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Information Requirements for Mechanism Design

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Abstract

Standard mechanism design begins with a statement of the problem, including knowledge on the designer's part about the distribution of the characteristics (preferences and information) of the participants who are to engage with the mechanism. There is a large literature on *robust* mechanism design, much of which aims to reduce the assumed information the designer has about the participants. In this paper we provide an auction mechanism that reduces the assumed information assumed of the seller, and, in addition, relaxes substantially the assumed information of the participants. In particular, the mechanism performs well when there are many buyers, even though there is no prior distribution over the accuracy of buyers' information on the part of the designer or the participants.

Keywords: Robustness, Optimal auctions, Incentive Compatibility, Mechanism Design, Interdependent Values, Informational Size, Common Knowledge

JEL Classifications: C70, D44, D60, D82

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1 Introduction

There is a large literature that addresses a concern raised in Wilson (1987) regarding mechanisms designed to implement desired social outcomes in the presence of asymmetric information. Typically, the construction of an incentive compatible mechanism designed to implement an efficient social outcome relies on strong common knowledge assumptions.

For many problems, this seems implausible, and has prompted researchers to search for mechanisms that are "robust" in the sense that they are less sensitive to the common knowledge assumptions typically made in the literature.[1] The robust implementation literature has made substantial strides toward understanding the degree to which the assumptions regarding the mechanism designer's information, as well as that of the agents, can be relaxed and the attendant characteristics of robust mechanisms.

In a private values auction problem in which buyers' know their own values, the second price auction is robust: irrespective of agents' beliefs, it is a dominant strategy for bidders to bid their values. While there are many auction problems for which this is the case, there are important problems for which it fails. Consider an auction for drilling rights on a particular oil tract. Bidders perform tests in order to estimate characteristics of the oil present in the tract. A bidder's signal resulting from the test may provide a very precise estimate of the amount of oil or the depth of the reservoir, while another bidder's signal may provide substantially less precise estimates. In essence, agents may have some relevant information about the tract to be auctioned that other bidders do not have. Hence, we have left the realm of private values problems: my value depends on other agents' information, their signals, as well as my own. It is known that in the interdependent value case (that is, when an agent may have both information of interest to other bidders and information of interest to her alone), second price auctions may perform poorly.[2]

Of particular interest to us is the fact that bidding one's value is *not* a dominant strategy in a second-price auction with interdependent values. Much of the robust implementation literature assumes that the mechanism designer

[1] See Bergemann and Morris (2012) and Borgers (2015), Chapter 10 for discussions of robust mechanism design.

[2] Jackson (2009) presents a simple example illustrating the problem with second price auctions when there is a mix of private and non-private information. In the example, the second price auction does not have either a symmetric equilibrium or an equilibrium in undominated strategies. The example shows that equilibrium exists only in the extremes of pure private and pure common values; existence in the private value model is not robust to a slight perturbation.

sets out the rules of the mechanism, following which the participating agents typically play a Bayes Nash equilibrium of the game induced by the mechanism. When an agent does not have a dominant strategy, her bidding strategy is a best response to other bidders' strategies given her beliefs regarding the private information of other bidders.

In this paper, we consider an auction model in which an agent's valuation for the object is the sum of a private value component and a common value component. More precisely, we assume that there are two payoff relevant states a and b (i.e., the possible amounts of oil) and bidder i receives a noisy signal s_i (i.e., a test result) from the set $\{\alpha, \beta\}$ correlated with the state. The probability that agent i receives signal α conditional on state a and the probability of β conditional on b are parameters of the problem. In addition, there are bounds on these accuracy parameters. The common value of the object is $v(a)$ in state a and $v(b)$ in state b. Bidder i also has a private valuation c_i. Consequently, an agent's payoff were he awarded the object would be $c_i + v(a)$ in state a and $c_i + v(b)$ in state b. The presence of the common value component gives rise to an auction with interdependent valuations.

If all of the data were common knowledge, then we could certainly propose a second price auction in which each player i, knowing his signal, updates his beliefs regarding the state and submits a bid for the object at auction. However, due to the interdependence of valuations in this auction game with asymmetric (but complete) information, it is no longer true that bidding one's true valuation is a dominant strategy and the Bayes-Nash equilibria may be significantly more complicated. If, in addition, the auctioneer knows the distributions of the agents' private values and signals, then a generalized VCG approach can deal with the interdependency and is reasonably satisfactory in the large numbers case that we have in mind in this paper.

McLean and Postlewaite (2004, 2017) (hereafter MP2004 and MP2017) analyze this generalized VCG approach in an interdependent value model similar to the model in this paper. These papers assume that the data of the game is common knowledge among the agents and the mechanism designer, and focuses on the role of "informational size" introduced in McLean and Postlewaite (2002). A given player's informational size in an asymmetric information problem is, roughly, the degree to which that player's information can affect, in expectation, the probability distribution over states of nature when other players truthfully reveal their private information. MP2004 shows that when each buyer's informational size is small, a seller can use a modified second price auction that generates nearly the same revenue as would be the case if the common value part of players' information were public. MP2017 shows how one can construct a two-stage mechanism for this kind of interdependent problem that extracts, and makes public, the common value part of private information in the first stage, transforming the problem in the second stage into a private value problem. The models in these papers follow the standard mechanism design approach in which there is a prior that is common knowledge among the mechanism designer and the participants in the problem and, knowing the distribution of the agents' signals, the mechanism designer can construct a reward scheme that induces agents to honestly report their types in a Bayes-Nash equilibrium of the game induced by the mechanism.

If on the other hand not all of the data are common knowledge, then the analysis is more delicate. In the model of this paper, the data essentially matches that of MP2017 but departs from MP2017 significantly in that here we do not assume that these "accuracy" parameters are common knowledge among the agents nor are they known to the mechanism. In particular, agent i's own accuracy parameter is not known to agent i. Consequently, the designer cannot construct a mechanism that induces honest reporting of signals, even if all of the other data of the problem were known by the mechanism, since the mechanism does not know the accuracy parameters. In particular, the aforementioned approaches of MP2004 and MP2017, based on VCG-like transfers, are no longer possible and a different approach is required.

To deal with the common knowledge problem, we take an epistemic viewpoint and augment the model with a type space structure in which agents hold beliefs regarding the value of the accuracy profile and the beliefs of other agents regarding the accuracy profile. It is important to distinguish between signals and types. If agent i has beliefs defined by type, say t_i, then agent i can compute the joint distribution of the accuracy profile, the state, the signals of other agents and the types of other agents.

Now, given a type space, we can take the standard game theoretic perspective and treat the problem as one of complete but asymmetric information. In this case, we would be dealing with a mechanism design problem to which the two stage methodology of MP2017 could potentially be applied in order to elicit the bidders' types. However, we are back to a situation in which the mechanism must know the probability structure defining of the type structure, an assumption we wish to avoid in the spirit of detail freeness. Consequently, we will propose a two stage mechanism in which agents report their signals, not their types, in the first stage. If the report of agent i is a majority report, then agent i, along with the other agents who have reported the same signal as i, move to stage 2. If n is the number of agents, then those who reach the second stage do not know the identities of the other second stage players. Agent i only knows that a set of agents of size at least $\frac{n}{2}$ reported the same signal that i reported.

In stage two the mechanism, with probability $1 - \varepsilon$, runs a second price auction in which only those who have advanced to the second stage can submit bids. With probability ε, the mechanism will randomly choose one of the second stage agents to receive the object outright. Players know the rules of this extensive form , including the value of ε.

A strategy for player i is a specification of a first stage report and a second stage bid if the mechanism runs an auction in stage 2. For each of i's types t_i, this (report, bid) pair is a function of i's private valuation c_i and i's signal s_i and a question remains: when bidders implement their strategies in this two stage game, will they honestly report their signals in the first stage?

In MP2004 and MP2017, the different mechanisms constructed in those papers accomplish exactly this in a perfect Bayesian equilibrium. In this paper we are interested in a mechanism that is detail free in two senses: (i) the mechanism designer need not know the type space structure and (ii) the mechanism "works" for a large class of type spaces. Given the definition of the mechanism, task (i) has been accomplished. We are able to accomplish task (ii) if we only require bidders to submit second stage bids that are not weakly dominated, i.e., that satisfy the condition of second stage admissibility defined below.

In the presence of many agents, we can use this admissibility assumption to show that truthful reporting of stage 1 signals is "approximately incentive compatible" for a large class of belief systems. Here, approximately incentive compatible means that there is a large set of agents for whom truthful announcement is optimal for every agent in the set if all other bidders report truthfully, and all bidders use undominated bids in the second stage.

Furthermore, our proposed mechanism is good for the auctioneer. In particular, for a broad class of type spaces, the undominated strategies that players use in the resulting Bayesian game of complete information yield to the seller almost all the surplus when there are many agents.

1.1 Literature review

MP2004 analyzed an interdependent value model similar to the model in this paper. That paper focused on the role of “informational size” introduced in McLean and Postlewaite (2002). A given player's informational size in an asymmetric information problem is, roughly, the degree to which that player's information can affect, in expectation, the probability distribution over states of nature when other players truthfully reveal their private information. MP2004 shows that when each buyer's informational size is small, a seller can use a modified second price auction that generates nearly the same revenue as would be the case if the common value part of players' information were public. McLean and Postlewaite MP2017 shows how one can construct two-stage mechanisms for this kind of interdependent problem that extract the common value part of private information in the first stage, transforming the problem in the second stage into a private value problem. The models in these papers follow the standard mechanism design approach in which there is a prior that is common knowledge among the mechanism designer and the participants in the problem. Knowing the distribution of the agents' signals, the mechanism designer can construct a transfer scheme that induces agents to honestly report their types in a Bayes-Nash equilibrium of the game induced by the mechanism.

Consequently, these mechanisms are not "detail free" in the sense of Wilson (1987). In this paper, we present a different two stage mechanism whose informational requirements are substantially weaker that those of MP2017. In particular, the mechanism in the current paper does *not* need to know that distribution, and indeed, does not need to know even the exact number of participating agents.

Du (2018) presents a mechanism to sell a common value object that maximizes the revenue guarantee when there is one buyer and shows that the revenue guarantee of that mechanism converges to full surplus as the number of buyers tends to infinity. Du assumes that the prior distribution of the common value is known. His mechanism, however, guarantees good revenue for every equilibrium, while as we discuss in the last section, our result focuses on "truthful revelation" outcomes.[3] Brooks and Du, *Econometrica* (2021) construct an auction mechanism for a common value problem that focusses on maxmin performance across all information structures. When the number of bidders is large, the profit guarantee is approximately the entire surplus. This takes care of the multiplicity problem.

These papers provide mechanisms for important auction problems that address the Wilson critique: the mechanism designer needs to know very little about the agents who will participate in the mechanism. While the designer's informational requirements are minimal, the participants' informational requirements often remain substantial. It is assumed that the participants will play

[3] See also a related paper by Bergemann, Brooks and Morris (2017).

a Bayes Nash equilibrium, which typically requires the participants to have substantial information about other agents. Our mechanism requires participants to know bounds on the accuracies of other agents' signals, but little more. In particular, agents are not assumed to have well-defined probabilistic beliefs about others. A negative aspect about our mechanism relative to these papers is that their results hold for all equilibria, while our result only guarantees the existence of at least one outcome with the desirable properties.

Wolitzky (2016) studies mechanism design and the possibility of weakening assumptions of agents' beliefs. Toward this end, he assumes that agents are maxmin expected utility maximizers a la Gilboa and Schmeidler (1989).[4] Our assumption about what agents know is substantially weaker, but Wolitzky's results hold for a fixed (possibly small) number of agents while our result holds for large numbers of agents.

Lastly, Yamashita (2015) studies weakly undominated strategy implementation. Unlike our analysis, Yamashita assumes the seller has a probability assessment for agents' private information and maximizes his expected utility.

2 The model and the main result

2.1 Preliminaries

Consider an auction model with n players and a single indivisible object. Player i's valuation for the object is the sum of a common value component and an idiosyncratic private value component. The private value component of player i is denoted c_i and we assume that $c_1, .., c_n$ are realizations of i.i.d. random variables taking values in $[0, 1]$. We assume that the distribution of this random variable admits a density g with corresponding distribution function G that is strictly increasing on $[0, 1]$. The common value component depends on the realization of one of two equally likely states of nature $\theta = a$ and $\theta = b$. In particular, player i's valuation for the object is given by $c_i + v(a)$ in state a and $c_i + v(b)$ in state b. Players learn the state only after the object has been allocated. However, each player receives a signal $s_i \in \{\alpha, \beta\}$ correlated with the state. The players' signals are independent conditional on the state, and i receives signal $s_i = \alpha$ (signal $s_i = \beta$) conditional on state a (state b) with

[4] Wolitzky also summarizes other recent papers examining the effect of weakening the common prior assumption.

probability $\lambda_i > \frac{1}{2}$. That is[5]

$$P_i(\alpha|a,\lambda) = \Pr ob(\tilde{s}_i = \alpha|\tilde{\theta} = a, \lambda) = \lambda_i = \Pr ob(\tilde{s}_i = \beta|\tilde{\theta} = b, \lambda) = P_i(\alpha|a,\lambda).$$

Furthermore, we assume that there exist numbers x and y such that $x < y$ and

$$\frac{1}{2} < x \le \lambda_i \le y < 1$$

for each i. We denote the set of vectors of accuracies $[x,y]^n = \{(\lambda_1, ..., \lambda_n) : \lambda_i \in [x,y]\}$, and by λ a generic element of $[x,y]^n$. For each $s = (s_1, .., s_n) \in \{\alpha, \beta\}^n$ and each i, let

$$f_\alpha^n(s_{-i}) := |\{j : s_j = \alpha \text{ and } j \neq i\}|$$

with a similar definition for $f_\beta^n(s_{-i})$. For each $s = (s_1, .., s_n) \in \{\alpha, \beta\}^n$ and each i, let

$$F_\alpha^n(s_{-i}) := \{j : t_j = \alpha \text{ and } j \neq i\}$$

with a similar definition for $F_\beta^n(s_{-i})$. Note that $|F_\alpha^n(s_{-i})| = f_\alpha^n(s_{-i})$ and $F_\beta^n(s_{-i})| = f_\beta^n(s_{-i})$.

The entire description of the game is common knowledge among the bidders and he seller **except** for the actual accuracy profile.

Consequently, the problem we treat in this paper is a significant departure from our earlier work on two stage models. Specifically, our previous work deals with a complete information environment. In this complete information setup, the seller is able to construct transfers that explicitly use the probability distribution relating states and bidders' signals and these transfers provide incentives for honest reporting in the first stage. In particular, the seller can construct a mechanism for which an equilibrium in the resulting game of asymmetric information played by the bidders implements honest first stage reporting. In the current paper, the presence of incomplete information does not allow for such a construction.

As mentioned in the introduction, we take an epistemic point of view and introduce a type space $([x,y]^n, T_1, .., T_n, \pi)$ where for each t_i, $\pi_i(\cdot|t_i) \in [x,y]^n \times T_{-i}$ defines the beliefs of player i regarding the accuracy profile and the types of other bidders. Note that T_i and π can depend on n but we suppress this dependence to lighten the notation. For simplicity, we will also assume that each T_i is finite to avoid measurability issues and that $\lambda \in [x,y]^n \mapsto \pi_i(\lambda, t_{-i}|t_i) \in \mathbb{R}_+$ is a pdf for each t_i and t_{-i}. In particular, for each $t_i \in T_i$,

$$\sum_{t_{-i} \in T_{-i}} \int_{[x,y]^n} \pi_i(\lambda, t_{-i}|t_i) = 1.$$

[5] We write $\tilde{s}_i, \tilde{s}, \tilde{\theta}$ etc for random variables whose realizations s_i, s, θ are elements of S_i, S, Θ.

Give a type space, we next propose a two stage mechanism with complete but asymmetric information whose extensive form is described as follows.

Stage 1: Buyer i of type $t_i \in T_i$ observes his signal $s_i \in \{\alpha, \beta\}$ and private value c_i and makes a (not necessarily honest) report of his signal to the auctioneer. If buyer i reports signal β and at least $\frac{n}{2}$ other buyers report β, then all buyers who have reported β (including i) advance to the second stage. If buyer i reports signal α and at least $\frac{n}{2}$ other buyers report α, then all buyers who have reported α (including i) advance to the second stage. If buyer i's report is not a majority report, then i exits the game with a payoff of 0.

Stage 2: Suppose that bidder i and k other bidders advance to the second stage where $k \geq \frac{n}{2}$. With probability ε, the auctioneer will randomly choose (with probability $\frac{1}{k+1}$) one of the second stage buyers to be awarded the object outright. With probability $1-\varepsilon$, the auctioneer will conduct a $k+1$ bidder second price auction. We assume that bidders who reach the second stage do not know the identities of the other second stage bidders. That is, bidder i who reaches the second stage only knows that at least $\frac{n}{2}$ other bidders submitted a first stage report matching that of bidder i.

2.2 Strategies, payoffs and the main result

To begin, we need to derive several probability expressions. For $\theta \in \{a, b\}$, $s = (s_1, .., s_n) \in \{\alpha, \beta\}^n$, and $(\lambda_1, ..., \lambda_n) \in [x, y]^n$, let $P(\theta, s|\lambda)$ denote the probability of (θ, s) conditional on λ. Then

$$P(a, s|\lambda) = \frac{1}{2} \prod_{j \in F_\alpha(s_{-i})} \lambda_j \prod_{j \in F_\beta(s_{-i})} (1-\lambda_j)$$

and

$$P(b, s|\lambda) = \frac{1}{2} \prod_{j \in F_\alpha(s_{-i})} (1-\lambda_j) \prod_{j \in F_\beta(s_{-i})} \lambda_j.$$

Therefore,

$$P_i(\alpha|\lambda) = P_i(\alpha|a,\lambda)\frac{1}{2} + P_i(\alpha|b,\lambda)\frac{1}{2} = \frac{1}{2}\lambda_i + \frac{1}{2}(1-\lambda_i) = \frac{1}{2}$$

and

$$P_i(\beta|\lambda) = P_i(\beta|a,\lambda)\frac{1}{2} + P_i(\beta|b,\lambda)\frac{1}{2} = \frac{1}{2}(1-\lambda_i) + \frac{1}{2}\lambda_i = \frac{1}{2}.$$

If i has type t_i, then given P and i's beliefs π_i, bidder i assigns a probability to $(\theta, s, \lambda, t_{-i})$ defined as

$$\mu_i(\theta, s, \lambda, t_{-i}|t_i) = P(\theta, s|\lambda)\pi_i(\lambda, t_{-i}|t_i).$$

If $s_i \in S_i$ and $t_i \in T_i$, then

$$\begin{aligned}
\mu_i(s_i|t_i) &= \sum_{s_{-i}\in\{\alpha,\beta\}^n} \sum_{\theta\in\{a,b\}} \sum_{t_{-i}\in T_{-i}} \int_{[x,y]^n} \mu_i(\theta, s, \lambda, t_{-i}|t_i) d\lambda \\
&= \int_{[x,y]^n} P_i(s_i|\lambda)\pi_i(\lambda|t_i)d\lambda \\
&= \int_{[x,y]^n} \frac{1}{2}\pi_i(\lambda|t_i)d\lambda \\
&= \frac{1}{2}
\end{aligned}$$

Since $P_i(s_i|\lambda) = \frac{1}{2} = \mu_i(s_i|t_i)$, it follows that

$$\begin{aligned}
\mu_i(\theta, s_{-i}, \lambda, t_{-i}|s_i, t_i) &= \frac{P(\theta, s|\lambda)\pi_i(\lambda, t_{-i}|t_i)}{\mu_i(s_i|t_i)} \\
&= \frac{P(\theta|s,\lambda)P(s|\lambda)\pi_i(\lambda, t_{-i}|t_i)}{\mu_i(s_i|t_i)} \\
&= \frac{P(\theta|s,\lambda)P(s_{-i}|s_i,\lambda)P_i(s_i|\lambda)\pi_i(\lambda, t_{-i}|t_i)}{\mu_i(s_i|t_i)} \\
&= \frac{P(\theta|s,\lambda)P(s_{-i}|s_i,\lambda)P_i(s_i|\lambda)\pi_i(\lambda, t_{-i}|t_i)}{\mu_i(s_i|t_i)} \\
&= P(\theta|s,\lambda)P(s_{-i}|s_i,\lambda)\pi_i(\lambda, t_{-i}|t_i)
\end{aligned}$$

Given n and the data defining the two stage game and a type space $([x, y]^n, T_1, .., T_n, \pi)$, we have a well defined Bayesian game. In this game, a (pure) strategy for bidder i of type t_i has two components: a first stage reporting function $(c_i, s_i) \mapsto r_i(c_i, s_i|t_i) \in \{\alpha, \beta\}$ and a second stage bidding function $(c_i, s_i) \mapsto b_i(c_i, s_i|t_i) \in \mathbb{R}$.

Definition: A strategy profile $(r_i, b_i)_{i=1}^n$ is *truthful* if for every i, $r_i(c_i, s_i|t_i) = s_i$ for all $c_i \in [0, 1], s_i \in \{\alpha, \beta\}$ and $t_i \in T_i$.

Our goal is to provide an incentive for bidders to honestly report their private signals in stage 1. That incentive depends, of course, on the way in which first stage reports affect second stage payoffs which, in turn, depends on bidders' second stage bidding strategies in the event that the mechanism designer (MD) conducts an auction in the second stage.

Fix a *truthful* strategy profile $(r_j, b_j)_{j=1}^n$. Suppose that player i of type t_i with private characteristic c_i observes signal $s_i = \beta$. If i reports signal β to the mechanism, what is the ex ante expected payoff of agent i ?

If $f^n_\beta(s_{-i}) < \frac{n}{2}$, then i's first stage report is a minority report so i leaves the game with payoff zero.

If $f^n_\beta(s_{-i}) \geq \frac{n}{2}$, then i moves to the second stage and i joins a set $F_\beta(s_{-i}) \subseteq N \backslash i$ of truthful agents who have also advanced to the second stage. Although i does not know the actual composition of $F^n_\beta(s_{-i})$, i does know that $|F^n_\beta(s_{-i})| = f^n_\beta(s_{-i}) = k$ for some $k \geq \frac{n}{2}$.

With probability $1-\varepsilon$, the designer conducts a second price auction. Conditional on type profile $(t_1, .., t_n)$ and signal profile s_{-i}, it follows that i's expected payoff in the auction, depends on i's bid $b_i(c_i, s_i|t_i)$, the bids of i's opponents $(b_j(c_j, s_j|t_j))_{j \in F_\beta(s_{-i})}$, the density g, i's private characteristic c_i, the parameter profile λ and i's signal $s_i = \beta$. We write this expected auction payoff as[6]

$$A_i(\beta|t_{-i}, \lambda, s_{-i}, c_i, t_i, s_i = \beta).$$

With probability ε, the designer conducts a lottery. Conditional on type profile $(t_1, .., t_n)$ and signal profile s_{-i}, it follows that i's expected payoff in the lottery depends on i's private characteristic c_i, the parameter profile λ and i's signal $s_i = \beta$. We write this expected lottery payoff as[7]

$$L_i(\beta|s_{-i}, \lambda, , c_i, s_i = \beta) = \frac{c_i + \sum_{\theta \in \{a,b\}} v(\theta) P(\theta, s_{-i}|s_i = \beta, \lambda)}{|F_\beta(s_{-i})|}.$$

Combining these, it follows that, if all other agents truthfully report their signals, then player i of type t_i with private value c_i who observes signal $s_i = \beta$ and reports β will have *ex ante* expected payoff equal to

$$\begin{aligned} U_i(\beta|(b_j)^n_{j=1}, c_i, s_i &= \beta, t_i) \\ = (1-\varepsilon) & \sum_{t_{-i}} \int_{[x,y]^n} \left[\sum_{k \geq \frac{n}{2}} \sum_{\substack{s_{-i} \\ :|F_\beta(s_{-i})| = k}} A_i(\beta|t_{-i}, \lambda, s_{-i}, c_i, t_i, s_i = \beta) P(s_{-i}|s_i = \beta, \lambda) \right] \pi_i(\lambda, t_{-i}|t_i) \\ & + \varepsilon \sum_{t_{-i}} \int_{[x,y]^n} \left[\sum_{k \geq \frac{n}{2}} \sum_{\substack{s_{-i} \\ :|F_\beta(s_{-i})| = k}} L_i(\beta|s_{-i}, \lambda, , c_i, s_i = \beta) P(s_{-i}|s_i = \beta, \lambda) \right] \pi_i(\lambda, t_{-i}|t_i)) \end{aligned}$$

Again, suppose that player i of type t_i with private value c_i observes signal $s_i = \beta$. Now suppose that i reports signal α to the mechanism. If $f^n_\alpha(s_{-i}) < \frac{n}{2}$,

[6] For the precise definition of $A_i(\beta|t_{-i}, \lambda, s_{-i}, c_i, t_i, s_i = \beta)$, see Section 5.2 below)

[7] Note that L_i does not depend on t_i since L_i does not depend on second stage bids and $r_j(c_j, s_j|t_j) = s_j$ if $j \neq i$.

then i's first stage report is a minority report so i leaves the game with payoff zero.

If $f_\beta^n(s_{-i}) \geq \frac{n}{2}$, then i moves to the second stage and joins a set $F_\alpha(s_{-i}) \subseteq N \backslash i$ of truthful agents who have also advanced to the second stage. Again, i does not know the actual composition of $F_\alpha^n(s_{-i})$ but i does know that $|F_\alpha^n(s_{-i})| = f_\alpha^n(s_{-i}) = k$ for some $k \geq \frac{n}{2}$.

Conditional on type profile t and signal profile $s_{-i},$, i has an analogous expected second stage auction payoff depending on i's bid $b_i(c_i, s_i|t_i)$, the bids of i's opponents $(b_j(c_j, s_j|t_j))_{j \in F_\alpha(s_{-i})}$, the density g, i's private characteristic c_i, the parameter profile λ and i's signal $s_i = \beta$ and we denote this payoff as

$$A_i(\alpha|t_{-i}, \lambda, s_{-i}, c_i, t_i, s_i = \beta).$$

Similarly, i has an analogous expected lottery payoff

$$L_i(\alpha|s_{-i}, \lambda, c_i, s_i = \beta) = \frac{c_i + \sum_{\theta \in \{a,b\}} v(\theta) P(\theta, s_{-i}|s_i = \beta, \lambda)}{|F_\alpha(s_{-i})|}.$$

Combining these, it follows that, if all other agents truthfully report their signals, then player i of type y_i with private value c_i who observes signal $t_i = \beta$ and reports α will have *ex ante* expected payoff equal to

$$\begin{aligned} U_i(\alpha|(b_j)_{j=1}^n, c_i, s_i &= \beta, t_i) \\ = (1-\varepsilon) & \sum_{t_{-i}} \int_{[x,y]^n} \left[\sum_{k \geq \frac{n}{2}} \sum_{\substack{s_{-i} \\ :|F_\alpha(s_{-i})|=k}} A_i(\alpha|t_{-i}, \lambda, s_{-i}, c_i, t_i, s_i = \beta) P(s_{-i}|s_i = \beta, \lambda) \right] \pi_i(\lambda, t_{-i}|t_i) \\ & + \varepsilon \sum_{t_{-i}} \int_{[x,y]^n} \left[\sum_{k \geq \frac{n}{2}} \sum_{\substack{s_{-i} \\ :|F_\alpha(s_{-i})|=k}} L_i(\alpha|s_{-i}, \lambda, , c_i, s_i = \beta) P(s_{-i}|s_i = \beta, \lambda) \right] \pi_i(\lambda, t_{-i}|t_i)) \end{aligned}$$

Definition: Suppose that Γ^n is a two stage game and $([x,y]^n, \pi, T_1, .., T_n)$ is a type space for Γ^n. Suppose that $c = (c_1, .., c_n)$ is a profile of private characteristics and $t = (t_1, .., t_n)$ is a profile of types. A truthful strategy profile $(r_i, b_i)_{i=1}^n$ is *incentive compatible for* $J \subseteq \{1, .., n\}$ *given* c *and* t if for every $i \in J$,

$$\begin{aligned} U_i(\beta|(b_j)_{j=1}^n, c_i, s_i &= \beta, t_i) \geq U_i(\alpha|(b_j)_{j=1}^n, c_i, s_i = \beta, t_i) \\ U_i(\alpha|(b_j)_{j=1}^n, c_i, s_i &= \alpha, t_i) \geq U_i(\beta|(b_j)_{j=1}^n, c_i, s_i = \alpha, t_i) \end{aligned}$$

In keeping with the typical approach to implementation, one would investigate the circumstances under which the Bayesian game admits an equilibrium that is truthful and incentive compatible. In this paper, we take a different tack. In particular, we show that there exists a strategy profile that is truthful and incentive compatible if players are restricted to submitting undominated bids in the second stage auction. The next definition formalizes our notion of weakly dominated.

Definition: Suppose that Γ^n is a two stage game and $([x,y]^n, \pi, T_1, .., T_n)$ is a type space for Γ^n. Suppose that (r_i, b_i) is a strategy for player i in the Bayesian game and suppose that $c_i \in [0,1]$. Then (r_i, b_i) is *ex-post weakly dominated* if there exists a bid b_i'' such that the following hold for every $Q \subseteq N \backslash i$, every $s_i \in \{\alpha, \beta\}$ and every $t_i \in T_i$ and every $c_i \in [0,1]$:

a. for every $(r_j, b_j)_{j \in Q}$ and for every s_{-i}, every t_{-i}, and every λ,

$$\left[c_i + \sum_{\theta \in \{a,b\}} v(\theta) P(\theta | s_{-i}, s_i, \lambda) - \max_{j \in Q} \{ b_j(c_j, s_j | t_j) \} \right] \chi(b_i'' > \max_{j \in Q} \{ b_j(c_j, s_j | t_j) \})$$

$$\geq \left[c_i + \sum_{\theta \in \{a,b\}} v(\theta) P(\theta | s_{-i}, s_i, \lambda) - \max_{j \in Q} \{ b_j(c_j, s_j | t_j) \} \right] \chi(b_i(c_i, s_i | t_i) > \max_{j \in Q} \{ b_j(c_j, s_j | t_j) \})$$

(b) there exists a $(r_j, b_j)_{j \in Q}$ and s_{-i} and t_{-i} such that

$$\left[c_i + \sum_{\theta \in \{a,b\}} v(\theta) P(\theta | s_{-i}, s_i, \lambda) - \max_{j \in Q} \{ b_j(c_j, s_j | t_j) \} \right] \chi(b_i'' > \max_{j \in Q} \{ b_j(c_j, s_j | t_j) \})$$

$$> \left[c_i + \sum_{\theta \in \{a,b\}} v(\theta) P(\theta | s_{-i}, s_i, \lambda) - \max_{j \in Q} \{ b_j(c_j, s_j | t_j) \} \right] \chi(b_i(c_i, s_i | t_i) > \max_{j \in Q} \{ b_j(c_j, s_j | t_j) \})$$

A profile of strategies $(r_i, b_i)_{i=1}^n$ is *second stage admissible* if for each i, (r_i, b_i) is not ex post weakly dominated.

Proposition: Suppose that $\frac{x}{1-x} > \min\{\frac{v(b)}{v(a)}, \frac{v(a)}{v(b)}\}$. Then for each $\varepsilon > 0$ there exists a $\overline{\delta}$ such that for each $\delta \in [\overline{\delta}, 1[$, there exists an N such that for each $n \geq N$ and every type space $([x,y]^n, \pi, S_1, .., S_n)$ for Γ^n the following holds: for every truthful strategy profile $(r_i, b_i)_{i=1}^n$ that is second stage admissible for c, the two stage mechanism is interim individually rational and incentive compatible for all $i \in J = \{j|c_j \in [0, \delta]\}$. Furthermore,

$$\text{Prob}(|J - G(\delta)n| \geq n^{-\frac{2}{3}}) < \exp[-2n^{\frac{1}{3}}]$$

Remark 1: To interpret the result, note that we assume that $\frac{x}{1-x} > \min\{\frac{v(b)}{v(a)}, \frac{v(a)}{v(b)}\}$. Consequently, we do require that the commonly known minimum accuracy x be sufficiently large. As a simple application of Hoeffding's Inequality, the result asserts that, in a sufficiently large auction game, a large fraction of the set of all players will report their true signals irrespective of their beliefs regarding the accuracy profile as long as their private valuations are bounded away from 1. To obtain this result, the players need only choose second stage bidding strategies that satisfy a weak requirement of admissibility.

Remark 2: For large n, the seller's expected revenue is close to

$$(1 - \varepsilon)\left(1 + \frac{v(a) + v(b)}{2}\right).$$

To see this, suppose that n is large. With probability $1-\varepsilon$, the second stage auction has $k \geq \frac{n}{2}$ bidders who have reported α and are choosing admissible second stage bids, then the bidders estimate the value of the common component to be approximately $v(a)$ so the winning bidder pays approximately $v(a)$ plus the second highest value of the private valuations of the other k bidders. For large n this is approximately $1 + v(a)$. If the second stage auction has $k \geq \frac{n}{2}$ bidders who have reported β and are choosing admissible second stage bids, then the bidders estimate the value of the common component to be approximately $v(b)$ so the winning bidder pays approximately $v(b)$ plus the second highest value of the private valuations of the other k bidders. For large n this is approximately $1 + v(b)$. Therefore the seller's expected revenue from the mechanism is approximately equal to

$$(1-\varepsilon)([1+v(a)]P(f_\alpha(\tilde{s}) \geq \frac{n}{2})+[1+v(b)]P(f_\beta(\tilde{s}) \geq \frac{n}{2})) = (1-\varepsilon)\left(1 + \frac{v(a) + v(b)}{2}\right).$$

Remark 3: Our mechanism can induce truthful reporting of signals without transfers designed by the auctioneer, transfers that would require the auctioneer to know the beliefs of the agents. While we view this as an advantage, this advantage comes at a price, namely, truthful reporting in the first stage by a large fraction of all participants.

Remark 4: The proof requires a number of complex estimates but the essential idea is relatively straightforward. Suppose that $xv(b) > (1-x)v(a)$. A player can estimate $A_i(\beta|t_{-i}, \lambda, s_{-i}, c_i, t_i, s_i = \beta)$ and $L_i(\beta|s_{-i}, \lambda, , c_i, s_i = \beta)$ and can also estimate $A_i(\alpha|t_{-i}, \lambda, s_{-i}, c_i, t_i, s_i = \beta)$ and $L_i(\beta|s_{-i}, \lambda, , c_i, s_i = \beta)$. For a player with $c_i \in [0, \delta]$, both $A_i(\beta|t_{-i}, \lambda, s_{-i}, c_i, t_i, s_i = \beta) - A_i(\alpha|t_{-i}, \lambda, s_{-i}, c_i, t_i, s_i = \beta)$ and $L_i(\beta|s_{-i}, \lambda, , c_i, s_i = \beta) - L_i(\beta|s_{-i}, \lambda, , c_i, s_i = \beta)$ converge to zero. The former is negative and converges to zero at an exponential rate while the latter is positive converges at a linear rate. Consequently, $U_i(\beta|(b_j)_{j=1}^n, c_i, s_i = \beta, t_i) - U_i(\alpha|(b_j)_{j=1}^n, c_i, s_i = \beta, t_i)$ is positive for sufficiently large n. Furthermore, these rates are valid irrespective of the type space associated with the underlaying data and the result follows. A completely symmetric argument establishes that $U_i(\alpha|(b_j)_{j=1}^n, c_i, s_i = \alpha, t_i) - U_i(\beta|(b_j)_{j=1}^n, c_i, s_i = \alpha, t_i)$ is positive for sufficiently large n if $xv(a) > (1-x)v(b)$.

3 Proof

Assume that i of type t_i receives signal $s_i = \beta$ and has private characteristic $c_i \in [0, \delta]$ where $\delta < 1$.

For a profile s of signals, note that

$$f_\alpha^n(s_{-i}) + f_\beta^n(s_{-i}) = n - 1$$

The dependence of $f_\theta^n(s_{-i})$ and $F_\theta^n(s_{-i})$ on n and $\lambda_1, .., \lambda_n$ is suppressed for notational ease.

Step 1: To begin, note that there exists an integer N_0 such that for each i and for all $n \geq N_0$, we have

$$\frac{n}{2} < x(n-1) - (n-1)^{\frac{2}{3}} \leq \lambda_i(n-1) - (n-1)^{\frac{2}{3}} < \lambda_i(n-1) + (n-1)^{\frac{2}{3}} \leq y(n-1) + (n-1)^{\frac{2}{3}} < n\ .$$

Applying Hoeffding's inequality, it follows that

$$P\left(\left|\frac{f_\beta(\tilde{s}_{-i})}{n-1} - \frac{\sum_{j\neq i}\lambda_j}{n-1}\right| > \frac{1}{(n-1)^{\frac{1}{3}}}|b\right) \leq 2\exp\left(-2(n-1)\frac{1}{(n-1)^{\frac{2}{3}}}\right).$$

Therefore,

$$P\left(f_\beta(\tilde{s}_{-i}) > y(n-1) + (n-1)^{\frac{2}{3}}\ |b\right) \leq P\left(f_\beta(\tilde{s}_{-i}) > \sum_{j\neq i}\lambda_j + (n-1)^{\frac{2}{3}}\ |b\right) \leq 2\exp[-2(n-1)^{\frac{1}{3}}]$$

and

$$P\left(f_\beta(\tilde{s}_{-i}) < x(n-1) - (n-1)^{\frac{2}{3}}|b\right) \leq P\left(f_\beta(\tilde{s}_{-i}) < \sum_{j\neq i}\lambda_j - (n-1)^{\frac{2}{3}}|b\right) \leq 2\exp(-2(n-1)^{\frac{1}{3}}).$$

Similarly,

$$P\left(\left|\frac{f_\alpha(\tilde{s}_{-i})}{n} - \frac{\sum_{j\neq i}\lambda_j}{n}\right| > \frac{1}{(n-1)^{\frac{1}{3}}}|a\right) \leq 2\exp(-2(n-1)\frac{1}{(n-1)^{\frac{2}{3}}})$$

implying that

$$P\left(f_\alpha(\tilde{s}_{-i}) > y(n-1) + (n-1)^{\frac{2}{3}}\ |a\right) \leq P\left(f_\alpha(\tilde{s}_{-i}) > \sum_{j\neq i}\lambda_j + (n-1)^{\frac{2}{3}}\ |a\right) \leq 2\exp(-2(n-1)^{\frac{1}{3}})$$

and

$$P\left(f_\alpha(\tilde{s}_{-i}) < x(n-1) - (n-1)^{\frac{2}{3}}|a\right) \leq P\left(f_\alpha(\tilde{s}_{-i}) < \sum_{j\neq i}\lambda_j - (n-1)^{\frac{2}{3}}|a\right) \leq 2\exp(-2(n-1)^{\frac{1}{3}}).$$

We also will need the following probability bounds that follow from the bounds computed above:

(i)

$$\begin{aligned} P\left(f_\alpha(\tilde{s}_{-i}) < x(n-1) - (n-1)^{\frac{2}{3}}, s_i = \beta|a\right) &= (1-\lambda_i)P\left(f_\alpha(\tilde{s}_{-i}) < x(n-1) - (n-1)^{\frac{2}{3}}|a\right) \\ &\leq 2(1-\lambda_i)\exp(-2(n-1)^{\frac{1}{3}}). \end{aligned}$$

(ii)

$$\begin{aligned} P\left(f_\alpha(\tilde{s}_{-i}) \geq \frac{n}{2}, t_i = \beta|b\right) &= \lambda_i P\left(f_\alpha(\tilde{s}_{-i}) \geq \frac{n}{2}|b\right) \\ &= \lambda_i P\left(f_\beta(\tilde{s}_{-i}) < \frac{n}{2}|b\right) \\ &\leq \lambda_i P\left(f_\beta(\tilde{s}_{-i}) < x(n-1) - (n-1)^{\frac{2}{3}}|b\right) \\ &\leq 2\lambda_i\exp(-2(n-1)^{\frac{1}{3}}). \end{aligned}$$

(iii)

$$\begin{aligned} P\left(f_\beta(\tilde{s}_{-i}) < x(n-1) - (n-1)^{\frac{2}{3}}, t_i = \beta|b\right) &= \lambda_i P\left(f_\beta(\tilde{s}_{-i}) < x(n-1) - (n-1)^{\frac{2}{3}}|b\right) \\ &\leq 2\lambda_i\exp(-2(n-1)^{\frac{1}{3}}) \end{aligned}$$

(iv)

$$\begin{aligned} P\left(f_\beta(\tilde{s}_{-i}) \geq \frac{n}{2}, t_i = \beta|a\right) &= (1-\lambda_i)P\left(f_\beta(\tilde{s}_{-i}) \geq \frac{n}{2}|a\right) \\ &= (1-\lambda_i)P\left(f_\alpha(\tilde{s}_{-i}) < \frac{n}{2}|a\right) \\ &\leq (1-\lambda_i)P\left(f_\alpha(\tilde{s}_{-i}) < x(n-1) - (n-1)^{\frac{2}{3}}|a\right) \\ &\leq 2(1-\lambda_i)\exp(-2(n-1)^{\frac{1}{3}}). \end{aligned}$$

Step 2: In this step, we compute bounds for

$$\sum_{\theta \in \{a,b\}} v(\theta) P(\theta | s_{-i}, s_i = \beta, \lambda)$$

if $F_\beta(s_{-i}) = Q$ and $|Q| = k$, and that hold for all sufficiently large n. To begin, note that

$$\sum_{\theta \in \{a,b\}} v(\theta) P(\theta | s_{-i}, s_i = \beta, \lambda) = v(b) - [v(b) - v(a)] P(\theta | s_{-i}, s_i = \beta, \lambda)$$

Since $F_\beta(s_{-i}) = Q$,

$$P(s_{-i}, s_i = \beta | a) = (1 - \lambda_i) \left[\prod_{j \in Q} (1 - \lambda_j) \right] \left[\prod_{j \notin Q \cup i} \lambda_j \right]$$

and

$$P(s_{-i}, s_i = \beta | b) = \lambda_i \left[\prod_{j \in Q} \lambda_j \right] \left[\prod_{j \notin Q \cup i} (1 - \lambda_j) \right]$$

we conclude that for all $n \geq N_0$,

$$\begin{aligned} P(a | s_{-i}, s_i = \beta) &= \frac{P(s_{-i}, s_i = \beta | a)}{P(s_{-i}, s_i = \beta | a) + P(s_{-i}, s_i = \beta | b)} \\ &= \frac{1}{1 + \frac{\lambda_i \left[\prod_{j \in Q} \lambda_j \right] \left[\prod_{j \notin Q \cup i} (1 - \lambda_j) \right]}{(1 - \lambda_i) \left[\prod_{j \in Q} (1 - \lambda_j) \right] \left[\prod_{j \notin Q \cup i} \lambda_j \right]}} \\ &\leq \frac{1}{1 + \left(\frac{x}{1-x} \right)^{2k - n + 2}}. \end{aligned}$$

Let $d = 2x - 1$. Then there exists an integer $N_1 > N_0$ such that $n \geq N_1$ and $k \geq x(n-1) - (n-1)^{\frac{2}{3}}$ imply that

$$\left(\frac{x}{1-x} \right)^{\frac{(n-1)d}{2}} \leq \left(\frac{x}{1-x} \right)^{2k - (n-1)}.$$

To see this choose $N_1 > N_0$ so that $d - 2(n-1)^{-\frac{1}{3}} > \frac{d}{2}$ for all $n \geq N_1$. Then $k \geq x(n-1) - (n-1)^{\frac{2}{3}}$ and $\frac{x}{1-x} > 1$ imply that

$$\left(\frac{x}{1-x} \right)^{2k - (n-1)} \geq \left(\frac{x}{1-x} \right)^{2(x(n-1) - (n-1)^{\frac{2}{3}}) - (n-1)}$$

and it follows that

$$\left(\frac{x}{1-x}\right)^{2k-(n-1)} \geq \left(\frac{x}{1-x}\right)^{2(x(n-1)-(n-1)^{\frac{2}{3}})-(n-1)} = \left(\frac{x}{1-x}\right)^{(n-1)\left[d-2(n-1)^{-\frac{1}{3}}\right]} \geq \left(\frac{x}{1-x}\right)^{\frac{(n-1)d}{2}}.$$

In particular,

$$\left(\frac{x}{1-x}\right)^{2k-n+2} \geq \left(\frac{x}{1-x}\right)^{\frac{(n-1)d}{2}+1}$$

Therefore, $n \geq N_1$ implies (since $v(a) < v(b)$) that for each $k \geq x(n-1) - (n-1)^{\frac{2}{3}}$ we have

$$\begin{aligned} v(b) &\geq \sum_{\theta\in\{a,b\}} v(\theta)P(\theta|s_{-i}, s_i = \beta, \lambda) \\ &= v(b) - [v(b) - v(a)]P(a|s_{-i}, s_i = \beta, \lambda) \\ &\geq v(b) - \left[\frac{1}{1+\left(\frac{x}{1-x}\right)^{\frac{(n-1)d}{2}+1}}\right][v(b) - v(a)]. \end{aligned}$$

Step 3: In this step, we next compute bounds for

$$\sum_{\theta\in\{a,b\}} v(\theta)P(\theta|s_{-i}, s_i = \beta, \lambda)$$

if $F_\alpha(s_{-i}) = Q$ and $|Q| = k$, and that hold for all sufficiently large n.

Since $F_\alpha(s_{-i}) = Q$,

$$P(s_{-i}, s_i = \beta|a) = (1-\lambda_i)\left[\prod_{j\in Q}\lambda_j\right]\left[\prod_{j\notin Q\cup i}(1-\lambda_j)\right]$$

and

$$P(s_{-i}, s_i = \beta|b) = \lambda_i\left[\prod_{j\in Q}(1-\lambda_j)\right]\left[\prod_{j\notin Q\cup i}\lambda_j\right]$$

we conclude that

$$\begin{aligned} P(b|s_{-i}, s_i &= \beta) = \frac{P(s_{-i}, s_i = \beta|b)}{P(s_{-i}, s_i = \beta|a) + P(s_{-i}, s_i = \beta|b)} \\ &= \frac{1}{1+\frac{(1-\lambda_i)\left[\prod_{j\in Q}\lambda_j\right]\left[\prod_{j\notin Q\cup i}(1-\lambda_j)\right]}{\lambda_i\left[\prod_{j\in Q}(1-\lambda_j)\right]\left[\prod_{j\notin Q\cup i}\lambda_j\right]}} \\ &\leq \frac{1}{1+\left(\frac{x}{1-x}\right)^{2k-n}}. \end{aligned}$$

If $n \geq N_1$ and $k \geq x(n-1)-(n-1)^{\frac{2}{3}}$ then we conclude from step 2 that

$$\left(\frac{x}{1-x}\right)^{2k-(n-1)} \geq \left(\frac{x}{1-x}\right)^{\frac{(n-1)d}{2}}$$

implying that

$$\left(\frac{x}{1-x}\right)^{2k-n} = \left(\frac{x}{1-x}\right)^{2k-(n-1)}\left(\frac{1-x}{x}\right) \geq \left(\frac{x}{1-x}\right)^{\frac{(n-1)d}{2}}\left(\frac{1-x}{x}\right) = \left(\frac{x}{1-x}\right)^{\frac{(n-1)d}{2}-1}.$$

Therefore,

$$\begin{aligned} v(a) &\leq \sum_{\theta\in\{a,b\}} v(\theta)P(\theta|s_{-i}, s_i=\beta,\lambda) \\ &= v(a)+[v(b)-v(a)]P(b|s_{-i}, s_i=\beta). \\ &\leq v(a)+\left[\frac{1}{1+\left(\frac{x}{1-x}\right)^{\frac{(n-1)d}{2}-1}}\right][v(b)-v(a)]. \end{aligned}$$

Step 4: For each n, define

$$\eta_n = \left[\frac{1}{1+\left(\frac{x}{1-x}\right)^{\frac{(n-1)d}{2}-1}}\right][v(b)-v(a)]$$

and note that

$$\eta_n \geq \left[\frac{1}{1+\left(\frac{x}{1-x}\right)^{\frac{(n-1)d}{2}+1}}\right][v(b)-v(a)].$$

Summarizing Steps 2 and 3, we conclude the following: for every $n \geq N_1$ and for each $k \geq x(n-1)-(n-1)^{\frac{2}{3}}$,

$$v(b) \geq \sum_{\theta\in\{a,b\}} v(\theta)P(\theta|s_{-i}, s_i=\beta,\lambda) \geq v(b)-\left[\frac{1}{1+\left(\frac{x}{1-x}\right)^{\frac{(n-1)d}{2}+1}}\right][v(b)-v(a)] \geq v(b)-\eta_n$$

if $|F_\beta(s_{-i}) = k$ and

$$v(a) \leq \sum_{\theta\in\{a,b\}} v(\theta)P(\theta|s_{-i}, s_i=\beta,\lambda) \leq v(a)+\left[\frac{1}{1+\left(\frac{x}{1-x}\right)^{\frac{(n-1)d}{2}-1}}\right][v(b)-v(a)] = v(a)+\eta_n.$$

if $|F_\alpha(s_{-i}) = k$.

Step 5: Let N_1 be defined as in Step 4. Suppose that $n \geq N_1$ and $k \geq x(n-1) - (n-1)^{\frac{2}{3}}$. Let $E[v|s_{-i}, s_i = \beta, \lambda] = \sum_{\theta \in \{a,b\}} v(\theta) P(\theta | s_{-i}, s_i = \beta, \lambda)$ and let η_n be defined as in Step 4.

5.1: We compute an upper bound on bidder i's the second stage auction payoff $A_i(\alpha | t_{-i}, \lambda, s_{-i}, t_i, s_i = \beta)$ if $F_\alpha(t_{-i}) = k$ and all bidders use ex post undominated bids. In this case,

$$v(a) \leq E[v|s_{-i}, s_i = \beta, \lambda] \leq v(a) + \eta_n$$

and we first prove the following lemma.

Lemma: Suppose that $v(a) \leq E[v|s_{-i}, s_i = \beta, \lambda] \leq v(a) + \eta_n$. Let $Q = F_\alpha(s_{-i})$ and let b_i' denote the second stage bid for player i. If $b_i' < c_i + v(a)$, then b_i' is weakly dominated.

Proof: Let $b_{-i}^* = \max_{j \neq i}\{b_j(c_j, s_j | t_j)\}$ and note that i's payoff in the auction is then

$$\left[c_i + E[v|s_{-i}, s_i = \beta, \lambda] - b_{-i}^*\}\right] \chi(b_i' \geq b_{-i}^*)$$

with a tie breaking rule if $b_i' = b_{-i}^*$. Suppose that $b_i' < c_i + v(a)$.

If $b_{-i}^* \geq c_i + v(a)$, then a bid of b_i' is a losing bid with payoff zero. However, a bid $c_i + v(a)$ has zero payoff if $b_{-i}^* > c_i + v(a)$ and a nonnegative payoff if $b_{-i}^* = c_i + v(a)$ since $c_i + E[v|s_{-i}, s_i = \beta, \lambda] - b_{-i}^* = E[v|s_{-i}, s_i = \beta, \lambda] - v(a) \geq 0$.

If $b_{-i}^* \leq b_i'$, then a bid of $c_i + v(a)$ is a winning bid with payoff $c_i + E[v|s_{-i}, s_i = \beta, \lambda] - b_{-i}^* > 0$. However, a bid of b_i' is a winning bid with payoff $c_i + E[v|s_{-i}, s_i = \beta, \lambda] - b_{-i}^* > 0$ if $b_{-i}^* < b_i'$ and a nonnegative payoff of at most $c_i + E[v|s_{-i}, s_i = \beta, \lambda] - b_{-i}^*$ if $b_{-i}^* = b_i'$.

If $b_i' < b_{-i}^* < c_i + v(a)$, then a bid of b_i' loses with payoff of zero while a bid of $c_i + v(a)$ wins with payoff

$$c_i + E[v|s_{-i}, s_i = \beta, \lambda] - b_{-i}^* > c_i + E[v|s_{-i}, s_i = \beta, \lambda] - c_i - v(a) > c_i + v(a) - c_i - v(a) = 0$$

5.2 Applying the lemma, note that $c_i + E[v|s_{-i}, s_i = \beta, \lambda] < c_i + v(a) + \eta_n$ and $\max_{j \in Q}\{c_i + v(a)\} \leq \max_{j \in Q}\{b_j(c_j, s_j | t_j)\}$. Let $Q = F_\alpha(s_{-i})$ and suppose that $|Q| = k$. If i participates in the auction and submits the bid $b_i' = b_i(c_i, s_i | t_i)$, then

$$
\begin{aligned}
&A_i(\alpha|t_{-i},\lambda,s_{-i},t_i,c_i,s_i=\beta)\\
&= \int_{[0,1]^k}\left[c_i+E[v|s_{-i},s_i=\beta,\lambda]-\max_{j\in Q}\{b_j(c_j,s_j|t_j)\}\right]\chi(b_i'\geq\max_{j\in Q}\{b_j(c_j,s_j|t_j)\})g_{-i}(c_{-i})dc_{-i}\\
&< \int_{[0,1]^k}\sum_{s_{-i}}\left[c_i+v(a)+\eta_n-\max_{j\in Q}\{c_j+v(a)\}\right]\chi(b_i'\geq\max_{j\in Q}\{b_j(c_j,s_j|t_j)\})g_{-i}(c_{-i})dc_{-i}\\
&= \int_{[0,1]^k}\sum_{s_{-i}}\left[c_i+v(a)+\eta_n-\max_{j\in Q}\{c_j\}-v(a)\right]\chi(b_i'\geq\max_{j\in Q}\{b_j(c_j,s_j|t_j)\})g_{-i}(c_{-i})dc_{-i}\\
&= \int_{c_{-i}}\left[c_i-\max_{j\in Q}\{c_j\}+\eta_n\right]\chi(b_i'\geq\max_{j\in Q}\{b_j(c_j,s_j|t_j)\})g_{-i}(c_{-i})dc_{-i}\\
&\leq \int_{[0,1]^k}\sum_{s_{-i}}\left[c_i-\max_{j\in Q}\{c_j\}+\eta_n\right]\chi(b_i'>\max_{j\in Q}\{b_j(c_j,s_j|t_j)\},c_i\geq\max_{j\in Q}\{c_i\}-\eta)g_{-i}(c_{-i})dc_{-i}\\
&\leq \int_{[0,1]^k}\left[c_i-\max_{j\in Q}\{c_j\}+\eta_n\right]\chi(c_i\geq\max_{j\in Q}\{c_j\}-\eta)g_{-i}(c_{-i})dc_{-i}\\
&= \int_0^{c_i+\eta_n}[c_i-\gamma+\eta_n]\,d[G(\gamma)^k]\leq G^k(c_i+\eta_n).
\end{aligned}
$$

5.3 There exists $N_2 > N_1$ such that $\eta_n < \frac{1-\delta}{2}$. Therefore, $n > N_2$ implies that

$$c_i+\eta_n<\delta+\frac{1-\delta}{2}=\frac{1+\delta}{2}<1.$$

If $n > N_2$,

$$A_i(\alpha|t_{-i},\lambda,s_{-i},t_i,c_i,s_i=\beta)\leq G^k(c_i+\eta_n)\leq G^k(\frac{1+\delta}{2}).$$

Step 6: Let N_2 be defined as in Step 5.3. In this step, we find an upper bound for $U_i(\alpha|(b_j)_{j=1}^n,c_i,s_i=\beta,t_i)$ if $n > N_2$.

6.1

$$\sum_{k\geq \frac{n}{2}} \sum_{\substack{s_{-i} \\ :|F_\alpha(s_{-i})|=k}} L_i(\alpha|s_{-i},\lambda,c_i,s_i=\beta)P(s_{-i}|s_i=\beta,\lambda)$$

$$\sum_{k\geq \frac{n}{2}} \sum_{\substack{s_{-i} \\ :|F_\alpha(s_{-i})|=k}} \left[\sum_{\theta\in\{a,b\}} (\frac{c_i+v(\theta)}{k+1})P(\theta|s_{-i},s_i=\beta,\lambda)\right] P(s_{-i}|s_i=\beta,\lambda)$$

$$\leq \sum_{k\geq \frac{n}{2}} \sum_{\substack{s_{-i} \\ :|F_\alpha(s_{-i})|=k}} \left[\frac{c_i+v(a)+\eta_n}{k+1}\right] P(s_{-i}|s_i=\beta,\lambda)$$

$$= \sum_{k\geq \frac{n}{2}} \left[\frac{c_i+v(a)+\eta_n}{k+1}\right] \left[\sum_{\substack{s_{-i} \\ :|F_\alpha(s_{-i})|=k}} P(s_{-i}|s_i=\beta,\lambda)\right]$$

$$= \sum_{k\geq \frac{n}{2}} \left[\frac{c_i+v(a)+\eta_n}{k+1}\right] P(f_\alpha(s_{-i})=k|s_i=\beta,\lambda)$$

6.2 Defining

$$B=\max\{c_i+v(b),1+c_i+v(a)\}$$

and

$$H_k=\sum_{\substack{S\subseteq N\setminus i \\ :|S|=k}} \left[\prod_{j\in S}\lambda_j\right]\left[\prod_{j\notin S\cup i}(1-\lambda_j)\right]$$

and recalling that

$$P\left(f_\alpha(\tilde{s}_{-i})\geq \frac{n}{2},t_i=\beta|b\right)\leq 2\lambda_i\exp(-2(n-1)^{\frac{1}{3}})$$

and

$$P\left(f_\alpha(\tilde{s}_{-i})<x(n-1)-(n-1)^{\frac{2}{3}},t_i=\beta|a\right)\leq 2(1-\lambda_i)\exp(-2(n-1)^{\frac{1}{3}})$$

we conclude that

$$\sum_{k\geq\frac{n}{2}}\left[(1-\varepsilon)G(\frac{1+\delta}{2})^k+\frac{\varepsilon}{k+1}\left[c_i+v(a)+\eta_n\right]\right]P(f_\alpha(\tilde{s}_{-i})=k|s_i=\beta)$$

$$=\sum_{\frac{n}{2}\leq k<x(n-1)-(n-1)^{\frac{2}{3}}}\left[(1-\varepsilon)\sum_{k\geq\frac{n}{2}}G(\frac{1+\delta}{2})^k+\frac{\varepsilon}{k+1}\sum_{k\geq\frac{n}{2}}\left[c_i+v(a)+\eta_n\right]\right]P(f_\alpha(\tilde{s}_{-i})=k,s_i=\beta|a)$$

$$+\sum_{k\geq x(n-1)-(n-1)^{\frac{2}{3}}}\left[(1-\varepsilon)G(\frac{1+\delta}{2})^k+\frac{\varepsilon}{k+1}\left[c_i+v(a)+\eta_n\right]\right]P(f_\alpha(\tilde{s}_{-i})=k,s_i=\beta|a)$$

$$+\sum_{k\geq\frac{n}{2}}\left[(1-\varepsilon)G(\frac{1+\delta}{2})^k+\frac{\varepsilon}{k+1}\left[c_i+v(a)+\eta_n\right]\right]P(f_\alpha(\tilde{s}_{-i})=k,s_i=\beta|b)$$

$$\leq\sum_{k\geq x(n-1)-(n-1)^{\frac{2}{3}}}\left[(1-\varepsilon)G(\frac{1+\delta}{2})^k+\frac{\varepsilon}{k+1}\left[c_i+v(a)+\eta_n\right]\right]P(f_\alpha(\tilde{s}_{-i})=k,s_i=\beta|a)+2B\exp(-2(n-1)^{\frac{1}{3}})$$

$$=\sum_{k\geq x(n-1)-(n-1)^{\frac{2}{3}}}\left[(1-\varepsilon)G(\frac{1+\delta}{2})^k+\frac{\varepsilon}{k+1}\left[c_i+v(a)+\eta_n\right]\right](1-\lambda_i)H_k+2B\exp(-2(n-1)^{\frac{1}{3}})$$

6.3 Consequently,

$$U_i(\alpha|(b_j)_{j=1}^n,c_i,s_i=\beta,t_i)$$

$$=(1-\varepsilon)\sum_{t_{-i}}\int_{[x,y]^n}\left[\sum_{k\geq\frac{n}{2}}\sum_{\substack{s_{-i}\\:|F_\alpha(s_{-i})|=k}}A_i(\alpha|t_{-i},\lambda,s_{-i},c_i,t_i,s_i=\beta)P(s_{-i}|s_i=\beta,\lambda)\right]\pi_i(\lambda,t_{-i}|t_i)$$

$$+\varepsilon\sum_{t_{-i}}\int_{[x,y]^n}\left[\sum_{k\geq\frac{n}{2}}\sum_{\substack{s_{-i}\\:|F_\alpha(s_{-i})|=k}}L_i(\alpha|s_{-i},\lambda,,c_i,s_i=\beta)P(s_{-i}|s_i=\beta,\lambda)\right]\pi_i(\lambda,t_{-i}|t_i))$$

$$\leq\sum_{t_{-i}}\int_{[x,y]^n}\left[\sum_{k\geq x(n-1)-(n-1)^{\frac{2}{3}}}\left[(1-\varepsilon)G(\frac{1+\delta}{2})^k+\frac{\varepsilon}{k+1}\left[c_i+v(a)+\eta_n\right]\right](1-\lambda_i)H_k+2B\exp(-2(n-1)^{\frac{1}{3}})\right]$$

Step 7 Let N_2 be defined as in Step 5.3. In this step, we find an lower bound for $U_i(\beta|(b_j)_{j=1}^n,c_i,s_i=\beta,t_i)$ if $n>N_2$.

7.1 Suppose that player i reports β . Then i's payoff in the auction is at least 0 while his payoff from the lottery

$$\sum_{k\geq\frac{n}{2}}\sum_{\substack{s_{-i}\\:|F_\beta(s_{-i})|=k}}\sum_{\theta\in\{a,b\}}(\frac{c_i+v(\theta)}{k+1})P(\theta,s_{-i}|s_i=\beta,\lambda)P(s_{-i}|s_i=\beta,\lambda)$$

$$\geq\sum_{k\geq\frac{n}{2}}\sum_{\substack{s_{-i}\\:|F_\beta(s_{-i})|=k}}(\frac{c_i+v(b)-\eta_n}{k+1})P(s_{-i}|s_i=\beta,\lambda)$$

$$=\sum_{k\geq\frac{n}{2}}(\frac{c_i+v(b)-\eta_n}{k+1})\left[\sum_{\substack{s_{-i}\\:|F_\beta(s_{-i})|=k}}P(s_{-i}|s_i=\beta,\lambda)\right]$$

$$=\sum_{k\geq\frac{n}{2}}(\frac{c_i+v(b)-\eta_n}{k+1})\left[P(f_\beta(\tilde{s}_{-i})=k|\beta,\lambda)\right]$$

Consequently, the payoff to player i is bounded from above by

$$\varepsilon\sum_{k\geq\frac{n}{2}}(\frac{c_i+v(b)-\eta_n}{k+1})\left[P(f_\beta(\tilde{s}_{-i})=k|s_i=\beta,\lambda)\right]$$

7.2 Again defining

$$B=\max\{c_i+v(b),1+c_i+v(a)\}$$

and

$$H_k=\sum_{\substack{S\subseteq N\setminus i\\:|S|=k}}\left[\prod_{j\in S}\lambda_j\right]\left[\prod_{j\notin S\cup i}(1-\lambda_j)\right]$$

and recalling that

$$P\left(f_\beta(\tilde{s}_{-i})\geq\frac{n}{2},t_i=\beta|a\right)\leq 2(1-\lambda_i)\exp(-2(n-1)^{\frac{1}{3}})$$

and

$$P\left(f_\beta(\tilde{s}_{-i})<x(n-1)-(n-1)^{\frac{2}{3}},s_i=\beta|b\right)\leq 2\lambda_i\exp(-2(n-1)^{\frac{1}{3}})$$

we conclude that

$$\sum_{k\geq\frac{n}{2}}\varepsilon(\frac{c_i+v(b)-\eta_n}{k+1})P(f_\beta(\tilde{s}_{-i})=k,s_i=\beta,\lambda)$$

$$=\sum_{\frac{n}{2}\leq k<x(n-1)-(n-1)^{\frac{2}{3}}}\varepsilon(\frac{c_i+v(b)-\eta_n}{k+1})P(f_\beta(\tilde{s}_{-i})=k|s_i=\beta|b,\lambda)$$

$$+\sum_{k\geq x(n-1)-(n-1)^{\frac{2}{3}}}\varepsilon(\frac{c_i+v(b)-\eta_n}{k+1})P(f_\beta(\tilde{s}_{-i})=k|s_i=\beta|b,\lambda)$$

$$+\sum_{k\geq\frac{n}{2}}\varepsilon(\frac{c_i+v(b)-\eta_n}{k+1})P(f_\beta(\tilde{s}_{-i})=k|s_i=\beta|a,\lambda)$$

$$\geq\sum_{k\geq x(n-1)-(n-1)^{\frac{2}{3}}}\varepsilon(\frac{c_i+v(b)-\eta_n}{k+1})P(f_\beta(\tilde{s}_{-i})=k|s_i=\beta|b,\lambda)-2B\exp(-2(n-1)^{\frac{1}{3}})$$

$$=\sum_{k\geq x(n-1)-(n-1)^{\frac{2}{3}}}\varepsilon(\frac{c_i+v(b)-\eta_n}{k+1})\lambda_iH_k-2B\exp(-2(n-1)^{\frac{1}{3}}).$$

7.3 Consequently,

$$U_i(\beta|(b_j)_{j=1}^n,c_i,s_i \;=\; \beta,t_i)$$

$$=(1-\varepsilon)\sum_{t_{-i}}\int_{[x,y]^n}\left[\sum_{k\geq\frac{n}{2}}\sum_{\substack{s_{-i}\\ :|F_\alpha(s_{-i})|=k}}A_i(\alpha|t_{-i},\lambda,s_{-i},c_i,t_i,s_i=\beta)P(s_{-i}|s_i=\beta,\lambda)\right]\pi_i(\lambda,t_{-i}|t_i)$$

$$+\varepsilon\sum_{t_{-i}}\int_{[x,y]^n}\left[\sum_{k\geq\frac{n}{2}}\sum_{\substack{s_{-i}\\ :|F_\alpha(s_{-i})|=k}}L_i(\alpha|s_{-i},\lambda,,c_i,s_i=\beta)P(s_{-i}|s_i=\beta,\lambda)\right]\pi_i(\lambda,t_{-i}|t_i))$$

$$\geq\sum_{t_{-i}}\int_{[x,y]^n}\left[\sum_{k\geq x(n-1)-(n-1)^{\frac{2}{3}}}\varepsilon(\frac{c_i+v(b)-\eta_n}{k+1})\lambda_iH_k-2B\exp(-2(n-1)^{\frac{1}{3}}).\right]\pi_i(\lambda,t_{-i}|t$$

Step 8: Suppose that $n\geq N_2$ and $k\geq x(n-1)-(n-1)^{\frac{2}{3}}$.

8.1 Combining Steps 6.2 and 7.2, it follows that

$$\sum_{k\geq\frac{n}{2}}\varepsilon\left(\frac{c_i+v(b)-\eta_n}{k+1}\right)P(f_\beta(s_{-i})=k|s_i=\beta,\lambda)$$

$$-\sum_{k\geq\frac{n}{2}}\left[(1-\varepsilon)G(\frac{1+\delta}{2})^k+\varepsilon\left(\frac{c_i+v(a)+\eta_n}{k+1}\right)\right]P(f_\alpha(t_{-i})=k|t_i=\beta)$$

$$\geq \sum_{k \geq x(n-1)-(n-1)^{\frac{2}{3}}} \varepsilon \left(\frac{c_i + v(b) - \eta_n}{k+1} \right) \lambda_i H_k$$
$$- \left[\sum_{k \geq x(n-1)-(n-1)^{\frac{2}{3}}} \left[(1-\varepsilon) G(\frac{1+\delta}{2})^k + \varepsilon \left(\frac{c_i + v(a) + \eta_n}{k+1} \right) \right] (1-\lambda_i) H_k \right]$$
$$- 4B \exp(-2(n-1)^{\frac{1}{3}})$$
$$= \sum_{k \geq x(n-1)-(n-1)^{\frac{2}{3}}} \varepsilon \left(\frac{c_i + v(b) - \eta_n}{k+1} \right) \lambda_i H_k - \varepsilon \left(\frac{c_i + v(a) + \eta_n}{k+1} \right) (1-\lambda_i) H_k (1-\varepsilon) G(\frac{1+\delta}{2})^k (1-\lambda_i) H_k$$
$$- 4B \exp(-2(n-1)^{\frac{1}{3}})$$
$$= \sum_{k \geq x(n-1)-(n-1)^{\frac{2}{3}}} \left[\varepsilon \left(\frac{c_i + v(b) - \eta_n}{k+1} \right) \lambda_i - \varepsilon \left(\frac{c_i + v(a) + \eta_n}{k+1} \right) (1-\lambda_i) - (1-\varepsilon) G(\frac{1+\delta}{2})^k (1-\lambda_i) \right] H_k$$
$$- 4B \exp(-2(n-1)^{\frac{1}{3}})$$
$$= \sum_{k \geq x(n-1)-(n-1)^{\frac{2}{3}}} \left[\varepsilon \left(\frac{c_i + v(b) - \eta_n}{k+1} \right) \lambda_i - \varepsilon \left(\frac{c_i + v(a) + \eta_n}{k+1} \right) (1-\lambda_i) - (1-\varepsilon) G(\frac{1+\delta}{2})^k (1-\lambda_i) \right] H_k$$
$$- 4B \exp(-2(n-1)^{\frac{1}{3}})$$
$$\geq \sum_{k \geq x(n-1)-(n-1)^{\frac{2}{3}}} \left[\frac{\varepsilon}{k+1} (xv(b) - (1-x)v(a) - \eta_n) - (1-\varepsilon) G(\frac{1+\delta}{2})^k \right] H_k$$
$$- 4B \exp(-2(n-1)^{\frac{1}{3}})$$

8.2: Suppose that $n \geq N_2$ and $k \geq x(n-1) - (n-1)^{\frac{2}{3}}$.
For all sufficiently large n,

$$xv(b) - (1-\lambda_i)v(a) - \eta_n \geq \frac{xv(b) - (1-x)v(a)}{2}$$

Since $G(\frac{1+\delta}{2}) < 1$, it follows that for k large enough,

$$\varepsilon \left[\frac{xv(b) - (1-x)v(a)}{2} \right] - (1-\varepsilon)(k+1) G(\frac{1+\delta}{2})^k > \varepsilon \left[\frac{xv(b) - (1-x)v(a)}{4} \right]$$

Furthermore, for n large enough,

$$\varepsilon \left[\frac{xv(b) - (1-x)v(a)}{4} \right] (1 - 2\exp(-2(n-1)^{\frac{1}{3}}) - 4B(n+1)\exp(-2(n-1)^{\frac{1}{3}}) > 0$$

Consequently, there exists an $N > N_2$ such that for all $n \geq N$ and $k \geq$

$x(n-1)-(n-1)^{\frac{2}{3}}$, and we conclude that

$$
\begin{aligned}
& \sum_{k\geq x(n-1)-(n-1)^{\frac{2}{3}}} \left[\frac{\varepsilon}{k+1}\left[xv(b)-(1-x)v(a)-\eta_n\right]-(1-\varepsilon)G(\frac{1+\delta}{2})^k\right] H_k \\
& -4B\exp(-2(n-1)^{\frac{1}{3}}) \\
\geq\ & \sum_{k\geq x(n-1)-(n-1)^{\frac{2}{3}}} \frac{1}{(k+1)}\left(\varepsilon\left[\frac{xv(b)-(1-x)v(a)}{2}\right]-(1-\varepsilon)(k+1)G(\frac{1+\delta}{2})^k\right) H_k \\
& -4B\exp(-2(n-1)^{\frac{1}{3}}) \\
\geq\ & \sum_{k\geq x(n-1)-(n-1)^{\frac{2}{3}}} \frac{1}{(k+1)}\left(\varepsilon\left[\frac{xv(b)-(1-x)v(a)}{4}\right]\right) H_k-4B\exp(-2(n-1)^{\frac{1}{3}}) \\
\geq\ & \frac{1}{(n+1)}\left[\sum_{k\geq x(n-1)-(n-1)^{\frac{2}{3}}}\left(\varepsilon\left[\frac{xv(b)-(1-x)v(a)}{4}\right]\right) H_k-4B(n+1)\exp(-2(n-1)^{\frac{1}{3}})\right] \\
\geq\ & \frac{1}{(n+1)}\left[\left(\varepsilon\left[\frac{xv(b)-(1-x)v(a)}{4}\right]\right)\left[\sum_{k\geq x(n-1)-(n-1)^{\frac{2}{3}}} H_k\right]-4B(n+1)\exp(-2(n-1)^{\frac{1}{3}})\right] \\
=\ & \frac{1}{(n+1)}\left[\left(\varepsilon\left[\frac{xv(b)-(1-x)v(a)}{4}\right]\right)\left[P(f_\beta(s_{-i})\geq x(n-1)-(n-1)^{\frac{2}{3}}|b)\right]-4B(n+1)\exp(-2(n-1)^{\frac{1}{3}})\right] \\
\geq\ & \frac{1}{(n+1)}\left[\left(\varepsilon\left[\frac{xv(b)-(1-x)v(a)}{4}\right]\right)(1-2\exp(-2(n-1)^{\frac{1}{3}})-4B(n+1)\exp(-2(n-1)^{\frac{1}{3}})\right] \\
>\ & 0
\end{aligned}
$$

8.3 Let N be defined as in Step 8.2. Combining steps 8.1, 8.2, 6.3, and 7.3 we conclude that for n>N,

$$U_i(\beta|(b_j)_{j=1}^n, c_i, s_i=\beta, t_i) - U_i(\alpha|(b_j)_{j=1}^n, c_i, s_i=\beta, t_i) \geq 0$$

4 Discussion

1. When the number of buyers is large, the information of a single agent will generally have a small influence on the expected value of the common component. As discussed above, this is related to the idea of informational size that we have employed in other papers but differs in important ways. Our previous work assumed common knowledge of the information structure. Thus, if we were able to induce truthful revelation of agents' private information about the common component and make that information public, there would be common

knowledge of the expected value of that common component. This turns the second stage auction into a private value auction. In the current paper there is no common knowledge prior over agents' information - no assumption is made about agents' beliefs about either the accuracy of their own signal or the signals of others. For every probability distribution over buyers' accuracies, one can compute the expected value of the common component. To prove our main result we show that there is a lower bound on these expected values that converges to the expected value given the true state.

2. We demonstrate that in our mechanism, if it is assumed that buyers do not make dominated bids should they reach the second stage auction, then it is optimal for a buyer to correctly reveal his state signal when there were many buyers and other buyers reported truthfully.[8] It would, however, also have been optimal for a buyer to misreport his signal if all other buyers did so, for more or less the same reasons that truthful revelation is often not the unique equilibrium in a standard direct mechanism. To get to the second stage in our model, a buyer wants to be in the majority; if all other buyers misreport, my doing so as well maximizes my chance to move to the second stage. It should be noted, however, that whether all buyers report truthfully or all buyers lie (that is, each buyer announces the opposite of her signal), the same set of buyers will advance to the second stage and, having advanced to the second stage, the constraints on the bids that are undominated is the same. Hence, the lower bound on the seller's expected revenue is the same whether buyers unanimously announce truthfully or untruthfully in the first stage. This does not, however, mean that the lower bound is the same for all first stage announcements. For example, it is incentive compatible for all buyers to report state a regardless of the signal they receive, and the lower bound on the seller's expected revenue would typically be lower in this case.

3. We assume two equally likely states. While it is not critical that the states be exactly equally likely, the analysis above will break down if the states have dramatically different probabilities. Suppose the probability of state a is p and buyers get a state signal that has accuracy .6. If $p = .5$ and my signal indicates that the state is a, my belief is that a is the more likely state, and consequently, other people are more likely to get the signal indicating state a than a signal

[8]Note that we do not say that correctly reporting the state signal is an equilibrium. Since a buyer who reaches the second stage does not necessarily have a well defined probability distribution over his possible values of the object, he does not have a well defined expected utility conditional on getting to the second stage.

indicating state b. However, if $p = .01$, my posterior beliefs are that state b is more likely than a, and I have a better chance of getting to the second stage by misreporting my state signal than by reporting truthfully. If the states are not equally likely, there will be a minimum accuracy ρ of the signal for which, when i observe a signal for state a, my belief is that a is the most likely state. It is necessary and sufficient that the signal accuracy be at least this high to elicit truthful reporting.

4. We demonstrate that, for a particular auction problem, the incentive problem stemming from interdependent values can be ameliorated when there are many buyers. The structure of the argument suggests a general message. A buyer gains by misreporting that part of his private information that affects other buyers' values. By doing so the buyer alters other buyers' values by distorting their beliefs. The information structure in our problem has the property that as the number of buyers gets large, the degree to which a buyer can distort others' beliefs gets small, hence small rewards for truthful revelation induce truthful reporting. When the number of buyers gets large, the *aggregate* reward necessary to induce truthful reporting is small because the amount by which a buyer can distort other buyers' beliefs decreases faster than rate at which the number of buyers increases.

While there are information structures for which this is not the case, many natural information structures share this property. When this property holds, an important part of agents' asymmetric information – the part leading to interdependent values – can be dealt with at small cost.

5. MP2017 constructs a two-stage mechanism that uses the first-stage announcements to convert the initial interdependent value problem into a private value problem in the second stage, assuming truthful reporting in the first stage. This makes the analysis of agents' second stage bidding behavior easier: in the standard second-price auction, bidding below one's expected value is weakly dominated. In the current paper the second period problem is *not* private value: agents do not have a probability distribution over the accuracies of the signals received, hence, they do not have a probability distribution over their value of the object being auctioned. However, the lower bound on the possible accuracies puts a lower bound on the probability of the correct state of nature over all possible accuracies. This, in turn, puts a lower bound for any agent on her expected values across all possible accuracies, and bidding below this lower bound is dominated. As the number of agents increases, this lower bound

converges (with probability one) to the value of the object had the underlying state of nature been known.

6. The first stage of our two stage mechanism functions as a way to provide information to agents in the second stage that is useful in constructing an accurate estimate of the true state $\theta \in \{a, b\}$. This estimate is then used to compute expected payoffs that determine those second stage bids that are undominated. In this paper, all agents report their signals and those making a majority report move to the second stage. In a two stage mechanism in which agents are truthful in the first stage, a player who advances to the second stage can compute the relative frequency vector and, consequently, construct an accurate estimate of the state θ as an application of the law of large numbers. Our choice of the first stage construction ensures strict interim individual rationality and strict incentive compatibility, properties that we view as desirable. If these strictness requirements are relaxed, then one can find alternative constructions of the first stage such that the information learned by second stage participants allows them to compute an accurate estimate of the state θ.

7. Our main result takes an asymptotic perspective as the number of bidders gets large. A "small numbers" result is possible if signals are sufficiently accurate. Suppose that there are at least three bidders and each bidder gets a noisy signal about theta with accuracy λ_i where $x^* \leq \lambda_i < 1$. Let x^* be close to 1, meaning that all agents are getting signals that are highly accurate, but not perfectly accurate. Agents as usual announce the state, a or b. The majority go to the second stage (ignoring ties). Given the assumptions on the signal structure, an agent's expected effect on possible posteriors is small when other agents are announcing truthfully. A small prize (get the object for free with probability ε) is enough to get truth as an equilibrium if x^* sufficiently close to 1.

8. We can extend the analysis to multidimensional states. Suppose for the oil field example, the state θ has two attributes that bidders (might) value, say, the amount of oil and the depth of oil. Suppose that each of the attributes is binary: the amount is High or Low and depth is Deep or Shallow. Bidders may care about these differentially, that is, some may care more about amount than depth while for others it is the reverse. Suppose now each agent is going to receive a signal correlated with one of the attributes, but not the other. This violates our assumption that for any state, an agent receives a signal that has accuracy above .5 that his signal is the true state; now agents won't know about

states that differ on the attribute signal they do not receive. Now, instead of asking an agent to "predict" the state, we ask him to predict the attribute with which his signal is correlated and the majority announcers go to the second stage. While not a private values problem in the second stage, our method of restricting bids to be undominated will still deliver the same result.

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