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Abstract

This paper characterizes the transition dynamics of a continuous-time neoclassical production economy with capital accumulation in which households face idiosyncratic income risk and cannot commit to repay their debt. Therefore, even though a full set of contingent claims that pay out conditional on the realization of idiosyncratic shocks is available, the equilibrium features imperfect insurance and a non-degenerate cross-sectional consumption distribution. When household labor productivity takes two values, one of which is zero, and the utility function is logarithmic, we characterize the entire transition dynamics induced by unexpected technology shocks, including the evolution of the consumption distribution, in closed form. Thus, the model constitutes an analytically tractable alternative to the standard incomplete markets general equilibrium Aiyagari (1994) model by retaining its physical environment, but replacing the incomplete asset markets structure with one in which limits to consumption insurance emerge endogenously due to limited commitment.

JEL Codes: E21, D11, D91, G22

Keywords: Idiosyncratic Risk, Limited Commitment, Transition Path, MIT Shock

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1 Introduction

Households face considerable idiosyncratic income and unemployment risk. Following the work of Bewley (1986), Huggett (1993) and Aiyagari (1994), a large literature has arisen studying the macroeconomic consequences of this risk on the micro level, both theoretically as well as empirically.[1] The key assumption in much of this work (henceforth denoted as standard incomplete markets, SIM) is that the idiosyncratic risk is uninsurable, in the sense that explicit market or informal insurance arrangements are by assumption absent, and the best households can do is to engage in self-insurance through the accumulation of assets whose payoff is non-contingent on the realization of the idiosyncratic risk.

However, there is now considerable evidence that households are able to smooth consumption better than what is implied by the standard approach of self-insurance. Blundell, Pistaferri and Preston (2008) developed a (by now standard) methodology to empirically measure the extent of consumption insurance against permanent and transitory income shocks, and Kaplan and Violante (2011) showed that, quantitatively a standard life-cycle version of the SIM model implies too little insurance especially against permanent income shocks. A substantial follow-up literature, which includes Arellano, Blundell and Bonhomme (2017), Chatterjee, Morley and Singh (2021), Eika, Mogstad and Vestad (2020), Braxton et al. (2023), Balke and Lamadon (2022), and Commault (2022) have largely confirmed these findings. Thus, alternatives to the conventional self-insurance approach encoded in the SIM model are needed.

To make a contribution to this goal, in this paper we introduce limited commitment in the tradition of Kehoe and Levine (1993, 2001), Kocherlakota (1996), and Alvarez and Jermann (2000) into the same physical environment that Aiyagari (1994) studied with standard incomplete markets. Specifically, we develop and study a continuous time general equilibrium neoclassical production economy with idiosyncratic income risk and explicit insurance contracts against these risks. We assume that households cannot honor their debts, and therefore cannot sell these contracts short, limiting the extent of insurance households can achieve. Effectively, therefore, ours is a model with a full set of Arrow securities and tight short-sale constraints at zero.[2]

[1]See, e.g., Krueger, Mitman and Perri (2016) for an overview of this literature. It may be appropriate to also point to Imrohoroglu (1989) and the PhD thesis by Uhlig (1990), the latter of which also featured a choice of households between risky and riskless investments. The thesis is available here: https://voices.uchicago.edu/haralduhlig/thesis/.

[2]As Krueger and Uhlig (2006) show, this asset market structure is equivalent to a one-sided limited commitment model in which perfectly competitive and perfectly committed insurance companies offer long-term

The purpose of this paper is to understand the consequences of introducing this limited insurance as cleanly as possible, by examining the most tractable scenario in which closed-form solution of the entire macroeconomic dynamics can be given. The analysis here thus seeks to serve as an important stepping stone and complements to a more quantitative and empirical, but ultimately less tractable investigation. To this end, we assume that household labor productivity takes two values, one of which is zero, and that the period utility function is logarithmic. In Krueger and Uhlig (2024), we analytically characterized the steady state of this model (for an arbitrary number of income states). In this paper, we show that the entire transition path of the economy induced by an MIT aggregate (transitory or permanent) productivity shock, including the dynamic evolution of the non-degenerate consumption and wealth distribution can be given in closed form as long as the aggregate shock is not too large.[3] This complete analytical tractability of the transition path not only sets our model apart from the standard Aiyagari (1994) SIM model, but also contrasts with the representative agent neoclassical growth model without any idiosyncratic income risk (or equivalently, with frictionless complete markets) for which no closed-form solution of its transitional dynamics is available.

This analytical tractability originates from the fact that under the assumptions made, the population endogenously separates into two groups: one group with only labor income but no capital income, and a second group with no labor income but heterogeneous asset holdings and thus asset incomes. Crucially, this latter group shares the same consumption growth rate and effective saving rate, which (given log utility) is a constant that does not depend on the current or future interest rates. This second group then aggregates exactly (both in steady state and along the transition), and the resulting macro economy is also characterized by a constant aggregate saving rate, as in the classic Solow model or as in the model of workers and entrepreneurs by Moll (2014), but unlike in the standard neoclassical growth model or the SIM model. As Sato (1963) and Jones (2000) have already shown, the nonlinear ordinary differential equation characterizing the aggregate dynamics of an economy with a constant saving rate is a Bernoulli differential equation with a closed-form solution – the same is then true in our economy. We wish to emphasize, though, that in contrast to the Solow model, the constant aggregate saving rate is a result rather than an assumption, and that this rate depends on the structural parameters of the model,

consumption insurance contracts to households that cannot commit to these long-term contracts and can switch to competing intermediaries without punishment. Without punishment, short-sale constraints at zero are then precisely Alvarez and Jermann's (2000) solvency constraints that are "not too tight."

[3]The transition path could also be induced by an initial capital stock that is not at its steady state value.

including the time preference rate as well as the parameters governing the idiosyncratic income process.

Given the dynamics of the aggregate capital stock, the speed of convergence to the new steady state and the entire transition path of the consumption distribution in response to the MIT shock can also be characterized in closed form. To demonstrate the potential usefulness of our tractable model for applied-quantitative work, we establish two results. First, we show that our model can slow down the speed of convergence of capital to its long-run steady state (relative to the standard neoclassical growth model), and therefore potentially contribute to a resolution of the puzzle originally identified by King and Rebelo (1993) that neoclassical convergence dynamics tends to be too fast, relative to what is observed in the data. Second, we show that in the model, consumption inequality is "procyclical:" it increases on impact in response to an (expansionary) positive productivity shock before converging back to its original level in the long run.

1.1 Related Literature

In this paper, we seek to integrate two foundational strands of the literature on macroeconomics with household heterogeneity. The first strand has developed and applied the standard incomplete markets model with uninsurable idiosyncratic income shocks and neoclassical production, as Bewley (1986), Imrohoroglu (1989), Uhlig (1990), Huggett (1993), and Aiyagari (1994). In a recent paper, Achdou et al. (2022) analyze a two-state continuous-time SIM model. As we do, they characterize the stationary equilibrium by two key differential equations: one governing the optimal solution of the consumption (self-)insurance problem, and one characterizing the associated stationary distribution. The papers complement each other by characterizing equilibria in the same physical environment, but with two different market structures. Furthermore, we achieve a full analytical characterization of the entire transition path of the economy, possibly opening a path for an analytical analysis of macroeconomic fluctuations.

The second branch is the literature on recursive contracts and endogenously incomplete markets which permits explicit insurance, but whose scope is limited by contract enforcement frictions.[4] More specifically, we incorporate explicit insurance contracts of-

[4]Recent work that builds on Kehoe and Levine (1993), Kocherlakota (1996), and Alvarez and Jermann (2000) includes Broer (2012), Abraham and Laczo (2018), and Sargent, Wang and Yang (2021). A common theme in this literature is the interaction between private and public insurance, see, e.g., Golosov and Tsyvinski (2007), Thomas and Worrall (2007), and Krueger and Perri (2011).

fered by competitive financial intermediaries, as analyzed previously in partial equilibrium by Krueger and Uhlig (2006), into a neoclassical production economy. In doing so, we seek to provide the macroeconomics profession with a novel, fully micro-founded yet analytically tractable model of neoclassical investment, production, and the cross-sectional consumption and wealth distribution, where the limits to cross-insurance are explicitly derived from first principles of contractual frictions. While the approach and formulation here are described from the perspective of a financial market, one can alternatively think of the insurance contracts offered by financial intermediaries as long-term wage contracts offered by firms that provide workers with partial insurance against productivity fluctuations to workers, in line with the formulation in Harris and Holmstrom (1982), Thomas and Worrall (1988), Guiso, Pistaferri and Schivardi (2005), Saporta-Eksten (2016), and Balke and Lamadon (2022).

Finally, our paper shares elements and insights with other work on dynamic macro models with limited commitment in endowment economies, such as Zhang (2013), Grochulski and Zhang (2011), and Miao and Zhang (2015), but models capital accumulation and production explicitly. In doing so, we provide a general equilibrium treatment, as do Gottardi and Kubler (2015), Hellwig and Lorenzoni (2009), and Martins-da-Rocha and Santos (2019), the latter two in the context of the sovereign debt literature.

In the next section, we describe the model and define the equilibrium. Section 3 characterizes the optimal household consumption-asset allocation for a given sequence of wages and interest rates. Section 4 aggregates these allocations to analytically characterize the equilibrium transition path, starting from an initial steady state with partial consumption insurance. It also provides sufficient conditions on the aggregate productivity process for a partial insurance transition equilibrium to exist, and contrasts the speed of convergence to the new steady state in our model to that of the representative agent neoclassical growth model. Section 5 traces out consumption and wealth inequality along the transition, both analytically and quantitatively through numerical simulations. Section 6 concludes. All proofs and additional technical details and results are contained in the Appendix.

2 The Model

2.1 Preferences, Endowments and Financial Markets

Time is continuous. There is a population of a continuum of infinitely lived agents of mass 1 who supply labor to the market, consume goods, and sign contracts. The labor productivity z_{it} of an individual agent i at time t follows a two-state Poisson process that is independent across agents. More precisely, productivity can either be high ($z_{it} = \zeta > 0$) or zero ($z_{it} = 0$). Let $Z = \{0, \zeta\}$. The transition from high to low productivity occurs at rate $\xi > 0$, whereas the transition from low to high productivity occurs at rate $\nu > 0$. The stationary productivity distribution associated with this process is given by

$$(\Psi_l, \Psi_h) = \left(\frac{\xi}{\xi + \nu}, \frac{\nu}{\xi + \nu} \right). \tag{1}$$

We assume that the initial distribution given in equation (1) – the same is then true for all $t > 0$.

Agents have log utility $u(c) = \log(c)$ and discount the future at rate $\rho > 0$. Then the expected utility of an agent from period t onward is given by

$$U_t = E_t \left[\int_t^{+\infty} e^{-\rho(\tau - t)} \log(c) \, d\tau \right],$$

where the expectation depends on the current idiosyncratic state and risk of the agent.[5]

There is a competitive sector of production firms which uses labor and capital to produce the final output good according to the Cobb-Douglas production function $A_t F(K, L) = A_t K^\theta L^{1-\theta}$, where $\theta \in (0, 1)$ denotes the capital share and $A_t > 0$ is a productivity parameter, evolving as an exogenous and non-stochastic function of time. More specifically, we always impose the following assumption.

Assumption 1. *A_t is differentiable as a function of time t for all $t > 0$ and converges to a finite and strictly positive limit,*

$$A_\infty = \lim_{t \to +\infty} A_t, A_\infty \in (0, +\infty). \tag{2}$$

Note that the above assumption permits productivity A_t to jump (or have a kink) at

[5] We abstract from aggregate risk in this paper. In addition, a number of our results for the deterministic transition analysis generalize to CRRA utility, see Online Appendix G.

$t = 0$. In fact, it merely stipulates that after this initial MIT shock, productivity evolves smoothly and converges to a finite limit. Capital depreciates at a constant rate $\delta \geq 0$. Production firms seek to maximize profits, taking as given the market spot wage w_t per efficiency unit of labor and rental rate of capital (net of depreciation) r_t per unit of capital. We normalize aggregate labor supply $L = \zeta \frac{\nu}{\xi+\nu}$ to unity, and thus

$$\zeta = \frac{\xi + \nu}{\nu}. \tag{3}$$

This normalization is without loss of generality given the CRTS production technology.

As in Krueger and Uhlig (2006, 2022, 2024), agents seek to insure themselves against their productivity fluctuations. We assume that a full set of individual-specific insurance contracts is available, but individuals cannot commit to honor these contracts and there is no punishment from default. There are two ways to formulate the resulting consumption insurance contracts and associated market structure, which turn out to be equivalent.

First, envision a market structure in which individuals buy long-term consumption insurance contracts from risk-neutral and perfectly competitive insurance companies. These financial intermediaries can fully commit to contracts, whereas individuals cannot commit and individuals can switch insurers without cost at any time, i.e., there is one-sided limited commitment. The intermediaries offer the utility-maximizing consumption allocation to individuals, subject to breaking even and subject to not losing an individual to the competition.[6]

In Krueger and Uhlig (2006), following the insights of Alvarez and Jermann (2000), they show that this is equivalent to an asset market-based formulation in which individuals own assets (in the form of physical capital), either by themselves or through an account at a financial intermediary, and given this capital, maximize lifetime utility by buying idiosyncratic shock-contingent Arrow securities subject to state-contingent short-sale constraints. This is the formulation we pursue here. The key result in Krueger and Uhlig (2006), reminiscent of Bulow and Rogoff (1989), is that limited commitment by households and no punishment from default implies that individuals cannot borrow at all in this capital account. Note that the capital account is state-contingent, and its balance can jump when productivity changes and otherwise evolves due to new (possibly negative) investment x, given the current agent-specific state and calendar time t. Our formulation is, therefore, quite different from a conventional borrowing constraint for state non-contingent assets.

[6] Krueger and Uhlig (2022) use this formulation in their analysis of stationary equilibria.

Denote by $U_t(k;z)$ the expected continuation lifetime utility of the agent, given the current capital account k, agent-specific productivity z and the aggregate state of the economy encapsulated by the time index t. This lifetime utility satisfies the Hamilton-Jacobi-Bellman (HJB) equation defining the optimal consumption-asset allocation.

Definition 1. *For $z \in Z$, wages w_t and interest rates r_t, let $\tilde{z}$ be the "other" state and $p_z \in \{\xi, \nu\}$ be the Poisson intensity for the transition from z to $\tilde{z}$. An optimal consumption allocation $\mathcal{C}_t = \left(U_t(k;z), c_t(k;z), x_t(k;z), \tilde{k}_t(k;z)\right)_{k\geq 0, z\in Z}$ is the solution to the program*

$$\rho U_t(k;z) = \max_{c,\tilde{k}\geq 0,x} u(c) + \dot{U}_t(k;z) + U_t'(k;z)\,x + p_z\left(U_t\left(\tilde{k},\tilde{z}\right) - U_t(k;z)\right) \quad (4)$$

$$\text{s.t.} \quad c + x + p_z\left(\tilde{k} - k\right) = r_t k + w_t z \quad (5)$$

$$x \geq 0 \text{ if } k = 0. \quad (6)$$

To build intuition for the HJB, consider the agent's (not planner's) problem in the standard deterministic neoclassical growth model where the agent receives a constant wage w and owns capital k, earning interest r. The HJB equation in that model reads as $\rho U(k) = \max_{c,x} u(c) + U'(k)\,x$ s.t. $c + x = rk + w$, or, substituting out investment x, simply $\rho U(k) = \max_{c\geq 0}\{u(c) + U'(k)[rk + w - c]\}$. The flow payoff $\rho U(k)$ of the value function $U(k)$ is the sum of the flow utility $u(c)$ from consuming c and the change in the value function $U'(k)\,x$ due to the investment $\dot{k}_t = x$. Investment and consumption have to respect the budget constraint $c + x = rk + w$. The agent chooses c (and x) so as to maximize the flow payoff $\rho U(k)$, given the budget constraint.

The fact that wages and interest rates (w_t, r_t) are time-varying adds a time subscript to the value function $U_t(\cdot)$, and the payoff now also includes the time derivative in the value function $\dot{U}_t(\cdot)$ due to changing factor prices. In the presence of idiosyncratic labor productivity risk, the current state of the household includes both capital as well as current productivity (k, z), and the value function becomes a function of both these individual state variables. In addition, labor income is now $w_t z$.

For Definition 1, two further crucial features are added that embed the financial market structure with explicit insurance but limited commitment. First, the flow payoff $\rho U_t(k;z)$ also accounts for the expected instantaneous change in utility $p_z\left(U_t\left(\tilde{k};\tilde{z}\right) - U_t(k;z)\right)$ due to a possible change in productivity from z to $\tilde{z}$. Note the crucial feature that the capital stock upon a productivity change, $\tilde{k}$, is allowed to differ from the current one, k. This is the feature of explicit insurance against idiosyncratic agent-specific shocks, in contrast to the standard incomplete markets model which requires $\tilde{k} = k$. The change in the capital stock has to be paid for, though, which explains the actuarially fair "insurance premium" $p_z \left(\tilde{k} - k\right)$ in the budget constraint[7] (5). Second, the lack of commitment is incorporated by the restriction that $\tilde{k} \geq 0$, as well as $x \geq 0$ when $k = 0$. Without punishment for walking away from a negative capital account (defaulting on an intermediary if the account is held with them), Krueger and Uhlig (2006) show that the state-contingent borrowing limits that are not too tight, in the sense of Alvarez and Jermann (2000), are exactly at zero.[8]

2.2 Equilibrium

In our model, agents hold capital to insure against a spell of low productivity. We will focus on equilibria in which agents never wish to purchase state-contingent capital for the high-productivity state – our definition of equilibrium below reflects that focus. For this to be optimal, the return on capital has to be sufficiently low and wage growth sufficiently high (in a way we make precise below). We will provide sufficient conditions on the parameters of the model such that this is indeed the case in equilibrium.[9]

The only reason for acquiring and subsequently holding capital is thus to finance the consumption stream of agents with zero productivity. High-productivity agents pay insurance premia to obtain a stock of capital should the transition to zero productivity occur, but hold no capital as long as they are productive. Thus, all these agents are identical and we do not need to keep track of their past productivity history. Low-productivity agents, in contrast, are distinguished by the length of time $\tau \geq 0$ elapsed since the transition from high to low productivity occurred. The distribution of these agent types is easy to characterize. The total mass of high- and low-productivity agents is given in equation (1). The density for low-productivity agents is given by

$$\psi_l(\tau) = \frac{\xi\nu}{\xi+\nu} e^{-\nu\tau}, \tau \geq 0, \tag{7}$$

which integrates to the total mass $\Psi_l = \frac{\xi}{\xi+\nu}$ of low-productivity agents. Low-productivity agents hold capital $k_{s,t}$ which depends on the date t and the time $s = t - \tau$ of the transition to

[7]We can think of insurance being offered by risk-neutral perfectly competitive intermediaries so that the price of one unit of capital upon a transition from z to $\tilde{z}$ costs p_z, which equals the transition rate from z to $\tilde{z}$.

[8]In Krueger and Uhlig's (2006) discrete-time environment, the interest rate is exogenous, but this does not change the fact that in the absence of any punishment, positive (possibly state-contingent) debt (a negative k) cannot be sustained, reminiscent of the classic Bulow and Rogoff (1989) result.

[9]Under these parameter restrictions, we conjecture this is the only equilibrium.

low productivity. Thus, rather than keeping track of the joint state distribution across capital and productivity states $(k;z)$, it is more convenient to keep track of the capital holding $k_{s,t}$ as a function of the transition time s and the calendar time t. Similarly, we denote by $c_{s,t}$ the consumption of an agent at time t who made the transition to low productivity at time $s<t$. In what follows, time derivatives are always with respect to calendar time.

Definition 2. *An equilibrium consists of household allocations $\mathcal{C}_t$, equilibrium wages w_t, interest rates r_t, aggregate capital K_t, and capital holdings of low-productivity agents $(k_{s,t})_{s\leq t}$, as functions of time $t\in(-\infty,+\infty)$, such that:*

1. *Given w_t and r_t, the household allocations $\mathcal{C}_t$ are optimal (see Definition 1).*

2. *The allocations $\mathcal{C}_t$ have the "only low-productivity agents hold capital" property that $\tilde{k}_t(k;0)=0$ for all $k=k_{t,\tau}$, $\tau\geq 0$ as well as $x_t(0;\zeta)=0$.*

3. *Capital holdings of low-productivity agents are consistent with the allocations $\mathcal{C}_t$, i.e.,*

$$\begin{aligned} k_{t,t} &= \tilde{k}_t(0;\zeta), && (8)\\ \dot{k}_{s,t} &= x_t(k_{s,t};0), && (9)\end{aligned}$$

   where $\dot{k}_{s,t}=\frac{\partial k_{s,t}}{\partial t}$.

4. *The interest rates and wages (r_t,w_t) satisfy*

$$\begin{aligned} r_t &= A_t F_K(K_t,1)-\delta, && (10)\\ w_t &= A_t F_L(K_t,1). && (11)\end{aligned}$$

5. *The goods market clears*

$$\int_0^{+\infty} c_t(k_{t-\tau,t};0)\,\psi_l(\tau)\,d\tau+\frac{\nu}{\xi+\nu}c_t(0;\zeta)+\delta K_t=A_tF(K_t,1). \qquad (12)$$

6. *The capital market clears*

$$\int_0^{+\infty} k_{t-\tau,t}\psi_l(\tau)\,d\tau=K_t. \qquad (13)$$

In the capital market clearing condition (13), the supply of capital comes from all agents with currently low productivity that were income rich τ periods ago, integrated over all τ. Similarly, aggregate consumption in (12) is composed of the integral over the heterogeneous consumption levels of the income-poor and the uniform consumption of the income-rich times their mass.

The thought experiment envisions the economy initially (for $t < 0$) in a *stationary* equilibrium in which all entities in Definition 2 indexed by time t are constant, where we establish that there is a unique stationary equilibrium in Section 4.1. Then, aggregate total factor productivity changes at time $t = 0$ from the constant A^* towards a time-varying path A_t with $\lim_{t\to+\infty} A_t = A_\infty$. Since this is a complete surprise to everyone (the typical "MIT shock"), the stationary equilibrium allocations chosen at $t < 0$ have not allowed for that contingency. We then characterize the dynamic transition path induced by this change in productivity. The aggregate capital stock and its distribution at $t = 0$ is predetermined by the steady state equilibrium associated with $A_t \equiv A^*$.

3 The Optimal Consumption Allocation

In this section, we characterize the optimal household consumption-asset allocation, given a path for wages and interest rates (r_t, w_t). We start with a graphical representation of the optimal allocation to provide intuition and to guide the ensuing theoretical analysis.

Figure 1 illustrates the consumption insurance allocation in the initial steady state (i.e., for $t < 0$) of an agent with productivity z_t and thus labor income $y_t = w^* z_t$ that switches idiosyncratic productivity at two Poisson dates from high to low and back to high productivity. In the high idiosyncratic income state of a simple equilibrium, the agent holds no capital, consumes less than his current income (see the upper panel) and uses the difference to make insurance payments against the possibility of a switch to low productivity. When the switch occurs, the agent receives a stock of capital as insurance payout (see the lower panel) and draws down this capital account to finance the consumption stream during the low-productivity (zero labor income) phase. Upon a transition back to high productivity, the capital account returns to its zero value and the allocation returns to the initial phase.

Figure 2 depicts what happens to the consumption insurance allocation on the impact of the MIT shock at time $t = 0$. Whereas the capital held by low-productivity agent remains unchanged on impact (since capital is a state variable and the transition was completely unexpected), a different path for consumption emerges, due to changed aggregate dynamics

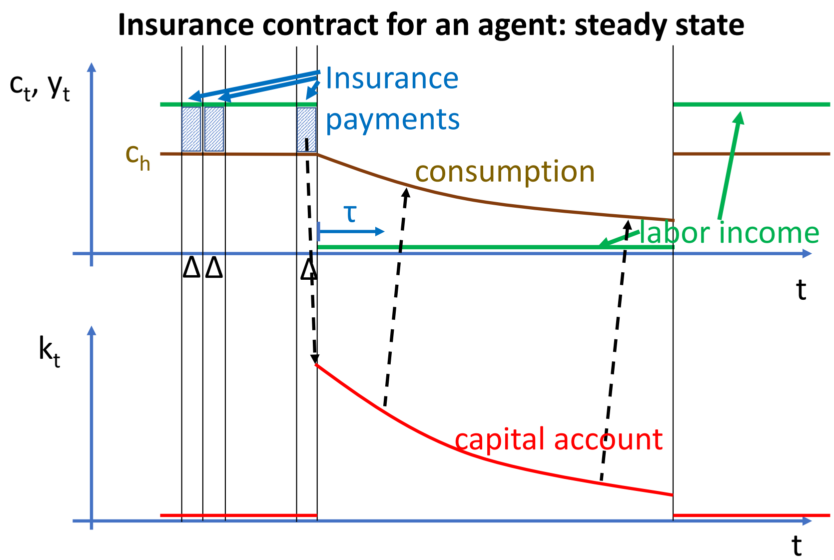


Figure 1: Consumption insurance allocations in stationary equilibrium. In the high-productivity state, the agent holds no capital, consumes less than current income, and pays for insurance against a productivity change. When the productivity state changes to zero, the agent receives a stock of capital as insurance payout, running it down while productivity is zero. When productivity switches to the high state again, the capital account returns to zero.

in wages and interest rates (w_t, r_t) and the resulting individual income process $y_t = w_t z_t$. In the high-productivity phase, consumption changes due to the changing wages along the transition, but as long as Assumption 3 below is satisfied, the capital account will still be zero and part of labor income will again be devoted to insurance payments against idiosyncratic productivity loss. During the low-productivity and thus zero labor income spell (assumed to encompass the instant $t = 0$ of the MIT shock), the aggregate shock will potentially induce an altered consumption path due to interest rate changes along the transition. Whereas consumption can in principle change discontinuously at $t = 0$ (as displayed in Figure 2), we will show that this is not the case when utility is logarithmic.[10]

We now proceed with the formal analysis. Given an allocation $\mathcal{C}_t$, we define the implied

[10]For a general CRRA utility function, a jump at $t = 0$ indeed occurs. See Online Appendix G.

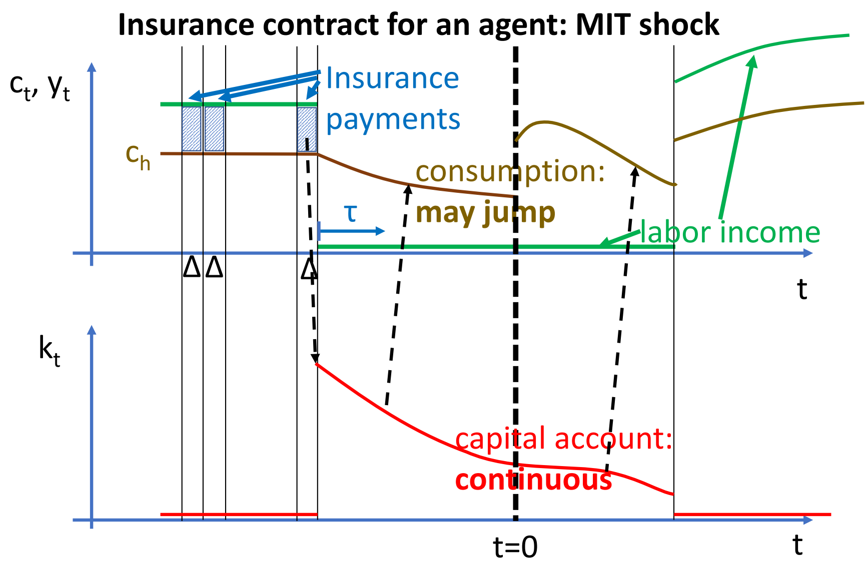


Figure 2: Consumption allocation around transition date $t = 0$. Low-productivity agents keep their capital. Since returns r_t and wages w_t have changed, a different consumption path might now be optimal, given this initial capital.

time derivative of consumption (assuming no idiosyncratic z-state transition) as[11]

$$\dot{c}_t\left(k;z\right) \equiv \frac{\partial c_t\left(k;z\right)}{\partial t} + \frac{\partial c_t\left(k;z\right)}{\partial k} x_t\left(k;z\right). \tag{14}$$

We make the following assumption, which involves equilibrium variables. In Proposition 2 below, we will show that the condition (in this assumption) holds under suitable assumptions about the exogenous parameters.

Assumption 2. *For some $T \geq 0$, $r_t < \rho$ for all $t \geq T$.*

Lemma 1 (The optimal allocation $\mathcal{C}_t$ for $z = 0$ and $k > 0$)**.** *For $k > 0$ and with assumption 2, the optimal contract of definition 1 is characterized by the consumption dynamics*

$$\frac{\dot{c}_t\left(k;z\right)}{c_t\left(k;z\right)} = r_t - \rho. \tag{15}$$

Furthermore, if $z = 0$ and, for some $\bar{k} > 0$ we have $\tilde{k}_t(k;0) = 0$ for all $k \leq \bar{k}$, then there

[11]To provide some intuition for this definition of the time derivative, suppose that productivity remains constant at z for some time interval. In that case note that $\dot{k}_t = x_t\left(k;z\right)$ and that consumption evolves as $c\left(t\right) = c_t\left(k_t;z\right)$ as a function of time only. Taking the derivative with respect to time yields the expression.

exists $\bar{\bar{k}}$ such that for all $k \leq \bar{\bar{k}}$,

$$c_t\left(k;0\right) \;=\; \left(\rho+\nu\right)k, \tag{16}$$

$$x_t\left(k;0\right) \;=\; \left(r_t-\rho\right)k, \tag{17}$$

The proof is in the Appendix. Note that we permit the possibility that $r_t > \rho$ for some period along the transition. When this happens, capital k will be temporarily increasing. In Lemma 1, we need to make sure that $k_t \leq \bar{k}$ for all t. This is assured by making the initial level of capital $k > 0$ small enough, i.e., $k \leq \bar{\bar{k}}$ for some suitable bound $\bar{\bar{k}}$. We now use this result to characterize the dynamics of consumption for individuals with currently high productivity. To do so, let us make the following assumption concerning equilibrium variables, to be replaced in Proposition 2 below by an assumption about exogenous parameters.

Assumption 3. *Suppose the aggregate wage and interest rate satisfy $\forall t \geq 0$,*

$$\frac{\dot{w}_t}{w_t} + \rho > r_t. \tag{18}$$

Generally, agents discount future payments at rate $\frac{\dot{c}_t}{c_t} + \rho$. Suppose that consumption is proportional to wages, as will be the case for individuals with high productivity. The assumption then says that this discount rate is higher than the interest rate r_t that can be earned on savings, i.e., these agents would rather not postpone consumption into the future by saving. Recall that they cannot borrow. Now define the constant

$$\alpha \equiv \frac{\rho+\nu}{\rho+\nu+\xi}. \tag{19}$$

For ease of notation, let $c_{s,t} = c_t(k_{s,t};0)$ denote consumption of a zero productivity agent at date t who switched from high to zero productivity at date $s \leq t$, and thus holds capital $k_{s,t}$. This notation implies that $c_{s,s}$ and $k_{s,s}$ are the consumption and capital holdings of an individual whose productivity has turned to zero this very instant. Denote by $c_{h,t} = c_t(0;\zeta)$ the date t consumption of a high-income individual with no assets.

Lemma 2 (The optimal allocation $\mathcal{C}_t$)**.** *Let assumption 3 be satisfied. Then the optimal contract of definition 1 implies the dynamics of consumption and investment as*

$$c_{h,t} = \alpha\zeta w_t, \tag{20}$$

$$c_{s,t} = c_{h,s} e^{\int_s^t (r_u - \rho) du}, \tag{21}$$

for all s, t, where the constant α is defined in equation (19) and where we recall that the instantaneous consumption growth rate of unconstrained agents is given by $r_t - \rho$.

The proof is in the Appendix. Equation (20) implies that high-productivity agents pay an insurance premium $(1 - \alpha)\zeta w_t$ against their productivity falling. Since the insurance contracts are actuarily fair, this finances initial capital

$$k_{t,t} = \frac{1 - \alpha}{\xi} \zeta w_t \tag{22}$$

after the switch to zero income. Equation (16) then implies

$$c_{t,t} = \frac{(\rho + \nu)(1 - \alpha)}{\xi} \zeta w_t = \alpha\zeta w_t \tag{23}$$

for the consumption following the switch, which coincides with $c_{h,t}$ and with the expression in (21). Finally, solving for α in equation (23) delivers (19).

Remarkably and as equation (16) shows, the wage-normalized entry level of consumption $c_{h,t}/w_t$ after switching to zero productivity is the same as in the initial steady state, despite the fact that the subsequent consumption path drifts down at a different (and time-varying) rate, see equation (21). However, since the rates at which future consumption is discounted also change, the present discounted value of this altered consumption stream remains the same with log-utility, and thus the wage-normalized entry level of consumption does not change along the transition path. This is a version of the well-known "income effect and substitution effect cancel" property of log preferences.

Applying the results above to the steady state, we have

$$c_h^* = \alpha\zeta w^*, \tag{24}$$

$$c_\tau^* = e^{(\rho - r^*)\tau} c_h^*, \tag{25}$$

$$k_\tau^* = \frac{1}{\rho + \nu} e^{(\rho - r^*)\tau} c_h^*. \tag{26}$$

4 Transition Dynamics

To compute the aggregate capital supply K_t at time t, we aggregate the capital holdings of low-productivity agents,

$$K_t = \int_{-\infty}^{t} k_{s,t}\psi_l(t-s)ds. \tag{27}$$

This allows us to characterize the evolution of the aggregate capital stock in closed form.

Lemma 3 (Dynamics of aggregate capital supply)**.** *Let the initial capital stock K_0 be given by the steady state capital stock associated with a steady state interest rate $r < \rho$ and*[12] *let Assumptions 1, 2 and 3 be satisfied. Then the aggregate law of motion for capital is given as*

$$\dot{K}_t = \hat{s}A_t K_t^{\theta} - \hat{\delta}K_t, \tag{28}$$

$$\text{where } \hat{s} = 1 - \alpha + \alpha\theta \text{ and } \hat{\delta} = \delta + \rho + \nu. \tag{29}$$

The proof is in the Appendix. The aggregation result is intuitive: high-productivity agents consume the fraction α of all wages and save the rest $(1-\alpha)w_t$. Low-productivity agents own all capital, earn r_t on their capital accounts, and deplete it at rate $\rho+\nu$. Adding up yields

$$\dot{K}_t = (1-\alpha)\, w_t + (r_t - (\rho+\nu))\, K_t. \tag{30}$$

Using the expressions for the wage and interest rate (r_t, w_t) from (10) and (11) delivers the law of motion (28) with saving rate $\hat{s}$ in (29).

Lemma 3 shows that the law of motion for aggregate capital is akin to that of a Solow model with the exogenous and constant saving rate $\hat{s}$ and depreciation rate $\hat{\delta}$. This is a consequence of the constant savings rates (see equations (16)) and (20)): with log utility, low-productivity agents consume a constant fraction of their capital account and high-productivity agents consume a constant fraction of their wage income.

As is well-known for the Solow model or from lemma 10 with $a_s = \hat{s}A_s$, the differential equation (28) has a closed-form solution for the equilibrium time path of capital, i.e.,

$$K_t = \left(e^{-(1-\theta)\hat{\delta}t}\,(K_0)^{1-\theta} + (1-\theta)\,\hat{s}\int_0^t e^{-(1-\theta)\hat{\delta}(t-s)}A_s ds\right)^{\frac{1}{1-\theta}}. \tag{31}$$

[12] Assumption S1 in Section 4.1 below guarantees such a unique partial insurance steady state with $r < \rho$.

Define the function

$$K(A) = \left(\frac{\hat{s}A}{\hat{\delta}}\right)^{\frac{1}{1-\theta}}. \tag{32}$$

Corollary 1. *Under the conditions of Lemma 3, we have* $K_t \to K(A_\infty)$.

This is a direct consequence of Assumption 1, Lemma 3 and equation (31).

With log utility, the right-hand side of (31) is exclusively a function of exogenous parameters and the exogenous time path of total factor productivity A_t. The explicit solution in (31), in principle, applies to any productivity path, but the requirement that Assumption 3 be satisfied imposes restrictions on the path for which (31) is a valid characterization of the equilibrium transition path for capital.

The time paths of all other aggregate variables, such as interest rate, wage and aggregate consumption directly follow from those of the aggregate capital stock, as in the standard neoclassical growth model, i.e.,

$$r_t = \theta A_t K_t^{\theta-1} - \delta, \tag{33}$$

$$w_t = (1-\theta) A_t K_t^{\theta}, \tag{34}$$

$$C_t = A_t K_t^{\theta} - \delta K_t - \dot{K}_t = (1-\hat{s}) A_t K_t^{\theta} + \left(\hat{\delta} - \delta\right) K_t. \tag{35}$$

A Solow model would yield $C_t = (1-\hat{s}) A_t K_t^{\theta}$. The difference here arises from the difference between $\hat{\delta}$ and δ. The expression (33) for interest rates allows us to rewrite (28) more conveniently as

$$\frac{\dot{K}_t}{K_t} = \frac{\hat{s}}{\theta}(r_t + \delta) - \hat{\delta}. \tag{36}$$

4.1 Partial Insurance Steady State

For a constant $A_t \equiv A^*$, the steady state K^* capital stock satisfying $\dot{K}_t = 0$ and associated interest rate r^* are given by

$$K^* = \left(\frac{\hat{s}A^*}{\hat{\delta}}\right)^{\frac{1}{1-\theta}} = K\left(A^*\right), \tag{37}$$

$$r^* = \theta\frac{\hat{\delta}}{\hat{s}} - \delta. \tag{38}$$

Define the constant

$$\chi \equiv \frac{\xi}{\nu\left(\rho+\nu+\xi\right)} - \frac{\theta}{\left(1-\theta\right)\left(\rho+\delta\right)}. \tag{39}$$

Krueger and Uhlig (2024) show that χ is the difference between steady-state wage-normalized capital supply and demand at $r = \rho$. We now impose the following assumption for the steady state (consequently labeled S1), which guarantees a unique stationary equilibrium with interest rate $r^* < \rho$.

Assumption S1. *Let the exogenous parameters of the model satisfy $\theta, \nu, \xi, \rho > 0$ and*

$$\chi > 0. \tag{40}$$

In Lemma 5 of Online Appendix B.1, we show that $r^* < \rho$ if and only if $\chi > 0$. The following proposition from Krueger and Uhlig (2024) follows from and summarizes our discussion above and fully characterizes the steady state of the model.

Proposition 1 (Krueger and Uhlig (2024))**.** *Let Assumption S1 be satisfied. Then there exists a unique stationary equilibrium. The unique equilibrium capital stock and interest rate (K^*, r^*) are given by equations (37) and (38), and r^* satisfies $r^* < \rho$. The equilibrium features partial insurance, i.e., consumption of the high-productivity agents is c_h and consumption of the low-productivity agents drifts downwards at rate $r^* - \rho < 0$. The equilibrium wage is given by $w^* = (1-\theta) A^* (K^*)^\theta$. The stationary consumption distribution has a mass point at $c_h^* = \alpha\zeta w^*$ for the mass $\frac{\nu}{\nu+\xi}$ of high-productivity agents, and*

$$k_\tau^* = e^{-(\rho - r^*)\tau} \frac{c_h^*}{\nu+\rho} \tag{41}$$

$$c_\tau^* = e^{-(\rho - r^*)\tau} c_h^* \tag{42}$$

for the low-productivity agents as a function of τ (the time elapsed since their last transition to low productivity), where $k_\tau^* = k_{t-\tau,t}$ and $c_\tau^* = c\left(k_{t-\tau,t}\right)$ is independent of t.

Note that k_τ^* is the net present value of the future zero-income consumptions $c_{\tau+s}^*$, taking into account the rate ν of switching out of the zero income state, i.e.,

$$k_\tau^* = \int_{s=0}^{+\infty} e^{-(\nu+r^*)s} c_{\tau+s}^* ds. \tag{43}$$

4.2 Sufficient Conditions

In this section, we derive sufficient conditions on the exogenous productivity path and the other model parameters under which equation (18) is satisfied in equilibrium. In what follows, we also impose Assumption S1 which guarantees the existence of a unique stationary equilibrium with partial insurance from which the transition path starts.

We begin by showing that Assumption 3 requires that the growth rate of productivity is not "too low" and the equilibrium interest rate is not "too high" along the transition path. Exploiting $\dot{w}_t/w_t = \dot{A}_t/A_t + \theta\dot{K}_t/K_t$ and using the dynamics of aggregate capital in equation (36), substituting $\hat{s}\delta - \theta\hat{\delta}$ with $\hat{s}r^*$ and rearranging terms, we rewrite condition (18) in Assumption 3 as

$$\frac{\dot{A}_t}{A_t} > (1-\hat{s})\,(r_t - r^*) + r^* - \rho. \tag{44}$$

Hence, if the growth rate of productivity $\dot{A}_t/A_t$ is bounded below and the equilibrium interest rate is bounded above, then a *sufficient* (but not necessary) condition for Assumption 3 to hold in equilibrium is that the inequality in equation (44) holds at these bounds for $\dot{A}_t/A_t$ and r_t. In that case, the incentives to save for the high idiosyncratic productivity state are sufficiently weak because future wage growth is sufficiently strong and interest rates are sufficiently low along the entire equilibrium path.

For a productivity process that is increasing over time, only an upper bound on the level of productivity is needed to establish that (44) holds. An increase in productivity has two countervailing effects. On the one hand, it leads to an increase in the equilibrium wage over time and thus *reduces* savings incentives, relaxing the inequality in (44). On the other hand, the increase in productivity leads to a temporary increase in the equilibrium interest rate along the transition and thus strengthens the incentives to save even for the high idiosyncratic productivity state. It is this second effect whose magnitude we have to bound with an upper bound on the level of productivity in order to ensure that (44) remains satisfied for all t. We formalize this argument in Section 4.2.1, and we provide two examples of productivity processes that satisfy this condition – one is a one-time permanent increase in productivity and the other is a monotonically increasing path of productivity.

The situation is asymmetric for a productivity decline, since it is negative wage *growth* now that might induce savings for the future for all contingencies and threatens to violate the no-savings condition (44). Now we require an upper bound on the speed of the productivity decline rather than on the level of productivity in order to bound the negative wage growth effect that is only partially offset by a decline in the interest rate along the transition. That a condition on the new level of productivity (as was the case for an increase in productivity) is now insufficient can be readily seen for a permanent decline in productivity (for which $\dot{A}_t/A_t = -\infty$ at $t = 0$ and thus (44) is necessarily violated since the interest rate remains finite at $t = 0$). We formalize this argument in Section 4.2.2 and give an example of a gradually declining productivity path that satisfies the no-savings condition (44).

Taken together, the sufficient conditions on the productivity paths for Assumption 3 to hold in equilibrium are "asymmetric" for increasing versus decreasing productivity paths – the former requires an upper bound on the *level* of productivity while the latter requires a lower bound on the negative *growth rate* of productivity. In Section 4.3, we show by example that *conditional* on the no-savings condition (44) being satisfied in both cases, a symmetric increase and decrease in productivity leads to a nearly symmetric transition path for aggregate capital and the other macroeconomic aggregates in the model. Finally, in Section 4.4, we compare the speed of convergence to a new steady state in our model to the speed in the standard (continuous time) representative agent neoclassical growth model.

4.2.1 Increase in Productivity

In this section, we consider a general productivity process $\{A_t\}_{t\geq 0}$ that is increasing over time. We can now replace Assumptions 2 and 3 on equilibrium wages and interest rates with Assumption 4, which is stated purely in terms of exogenous parameters.

Assumption 4 (Upper bound on the productivity level)**.** *The productivity process* $\{A_t\}_{t\geq 0}$ *satisfies* $A_t < \bar{A}$ *for all* $t \geq 0$ *and the bound* $\bar{A} > A^*$ *satisfies*

$$\frac{\bar{A}}{A^*} = 1 + \frac{\rho - r^*}{(r^* + \delta)(1 - \hat{s})}, \tag{45}$$

where A^* *is the initial steady state productivity level and* r^* *is the stationary interest rate.*

Given Assumption S1, Proposition 1, equation (38) and $0 < \hat{s} < 1$ imply that $\bar{A} > A^*$. Broadly speaking, Assumption 4 imposes an upper bound on the level of productivity so that the interest rate does not rise "too high" along the transition path. This discourages the high-productivity agent from accumulating capital.[13]

[13] The upper bound $\bar{A}$ follows from the following consideration, to be made precise in the proof of Proposition 2 in the Appendix. Since A_t is increasing, K_t is increasing and $r_t < \bar{r} = \bar{A}\,(K_0^*)^{\theta-1} - \delta$, the interest

With Assumption 4, we can now characterize the dynamics of aggregate capital in closed form in Proposition 2. This proposition is the counterpart of Lemma 9 in Online Appendix G.3, but with logarithmic utility and with an assumption exclusively on the exogenous parameters of the model.

Proposition 2 (Transition dynamics with log utility after a productivity increase)**.** *Suppose agents have log utility and the productivity process $\{A_t\}_{t\geq 0}$ is weakly increasing over time with $A_0 \geq A^*$. Furthermore impose Assumptions 1, S1 and 4. Then the dynamics of aggregate capital is given in* (28)*, capital is weakly increasing and the equilibrium wage and interest rate processes jointly satisfy Assumptions 2 and 3.*

The proof is in the Appendix. An immediate and permanent increase in productivity from A^* to $\tilde{A}$ is a special case of Proposition 2 and further simplifies the closed-form transitional dynamics, as we show in Corollary 2 below. Note that in this case, Assumption S1 not only guarantees the existence of a unique stationary equilibrium with partial insurance from which the transition path starts, but also insures that the interest rate is monotonically decreasing along the transition path induced by the permanent productivity increase.

Corollary 2 (Transition dynamics after a permanent productivity increase)**.** *Suppose the agents have log utility and a permanent shock raises productivity from A^* to $\tilde{A}$. Further impose Assumptions S1 and 4.*

1. *Recall $\tilde{K}$ from equation* (37)*. For all $t \geq 0$, the aggregate capital stock is*

$$K_t = \left(\tilde{K}^{1-\theta} + \left((K^*)^{1-\theta} - \tilde{K}^{1-\theta}\right) e^{-(1-\theta)\hat{\delta}t}\right)^{\frac{1}{1-\theta}}. \tag{46}$$

2. *The aggregate capital stock and the wage are strictly increasing over time, and the equilibrium interest rate is strictly decreasing over time.*

3. *The equilibrium wage and interest rate processes jointly satisfy Assumption 3.*

The proof is in the Appendix. We display the essence of Corollary 2 graphically in Figure 3 for a specific set of parameters (see the caption of the figure for their values).[14] This serves to further clarify the intuition behind our results. Panel (a) shows that the

rate, which would prevail at $t = 0$, if $A_0 = \bar{A}$. Note that $\bar{r} + \delta = \left(\bar{A}/A^*\right)(r^* + \delta)$. Since $\dot{A}_t/A_t \geq 0$, condition (44) is implied by $0 = (1 - \hat{s})\left(\left(\bar{A}/A^*\right) - 1\right)(r^* + \delta) + r^* - \rho$ or (45).

[14] In Figure E.1 of Online Appendix E we consider a continuous increase in productivity and show that the dynamics of the aggregate variables are qualitatively similar.

interest rate jumps up on impact and then drifts down to its new (equal to the old) steady state level. Panels (b) and (d) display that the wage and aggregate consumption jump up on impact and then continue to increase to their new steady state levels. The aggregate capital stock in panel (c) increases monotonically over time towards the new and larger steady state level associated with a permanently higher productivity.

In panel (e), we depict the growth rate of wages and the (negative) interest-discount rate differential, the two key ingredients of Assumption 3. Recall that in order for agents to not want to save for the high-productivity state, the sum between the two terms has to be negative for all t. The figure shows that even though the interest rate rises on impact, this effect is not strong enough: $r_t - \rho$ remains negative throughout the transition, which, coupled with positive wage growth, ensures that savings incentives remain sufficiently low.

The figure also clarifies the role Assumption 4 plays: if the increase in productivity is too large, the interest rate jump on impact might be so large that, temporarily, agents might want to save for the high-productivity state as well. Assumption 4 insures precisely that this does not happen at any time during the transition.[15] An alternative way to see this is to plot all combinations of $\left(r_t, \frac{\dot{w}_t}{w_t} + \rho\right)$ attained along the transition. Panel (f) (with r on the x-axis) shows that all these combinations satisfy $r_t - \rho < \frac{\dot{w}_t}{w_t}$.

4.2.2 Decline in Productivity

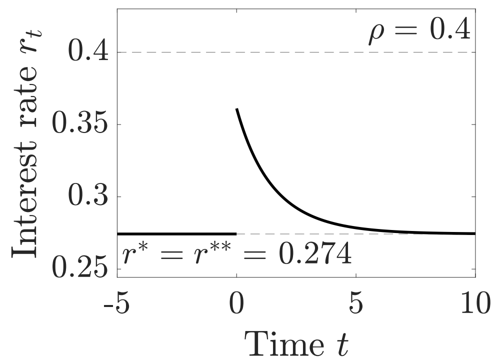
In this section, we consider a general productivity process $\{A_t\}_{t\geq 0}$ that is continuous and decreasing over time. Proposition 3 shows that the no-savings Assumption 3 or, equivalently, equation (44) is satisfied as long as the exogenous (and possibly time-varying) growth rate of productivity is not too negative, as made precise in Assumption 5 below.

Assumption 5 (Lower bound on the growth rate of productivity)**.** *The growth rate $\frac{\dot{A}_t}{A_t}$ of the productivity process $\{A_t\}_{t\geq 0}$ satisfies*

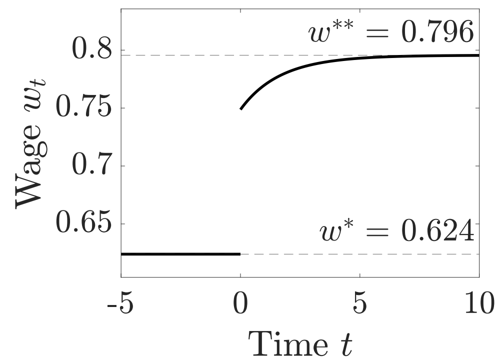
$$\frac{\dot{A}_t}{A_t} > r^* - \rho. \tag{47}$$

To see this, note that a decreasing path for A_t implies $r_t \leq r^*$ (see Lemma 4). Thus, the right hand side of (44) is bounded above by $r^* - \rho < 0$.

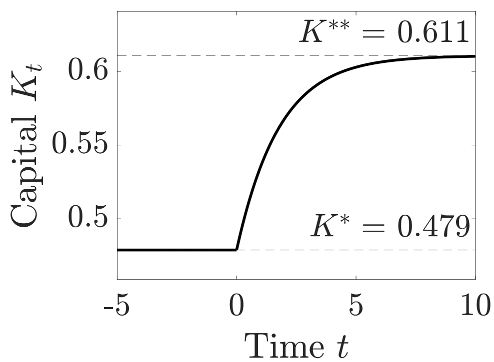
[15] In this example, the interest rate satisfies $r_t < \rho$ for all t along the transition path. We can also find productivity processes that satisfy the conditions in Corollary 2 for which the interest rate is not always below ρ along the transition. We provide an example in Figure E.2, which features a "large" permanent increase in productivity. However, Assumption 3 still holds for this example since $r_t < \frac{\dot{w}_t}{w_t} + \rho$ in equilibrium.



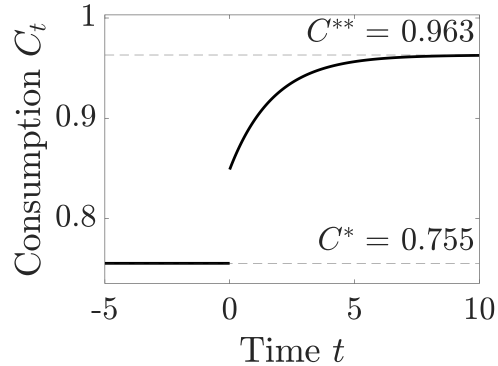
(a) Interest rate



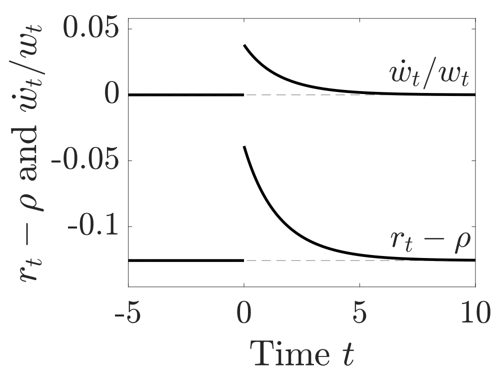
(b) Wage



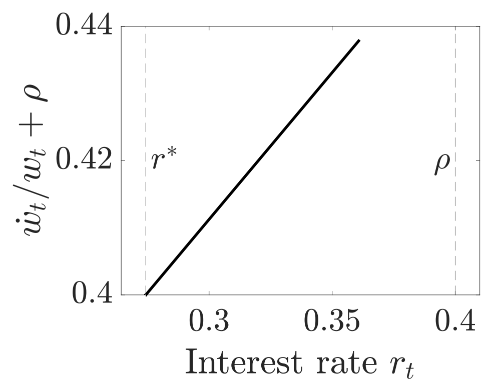
(c) Capital



(d) Consumption



(e) Comparing $r_t - \rho$ with $\frac{\dot{w}_t}{w_t}$



(f) $\frac{\dot{w}_t}{w_t} + \rho$ as a function of r_t

Figure 3: Transitional dynamics with a permanent increase in productivity. The figure plots the transition dynamics when productivity permanently increases from $A^* = 1$ to $\tilde{A} = 1.2$. Agents have log utility and $\delta = 0.16, \nu = 0.2, \rho = 0.4, \theta = 0.25, \xi = 0.2$.

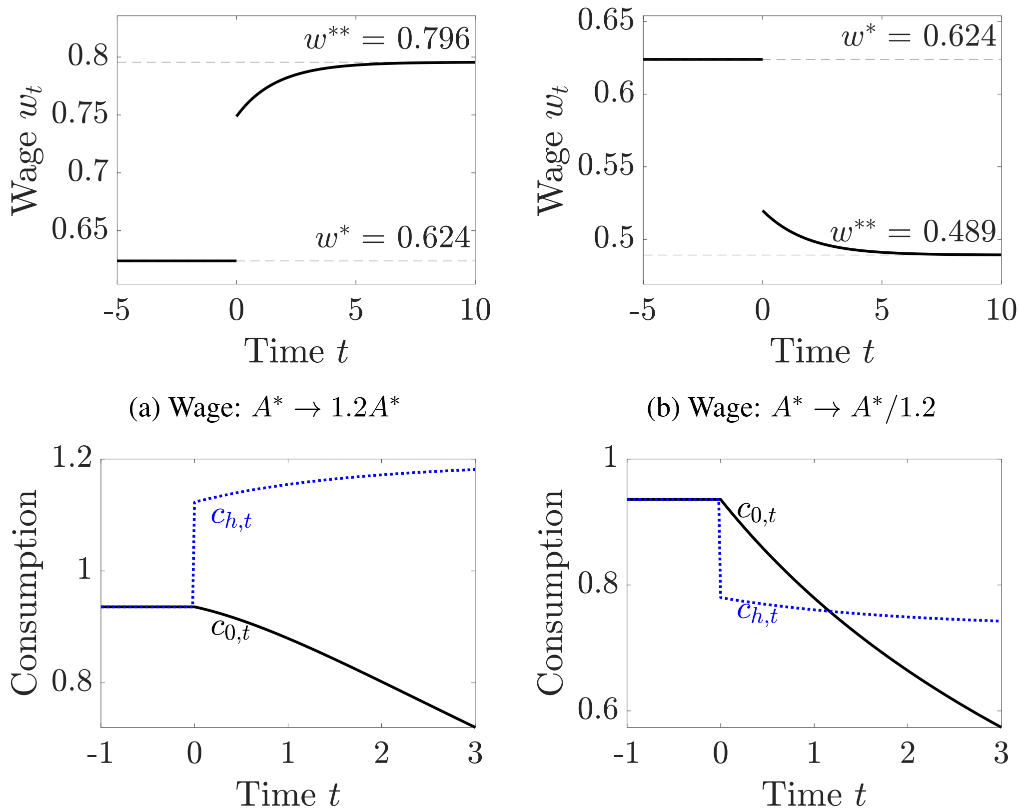
Proposition 3 (Transition dynamics after a continuous productivity decline)**.** *Impose Assumptions 1, S1, 5, $A_0 = A^*$ and $\dot{A}_t \leq 0$ for all $t > 0$. Then the dynamics of aggregate capital is given in* (28)*, and the equilibrium wage and interest rate processes jointly satisfy Assumptions 2 and 3.*

The proof is in the Appendix. Proposition 3 states that in order to discourage agents from accumulating capital for the high idiosyncratic productivity state, we need to bound the growth rate of the productivity decrease. This insures that wages do not fall too fast, otherwise even high-productivity agents would want to hold some capital to fund future consumption in light of low future wages. Note that this requirement contrasts with the case of a productivity increase for which we required a bound on the level of productivity in Assumption 4 to limit the increase in the interest rate.

To demonstrate why we need a bound on the speed of the productivity decline for our equilibrium characterization, we now continue the numerical example from the previous section, but now consider a symmetric permanent decline in productivity for which condition (47) in Assumption 5 is violated. Let productivity permanently decrease from $A^* = 1$ to $\tilde{A} = 1/1.2$, and suppose the proposed consumption insurance contract is optimal (which we will argue it is not). The implied interest rate and wage paths violate Assumption 3 since the wage jumps down at the time of the shock (see Figure 4 panel (b)), which implies that $\frac{\dot{w}_t}{w_t} = -\infty$ at $t = 0$ (and together with a finite r_0) leads to a violation of Assumption 3.

Figure 4 displays the path of wages (upper panels) and consumption of high-productivity agents ($c_{h,t}$) and of agents that have experienced an idiosyncratic productivity decline exactly at time 0, that is $c_{0,t}$, in the lower panels. The left panels correspond to the productivity increase from the previous subsection, and the right panels are for the corresponding permanent decline in aggregate productivity. In the left panels, wage growth is high, consumption of high-productivity agents jumps on impact, whereas consumption of low-productivity agents remains continuous. Of course, this indicates inefficient consumption insurance since $c_{h,t} > c_{0,t}$ at time $t = 0$. Ideally, consumption $c_{0,t}$ should also jump up after the positive aggregate productivity shock. However, this would require the low-productivity agents to borrow against future higher income, which is precisely what the limited commitment constraint prevents. Thus, the depicted consumption paths are indeed optimal as the permanent productivity increase satisfies Assumption 4.

The same is not true for a sudden decline in productivity (right panel) and associated drop in wages. In the associated conjectured consumption allocation, $c_{h,t}$ drops immediately in response to the collapse in wages and remains below $c_{0,t}$ for a sustained period



(a) Wage: $A^* \to 1.2A^*$

(b) Wage: $A^* \to A^*/1.2$

(c) Household consumption: $A^* \to 1.2A^*$

(d) Household consumption: $A^* \to A^*/1.2$

Figure 4: Difference between permanent increase and decrease in productivity. We plot the dynamics of wages and individual agents' consumption, assuming that agents behave according to the allocation from Lemmas 1 and 2. Panels (a) and (c) plot the dynamics when productivity permanently increases from $A^* = 1$ to $\tilde{A} = 1.2$. Panels (b) and (d) do the same when productivity permanently decreases from $A^* = 1$ to $\tilde{A} = 1/1.2$. In panels (c) and (d), the solid black line plots consumption of a low-productivity agent who just transitioned from high productivity at time t, while the dotted blue line denotes the consumption of high-productivity agent at time t. Agents have log utility and $\delta = 0.16, \nu = 0.2, \rho = 0.4, \theta = 0.25, \xi = 0.2$.

of time, see panel (d) of Figure 4. That is, if the low-productivity agent has a reversal towards high productivity at some $t > 0$ (when $c_{h,t} < c_{0,t}$ is still the case), she would see consumption decline on impact of this event, again an instance of inefficient consumption insurance. In this case, however, this *can* be avoided, by the low-productivity agent *saving* for the high-productivity reversal. That is, the conjectured consumption allocation is sub-optimal precisely because the no-savings condition of Assumption 3 is violated.

Although it is clear that a discontinuous fall in productivity will always violate Assumption 3, a number of continuously declining productivity processes satisfy the assumption. In the next corollary, we provide such an example.

Corollary 3. *Consider the following log-linearly decreasing productivity path* $\{A_t\}_{t\geq 0}$

$$\log A_t = \log A^* + \frac{t + (T-t)\cdot \mathbf{1}\{t > T\}}{T}\left(\log \tilde{A} - \log A^*\right) \tag{48}$$

with $A_0 = A^* > \tilde{A}$ *and* $T > 0$. *Impose Assumption S1. If the parameters* $\tilde{A}$ *and* T *satisfy*[16]

$$\frac{\log A^* - \log \tilde{A}}{T} < \rho - r^*, \tag{49}$$

then the dynamics of aggregate capital is given in (28)*, the equilibrium wage and interest rate processes jointly satisfy Assumptions 2 and 3.*

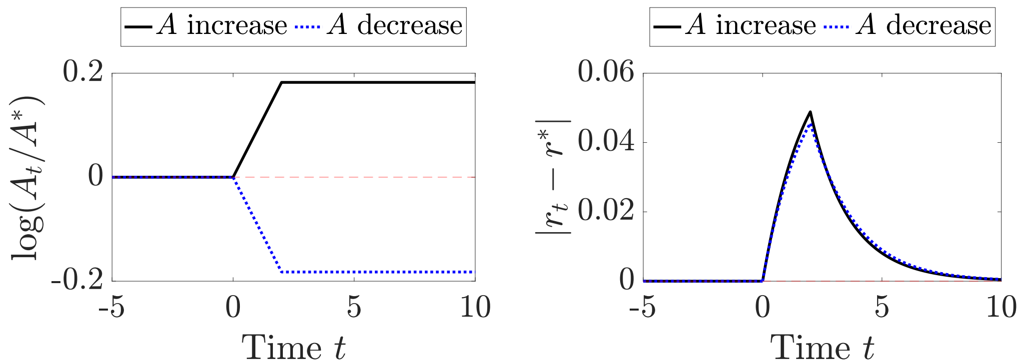
This follows directly from Proposition 3. That is, if the eventual decline from A_0 to $\tilde{A}$ is not too large and the decay rate parameterized by T is not too fast, then Assumption 5 and thus Assumption 3 are satisfied for this continuously declining productivity process.

4.3 Symmetric Continuous Productivity Increase and Decrease

In the last section, we show that the sufficient conditions needed to guarantee that household allocations are "simple" are asymmetric between productivity increases and productivity declines. We now argue that *conditional* on these conditions being satisfied, and conditional on the productivity increase and decrease being symmetric, equilibrium allocations and prices are nearly symmetric as well.

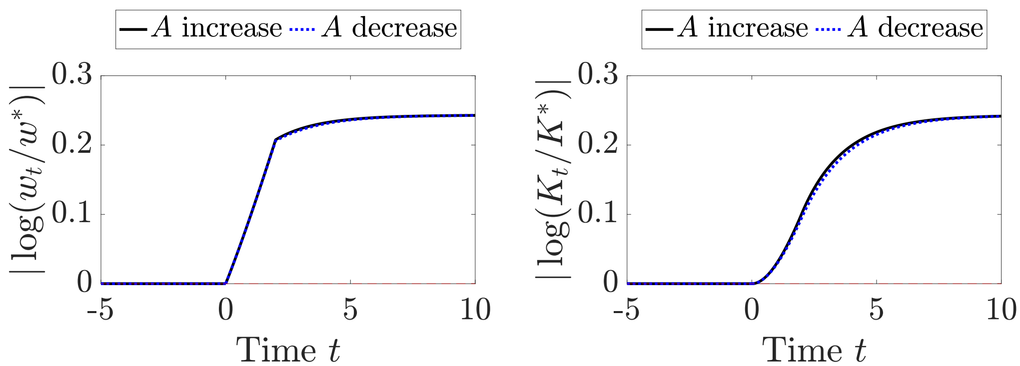
To do so, in Figure 5, we compare the transitional dynamics of the aggregate variables under the log-linear productivity process from Corollary 3 with parameters $T = 2$ and

[16] The right-hand side of equation (49) is the negative of the right-hand side of equation (47).



(a) Log productivity

(b) Interest rate (absolute deviation)



(c) Log wage (absolute deviation)

(d) Log capital (absolute deviation)

Figure 5: This figure compares the dynamics of aggregate variables with a continuous increase and decrease in productivity, as described in the main text. The solid black line plots the case of the productivity increase and the dotted blue line displays the productivity decrease. All variables in log deviation from their initial steady state values. Agents have log utility and $\delta = 0.16, \nu = 0.2, \rho = 0.4, \theta = 0.25, \xi = 0.2$.

$A^* = 1$, and where the final steady state productivity is either $\tilde{A} = 1/1.2$ or $\tilde{A} = 1.2$. That is, log productivity linearly increases (decreases) from $\log(A^*)$ to the new steady state value $\log \tilde{A}$ in T periods. Both paths satisfy the respective sufficient conditions from the previous sections so that in each case, Assumption 3 is satisfied. In each panel of the figure, the solid black line plots the case when productivity increases and the dotted blue line plots the case when productivity decreases.

Panel (a) plots the two productivity paths (in log deviation from the initial steady state values) and shows that, by construction, the increase and decline are symmetric about the old steady state. The remaining panels (b)-(d) display paths of the interest rate, the wage and the aggregate capital stock. The figure shows that all macroeconomic aggregates (as well as aggregate output and consumption, since these follow directly from the dynamics of the aggregate capital stock) respond symmetrically to a symmetric productivity change. In fact, as one can see from equation (31) characterizing the capital stock in closed form, $K_t^{1-\theta}$ responds to productivity movements (the terms in the last integral of equation (31)) fully symmetrically. Since wages equal $w_t = A_t K_t^{1-\theta}$, they evolve symmetrically, too, see panel (c). For the interest rate and capital, the transition path is not exactly symmetric (since these variables depend on K_t^{θ} or $K_t^{\theta-1}$), but as the figure shows, the deviations from complete symmetry are quantitatively very small.

We conclude that the main asymmetry induced by limited commitment is one of the conditions required so that high-productivity agents do not want to save. The economy then responds symmetrically to positive and negative aggregate productivity shocks.

4.4 Speed of Convergence

So far we have characterized the transition path induced by a shock to productivity. This begs the question of how rapid the convergence to the new steady state is, given the exogenous shock. We can think of the initial steady state as describing an originally poor country that, all of a sudden, obtains access to frontier production technologies. The speed of convergence question then asks how quickly such a country will catch up with frontier economies if it is described by our model. In this section, we will answer this question and compare our model to the standard neoclassical growth model along this dimension.[17]

Barro and Sala-i-Martin (2004) formally define the speed of convergence β_t as

$$\beta_t \equiv -\frac{\partial\left(\dot{K}_t/K_t\right)}{\partial \log\left(K_t\right)}. \tag{50}$$

It measures how much the growth rate of capital declines (increases) as the capital stock increases (declines) towards its new steady state. One expects the partial derivative to be negative and the minus sign in the definition turns the speed of convergence positive. The following proposition characterizes the speed of convergence in our model.

Proposition 4 (Speed of convergence)**.** *Suppose Assumptions 1, 2 and 3 hold. Then the speed of convergence in our model is characterized as follows.*

[17] See Barro and Sala-i-Martin (2004) for a summary of the empirical evidence on this issue, and King and Rebelo (1993) for a quantitative assessment of the standard neoclassical growth model for this question.

1. *Along the transition,*

$$\beta_t = \frac{(1-\theta)}{\theta} \hat{s} \left(r_t + \delta \right). \tag{51}$$

2. *In the long run,*

$$\beta_t \to \beta = (1-\theta)\, \hat{\delta}. \tag{52}$$

Thus the speed of convergence in the long run is independent of the path for productivity A_t or the risk of losing productivity ξ.[18] Note that this proposition holds rather generally. In particular, its conditions are satisfied under the conditions of Proposition 2 or Proposition 3.

The proof is in the Appendix. We now study the speed of convergence numerically, both to show Proposition 4 in action as well as to contrast it to the speed of convergence in the standard neoclassical growth model.[19] Since the neoclassical growth model has no closed-form solution, it is a priori unclear how its speed of convergence compares to our model. Proposition 5 below provides an answer, using a log-linearization of the neoclassical growth model around the steady state. No such approximation is needed for our model since its speed of convergence around the steady state can be given without it, as shown in the previous proposition.

Proposition 5 (Comparison of speed of convergence)**.** *Suppose a permanent shock raises productivity from A^* to $\tilde{A}$. Furthermore impose Assumptions 1, S1 and 4. Around the new steady state, our model exhibits a slower speed of convergence than the neoclassical growth model if and only if*

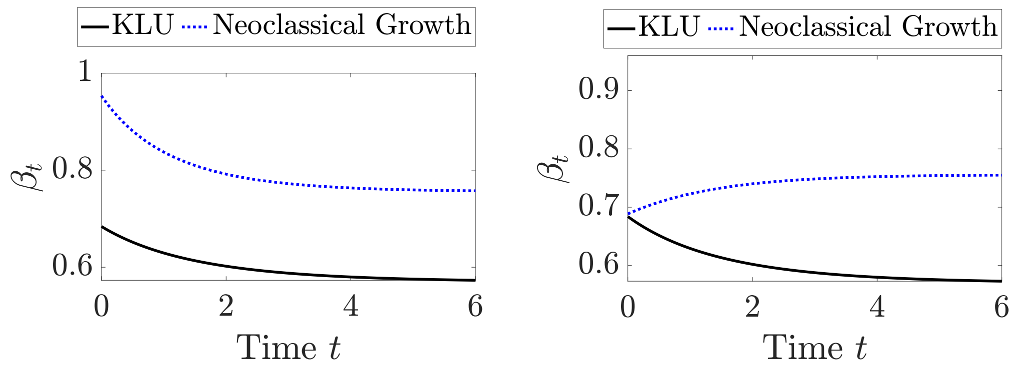
$$\theta \left(1 + \frac{\nu}{\rho + \delta} \right) \left(1 + \frac{\rho + \nu}{\frac{\rho}{1-\theta} + \delta} \right) < 1. \tag{53}$$

The proof involves some tedious calculations and is in Online Appendix C.2. Figure 6 plots the speed of convergence in both models under the baseline parameters of Figure 3.

These parameter values satisfy equation (53) and thus in the long run our model displays slower convergence to the new steady state than the neoclassical growth model. Panel (a) and panel (b) differ in the initial levels of capital in the two models. In panel (a), each model starts from its own steady state capital stock which is lower in the neoclassical growth model since in that model the steady state interest rate is $r = \rho$, whereas our model

[18]The expression in equation (52) is also the speed of convergence along the transition if one log-linearizes the model around the new steady state.

[19]Online Appendix C.1.2 includes the computational details for our solution of that model.



(a) Starting from steady state (b) Starting from same capital

Figure 6: This figure compares the speed of convergence in our model with that in the representative agent neoclassical growth model, following a permanent increase in productivity. In each panel, the solid black line represents our model and the dotted blue line is neoclassical growth model. Productivity permanently increases from $A^* = 1$ to $\tilde{A} = 1.2$. Agents have log utility and $\delta = 0.16, \nu = 0.2, \rho = 0.4, \theta = 0.25, \xi = 0.2$.

features partial insurance and thus $r < \rho$. In panel (b), we instead assume that both models start from the *same* capital stock, equal to the initial steady state capital stock in our model.

The figure shows that our model displays slower convergence than the neoclassical growth model, both in the short run and the long run. As King and Rebelo's (1993) classic paper on this issue shows, the speed of convergence in the neoclassical growth model is fast initially if the capital stock is far below its new long-run steady state level (as is the case when there is a large permanent increase in productivity) since this means temporarily high returns to capital, and temporarily high investment rates (and thus high endogenous savings rates of the representative agent). In our model, akin to the classical Solow model, the saving rate of those making positive saving decisions is constant (but endogenous). Thus convergence is slower in the short run, and, with the inequality in Proposition 5 also in the long run. As King and Rebelo (1993) and Barro and Sala-i-Martin (2004) argue, the neoclassical growth model implies unreasonably fast convergence for commonly used calibrations. Thus our model can potentially alleviate this fast convergence "puzzle." Of course, this requires that the parameters satisfy the condition of Proposition 5.

5 Consumption Inequality along the Transition

In this section, we study the evolution of consumption inequality along the transition induced by a (permanent or transitory) productivity shock. This analysis shows how the distribution of consumption changes over the business cycle, according to our model, assuming that the unexpected MIT productivity shock can be treated as a good approximation of aggregate fluctuations. Our model is tractable enough to do this analytically, to a large degree, and uses numerical analysis only to illustrate the theoretical results.

5.1 Theory: Inequality in Steady State and along Transition

At any point in time t, we can theoretically characterize the (percentage) consumption gap between an agent with currently high productivity $c_{h,t}$ and an agent that had high productivity last $\tau \geq 0$ periods ago, $c_{t-\tau,t}$, as

$$\log\left(\frac{c_{h,t}}{c_{t-\tau,t}}\right). \tag{54}$$

From the characterization in equation (20), $c_{h,t}$ is proportional to the current wage

$$c_{h,t} = \alpha\zeta w_t = \frac{\nu+\rho}{\nu+\rho+\xi}\zeta w_t.$$

Hence, high-productivity agents' consumption normalized by the wage, remains constant over time, and absolute consumption of this group changes proportionally with the aggregate wage rate w_t in the economy. Since the share of high-productivity agents is $\Psi_h = \frac{\nu}{\nu+\xi}$ and the distribution of waiting times τ for low-productivity agents is $\psi_l(\tau) = \xi\nu e^{-\nu\tau}/(\xi+\nu)$ (see equation (7)), a characterization of the consumption gap $\log(c_{h,t}/c_{t-\tau,t})$ for all $\tau \geq 0$ fully characterizes the consumption distribution at each time t. Since $c_{t-\tau,t}$ is strictly decreasing in τ, Online Appendix D demonstrates the (obvious) fact that the index τ maps into a specific quantile of the consumption distribution.

Proposition 6 below decomposes the gap $\log(c_{h,t}/c_{t-\tau,t})$ into a wage component reflecting the fact that wages might have been different when agent τ had high productivity, relative to today, and a discounting component capturing the fact that this agent's consumption has drifted down between period $t-\tau$ and period t at rate $\rho - r_u$ with $u \in [t-\tau, t]$.

Define the average interest rate $r^a_{t,\tau}$ over this interval as

$$r^a_{t,\tau} = \frac{1}{\tau} \int_{t-\tau}^{t} r_u du. \tag{55}$$

Proposition 6 (Consumption inequality and decomposition)**.** *Suppose Assumptions S1 and 4 hold. Then at any time t, the consumption gap can be expressed as the sum of a "wage gap" and a "discounting gap:"*

$$\underbrace{\log\left(\frac{c_{h,t}}{c_{t-\tau,t}}\right)}_{\text{consumption gap}} = \underbrace{\log\left(\frac{w_t}{w_{t-\tau}}\right)}_{\text{wage gap}} + \underbrace{\tau\left(\rho - r^a_{t,\tau}\right)}_{\text{discounting gap}} > 0. \tag{56}$$

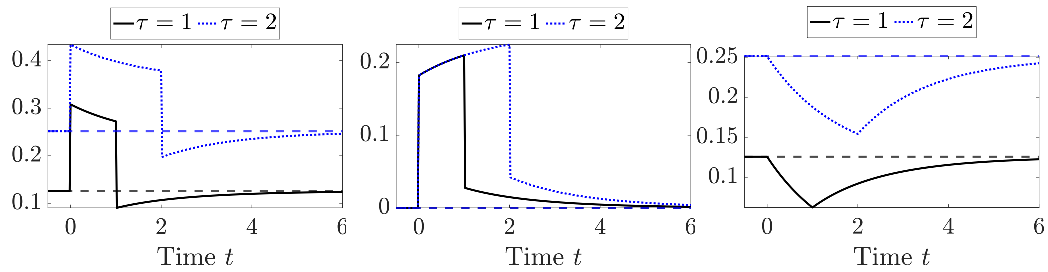
The proof is straightforward and can be found in Online Appendix B.2 for completeness. Equations (36) and (38) imply that $r^a_{t,\tau} = r^* + (\theta/(\tau\hat{s}))\log(K_t/K_{t-\tau})$. Equation (34) implies that $\log(w_t/w_{t-\tau}) = \log(A_t/A_{t-\tau}) + \theta\log(K_t/K_{t-\tau})$. Therefore, one can calculate the consumption gap explicitly as

$$\log\left(\frac{c_{h,t}}{c_{t-\tau,t}}\right) = \log\left(\frac{A_t}{A_{t-\tau}}\right) - \frac{1-\hat{s}}{\hat{s}}\theta\log\left(\frac{K_t}{K_{t-\tau}}\right) + \tau\left(\rho - r^*\right). \tag{57}$$

We can use the decomposition (56) and the consumption gap formula (57) to characterize the evolution of consumption inequality along the transition following a permanent upward jump in productivity. In Section 5.2, we consider one-time permanent productivity changes and in Section 5.3, we discuss gradual shifts in productivity.

5.2 Permanent Productivity Shocks

Suppose a permanent shock raises productivity from A^* to $\tilde{A}$ at $t = 0$ and suppose the conditions of Corollary 2 hold. Then, for a given agent characterized by $\tau \geq 0$, the wage gap and the discounting gap, and thus the overall consumption gap have the following properties (see Figure 7 for a numerical illustration). At $t = 0$, the wage gap and consumption gap jump up, since $A_t/A_{t-\tau}$ jumps from 1 to $\tilde{A}/A^*$ in (57). There is no change in the discounting gap. For $t \in (0, \tau)$, the wage gap continuously widens while the discounting gap continuously shrinks. This is a consequence of wages jumping up at $t = 0$ and further increasing along the transition, whereas interest rates also jump up at $t = 0$ but then decline towards the new (equal to the old) stationary equilibrium r^* over time. The latter effect



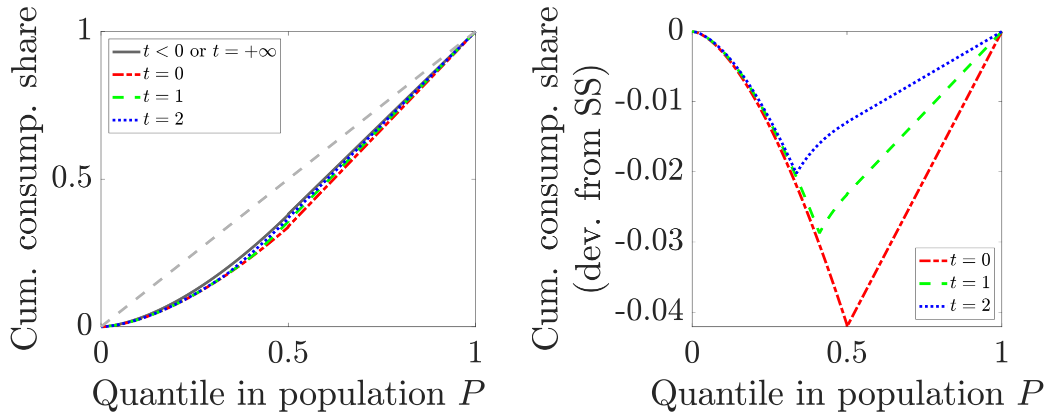
(a) Consumption gap (b) Wage gap (c) Discounting gap

Figure 7: Transition dynamics of consumption inequality with a permanent increase in productivity from $A^* = 1$ to $\tilde{A} = 1.2$. Panels (a)-(c) plot the consumption gap, wage gap, and discounting gap defined in Proposition 6. See Figure 3 for parameter values.

dominates, so the consumption gap shrinks continuously, as one can see from (57) and increasing capital K_t. At $t = \tau$, $A_t/A_{t-\tau}$ jumps from $\tilde{A}/A^*$ to 1 in (57). Thus, the wage gap declines discontinuously and the discounting gap shrinks continuously, so the consumption gap falls discontinuously at $t = \tau$. For $t > \tau$, the wage gap shrinks further and the discounting gap widens continuously. The latter effect dominates, so the consumption gap widens continuously. When the economy has converged to the new stationary equilibrium, the wage gap is zero, the discounting gap and thus the consumption gap revert back to their original levels in the old stationary equilibrium.

We now illustrate the consequences for the overall consumption distribution, as summarized by the Lorenz curve, using the numerical example from Section 4.2.1. Figure 8 displays the Lorenz curve at various points in time, at $t < 0$ (initial stationary equilibrium) as well as three points of time along the transition (including $t = 0$, the instant after the surprise MIT shock has occurred). Note that the Lorenz curve in the final steady state following a permanent shock to productivity is identical to that in the initial steady state. Panel (a) shows the Lorenz curves and panel (b) presents them in deviation from the the initial (and final) stationary equilibrium Lorenz curve. For example, a value of -0.04 at the 50th quantile in panel (b) for period $t = 0$ means that on impact, the consumption share of the bottom half of the population falls by 4 percentage points.

We observe that in response to a positive permanent productivity shock, consumption inequality increases before converging over time to the initial distribution. We will show below this is the consequence of the consumption of high-productivity agents immediately jumping up with the higher wage implied by higher aggregate productivity, whereas the low-productivity agents that finance their consumption from their (contingent) wealth



(a) Cumulative consumption share

(b) Cumulative consumption share (deviation from steady state)

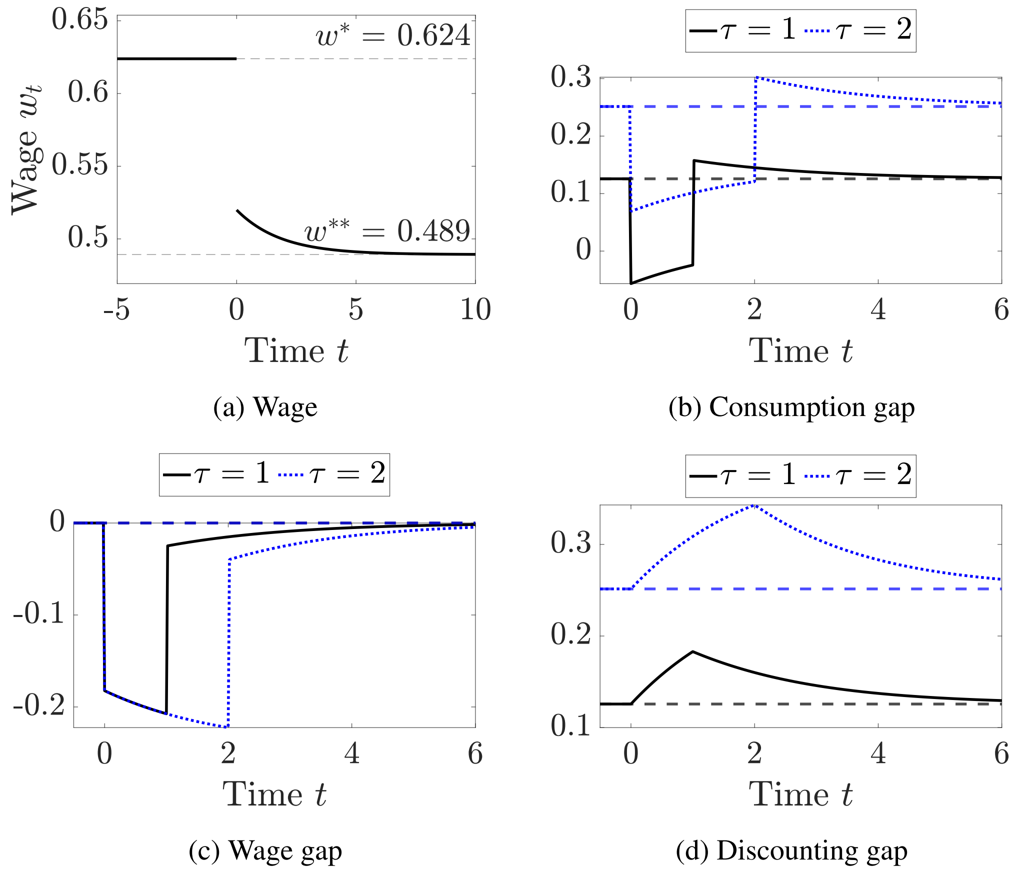
Figure 8: Evolution of the Lorenz curve with a permanent productivity increase from $A^* = 1$ to $\tilde{A} = 1.2$. In each panel, the x-axis corresponds to the quantile in the population and each line corresponds to a different time t. Panel (a) plots the cumulative consumption share (defined in Online Appendix D.3) on the y-axis and panel (b) plots the deviation of the cumulative consumption share from the steady state. See Figure 3 for parameter values.

holdings initially fall behind as their consumption is continuous in time (but now falls at a slower rate over time as the interest rate increases with aggregate TFP). Over time, the consumption of capital owners catches up to that of wage earners and the Lorenz curve slowly converges back to the initial steady state curve.

We now return to the question of why a sudden decline in productivity leads to a violation of the no-savings condition. In Section 4.2.2, we argued that the proposed consumption insurance contract is not optimal when A_t falls discretely. In this case, the wage would immediately drop under the proposed allocation, inducing the high-productivity agent to accumulate capital and thus violating Assumption 3.

We can also see this from the dynamics of the consumption gap. Figure 9 panel (b) plots the hypothetical transition dynamics of the consumption gap assuming that individual agents consume according to the consumption allocation in Lemmas 1 and 2. The solid black line corresponds to an agent that had low productivity for one time unit ($\tau = 1$). The consumption gap for this agent turns negative upon the permanent decline in productivity, implying the agent would consume more than the high-productivity agent (which is suboptimal consumption smoothing). This result is driven by the discrete drop in wage (panel (a) of Figure 9) and the widening of the wage gap (panel (c) of Figure 9).

The discrete drop in the consumption gap also occurs for a low-productivity agent who



(a) Wage (b) Consumption gap

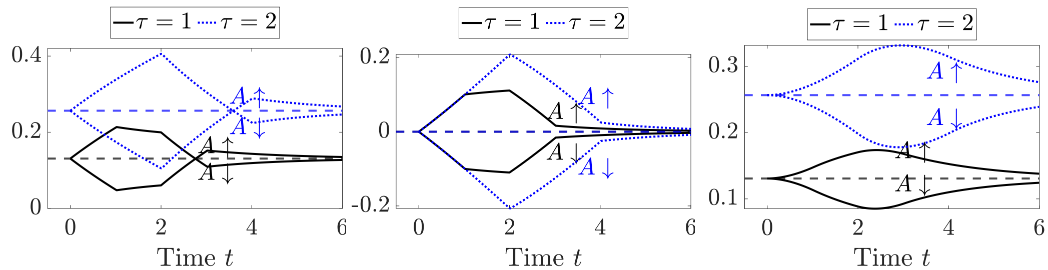
(c) Wage gap (d) Discounting gap

Figure 9: Transition dynamics of consumption inequality with a permanent productivity decrease, assuming the proposed contract is optimal. We plot the transition dynamics of wages and consumption inequality when productivity permanently decreases from $A^* = 1$ to $\tilde{A} = 1/1.2$, assuming that agents consume according to the optimal allocation in Lemmas 1 and 2. Panel (a) plots the wage and panels (b)-(d) plot the consumption gap, wage gap, and discounting gap in Proposition 6. See Figure 3 for parameter values.

just transitioned from high to low productivity at time $t = 0$. This agent would experience a discrete drop in consumption under the proposed consumption contract. This is not optimal as the agent would rather save more when he last had high productivity and avoid the discrete drop in consumption once he switches to high productivity in the future.

5.3 A Continuous Change in Productivity

Finally, we study the dynamics of consumption inequality after a continuous increase and decline in productivity that satisfies the no-savings condition. Consider the transition dynamics in the numerical example in Section 4.3. Recall from that section that the new steady state values of the aggregate variables are "symmetric" in the two cases. In this section, we examine the "symmetry" of the consumption gap and its components.



(a) Consumption gap (b) Wage gap (c) Discounting gap

Figure 10: Consumption inequality along transition with continuous productivity increase and decrease. Productivity either increases from $A^* = 1$ to $\tilde{A} = 1.2$ following the continuous path in equation (48) with parameter $T = 2$ or it decreases from $A^* = 1$ to $\tilde{A} = 1/1.2$ following the path in equation (48) with parameter $T = 2$. Panels (a)-(c) plot the consumption gap, wage gap, and discounting gap defined in Proposition 6. Agents have log utility. See Section 4.3 for parameter values.

Figure 10 plots the transition dynamics of the consumption gap (panel (a)) and its components (panels (b) and (c)) after a continuous change in productivity. In each panel, the solid black line corresponds to a low-productivity agent who last had high productivity for $\tau = 1$ period ago, while the dotted blue line corresponds to a low-productivity agent who last had high productivity for $\tau = 2$ periods ago.

Panel (a) shows that the consumption gap is monotonic in τ for both the increase and decrease in productivity, i.e., the gap is the larger the bigger is τ in both cases. However, it is noteworthy that the ordering of the wage gap is not symmetric while the ordering of the discounting gap is symmetric. This is most transparent by observing that for a positive productivity shock, the wage gap is monotonic in τ, while for a productivity decline this is no longer true (see panel (b) of Figure 10). Since the two gaps go in opposite directions for a decline in productivity and since they are of different magnitudes, the combined effect on the consumption gap remains symmetric, in line with the symmetry result for aggregate variables (conditional on the no-savings condition being satisfied) in Section 4.3.

6 Conclusion

In this paper, we have analytically characterized the transition dynamics in a neoclassical production economy with idiosyncratic income shocks and long-term one-sided limited commitment contracts. When income can only take two values one of which is zero (i.e., unemployment) and the utility function is logarithmic, the transition path induced by an unexpected productivity shock can be given in closed form, both for the macroeconomic variables as well as the non-degenerate consumption distribution which displays partial consumption insurance against the idiosyncratic income shocks.

Given these findings, we would identify two immediately relevant next questions. First, on account of our use of continuous time, the endogenous optimal contract length is analytically tractable even outside the special case we have focused on thus far, and it will be important to generalize our findings to the more general case. Second, thus far we have focused on an environment that has idiosyncratic but no aggregate shocks, rendering the macroeconomic dynamics deterministic. Given our sharp analytical characterization of the equilibrium in the absence of aggregate shocks, we conjecture that the economy with aggregate shocks might be at least partially analytically tractable as well. We view these questions as important topics for future research.[20]

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APPENDIX

Proof of Lemma 1. Let $\mu \geq 0$ denote the Lagrange multiplier (LM) on the budget constraint (5), λ the LM on the borrowing constraint (6), and $\omega \geq 0$ the LM on the constraint $\tilde{k} \geq 0$. Then the Lagrangian for the maximization problem in Definition 1 is

$$\begin{aligned}\mathcal{L} &= u(c) + \dot{U}_t(k;z) + U_t'(k;z)\,x + p_z\left(U_t\left(\tilde{k};\tilde{z}\right) - U_t(k;z)\right) \\ &\quad -\mu\left(c + x + p_z\left(\tilde{k} - k\right) - r_t k - w_t z\right) + \lambda x + \omega\tilde{k}.\end{aligned} \tag{A.1}$$

The FOCs wrt to c, k and $\tilde{k}$ are

$$u'(c) = \mu, \;\; U_t'(k;z) = \mu - \lambda \text{ and } U_t'\left(\tilde{k};\tilde{z}\right) = \mu - \frac{\omega}{p_z}. \tag{A.2}$$

Consider an agent with $k > 0$. In this case, condition (6) does not apply and $\lambda = 0$. Then the FOCs imply $u'(c) = U_t'(k;z)$. When productivity stays unchanged for an interval of time, we differentiate both sides wrt time t and use $\dot{k}_t = x$ to obtain

$$u''(c)\,\dot{c} = \dot{U}_t'(k;z) + U_t''(k;z)\,x \qquad \text{where} \tag{A.3}$$

$$\dot{c}_t(k;z) \equiv \frac{\partial c_t(k;z)}{\partial t} + \frac{\partial c_t(k;z)}{\partial k} x_t(k;z), \; U''_t(k;z) \equiv \frac{\partial^2 U_t(k;z)}{\partial k^2}, \; \dot{U}'_t(k;z) \equiv \frac{\partial^2 U_t(k;z)}{\partial k \partial t}.$$

Differentiating the objective in equation (4) wrt the state k delivers the envelope condition

$$\rho U'_t(k;z) = \dot{U}'_t(k;z) + U''_t(k;z)\, x - p_z U'_t(k;z) + \mu\,(p_z + r_t)\,. \tag{A.4}$$

Using the first order conditions at $k > 0$ and (A.3), we have $\rho u'(c) = u''(c)\,\dot{c} + u'(c)\, r_t$. With $u(c) = \log(c)$, the optimal consumption therefore follows equation (15), when $k > 0$.

When $z = 0$ and $\tilde{k}_t(k;0) = 0$ for all $k \leq \bar{k}$ and some $\bar{k}$, the consumption dynamics (15) and the budget constraint (5) can be rewritten as the linear system of differential equations[21] $\dot{c}_t = (r_t - \rho)\, c_t$ and $\dot{k}_t = (r_t + \nu)\, k_t - c_t$ in the unknown functions c_t and k_t with the boundary condition[22] $\lim_{t \to +\infty} k_t = 0$, provided $k_t \leq \bar{k}$ for all t. Such a system of linear ODEs has a unique solution. With $x_t(k;0) = \dot{k}_t$, it is easy to verify that the solution is (16) and (17). The solution is valid, as long as the implied path for k_s for $s \geq t$ does not cross the upper bound $\bar{k}$, since (17) is $\dot{k}_t = (r_t - \rho)\, k$ and since $r_t < \rho$ for $t \geq T$ per assumption 2. This will be true for all $k_t \in \left(0, \bar{\bar{k}}\right)$ and some suitable $\bar{\bar{k}}$. □

Proof of Lemma 2. The lemma is a version of Section 3.3 in the Online Appendix of Krueger and Uhlig (2022), generalized to the case, where aggregate wages and interest rates are functions of time. Rather than replicating the steps, here we provide the logic of the argument and point to the results and proofs in Krueger and Uhlig (2022) for the details.

For a high-z agent, the Lagrangian, first-order and envelope conditions are as in the proof for Lemma 1 (see (A.1), (A.2) and (A.4)) but applied to $k = 0$, $z = \zeta$, and $p_z = \xi$.

1. We first consider the choice of $\tilde{k}$. The solution in Lemma 1 implies that

$$U'_t\left(\tilde{k};0\right) = u'\left((\rho+\nu)\,\tilde{k}\right) = \frac{1}{(\rho+\nu)\tilde{k}},$$

which increases to infinity, as $\tilde{k} \to 0$. The third first-order condition in (A.2) therefore implies that $\tilde{k} > 0$ and thus $\omega = 0$. With the first and third first-order conditions in (A.2), we obtain consumption smoothing $u'\left((\rho+\nu)\,\tilde{k}\right) = u'(c)$ and thus

[21] Note that equation (15) implies $c_s = e^{\int_t^s (r_u - \rho) du} c_t$. A less formal, but more meaningful argument in terms of economic theory is thus to recognize that the budget constraint and utility maximization implies that the current capital k is equal to the net present value of all future consumption, as long as the productivity state stays unchanged. This yields $k = \int_t^{+\infty} e^{-\int_t^s (r_u+\nu)du} c_s ds = c_t / (\rho+\nu)$.

[22] The boundary condition ensures that no capital gets wasted and this follows from utility maximization.

$(\rho + \nu)\, \tilde{k} = c$. Therefore, equation (20) follows from the budget constraint (5), provided that $x = 0$.

2. Then we need to show that $x > 0$ is not optimal. Suppose otherwise, $x > 0$ were optimal, then $(\rho + \nu)\, \tilde{k} = c$ together with the budget constraint (5) implies that $c_t < \alpha\zeta w_t$, i.e. consumption is less than the right-hand side of equation (20). Furthermore, constraint (6) would not be binding, $\lambda = 0$, and consumption growth would satisfy equation (15). Let $[t, t + \Delta]$ be a time interval for some $\Delta > 0$, during which this is the case and along a path where no productivity switch occurs. Assumption 3 then implies $c_s \leq \alpha\zeta w_s$ during the interval $s \in [t, t + \Delta]$, i.e. consumption is less than the consumption level proposed in Lemma 2 for that episode. The integral of utility during that time interval is then smaller than the utility of the solution proposed in Lemma 2. This loss in utility can only be justified by the additional utility gained from consuming the accumulated capital after a switch to lower productivity for $s > 0$, or, alternatively, for $s > \Delta$ in case there is no switch to lower productivity. This amounts to postponing consumption compared to the solution proposed in Lemma 2. But this contradicts the impatience of the agent relative to wage growth, as expressed in Assumption 3. A precise formulation of that contradiction requires replicating the arguments in Section 3.3 of Krueger and Uhlig (2022), allowing for the additional time evolution of r_t and w_t.

For a low-productivity agent, given his consumption dynamics in equation (15) and the consumption of a high-productivity agent in equation (20), we have

$$c_{s,t} = e^{\int_s^t (r_u - \rho)du} c_{s,s} = c_{h,s} e^{-\int_s^t (\rho - r_u)du}.$$

Hence, the consumption of a low-productivity agent is given equation (21). □

Proof of Lemma 3. We differentiate both sides of equation (27) wrt time t,

$$\dot{K}_t = k_{t,t}\psi_l\,(0) + \int_{-\infty}^{t} \left(\dot{k}_{s,t}\psi_l\,(t-s) + k_{s,t}\psi_l'\,(t-s)\right) ds. \tag{A.5}$$

Equation (7) implies $\psi_l'\,(t-s) = -\nu\psi_l\,(t-s)$. Rewrite equation (17) with $\dot{k}_{s,t} = x_t$ as $\dot{k}_{s,t} = (r_t - \rho)\, k_{s,t}$. Equations (3), (7), and (22) yield

$$k_{t,t}\psi_l\,(0) = \frac{1-\alpha}{\xi}\zeta w_t \frac{\xi\nu}{\xi + \nu} = (1-\alpha)\, w_t. \tag{A.6}$$

Substituting into equation (A.5), we get

$$\begin{aligned}\dot{K}_t &= k_{t,t}\psi_l(0) + \int_{-\infty}^{t} \left((r_t-\rho)k_{s,t}\psi_l(t-s) - \nu k_{s,t}\psi_l(t-s)\right)ds \\ &= (1-\alpha)w_t + (r_t-\rho-\nu)K_t \\ &= (1-\alpha)(1-\theta)A_tK_t^{\theta} + \left(\theta A_tK_t^{\theta-1} - \delta - \rho - \nu\right)K_t = \hat{s}A_tK_t^{\theta} - \hat{\delta}K_t,\end{aligned}$$

where the third line above uses the interest rate and wage in equations (10) and (11), and the last line above uses the definition of $\hat{s}$ and $\hat{\delta}$ in equation (29). □

Recall that r^* is the steady state interest rate in equation (38) that K^* is the steady state level of capital, when productivity $A_t \equiv A^*$ for $t < 0$.

Lemma 4. *Consider any productivity process $\{A_t\}_{t\geq 0}$ such that in equilibrium, the aggregate capital evolves according to equation (28).*

- *If $A_t > A^*$ and is weakly increasing for all $t > 0$, then $r_t > r^*$ and $\dot{K}_t > 0$.*
- *If $A_t < A^*$ and is weakly decreasing for all $t > 0$, then $r_t < r^*$ and $\dot{K}_t < 0$.*

Proof. It suffices to show this for the first case of an increasing A_t, since the proof for the second case of a decreasing A_t is entirely symmetric.

Recall the function $K(A) = \left(\hat{s}A_t/\hat{\delta}\right)^{1/(1-\theta)}$ from equation (32), where $\hat{s}$ and $\hat{\delta}$ are defined in equation (29) of Lemma 3. $K(A)$ is the steady state level of capital, if productivity was constant at A. Since $r^* = \theta\hat{\delta}/\hat{s} - \delta$ per equation (38) and since

$$\frac{\dot{K}_t}{K_t} = \hat{s}A_tK_t^{\theta-1} - \hat{\delta} = \frac{\hat{s}}{\theta}(r_t+\delta) - \hat{\delta} \tag{A.7}$$

per (28) and (36), it follows that

$$r_t > r^* \iff \dot{K}_t > 0 \iff K_t < K(A_t). \tag{A.8}$$

For any $\tilde{t} > 0$, consider the solution $\tilde{K}_t$ to the ODE (A.7) starting at $\tilde{K}_0 = K^*$, but with $A_t \equiv A_{\tilde{t}}$ for $t \in [0, \tilde{t}]$ instead. It is clear that $\tilde{K}_t < K(A_{\tilde{t}})$: convergence to the new steady state $K(A_{\tilde{t}})$ for $A_t \equiv A_{\tilde{t}}$ does not happen in finite time and is strictly monotone, as one can see by examining the solution in (46). Given any $t \in [0, \tilde{t}]$ and any K_t, we have $\dot{K}_t \leq \dot{\tilde{K}}_t$, since the right-hand side of (A.7) weakly increases, when A_t is replaced by $A_{\tilde{t}}$. Since $K_0 = \tilde{K}_0 = K^*$, this implies that $K_t \leq \tilde{K}_t$ for $t \in [0, \tilde{t}]$. In particular now, $K_{\tilde{t}} \leq \tilde{K}_{\tilde{t}} < K(A_{\tilde{t}})$ and we must have $r_{\tilde{t}} > r^*$ as well as $\dot{K}_{\tilde{t}} > 0$ per equation (A.8).[23] Since $\tilde{t} > 0$ is arbitrary, this establishes the claim. □

Proof of Proposition 2. We proceed in two steps. We first conjecture that Assumptions 2 and 3 hold and use the upper bound $\bar{A}$ from Assumption 4 to obtain an upper bound $\bar{r}$ for the interest rates r_t along the transition path. We then verify Assumptions 2 and 3 by showing this upper bound $\bar{r}$ is sufficiently low.

Without loss of generality, we focus on the case where $A_t > A^*$ for all $t > 0$ and is weakly increasing.[24] Let $\bar{r}$ be the equilibrium interest rate that would prevail if capital was at its initial or steady state value $K_t = K_0 = K^*$ and productivity was at its upper bound $\bar{A}$ of Assumption 4, i.e., define

$$\bar{r} = \theta \frac{\bar{A}}{(K^*)^{1-\theta}} - \delta = \frac{\bar{A}}{A^*}(r^* + \delta) - \delta. \tag{A.9}$$

1. Conjecture that Assumptions 2 and 3 hold . Lemma 3 implies (28). Lemma 4 implies $r_t > r^*$ and $\dot{K}_t > 0$ for all $t > 0$. Hence, K_t is increasing and $K_t > K_0 = K^*$ for $t > 0$. Comparing equation (33) and (A.9), we get $r_t = \theta \frac{A_t}{K_t^{1-\theta}} - \delta \leq \bar{r}$.

2. Since A_t is weakly increasing and bounded, $A_t \to \tilde{A}$, $K_t \to K\left(\tilde{A}\right)$ and $r_t \to r^*$. Thus, Assumption 2 holds. Use the expression for equilibrium wage in equation (34) as well as equation (36) and calculate

$$\begin{aligned}
&\frac{\dot{w}_t}{w_t} + \rho - r_t = \frac{\dot{A}_t}{A_t} + \theta\frac{\dot{K}_t}{K_t} + \rho - r_t = \frac{\dot{A}_t}{A_t} - (1-\hat{s})\, r_t + \hat{s}\delta - \theta\hat{\delta} + \rho \\
\geq \;& 0 - (1-\hat{s})\,\bar{r} - \hat{s}r^* + \rho \\
= \;& (1-\hat{s})\left(\delta - \left(1 + \frac{\rho - r^*}{(r^*+\delta)(1-\hat{s})}\right)(r^*+\delta)\right) + (1-\hat{s})\, r^* + \rho - r^* = 0,
\end{aligned}$$

where the third line above uses (A.9) and (45). □

Proof of Corollary 2. Proposition 2 implies the aggregate capital can be solved from equation (28), which is a Bernoulli differential equation. Given an initial condition K_0, it can

[23] An alternative way to see this is to examine the closed-form solution (31).

[24] For the case where $A_t = A^*$ is constant for $t \in [0, \hat{t})$ and $A_t > A^*$ for $t > \hat{t}$, the economy will remain at its initial steady state for $t \in [0, \hat{t})$. The proof then goes through, starting at $\hat{t}$ rather than at $t = 0$, and the proposition claim holds for all t.

be solved by replacing $K_t^{1-\theta}$ with a new variable X_t and solving the resulting linear ODE in X_t as usual, see the Online Appendix G.4.4 for details. One obtains equation (31) in the paper. For the special case where a permanent shock implies $A_t \equiv \tilde{A}$, replace $\hat{s}\tilde{A}$ with $\hat{\delta}\tilde{K}^{1-\theta}$ and integrate the exponential function in (31) to obtain (46). K_t is increasing per Proposition 2. The remaining claims now follow directly from Proposition 2 as well as the expressions for w_t and r_t given in (34) and (33). □

Proof of Proposition 3. WLOG, we focus on the case where A_t is strictly decreasing at $t = 0$ and weakly decreasing for any $t > 0$.[25] Conjecture that Assumptions 2 and 3 hold in equilibrium. Then Lemma 3 gives the dynamics of aggregate capital in equation (28) and implies Assumption 2. Combining the expression for the equilibrium wage in equation (34) with the dynamics of aggregate capital in equation (36), we calculate

$$\begin{aligned}
&\frac{\dot{w}_t}{w_t} + \rho - r_t = \frac{\dot{A}_t}{A_t} + \theta\frac{\dot{K}_t}{K_t} + \rho - r_t = \frac{\dot{A}_t}{A_t} - (1-\hat{s})\, r_t - \hat{s}\left(\theta\frac{\hat{\delta}}{\hat{s}} - \delta\right) + \rho \\
> \quad & (r^* - \rho) - r^* + \rho = 0.
\end{aligned}$$

□

Proof of Proposition 4. From Proposition 2, rewrite aggregate capital dynamics (28) as

$$\frac{\dot{K}_t}{K_t} = \hat{s}A_t e^{(\theta-1)\log K_t} - \hat{\delta}. \tag{A.10}$$

Thus, the speed of convergence defined in equation (50) is

$$\beta_t = -\frac{\partial\left(\dot{K}_t/K_t\right)}{\partial \log\left(K_t\right)} = (1-\theta)\,\hat{s}A_t K_t^{\theta-1} = \frac{1-\theta}{\theta}\hat{s}\left(r_t + \delta\right),$$

where the last equality uses the interest rate in equation (33). Equation (52) follows from Corollary 1, implying $r_t \to r^* = \theta\hat{\delta}/\hat{s} - \delta$. □

[25] For the case where A_t is constant for $t \in [0, \hat{t})$ and strictly decreasing at $t = \hat{t}$ with $\hat{t} > 0$, the economy will remain at its initial steady state for $t \in [0, \hat{t})$. Then we just need to show the transition dynamics for $t \geq \hat{t}$, the proof of which is the same as the case where A_t is strictly decreasing at $t = 0$.

ONLINE APPENDIX

B More Proofs

B.1 Lemma 5 and Proof

Lemma 5. $r^* < \rho \iff \chi > 0$.

Proof. Using the definition of $\hat{s}$ and $\hat{\delta}$ in equation (29) of Lemma 3, we can rewrite the steady state interest rate r^* in equation (38) as

$$\begin{aligned} r^* &= \theta \frac{\hat{\delta}}{\hat{s}} - \delta \\ &= \theta \frac{(\delta + \rho + \nu)(\rho + \nu + \xi)}{\xi + \theta(\rho + \nu)} - \delta \\ &= \frac{\theta(\rho + \nu)(\rho + \nu + \xi) - \delta\xi(1 - \theta)}{\xi + \theta(\rho + \nu)}. \end{aligned}$$

Then

$$\begin{aligned} & \chi > 0 \\ \iff & \theta(\rho + \nu + \xi)\nu < \xi(1 - \theta)(\rho + \delta) \\ \iff & \theta(\rho + \nu + \xi)(\rho + \nu) < \xi(1 - \theta)(\rho + \delta) + \theta(\rho + \nu + \xi)\rho \\ \iff & \frac{\theta(\rho + \nu)(\rho + \nu + \xi) - \delta\xi(1 - \theta)}{\xi + \theta(\rho + \nu)} < \rho \\ \iff & r^* < \rho. \end{aligned}$$

□

B.2 Proof of Proposition 6

Proof. Using the optimal consumption allocation in equations (20) and (21), we compute the consumption ratio between the low-productivity agent and the high-productivity agent

$$\frac{c_{h,t}}{c_{t-\tau,t}} = \frac{w_t}{w_{t-\tau} e^{\int_{t-\tau}^{t} (r_u - \rho) du}}. \tag{B.1}$$

Taking logs and arranging terms delivers the equality in (56). Integrating equation (18) from $t-\tau$ to t implies that $\log(w_t) - \log w_{t-\tau} + \tau\rho > \tau r^a_{t,\tau}$. Therefore, the inequality in (56) follows and low-productivity agents consume less than the high income agent. □

C Details of the Speed of Convergence Analysis

C.1 Neoclassical Growth Model

In this section, we consider the neoclassical growth model with representative agent and complete market in continuous time. We first derive the speed of convergence along the transition using the definition in equation (50) and then compute its long-run value by log-linearizing the model.

C.1.1 Setup

There is a representative agent with utility function

$$\int_0^{+\infty} e^{-\rho t} u(C_t)\, dt,$$

where $u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$. The production technology is

$$\begin{aligned} Y_t &= A_t K_t^{\theta} L_t^{1-\theta}, \\ C_t + I_t &= Y_t, \\ \dot{K}_t &= I_t - \delta K_t, \\ C_t &\geq 0, K_t \geq 0, \end{aligned}$$

where A_t, Y_t, K_t, C_t, I_t are the productivity, output, capital stock, consumption, and investment at time t, respectively. The economy is endowed with capital K_0 at time $t = 0$ and there is unit labor supply $L_t = 1$.

The first order conditions give the interest rate and wage

$$\begin{aligned} r_t &= \theta A_t K_t^{\theta-1} - \delta, \\ w_t &= (1-\theta) A_t K_t^{\theta}. \end{aligned}$$

This yields the following consumption and capital dynamics

$$\begin{aligned}\frac{\dot{C}_t}{C_t} &= -\frac{\rho - r_t}{\sigma} = \frac{\theta A_t K_t^{\theta-1} - \rho - \delta}{\sigma}, \\ \dot{K}_t &= A_t K_t^{\theta} - \delta K_t - C_t.\end{aligned}$$

In the steady state under productivity level A^*, capital and consumption are

$$\begin{aligned}K^* &= \left(\frac{\theta A^*}{\rho + \delta}\right)^{\frac{1}{1-\theta}}, \\ C^* &= A^* \left(K^*\right)^{\theta} - \delta K^*.\end{aligned}$$

C.1.2 Numerical Solution of the Speed of Convergence

We consider the transition dynamics under the productivity process $\{A_t\}_{t\geq 0}$. Let s_t denote the savings rate at time t. By definition,

$$C_t = (1 - s_t) Y_t = (1 - s_t) A_t K_t^{\theta}.$$

This implies

$$\begin{aligned} & \dot{K}_t = s_t A_t K_t^{\theta} - \delta K_t \\ \implies \quad & \frac{\dot{K}_t}{K_t} = s_t A_t K_t^{\theta-1} - \delta = \frac{s_t (r_t + \delta)}{\theta} - \delta = s(\log K_t) A_t e^{(\theta-1)\log K_t} - \delta,\end{aligned}$$

where we view the savings rate as a function of log capital, i.e. $s_t = s(\log K_t)$.

Using the definition in equation (50), the speed of convergence can be expressed as

$$\beta_t = (1-\theta) s_t A_t K_t^{\theta-1} - \frac{\partial s(\log K_t)}{\partial \log K_t} A_t K_t^{\theta-1} = \frac{1-\theta}{\theta} s_t (r_t + \delta) - \frac{\partial s(\log K_t)}{\partial \log K_t} \frac{r_t + \delta}{\theta}.$$

We compute the above expression numerically for Figure 6 of the paper.

C.1.3 Speed of Convergence in the Long Run

In this section, we derive the speed of convergence in the long run for a particular productivity process. Specifically, consider a permanent shock that raises productivity from A^* to $\tilde{A}$.

We first log-linearize the model. Note that

$$\begin{aligned}
\frac{d\log K_t}{dt} &= \tilde{A}e^{-(1-\theta)\log K_t} - e^{\log\left(\frac{C_t}{K_t}\right)} - \delta, \\
\frac{d\log C_t}{dt} &= \frac{1}{\sigma}\left(\theta\tilde{A}e^{-(1-\theta)\log K_t} - \rho - \delta\right).
\end{aligned}$$

In the new steady state, where $\frac{d\log K_t}{dt} = \frac{d\log C_t}{dt} = 0$, we have

$$\begin{aligned}
&\tilde{A}e^{-(1-\theta)\log K^{**}} - e^{\log\left(\frac{C^{**}}{K^{**}}\right)} = \delta, \\
&\theta\tilde{A}e^{-(1-\theta)\log K^{**}} = \rho + \delta,
\end{aligned}$$

where C^{**} and K^{**} denote consumption and capital in the new steady state (under productivity level $\tilde{A}$). Taking a first-order Taylor expansion, we get

$$\begin{bmatrix} \frac{d\log K_t}{dt} \\ \frac{d\log C_t}{dt} \end{bmatrix} = \begin{bmatrix} \rho & \delta - \frac{\rho+\delta}{\theta} \\ -(1-\theta)\frac{\rho+\delta}{\sigma} & 0 \end{bmatrix} \begin{bmatrix} \log\left(\frac{K_t}{K^{**}}\right) \\ \log\left(\frac{C_t}{C^{**}}\right) \end{bmatrix}.$$

Let ϵ denote the eigenvalues of the first matrix on the right hand side, i.e.

$$\det\begin{bmatrix} \rho - \epsilon & \delta - \frac{\rho+\delta}{\theta} \\ -(1-\theta)\frac{\rho+\delta}{\sigma} & -\epsilon \end{bmatrix} = 0.$$

This implies

$$2\epsilon = \rho \pm \left(\rho^2 + 4\left(\frac{\rho+\delta}{\theta} - \delta\right)(1-\theta)\frac{\rho+\delta}{\sigma}\right)^{\frac{1}{2}}.$$

Let ϵ_1 denote the positive root and ϵ_2 denote the negative root, then

$$\log K_t = \log K^{**} + \psi_1 e^{\epsilon_1 t} + \psi_2 e^{\epsilon_2 t}.$$

$\psi_1 = 0$ must hold and ψ_2 is determined from the initial condition

$$\psi_2 = \log(K^*) - \log(K^{**}).$$

Hence, in the log-linearized model, capital evolves according to

$$\log(K_t) = \left(1 - e^{-\beta t}\right)\log(K^{**}) + e^{-\beta t}\log(K_0),$$

where β is defined as

$$\beta \equiv -\epsilon_2 = -\frac{\rho - \left(\rho^2 + 4\left(\frac{\rho+\delta}{\theta} - \delta\right)(1-\theta)\frac{\rho+\delta}{\sigma}\right)^{\frac{1}{2}}}{2}. \tag{C.1}$$

The capital dynamics imply

$$\frac{\dot{K}_t}{K_t} = -\beta e^{-\beta t}\left(\log\left(K_0\right) - \log\left(K^{**}\right)\right) = -\beta\left(\log K_t - \log K^{**}\right).$$

Using the definition in equation (50), the speed of convergence in the log-linearized model is

$$-\frac{\partial\left(\frac{\dot{K}_t}{K_t}\right)}{\partial\left(\log K_t\right)} = -\frac{\frac{\dot{K}_t}{K_t}}{\log K_t - \log K^{**}} = \beta.$$

Hence, β, which is defined in equation (C.1), is the long-run speed of convergence in the neoclassical growth model.

C.2 Proof of Proposition 5

Proof. Let β^{NG} and β denote the long-run speed of convergence in the neoclassical growth model and the Krueger-Li-Uhlig model, respectively. According to equation (C.1) in Online Appendix C.1.3 and Proposition 4,

$$\begin{aligned}\beta^{NG} &= \frac{\left(\rho^2 + 4\left(\frac{\rho+\delta}{\theta} - \delta\right)(1-\theta)(\rho+\delta)\right)^{\frac{1}{2}} - \rho}{2}, \\ \beta &= (1-\theta)\hat{\delta} = (1-\theta)(\delta+\rho+\nu).\end{aligned}$$

We want to find the necessary and sufficient condition for $\beta < \beta^{NG}$. Note that both coefficients are positive. Then

$$
\begin{aligned}
& \beta < \beta^{NG} \\
\iff \quad & (\rho + 2(1-\theta)(\delta+\rho+\nu))^2 < \rho^2 + 4\left(\frac{\rho+\delta}{\theta} - \delta\right)(1-\theta)(\rho+\delta) \\
\iff \quad & \rho(\delta+\rho+\nu) + (\delta+\rho+\nu)^2(1-\theta) < \left(\frac{\rho+\delta}{\theta} - \delta\right)(\rho+\delta) \\
\iff \quad & (\rho+\delta+\nu)(\rho + (\rho+\delta+\nu)(1-\theta)) < (\rho+\delta)\frac{1}{\theta}(\rho+\delta(1-\theta)) \\
\iff \quad & \theta\left(1+\frac{\nu}{\rho+\delta}\right)\left(1+\frac{(\rho+\nu)(1-\theta)}{\rho+\delta(1-\theta)}\right) < 1 \\
\iff \quad & \theta\left(1+\frac{\nu}{\rho+\delta}\right)\left(1+\frac{\rho+\nu}{\frac{\rho}{1-\theta}+\delta}\right) < 1.
\end{aligned}
$$

□

D Details of the Consumption Inequality Analysis

In this section, we establish the mappings between the consumption inequality measures, the individual characteristic τ, the consumption ratio between a low-productivity agent and a high-productivity agent, and the quantiles in the population and consumption distribution. We consider a permanent shock that raises productivity from A^* to $\tilde{A}$.

We first introduce the following notations. Let ι denote the consumption ratio of a low-productivity agent to a high-productivity agent, P denote the quantile in the population of agents ranked by consumption level, and G denote the quantile in the consumption distribution.

Symbol	Definition
τ	Length of time elapsed since the low-productivity agent's last transition to low-productivity
ι	Consumption ratio of a low-productivity agent to a high-productivity agent
P	Quantile in the population ranked by consumption level
G	Quantile in the consumption distribution

D.1 Mapping between τ and P

We establish a one-to-one mapping between τ and the quantiles in the population ranked from the lowest to the highest consumption level. Using the optimal consumption allocation in equations (20) and (21) of Lemma 2, for $\forall \tau, \tau'$ such that $0 < \tau' < \tau$, we have

$$\frac{c_{t-\tau,t}}{c_{t-\tau',t}} = \frac{w_{t-\tau}}{w_{t-\tau'}} e^{-\int_{t-\tau}^{t-\tau'} g_u du} < 1.$$

Hence, τ fully characterizes an agent's rank in the consumption distribution.

At any time t, we compute the fraction of agents with consumption level lower than the agent who last had high productivity at time $t-\tau$. We use $P(\tau)$ to denote this metric, then

$$P(\tau) = \int_{\tau}^{+\infty} \psi_l(x)\, dx = \int_{\tau}^{+\infty} \frac{\xi\nu}{\xi+\nu} e^{-\nu x} dx = \frac{\xi}{\xi+\nu} e^{-\nu\tau}.$$

Hence, $P(\tau)$ is the one-to-one mapping from τ to the quantiles in the population ranked by consumption level.

Inversely, we can also establish the mapping from the quantiles in the population to τ. Given a P, we find the τ such that P fraction of the agents have consumption level lower than the agent who last had high productivity at time $t-\tau$.

$$\tau(P) = \frac{1}{\nu} \log\left(\frac{\xi}{(\xi+\nu)\, P}\right).$$

D.2 Mapping from ι to P

Given an ι, we compute the fraction of agents whose consumption ratio is below ι.

For $\iota < 1$,

$$\begin{aligned} P(\iota, t) &= Pr\left(\frac{c_{t-\tau,t}}{c_{h,t}} \leq \iota\right) = Pr\left(-\log\left(\frac{c_{t-\tau,t}}{c_{h,t}}\right) \geq -\log(\iota)\right) \\ &= \int_{\underline{\tau}}^{+\infty} \psi_l(x)\, dx = \int_{\underline{\tau}}^{+\infty} \frac{\xi\nu}{\xi+\nu} e^{-\nu x} dx \\ &= \frac{\xi}{\xi+\nu} e^{-\nu \underline{\tau}(\iota,t)}. \end{aligned}$$

where $\underline{\tau}(\iota, t)$ is the solution to the equation $\frac{c_{t-\underline{\tau},t}}{c_{h,t}} = \iota$.

For $\iota = 1$,

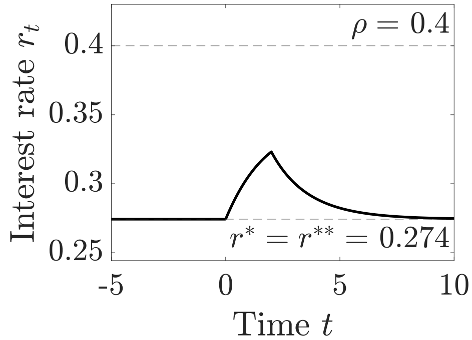
$$P(1,t) = 1.$$

D.3 Mapping from P to G

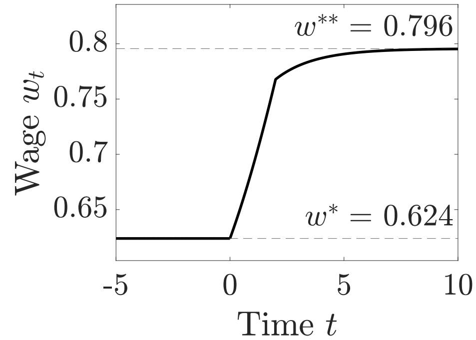
Given the quantile P for a group of agents, we compute their cumulative consumption share.

$$\begin{aligned}
G(P,t) &= \frac{1}{C_t}\int_{\tau(P)}^{+\infty} c_{t-x,t}\psi_l(x)dx \\
&= \frac{1}{C_t}\int_{\tau(P)}^{+\infty} c_{h,t-x}e^{-\int_{t-x}^{t} g_u du}\frac{\xi\nu}{\xi+\nu}e^{-\nu x}dx \\
&= \frac{1}{C_t}\frac{\xi\nu}{\xi+\nu}\alpha\zeta\int_{\tau(P)}^{+\infty} w_{t-x}e^{-\int_{t-x}^{t} g_u du}e^{-\nu x}dx \\
&= \alpha\xi\frac{w_t}{C_t}\int_{\tau(P)}^{+\infty}\frac{w_{t-x}}{w_t}e^{-\int_{t-x}^{t} g_u du}e^{-\nu x}dx.
\end{aligned}$$

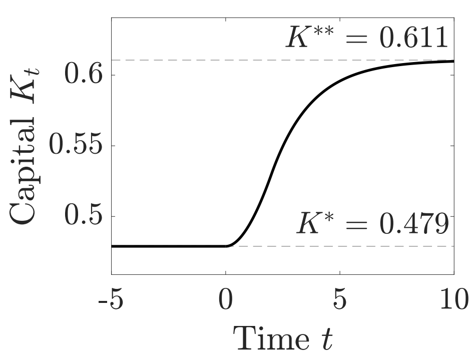
E Additional Figures



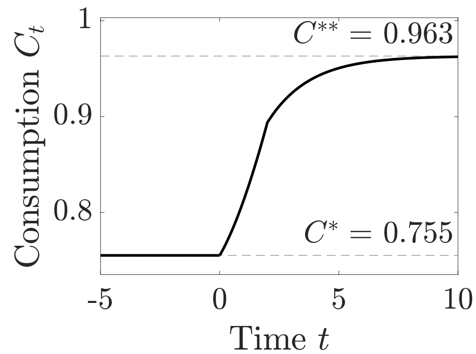
(a) Interest rate



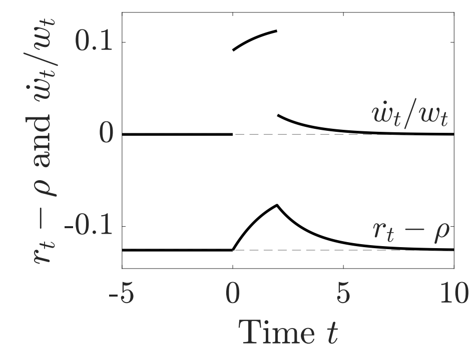
(b) Wage



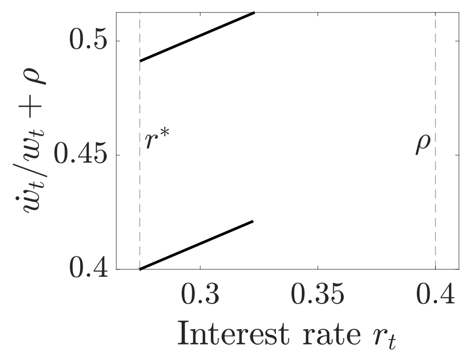
(c) Capital



(d) Consumption

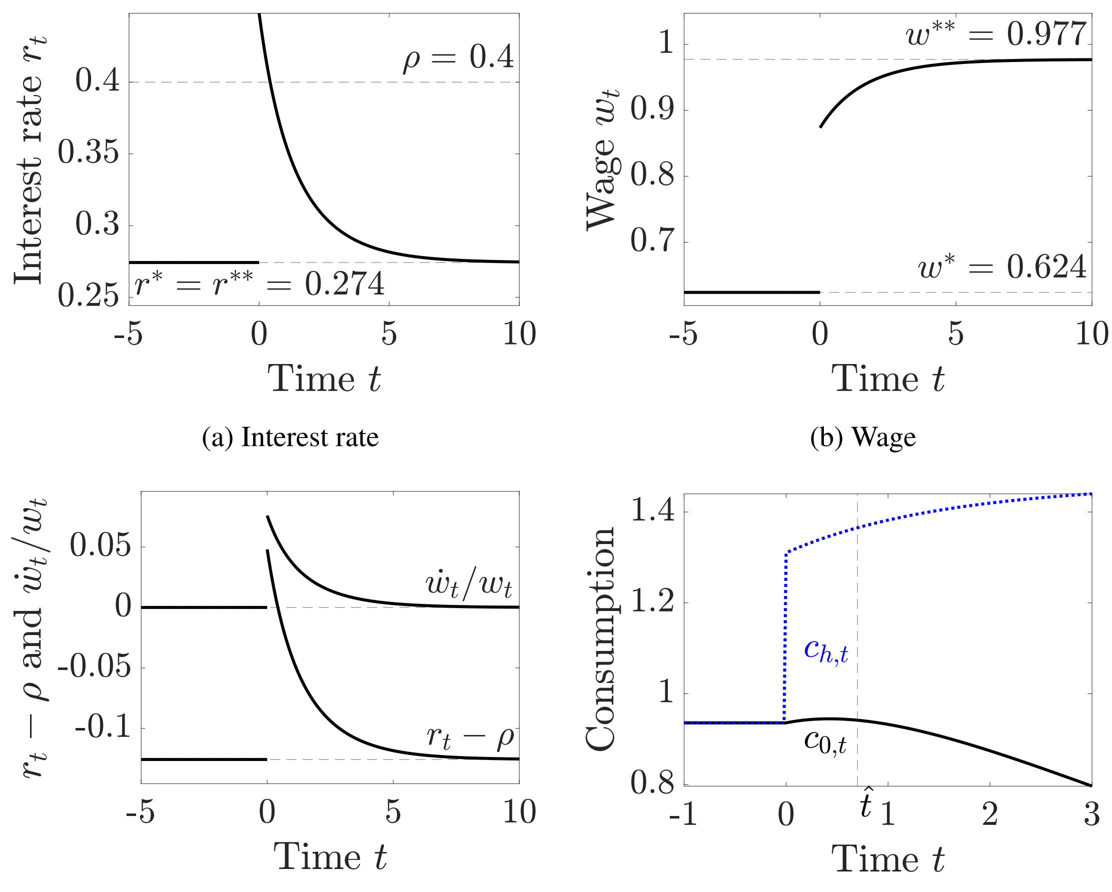


(e) Comparing $r_t - \rho$ with $\frac{\dot{w}_t}{w_t}$



(f) $\frac{\dot{w}_t}{w_t} + \rho$ as a function of r_t

Figure E.1: Transitional dynamics of the aggregate variables with a continuous increase in productivity. This figure plots the transition dynamics of the aggregate variables when productivity increases from $A^* = 1$ to $\tilde{A} = 1.2$ following the continuous path in equation (48) with parameter $T = 2$. Agents have log utility and $\delta = 0.16, \nu = 0.2, \rho = 0.4, \theta = 0.25, \xi = 0.2$.



(a) Interest rate

(b) Wage

(c) Comparing $r_t - \rho$ with $\frac{\dot{w}_t}{w_t}$

(d) Households' consumption

Figure E.2: Transitional dynamics of the aggregate variables and individual agents' consumption with a large permanent increase in productivity. This figure plots the transition dynamics of the aggregate variables and individual agents' consumption when productivity permanently increases from $A^* = 1$ to $\tilde{A} = 1.4$. Agents have log utility and $\delta = 0.16, \nu = 0.2, \rho = 0.4, \theta = 0.25, \xi = 0.2$.

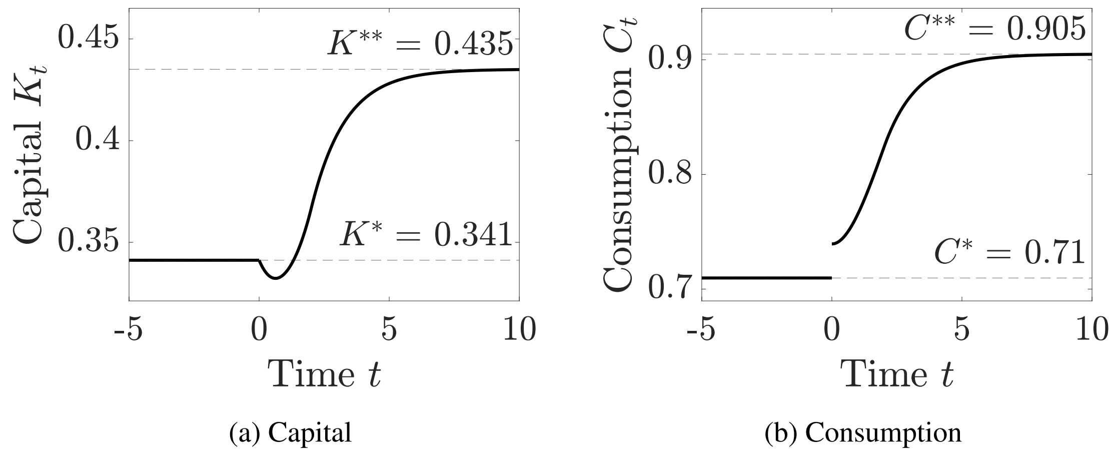


Figure E.3: Neoclassical growth model transition dynamics with a continuous increase in productivity. This figure plots the transition dynamics of the aggregate variables in the neoclassical growth model with representative agent and complete market. We consider the transition dynamics when productivity increases from $A^* = 1$ to $\tilde{A} = 1.2$ following the continuous path in equation (48) with parameter $T = 2$. The representative agent has log utility and $\delta = 0.16, \rho = 0.4, \theta = 0.25$.

F Computational Details for Figures

This section includes the computational details for the figures. We assume log utility when creating the figures.

- Figure 3: This figure plots the transition dynamics of the aggregate variables with a permanent increase in productivity. The steady state values and time paths of these variables are computed according to Section 4.
- Figure 4: This figure plots the transition dynamics of wage and individual households' consumption assuming that individual households consume according to the consumption insurance contracts in equations (20) and (21) of Lemma 2. The time path of wage is computed according to Section 4. Individual households' consumption in panels (c) and (d) are computed from equations (20) and (21) of Lemma 2.
- Figure 5: This figure plots the transition dynamics of the aggregate variables with a continuous increase in productivity and a continuous decrease in productivity. The computational details for these two cases are the same as in Figure 3.

- Figure 6: This figure compares the speed of convergence to the new steady state in our model with that in the neoclassical growth model. We consider a permanent increase in productivity. The speed of convergence in our model is computed from equation (51) of Proposition 4, and that in the neoclassical growth model is computed in Online Appendix C.1.2.

- Figure 8: This figure plots the evolution of the Lorenz curve with a permanent increase in productivity. We create the figure in the following steps: At each time t,

  1. A low income household i is characterized by the time since he last had high income, which is denoted as τ_i. We create an equally spaced vector $\boldsymbol{\tau} = (\tau_1, \tau_2, \cdots, \tau_n)$, which represents the cross section of low income households at time t.

  2. For each household i, we compute the corresponding quantile in the population ranked by the consumption level, i.e., $P(\tau_i)$, using the definition of $P(\tau)$ in Section D.1.

  3. For each household i, we compute the cumulative consumption share (as a fraction of aggregate consumption) for those households with consumption level lower than i. The individual consumption level is computed from equations (20) and (21) of Lemma 2.

  4. Finally, we plot the population quantiles of these households (obtained from Step 2) on the x-axis and the cumulative consumption share from Step 3 on the y-axis.

- Figure 7: This figure plots the transition dynamics of consumption inequality with a permanent increase in productivity. The consumption gap, wage gap, and discounting gap are computed according to Proposition 6. The time path of individual consumption is computed from equations (20) and (21) of Lemma 2 and the time paths of the equilibrium wage and interest rate are computed according to in Section 4.

- Figure 9: This figure plots the transition dynamics of wage and the consumption inequality with a permanent decrease in productivity, assuming that individual households consume according to the consumption insurance contracts in equations (20) and (21) of Lemma 2. The computational details are the same as in Figure 7.

- Figure 10: This figure plots the transition dynamics of consumption inequality with a continuous increase in productivity and a continuous decrease in productivity. The computational details are the same as in Figure 7.
- Figure E.1: This figure plots the transition dynamics of the aggregate variables with a continuous increase in productivity. The computational details are the same as in Figure 3.
- Figure E.2: This figure plots the transition dynamics of the aggregate variables and individual households' consumption with a large permanent increase in productivity. The computational details are the same as in Figure 3.
- Figure E.3: This figure plots the transition dynamics of aggregate capital and consumption in the neoclassical growth model. We consider a continuous increase in productivity. The steady state values and time paths of the variables are computed according to Online Appendix C.1.1.

We summarize the features of the above figures in Table F.1.

Table F.1: Summary of figures

Item	Productivity shock	x-axis	y-axis
Figure 3	Permanent $\uparrow$	Time	Aggregate variables
Figure 4	Permanent $\uparrow, \downarrow$	Time	Aggregate variables, individual consumption
Figure 5	Continuous $\uparrow, \downarrow$	Time	Aggregate variables
Figure 6	Permanent $\uparrow$	Time	Capital, speed of convergence
Figure 8	Permanent $\uparrow$	Population quantile	Cumulative consumption share
Figure 7	Permanent $\uparrow$	Time	Consumption inequality
Figure 9	Permanent $\downarrow$	Time	Consumption inequality
Figure 10	Continuous $\uparrow, \downarrow$	Time	Consumption inequality
Figure E.1	Continuous $\uparrow$	Time	Aggregate variables
Figure E.2	Permanent $\uparrow$	Time	Aggregate variables, individual consumption
Figure E.3	Continuous $\uparrow$	Time	Capital, consumption

G Transition Dynamics When Agents Have CRRA Utility

In this section, we show how to generalize our analysis, when agents have CRRA utility. Specifically, we assume agents have the period CRRA utility function

$$u(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma}$$

instead of log-utility $u(c) = \log(c)$, so that the time-t expected utility of an agent is given by

$$U_t = E_t \left[\int_t^{+\infty} e^{-\rho(\tau - t)} \frac{c_\tau^{1-\sigma} - 1}{1 - \sigma} d\tau \right].$$

While much can still be calculated analytically, there is no longer a direct calculation of the equilibrium path. Rather, we obtain a fixed-point problem, which one would need to solve numerically.

More precisely, a transition equilibrium is characterized by the following fixed-point problem.

1. Conjecture a path for aggregate capital K_t along the transition, given the initial condition $K_0 = K^*$. Calculate r_t and w_t, using the first-order condition of production firms, i.e. capital and labor demand. See Section G.1 below.

2. Characterize the optimal allocation $\mathcal{C}_t$, given the paths for r_t and w_t. See Section G.2 below.

3. Compute the path of aggregate capital supply K_t^S by aggregating the capital holdings across individual agents. See Section G.3 below.

4. Check whether the path of aggregate capital supply K_t^S matches the conjectured path of aggregate capital stock K_t in step 1.

G.1 Conjecture a Path for Aggregate Capital

We conjecture a path for aggregate capital $\{K_t\}_{t \geq 0}$ and then calculate the path for interest rate and wage using the first-order conditions (33) and (34) of production firms. Since the interest rate in stationary equilibrium does not depend on the value of aggregate productivity (exactly as in the standard neoclassical growth model), the interest rate will converge to the original steady state value r^*, i.e.,

$$\lim_{t \to +\infty} r_t = r^*.$$

This together with equation (33) implies the aggregate capital K_t must converge to

$$K_\infty = \left(\frac{\theta A_\infty}{r^* + \delta} \right)^{\frac{1}{1-\theta}}.$$

We therefore see that the conjectured capital path $\{K_t\}_{t\geq 0}$ is constrained by the two boundary conditions: $K_0 = K^*$ (i.e. the original steady state as initial condition for $t = 0$) and $\lim_{t\to+\infty} K_t = K_\infty$.

G.2 Characterize the Optimal Allocation $\mathcal{C}_t$

The interest rate and wage are the only aggregate variables relevant for the dynamic insurance problem at the agent level. Equipped with a conjectured path of interest rate and wage from Section G.1, we can now characterize the optimal allocation by deriving conditions for the evolution of (individual) consumption and (individual) capital over time as functions of productivity. Define

$$g_t \equiv \frac{\rho - r_t}{\sigma}.$$

As might be expected from the standard consumption-savings problem with CRRA utility, g_t will turn out to be the negative of the consumption growth rate, if capital holdings are strictly positive. We define the discounting term

$$D_t \equiv \int_t^{+\infty} e^{-\int_t^s (r_u+\nu+g_u)du} ds. \tag{G.1}$$

This is the net present value of a consumption spell that starts at a level of 1, falls at rate $-g_t < 0$ defined above, and ends at the Poisson rate ν. This expression will be useful for calculating the cost of a consumption allocation for a newly unproductive agent.

Given the allocation $\mathcal{C}_t$, we define the implied time derivative of consumption as[26]

$$\dot{c}_t\left(k;z\right) \equiv \frac{\partial c_t\left(k;z\right)}{\partial t} + \frac{\partial c_t\left(k;z\right)}{\partial k} x_t\left(k;z\right).$$

Lemma 6 (The optimal allocation $\mathcal{C}_t$ for $z = 0$ and $k > 0$.)**.** *For $k > 0$, the optimal contract of Definition 1 implies the consumption dynamics*

$$\frac{\dot{c}_t\left(k;z\right)}{c_t\left(k;z\right)} = -g_t. \tag{G.2}$$

[26]As a heuristic for this definition of the time derivative, suppose that productivity remains constant at z for some interval of time. In that case, note that $\dot{k}_t = x_t\left(k;z\right)$ and that consumption evolves as $c_t = c_t\left(k_t;z\right)$ which is a function of time only. Taking the derivative with respect to time yields the expression here.

Furthermore, if $z = 0$ and $\tilde{k}_t(k;0) = 0$ for all $k \leq \bar{k}$ and some $\bar{k}$, then

$$c_t(k;0) = \frac{k}{D_t}, \tag{G.3}$$

$$x_t(k;0) = \left(r_t + \nu - \frac{1}{D_t}\right)k, \tag{G.4}$$

for all $k \leq \bar{\bar{k}}$ and some suitably chosen $\bar{\bar{k}}$.

The proof is in Online Appendix G.4.1. We now use this result to characterize the consumption dynamics of agents with currently high productivity. To do so, we make the following assumption.

Assumption 6. *Suppose the aggregate wage and interest rate satisfy the following condition, $\forall t \geq 0$,*

$$0 < g_t + \frac{\dot{w}_t}{w_t} - \frac{\xi \dot{D}_t}{\zeta w_t}. \tag{G.5}$$

Lemma 7 (The contract $\mathcal{C}_t$ for $z = \zeta$ and $k = 0$)**.** *Let Assumption 6 be satisfied. Then the optimal contract of Definition 1 implies the following consumption dynamics of low-productivity agents*

$$c_t(0;\zeta) = \frac{w_t\zeta}{1+\xi D_t}, \tag{G.6}$$

$$x_t(0;\zeta) = 0. \tag{G.7}$$

Furthermore,

$$\tilde{k}_t(0;\zeta) = \frac{w_t\zeta}{1+\xi D_t} D_t. \tag{G.8}$$

The proof is in Online Appendix G.4.2. The term ξD_t in the denominator of the right-hand side of (G.6) is the insurance premium to obtain the capital stock $\tilde{k}_t(0;\zeta)$ in case of a transition to the low-productivity state and to assure desirable continuity of consumption via (G.3), if so. Indeed, the equality

$$\xi \tilde{k}_t(0;\zeta) = w_t\zeta - c_t(0;\zeta)$$

shows that this is an actuarially fair contract. Note that (G.6) implies

$$\frac{\dot{c}_t(0;\zeta)}{c_t(0;\zeta)} = \frac{\dot{w}_t}{w_t} - \frac{\xi \dot{D}_t}{\zeta w_t}. \tag{G.9}$$

Equation (G.9) rationalizes why we need Assumption 6. If consumption could be chosen in an unconstrained fashion, then we would obtain (G.2). With (G.5), consumption would grow more slowly than the right-hand side of (G.9), but this can now only be accomplished per borrowing against future wages and choosing $x < 0$, subject to making the consumption-smoothing insurance payments against the transition to low productivity. But this is ruled out by the borrowing constraint (6). Put differently, Assumption 6 assures that the high-productivity agent has no desire to accumulate capital.

For ease of notation, let $c_{s,t} = c_t(k_{s,t};0)$ denote consumption of a low-productivity agent at date t who switched from high to low productivity at time $s \leq t$. This agent holds capital $k_{s,t}$ at time t. This notation implies that $c_{s,s}$ and $k_{s,s}$ are the consumption and capital holdings of an agent who has switched to low productivity this very instant. Finally, we also denote by $c_{h,t} = c_t(0;\zeta)$ the date-t consumption of a high-productivity agent with no assets.

From equation (G.3), capital holdings are proportional to consumption for low-productivity agents,

$$k_{s,t} = D_t c_{s,t}. \tag{G.10}$$

Equations (G.6) and (G.8) imply that

$$c_{s,s} = c_s(0;\zeta) = \frac{w_s \zeta}{1 + \xi D_s}, \tag{G.11}$$

$$k_{s,s} = \tilde{k}_s(0;\zeta) = \frac{w_s \zeta}{1 + \xi D_s} D_s. \tag{G.12}$$

Equation (G.11) is due to the fact that consumption is continuous and does not jump upon receiving a negative productivity shock (it in principle could, since it is a jump variable). For low-productivity agents, the consumption growth equation (G.2) or equivalently

$$\frac{\dot{c}_{s,t}}{c_{s,t}} = -\frac{\rho - r_t}{\sigma} = -g_t \tag{G.13}$$

holds except for the economy-wide "MIT-shock" date $t = 0$ (on which the economy transitions to the productivity path $\{A_t\}_{t\geq 0}$). If an agent last switched from high to low productivity after that transition date, i.e. if $s > 0$, then equation (G.13) characterizes his consumption dynamics since that date. If the switch last happened at some date $s \leq 0$, this low-productivity agent will have started at some steady state capital k^*_{-s}, characterized by equation (41) in the log-utility case. More generally, using the results above applied to the steady state together with

$$g^* = \frac{\rho - r^*}{\sigma}, \tag{G.14}$$

$$D^* = \frac{1}{\nu + r^* + g^*}, \tag{G.15}$$

we have

$$c_h^* = \frac{w^* \zeta}{1 + \xi D^*}, \tag{G.16}$$

$$c_\tau^* = e^{-g^* \tau} c_h^*, \tag{G.17}$$

$$k_\tau^* = D^* e^{-g^* \tau} c_h^*. \tag{G.18}$$

The above two cases for s yield the consumption dynamics of low-productivity agents in Lemma 8 below.

Lemma 8 (Consumption dynamics for low productivity agents)**.** *Consider the time-t consumption $c_{s,t}$ of a low-productivity agent who last switched from $z = \zeta$ to $z = 0$ at time $s \leq t$.*

1. *If $s > 0$, then*

$$c_{s,t} = e^{-\int_s^t g_u du} \frac{w_s \zeta}{1 + \xi D_s}. \tag{G.19}$$

2. *If $s \leq 0$, then*

$$c_{s,t} = e^{-\int_0^t g_u du} \frac{k^*_{-s}}{D_0}. \tag{G.20}$$

Equation (G.19) can be rewritten with equation (G.12) as

$$c_{s,t} = e^{-\int_s^t g_u du} \frac{k_{s,s}}{D_s}, \tag{G.21}$$

or more generally, as

$$c_{s,t} = e^{-\int_q^t g_u du} \frac{k_{s,q}}{D_q}, \tag{G.22}$$

for any $s \leq q \leq t$. Comparing equations (G.20) and (G.22), we see that the consumption of agents with $s < t$ will jump, if and only if $D_0 \neq D^*$: the change in the path of future interest rates may induce the agent to reduce or to increase current consumption, compared to the steady state and given the same budget or net present value at time $t = 0$.

G.3 Compute the Path of Aggregate Capital Supply

To compute the aggregate capital supply K_t at time t, we aggregate the capital holdings of low-productivity agents,

$$K_t = \int_{-\infty}^{t} k_{s,t} \psi_l (t-s)\, ds. \tag{G.23}$$

Lemma 9 (Dynamics of aggregate capital supply)**.** *Capital supply evolves according to*

$$\dot{K}_t \quad = \quad \left(\frac{\xi D_t}{1+\xi D_t} (1-\theta) + \theta \right) A_t K_t^{\theta} - \left(\delta + \frac{1}{D_t} \right) K_t, \tag{G.24}$$

where D_t is defined in equation (G.1).

The proof is in Online Appendix G.4.3. Given the initial capital stock $K_0 = K^*$, we can use the capital dynamics in equation (G.24) to solve for the path of aggregate capital supply. Lemma 10 summarizes the results with proof in Online Appendix G.4.4.

Lemma 10 (Aggregate capital supply)**.** *Suppose the aggregate capital evolves according to equation (G.24) for all $t \geq 0$, given the initial condition $K_0 = K^*$. Then the aggregate capital supply at any time $t \geq 0$ takes the following form*

$$K_t^S = \left(e^{-(1-\theta)\int_0^t b_u du} K_0^{1-\theta} + (1-\theta) \int_0^t e^{-(1-\theta)\int_s^t b_u du} a_s ds \right)^{\frac{1}{1-\theta}}, \tag{G.25}$$

where

$$a_t \equiv \left(\frac{\xi D_t}{1+\xi D_t}(1-\theta)+\theta\right)A_t, \tag{G.26}$$

$$b_t \equiv \delta + \frac{1}{D_t}, \tag{G.27}$$

and D_t is defined in equation (G.1).

G.4 Proofs

G.4.1 Proof of Lemma 6

Proof. The proof expands the proof of Lemma 1 in the Appendix. It is verbatim the same except that $\rho u'(c) = u''(c)\dot{c} + u'(c)r_t$ now implies

$$\frac{\dot{c}}{c} = \frac{r_t - \rho}{\sigma} = -g_t, \tag{G.28}$$

which is the consumption dynamics in equation (G.2).

When $z = 0$ and $\tilde{k}_t(k;0) = 0$ for all $k \leq \bar{k}$ and some $\bar{k}$, the consumption dynamics (G.2)[27] and the budget constraint (5) can be rewritten as the linear system of differential equations

$$\dot{c}_t = -g_t c_t, \tag{G.29}$$

$$\dot{k}_t = (r_t + \nu)k_t - c_t \tag{G.30}$$

in the unknown functions c_t and k_t with the boundary condition[28] $\lim_{t\to+\infty} k_t = 0$, provided $k_t \leq \bar{k}$ for all t.[29] The solution is valid, as long as the implied path for k_s for $s \geq t$

[27]Note that equation (G.2) implies $c_s = e^{-\int_t^s g_u du}c_t$. A less formal, but more meaningful argument in terms of economic theory is thus to recognize that the budget constraint and utility maximization implies that the current capital k is equal to the net present value of all future consumption, as long as the productivity state stays unchanged. This yields $k = \int_t^{+\infty} e^{-\int_t^s (r_u+\nu)du}c_s ds = D_t c_t$.

[28]The boundary condition ensures that no capital gets wasted and this follows from utility maximization.

[29]To derive equation (G.30), conjecture that the solution satisfies equation (G.3), which can be rewritten as

$$k_t = D_t c_t. \tag{G.31}$$

Note that

$$\dot{D}_t = -1 + (r_t + \nu + g_t)D_t. \tag{G.32}$$

does not cross the upper bound $\bar{k}$. This will be true for all $k_t \in \left(0, \bar{\bar{k}}\right)$ and some suitable $\bar{\bar{k}}$. □

G.4.2 Proof of Lemma 7

Proof. The lemma is a version of Section D.3 (Lemma 7) in the Online Appendix of Krueger and Uhlig (2022), generalized to the case, where aggregate wages and interest rates are functions of time. Rather than replicating the steps of that section, we provide the key logic of the argument here and point to the results and proofs in Krueger and Uhlig (2022) mentioned above for a deeper foundation.

For a high-productivity agent, the Lagrangian, first-order conditions, and envelope condition are as in the proof for Lemma 1, see equations (A.1), A.2 and (A.4), but applied to $k = 0$, $z = \zeta$, and $p_z = \xi$.

1. We first consider the choice of $\tilde{k}$. The solution in Lemma 1 implies that

$$U_t'(\tilde{k}; 0) = u'\left(\frac{\tilde{k}}{D_t}\right),$$

which increases to infinity, as $\tilde{k} \to 0$. The third first-order condition in (A.2) therefore implies that $\tilde{k} > 0$ and thus $\omega = 0$. With the first and third first-order conditions in (A.2), we obtain consumption smoothing

$$u'\left(\frac{\tilde{k}}{D_t}\right) = u'(c) \implies \tilde{k} = D_t c. \tag{G.33}$$

Therefore, equation (G.6) follows from the budget constraint (5), provided that $x = 0$.

2. Then we need to show that $x > 0$ is not optimal. Suppose otherwise, $x > 0$ were optimal, then equation (G.33) together with the budget constraint (5) implies that $c_t < w_t\zeta/(1 + \xi D_t)$, i.e. consumption is less than the right hand side of equation (G.6). Furthermore, constraint (6) would not be binding, $\lambda = 0$, and consumption growth would satisfy equation (G.2). Let $[t, t + \Delta]$ be a time interval for some $\Delta > 0$, during which this is the case and along a path where no productivity

Differentiate equation (G.31) w.r.t. time t and use equations (G.29) and (G.32) to derive equation (G.30).

switch occurs. Assumption 6 then implies $c_s \leq w_s\zeta/\left(1+\xi D_s\right)$ during the interval $s \in [t, t+\Delta]$, i.e. consumption is less than the consumption level proposed in Lemma 7 for that episode. The integral of utility during that time interval is then smaller than the utility of the solution proposed in Lemma 7. This loss in utility can only be justified by the additional utility gained from consuming the accumulated capital after a switch to lower productivity for $s > 0$, or, alternatively, for $s > \Delta$ in case there is no switch to lower productivity. This amounts to postponing consumption compared to the solution proposed in Lemma 7. But this can be seen to contradict the impatience of the agent relative to wage growth, as expressed in Assumption 6. A precise formulation of that contradiction requires replicating the arguments in Section D.3 of Krueger and Uhlig (2022), allowing for the additional time evolution of r_t and w_t.

□

G.4.3 Proof of Lemma 9

Proof. We differentiate both sides of equation (G.23) wrt time t,

$$\dot{K}_t = k_{t,t}\psi_l\left(0\right) + \int_{-\infty}^{t}\left(\dot{k}_{s,t}\psi_l\left(t-s\right) + k_{s,t}\psi_l'\left(t-s\right)\right)ds. \tag{G.34}$$

We derive $\psi_l'\left(t-s\right)$ from equation (7),

$$\psi_l'\left(t-s\right) = \frac{\xi\nu}{\xi+\nu}e^{-\nu(t-s)}\left(-\nu\right) = -\nu\psi_l\left(t-s\right). \tag{G.35}$$

We derive $\frac{\dot{k}_{s,t}}{k_{s,t}}$ from equation (G.10),

$$\dot{k}_{s,t} = \dot{D}_t c_{s,t} + D_t\dot{c}_{s,t} \implies \frac{\dot{k}_{s,t}}{k_{s,t}} = \frac{\dot{D}_t}{D_t} + \frac{\dot{c}_{s,t}}{c_{s,t}}. \tag{G.36}$$

Using the definition of D_t in equation (G.1), we get

$$\begin{aligned} &\dot{D}_t = -1 + \left(r_t+\nu+g_t\right)\int_t^{+\infty} e^{-\int_t^s (r_u+\nu+g_u)du}ds = -1 + \left(r_t+\nu+g_t\right)D_t \\ \implies \quad &\frac{\dot{D}_t}{D_t} = r_t+\nu+g_t - \frac{1}{D_t}. \end{aligned}$$

Substituting the above equation and equation (G.13) for consumption growth into equation (G.36), we get

$$\frac{\dot{k}_{s,t}}{k_{s,t}} = r_t + \nu + g_t - \frac{1}{D_t} - g_t = r_t + \nu - \frac{1}{D_t}. \tag{G.37}$$

Substituting equations (G.35) and (G.37) into equation (G.34),

$$\begin{aligned}
\dot{K}_t &= k_{t,t}\psi_l(0) + \int_{-\infty}^{t} \left(\left(r_t + \nu - \frac{1}{D_t} \right) k_{s,t}\psi_l(t-s) - \nu k_{s,t}\psi_l(t-s) \right) ds \\
&= \frac{w_t \zeta}{1+\xi D_t} D_t \frac{\xi \nu}{\xi + \nu} + \left(r_t - \frac{1}{D_t} \right) K_t \\
&= \frac{\xi w_t}{1+\xi D_t} D_t + \left(r_t - \frac{1}{D_t} \right) K_t \\
&= \frac{\xi (1-\theta) A_t K_t^{\theta}}{1+\xi D_t} D_t + \left(\theta A_t K_t^{\theta-1} - \delta - \frac{1}{D_t} \right) K_t \\
&= \left(\frac{\xi D_t}{1+\xi D_t} (1-\theta) + \theta \right) A_t K_t^{\theta} - \left(\delta + \frac{1}{D_t} \right) K_t.
\end{aligned}$$

The second line above uses the density and the capital holdings of the low productivity agents in equations (7) and (G.12), the third line above uses the normalization in equation (3), and the fourth line above uses the interest rate and wage in equations (33) and (34). □

G.4.4 Proof of Lemma 10

Proof. Equation (G.24) is a Bernoulli differential equation. Given an initial condition K_0, it can be solved as follows.

1. Rewrite equation (G.24) as a linear differential equation. Define $X_t \equiv K_t^{1-\theta}$, $a_t \equiv \left(\frac{\xi D_t}{1+\xi D_t} (1-\theta) + \theta \right) A_t$, and $b_t \equiv \delta + \frac{1}{D_t}$. We can rewrite equation (G.24) as

$$\dot{X}_t + (1-\theta) b_t X_t = (1-\theta) a_t. \tag{G.38}$$

2. Solve the linear differential equation (G.38). Multiply both sides of equation (G.38) by $e^{(1-\theta)\int_0^t b_u du}$,

$$\frac{d\left(e^{(1-\theta)\int_0^t b_u du} X_t \right)}{dt} = (1-\theta) e^{(1-\theta)\int_0^t b_u du} a_t.$$

Integrate both sides of the above equation from time 0 to time t,

$$e^{(1-\theta)\int_0^t b_u du} X_t - X_0 = \int_0^t (1-\theta)\, e^{(1-\theta)\int_0^s b_u du} a_s ds,$$

which yields

$$X_t = e^{-(1-\theta)\int_0^t b_u du} X_0 + (1-\theta)\int_0^t e^{(1-\theta)\int_t^s b_u du} a_s ds. \tag{G.39}$$

3. Substitute the definition of X_t into equation (G.39),

$$\begin{aligned} K_t &= \left(e^{-(1-\theta)\int_0^t b_u du} K_0^{1-\theta} + (1-\theta)\int_0^t e^{(1-\theta)\int_t^s b_u du} a_s ds \right)^{\frac{1}{1-\theta}} \\ &= \left(e^{-(1-\theta)\int_0^t b_u du} K_0^{1-\theta} + (1-\theta)\int_0^t e^{-(1-\theta)\int_s^t b_u du} a_s ds \right)^{\frac{1}{1-\theta}}. \end{aligned}$$

□