in the analysis. A deterministic contract w is *linear* if $w = \alpha y$ for some slope $\alpha \in [0,1]$. A random contract $p \in \bar{\Delta}(\mathbb{R}^n_+)$ is *linear*, or a *random linear contract*, if every contract in the support of p is linear, i.e., for all $w \in \text{supp}(p)$, there exists a slope $\alpha \in [0,1]$ such that $w = \alpha y$. Whenever convenient, we identify each (deterministic) linear contract with its slope, and then identify the set of random linear contracts with $\bar{\Delta}([0,1])$.

Some remarks are in order regarding the formulation of the problem:

1. Because we assume that the agent observes the realized contract before choosing an action, the agent need not know the underlying random contract, or even be aware that the contract was generated via randomization. In fact, because of bilateral risk-neutrality, giving the agent a random contract would not be useful: Given any action (π, c), the principal's and the agent's payoffs from a randomized contract p would be $\pi(y - \mathbb{E}_p[w])$ and $\pi\mathbb{E}_p[w] - c$, and thus we could equivalently use the deterministic contract $\tilde{w} = \mathbb{E}_p[w]$.

2. Our contracting environment is the same as in Carroll (2015), save for the following minor differences. First, we take the set of outputs, Y, to be finite (rather than just compact) to minimize technicalities. This is not required to prove Proposition 3, but it allows for simple duality-based proofs in Section 3. Second, we assume adversarial tie-breaking in (2.3), whereas Carroll broke ties in the principal's favor. Adversarial tie-breaking could be argued to better capture the spirit of the robustness exercise,

and it simplifies the analysis as the infimum in (2.6) becomes a minimum. However, as we will see, both assumptions lead to the same optimal guarantee and the optimal contract is similarly unaffected by tie-breaking except for an uninteresting corner case.

3 Linear improvement argument

The duality approach we adopt to prove the optimality of random linear contracts also yields a short proof of the optimality of linear contracts within the class of deterministic contracts. It is instructive to see the argument first in this simpler case. We then generalize from the deterministic case by showing that, for any finite random contract, there exists a random linear contract with weakly smaller support that obtains at least the same payoff guarantee.

3.1 Deterministic case

Consider a deterministic contract w. We formulate the guarantee (2.4) of w as a linear program. To this end, note that, given any technology $A \supseteq A^0$, only the action chosen by the agent matters for the principal's payoff $V(w|A)$ in (2.3).[1] It thus suffices to take the infimum in (2.4) over technologies that add at most one new action to the known technology A^0. It follows that the contract's guarantee is characterized by the following linear program:

$$V(w) = \min_{\pi, c} \pi(y - w) \tag{3.1}$$

$$\text{s.t.} \quad \pi w - c \geq U^0(w), \tag{3.2}$$

$$\sum_{i=1}^{n} \pi_i = 1, \tag{3.3}$$

$$c, \pi_i \geq 0 \quad \forall i \in I. \tag{3.4}$$

That is, Nature designs an action (π, c) to minimize the principal's profit, with constraint (3.2) ensuring that (π, c) is a best response for the agent since, by (2.2), $U^0(w)$ is the agent's maximum payoff from the known technology A^0. Problem (3.1–3.4) is feasible because taking (π, c) to be an action in A^0 that attains $U^0(w)$ satisfies all constraints.

We can now show that any contract can be weakly improved upon by some linear contract.

Proposition 1. *For every contract w, there is a linear contract α such that $V(w) \leq V(\alpha)$.*

Proof. Fix a contract w. The guarantee $V(w)$ is clearly bounded from below by $-\max_i w_i$.

[1] I.e., if (π^*, c^*) attains the minimum in (2.3) given w and $A \supseteq A^0$, then $V(w|A) = V(w|A^0 \cup \{(\pi^*, c^*)\})$.

By strong duality, the following dual to problem (3.1–3.4) is then feasible and bounded:

$$V(w) = \max_{\lambda,\mu} \lambda U^0(w) + \mu \tag{3.5}$$

$$\text{s.t.} \quad \lambda w_i + \mu \le y_i - w_i \quad \forall i \in I, \tag{3.6}$$

$$\lambda \ge 0. \tag{3.7}$$

Let (λ^*, μ^*) be an optimal solution. Evaluating (3.6) at $i = 1$ gives $\mu^* \le -(1+\lambda^*)w_1 \le 0$ as $y_1 = 0$. Define a new contract w' as follows. For each i, choose w'_i to satisfy (3.6) as an equality, i.e., let

$$w'_i := \frac{y_i - \mu^*}{\lambda^* + 1}. \tag{3.8}$$

By inspection, w' is an affine function of output. As the original contract w also satisfies (3.6), we have $w' \ge w \ge 0$. Thus, w' is a well-defined contract. Moreover, we have

$$V(w') \ge \lambda^* U^0(w') + \mu^* \ge \lambda^* U^0(w) + \mu^* = V(w), \tag{3.9}$$

where the first inequality follows because (λ^*, μ^*) is by construction of w' still feasible when w is replaced with w' in the dual, and the second inequality follows because $w' \ge w$ and $U^0(w)$ is weakly increasing in w by inspection of (2.2). We conclude that the original contract w is outperformed by the affine contract w'.

To get a linear contract, we may drop the lump-sum payment $-\mu^*/(\lambda^* + 1) \ge 0$ from w' by letting $w'' := y/(\lambda^* + 1)$. This leaves the agent's choice from every technology unchanged, but weakly increases the principal's payoff. Thus, $V(w'') \ge V(w') \ge V(w)$. □

A reader familiar with Carroll's (2015) proof will see the parallels between the arguments, with duality here replacing the separating hyperplane theorem. Given the equivalence of linear programming duality and the separating hyperplane theorem, at a deeper level the proofs are the same. However, adopting a linear programming perspective seems to allow a somewhat more concise argument.[2]

[2] The proof of Proposition 1 extends mutatis mutandis to any compact set $Y \subset \mathbb{R}_+$ such that $\min Y = 0$, with a contract then defined to be a continuous function $w : Y \to \mathbb{R}_+$. The primal problem (3.1–3.4) then becomes one of choosing a regular nonnegative Borel measure π on Y and a cost $c \in \mathbb{R}$ to solve

$$V(w) = \min \int_Y (y - w(y))\, d\pi(y) \ \text{ s.t. } \int_Y w(y)\, d\pi(y) - c \ge U^0(w), \ \int_Y 1\, d\pi(y) = 1, \text{ and } c \ge 0.$$

The dual problem (3.5–3.7) in turn becomes one of choosing numbers $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{R}$ to

$$\max \ \lambda U^0(w) + \mu \ \text{ s.t. } \lambda w(y) + \mu \le y - w(y) \ \forall y \in Y, \text{ and } \lambda \ge 0.$$

As the zero contract has $V(0) \ge 0$, we may assume that $V(w) > 0$. By weak duality, this implies that the

Proposition 1 implies that linear contracts are optimal, in the following sense:

Corollary 1. *The optimal deterministic guarantee satisfies* $\sup_{w \in \mathbb{R}^n_+} V(w) = \sup_{\alpha \in [0,1]} V(\alpha)$.

Proof. Clearly $\bar{v} := \sup_{w \in \mathbb{R}^n_+} V(w) \geq \sup_{\alpha \in [0,1]} V(\alpha)$. For the converse, let w_n be a sequence such that $V(w_n) \to \bar{v}$. By Proposition 1, there is a sequence of linear contracts α_n such that $V(w_n) \leq V(\alpha_n) \leq \sup_{m \geq n} V(\alpha_m)$ for all n. Hence, $\bar{v} \leq \limsup_n V(\alpha_n) \leq \sup_{\alpha \in [0,1]} V(\alpha)$. □

Having established the optimality of linear contracts as a class, it only remains to characterize the optimal deterministic guarantee and the optimal linear contracts to conclude the analysis of deterministic contracts. This task is accomplished in Appendix A.1, which largely follows from the analysis in Carroll (2015), but serves to illustrate that adversarial and principal-optimal tie-breaking yield essentially the same results.

3.2 Random contracts

We now consider random contracts. To facilitate our duality-based arguments, we temporarily restrict attention to finitely supported random contracts (which are dense in the space of all contracts). The following results generalize Proposition 1 and Corollary 1 to such contracts.

Proposition 2. *For every finitely supported random contract* $p \in \Delta(\mathbb{R}^n_+)$*, there is a finitely supported random linear contract* $q \in \Delta(\mathbb{R}^n_+)$ *such that*

$$V(p) \leq V(q) \ \text{and} \ |\text{supp}(p)| \geq |\text{supp}(q)|\,.$$

Corollary 2. *The optimal guarantee from finitely supported random contracts satisfies*

$$\sup_{p \in \Delta(\mathbb{R}^n_+)} V(p) = \sup_{p \in \Delta([0,1])} V(p).$$

Proof. Same as Corollary 1, mutatis mutandis. □

We note that because of the conclusion regarding supports, Proposition 2 nests Proposition 1 as a special case by taking p to be a degenerate random contract. More generally, Proposition 2 implies that random linear contracts are optimal given any upper bound on the size of a contract's support (e.g., because of computational or complexity considerations).

dual is also bounded. Moreover, if we let $\lambda' = 1$ and $\mu' = \min\{y - 2w(y) : y \in Y\} - 1$, then (λ', μ') satisfies all dual constraints as strict inequalities. Therefore, strong duality holds for this pair of semi-infinite linear programs (see, e.g., Lai and Wu, 1992, Theorem 2.1). The rest of the argument now proceeds as above, with the affine improvement contract constructed analogously to (3.8) given some optimal dual solution.

In order to establish Proposition 2, the first step is to formulate the guarantee (2.6) as a linear program. Let $p \in \Delta(\mathbb{R}^n_+)$ be a finitely supported random contract. Enumerate the contracts in the support of p so that $\text{supp}(p) = \{w^1, \ldots, w^k\}$ and denote the index set by $T := \{1, \ldots, k\}$. We will write $p_t := p(w^t)$ for the probability of contract t. In searching for a worst-case technology against p, it suffices to consider technologies $A \supseteq A^0$ that have at most k new actions, one for each possible realized contract.[3] The guarantee $V(p)$ is thus characterized by the following finite linear program, which we refer to as the primal problem:

$$V(p) = \min_{\{(\pi^t, c_t)\}} \sum_{t \in T} p_t \pi^t (y - w^t) \tag{3.10}$$

$$\text{s.t.} \quad \pi^t w^t - c_t \geq \pi^s w^t - c_s \qquad \forall t, s \in T : t \neq s, \qquad (\kappa_{ts}) \tag{3.11}$$

$$\pi^t w^t - c_t \geq U^0(w^t) \qquad \forall t \in T, \qquad (\lambda_t) \tag{3.12}$$

$$\sum_{i \in I} \pi^t_i = 1 \qquad \forall t \in T, \qquad (\mu_t) \tag{3.13}$$

$$c_t, \pi^t_i \geq 0 \qquad \forall i \in I, \, t \in T. \tag{3.14}$$

That is, Nature designs, for each contract w^t in the support of p, an action (π^t, c_t) the agent will take if contract w^t is realized. This problem can be interpreted as a multi-dimensional mechanism design problem where the agent's type t corresponds to a contract $w^t \in \mathbb{R}^n_+$, the allocation is an output distribution $\pi \in \Delta(Y)$, and the transfer is a cost c, constrained to be nonnegative. The constraint (3.12) is a participation constraint that ensures that each type t prefers its own action (π^t, c_t) to any of the known actions in A^0. (Note that the utility from this outside option depends on the type t.) The new constraint (3.11), absent from Nature's problem studied in the deterministic case, is an incentive compatibility constraint that ensures that each type t prefers its own action (π^t, c_t) to the action (π^s, c_s) of every other type s.

Problem (3.10–3.14) is feasible because taking each (π^t, c_t) to be an action in A^0 that attains $U^0(w^t)$ satisfies all constraints. (Put differently, Nature can always just give the agent the known technology A^0.) Moreover, its value is clearly bounded from below by $-\max_{i,t} w^t_i$, and thus an optimal solution exists.

As in the deterministic case, we will study the dual to Nature's problem. Strong duality

[3] To see this, given any technology $A \supseteq A^0$, let $a^t := (\pi^t, c_t)$ denote the action the agent chooses from A if contract w^t is realized. (That is, a^t attains $V(w^t|A)$.) It is straightforward to verify that, for all $t \in T$, we have $V(w^t|A) = V(w^t|A^0 \cup \{a^1, \ldots, a^k\})$. This implies that

$$V(p|A) = \sum_{t \in T} p_t V(w^t|A) = \sum_{t \in T} p_t V(w^t|A^0 \cup \{a^1, \ldots, a^k\}) = V(p|A^0 \cup \{a^1, \ldots, a^k\}).$$

Therefore, for any $A \supseteq A^0$, there exists a technology $A' \supseteq A^0$ with $\left|A' \setminus A^0\right| \leq k$ such that $V(p|A') = V(p|A)$.

implies that the following dual is feasible and bounded, with $V(p)$ the optimal value:

$$V(p) = \max_{\kappa,\lambda,\mu} \sum_{t\in T} \lambda_t U^0(w^t) + \sum_{t\in T} \mu_t \tag{3.15}$$

$$\text{s.t.} \quad \lambda_t w_i^t + \mu_t + \sum_{s\neq t} \kappa_{ts} w_i^t - \sum_{s\neq t} \kappa_{st} w_i^s \leq p_t(y_i - w_i^t) \qquad \forall i \in I, t \in T, \qquad (\pi_i^t) \tag{3.16}$$

$$\lambda_t + \sum_{s\neq t} \kappa_{ts} - \sum_{s\neq t} \kappa_{st} \geq 0 \qquad \forall t \in T, \qquad (c_t) \tag{3.17}$$

$$\kappa_{ts}, \lambda_t \geq 0 \qquad \forall t, s \in T : t \neq s. \tag{3.18}$$

We will use *dual solution* to refer to any vector $(\kappa, \lambda, \mu) \in \mathbb{R}^{T(T-1)} \times \mathbb{R}^T \times \mathbb{R}^T$ satisfying constraints (3.17) and (3.18), and use the qualifiers *feasible* or *optimal* to indicate, respectively, that the dual solution is feasible or optimal in (3.15–3.18) for a given random contract $p \in \Delta(\mathbb{R}^n_+)$.

In order to make duality between problems (3.10–3.14) and (3.15–3.18) easier to verify, each non-trivial primal constraint in (3.10–3.14) has the associated dual variable displayed next to it in parenthesis. Similarly, the associated primal variable is displayed next to each non-trivial dual constraint in (3.15–3.18). It may also be instructive to compare the above dual problem to the dual (3.5–3.7) from the deterministic case; by inspection, the problems coincide if the random contract p is degenerate so that T is a singleton.

The general idea in the proof of Proposition 2 is analogous to the proof of Proposition 1 in the deterministic case: Given a random contract p, we take an optimal dual solution $(\kappa^*, \lambda^*, \mu^*)$ and use constraint (3.16) to construct, for each contract w^t in the support of p, an affine contract that dominates it from above. It will then be shown that a randomization over these affine contracts improves on p, and that a further improvement is obtained by a random linear contract.

We will need a preliminary result about the structure of optimal dual solutions. Given a dual solution (κ, λ, μ), let $G(\kappa)$ be a directed graph with vertex set $T = \{1, \ldots, k\}$ and an arc directed from t to s whenever $\kappa_{ts} > 0$. Note that if (κ, λ, μ) is an optimal solution to (3.15–3.18), then by complementary slackness, each arc in $G(\kappa)$ corresponds to a binding incentive compatibility constraint in (3.11) in the primal problem.

Lemma 1. *There is an optimal dual solution $(\kappa^*, \lambda^*, \mu^*)$ for which the graph $G(\kappa^*)$ is acyclic.*

We prove Lemma 1 in the Appendix by ε-relaxing the incentive compatibility constraints in (3.11) and using the continuity of the corresponding dual solution with respect to ε.

The next two lemmas establish the existence of affine contracts that outperform the contracts in the support of p. In showing this, we will view the dual constraint (3.16) as a system of linear inequalities in the contracts $\{w^1, \ldots, w^k\}$, in the following sense.

Definition 1. Given a dual solution (κ, λ, μ) and probabilities $(p_1, \dots, p_k) \in [0,1]^k$, the *w-system* is the system of linear inequalities in $(w^1, \dots, w^k) \in \mathbb{R}_+^{kn}$ defined by (3.16).

Given a dual solution (κ, λ, μ), define the set of *in-neighbors* of $t \in T$ in the graph $G(\kappa)$ as $N(t) := \{s \in T : \kappa_{st} > 0\}$, and the set of *out-neighbors* as $O(t) := \{s \in T : \kappa_{ts} > 0\}$. We say that contract $w \in \mathbb{R}_+^n$ is *positive affine* if $w_i = \alpha y_i + \beta$ for all $i \in I$ for some $\alpha \in (0,1]$ and $\beta \geq 0$ independent of i. (Because $y_1 = 0$, every affine contract has $\beta \geq 0$, so being positive affine is a restriction on α.)

Lemma 2. *Let $(w^1, \dots, w^k) \in \mathbb{R}_+^{kn}$ be feasible in the w-system given (κ, λ, μ) and $(p_1, \dots, p_k)$. If all in-neighbors of a contract $t \in T$ are positive affine, i.e., if the contract w^s in $(w^1, \dots, w^k)$ is positive affine for all $s \in N(t)$, then there is a positive affine contract x^t satisfying the following properties:*

1. $x^t \geq w^t$,
2. (x^t, w^{-t}) *is feasible in the w-system.*[4]

Proof. Let $t \in T$ be a contract whose every in-neighbor $s \in N(t)$ is a positive affine contract. The (i,t)-instance of constraint (3.16) then takes the form

$$\lambda_t w_i^t + \mu_t + \sum_{s \in O(t)} \kappa_{ts} w_i^t - \sum_{s \in N(t)} \kappa_{st}(\alpha_s y_i + \beta_s) \leq p_t(y_i - w_i^t),$$

or, equivalently,

$$w_i^t \leq \frac{p_t + \sum_{s \in N(t)} \kappa_{st}\alpha_s}{\lambda_t + p_t + \sum_{s \in O(t)} \kappa_{ts}} y_i + \frac{\sum_{s \in N(t)} \kappa_{st}\beta_s - \mu_t}{\lambda_t + p_t + \sum_{s \in O(t)} \kappa_{ts}} =: x_i^t. \tag{3.19}$$

As i ranges over I, the right-hand side defines a vector x^t. By construction, $x^t \geq w^t \geq 0$, and thus x^t is a contract. It is affine by inspection. To verify that x^t is positive affine, note that

$$0 < \frac{p_t + \sum_{s \in N(t)} \kappa_{st}\alpha_s}{\lambda_t + p_t + \sum_{s \in O(t)} \kappa_{ts}} \leq \frac{p_t + \sum_{s \in N(t)} \kappa_{st}}{\lambda_t + p_t + \sum_{s \in O(t)} \kappa_{ts}} \leq \frac{p_t}{p_t} = 1,$$

where the first inequality follows because $p_t, \alpha_s > 0$ and $\kappa_{ts}, \kappa_{st}, \lambda_t \geq 0$, the second follows because $\alpha_s \leq 1$ as every in-neighbor s is positive affine, and the third inequality follows because the dual solution (κ, λ, μ) satisfies (3.17) by definition.

For property 2, note that (x^t, w^{-t}) satisfies the (i,t)-instance of (3.16) with equality for all $i \in I$ by construction of x^t. Moreover, because $x^t \geq w^t$ and $\kappa_{st} \geq 0$ for all $s \neq t$, replacing w^t with x^t weakly relaxes all other instances of (3.16). ☐

[4] We use the standard shorthand $(x^t, w^{-t}) := (w^1, \dots, w^{t-1}, x^t, w^{t+1}, \dots, w^k)$.

Lemma 3. *Let* $(w^1, \dots, w^k) \in \mathbb{R}_+^{kn}$ *be feasible in the* w*-system given* (κ, λ, μ) *and* $(p_1, \dots, p_k)$*. If the graph* $G(\kappa)$ *is acyclic, then there exists a vector of positive affine contracts* $(x^1, \dots, x^k)$ *that is feasible in the* w*-system and satisfies* $x^t \geq w^t$ *for every* $t \in T$*.*

Proof. Let $(w^1, \dots, w^k) \in \mathbb{R}_+^{kn}$ be feasible in the w-system and suppose $G(\kappa)$ is acyclic. Partition T recursively as follows:

- Let $L_0 := \{t \in T : N(t) = \emptyset\}$.
- For $\ell \geq 1$, let $L_\ell := \{t \in T \setminus \cup_{m<\ell} L_m : N(t) \subseteq \bigcup_{m<\ell} L_m\}$.

That is, the zero layer L_0 consists of all contracts that have no in-neighbors in the graph $G(\kappa)$. For $\ell > 0$, the ℓ-th layer L_ℓ consists of all contracts not contained in any of the lower layers L_m, $m < \ell$, but whose in-neighbors are in these layers. Because $G(\kappa)$ is acyclic, it is straightforward to verify that there exists $\bar{\ell} \in \mathbb{N}_0$ such that L_ℓ is nonempty if and only if $0 \leq \ell \leq \bar{\ell}$, and that $\{L_1, \dots, L_{\bar{\ell}}\}$ is a partition of T.

By relabeling if necessary, we can assume without loss of generality that the contracts $(w^1, \dots, w^k)$ are labeled monotonically so that contracts with higher indices are in higher layers, in the following sense: If $t < t'$, $t \in L_\ell$, and $t' \in L_{\ell'}$, then $\ell \leq \ell'$.

We now construct the positive affine contracts $(x^1, \dots, x^k) \in \mathbb{R}_+^{kn}$ by induction on t.

Base case: Because the contracts are labeled monotonically, we have $1 \in L_0$ and thus contract 1 has no in-neighbors. Moreover, $(w^1, \dots, w^k)$ is feasible in the w-system given (κ, λ, μ) and $(p_1, \dots, p_k)$ by assumption. Thus, by Lemma 2, there is a positive affine contract $x^1 \geq w^1$ such that $(x^1, w^2, \dots, w^k)$ is feasible in the w-system.

Induction step: Let $1 < t \leq k$. Suppose there exist positive affine contracts $x^1, \dots, x^{t-1}$ with $x^s \geq w^s$ for all $1 \leq s \leq t-1$, and $(x^1, \dots, x^{t-1}, w^t, \dots, w^k)$ is feasible in the w-system. Let L_ℓ be the layer containing contract t. By definition of L_ℓ, we have $N(t) \subseteq \cup_{m<\ell} L_m$. The induction hypothesis and monotonicity of labeling imply that t's in-neighbors are positive affine. (Note that this subsumes the special case $\ell = 0$, in which case $N(t) \subseteq \cup_{m<0} L_m = \emptyset$ and the conclusion holds vacuously.) Thus, Lemma 2 again gives us a positive affine contract $x^t \geq w^t$ such that $(x^1, \dots, x^t, w^{t+1}, \dots, w^k)$ is feasible in the w-system. □

With the above results in hand, we are now ready to prove Proposition 2.

Proof of Proposition 2. Let $p \in \Delta(\mathbb{R}_+^n)$ be a random contract with $\operatorname{supp}(p) = \{w^1, \dots, w^k\}$ and let $(\kappa^*, \lambda^*, \mu^*)$ be an optimal solution to the dual (3.15–3.18) such that $G(\kappa^*)$ is acyclic, the existence of which is ensured by Lemma 1.

Because $G(\kappa^*)$ is acyclic and $(w^1, \dots, w^k)$ is obviously feasible in the w-system given $(\kappa^*, \lambda^*, \mu^*)$ and $(p_1, \dots, p_k)$, Lemma 3 gives us a vector of positive affine contracts $(x^1, \dots, x^k)$ that is feasible in the w-system and satisfies $x^t \geq w^t$ for all $t \in T$.

Consider now the dual (3.15–3.18) where we replace the contracts $(w^1, \ldots, w^k)$ with the positive affine contracts $(x^1, \ldots, x^k)$ while keeping the probabilities $(p_1, \ldots, p_k)$ unchanged:

$$\begin{aligned} V(p_1, \ldots, p_k; x^1, \ldots, x^k) := \max_{\kappa,\lambda,\mu} & \sum_{t\in T} \lambda_t U^0(x^t) + \sum_{t\in T} \mu_t \\ \text{s.t.} \quad & \lambda_t x_i^t + \mu_t + \sum_{s\neq t} \kappa_{ts} x_i^t - \sum_{s\neq t} \kappa_{st} x_i^s \leq p_t(y_i - x_i^t) && \forall i \in I, t \in T, \\ & \lambda_t + \sum_{s\neq t} \kappa_{ts} - \sum_{s\neq t} \kappa_{st} \geq 0 && \forall t \in T, \\ & \kappa_{ts}, \lambda_t \geq 0 && \forall t, s \in T : t \neq s. \end{aligned}$$

Because $(x^1, \ldots, x^k)$ is feasible in the w-system given $(\kappa^*, \lambda^*, \mu^*)$ and $(p_1, \ldots, p_k)$, the dual solution $(\kappa^*, \lambda^*, \mu^*)$ is feasible in the above problem. Therefore,

$$V(p_1, \ldots, p_k; x^1, \ldots, x^k) \geq \sum_{t\in T} \lambda_t^* U^0(x^t) + \sum_{t\in T} \mu_t^* \geq \sum_{t\in T} \lambda_t^* U^0(w^t) + \sum_{t\in T} \mu_t^* = V(p), \tag{3.20}$$

where the second inequality follows because $x^t \geq w^t$ for all t and U^0 is weakly increasing in the contract (in the pointwise order) by inspection of (2.2).

Moreover, strong duality implies that $V(p_1, \ldots, p_k; x^1, \ldots, x^k)$ is the optimal value of the primal problem (3.10–3.14), which using the positive affine form $x_i^t = \alpha_t y_i + \beta_t$ $(i \in I)$ of each contract t now becomes

$$V(p_1, \ldots, p_k; x^1 \ldots, x^k) = \min_{\{(\pi^t, c_t)\}} \sum_{t\in T} p_t(1 - \alpha_t)\pi^t y - \sum_{t\in T} p_t \beta_t \tag{3.21}$$

$$\text{s.t.} \quad \alpha_t \pi^t y - c_t \geq \alpha_t \pi^s y - c_s \qquad \forall t, s \in T : t \neq s, \tag{3.22}$$

$$\alpha_t \pi^t y - c_t \geq U^0(\alpha_t) \qquad \forall t \in T, \tag{3.23}$$

$$\sum_{i\in I} \pi_i^t = 1 \qquad \forall t \in T, \tag{3.24}$$

$$c_t, \pi_i^t \geq 0 \qquad \forall i \in I,\ t \in T. \tag{3.25}$$

Note that the feasible set is independent of the constants $\beta_t \geq 0$, $t \in T$. (They enter both sides of the incentive compatibility and participation constraints (3.22) and (3.23), and cancel out.) By inspection of (3.21–3.25), replacing each x^t with the linear contract α_t gives

$$V(p_1, \ldots, p_k; \alpha_1, \ldots, \alpha_k) = V(p_1, \ldots, p_k; x^1, \ldots, x^k) + \sum_{t\in T} p_t \beta_t \geq V(p_1, \ldots, p_k; x^1, \ldots, x^k),$$

which together with (3.20) implies $V(p_1, \ldots, p_k; \alpha_1, \ldots, \alpha_k) \geq V(p)$.

To conclude the proof, define the random linear contract q by setting $q(\alpha) = \sum_{t:\alpha_t=\alpha} p_t$ for all $\alpha \in [0,1]$, with the sum over the empty set equal to zero, as usual.[5] It is then immediate that $|\text{supp}(q)| \leq |\text{supp}(p)|$. Furthermore, $V(q)$ is the optimal value of the problem (3.21–3.25) with $\beta_t \equiv 0$ for all t and with the additional constraint that $(\pi^t, c_t) = (\pi^s, c_s)$ for all $s, t \in T$ such that $\alpha_s = \alpha_t$. Therefore, $V(q) \geq V(p_1, \ldots, p_k; \alpha_1, \ldots, \alpha_k) \geq V(p)$. $\square$

Proposition 2 shows that every random contract $p \in \Delta(\mathbb{R}^n_+)$ is weakly improved by a random linear contract. The proof shows that p can be weakly improved by linearizing each contract in its support while keeping the probabilities fixed, thus generalizing the proof of Proposition 1 from the deterministic case. The new challenge in the random case is the presence of the incentive compatibility constraints (3.11), which create interlinkages between contracts as the action (π^t, c_t) Nature targets to contract w^t will be available to the agent also when any other contract is realized. (This is why randomization is useful in the first place.) Because the primal (3.10–3.14) is a multi-dimensional screening problem, the structure of binding incentive compatibility constraints (i.e., those with a positive shadow price) is not clear a priori. Our proof deals with this by showing that there nevertheless is an optimal solution under which the binding constraints do not form a cycle. This is enough to then use the dual to Nature's problem to construct the improvement contract.

4 Identifying optimal contracts

We now sketch how to identify an optimal randomization over a grid of linear contracts.[6] As the grid becomes dense in the unit interval, the limit of the corresponding optimal randomizations is a natural guess for the optimal random contract. (A rough intuition for why optimality is only obtained in the limit is that each contract adds an additional incentive compatibility constraint in Nature's minimization problem, thereby reducing its value.) This guess is verified to be optimal in Section 5. Because the verification proof does not formally rely on any results in this section, we proceed here somewhat informally.

We passed through Nature's problem against a randomization over linear contracts in the proof of Proposition 2. Specifically, setting $\beta_t \equiv 0$ for all $t \in T$ in problem (3.21–3.25) gives the primal problem for a random linear contract p with $\text{supp}(p) = \{\alpha_1, \ldots, \alpha_k\}$. It will be convenient to let the support of the contracts lie on an equally-spaced grid, i.e., $0 < \alpha_1 < \cdots < \alpha_k \equiv 1$ are defined by $\alpha_t := t/k$ for $t = 0, \ldots, k$. (Thus, $\alpha_{t+1} - \alpha_t = k^{-1}$.) By inspection, the problem depends on each output distribution π^t only via the expected output $e_t := \pi^t y \in [0, y_n]$. With this change of variables, we can write the principal's guarantee as

[5] If the list $\alpha_1, \ldots, \alpha_k$ includes multiple copies of some slope, then the tuple $(p_1 \ldots, p_k; \alpha_1, \ldots, \alpha_k)$ is itself not a random linear contract according to our definition as it is not a finitely supported probability on $[0,1]$.

[6] See the working paper Kambhampati, Toikka, and Vohra (2024) for further details.

follows:

$$V(p) = \min_{\{(e_t,c_t)\}} \sum_{t\in T} p_t(1-\alpha_t)e_t \tag{4.1}$$

$$\text{s.t.} \quad \alpha_t e_t - c_t \geq \alpha_t e_s - c_s \qquad \forall t,s \in T : t \neq s, \tag{4.2}$$

$$\alpha_t e_t - c_t \geq U^0(\alpha_t) \qquad \forall t \in T, \tag{4.3}$$

$$e_t \leq y_n \qquad \forall t \in T, \tag{4.4}$$

$$e_t, c_t \geq 0 \qquad \forall t \in T. \tag{4.5}$$

The agent's type is now a slope α, the allocation is an expected output $e \in [0, y_n]$, the "transfer" is a cost $c \geq 0$, and the outside option in (4.3) is still type-dependent.

Standard arguments allow us to simplify the constraints (4.2–4.5). Specifically, the upward adjacent constraints in (4.2) can be taken to hold with equality without loss of optimality. This allows us to recursively eliminate the costs up to c_1, which can then be set to zero without loss of optimality. Adding in the usual monotonicity constraint on the allocation rule to ensure incentive compatibility and dropping the upper-bound constraints (without loss of optimality) yields a simplified program:

$$V(p) = \min_{\{e_t\}} \sum_{t\in T} p_t(1-\alpha_t)e_t \tag{4.6}$$

$$\text{s.t.} \quad k^{-1}\sum_{s=1}^{t} e_s \geq U^0(\alpha_t) \qquad \forall t \in T, \qquad (\lambda_t) \tag{4.7}$$

$$e_{t+1} - e_t \geq 0 \qquad \forall t \in T \setminus \{k\}, \qquad (\theta_t) \tag{4.8}$$

$$e_t \geq 0 \qquad \forall t \in T. \tag{4.9}$$

Because the principal chooses p to maximize V, it is useful to take the dual of Nature's minimization problem to arrive at a single maximization problem characterizing the principal's optimal guarantee over the grid:

$$\max_{p,\lambda,\theta} \sum_{t\in T} \lambda_t U^0(\alpha_t) \tag{4.10}$$

$$\text{s.t.} \quad k^{-1}\sum_{s=t}^{k} \lambda_s + \theta_{t-1} - \theta_t \leq p_t(1-\alpha_t) \qquad \forall t \in T, \qquad (e_t) \tag{4.11}$$

$$\sum_{t\in T} p_t = 1 \qquad (\delta) \tag{4.12}$$

$$\lambda_t, p_t \geq 0 \qquad \forall t \in T, \tag{4.13}$$

$$\theta_t \geq 0 \qquad \forall t \in T \setminus \{k\}, \tag{4.14}$$

where $\alpha_0 \equiv 0$, $\theta_0 \equiv 0$, and $\theta_k \equiv 0$. By strong duality, the value of (4.10–4.14), i.e., the optimal guarantee over the grid, is simply

$$\min_{\{e_t\},\delta} \ \delta \tag{4.15}$$

$$\text{s.t.} \quad \delta - (1-\alpha_t)e_t \geq 0 \qquad \forall t \in T, \qquad (p_t) \tag{4.16}$$

$$k^{-1}\sum_{s=1}^{t} e_s \geq U^0(\alpha_t) \qquad \forall t \in T, \qquad (\lambda_t) \tag{4.17}$$

$$e_{t+1} - e_t \geq 0 \qquad \forall t \in T \setminus \{k\}, \qquad (\theta_t) \tag{4.18}$$

$$e_t \geq 0 \qquad \forall t \in T. \tag{4.19}$$

Identifying a solution to (4.15–4.19) is straightforward. Note first that (4.16) can be taken to hold with equality for all $t < k$ without loss of optimality; increasing each e_t so that the constraint binds relaxes the participation constraints (4.17) and satisfies the monotonicity constraints (4.18) with strict inequality (recall, $\alpha_1 < \cdots < \alpha_k$).[7] Then, expressing each allocation e_t in terms of the single parameter δ and substituting into the t-th participation constraint (4.17) yields

$$\delta \geq \frac{U^0(\frac{t}{k})}{\sum_{s=1}^{t}\frac{1}{k-s}},$$

where α_t has been replaced with t/k to simplify the right-hand side expression. To minimize δ subject to the participation constraints, it must therefore be that

$$\delta = \max_{t\in T}\frac{U^0(\frac{t}{k})}{\sum_{s=1}^{t}\frac{1}{k-s}}. \tag{4.20}$$

The value of (4.15–4.19), and hence (4.10–4.14), is thus given by (4.20).

To find the random linear contract that attains the guarantee, we can use the complementary slackness conditions, together with (4.11) and (4.12). Because all monotonicity constraints (4.18) hold with strict inequality, complementary slackness implies that $\theta_t = 0$ for all t in the corresponding solution to (4.10–4.14). Moreover, (4.17) holds with equality only at the value $t = t^*$ that solves the optimization problem in (4.20). Thus, by complementary slackness, $\lambda_t = 0$ for all $t \neq t^*$. Because (4.11) must bind for each t, it follows that $p_t = 0$ for all $t > t^*$. Moreover, for $t \leq t^*$, (4.11) yields

$$p_t = \frac{\lambda_{t^*}}{k-t},$$

[7] Because $\alpha_k = 1$, we can take $e_k = e_{k-1} + 1$ without loss of optimality.

again using $\alpha_t = t/k$. Substituting each p_t for the right-hand side expression and using the probability constraint (4.12) pins down the exact value of λ_{t^*}. It follows from simple arithmetic that the the optimal grid-based contract sets

$$p_t = \begin{cases} \frac{1}{k-t}\left(\sum_{s=1}^{t^*} \frac{1}{k-s}\right)^{-1} & \text{if } 1 \leq t \leq t^*, \\ 0 & \text{if } t^* < t \leq k. \end{cases} \tag{4.21}$$

It can be shown that as the number of grid points k grows to infinity, (4.20) converges to (5.1), the value proven to be the optimal guarantee. When there exists a positive solution to the maximization problem in (5.1), then the limit of (4.21) is the random linear contract corresponding to the cumulative distribution function satisfying (5.2), the contract proven to be optimal. Otherwise, the sequence of solutions converges to the point mass on the zero contract. We show below that this is precisely the corner case in which randomization does not have value (see Corollary 3).

5 Saddle point

To identify the optimal guarantee and an optimal contract, we construct a saddle point in the zero-sum game played between the principal and Nature using our guess from Section 4.

Proposition 3 (Optimal Guarantee and Contract)**.**

1. *The optimal guarantee is*[8]

$$\bar{v} := \max_{\alpha \in [0,1]} \frac{U^0(\alpha)}{-\ln(1-\alpha)} > 0. \tag{5.1}$$

2. *If there exists $\alpha^* > 0$ attaining the maximum in (5.1), then the optimal guarantee is achieved by the random linear contract $p^* \in \bar{\Delta}([0,1])$ corresponding to the cumulative distribution function $G_{p^*} : \mathbb{R} \to [0,1]$ defined by*

$$G_{p^*}(\alpha) := \frac{\ln(1-\alpha)}{\ln(1-\alpha^*)} \quad \textit{for } \alpha \in [0, \alpha^*]. \tag{5.2}$$

[8]By convention, we extend the quotient to all of $[0,1]$ by left and right limits. That is, let

$$\frac{U^0(0)}{-\ln(1-0)} := \lim_{\alpha \to 0^+} \frac{U^0(\alpha)}{-\ln(1-\alpha)} \in [-\infty, \infty) \quad \text{and} \quad \frac{U^0(1)}{-\ln(1-1)} := \lim_{\alpha \to 1^-} \frac{U^0(\alpha)}{-\ln(1-\alpha)} = 0,$$

where the first limit is bounded from above (because $U^0(0) \leq 0$), ensuring the existence of a maximum in (5.1). The maximum is positive because $U^0(\alpha) > 0$ for α sufficiently close to one by the existence of a known surplus-generating action. Hence, any maximizer must be strictly smaller than one.

Otherwise, the optimal guarantee is attained in the limit of a sequence of deterministic linear contracts converging to the zero contract.

The rest of this section is dedicated to the proof of Proposition 3.

5.1 Attaining $\bar{v}$

We show that the conjectured optimal contract attains $\bar{v}$, defined in (5.1) in Proposition 3. Suppose first that there exists $\alpha^* \in (0, 1)$ attaining the maximum in (5.1). Let p^* be the random linear contract corresponding to (5.2). We show that p^* obtains a guarantee of at least $\bar{v}$.

By applying the Revelation Principle, we again formulate Nature's problem of choosing a technology A to minimize the profit $V(p, A)$ as a mechanism design problem where the realized contract α is the agent's type, the expected output e is the allocation, the cost c plays the role of a transfer, and the agent's outside option is to play a known action. For any random linear contract p, let

$$L(p) := \min_{e(\cdot),c(\cdot)} \int_0^1 (1-\alpha)e(\alpha)\, dG_p(\alpha) \quad \text{s.t.} \tag{5.3}$$

$$\alpha e(\alpha) - c(\alpha) \geq \alpha e(\alpha') - c(\alpha') \qquad \forall \alpha, \alpha' \in [0,1] : \alpha \neq \alpha', \tag{5.4}$$

$$\alpha e(\alpha) - c(\alpha) \geq U^0(\alpha) \qquad \forall \alpha \in [0,1], \tag{5.5}$$

$$c(\alpha) \geq 0,\ 0 \leq e(\alpha) \leq y_n \qquad \forall \alpha \in [0,1]. \tag{5.6}$$

Note that the minimization problem (5.3–5.6) is well-defined and the existence of a minimum follows by general existence results for principal-agent problems with adverse selection (e.g., Nöldeke and Samuelson (2018), Proposition 9). It is also worth noting that constraints (5.4–5.6) are taken to hold for all $\alpha, \alpha' \in [0, 1]$, not just for $\alpha, \alpha' \in \text{supp}(p)$. This is convenient as it allows us to work with functions $e(\cdot)$ and $c(\cdot)$ defined on the full interval $[0, 1]$ even when the support of p is only a subset thereof.

The following Lemma uses standard arguments to simplify constraints (5.4–5.6). It also verifies that $L(p)$ is, in fact, the principal's payoff guarantee, taking into account the requirement that Nature's technology must be compact and the assumption of adversarial tie-breaking.

Lemma 4. *For any random linear contract p,*

$$V(p) = L(p) = \min_{e(\cdot)} \int_0^1 (1-\alpha)e(\alpha)\, dG_p(\alpha) \quad s.t. \tag{5.7}$$

$$e(\cdot) \text{ is nondecreasing,} \tag{5.8}$$

$$\int_0^\alpha e(t)dt \geq U^0(\alpha) \quad \forall \alpha \in [0,1], \tag{5.9}$$

$$e(0) \geq 0,\ e(1) \leq y_n. \tag{5.10}$$

Now, suppose $e^*(\cdot)$ is a feasible solution to (5.8–5.10) and attains the minimum in (5.3). By Lemma 4 and the definition of G_{p^*}, the guarantee of p^* is given by

$$\begin{aligned} V(p^*) = \int_0^1 (1-\alpha)e^*(\alpha)dG_{p^*}(\alpha) &= \int_0^{\alpha^*} (1-\alpha)e^*(\alpha)d\left(\frac{\ln(1-\alpha)}{\ln(1-\alpha^*)}\right) \\ &= \int_0^{\alpha^*} \frac{e^*(\alpha)}{-\ln(1-\alpha^*)}d\alpha = \frac{\int_0^{\alpha^*} e^*(\alpha)d\alpha}{-\ln(1-\alpha^*)} \geq \frac{U^0(\alpha^*)}{-\ln(1-\alpha^*)} = \bar{v}, \end{aligned}$$

where the inequality is from (5.9). This establishes the desired result.

We now consider the corner case in which the unique maximizer of (5.1) is $\alpha^* = 0$. We show that $\bar{v}$ is attained in the limit of a sequence of deterministic contracts converging to the zero contract. From the formula for the deterministic guarantee given in (A.2), this is equivalent to showing that

$$\lim_{\alpha \to 0^+} \frac{1-\alpha}{\alpha} U^0(\alpha) \geq \bar{v}.$$

Because the left-hand side expression is larger than zero (by (A.1)), the inequality holds if $\bar{v} < 0$. If $\bar{v} \geq 0$, then it must be that $U^0(0) = 0$. Hence, the known technology A^0 contains a zero cost action. Let e^0 denote the maximum expected output among actions with zero cost. Suppose $(\alpha_k)_k$ is a positive sequence that converges to zero. Take any corresponding sequence of (principal least-preferred) best-responses in A^0, $(e(\alpha_k), c(\alpha_k))_k$. Then,

$$\alpha_k e^0 \leq \alpha_k e(\alpha_k) - c(\alpha_k) \iff c(\alpha_k) \leq \alpha_k(e(\alpha_k) - e^0) \leq \alpha_k y_n,$$

where the last inequality follows because output is bounded below by 0 and above by y_n. Hence, as $k \to \infty$, it must be that $c(\alpha_k) \to 0$. Because the tail of $(e(\alpha_k), c(\alpha_k))_k$ belongs to a compact set $[e^0, y_n] \times [0, \bar{c}]$ for some $\bar{c} > 0$, we may extract a convergent subsequence $(\alpha_{k_\ell})_{k_\ell}$ with $e(\alpha_{k_\ell}) \to \bar{e}$ and $c(\alpha_{k_\ell}) \to 0$. Thus,

$$\bar{v} = \lim_{\alpha \to 0^+} \frac{U^0(\alpha)}{-\ln(1-\alpha)} = \lim_{k_\ell \to \infty} \frac{U^0(\alpha_{k_\ell})}{-\ln(1-\alpha_{k_\ell})} = \lim_{k_\ell \to \infty} \frac{\alpha_{k_\ell} e(\alpha_{k_\ell})}{-\ln(1-\alpha_{k_\ell})} = \bar{e}.$$

On the other hand,

$$\lim_{\alpha\to 0^+}\frac{1-\alpha}{\alpha}U^0(\alpha)=\lim_{k_\ell\to\infty}\frac{1-\alpha_{k_\ell}}{\alpha_{k_\ell}}U^0(\alpha_{k_\ell})=\lim_{k_\ell\to\infty}\frac{1-\alpha_{k_\ell}}{\alpha_{k_\ell}}\left(\alpha_{k_\ell}e(\alpha_{k_\ell})-c(\alpha_{k_\ell})\right)=\bar{e}\geq\bar{v}.$$

5.2 $\bar{v}$ is an upper bound

We conclude the proof of Proposition 3 by showing that no contract can obtain a guarantee higher than $\bar{v}$, again defined in (5.1) in Proposition 3. It suffices to exhibit a technology $A \supseteq A^0$ such that no *deterministic* contract can attain a payoff higher than $\bar{v}$ against A (no matter how ties are broken among optimal actions for the agent). For the purposes of constructing such a technology, let $\alpha^* \in [0, 1)$ attain the maximum in (5.1) (by footnote 8, $\alpha^* < 1$). Moreover, let e^* be given by

$$e^*(\alpha) := \min\left\{y_n, \frac{U^0(\alpha^*)}{-(1-\alpha)\ln(1-\alpha^*)}\right\}, \quad \forall\alpha\in[0,1], \tag{5.11}$$

and let c^* be given by

$$c^*(\alpha) := \alpha e^*(\alpha) - \int_0^\alpha e^*(t)dt, \quad \forall\alpha\in[0,1].$$

Finally, let $\bar{\alpha}$ satisfy $e^*(\bar{\alpha}) = \max_{(\pi_0,c_0)\in A^0}\pi_0 y$.[9] (To see where (5.11) comes from, observe that it corresponds to making the dual constraint (4.16) bind, save for the upper bound on output.)

We now identify a worst-case action set. For every $\alpha \in [0, \bar{\alpha}]$, let

$$\begin{aligned}A(0) &:= \{(\pi,c)|\pi y \leq e^*(0), \max\{y_n,\bar{c}\}\geq c\geq 0\}\\ A(\alpha) &:= \{(\pi,c)|\pi y = e^*(\alpha), \max\{y_n,\bar{c}\}\geq c\geq c^*(\alpha)\}, \quad \forall\alpha\in(0,\bar{\alpha}],\end{aligned}$$

where $\bar{c}$ denotes the maximum cost of any action in A^0. By construction, $A := \cup_{\alpha\in[0,\bar{\alpha}]}A(\alpha)$ is compact, i.e., a technology. We show, in addition, that it contains A^0.

Lemma 5. *$A \supseteq A^0$.*

Proof. For an arbitrary $(\pi_0, c_0) \in A^0$, if $\pi_0 y \leq e^*(0)$, we have that $(\pi_0, c_0) \in A(0)$. Otherwise,

[9] Note that such an $\bar{\alpha}$ exists because $\max_{(\pi_0,c_0)\in A^0}\pi_0 y \leq y_n = e^*(1)$. Moreover,

$$e^*(0)\leq\frac{U^0(\alpha^*)}{-\ln(1-\alpha^*)}\leq\lim_{\alpha\to\alpha^*}\frac{\alpha\left(\max_{(\pi_0,c_0)\in A^0}\pi_0 y\right)}{-\ln(1-\alpha)}\leq\max_{(\pi_0,c_0)\in A^0}\pi_0 y.$$

The final inequality follows because $\alpha^* \leq -\ln(1-\alpha^*)$ for $\alpha^* \in (0, 1)$ and $\lim_{\alpha\to 0}(-\alpha/\ln(1-\alpha)) = 1$.

let $\pi_0 y = e^*(\alpha_0)$ for some $\alpha_0 \in (0, \bar{\alpha}]$. We then have

$$\begin{aligned}\alpha_0 e^*(\alpha_0) - c^*(\alpha_0) &= \int_0^{\alpha_0} e^*(t)dt = \int_0^{\alpha_0} \frac{U^0(\alpha^*)}{-(1-t)\ln(1-\alpha^*)}dt = \frac{-\ln(1-\alpha_0)U^0(\alpha^*)}{-\ln(1-\alpha^*)} \\ &\geq U^0(\alpha_0) = \max_{(\pi,c)\in A^0}(\alpha_0 \pi y - c) \geq \alpha_0 \pi_0 y - c_0 = \alpha_0 e^*(\alpha_0) - c_0,\end{aligned}$$

where the first inequality follows from the definition of α^*. Consequently, $\bar{c} \geq c_0 \geq c^*(\alpha_0)$ and $(\pi_0, c_0) \in A(\alpha_0)$. □

For any (potentially nonlinear) deterministic contract w, we show that

$$V(w, A) \leq \bar{v}.$$

Let $a_w = (\pi_w, c_w) \in A$ be a best-response of the agent. Then, a_w maximizes the agent's utility, i.e.,

$$\pi_w w - c_w = \max_{(\pi,c)\in A}(\pi w - c).$$

Suppose $a_w \in A(\alpha_w)$. Then $\pi_w y \leq e^*(\alpha_w)$ and $c_w = c^*(\alpha_w)$. Consider the following subset of A:

$$\left\{\left(t\pi_w + (1-t)0, c^*\left((e^*)^{-1}\left(te^*(\alpha_w)\right)\right)\right)\right\}_{t\in[0,1]},$$

where 0 is the n-dimensional zero vector and, by convention, $(e^*)^{-1}(y_n) := 1$. For each $t \in [0,1]$, the agent's corresponding utility is

$$u_w(t) := t\pi_w w - c^*((e^*)^{-1}(te^*(\alpha_w))).$$

Observe that $t = 1$ corresponds to the agent's best action. Hence, taking $u_w'(1)$ to be the left-derivative of u_w at 1,[10]

$$\begin{aligned}0 \leq u_w'(1) &= \pi_w w - (c^*)'\left((e^*)^{-1}(te^*(\alpha_w))\right)\left((e^*)^{-1}\right)'(te^*(\alpha_w))e^*(\alpha_w)\Big|_{t=1} \\ &= \pi_w w - (c^*)'(\alpha_w)\frac{1}{(e^*)'(\alpha_w)}e^*(\alpha_w) \\ &= \pi_w w - \alpha_w e^*(\alpha_w).\end{aligned}$$

Here, the first equality follows from the chain rule; the second equality follows from the inverse function theorem; and the last equality follows from the definition of c^*. Notice that

$$0 \leq \pi_w w - \alpha_w e^*(\alpha_w) \iff \alpha_w e^*(\alpha_w) \leq \pi_w w.$$

[10]The left-derivative is well-defined because $(e^*)^{-1}(\cdot)$ is a differentiable bijection on $(0,1)$.

Consequently, the expected payoff of the principal is at most

$$\pi_w(y - w) \leq (1 - \alpha_w)e^*(\alpha_w) = \frac{U^0(\alpha^*)}{-\ln(1 - \alpha^*)} = \bar{v}.$$

6 Discussion

We have shown that linear contracts remain robustly optimal when randomization is allowed. We have also identified an optimal random contract described by a single-parameter cumulative distribution function. Though we have focused attention on the Carroll (2015) model, we suspect both that randomization improves the principal's guarantee and that our techniques will be useful in characterizing optimal contracts in related models.[11]

We conclude by discussing corollaries and extensions of our results.

6.1 Gain from randomization

Define the *gain from randomization* as the ratio between the optimal guarantee and the optimal deterministic guarantee, or $\sup_{p \in \Delta(\mathbb{R}^n_+)} V(p) / \sup_{w \in \mathbb{R}^n_+} V(w)$. We say that the gain is *positive* if the ratio is strictly greater than 1.

Kambhampati (2023) showed that a sufficient condition for the gain from randomization to be positive is that there is an optimal deterministic contract different from the zero contract. We provide a weaker necessary and sufficient condition.

Corollary 3. *The gain from randomization is positive if and only if*

$$\frac{U^0(0)}{-\ln(1 - 0)} < \max_{\alpha \in [0,1]} \frac{U^0(\alpha)}{-\ln(1 - \alpha)}.$$

In particular, a sufficient condition for the gain from randomization to be positive is that there exists an optimal deterministic contract different from the zero contract.

Proof. Denote the maximands in the optimal guarantee (5.1) and the optimal deterministic guarantee (A.2) by

$$f(\alpha) := \frac{U^0(\alpha)}{-\ln(1 - \alpha)} \quad \text{and} \quad g(\alpha) := \frac{1 - \alpha}{\alpha} U^0(\alpha).$$

Recalling that these quotients are extended to all of $[0, 1]$ by left- and right-continuity, it is straightforward to verify that this defines continuous functions $f, g : [0, 1] \to \mathbb{R} \cup \{-\infty\}$ such

[11] For instance, the working paper by Peng and Tang (2024) considered a setting with multiple agents.

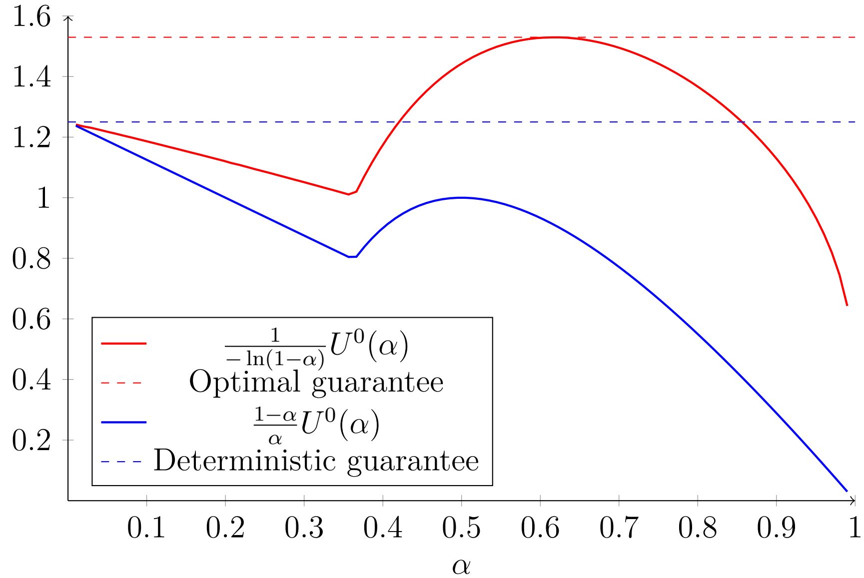

Figure 6.1. Optimal guarantee versus deterministic guarantee.

that $f(0) = g(0)$ and $f(1) = g(1) = 0$. (The first equality follows because $f(\alpha)/g(\alpha) \to 1$ as $\alpha \to 0$ by l'Hospital's rule.) Moreover, because $(1-\alpha)/\alpha < -1/\ln(1-\alpha)$ for all $\alpha \in (0,1)$, we have

$$g(\alpha) > 0 \quad \Longrightarrow \quad f(\alpha) > g(\alpha) \quad \forall \alpha \in (0,1). \tag{6.1}$$

Suppose now that the inequality in Corollary 3 is not satisfied. Then there is no gain from randomization, because $\max_\alpha f(\alpha) = f(0) = g(0) \le \max_\alpha g(\alpha)$.

In the other direction, suppose the inequality in Corollary 3 is satisfied. Let $\alpha^* > 0$ attain the maximum on the right-hand side (and hence that of f) and let α_D^* maximize g. Because the known technology contains a surplus-generating action, we have $\alpha_D^* < 1$ and $g(\alpha^*) > 0$. If $\alpha_D^* = 0$, then $g(\alpha^*) = g(0) = f(0) < f(\alpha^*)$. If instead $\alpha_D^* > 0$, then (6.1) implies that $f(\alpha^*) \ge f(\alpha_D^*) > g(\alpha_D^*)$. We conclude that either way, $\max_\alpha f(\alpha) > \max_\alpha g(\alpha)$, i.e., the gain from randomization is positive.

Finally, if there is an optimal deterministic contract, then by Proposition A.1 there is an optimal linear contract $\alpha_D^* \in (0,1)$ that maximizes g. Hence, $\max_\alpha f(\alpha) \ge f(\alpha_D^*) > g(\alpha_D^*)$ by (6.1). $\square$

The following example shows that the gain from randomization can be positive even when the optimal deterministic guarantee is obtained only in the limit of a sequence of contracts converging to the zero contract.

Example 1. Suppose the known technology A^0 consists of two actions, $(\pi, 1)$ and $(\pi', 0)$, with expected outputs $\pi y = 4$ and $\pi' y = 1+\varepsilon$, where $\varepsilon \ge 0$. Then, $U^0(\alpha) = \max\{4\alpha - 1, (1+\varepsilon)\alpha\}$.

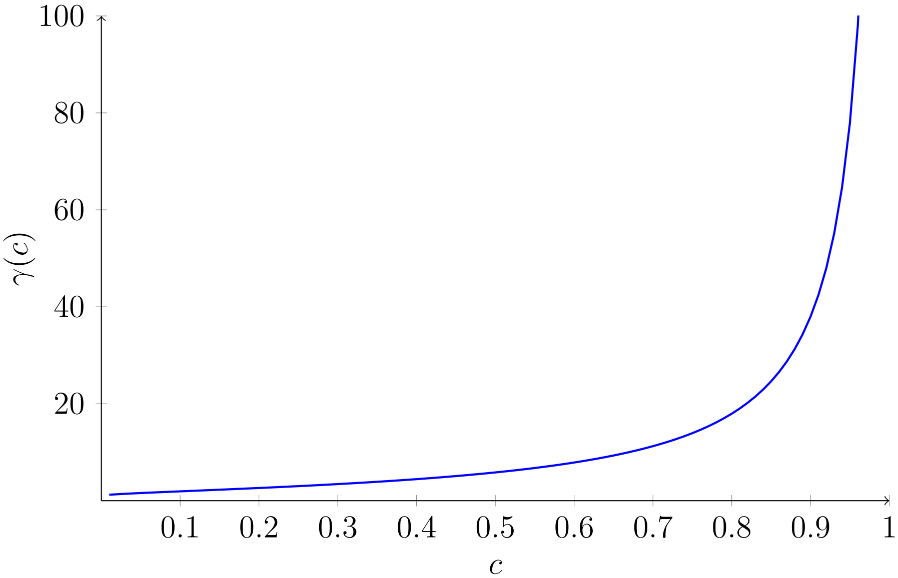

Figure 6.2. Gain from randomization.

By (A.1), when $\varepsilon = 0$, the principal is indifferent between the deterministic contract $\alpha = 1/2$ and any limit of contracts converging to the zero contract, but strictly prefers the limiting sequence when $\varepsilon > 0$. The optimal deterministic guarantee is thus $1 + \varepsilon$. Plotting the functions f and g from the proof of Corollary 3, Figure 6.1 shows, for $\varepsilon = 1/4$, that the optimal guarantee is strictly larger than the optimal deterministic guarantee. (The result can be shown to hold for any $\varepsilon > 0$ sufficiently small.) The basic intuition is that the principal optimally "targets" the positive-cost known action to increase efficiency, while using the randomization to extract enough rent from the agent to make incentivization worthwhile.

The following example shows that the gain from randomization can be arbitrarily large.

Example 2. Suppose the known technology A^0 consists of a single action (π, c), whose expected output πy is normalized to 1 and $c < 1$. Then $U^0(\alpha) = \alpha - c$. A straightforward calculation using (A.2) gives the optimal deterministic guarantee $(1 - \sqrt{c})^2$. On the other hand, the optimal guarantee is

$$\max_{\alpha \in [0,1]} \frac{\alpha - c}{-\ln(1-\alpha)}.$$

The necessary first-order condition to this problem is $-\ln(1-\alpha) - (1-\alpha)^{-1}(\alpha - c) = 0$, which can readily be verified to have a unique solution, $\alpha^*(c)$, which is a continuous increasing function of c, with $\lim_{c \to 0} \alpha^*(c) = 0$ and $\lim_{c \to 1} \alpha^*(c) = 1$. The implicit function theorem implies that its derivative on $(0, 1)$ is $(\alpha^*)'(c) = -1/\ln(1 - \alpha^*(c))$. Substituting back into the objective then gives the optimal guarantee $1 - \alpha^*(c)$. The gain from randomization is thus

$$\gamma(c) := \frac{1-\alpha^*(c)}{(1-\sqrt{c})^2}.$$

From the properties of $\alpha^*(c)$ it follows that the gain $\gamma(c)$ is a continuous function of c with $\gamma(0) = 1$. Using l'Hôpital's rule twice shows that $\lim_{c\to 1} \gamma(c) = \infty$ (see Figure 6.2). This example thus exhibits the entire range of possibilities as c varies.

6.2 Commitment

The proof of Proposition 3 constructs a saddle-point (p, A) for the principal's max-min problem, i.e., p is a best-response for the principal against A and A is a best-response for Nature against p. Hence, given the worst-case technology A, the principal is indifferent among all contracts in the support of p. It follows that commitment to randomization is not needed. That is, the principal has no incentive to deviate from any of her realized contracts.

6.3 Screening

Observe, also, that randomizing over menus of contracts cannot increase the principal's guarantee. This is again because the proof of Proposition 3 constructs a saddle-point (p, A) for the principal's max-min problem. Thus, the guarantee of any randomization over menus of contracts is no better than its performance against this particular technology A. Specifically, against A, the randomization over menus reduces to a randomization over the contracts the agent chooses from each menu given A. The resulting payoff is thus no higher than $V(p)$, because p is an optimal randomization against A.

6.4 Extensions

We have deliberately kept the model streamlined to allow for a simple and transparent analysis of the merits of linear contracts, but we note here two extensions that are immediate.

First, the improvement argument can easily accomodate the introduction of a participation constraint. The simplest way to impose a participation constraint is to assume that the known technology A^0 contains an action that results in zero output at no cost to the agent—this is subsumed by our general analysis. If instead we assume that any contract has to deliver the agent an expected payoff (net of cost) of at least $\bar{u} > 0$, then this can be handled by replacing $U^0(w)$ with $\max\{U^0(w), \bar{u}\}$ on the right-hand side of (3.12). Because each affine contract x^t constructed in the proof of Proposition 2 pointwise increases the corresponding original contract w^t , it satisfies the so-modified participation constraint (3.12).

It then follows that for every random contract, there exists a random affine contract that (weakly) outperforms it, and thus random affine contracts are optimal as a class.

Second, it is straightforward to allow for multi-dimensional output y from any finite set Y, with the payoff to the principal being $v(y)$ for some function $v : Y \to \mathbb{R}_+$. Modifying the primal problem (3.10–3.14) in the obvious way, the same argument gives an improvement from moving to a random contract that is linear in $v(y)$. Specifically, replace each y_i with $v(y_i)$ in the primal objective (3.10) and on the right-hand side of the dual constraint (3.16).

6.5 Tie-breaking

We assume that, when the agent is indifferent among multiple actions, he chooses the one least preferred by the principal. The value (5.1) remains the optimal guarantee if instead we assume principal-preferred tie-breaking. This follows because the guarantee of the optimal contract (5.2) can only increase and the proof given in Section 5.2 shows that the value (5.1) remains an upper bound. The only modification required in the statement of Proposition 3 is in the corner case in which randomization has no value. With principal-preferred tie-breaking, the optimal guarantee is attained by the zero contract with no need for approximation.

Appendix

A.1 Deterministic analysis

Consider Nature's problem (3.1–3.4) given a linear contract with slope α:

$$V(\alpha) = \min_{\pi,c} \, (1-\alpha)\pi y \quad \text{s.t.} \quad \alpha\pi y - c \geq U^0(\alpha), \quad \sum_{i=1}^{n} \pi_i = 1, \quad \text{and} \quad c, \pi_i \geq 0 \quad \forall i \in I.$$

It is clearly optimal to set $c = 0$ and choose π such that $\alpha\pi y = U^0(\alpha)^+$, where our convention is to write $b^+ = \max\{b, 0\}$ for any $b \in \mathbb{R}$. If $\alpha = 0$, then it is optimal to put $\pi y = U^0(0)^+ = 0$, and thus the guarantee from the zero contract is zero. (This is of course immediate given adversarial tie-breaking.) On the other hand, the guarantee for any positive slope $\alpha > 0$ is $V(\alpha) = (1-\alpha)U^0(\alpha)^+/\alpha$. Recalling the definition of U^0 from (2.2), we thus have

$$V(\alpha) = \begin{cases} 0 & \text{if } \alpha = 0, \\ (1-\alpha) \max_{(\pi,c)\in A^0} \left(\pi y - \frac{c}{\alpha}\right)^+ & \text{if } \alpha \in (0,1]. \end{cases} \tag{A.1}$$

By Corollary 1, the optimal deterministic guarantee is given by the supremum of the function $V : [0,1] \to \mathbb{R}_+$ defined by (A.1). The non-triviality assumption implies that $V(\alpha) > 0$ for α sufficiently close to 1, and thus the optimal guarantee is positive. It can also readily be verified to coincide with Carroll's (2015) guarantee for principal-optimal tie-breaking.[12] In what follows, we will use the fact that the optimal deterministic guarantee can also be written as

$$\sup_{w \in \mathbb{R}^n_+} V(w) = \max_{\alpha \in [0,1]} \frac{1-\alpha}{\alpha} U^0(\alpha), \tag{A.2}$$

provided that $\frac{(1-0)}{0} U^0(0)$ is taken to mean the limit as $\alpha \to 0$.

An optimal deterministic contract, which is linear, can be found by simply maximizing the guarantee (A.1). However, there is a small wrinkle as V, which is continuous on $(0,1]$, may have a downward jump in the limit as $\alpha \to 0$. This leads to the following characterization.[13]

Proposition A.1 (Optimal Deterministic Guarantee and Contract)**.** *A linear contract α is an optimal deterministic contract if and only if it maximizes the function $V : [0,1] \to \mathbb{R}_+$ defined by* (A.1)*. (Because of the existence of a known surplus-generating action, any maximizer must be interior.) If no such maximizer exists, then there does not exist an optimal deterministic contract. In this case, the optimal deterministic guarantee is attained in the limit of any sequence of linear contracts $\alpha_n > 0$ such that $\alpha_n \to 0$.*

Proof. The first claim characterizing optimal linear contracts follows by Corollary 1 and the construction of V. That there is no optimal deterministic contract when there is no linear optimal contract follows by Proposition 1. Finally, the only possible point of discontinuity of V is 0, and hence if a maximizer does not exist, then $\sup_{\alpha \in [0,1]} V(\alpha) = \lim_{\alpha \to 0^+} V(\alpha)$. □

While adversarial tie-breaking generates an existence problem relative to the case of principal-optimal tie-breaking where an optimal (linear) contract always exists, the issue is arguably minor: It only arises in the corner case where under principal-optimal tie-breaking, the zero contract is the uniquely optimal linear contract. This case seems uninteresting from the perspective of optimal incentive provision as none is required. It is incidentally also the only case where randomization does not improve the principal's payoff—see Corollary 3. A simple sufficient condition to rule it out and to ensure the existence of an optimal contract is for any known action generating a positive surplus to have a non-zero cost.

[12] Principal-optimal tie-breaking leads in general to a weakly higher guarantee from the zero contract than adversarial tie-breaking, but the guarantees are the same for any positive slope. As the guarantee under principal-optimal tie-breaking is continuous in α, this implies that the optimal guarantees are the same.

[13] It is easy to show that all optimal deterministic contracts are linear under Carroll's (2015) full support condition by verifying that then certain inequalities in the proof of Proposition 1 are strict. As this argument is similar to Carroll's, we omit it in the interest of space.

A.2 Proof of Lemma 1

We first prove a stronger result for an ε-perturbation of the problem and then establish Lemma 1 via taking the limit $\varepsilon \to 0$.

Given $\varepsilon \geq 0$, we introduce a relaxation of the incentive compatibility constraint (3.11):

$$\pi^t w^t - c_t \geq \pi^s w^t - c_s - \varepsilon \quad \forall t, s \in T : t \neq s. \tag{A.3}$$

The *ε-perturbed primal problem* consists of solving (3.10) subject to (A.3), (3.12), (3.13), and (3.14). It is feasible for all $\varepsilon \geq 0$, because it is a relaxation of the feasible primal problem (3.10–3.14). Let $V(p;\varepsilon)$ denote the value of the ε-perturbed primal problem. We clearly have $V(p;\varepsilon) \geq -\max_{i,t} w_i^t$, and hence an optimal solution exists.

Because the perturbation ε only enters the right-hand sides of some primal constraints, in the dual it only enters the objective. The *ε-perturbed dual problem* consists of solving

$$V(p;\varepsilon) = \max_{\kappa,\lambda,\mu} \sum_{t\in T} \lambda_t U(w^t) + \sum_{t\in T} \mu_t - \varepsilon \sum_{t\in T} \sum_{s\neq t} \kappa_{st} \tag{A.4}$$

subject to (3.16–3.18).

Lemma A.1. *Let $\varepsilon > 0$. If $(\kappa^*, \lambda^*, \mu^*)$ is an optimal solution to the ε-perturbed dual problem, then $G(\kappa^*)$ is acyclic.*

Proof. Let $\varepsilon > 0$ and let $(\kappa^*, \lambda^*, \mu^*)$ be an optimal solution to the ε-perturbed dual. Suppose toward contradiction that $G(\kappa^*)$ contains a cycle. By relabeling if necessary, we can assume without loss of generality that the cycle involves vertices $C := \{1, \ldots, m\}$ for $2 \leq m \leq k$, with $\kappa^*_{t,t+1} > 0$ for all $t \in C$, where $m + 1 = 1$ by convention. Similarly, we can assume without loss of generality that $p_1 = \min_{t\in C} p_t$.

By complementary slackness, the incentive compatibility constraints in (A.3) corresponding to $\kappa^*_{t,t+1}$ for $t \in C$ hold with equality. Letting $\{(\pi^t, c_t)\}_{t\in T}$ denote an optimal solution to the ε-perturbed primal, we thus have

$$\pi^t w^t - c_t - \pi^{t+1} w^t + c_{t+1} = -\varepsilon \quad \forall t \in C. \tag{A.5}$$

Summing the equalities over $t \in C$ and recalling that $m + 1 = 1$ gives

$$\sum_{t\in C} (\pi^t - \pi^{t+1}) w^t = -m\varepsilon. \tag{A.6}$$

We will show that it is then possible to strictly lower the value of the ε-perturbed primal, contradicting optimality.

Construct a new solution for the ε-perturbed primal as follows. Let $(\hat{\pi}^t, \hat{c}_t) := (\pi^t, c_t)$ for all $t \notin C$. For each $t \in C$, let

$$(\hat{\pi}^t, \hat{c}_t) := (1 - \delta_t)(\pi^t, c_t) + \delta_t(\pi^{t+1}, c_{t+1}),$$

where $\delta_t := p_1/p_t \in (0, 1]$. As the action assigned to each contract $t \notin C$ is the same as before, their contribution to the objective is unchanged. But the contracts in C now yield

$$\begin{aligned}\sum_{t\in C} p_t\hat{\pi}^t(y - w^t) &= \sum_{t\in C} p_t[(1 - \delta_t)\pi^t + \delta_t\pi^{t+1}](y - w^t)\\ &= \sum_{t\in C}[(p_t - p_1)\pi^t + p_1\pi^{t+1}](y - w^t)\\ &= \sum_{t\in C} p_t\pi^t(y - w^t) + p_1\sum_{t\in C}(\pi^t - \pi^{t+1})(w^t - y)\\ &= \sum_{t\in C} p_t\pi^t(y - w^t) - p_1 m\varepsilon,\end{aligned}$$

where the last equality follows by (A.6) and the fact that $\sum_{t\in C}(\pi^t - \pi^{t+1})y = 0$ because the sum is cyclical. Therefore, the new solution $\{(\hat{\pi}^t, \hat{c}_t)\}_{t\in T}$ is a strict improvement over the supposed optimal solution $\{(\pi^t, c_t)\}_{t\in T}$, provided we show that it is feasible.

To show feasibility, we note first that each $(\hat{\pi}^t, \hat{c}_t)$ satisfies the probability and non-negativity constraints (3.13) and (3.14). For $t \notin C$ this is trivial as $(\hat{\pi}^t, \hat{c}_t) = (\pi^t, c_t)$. For $t \in C$ this follows because $(\hat{\pi}^t, \hat{c}_t)$ is by definition a convex combination of two actions, each of which satisfies (3.13) and (3.14).

Note then that, given any contract w^t, the agent's payoff from the new action $(\hat{\pi}^t, \hat{c}_t)$ is weakly higher than from the old action (π^t, c_t) by construction:

$$\hat{\pi}^t w^t - \hat{c}_t = \begin{cases}\pi^t w^t - c_t + \delta_t(\pi^{t+1}w^t - c_{t+1} - \pi^t w^t + c_t) = \pi^t w^t - c_t + \delta_t\varepsilon & \text{if } t \in C,\\ \pi^t w^t - c_t & \text{if } t \notin C,\end{cases}$$

where the first case follows by the definition of $(\hat{\pi}^t, \hat{c}_t)$ and equation (A.5). This implies that the new solution satisfies the participation constraint (3.12) for all t, since

$$\hat{\pi}^t w^t - \hat{c}_t \geq \pi^t w^t - c_t \geq U^0(w^t).$$

It remains to verify the incentive compatibility constraint (A.3). Fix contracts t and $s \neq t$. Observe that by construction of $(\hat{\pi}^s, \hat{c}_s)$, we have $(\hat{\pi}^s, \hat{c}_s) = (1 - \beta)(\pi^s, c_s) + \beta(\pi^{s+1}, c_{s+1})$ for

$\beta \in (0,1]$ (if $s \in C$) or for $\beta = 0$ (if $s \notin C$). Therefore,

$$\hat{\pi}^t w^t - \hat{c}_t \geq \pi^t w^t - c_t \geq (1-\beta)(\pi^s w^t - c_s - \varepsilon) + \beta(\pi^{s+1} w^t - c_{s+1} - \varepsilon) = \hat{\pi}^s w^t - \hat{c}_s - \varepsilon,$$

where the second inequality follows because $\{(\pi^t, c_t)\}_{t \in T}$ satisfies (A.3) by assumption.

We conclude that the new solution $\{(\hat{\pi}^t, \hat{c}_t)\}_{t \in T}$ is feasible, a contradiction. $\square$

To prove Lemma 1, for each $n \in \mathbb{N}$, let $(\kappa_n^*, \lambda_n^*, \mu_n^*)$ be an optimal extreme point solution to the $1/n$-perturbed dual.[14] The feasible set is independent of the perturbation parameter and it contains only finitely many extreme points. Hence, one of them appears infinitely often as $n \to \infty$. Thus, by extracting a subsequence if necessary, we may assume that the sequence is constant. That is, there is an extreme point $(\kappa^*, \lambda^*, \mu^*)$ such that $(\kappa_n^*, \lambda_n^*, \mu_n^*) = (\kappa^*, \lambda^*, \mu^*)$ for all n. By Lemma A.1, the associated graph $G(\kappa^*)$ is acyclic.

We claim that $(\kappa^*, \lambda^*, \mu^*)$ is optimal in the 0-perturbed dual (3.15–3.18). To see this, note that the perturbed dual objective $f(\kappa, \lambda, \mu, \varepsilon) := \sum_{t \in T} \lambda_t U(w^t) + \sum_{t \in T} \mu_t - \varepsilon \sum_{t \in T} \sum_{s \neq t} \kappa_{st}$ is jointly continuous in $(\kappa, \lambda, \mu, \varepsilon)$. Therefore, given any feasible solution (κ, λ, μ), we have

$$f(\kappa, \lambda, \mu, 0) = \lim_n f(\kappa, \lambda, \mu, 1/n) \leq \lim_n f(\kappa^*, \lambda^*, \mu^*, 1/n) = f(\kappa^*, \lambda^*, \mu^*, 0),$$

where the inequality is because $(\kappa^*, \lambda^*, \mu^*)$ is optimal for all n by construction. We conclude that $(\kappa^*, \lambda^*, \mu^*)$ is an optimal solution to (3.15–3.18) such that $G(\kappa^*)$ is acyclic.

A.3 Proof of Lemma 4

We prove that $L(p)$ is equivalent to the problem of finding $e(\cdot)$ to attain the minimum in (5.7) subject to (5.8–5.10). Let $(e(\cdot), c(\cdot))$ be any feasible solution to (5.4–5.6). Write $U(\alpha) := \alpha e(\alpha) - c(\alpha)$ for the agent's indirect utility given contract α. Because $(e(\cdot), c(\cdot))$ satisfies (5.4), $e(\cdot)$ satisfies (5.8) and the envelope theorem (Myerson, 1981; Milgrom and Segal, 2002) implies that

$$U(\alpha) = U(0) + \int_0^\alpha e(t)dt \quad \forall \alpha \in [0,1]. \tag{A.7}$$

The feasibility constraints in (5.6) imply

$$U(0) \leq 0, \; e(0) \geq 0, \; e(1) \leq y_n, \tag{A.8}$$

[14] To see that the feasible set of the ε-perturbed dual has an extreme point (and thus an extreme point that is optimal, because the optimal value is finite) even though there are free variables, let $\mu'_t := \min_{i \in I} p_t(y_i - w_i^t)$ for all $t \in T$. Then $(\kappa, \lambda, \mu) = (0, 0, \mu')$ is an extreme point.

which clearly imply (5.10). (Here $U(0) \leq 0$ follows because $U(0) = -c(0) \leq 0$.) Finally, (A.7) and the participation constraints in (5.5) imply that

$$U(0) + \int_0^{\alpha} e(t)dt \geq U^0(\alpha) \quad \forall \alpha \in [0,1], \tag{A.9}$$

which in turn implies (5.9) because $U(0) = -c(0) \leq 0$.

We next show that $V(p) = L(p)$. The basic idea is to use the Revelation Principle to justify direct mechanisms, treating the known action as the agent's outside option. However, the requirement that any feasible technology be compact and our adversarial tie-breaking assumption create some extra work. The inequality $V(p) \geq L(p)$ follows, because any feasible technology A gives rise to a feasible solution to the minimization problem (5.3–5.6), and hence we have $V(p, A) \geq L(p)$. We will thus show that $V(p) \leq L(p)$ by using an optimal solution to (5.3–5.6) to construct a feasible technology A such that $V(p, A) = L(p)$.

Fix an optimal solution $(e^*(\cdot), c^*(\cdot))$. The obvious candidate for A is then the set $A' := \{(e^*(\alpha), c^*(\alpha)) : \alpha \in [0,1]\} \cup A_0$. However, A' need not be compact if $(e^*(\cdot), c^*(\cdot))$ is not continuous. So to deal with this, we will take A to be the closure of A'. More specifically, we will complete the proof in three steps: 1) As a preliminary result, we show that $e^*(\cdot)$ can be taken to be lower semi-continuous. 2) We then define A as the closure of A' and verify that this gives a feasible technology where $(e^*(\alpha), c^*(\alpha))$ remains a best-response to α for all $\alpha \in [0,1]$. Finally, 3) we verify that each $(e^*(\alpha), c^*(\alpha))$ satisfies the adversarial tie-breaking assumption.

Lemma A.2. *The minimization problem (5.3–5.6) has an optimal solution* $(e^*(\cdot), c^*(\cdot))$ *such that* $e^*(\cdot)$ *is lower semi-continuous.*

Proof. By Lemma 4, we may consider the minimization problem (5.7–5.10) instead. Let $e(\cdot)$ be a minimizer and denote by $D \subset [0,1]$ the set of points at which $e(\cdot)$ is discontinuous. Because $e(\cdot)$ is nondecreasing, D has at most countably many elements.

Define $e^*(\cdot) : [0,1] \to [0, y_n]$ by letting $e^*(\alpha) = e(\alpha)$ for all $\alpha \notin D$ and $e^*(\alpha) = \lim_{\alpha' \uparrow \alpha} e(\alpha')$ for all $\alpha \in D$. By construction, $e^*(\cdot)$ is nondecreasing and lower semi-continuous, and it satisfies (5.10). Moreover, because $e(\cdot)$ satisfies (5.9) and D is countable, $e^*(\cdot)$ satisfies (5.9) as well. Finally, because we have $e^*(\alpha) \leq e(\alpha)$ for all $\alpha \in [0,1]$ by construction, the allocation rule $e^*(\cdot)$ yields a weakly lower profit than the minimizer $e(\cdot)$. Therefore, $e^*(\cdot)$ is a lower semi-continuous minimizer to problem (5.7–5.10). By Lemma 4, there then exists $c^*(\cdot)$ such that $(e^*(\cdot), c^*(\cdot))$ is an optimal solution to (5.3–5.6). ☐

Lemma A.3. *Let* $(e^*(\cdot), c^*(\cdot))$ *be an optimal solution to the minimization problem* (5.3)–(5.6). *Let A be the closure of* $A' := \{(e^*(\alpha), c^*(\alpha)) : \alpha \in [0,1]\} \cup A_0$. *Then A is compact and* *thus it is a feasible technology. Furthermore, for all $\alpha \in [0,1]$, the action $(e^*(\alpha), c^*(\alpha))$ is a best response for the agent to contract α given technology A, i.e., $(e^*(\alpha), c^*(\alpha)) \in B(\alpha, A)$.*

Proof. To see that A is compact, note that it is closed by definition, and it is bounded because we have $e^*(\alpha) \in [0, y_n]$ for all α by (5.6) and $c^*(\alpha) \in [0, y_n - U^0(\alpha)]$ for all α by (5.5) and (5.6). We also have $(e_0, c_0) \in A$ by construction. Thus, A is a feasible technology.

To prove the second part, fix any $\alpha \in [0,1]$. Then

$$\alpha e^*(\alpha) - c^*(\alpha) = \max_{(e,c)\in A'} \alpha e - c = \max_{(e,c)\in A} \alpha e - c,$$

where the first equality follows by (5.4) and (5.5), and the second equality follows because the payoff $\alpha e - c$ is a continuous function of (e, c) that attains its maximum on A', and therefore the maximum cannot increase by taking the closure of A'. □

Lemma A.4. *Let $(e^*(\cdot), c^*(\cdot))$ be an optimal solution to the minimization problem (5.3–5.6) such that $e^*(\cdot)$ is lower semi-continuous. Define the feasible technology A as in Claim A.3. Then for all $\alpha \in [0,1]$, the action $(e^*(\alpha), c^*(\alpha))$ yields the lowest profit to the principal across all best responses to contract α given technology A, i.e.,*

$$(1-\alpha)e^*(\alpha) = \min_{(e,c)\in B(\alpha,A)} (1-\alpha)e \quad \forall \alpha \in [0,1],$$

and hence $V(p, A) = L(p)$.

Proof. Suppose toward contradiction that for some $\bar{\alpha} \in [0,1]$, the best response set $B(\bar{\alpha}, A)$ contains an action $(\bar{e}, \bar{c})$ such that $\bar{e} < e^*(\bar{\alpha})$.

Suppose first that $\bar{\alpha} = 0$ so that $e^*(0) > \bar{e}$. Because $e^*(\cdot)$ is nondecreasing, this implies that $e^*(\alpha) > \bar{e}$ for all $\alpha \in [0,1]$. Thus, the only way the action $(\bar{e}, \bar{c})$ can be in A is that it is a known action $(e_0, c_0) \in A_0$. This in turn implies that $e^*(\alpha) > e_0$ for all α. But this contradicts the optimality of $(e^*(\cdot), c^*(\cdot))$, because we can then strictly lower the value of the objective function by instead using the clearly feasible solution $(e(\alpha), c(\alpha)) = (e_0, c_0)$ for all $\alpha \in [0,1]$. Therefore, we must have $\bar{\alpha} > 0$.

Because the agent's payoff $\alpha e - c$ has strictly increasing differences in (α, e), the Monotone Selection Theorem implies that the selections $(e^*(\alpha), c^*(\alpha)) \in B(\alpha, A)$ and $(\bar{e}, \bar{c}) \in B(\bar{\alpha}, A)$ satisfy $e^*(\alpha) \leq \bar{e}$ for all $\alpha < \bar{\alpha}$. Therefore, we have $\lim_{\alpha\uparrow\bar{\alpha}} e^*(\alpha) \leq \bar{e} < e^*(\bar{\alpha})$, and hence $e^*(\cdot)$ is not lower semi-continuous at $\bar{\alpha}$, a contradiction. □

By Lemma A.2 there exists an optimal solution $(e^*(\cdot), c^*(\cdot))$ to (5.3–5.6) where $e^*(\cdot)$ is lower semi-continuous. Lemma A.4 then gives the existence of a feasible technology A such that $V(p, A) = L(p)$, which implies $V(p) \leq L(p)$.

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