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An Assignment Problem with Interdependent Valuations and Externalities

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Abstract

In this paper, we take a mechanism design approach to optimal assignment problems with asymmetrically informed buyers. In addition, the surplus generated by an assignment of a buyer to a seller may be adversely affected by externalities generated by other assignments. The problem is complicated by several factors. Buyers know their own valuations and externality costs but do not know this same information for other buyers. Buyers also receive private signals correlated with the state and, consequently, the implementation problem exhibits interdependent valuations. This precludes a naive application of the VCG mechanism and to overcome this interdependency problem, we construct a two-stage mechanism. In the first stage, we exploit correlation in the firms signals about the state to induce truthful reporting of observed signals. Given that buyers are honest in stage 1, we then use a VCG-like mechanism in stage 2 that induces honest reporting of valuation and externality functions.

Keywords: mechanism design, assignment problem, asymmetric information, two stage mechanism

JEL Classification: D82

1 Introduction

Suppose a government authority is tasked with allocating sections of the spectrum to a given set of telecommunications firms. A central issue in their task concerns externalities: the bandwidth allocated to a given firm may interfere with other firms' spectrum usage, imposing negative externalities on those firms. A firm's valuation for different portions of the spectrum also depends on other factors such as the firm's demand and operating costs. To complicate matters, these valuations and externality costs may depend on factors unobserved at the time that economic transactions take place. For example, there may be industry-wide uncertainty about regulation, such as the recent *Net Neutrality* debate.[1] The value of any part of the spectrum depends substantially on whether a bill to mandate net neutrality is passed. Consequently, the firms' valuations for spectrum segments, as well as their externality costs, depend on a state variable unobservable at the time that economic decisions must be made.[2]

In this paper, we take a mechanism design approach to assignment problems like those described above. The model will consist of "buyers" and "sellers" (or "objects") with the goal of assigning buyers to objects so as to maximize aggregate expected surplus. The surplus enjoyed by a telecom firm that is allocated a piece of the spectrum is the expected value of that piece to the firm less the expected interference costs the firm incurs from the allocation of spectrum pieces to other firms. The authority could auction off the spectrum pieces but there are difficulties with standard auction protocols such as first or second price auctions. First, this is not a standard private values auction problem. A firm's value for a particular piece of the spectrum depends on what other firms are allocated because of the externalities.

The problem is complicated by several factors. We will assume that buyers know how their own valuations and the externality costs they incur depend on the unobservable state but do not know this same information for other buyers, nor will we posit that a buyer has beliefs about other buyers' valuations and externality costs. Buyers also receive private signals correlated with the state (e.g., as a result of their lobbying with respect to the net neutrality question).[3] Consequently, the problem is an interdependent value problem due to the firms' informative signals about the state of the world. This precludes a naive application of the VCG mechanism.

To overcome the interdependency problem, we construct a two-stage mechanism. In the first stage, we exploit correlation in the firms signals about the state. In particular, firms' signals are truthfully elicited by giving each firm a positive transfer that depends on the relation of it's announced signal to the other firms' announced signals. Given a profile of reported first stage signals, the mechanism then computes the associated posterior distribution on the state space and makes this public. In stage 2, firms observe this posted posterior distribution and then make not necessarily honest reports of their valuations and externality costs to the mechanism. Once these are reported,

[1]Net neutrality is the principle that an internet service provider (ISP) has to provide access to all sites, content and applications at the same speed, under the same conditions without blocking or giving preference to any content.

[2]Similarly, for the problem of allocating airplane gate slots to airlines, the costs and benefits of assigning a particular type of aircraft departing from a particular city to a particular gate in another city can depend on other aircraft-gate pairings throughout the air traffic system as well as unobservables like the weather.

[3]See [1], Chapter 5.5, for evidence of shared uncertainty about spectrum values and the sources of the uncertainty.

the mechanism can identify a surplus-maximizing assignment. We show that, if buyers honestly report their observed signals in the first stage, then this second stage problem is a private values problem and we are able to use VCG transfers in the second stage to elicit firms' private expected bandwidth valuations and externality costs. We show that there exist first stage rewards and second stage VCG transfers such that firms honestly report their signals in Stage 1 and their expected valuations and externality costs in Stage 2 in an equilibrium. Given the weak informational assumptions of this paper, we propose a behavior strategy profile and beliefs at each information set that need not be a perfect Bayesian equilibrium but does satisfy a certain dominant strategy property along the equilibrium path of play.

The VCG mechanism used at the second stage is attractive in that the transfers are non-negative – any transfers go from the firms to the authority. It is not surprising that one can structure the VCG mechanism so that aggregate payments are negative. What is important, though, is that the outcome of the VCG mechanism is individually rational. Again, it is not surprising that one can construct the VCG mechanism to deliver individually rational outcomes, but it is *not* generally true that one can simultaneously satisfy individual rationality and non-positive surplus.

Offsetting (at least partially) the VCG surplus in the second stage are the first period rewards that the authority pays to the firms to elicit the information about the state of the world. If the sum of those payments is larger than the VCG surplus in the second stage, the authority would have an aggregate deficit. We show that if there is a deficit, that deficit goes to zero as the number of buyers and objects goes to infinity.

2 Related Work

Our work is related to matching problems with asymmetric information. [2] illustrated that even a small infusion of uncertainty about preferences of other agents into ordinal matching problems can undermine the results achieved under a complete information framework. [3] elaborated on the notion of incomplete information stability concept in cardinal matching problems, and introduced an iterative belief-formation process, used by agents in allocation blocking decision making. Complementing this work, [4] points out that some results in a complete information framework that are related to dominant strategy behavior, can be transferred with no change to incomplete information framework. In particular, within the context of marriage problems, every man has a dominant strategy to report his value truthfully in an M-optimal stable matching mechanism. [5] and [6] draw parallel conclusions for trading settings under asymmetric information cast as cardinal matching problems. They show that buyer-optimal stable outcomes can be supported by VCG transfers.

Similar to our work, in addition to asymmetric information, [7] incorporate interdependent valuations in a centralized matching mechanism. The authors introduce a mediator who controls the amount of information available to the agents by varying information structures of the game. Different information structures define different notions of stability and corresponding existence results. Our paper is also reminiscent of the work by [8], who assign agents to objects and focus on the questions of efficiency rather than stability. The authors show that there is no Pareto efficient and ex-post incentive compatible mechanisms in assignment games with interdependent information. This negative result is remedied by relaxing ex-post requirements to Bayes incentive compatibility constraints, and by introducing a single-crossing property assumption. The latter is not required for our efficiency results.

This paper is also related to work on auctions with heterogeneous goods under interdependent information. In the environment of independent signals and interdependent valuations, [9] study multi-unit one-sided auctions, as collections of two-person single-unit second price auctions. Under a set of assumptions on valuation functions Perry and Reny construct a two-round mechanism, where agents reveal their private signals in the first round, and, given this aggregated information, estimate their valuations; while in the second round they engage in corresponding second price auctions. Our mechanism does not require the assumptions used for Perry and Reny's results. [10] investigates a dynamic one-sided auction, for heterogeneous goods, with a single seller, who seeks to allocate a finite number of goods to multiple potential buyers. He introduces an adjustment system to compute agents' allocations and payments. In this system, the adjustments are driven by aggregate reports of the opponents, making the truthful reporting a dominant strategy. The resulting transaction price converges to a competitive Walrasian price. While Ausubel focuses on the trajectory properties of the model in the time limit, we examine the system's behavior through replica economies.

Unlike the aforementioned papers, our model includes allocative externalities. Specifically, given an assignment, each agent experiences externalities from other matches. Hence, each agent cares not only about who he is paired with, but is also concerned with the other agents' matches. Our definition of externalities is related to that in [11] who consider allocative externalities in an implementation problem with interdependent valuations. Each player's private information is his object valuation and the amount of allocative externality he imposes on others. Their modeling of externalities and interdependency is different from that in this paper and the common knowledge assumptions are also different. They explore the limits of Bayesian incentive compatible implementation of ex post efficient social choice rules within their specific framework by showing that special conditions on marginal rates of information substitution in agents' signals are necessary for efficient, incentive compatible implementation.

We implement an efficient and asymptotically budget balanced assignment via a voluntary and incentive compatible mechanism in the presence of incomplete and interdependent information. Our solution is based on a two-stage approach developed by [12]. However, in this paper we work in the environment where the set of feasible outcomes is not fixed and can vary with the set of players [4]. The distinguishing feature of this paper is that the model requires less information on the part of the mechanism designer and the players. In particular, the agents are not assumed to know the payoff functions or externalities of other agents.

[4] See discussion in [13]

3 Preliminaries

3.1 States, Signals and Payoffs

If K is a finite set, let $|K|$ denote the cardinality of K and let $\Delta(K)$ denote the set of probability measures on K. Let $\Delta^*(K)$ denote the subset of $\Delta(K)$ with full support. Throughout the paper, $||\cdot||_2$ will denote the 2-norm and $||\cdot||$ will denote the 1-norm. The real vector spaces on which these norms are defined will be clear from the context.

We will be concerned with a two sided market consisting of n "buyers" that are to be matched with n "sellers" or "objects."[5] Let $N = \{1, .., n\}$. Buyers and sellers will be paired by a mechanism and will then engage in a transaction. A transaction might involve a transfer of an object, e.g., a portion of the spectrum to the buyer, or the provision of a service to the buyer by the seller.

Let Θ represent a finite set of states of nature. Neither buyers nor the mechanism know the value of θ prior to the conclusion of all transactions.

Buyers receive signals correlated with the state and these signals are private information. Each buyer i receives a signal in a finite set B_i. Let $B = B_1 \times \cdots \times B_n$ and let $b = (b_1, .., b_n) \in B$ denote a generic signal profile in B.

The surplus generated when buyer i is matched with object/seller j depends on the state $\theta \in \Theta$ and externalities that adversely affect this surplus. We denote the value generated when buyer i is matched with seller j in state θ as $u_{ij}(\theta)$ and we assume that this value accrues to the buyer. We could write this value more explicitly as $u_{ij}(\theta) = f_{ij}(\theta) - g_{ij}(\theta)$ where $f_{ij}(\theta)$ is the benefit accruing to buyer i when matched with seller j in state θ and $g_{ij}(\theta)$ is the cost incurred by seller j when matched with buyer i in state θ. However, sellers are non-strategic actors in our model so we work directly with $u_{ij}(\theta)$.

In our model, we allow for the possibility of externalities that negatively impact the surplus generated by a matching. To incorporate these as a factor in the implementation problem, let $c_{ij}^{pq}(\theta)$ denote the externality cost imposed on buyer i if buyer i and seller j are matched when buyer $p \neq i$ is matched with seller $q \neq j$ and the state is θ. The number $c_{ij}^{pq}(\theta)$ is assumed to be nonnegative. We assume that for all (i, j), (p, q) and θ, $0 \leq u_{ij}(\theta), c_{ij}^{pq}(\theta) \leq M$ for some $M > 0$. We will write $u_i(\theta) = (u_{i1}(\theta), .., u_{in}(\theta))$, $u(\theta) = (u_1(\theta), .., u_n(\theta))$ and $u_{-i}(\theta) = (u_k(\theta))_{k \in N \setminus i}$. If $\pi \in \Delta(\Theta)$, define $u_i(\pi) = (u_{i1}(\pi), .., u_{in}(\pi)) \in \mathbb{R}^n$ as $u_i(\pi) = \sum_{\theta \in \Theta} u_i(\theta)\pi(\theta)$ and let $u(\pi) = (u_1(\pi), .., u_n(\pi))$. Define $c_{ij}(\theta) \in R^{(n-1)(n-1)}$ where $c_{ij}(\theta) = (c_{ij}^{pq}(\theta))_{p \neq i, q \neq j}$. We will write $c_i(\theta) = (c_{i1}(\theta), .., c_{in}(\theta))$, $c(\theta) = (c_1(\theta), .., c_n(\theta))$ and $c(\pi) = (c_1(\pi), .., c_n(\pi))$ where $c_{ij}^{pq}(\pi) = \sum_{\theta \in \Theta} c_{ij}^{pq}(\theta)\pi(\theta)$.

3.2 Stochastic Structure

Next, let $\Delta^*(\Theta \times B)$ denote the set of $P \in \Delta(\Theta \times B)$ satisfying $P(\theta, b) > 0$ for each $(\theta, b) \in \Theta \times B$. For $P \in \Delta^*(\Theta \times B)$ and $b \in B$, the conditional distribution induced by P on Θ given $b \in B$ is denoted $P_\Theta(\cdot|b)$. In the interest of notational simplicity, we will often write $\rho(b)$ for $P_\Theta(\cdot|b)$.

[5] At the cost of more complex notation, we could allow the number of buyers and sellers to be different.

For each i and $b_i \in B_i$, the conditional distribution induced by P on B_{-i} given $b_i \in B$ is denoted $P_{-i}(\cdot|b_i)$. That is,

$$P_{-i}(b_{-i}|b_i) = \sum_{\theta \in \Theta} P(\theta, b_{-i}|b_i).$$

Finally, we assume that $P_{-i}(\cdot|b_i) \neq P_{-i}(\cdot|b_i')$ if $b_i \neq b_i'$.

3.3 Informational Assumptions

As stated above, the observed signals of buyers are private information. In addition, the value functions and the externality functions are also private information. That is, the value-externality profile $(u_i(\theta), c_i(\theta))_{\theta \in \Theta}$ is known only to buyer i. On the other hand, the prior $P \in \Delta^*(\Theta \times B)$ is known to all buyers as well as the mechanism. In addition, all buyers, as well as the mechanism, know the bound M on buyer values and buyer externalities.

3.4 The Implementation Problem

An *assignment problem with interdependent values* is a collection $(u_1, .., u_n, c_1, .., c_n, P) = (u, c, P)$ where $P \in \Delta^*(\Theta \times B)$. Let Z denote the set of all feasible matchings of buyers and sellers. That is, Z is the set of all $n \times m$ arrays z such that $z_{ij} \in \{0, 1\}$ and

$$\sum_{i=1}^{n} z_{ij} \leq 1 \text{ for all } j \text{ and } \sum_{j=1}^{m} z_{ij} \leq 1 \text{ for all } i.$$

The presence of externalities complicates the task of efficiently matching buyers to sellers in the presence of asymmetric information since buyer i in an (i, j) match incurs externality costs only from those different (p, q) matches that actually take place in an optimal assignment.

If $z \in Z$ is a feasible assignment, then the net surplus accruing to buyer i when matched with j in state θ given by

$$u_{ij}(\theta) - \sum_{p \neq i} \sum_{q \neq j} c_{ij}^{pq}(\theta) z_{pq}.$$

An *assignment function* is a mapping $((u(\theta), c(\theta))_{\theta \in \Theta}, b) \mapsto F((u(\theta), c(\theta))_{\theta \in \Theta}, b) \in Z$ that specifies an outcome in Z for each profile $((u(\theta), c(\theta))_{\theta \in \Theta}, b)$. An assignment function is outcome efficient if for each $(u(\theta), c(\theta))_{\theta \in \Theta}, b)$, the assignment $F((u(\theta), c(\theta))_{\theta \in \Theta}, b)$ solves the quadratic assignment problem

$$\max_{z \in Z} \sum_{i=1}^{n} \sum_{j=1}^{m} \left[\sum_{\theta \in \Theta} \left[u_{ij}(\theta) - \sum_{p \neq i} \sum_{q \neq j} c_{ij}^{pq}(\theta) z_{pq} \right] P_{\Theta}(\theta|b) \right] z_{ij}.$$

Given a matching problem, our goal in this paper is to implement an outcome efficient assignment function via an individually rational, incentive compatible mechanism. Furthermore, we wish to design the mechanism so as to require as little information as possible on the part of buyers and the mechanism designer. To accomplish this, we present a two-stage mechanism that borrows its structure from [12]. To simplify the presentation that follows, we introduce one more piece of notation. For each i, let $w_i = (w_{i1}, .., w_{in}) \in \mathbb{R}^n$ (interpreted as a valuation vector) and let $d_i = (d_{i1}, .., d_{in})$ where each $d_{ij} \in \mathbb{R}_+^{(n-1)(n-1)}$ (interpreted as an externality vector). For each (i, j) pair and $z \in Z$, define buyer $i's$ payoff for a given assignment $z \in Z$ as

$$g_i(z; w_i, d_i) = \sum_{j=1}^{n} \left[w_{ij} - \sum_{p \neq i} \sum_{q \neq j} d_{ij}^{pq} z_{pq} \right] z_{ij}.$$

Note that

$$\left| g_i(z; w_i, d_i) \right| \leq (1 + n)M.$$

In this notation, an assignment function is outcome efficient if for each $((u(\theta), c(\theta))_{\theta \in \Theta}, b)$, the assignment $F((u(\theta), c(\theta))_{\theta \in \Theta}, b)$ solves the quadratic assignment problem

$$\max_{z \in Z} \sum_{i=1}^{n} \left[\sum_{\theta \in \Theta} g_i(z; u_i(\theta), c_i(\theta)) P_\Theta(\theta | b) \right]$$

4 The Two-Stage Implementation Game

4.1 Preliminaries

We begin by considering a simpler implementation problem that will provide the basic structure of the second stage of the more complex two-stage model to follow. The mechanism seeks to match buyers and sellers so as to maximize total surplus generated by their true values. We will assume that w_i and d_i are known only to buyer i. In this simple problem, each buyer makes a (not necessarily honest) report of his value-externality vector to the mechanism. If $(w, d) = (w_1, .., w_n, d_1, .., d_n)$ is the profile of buyers' reports, the mechanism then computes payoffs and classical VCG transfers for the buyers by solving the quadratic assignment problem that maximizes total surplus. We show that these transfers yield a mechanism that is dominant strategy incentive compatible, individually rational and incurs no budget deficit. Define for each (w, d) the outcome

$$\hat{\varphi}(w, d) \in \arg \max_{z \in Z} \sum_{i=1}^{n} g_i(z; w_i, d_i).$$

The resulting payoff to buyer i in the optimal solution measured as "net benefit" is then

$$g_i(\hat{\varphi}(w, d); w_i, d_i) = \sum_{j=1}^{m} \left[w_{ij} - \sum_{p \neq i} \sum_{q \neq j} d_{ij}^{pq} \hat{\varphi}_{pq}(w, d) \right] \hat{\varphi}_{ij}(w, d)$$

while the total payoff to buyers different from i is

$$\sum_{k:k\neq i} g_k(\hat{\varphi}(w,d);w_k,d_k) = \sum_{k:k\neq i}\sum_{j=1}^{m}\left[w_{kj} - \sum_{p\neq k}\sum_{q\neq j} d_{kj}^{pq}\hat{\varphi}_{pq}(w,d)\right]\hat{\varphi}_{kj}(w,d).$$

Next, let $Z_{-i} = \{z \in Z | z_{ij} = 0 \text{ for all } j\}$ and note that $Z_{-i} \subseteq Z$. Define transfers as follows:

$$x_i(w,d) = \sum_{k:k\neq i} g_k(\hat{\varphi}(w,d);w_k,d_k) - \max_{z\in Z_{-i}}\left[\sum_{k:k\neq i} g_k(z;w_k,d_k)\right].$$

Since these transfers define a standard Groves mechanism, dominant strategy incentive compatibility is immediately obtained. That is, for any buyer i and any value-externality vectors (w_i,d_i) and (w_i',d_i') and any profile of valuation-externality vectors (w_{-i},d_{-i}) of other buyers we have

$$g_i(\hat{\varphi}(w_{-i},d_{-i},w_i,d_i);w_i,d_i)+x_i(w_{-i},d_{-i},w_i,d_i) \geq g_i(\hat{\varphi}(w_{-i},d_{-i},w_i',d_i');w_i,d_i)+x_i(w_{-i},d_{-i},w_i',d_i').$$

To verify individual rationality, suppose that

$$z^* \in \arg\max_{z\in Z_{-i}}\left[\sum_{k:k\neq i} g_k(z;w_k,d_k)\right].$$

Since $z^* \in Z$ and $g_i(z^*;w_i,d_i) = 0$, it follows that

$$\begin{aligned} g_i(\hat{\varphi}(w,d);w_i,d_i) + x_i(w,d) &= \sum_{k=1}^{n} g_k(\hat{\varphi}(w,d);w_k,d_k) - \left[\sum_{k:k\neq i} g_k(z^*;w_k,d_k)\right] \\ &= \sum_{k=1}^{n} g_k(\hat{\varphi}(w,d);w_k,d_k) - \left[\sum_{k=1}^{n} g_k(z^*;w_k,d_k)\right] \\ &\geq 0. \end{aligned}$$

To verify that transfers $x_i(w,d)$ are nonpositive, let $z_{kj}' = \hat{\varphi}_{kj}(w,d)$ for all j and all $k \neq i$ and $z_{ij}' = 0$ for all j. Then $z' \in Z_{-i}$. Since

$$\sum_{q\neq j} d_{kj}^{iq}\hat{\varphi}_{iq}(w,d) \geq 0$$

we conclude that

$$\sum_{k:k\neq i} g_k(\hat{\varphi}(w,d);w_k,d_k) = \sum_{k\neq i}\sum_j \left[w_{kj} - \sum_{p\neq k,i}\sum_{q\neq j} d_{kj}^{pq}\hat{\varphi}_{pq}(w,d) - \sum_{q\neq j} d_{kj}^{iq}\hat{\varphi}_{iq}(w,d)\right]\hat{\varphi}_{kj}(w,d) \leq$$

$$\leq \sum_{k\neq i}\sum_j \left[w_{kj} - \sum_{p\neq k}\sum_{q\neq j} d_{kj}^{pq} z'_{pq}\right] z'_{kj} \leq \max_{z\in Z_{-i}} \left[\sum_{k:k\neq i} g_k(z;w_k,d_k)\right]$$

and transfers are nonpositive.

4.2 The Two-Stage Game

We wish to formulate our implementation problem with interdependent valuations as a two-stage problem in which honest reporting of the agents' signals in stage one resolves the "interdependency" problem so that the second stage problem is a simple implementation problem with private values of the type presented in Section 4.1 above, to which the VCG mechanism can be immediately applied. We now define an extensive form game that formalizes the two-stage game that lies behind this idea. Let $\xi = (\xi_i)_{i\in N}$ be an n-tuple of functions $\xi_i : B \to \mathbb{R}_+$ each of which assigns to each $b \in B$ a nonnegative number $\xi_i(b)$ interpreted as a "reward" to buyer i. These rewards are designed to induce buyers to honestly report their signals in stage 1. We now describe the extensive form of a two-stage game $\Gamma(u, c, P, \xi)$.

Stage 1: Buyer i learns his signal b_i and let $b \in B$ denote the resulting profile of observed signals. Buyer i then makes a (not necessarily honest) report of r_i to the mechanism. If $r = (r_1, .., r_n)$ is the profile of stage 1 reports, then buyer i receives a nonnegative payment $\xi_i(r)$, and the game moves to stage 2.

Stage 2: If r is the reported signal profile in stage 1, the mechanism publicly posts the conditional distribution $P_\Theta(\cdot|r) = \rho(r)$ but not the reported profile r. Upon observing $\rho(r)$, each buyer i reports a valuation-externality vector $(w_i(\theta), d_i(\theta))_{\theta\in\Theta}$ to the mechanism. Given the reported profile $(w(\theta), d(\theta))_{\theta\in\Theta} = (w_i(\theta), d_i(\theta))_{\theta\in\Theta, i\in N}$ and the posted $\rho(r)$, the mechanism then computes

$$(w_k(\rho(r)), d_k(\rho(r))) = \sum_\theta (w_k(\theta), d_k(\theta))P(\theta|r)$$

for each buyer k. Next, the mechanism computes the social outcome

$$\hat{\varphi}(w(\rho(r)), d(\rho(r))) \in \arg\max_{z\in Z} \sum_{k=1}^n g_k(z; w_k(\rho(r)), d_k(\rho(r)))$$

and VCG transfers $x_i(w(\rho(r)), d(\rho(r)))$ for each buyer i where

$$x_i(w(\rho(r)), d(\rho(r))) = \sum_{k:k\neq i} g_k(\hat{\varphi}(w(\rho(r)), d(\rho(r))); w_k(\rho(r)), d_k(\rho(r))) -$$

$$- \max_{z \in Z_{-i}} \left[\sum_{k:k \neq i} g_k(z; w_k(\rho(r)), d_k(\rho(r))) \right].$$

If $\hat{\varphi}_{ij}(w(\rho(r)), d(\rho(r))) = 1$, then buyer i receives transfer $x_i(w(\rho(r)), d(\rho(r)))$ and j's object is given (or j's service is provided) to buyer i.

Since $\rho(b) = P_\Theta(\cdot|b)$ is the conditional distribution on states given the realized signal profile b, the resulting expected payoff to buyer i at the end of stage 2 (but before the state θ is known) is

$$g_i(\hat{\varphi}(w(\rho(r)), d(\rho(r))); u_i(\rho(b)), c_i(\rho(b))) + x_i(w(\rho(r)), d(\rho(r)) + \xi_i(r).$$

We wish to design the rewards ξ_i so as to accomplish two goals. In stage 1, we want to induce agents to report honestly so that the reported stage 1 profile is exactly b when the realized signal profile is b. Then, upon observing the posted posterior distribution $P_\Theta(\cdot|b) = \rho(b)$ in stage 2, we want each buyer i to report the true valuation-externality vector $(u_i(\theta), c_i(\theta))_{\theta \in \Theta}$. If these twin goals are accomplished in an equilibrium, then the social outcome is

$$\hat{\varphi}(u(\rho(b)), c(\rho(b))) \in \arg\max_{z \in Z} \sum_{i=1}^{n} g_i(z; u_i(\rho(b)), c_i(\rho(b))).$$

In particular,

$$\hat{\varphi}(u(\rho(b)), c(\rho(b))) \in \arg \max_{z \in Z} \sum_{i=1}^{n} \left[\sum_{\theta \in \Theta} g_i(z; u_i(\theta), c_i(\theta)) P_\Theta(\theta|b) \right]$$

so that $\hat{\varphi}(u(\rho(b)), c(\rho(b)))$ is an outcome efficient assignment.

The resulting nonnegative expected payoff to buyer i at the end of stage 2 (but before the state θ is known) is

$$g_i(\hat{\varphi}(u(\rho(b)), c(\rho(b))); u_i(\rho(b)), c_i(\rho(b))) + x_i(u(\rho(b)), c(\rho(b))) + \xi_i(b)$$

and the net payment to the mechanism is

$$\sum_i x_i(u(\rho(b)), c(\rho(b))) + \sum_i \xi_i(b).$$

While the sum of VCG transfers is nonpositive, this net payment could be positive since the first stage rewards are nonnegative. Consequently, it is important to identify situations in which the sum of the first period rewards is small and this issue is considered in Section 5 below in the case of large n.

4.3 Strategies and Equilibria in the Two-Stage Game

For each buyer i and each $b_i \in B_i$, let

$$D(b_i) := \{P_\Theta(\cdot|b_{-i}, b_i) : b_{-i} \in B_{-i}\}$$

and define a partition $\Pi_i(b_i) = D(b_i)/\sim$ where $P_\Theta(\cdot|b'_{-i}, b_i) \sim P_\Theta(\cdot|b''_{-i}, b_i)$ if and only if $P_\Theta(\cdot|b'_{-i}, b_i) = P_\Theta(\cdot|b''_{-i}, b_i)$.

Given the specification of the extensive form, the second stage information sets of buyer i are indexed by triples (r_i, π, b_i) where $b_i \in B_i$ is the privately observed signal of buyer i in stage 1, $r_i \in B_i$ is the reported signal of buyer i in stage 1, and $\pi \in \Pi_i(r_i)$ is the posted posterior distribution. Consequently, a behavior strategy for buyer i in this game is a pair (α_i, β_i) where $b_i \in B_i \mapsto \alpha_i(b_i) \in B_i$ specifies a first stage report as a function of i's observed signal b_i. At each second stage information set, β_i specifies for each θ and j a valuation $w_{ij}(\theta)$ and an externality vector $d_{ij}(\theta) \in \mathbb{R}^{(n-1)(n-1)}$. More formally, β_i is a function

$$(r_i, \pi, b_i) \in B_i \times \Pi_i(r_i) \times B_i \mapsto \beta_i(r_i, \pi, b_i) \in (\mathbb{R}^n \times \mathbb{R}^{n(n-1)(n-1)})^n$$

that specifies a second stage report $([\beta_i(r_i, \pi, b_i)](\theta))_{\theta\in\Theta}$ as a function of i's private signal $b_i \in B_i$, i's first stage report r_i and the posted distribution $\pi \in \Pi_i(r_i)$.

We are interested in an equilibrium assessment for the two-stage implementation game consisting of a strategy profile $(\alpha_i, \beta_i)_{i\in N}$ and a system of second stage beliefs for each buyer i, in which buyers truthfully report their private information at each stage.

Definition 1: A strategy (α_i, β_i) for buyer i is *truthful* for i in $\Gamma(u, c, P, \xi)$ if $\alpha_i(b_i) = b_i$ for all $b_i \in B_i$ and

$$\beta_i(b_i, \pi, b_i) = (u_i(\theta), c_i(\theta))_{\theta\in\Theta}$$

for all $b_i \in B_i$ and all $\pi \in \Pi_i(b_i)$.

A strategy profile $(\alpha_i, \beta_i)_{i\in N}$ is *truthful* in $\Gamma(u, c, P, \xi)$ if (α_i, β_i) is truthful for each buyer i. In a truthful strategy, each buyer i honestly reports the observed signal in stage 1. Then, in stage 2, buyer i honestly reports his true valuation-externality function at any second stage information set corresponding to a reported signal that matches i's observed signal in stage 1.

Formally, a *system of beliefs* for buyer i is a collection of probability measures in $\Delta(\Theta \times B_{-i})$ indexed by (r_i, π, b_i) with $\pi \in \Pi_i(r_i)$ where $\mu_i(\cdot|r_i, \pi, b_i) \in \Delta(\Theta \times B_{-i})$ has the following interpretation: when player i observes signal b_i reports r_i in stage 1 and observes the posted distribution $\pi \in \Pi_i(r_i)$ in stage 2, then buyer i assigns probability mass $\mu_i(\theta, b_{-i}|r_i, \pi, b_i)$ to the event that other buyers have observed signals b_{-i} and the state of nature is θ. As usual, an *assessment* is a collection $\{(\alpha_i, \beta_i)_{i\in N}, (\mu_i)_{i\in N}\}$ consisting of a behavior strategy and a system of beliefs for each buyer i.

Definition 2: An assessment $\{(\alpha_i^*, \beta_i^*)_{i\in N}\ , (\mu_i^*)_{i\in N}\}$ is a *sequential dominant strategy assessment* in $\Gamma(u, c, P, \xi)$ if $\{(\alpha_i^*, \beta_i^*)_{i\in N}\ , (\mu_i^*)_{i\in N}\}$ is a Nash equilibrium assessment satisfying (i) the profile $((\alpha_i^*, \beta_i^*)_{i\in N}$ is truthful, (ii) for any profile $(\alpha_j, \beta_j)_{j\in N\backslash i}$ of behavior strategies of other players, (α_i^*, β_i^*) is sequentially rational for i at each first stage information set $b_i \in B_i$ and (iii) for each $b_i \in B_i$ and each $\pi \in \Pi_i(b_i)$ and for any profile $(\beta_j)_{j\in N\backslash i}$ of second stage strategies of other players, (α_i^*, β_i^*) is sequentially rational at each second stage information set given $(\alpha_j^*, \beta_j)_{j\in N\backslash i}$.

Informally, $\{(\alpha_i^*, \beta_i^*)_{i\in N}\ , (\mu_i^*)_{i\in N}\}$ is a sequential dominant strategy assessment if for each i, (α_i^*, β_i^*) is a dominant strategy "along the equilibrium path of play." Condition (ii) states that, (α_i^*, β_i^*) is a best response against any $(\alpha_j, \beta_j)_{j\in N\backslash i}$ at each first stage information set and condition (iii) states that (α_i^*, β_i^*) is a best response against any $(\alpha_j^*, \beta_j)_{j\in N\backslash i}$ given that all players use $(\alpha_j^*)_{j\in N}$ in stage 1 and beliefs are derived from Bayes rule.

4.4 The Main Result

Theorem 1. Let (u, c, P) be an assignment problem with interdependent valuations. For each i, define a behavior strategy (α_i^*, β_i^*) for i where $\alpha_i^*(b_i) = b_i$ for each $b_i \in B_i$ and $\beta_i^*(r_i, \pi, b_i) = (u_i(\theta), c_i(\theta))_{\theta\in\Theta}$ for each (r_i, π, b_i) such that $r_i \in B_i, \pi \in \Pi_i(r_i)$ and $b_i \in B_i$. Then there exists a reward system $\xi = (\xi_i)_{i\in N}$ and second stage beliefs $(\mu_i^*)_{i\in N}$ for each i such that $(\alpha_i^*, \beta_i^*, \mu_i^*)_{i\in N}$ is a sequential dominant strategy assessment in the two-stage game $\Gamma(u, c, P, \xi)$.

Proof. To prove this result, we proceed in several steps which we outline here. Let $(\alpha_i^*, \beta_i^*)_{i\in N}$ be defined as in the statement of Theorem 1. Clearly, $(\alpha_i^*, \beta_i^*)_{i\in N}$ is a truthful profile.

Step 1: An equilibrium assessment requires that second stage beliefs be specified for each buyer i at each of buyer i's second stage information sets. Furthermore, given $(\alpha_i^*, \beta_i^*)_{i\in N}$, these should be "consistent with Bayes rule whenever possible." Suppose that buyer i receives signal b_i, the other players receive signal profile $b_{-i} \in B_{-i}$, and buyer i reports r_i in stage 1. Then $\alpha_k^*(b_k) = b_k$ for each $k \neq i$. Therefore, buyer i with signal b_i who has submitted report r_i in stage 1 and who observes $\pi \in \Pi(r_i)$ at stage 2 will assign positive probability

$$\sum_{\hat{b}_{-i}:\rho(\hat{b}_{-i}, r_i)=\pi} P_{-i}(\hat{b}_{-i}|b_i)$$

to the event

$$\{\hat{b}_{-i} \in B_{-i} : \rho(\hat{b}_{-i}, r_i) = \pi\}.$$

Therefore, i's updated beliefs regarding (θ, b_{-i}) consistent with (α^*, β^*) are given by

$$\mu_i^*(\theta, b_{-i}|r_i, \pi, b_i) = \begin{cases} \frac{P_\Theta(\theta|b_{-i}, b_i)P_{-i}(b_{-i}|b_i)}{\sum\limits_{\hat{b}_{-i}:\rho(\hat{b}_{-i}, r_i)=\pi} P_{-i}(\hat{b}_{-i}|b_i)}, & \text{if } \rho(b_{-i}, r_i) = \pi \\ 0, & \text{otherwise.} \end{cases}$$

Step 2: To verify part (iii) of the definition of sequential dominant strategy assessment, we must show that, if all buyers are truthful in stage 1, then β_i^* is a best response against any collection $(\beta_j)_{j\neq i}$ of second stage strategies of i's opponents. For each $b_{-i} \in B_{-i}, \pi \in \Pi(b_i)$ and each $k \neq i$, let

$$\sum_{\theta} [\beta_k(b_k, \pi, b_k)](\theta)\pi(\theta) = \gamma_k(\pi, b_k)$$

and

$$(\gamma_j(\pi, b_j))_{j\neq i} = \gamma_{-i}(\pi, b_{-i}).$$

To verify part (iii), we show that for each $b_i \in B_i$ and $\pi \in \Pi(b_i)$ and each $(w_i(\theta), d_i(\theta))_{\theta\in\Theta}$, we have

$$\sum_{b_{-i}\in B_{-i}} \sum_{\theta\in\Theta} \Bigg[g_i(\hat{\varphi}(\gamma_{-i}(\pi, b_{-i}), u_i(\pi), c_i(\pi)); u_i(\theta), c_i(\theta)) + \hat{x}_i(\gamma_{-i}(\pi, b_{-i}), u_i(\pi), c_i(\pi))) \Bigg] \mu_i(\theta, b_{-i}|b_i, \pi, b_i) \geq$$

$$\geq \sum_{b_{-i}\in B_{-i}} \sum_{\theta\in\Theta} \Bigg[g_i(\hat{\varphi}(\gamma_{-i}(\pi, b_{-i}), w_i(\pi), d_i(\pi)); u_i(\theta), c_i(\theta)) + \hat{x}_i(\gamma_{-i}(\pi, b_{-i}), w_i(\pi), d_i(\pi))) \Bigg] \mu_i(\theta, b_{-i}|b_i, \pi, b_i).$$

Step 3: To verify part (ii) of the definition of sequential dominant strategy assessment, we must show that $((\alpha^*, \beta^*), \mu^*)$ is sequentially rational for buyer i at his first stage information sets given any behavior strategy profile $(\alpha_k, \beta_k)_{k\neq i}$ of the other players. In particular, we must show that a coordinated deviation by buyer i in which buyer i lies in stage 1 and reports optimally in the second stage given the first stage lie, is not profitable when the sellers different from i use any behavior strategy profile $(\alpha_k, \beta_k)_{k\neq i}$. In this step, the first stage rewards play a crucial role.

To see this, choose $b = (b_{-i}, b_i)$, $r_i \in B_i$ and let $\pi = \rho(b)$ and $\pi' = \rho(b_{-i}, r)$. For each $j \neq i$, let

$$\sum_{\theta\in\Theta} [\beta_j(\alpha_j(b_j), \pi, b_j)](\theta)\pi(\theta) = \gamma_j(\pi, b_j)$$

and

$$\sum_{\theta\in\Theta} [\beta_j(\alpha_j(b_j), \pi', b_j)](\theta)\pi'(\theta) = \gamma_j(\pi', b_j).$$

Let $(\gamma_j(\pi, b_j))_{j\neq i} = \gamma_{-i}(\pi, b_{-i})$ and $(\gamma_j(\pi', b_j))_{j\neq i} = \gamma_{-i}(\pi', b_{-i})$. We first show that, for all $(w_i(\theta), d_i(\theta))_{\theta\in\Theta}$, we have

$$\begin{aligned} & g_i(\hat{\varphi}(\gamma_{-i}(\pi, b_{-i}), u_i(\pi), c_i(\pi)); u_i(\pi), c_i(\pi)) + \hat{x}_i(\gamma_{-i}(\pi, b_{-i}), u_i(\pi), c_i(\pi)) - \\ & - \max_{(w_i(\theta), d_i(\theta))_{\theta\in\Theta}} [g_i(\hat{\varphi}(\gamma_{-i}(\pi', b_{-i}), w_i(\pi'), d_i(\pi')); u_i(\pi), c_i(\pi)) + \hat{x}_i(\gamma_{-i}(\pi', b_{-i}), w_i(\pi'), d_i(\pi'))] \geq \\ \geq\ & -(1+n)(2n+1)M||\pi - \pi'|| = \\ =\ & -(1+n)(2n+1)M||\rho(b) - \rho(b_{-i}, r_i)||. \end{aligned}$$

Consequently, a first stage deviation will be unprofitable if

$$-(1+n)(2n+1)M\sum_{b_{-i}}||\rho(b)-\rho(b_{-i},r_i)||P_{-i}(b_{-i}|b_i)+\sum_{b_{-i}}[\xi_i(b)-\xi_i(b_{-i},r_i)]P_{-i}(b_{-i}|b_i)>0.$$

To complete the proof , we construct a reward system $(\xi_i)_{i\in N}$ defined by spherical scoring rules: for each b, and i,

$$\xi_i(b)=\delta\frac{P_{-i}(b_{-i}|b_i)}{||P_{-i}(\cdot|b_i)||_2}.$$

Since

$$\sum_{b_{-i}\in B_{-i}}[\xi_i(b)-\xi_i(b_{-i},r_i)]P_{-i}(b_{-i}|b_i))>0$$

we can choose δ so as to ensure that all agents report their first stage signals honestly.

□

4.5 Remarks

Unlike the previous paper, [12], we do not claim that the sequential dominant strategy assessment (α^*,β^*,μ^*), defined in Theorem 1, is a perfect Bayesian equilibrium assessment. Note that for $r_i\neq b_i$, the equilibrium path of play does not pass through the information sets indexed by (r_i,π,b_i) with $\pi\in\Pi_i(r_i)$. There is no difficulty in identifying beliefs at these information sets compatible with Bayes rule and we have computed $\mu_i^*(\theta,b_{-i}|r_i,\pi,b_i)$ above. There is also no difficulty in defining second stage strategies at these information sets and we define $\beta_i^*(r_i,\pi,b_i)=(u_i(\theta),c_i(\theta))_{\theta\in\Theta}$ as above. The difficulty arises when i wishes to compute his best response at one of these unreached information sets. If i knew $(u_j(\theta),c_j(\theta))_{j\neq i}$ (as in [12]), then upon receiving signal b_i and reporting r_i and then observing the posted $\pi\in\Pi_i(r_i)$, buyer i could first compute $(u_j(\pi),c_j(\pi)_{j\neq i})$ and then compute the best response given beliefs at this information set. That is, i could then compute $(w_i(\theta),d_i(\theta))_{\theta\in\Theta}$ that maximizes

$$\sum_{b_{-i}\in B_{-i}}\sum_{\theta\in\Theta}\Bigg[g_i(\hat{\varphi}(\gamma_{-i}(\pi,b_{-i}),w_i(\pi),d_i(\pi));u_i(\theta),c_i(\theta)+ \\ +\hat{x}_i(\gamma_{-i}(\pi,b_{-i}),w_i(\pi),d_i(\pi)))\Bigg]\mu_i(\theta,b_{-i}|r_i,\pi,b_i)$$

for $\pi\in\Pi_i(r_i)$. Note that, even if i could compute $(u_j(\pi),c_j(\pi))_{j\neq i}$, it need not be true that $(u_i(\theta),c_i(\theta))_{\theta\in\Theta}$ is buyer i's best response.

In this paper, we are assuming that i does not know the payoff-externality profile of the other players nor are we assuming that i has beliefs regarding these profiles. Along the equilibrium path of play however, buyer i need not know the true values of $(u_j(\theta),c_j(\theta))_{j\neq i}$. As long as all buyers honestly report their first stage signals, buyer i can determine that he should truthfully report $(u_i(\theta),c_i(\theta))_{\theta\in\Theta}$ in the second stage for any second stage strategies of i's opponents.

5 The Problem for Large n

The rewards that ensure honest first stage reporting by buyers may be quite large so it is useful to identify a model in which the individual stage 1 rewards, or even the sum of these rewards, is small. To that end, we define a special sequence of assignment problems (u^n, c^n, P^n) with interdependent valuations. Assume that there exist a finite set X such that $B_i = X$ for all i. Consequently, we write $B = X^n$ where X^n is the Cartesian product of n copies of X. For each n and each $i \in N = \{1, .., n\}$, let $u^n_{ij}(\theta)$ and $c^{pq,n}_{ij}(\theta)$ denote the valuations and externalities when i is matched with j in state θ. Furthermore, assume that for all n and for all $i, j \in N$, $u^n_{ij}(\theta) \leq M$ and $c^{pq,n}_{ij}(\theta) \leq M$ for some $M > 0$.

Definition 3: A sequence of triples (u^n, c^n, P^n) is a *conditionally independent sequence* of assignment problems with interdependent valuations if the sequence $P^n \in \Delta^*_{\Theta \times X^n}$ satisfies:

(i) There exists $\lambda \in \Delta^*(\Theta)$ and for each $\theta \in \Theta$, there exist a $Q(\cdot|\theta) \in \Delta^*(X)$ such that for each $b \in X^n$,

$$P^n(\theta, b) = \left[\prod_{i=1}^{n} Q(b_i|\theta)\right] \lambda(\theta).$$

(ii) Let $\hat{P}$ denote the common marginal of P^n on $B_i \times B_j (= X \times X)$ for $i \neq j$. For each pair (i, j) and b_i, b'_i in X with $b_i \neq b'_i$, there exists $b_j \in X$ such that $\hat{P}(b_j|b_i) \neq \hat{P}(b_j|b'_i)$.

Theorem 2: Let (u^n, c^n, P^n) be a sequence of conditionally independent matching problems. There exists an $\hat{n}$ such that, for all $n > \hat{n}$, there exists a reward system $\xi^n = (\xi^n_i)_{i \in N}$ such that the two-stage game $\Gamma(u^n, c^n, P^n, \xi^n)$ admits a sequential dominant strategy assessment. Furthermore, for each k we may choose ξ^n so that $\sum_{i=1}^n \xi^n_i(b) \sim O\left(\frac{1}{n^k}\right)$ for every $b \in X^n$.

6 Proof of Theorem 1

6.1 Preparatory Lemmas

Lemma A: Let M be a positive number and let $w_{ij}(\theta), d^{pq}_{ij}(\theta)$ be a collection of numbers satisfying $0 \leq w_{ij}(\theta), d^{pq}_{ij}(\theta) \leq M$ for all i, j, p, q, θ. For each $\pi \in \Delta(\Theta)$, define $w_i(\pi)$ and $d_i(\pi)$ as in Section 1.1. For each $\pi \in \Delta(\Theta)$, let

$$F(\pi) = \max_{z \in Z} \sum_{i=1}^{n} g_i(z; w_i(\pi), d_i(\pi)).$$

Then for each $\pi, \pi' \in \Delta(\Theta)$,

$$|F(\pi) - F(\pi')| \leq (1 + n)nM||\pi - \pi'||.$$

Proof. For each $z \in Z$, let

$$G_\pi(z) = \sum_{i=1}^{n} \left[\sum_{\theta \in \Theta} g_i(z; w_i(\theta), d_i(\theta))\pi(\theta) \right]$$

and let

$$\xi(\pi) \in \arg\max_{z \in Z} G_\pi(z).$$

Then

$$\begin{aligned} F(\pi) - F(\pi') &= G_\pi(\xi(\pi)) - G_{\pi'}(\xi(\pi')) \\ &= G_\pi(\xi(\pi)) - G_{\pi'}(\xi(\pi')) + G_{\pi'}(\xi(\pi)) - G_{\pi'}(\xi(\pi)) \\ &\leq G_\pi(\xi(\pi)) - G_{\pi'}(\xi(\pi)) \\ &= \sum_{\theta \in \Theta} \left[\sum_{i=1}^{n} g_i(\xi(\pi); w_i(\theta), d_i(\theta)) \right] [\pi(\theta) - \pi'(\theta)] \\ &\leq (1+n)nM||\pi - \pi'||. \end{aligned}$$

The result follows by reversing the roles of π and π'. $\square$

Lemma B: Let M be a positive number and let $w_{ij}(\theta), d_{ij}^{pq}(\theta)$ be a collection of numbers satisfying $0 \leq w_{ij}(\theta), d_{ij}^{pq}(\theta) \leq M$ for all i, j, p, q, θ. For each $\pi \in \Delta(\Theta)$, define $w_i(\pi)$ and $d_i(\pi)$ as in Section 3.1. Define

$$\xi(\pi) \in \arg\max_{z \in Z} \left\{ \sum_{i=1}^{n} g_i(z; w_i(\pi), d_i(\pi)) \right\}$$

and VCG transfers $\eta_i(\pi)$ for each buyer, i.e.,

$$\eta_i(\pi) = \sum_{\substack{k \\ :k \neq i}} g_k(\xi(\pi); w_k(\pi), d_k(\pi)) - \max_{z \in Z_{-i}} \sum_{\substack{k \\ :k \neq i}} g_k(z; w_k(\pi), d_k(\pi)).$$

Then for all $\pi, \pi' \in \Delta(\Theta)$,

$$\left| g_i(\xi(\pi'); w_i(\pi'), d_i(\pi')) + \eta_i(\pi') - [g_i(\xi(\pi); w_i(\pi), d_i(\pi)) + \eta_i(\pi)] \right| \leq 2(1+n)nM||\pi - \pi'||.$$

Proof. Applying Lemma A, it follows that

$$\begin{aligned} &g_i(\xi(\pi'); w_i(\pi'), d_i(\pi')) + \eta_i(\pi') - [g_i(\xi(\pi); w_i(\pi), d_i(\pi)) + \eta_i(\pi)] = \\ &= \max_{z \in Z} \left[\sum_{i=1}^{n} g_i(z; w_i(\pi'), d_i(\pi')) \right] - \max_{z \in Z} \left[\sum_{i=1}^{n} g_i(z; w_i(\pi), d_i(\pi)) \right] + \end{aligned}$$

$$+\max_{z\in Z_{-i}}\left[\sum_{\substack{k\\:k\neq i}} g_k(z;w_k(\pi),d_k(\pi))\right]-\max_{z\in Z_{-i}}\left[\sum_{\substack{k\\:k\neq i}} g_k(z;w_k(\pi'),d_k(\pi'))\right]\leq$$
$$\leq (1+n)nM||\pi-\pi'||+(1+n)nM||\pi-\pi'||$$
$$= 2(1+n)nM||\pi-\pi'||.$$

□

6.2 Proof of Theorem 1

Let $\alpha_i^*(b_i)=b_i$ for each i and b_i and recall that β_i^* is defined for buyer i as follows: $\beta_i(r_i,\pi,b_i)=(u_i(\pi),c_i(\pi))$ for each (r_i,π,b_i) with $r_i\in B_i.\pi\in\Pi_i(r_i)$, and $b_i\in B_i$.

Define beliefs $\mu_i^*(\cdot|r_i,\pi,b_i)\in\Delta(\Theta\times B_{-i})$ for agent i at each information set (r_i,π,b_i) as in Section 2.4 so that, along the equilibrium path of play, we have

$$\mu_i(\theta,b_{-i}|b_i,\pi,b_i)=\frac{P_\Theta(\theta|b_{-i},b_i)P_{-i}(b_{-i}|b_i)}{\sum_{\hat b_{-i}:\rho(\hat b_{-i},b_i)=\pi}P_{-i}(\hat b_{-i}|b_i)}\text{ if }\rho(b_{-i},b_i)=\pi$$
$$=0\text{ otherwise.}$$

Next, let ξ_i be first stage rewards defined by spherical scoring rules: for each b and i, let

$$\xi_i(b_{-i},b_i)=\delta\frac{P_{-i}(b_{-i}|b_i)}{||P_{-i}(\cdot|b_i)||_2}.$$

We will show that δ can be chosen so that $((\alpha^*,\beta^*),\mu^*)$ is a sequential dominant strategy assessment in the game $\Gamma(u,c,P,\xi)$.

Part 1: The strategy profile (α^*,β^*) is truthful. To show that part (ii) of the definition of sequential dominant strategy assessment is satisfied, suppose that $(\beta_j)_{j\neq i}$ is a collection of second stage strategies of i's opponents. We first lighten the notation and for each $b_{-i}\in B_{-i}$ and each $k\neq i$, will write

$$\sum_\theta[\beta_k(b_k,\pi,b_k)](\theta)\pi(\theta)=\gamma_k(\pi,b_k)$$

and

$$(\gamma_j(\pi,b_j))_{j\neq i}=\gamma_{-i}(\pi,b_{-i}).$$

We are assuming that all buyers are truthful in stage 1 and we must show that for each $b_i\in B_i$ and $\pi\in\Pi(b_i)$ and each $(w_i(\theta),d_i(\theta))_{\theta\in\Theta}$, we have

$$\sum_{b_{-i}\in B_{-i}}\sum_{\theta\in\Theta}\left[g_i(\hat\varphi(\gamma_{-i}(\pi,b_{-i}),u_i(\pi),c_i(\pi));u_i(\theta),c_i(\theta))+\hat x_i(\gamma_{-i}(\pi,b_{-i}),u_i(\pi),c_i(\pi)))\right]\mu_i(\theta,b_{-i}|b_i,\pi,b_i)\geq$$
$$\geq\sum_{b_{-i}\in B_{-i}}\sum_{\theta\in\Theta}\left[g_i(\hat\varphi(\gamma_{-i}(\pi,b_{-i}),w_i(\pi),d_i(\pi));u_i(\theta),c_i(\theta))+\hat x_i(\gamma_{-i}(\pi,b_{-i}),w_i(\pi),d_i(\pi))\right]\mu_i(\theta,b_{-i}|b_i,\pi,b_i).$$

To see this, first note that the dominant strategy property of the mechanism implies that for each π and b_{-i},

$$\begin{aligned} & g_i(\hat{\varphi}(\gamma_{-i}(\pi, b_{-i}), w_i(\pi), d_i(\pi)); u_i(\pi), c_i(\pi)) + x_i(\gamma_{-i}(\pi, b_{-i}), w_i(\pi), d_i(\pi)) \leq \\ \leq\ & g_i(\hat{\varphi}(\gamma_{-i}(\pi, b_{-i}), u_i(\pi), c_i(\pi)); u_i(\pi), c_i(\pi)) + x_i(\gamma_{-i}(\pi, b_{-i}), u_i(\pi), c_i(\pi)). \end{aligned}$$

Consequently,

$$\begin{aligned} & \sum_{b_{-i} \in B_{-i}} \sum_{\theta \in \Theta} \Bigg[g_i(\hat{\varphi}(\gamma_{-i}(\pi, b_{-i}), w_i(\pi), d_i(\pi)); u_i(\theta), c_i(\theta)) + \\ & \qquad + \hat{x}_i(\gamma_{-i}(\pi, b_{-i}), w_i(\pi), d_i(\pi)) \Bigg] \mu_i(\theta, b_{-i} | b_i, \pi, b_i) \\ = & \sum_{\substack{b_{-i} \in B_{-i} \\ : \rho(b_{-i}, b_i) = \pi}} \sum_{\theta \in \Theta} \Bigg[g_i(\hat{\varphi}(\gamma_{-i}(\pi, b_{-i}), w_i(\pi), d_i(\pi)); u_i(\theta), c_i(\theta)) + \\ & \qquad + \hat{x}_i(\gamma_{-i}(\pi, b_{-i}), w_i(\pi), d_i(\pi)) \Bigg] \frac{P_\Theta(\theta | b_{-i}, b_i) P_{-i}(b_{-i} | b_i)}{\sum\limits_{\substack{\hat{b}_{-i} \in B_{-i} \\ : \rho(\hat{b}_{-i}, b_i) = \pi}} P_{-i}(\hat{b}_{-i} b_i)} = \\ = & \sum_{\substack{b_{-i} \in B_{-i} \\ : \rho(b_{-i}, b_i) = \pi}} \Bigg[\sum_{\theta \in \Theta} g_i(\hat{\varphi}(\gamma_{-i}(\pi, b_{-i}), w_i(\pi), d_i(\pi)); u_i(\theta), c_i(\theta)) P_\Theta(\theta | b_{-i}, b_i) + \\ & \qquad + \hat{x}_i(\gamma_{-i}(\pi, b_{-i}), w_i(\pi), d_i(\pi)) \Bigg] \frac{P_{-i}(b_{-i} | b_i)}{\sum\limits_{\substack{\hat{b}_{-i} \in B_{-i} \\ : \rho(\hat{b}_{-i}, b_i) = \pi}} P_{-i}(\hat{b}_{-i} b_i)} = \\ = & \sum_{\substack{b_{-i} \in B_{-i} \\ : \rho(b_{-i}, b_i) = \pi}} \Bigg[g_i(\hat{\varphi}(\gamma_{-i}(\pi, b_{-i}), w_i(\pi), d_i(\pi)); u_i(\pi), c_i(\pi)) + \\ & \qquad + \hat{x}_i(\gamma_{-i}(\pi, b_{-i}), w_i(\pi), d_i(\pi)) \Bigg] \frac{P_{-i}(b_{-i} | b_i)}{\sum\limits_{\substack{\hat{b}_{-i} \in B_{-i} \\ : \rho(\hat{b}_{-i}, b_i) = \pi}} P_{-i}(\hat{b}_{-i} b_i)} \leq \\ \leq & \sum_{\substack{b_{-i} \in B_{-i} \\ : \rho(b_{-i}, b_i) = \pi}} \Bigg[g_i(\hat{\varphi}(\gamma_{-i}(\pi, b_{-i}), u_i(\pi), c_i(\pi)); u_i(\pi), c_i(\pi)) + \end{aligned}$$

$$+ \hat{x}_i(\gamma_{-i}(\pi, b_{-i}), u_i(\pi), c_i(\pi))\Bigg] \frac{P_{-i}(b_{-i}|b_i)}{\sum\limits_{\substack{\hat{b}_{-i} \in B_{-i} \\ :\rho(\hat{b}_{-i}, b_i) = \pi}} P_{-i}(\hat{b}_{-i} b_i)} =$$

$$= \sum_{b_{-i} \in B_{-i}} \sum_{\theta \in \Theta} \Bigg[g_i(\hat{\varphi}(\gamma_{-i}(\pi, b_{-i}), u_i(\pi), c_i(\pi); u_i(\theta), c_i(\theta)) + \hat{x}_i(\gamma_{-i}(\pi, b_{-i}), u_i(\pi), c_i(\pi)) \Bigg] \mu_i(\theta, b_{-i}|b_i, \pi, b_i),$$

where the penultimate inequality follows from the dominant strategy property of the VCG mechanism.

Part 2: To complete the proof, we must show that first stage rewards ξ_i can be chosen so that coordinated deviations across stages are unprofitable for buyer i given any behavior strategy profile $(\alpha_{-i}, \beta_{-i})$ of the other players. To see this, choose $b = (b_{-i}, b_i)$, $r_i \in B_i$ and let $\pi = \rho(b)$ and $\pi' = \rho(b_{-i}, r)$. For each $j \neq i$ and θ, let

$$\sum_{\theta \in \Theta} [\beta_j(\alpha_j(b_j), \pi, b_j)](\theta)\pi(\theta) = \gamma_j(\pi, b_j)$$

and

$$\sum_{\theta \in \Theta} [\beta_j(\alpha_j(b_j), \pi', b_j)](\theta)\pi'(\theta) = \gamma_j(\pi', b_j).$$

Let $(\gamma_j(\pi, b_j))_{j \neq i} = \gamma_{-i}(\pi, b_{-i})$ and $(\gamma_j(\pi', b_j))_{j \neq i} = \gamma_{-i}(\pi', b_{-i})$. We claim that, for all $(w_i(\theta), d_i(\theta))_{\theta \in \Theta}$, we have

$$\begin{aligned} & g_i(\hat{\varphi}(\gamma_{-i}(\pi, b_{-i}), u_i(\pi), c_i(\pi)); u_i(\pi), c_i(\pi)) + \hat{x}_i(\gamma_{-i}(\pi, b_{-i}), u_i(\pi), c_i(\pi)) - \\ & - \Bigg[g_i(\hat{\varphi}(\gamma_{-i}(\pi', b_{-i}), w_i(\pi'), d_i(\pi')); u_i(\pi), c_i(\pi)) + \hat{x}_i(\gamma_{-i}(\pi', b_{-i}), w_i(\pi'), d_i(\pi')) \Bigg] \geq \\ \geq\ & -(1+n)(2n+1)M||\pi - \pi'||. \end{aligned}$$

First, note that

$$\begin{aligned} & \left| g_i(\hat{\varphi}(\gamma_{-i}(\pi', b_{-i}), w_i(\pi'), d_i(\pi')); u_i(\pi'), c_i(\pi')) - \Big[g_i(\hat{\varphi}(\gamma_{-i}(\pi', b_{-i}), w_i(\pi'), d_i(\pi')); u_i(\pi), c_i(\pi)) \Big] \right| = \\ =\ & \left| \sum_{\theta} g_i(\hat{\varphi}(\gamma_{-i}(\pi', b_{-i}), w_i(\pi'), d_i(\pi')); u_i(\theta), c_i(\theta))[\pi'(\theta) - \pi(\theta)] \right| \leq \\ \leq\ & (1+n)M||\pi' - \pi|| \end{aligned}$$

Since the dominant strategy property of the VCG mechanism implies that

$$\begin{aligned} & g_i(\hat{\varphi}(\gamma_{-i}(\pi', b_{-i}), u_i(\pi'), c_i(\pi')); u_i(\pi'), c_i(\pi')) + \hat{x}_i(\gamma_{-i}(\pi', b_{-i}), u_i(\pi'), c_i(\pi')) \geq \\ \geq\ & g_i(\hat{\varphi}(\gamma_{-i}(\pi', b_{-i}), w_i(\pi'), d_i(\pi'); u_i(\pi'), c_i(\pi')) + \hat{x}_i(\gamma_{-i}(\pi', b_{-i}), w_i(\pi'), d_i(\pi')) \end{aligned}$$

it then follows from Lemmas A and B that

$$
\begin{aligned}
& g_i(\hat{\varphi}(\gamma_{-i}(\pi, b_{-i}), u_i(\pi), c_i(\pi)); u_i(\pi), c_i(\pi)) + \hat{x}_i(\gamma_{-i}(\pi, b_{-i}), u_i(\pi), c_i(\pi)) - \\
& -[g_i(\hat{\varphi}(\gamma_{-i}(\pi', b_{-i}), w_i(\pi'), d_i(\pi')); u_i(\pi), c_i(\pi)) + \hat{x}_i(\gamma_{-i}(\pi', b_{-i}), w_i(\pi'), d_i(\pi'))] = \\
= \; & g_i(\hat{\varphi}(\gamma_{-i}(\pi, b_{-i}), u_i(\pi), c_i(\pi)); u_i(\pi), c_i(\pi)) + \hat{x}_i(\gamma_{-i}(\pi, b_{-i}), u_i(\pi), c_i(\pi)) - \\
& - \Big[g_i(\hat{\varphi}(\gamma_{-i}(\pi', b_{-i}), u_i(\pi'), c_i(\pi')); u_i(\pi'), c_i(\pi')) + \hat{x}_i(\gamma_{-i}(\pi', b_{-i}), u_i(\pi'), c_i(\pi')) \Big] + \\
& + g_i(\hat{\varphi}(\gamma_{-i}(\pi', b_{-i}), u_i(\pi'), c_i(\pi')); u_i(\pi'), c_i(\pi')) + \hat{x}_i(\gamma_{-i}(\pi', b_{-i}), u_i(\pi'), c_i(\pi')) - \\
& - \Big[g_i(\hat{\varphi}(\gamma_{-i}(\pi', b_{-i}), w_i(\pi'), d_i(\pi')); u_i(\pi'), c_i(\pi')) + \hat{x}_i(\gamma_{-i}(\pi', b_{-i}), w_i(\pi'), d_i(\pi')) \Big] + \\
& + g_i(\hat{\varphi}(\gamma_{-i}(\pi', b_{-i}), w_i(\pi'), d_i(\pi')); u_i(\pi'), c_i(\pi')) - \\
& - g_i(\hat{\varphi}(\gamma_{-i}(\pi', b_{-i}), w_i(\pi'), d_i(\pi')); u_i(\pi), c_i(\pi)) \geq \\
\geq \; & - \Big[2(1+n)nM||\pi - \pi'|| + (1+n)M||\pi' - \pi|| \Big] = \\
= \; & -(1+n)(2n+1)M||\pi - \pi'||.
\end{aligned}
$$

Part 3: To complete the proof, we must show that the first stage rewards ξ_i can be chosen so that for all $r_i \in B_i$ and all $(w_i(\theta), d_i(\theta))_{\theta \in \Theta}$, we have

$$
\begin{aligned}
& \sum_{b_{-i} \in B_{-i}} \Big[g_i(\hat{\varphi}(\gamma_{-i}(\rho(b), b_{-i}), u_i(\rho(b)), c_i(\rho(b)); u_i(\rho(b)), c_i(\rho(b))) + \\
& + \hat{x}_i(\gamma_{-i}(\rho(b), b_{-i}), u_i(\rho(b)), c_i(\rho(b))) + \xi_i(b) \Big] P(b_{-i}|b_i) \geq \\
\geq \; & \sum_{b_{-i} \in B_{-i}} \Big[g_i(\hat{\varphi}(\gamma_{-i}(\rho(b_{-i}, r_i), b_{-i}), w_i(\rho(b_{-i}, r_i)), d_i(\rho(b_{-i}, r_i)); u_i(\rho(b)), c_i(\rho(b))) + \\
& + \hat{x}_i(\gamma_{-i}(\rho(b_{-i}, r_i), b_{-i}), w_i(\rho(b_{-i}, r_i)), d_i(\rho(b_{-i}, r_i))) + \xi_i(b_{-i}, r_i) \Big] P(b_{-i}|b_i)
\end{aligned}
$$

If $\delta > 0$ and

$$\xi_i(b_{-i}, b_i) = \delta \frac{P_{-i}(b_{-i}|b_i)}{||P_{-i}(\cdot|b_i)||_2}$$

then

$$\sum_{b_{-i}} [\xi_i(b_{-i}, b_i) - \xi_i(b_{-i}, r_i)] P_{-i}(b_{-i}|b_i) > 0$$

whenever $b_i \neq r_i$ (recall that $P_{-i}(\cdot|b_i) \neq P_{-i}(\cdot|r_i)$ whenever $b_i \neq r_i$). Therefore, we can choose $\delta > 0$ so that

$$\sum_{b_{-i}\in B_{-i}}\Bigg[g_i(\hat{\varphi}(\beta_j(\alpha_j(b_j),\rho(b),b_j)_{j\neq i},u_i(\rho(b)),c_i(\rho(b)));u_i(\rho(b)),c_i(\rho(b)))+$$

$$+\hat{x}_i(\beta_j(\alpha_j(b_j),\rho(b),b_j)_{j\neq i},u_i(\rho(b)),c_i(\rho(b)))+\xi_i(b)\Bigg]P(b_{-i}|b_i)-$$

$$-\Bigg[\sum_{b_{-i}\in B_{-i}}[g_i(\hat{\varphi}(\beta_j(\alpha_j(b_j),\rho(b_{-i},r_i),b_j)_{j\neq i},w_i,d_i);u_i(\rho(b)),c_i(\rho(b)))+$$

$$+\hat{x}_i(\beta_j(\alpha_j(b_j),\rho(b_{-i},r_i),b_j)_{j\neq i},w_i,d_i)+\xi_i(b_{-i},r_i)]P(b_{-i}|b_i)\Bigg]\geq$$

$$\geq -(1+n)(2n+1)M\sum_{b_{-i}}||\rho(b)-\rho(b_{-i},r_i)||P_{-i}(b_{-i}|b_i)+\sum_{b_{-i}}[\xi_i(b)-\xi_i(b_{-i},r_i)]P_{-i}(b_{-i}|b_i)$$

$$> 0.$$

This completes the proof of Theorem 1.

7 Proof of Theorem 2

7.1 Preliminaries

The proof of Theorem 2 is a close adaptation of the proof of Theorem 3 in [14] so we omit most of the details. As in that proof, we will treat $P^n \in \Delta^*_{\Theta\times X^n}$ as the distribution of a $(\Theta \times X^n)$-valued random variable which we denote $(\tilde{\theta},\tilde{b})$, i.e.,

$$\mathrm{Prob}\{(\tilde{\theta},\tilde{b})=(\theta,b)\}=P^n(\theta,b)$$

for each $(\theta,b)\in\Theta\times X^n$. For each n and each i and j, define

$$\nu_i^n=\max_{b_i\in X}\max_{r_i\in X}\min\{\varepsilon\geq 0|||\mathrm{Prob}\{P^n_\Theta(\cdot|\tilde{b}_{-i},b_i)-P^n_\Theta(\cdot|\tilde{b}_{-i},r_i)||>\varepsilon|\tilde{b}_i=b_i\}\leq\varepsilon\}.$$

From the proof of Lemma B in [14] we conclude that $\nu_i^n \sim O\left(\frac{1}{n^k}\right)$ for every positive integer k.

Next, suppose that $b=(b_{-i},b_i)\in X^n$ and $r_i\in X$. Define $\rho^n(b)=P^n_\Theta(\cdot|b)$ and $\rho^n(b_{-i},r_i)=P^n_\Theta(\cdot|b_{-i},r_i)$. Let M be the bound defined in the statement of Theorem 2. From step 2 in the proof of Theorem 1, it follows that

$$g_i(\hat{\varphi}(\gamma_{-i}(\rho^n(b),b_{-i}),u_i^n(\rho^n(b)),c_i^n(\rho^n(b)));u_i^n(\rho^n(b)),c_i^n(\rho^n(b)))+$$
$$+\hat{x}_i(\gamma_{-i}(\rho^n(b),b_{-i}),u_i^n(\rho^n(b)),c_i^n(\rho^n(b)))-$$

$$-\Big[g_i(\hat{\varphi}(\gamma_{-i}(\rho^n(b_{-i},r_i),b_{-i}),w_i(\rho^n(b_{-i},r_i)),d_i(\rho^n(b_{-i},r_i)));u_i^n(\rho^n(b)),c_i^n(\rho^n(b)))+$$

$$+\hat{x}_i(\gamma_{-i}(\rho^n(b_{-i},r_i),b_{-i}),w_i(\rho^n(b_{-i},r_i)),d_i(\rho^n(b_{-i},r_i)))\Big]\geq -(1+n)(2n+1)M||\rho^n(b)-\rho^n(b_{-i},r_i)||.$$

To prove Theorem 2, it suffices to show that we can find for each k first stage rewards ξ_i such that

$$n^k\sum_{i=1}^n \xi_i(b)\underset{n\to\infty}{\to} 0$$

and

$$-(1+n)(2n+1)M\sum_{b_{-i}\in X^{n-1}}||\rho^n(b)-\rho^n(b_{-i},r_i)||P^n_{-i}(b_{-i}|b_i)+\sum_{b_{-i}\in X^{n-1}}[\xi_i(b)-\xi_i(b_{-i},r_i)]P^n_{-i}(b_{-i}|b_i)>0$$

for all sufficiently large n.

Choose $k>1$ and for each $b=(b_1,..,b_n)\in X^n$, define

$$\begin{aligned}\xi_i(b_{-i},b_i) &= \frac{1}{n^{k+2}}\frac{\hat{P}(b_{i+1}|b_i)}{||\hat{P}(\cdot|b_i)||_2} \text{ if } i=1,..,n-1\\ &= \frac{1}{n^{k+2}}\frac{\hat{P}(b_1|b_i)}{||\hat{P}(\cdot|b_i)||_2} \text{ if } i=n\end{aligned}$$

where $\hat{P}$ is defined as in Theorem 2. Therefore,

$$0\leq\sum_{i=1}^n\xi_i(b_{-i},b_i)\leq\frac{1}{n^{k+1}}$$

for all i, b_{-i} and b_i.

Let $|X|$ denote the cardinality of X and let $K=\frac{|X|^{-\frac{5}{2}}}{4(|X|-1)}$. Applying Lemma A.3 in [15], we conclude that

$$\begin{aligned}&\sum_{b_{-i}}\left(\xi_i(b)-\xi_i(b_{-i},r_i)\right)P^n_{-i}(b_{-i}|b_i)\\ =&\sum_{b_{-i}}\frac{1}{n^{k+2}}\left[\frac{\hat{P}(b_{i+1}|b_i)}{||\hat{P}^n(\cdot|b_i)||_2}-\frac{\hat{P}(b_{i+1}|r_i)}{||\hat{P}^n(\cdot|r_i)||_2}\right]P^n_{-i}(b_{-i},|b_i)\\ =&\sum_{x\in X}\frac{1}{n^{k+2}}\left[\frac{\hat{P}(x|b_i)}{||\hat{P}(\cdot|b_i)||_2}-\frac{\hat{P}(x|r_i)}{||\hat{P}(\cdot|r_i)||_2}\right]\hat{P}^n(x|b_i)\\ >&\ \frac{1}{n^{k+2}}K(||\hat{P}(\cdot|b_i)-\hat{P}(\cdot|r_i)||)^2.\end{aligned}$$

To complete the argument, define

$$A_i(r_i, b_i) = \{b_{-i} \in B_{-i} |\ ||P^n_\Theta(\cdot|b_{-i}, b_i) - P^n_\Theta(\cdot|b_{-i}, r_i)|| > \hat{\nu}^n_i\}$$

and

$$C_i(r_i, b_i) = \{b_{-i} \in B_{-i} |\ ||P^n_\Theta(\cdot|b_{-i}, b_i) - P^n_\Theta(\cdot|b_{-i}, r_i)|| \leq \hat{\nu}^n_i\}.$$

Since

$$\text{Prob}\{\tilde{b}_{-i} \in A_i(r_i, b_i)|\tilde{b}_i = b_i\} \leq \nu^n$$

we conclude that

$$\begin{aligned}
&\sum_{b_{-i}, \in X^{n-1}} ||\rho^n(b) - \rho^n(b_{-i}, r_i)||P^n(b_{-i}|b_i) \\
= &\sum_{b_{-i} \in A_i(r_i, b_i)} ||\rho^n(b) - \rho^n(b_{-i}, r_i)||P^n(b_{-i}|b_i) + \sum_{b_{-i} \in C_i(r_i, b_i)} ||\rho^n(b) - \rho^n(b_{-i}, r)||P^n(b_{-i}|b_i) \\
\leq &\ 2\nu^n
\end{aligned}$$

Therefore, for all sufficiently large n we have

$$\begin{aligned}
&-(1+n)(2n+1)M \sum_{b_{-i}} ||\rho^n(b) - \rho^n(b_{-i}, r_i)||P^n_{-i}(b_{-i}|b_i) + \sum_{b_{-i}} [\xi_i(b) - \xi_i(b_{-i}, r_i)]P^n_{-i}(b_{-i}|b_i) \\
\geq &-(1+n)(2n+1)M2\nu^n + \frac{1}{n^{k+2}} K(||\hat{P}(\cdot|b_i) - \hat{P}(\cdot|r_i)||)^2 \\
= &\ \frac{1}{n^{k+2}} \left[K(||\hat{P}(\cdot|b_i) - \hat{P}(\cdot|r_i)||)^2 - n^{k+2}(1+n)(2n+1)M2\nu^n\right] \\
> &\ 0.
\end{aligned}$$

8 Discussion

1. We have assumed that the valuations u_i and externality costs c_i of buyer i are functions only of the unobserved state θ. At the cost of more complex notation, we could allow these functions to depend also on the signal b_i.

2. In our model, buyers receive private signals that are correlated with the state θ. Furthermore, the valuations $(u_i(\theta))_{\theta \in \Theta}$ and externalities $(c_i(\theta))_{\theta \in \Theta}$ of each buyer i are also that buyer's private information. As a result, sellers are not strategic actors in our model. We could allow each seller j to also receive a private signal correlated with the state. Consequently, the mechanism would now want to elicit the privately observed signals of both buyers and sellers in order to compute an efficient allocation. If we retain the assumption that the surplus generated by each pairing in an optimal assignment goes to the buyer (as we do in this paper) and assume that the sellers' costs (but not their signals) are known to all agents and the mechanism, then we can find first-stage rewards for both buyers and sellers that induce honest reporting in the first stage.

3. Our two-stage formulation allows us to resolve the interdependency resulting from private signals so that the second stage problem is reduced to a simpler implementation problem to which the classic private values VCG transfers can be applied to elicit the buyers' valuations and externalities. Two natural extensions of the model would pose challenges.

   For example, we could allow the surplus generated by each pairing in an optimal assignment to go to the buyer (as we do in this paper) but assume that the sellers' valuations, as well as buyers' valuations, to be private information. The mechanism would then have to elicit the valuations of buyers and the costs of sellers. In this case, the second stage following the announcement of first stage signals would no longer be an implementation problem with private values since the payoff of buyer i depends on the valuation of the seller with whom he is matched and that valuation is private information of the seller. Consequently, the classical private values VCG transfers do not typically provide incentives for honest reporting.

4. In this paper we have made weak assumptions concerning what players and the mechanism know regarding the data of the game. All agents and the mechanism know the probability measure $P \in \Delta^*(\Theta \times B)$, the sellers' costs $(v_i(\theta))_{\theta\in\Theta}$, and the bound M on valuations, externalities and costs. However, buyer i's valuations and externality parameters $(u_i(\theta), c_i(\theta))_{\theta\in\Theta}$ are known only to i. As we have emphasized, we do not take a Bayesian viewpoint and propose beliefs for i regarding the valuations and externalities of other buyers. There are drawbacks and advantages to these assumptions. Our model is arguably best tailored to situations in which n is large for two reasons. First, it is the case of large n where the assumption of common knowledge of valuations and externalities seems least plausible. Second, as per Theorem 2, we can find for every k in the conditionally independent framework a sequence of first-period rewards $(\xi_i^n)_{i=1}^n$ such that $n^k \sum_{i=1}^n \xi_i^n(b) \to 0$ as $n \to \infty$. Consequently, we obtain a sequential dominant strategy assessment with asymptotically negligible aggregate first period payments under a weak informational assumption. There is a cost associated with this weak informational assumption, with or without large numbers. At second stage information sets through which the equilibrium path does not pass, we cannot determine if our proposed equilibrium assessment of Theorem 1 is sequentially rational. If valuations and externalities are common knowledge and signals provide the only source of asymmetric information (as in [12]), then we can in that case propose a truthful equilibrium that is sequentially rational at every information set thus yielding a Perfect Bayesian equilibrium.

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