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Stochastic Impatience and the Separation of Time and Risk Preferences

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Stochastic Impatience and the Separation of Time and Risk Preferences*

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Abstract

We study how the separation of time and risk preferences relates to a behavioral property that generalizes impatience to stochastic environments: Stochastic Impatience. We show that, within a broad class of models, Stochastic Impatience holds if and only if risk aversion is not too high relative to the inverse of the elasticity of intertemporal substitution. In particular, in the models of Epstein and Zin (1989) and Hansen and Sargent (1995), Stochastic Impatience is violated for all commonly used parameters.

Key words: Stochastic Impatience, Epstein-Zin preferences, Separation of Time and Risk preferences, Risk Sensitive preferences, Non-Expected Utility

JEL: D81, D90, G11, E7.

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1 Introduction

In the standard Expected Discounted Utility model, the inverse of the elasticity of intertemporal substitution (EIS) coincides with the coefficient of relative risk aversion. However, an enormous literature in macroeconomics, finance, and behavioral economics has pointed to the need to separate these two coefficients both on empirical and on conceptual grounds. Empirically, observations from lab experiments, longitudinal microdata, and the desire to fit macroeconomic and financial data require a higher coefficient of risk aversion than the inverse of EIS.[1] Conceptually, attitudes towards risk and towards intertemporal smoothing belong to different domains, and there is no compelling reason why they must be equal. These observations led to models that separate risk attitudes from EIS: prominent examples include the CRRA-CES version of Epstein and Zin (1989) (henceforth EZ) and the Risk Sensitive preferences of Hansen and Sargent (1995) (henceforth HS). Given the fundamental role of risk aversion and EIS in economics, evaluating these models qualitatively and quantitatively is an issue of primary relevance.

In this paper, we show that a behavioral postulate we call *Stochastic Impatience* imposes a bound on how high risk aversion can be relative to EIS. Consider the choice between:

A. With equal probability, permanently increase consumption by either 20% starting today or by 10% starting next year;

B. With equal probability, permanently increase consumption by either 10% starting today or by 20% starting next year.

Both options involve identical benefits, odds, and dates. However, option A pairs the highest increase (20%) with the earlier date, whereas B pairs it with the later date. What would, or should, an individual choose?

To the extent that an individual prefers higher payments sooner, she may choose option A. One way to see this is by decomposing each alternative into two parts. Both A and B offer a basic lottery with an increase of 10% either today or next year, as well as a 50-50 chance of an additional increase of 10%. The difference is when this additional increase is made: today in option A, next year in option

[1]For example, Barsky et al. (1997) study a cross section of American households and find that risk aversion and EIS are uncorrelated. See Bansal and Yaron (2004); Hansen et al. (2007); Barro (2009); Andreoni and Sprenger (2012); Nakamura et al. (2017) and references therein.

B. An individual who prefers obtaining a payment sooner would prefer option A. This property is a version of impatience (preference for earlier payments) for risky environments. Impatience and Stochastic Impatience are equivalent under Expected Discounted Utility, but not when time and risk preferences are separate. Our main result shows that under general conditions, Stochastic Impatience is violated whenever risk aversion is high enough for a fixed EIS.

We first consider the widely used CRRA-CES version of EZ. We show that Stochastic Impatience fails if the coefficient of risk aversion exceeds both the inverse of EIS and one. All applications of EZ that we are aware of use parameters in this range: assuming risk aversion above the inverse of EIS is the main reason to use EZ in the first place. For example, with the parameter values used by Bansal and Yaron (2004), the individual would prefer option B in the example above, violating Stochastic Impatience. For HS preferences, we show that Stochastic Impatience always fails if the range of utilities of consumption is large enough. For example, with the parameters of Tallarini Jr (2000), option B is again chosen.

We then move beyond EZ and HS, establishing a more general result. Consider any preference relation over lotteries over consumption streams.[2] Assume that (i) without risk, preferences coincides with Discounted Utility and admit a representation $\sum D(t)u(x(t))$ for a decreasing D and an increasing u (note that little is assumed about D); and (ii) with risk, preferences satisfy the Expected Utility postulates. These assumptions hold for the vast majority of models, including EZ and HS.

We observe that, in the space we consider, any preference relation that satisfies these two assumptions can be represented by the expectation of $\phi\big(\sum D(t)u(x(t))\big)$, where ϕ is some increasing function over discounted utils. Known as the Kihlstrom-Mirman (KM) representation, this model is similar to Expected Discounted Utility except for the additional curvature ϕ. Since ϕ affects risk aversion but not EIS, it captures the separation between the two. The KM representation thus gives a convenient way to discuss the gap between time and risk attitudes.[3]

Our general result shows that Stochastic Impatience imposes a bound on how

[2] Instead of considering dynamic preferences over temporal lotteries, as in EZ, it suffices to consider their static implications—how they evaluate lotteries over consumption streams at a given point in time and for a given date of resolution of uncertainty. This contains all the information on the separation between time and risk preferences.

[3] The preferences implied in our setup by EZ and HS admit a KM representation with convenient functional forms (ϕ being CRRA or CARA, respectively). This *does not* mean that EZ and HS are special cases of KM—this is not the case in the full setup of temporal lotteries.

high risk aversion can be holding EIS fixed. If ϕ is more concave than the log, we can always construct a violation of Stochastic Impatience. Conversely, Stochastic Impatience holds if ϕ is less concave than the log. This result unifies the findings described previously, as they hold in any model that satisfies our two basic assumptions.[4] These results are expressed both in terms of properties of the representation and its behavioral counterpart, using the notion of *Residual Risk Aversion*.

There are two implications of our results. First, if Stochastic Impatience is taken as an appealing property, our results highlight issues with current modeling, including all common parametrizations of leading approaches. In Section 5 we discuss possible ways to maintain Stochastic Impatience without sacrificing the fit of empirical data. Second, our results provide simple empirical tests. To verify that risk aversion exceeds the inverse of EIS, it is sufficient to document a violation of Stochastic Impatience. This is a more direct test than existing ones, which involve estimating the two parameters separately using multiple questions and assuming specific functional forms.

We are not the first to point out implications of how the separation between time and risk is modeled. Epstein et al. (2014) argue that common parameterizations of EZ imply unrealistic preferences for early resolution of uncertainty. We show that they also violate Stochastic Impatience, a property distinct from preference over the timing of resolution of uncertainty. Bommier et al. (2017) show that many models that separate time and risk preferences, including common specifications of EZ, violate a monotonicity property. The latter is unrelated to Stochastic Impatience: for example, EZ with both risk aversion and the inverse of EIS less than 1 satisfies Stochastic Impatience but not Monotonicity; conversely, HS always satisfies Monotonicity but violates Stochastic Impatience when the utility range is large enough. Lastly, a companion paper, Dejarnette et al. (2020), studies theoretically and experimentally risk attitudes towards time lotteries, including their relationship with Stochastic Impatience.[5]

[4]In Appendix A we provide an extension to continuous time, showing that equivalent results hold, and to non-Expected Utility, where we show that First-Order Risk Aversion implies violations of Stochastic Impatience.

[5]Dejarnette et al. (2020) shows in an experiment that the vast majority of individuals are not risk seeking over time lotteries, as implied by EDU; but also shows that this is incompatible with Stochastic Impatience within a broad class of models and suggests generalizations to accommodate both.

2 Framework

We study preferences on lotteries over consumption streams. Consider an interval of per-period consumption $C \subset \mathbb{R}_+$ and a set of dates $T = \{1, \ldots, \bar{t}\}$, where $\bar{t}$ is finite or infinite.[6] A consumption program $\mathbf{x} = (x(1), x(2), \ldots, x(\bar{t}))$ yields consumption $x(t) \in C$ in period $t \in T$. Let $\mathcal{X} = C^T$ be the set of consumption programs and Δ be the set of all probability measures over $\mathcal{X}$ with finite support. Let $\succcurlyeq$ be a complete and transitive preference relation over Δ.

Abusing notation, $\mathbf{x} \in \mathcal{X}$ refers both to the consumption program and the lottery that gives it with certainty; $x \in C$ denotes both the consumption x and the constant stream that gives it in every period; and for, any $t > 1$, $(c, t, x) \in C \times T \times C$ denotes the stream that gives c in every period until $t-1$ and x from t onwards, that is, $(c, c, ..., \underbrace{c}_{t-1}, \underbrace{x}_{t}, x, ..., \underbrace{x}_{\bar{t}})$.

We consider the static space of lotteries over streams, and not the more complex space of temporal lotteries used in Kreps and Porteus (1978) or EZ, because this subdomain is sufficient for our purposes and allows a simpler treatment that does not sacrifice generality. Any model over temporal lotteries induces preferences over Δ.[7] Crucially, these static preferences contain all the information on the separation of time and risk preferences relevant for our analysis. Moreover, restricting attention to this subdomain allows to derive results for a richer class of models, independently of how they are defined dynamically.

We now introduce our main property:

Definition 1 (Stochastic Impatience)**.** The relation $\succcurlyeq$ satisfies Stochastic Impatience if for any $t_1, t_2 \in T$ with $t_1 < t_2$, and any $c, x_1, x_2 \in C$ with $x_1 > x_2 > c$,

$$\frac{1}{2}(c, t_1, x_1) + \frac{1}{2}(c, t_2, x_2) \succcurlyeq \frac{1}{2}(c, t_2, x_1) + \frac{1}{2}(c, t_1, x_2). \tag{1}$$

Stochastic Impatience states that the individual prefers the lottery in which she either starts receiving higher payments earlier or lower payments later. It can be

[6] We focus on real-valued consumption and discrete time for simplicity. We start from date one to allow for additional consumption at time zero not subject to uncertainty. Appendix A presents the extension to continuous time.

[7] Formally, lotteries over streams are embedded within temporal lotteries once we fix a start date and a time of resolution of uncertainty. For example, if we start from preferences over temporal lotteries, hold the time-zero consumption fixed and assume that uncertainty is resolved between periods zero and one, we obtain preferences over Δ.

seen as a stochastic counterpart of standard impatience—preferring higher payments sooner. As mentioned in the introduction, Stochastic Impatience can be interpreted by decomposing each lottery in two parts. Both options offer the same basic lottery that pays x_2 starting at either t_1 or t_2 and an increment of $x_1 - x_2$. The difference is when the increment is paid: on the left, it is paired with the earlier date t_1; on the right, with the later date t_2. Insofar as the individual prefers to obtain it sooner, the option on the left is preferred. We say that Stochastic Impatience fails if (1) fails for some $t_1 < t_2$ and $x_1 > x_2 > c$.

Lanier et al. (2020) presents the first empirical test of Stochastic Impatience, finding evidence in favor of it in an experiment with assets that pay in different dates and in different states.

Stochastic Impatience is related to multivariate risk aversion (Richard, 1975; Wakker et al., 2004) and, more generally, to supermodularity. What distinguishes it is the specification of one dimension as prize and the other as time, giving it a different appeal. To our knowledge, Stochastic Impatience is a new property—except that a version of it appears in a companion paper (Dejarnette et al., 2020).[8] The most related, albeit different, condition appears in Bommier (2007), where multivariate risk aversion is applied to an intertemporal context with the goal of capturing correlation aversion.

In Expected Discounted Utility (henceforth EDU), impatience—a preference for earlier rewards—and Stochastic Impatience are equivalent.

Observation 1. Suppose $\succcurlyeq$ is represented by $\mathbb{E}\big[\sum_T D(t)u(x(t))\big]$ with u strictly increasing. Then, $\succcurlyeq$ satisfies Stochastic Impatience if and only if D is weakly decreasing.

All proofs are relegated to Appendix C.

Definition 1 considers prizes that raise consumption permanently. In different formulations, the increase in consumption could last any number of periods. In Appendix A we show that, under general conditions, these formulations are equivalent.[9]

[8] Dejarnette et al. (2020) considers a setup of prize-date pairs instead of streams and a corresponding version of Stochastic Impatience.

[9] Specifically, we show that under the conditions of Proposition 4, these properties are equivalent in continuous time and in discrete time as time intervals become small. In general, they are not equivalent in discrete time although qualitative conclusions remain unchanged.

3 Stochastic Impatience in EZ and HS

3.1 Epstein-Zin preferences

We first consider the most widely used model that separates time and risk preferences: the CRRA-CES version of EZ. Let $C = \mathbb{R}_{++}$ and $\bar{t} = +\infty$. This model admits the following recursive representation:

$$V_t = \left\{ (1-\beta)\, x(t)^{1-\frac{1}{\psi}} + \beta \left[E_t \left(V_{t+1}^{1-\alpha} \right) \right]^{\frac{1-\frac{1}{\psi}}{1-\alpha}} \right\}^{\frac{1}{1-\frac{1}{\psi}}} \tag{2}$$

where $\alpha \in \mathbb{R}_+ \backslash \{1\}$ is the coefficient of relative risk aversion and $\psi \in \mathbb{R}_+ \backslash \{1\}$ is the EIS. When $\alpha = \frac{1}{\psi}$, the model coincides with EDU.

The following result characterizes Stochastic Impatience in this model.

Proposition 1. *Suppose $\succcurlyeq$ admits the representation in (2). Stochastic Impatience holds if and only if either (i) $\alpha \leq \frac{1}{\psi}$, or (ii) $\alpha < 1$.*

To understand Proposition 1, consider again the choice between lotteries A and B given in the introduction:

A. With equal probability, permanently increase consumption by either 20% starting today, or by 10% starting next year;

B. With equal probability, permanently increase consumption by either 10% starting today, or by 20% starting next year.

Stochastic Impatience prescribes that A should be preferred. Of the four possible outcomes, the best is 20% starting today, the worst is 10% next year, while the other two are intermediate. Option A involves the best and the worst outcomes, while option B features the two 'intermediate' ones. Thus, option A has more spread in discounted utility but also has a higher expected discounted utility—since the higher discounting is applied to the smaller amount. When $\alpha = \frac{1}{\psi}$, i.e., with EDU, the agent cares only about the expected discounted utility, thus strictly prefers option A. But when risk aversion is increased fixing EIS, the individual starts disliking spread in discounted utilities, which favors option B. When risk aversion is high enough, this second effect prevails, leading the individual to prefer option B and violate Stochastic Impatience. Proposition 1 provides the exact condition: $\alpha > \max\{\frac{1}{\psi}, 1\}$.

The relevance of this result should be understood in light of the parameters used in the wide literature that adopts EZ. All applications we are aware of assume $\alpha > \max\{\frac{1}{\psi}, 1\}$. Indeed, the possibility of incorporating risk aversion greater than the inverse of EIS is a primary reason for adopting this model, and relative risk aversion above one is also typically imposed to fit finance data. For example, Bansal et al. (2016) estimate $\alpha = 9.67$ and $\psi = 2.18$ (see Example 1 below for other references). By Proposition 1, Stochastic Impatience fails in this range.

Another strand of the literature (typically not adopting EZ) has instead argued for $\psi < 1$.[10] With this restriction, EZ necessarily violates Stochastic Impatience if $\alpha > \frac{1}{\psi}$.

Proposition 1 shows when there exist violations of Stochastic Impatience. We now provide an example of such violation.

Example 1. Consider again lotteries A and B described above. Stochastic Impatience implies A preferred to B. However, B is preferred adopting the EZ model with the parameters of many known papers: Bansal and Yaron (2004) ($\alpha = 10$, $\beta = 0.998$, $\psi = 1.5$), Bansal et al. (2016) ($\alpha = 9.67$, $\beta = .999$, $\psi = 2.18$), Nakamura et al. (2017) ($\alpha = 9$, $\beta = 0.99$, $\psi = 1.5$), and Colacito et al. (2018) ($\alpha = 10$, $\beta = 0.97$, $\psi = 1.1$).[11]

Finally, in EZ α and ψ also determine the preference over the timing of resolution of uncertainty: The individual prefers early (late) resolution whenever α is higher (smaller) than $\frac{1}{\psi}$ (Epstein and Zin, 1989). By imposing an upper bound on α given ψ, Stochastic Impatience limits the strength of the preference for early resolution, even though these are conceptually independent notions. This links our results to Epstein et al. (2014), who argue that the parameters used in much of the literature generate an implausibly high preference for early resolution. Here we show that these same parameters imply a violation of Stochastic Impatience.

To summarize, with the parameters commonly used to fit macroeconomic and financial data, the CRRA-CES specification of EZ violates Stochastic Impatience.

[10]See Campbell (1999), Attanasio and Weber (2010), Campbell (2003) and, more recently, Gruber (2013); Ortu et al. (2013); Crump et al. (2015); Best et al. (2017).

[11]Even with $\psi < 1$, B is preferred to A if α is high enough. With $\alpha = 10$ and $\beta = 0.998$, B is preferred if $\psi > 0.2576$. With lower risk aversion, violations of Stochastic Impatience require higher prizes. For example, with the parameters of Nakamura et al. (2013) ($\alpha = 6.4$, $\beta = 0.967$, $\psi = 2$), a violation is observed with low prize of 20% and high prize of 30% of per-period consumption. With the parameters of Barro (2009) ($\alpha = 4$, $\beta = 0.948$, and $\psi = 2$), with low prize of 35% and high prize of 40%. With even less risk aversion, closer to one, violations require higher and higher prizes.

3.2 Risk Sensitive preferences

We now consider the Risk Sensitive preferences of HS, which admit the recursive representation:

$$V_t = u(x(t)) - \beta \cdot \frac{1}{k} \cdot \ln \left(E \left[e^{-kV_{t+1}} \right] \right). \tag{3}$$

for $t = 1, 2, ...$, where k captures risk sensitivity: $k > 0$ increases aversion to risk compared to standard expected utility.

Proposition 2. *Suppose $\succcurlyeq$ admits the representation in (3). Stochastic Impatience holds if and only if* $\sup_{x\in C}\{u(x)\} - \inf_{x\in C}\{u(x)\} \leq -\frac{\ln(\beta)}{k\beta(1-\beta)}$.

Under HS, Stochastic Impatience is violated if the utility range of prizes is large enough. This is necessarily the case if u is unbounded above or below (such as with a CARA utility and an unbounded consumption space). Otherwise, Stochastic Impatience requires $(\sup u(x) - \inf u(x))$ and k to be small enough.

We now illustrate examples of violations using an influential parameterization:

Example 2. Tallarini Jr (2000) uses HS preferences with $C = \mathbb{R}_{++}$, $u(x) = \ln(x)$, and $k = (1-\beta)(\xi-1)$. Since u is unbounded, Stochastic Impatience fails. Tallarini Jr (2000) shows that the model can match key moments in asset pricing for some $(\xi, \beta) \in [46, 180] \times [.991, .999]$. Consider again options A and B used in Example 1. With any of the parameters above, B is preferred, violating Stochastic Impatience.

4 A general result

We now generalize our previous results beyond the models of EZ and HS. First, we present a convenient functional form to analyze the separation of time and risk preferences. Next, we introduce a behavioral counterpart of this separation, which we call Residual Risk Aversion. Finally, we present our results on the bounds imposed by Stochastic Impatience both in terms of the functional form and behaviorally.

4.1 The KM model

For simplicity, in the remainder we assume that the space of per-period consumption is a compact interval: $C = [\underline{x}, \overline{x}] \subset \mathbb{R}_+$.[12] We focus on preferences that satisfy the following two assumptions.

[12] All results hold when C is unbounded above but the per-period utility over outcomes u is bounded (so discounted sums are well-defined).

Assumption 1 (Discounted Utility without risk)**.** *There exist a strictly increasing and continuous function* $u : [\underline{x}, \overline{x}] \to \mathbb{R}_+$ *and a strictly decreasing function* $D : T \to [0, 1]$ *such that for all* $\mathbf{x}, \mathbf{y} \in \mathcal{X}$

$$\mathbf{x} \succcurlyeq \mathbf{y} \quad \Leftrightarrow \quad \sum_{t \in T} D(t) u(x(t)) \geq \sum_{t \in T} D(t) u(y(t)).$$

Assumption 2 (Expected Utility)**.** *The following hold:*

1. *For all* $p, q, r \in \Delta$ *and* $\lambda \in (0, 1)$*,* $p \succcurlyeq q \Leftrightarrow \lambda p + (1 - \lambda) r \succcurlyeq \lambda q + (1 - \lambda) r$*;*
2. *For all* $p, q, r \in \Delta$ *with* $p \succ q \succ r$*, there exist* $\lambda, \gamma \in (0, 1)$ *such that* $\lambda p + (1 - \lambda) r \succ q \succ \gamma p + (1 - \gamma) r$*.*

Assumption 1 posits that in the absence of risk, preferences can be modeled using Discounted Utility with a generic discount function D. This allows for many types of discounting (exponential, hyperbolic, quasi-hyperbolic). Assumption 2 posits the postulates of Expected Utility, satisfied by most models in the literature.

Assumptions 1 and 2 yield the following representation:

Observation 2. The relation $\succcurlyeq$ satisfies Assumptions 1 and 2 if and only if there exist a strictly increasing and continuous $u : [\underline{x}, \overline{x}] \to \mathbb{R}$, a strictly decreasing $D : T \to [0, 1]$, and a strictly increasing $\phi : u(C) \to \mathbb{R}$ such that $\succcurlyeq$ is represented by

$$V(p) = \mathbb{E}_p \Big[\phi \Big(\frac{1}{\sum_T D(t)} \sum_T D(t) u(x(t)) \Big) \Big]. \tag{4}$$

Conditional on u and D, ϕ is unique up to a positive affine transformation.[13]

This representation is known as the Kihlstrom-Mirman (KM) representation, as it can be seen as an application of the multi-attribute function of Kihlstrom and Mirman (1974) to time.[14] Fixing D, the curvature of u captures EIS; risk aversion is captured by $\phi \circ u$. Thus, ϕ is the additional curvature that separates between risk aversion and EIS.

[13] Assumption 1 guarantees a Discounted Utility representation without risk; Assumption 2 guarantees an Expected Utility representation with a given Bernoulli utility V. Since V and the Discounted Utility representation must be ordinally equivalent, there exists a strictly increasing function ϕ that makes them equal.

[14] See, for example, Epstein and Zin (1989). This functional form is derived, in a different setup, in Dejarnette et al. (2020). A similar functional form is used in Edmans and Gabaix (2011); Garrett and Pavan (2011); Abdellaoui et al. (2017); Andersen et al. (2017); Apesteguia et al. (2019).

Observation 2 highlights that the KM model is characterized by Assumptions 1 and 2 and thus provides a convenient functional form to study the static implications of commonly used models, including EZ and HS—as their static implications satisfy both assumptions. Importantly, the function ϕ contains all the information on the separation of time and risk preferences. As we show next, EDU, EZ, and HS correspond to linear, CRRA, and CARA functions ϕ, respectively.[15]

Example 3 (Expected Discounted Utility)**.** If ϕ is affine, $\succcurlyeq$ can be represented by $\mathbb{E}\big[\sum_T D(t)u(x(t))\big]$.

Example 4 (EZ with CRRA-CES)**.** Fix a consumption for time zero and consider a preference relation $\succcurlyeq$ over Δ generated by (2). Then, $\succcurlyeq$ admits a KM representation, with $u(x) = \frac{x^{1-\frac{1}{\psi}}}{1-\frac{1}{\psi}}$, $D(t) = \beta^t$, and

$$\phi(z) = \begin{cases} z^{\frac{1-\alpha}{1-\frac{1}{\psi}}} & \text{if } \alpha < 1 < \psi \\ -(-z)^{\frac{1-\alpha}{1-\frac{1}{\psi}}} & \text{if } \alpha > 1 > \psi \\ -z^{\frac{1-\alpha}{1-\frac{1}{\psi}}} & \text{if } 1 < \alpha,\ 1 < \psi \\ (-z)^{\frac{1-\alpha}{1-\frac{1}{\psi}}} & \text{if } \alpha < 1,\ \psi < 1 \end{cases}. \tag{5}$$

Example 5 (HS)**.** Fix a consumption for time zero and consider a preference relation $\succcurlyeq$ over Δ generated by (3). Then, $\succcurlyeq$ admits a KM representation with $D(t) = \beta^t$ and $\phi(x) = -\exp(-kx)$.

4.2 Residual Risk Aversion

We now introduce a behavioral notion which captures the additional risk aversion relative to the level implied by EIS. We call this *Residual Risk Aversion.* This property

[15] This does not imply that the KM model includes EZ and HS as special cases. Rather, it implies that the static implications of EZ and HS on Δ admit a KM representation. If applied dynamically on the space of temporal lotteries with different current-period consumption, the models are not nested (e.g., EZ and HS are dynamically consistent, while KM is not if applied dynamically without modification). These differences are inconsequential in our setup, and the observation highlights that one can view EZ or HS as being composed of a *collection* of KM representations, where ϕ varies with the timing of resolution of uncertainty and with current consumption in order to allow for recursivity and dynamic consistency.

is an adaptation to our framework of Traeger (2014)'s notion of Intertemporal Risk Aversion.[16] It is also related to notions of multivariate risk aversion (Richard, 1975).

Definition 2. The relation $\succcurlyeq$ displays Residual Risk Aversion if for any $a, b, c, d, x \in [\underline{x}, \overline{x}]$ such that

$$(a, d, x, x, \dots) \sim (b, b, x, x, \dots) \text{ and } (d, a, x, x, \dots) \sim (c, c, x, x, \dots)$$

we have

$$\frac{1}{2}(b, b, \dots) + \frac{1}{2}(c, c, \dots) \succcurlyeq \frac{1}{2}(a, a, \dots) + \frac{1}{2}(d, d, \dots).$$

Similarly, $\succcurlyeq$ displays Residual Risk Seeking/Neutrality if the above is instead $\preccurlyeq / \sim$.

Suppose $\succcurlyeq$ satisfies Assumption 1 with u and D. If $(a, d, x, x, \dots) \sim (b, b, x, x, \dots)$ and $(d, a, x, x, \dots) \sim (c, c, x, x, \dots)$, then

$$D(1)u(a)+D(2)u(d) = [D(1)+D(2)]u(b) \text{ and } D(1)u(d)+D(2)u(a) = [D(1)+D(2)]u(c).$$

Thus, $u(a) + u(d) = u(b) + u(c)$. Moreover, either $a > b > c > d$ or $a < b < c < d$, and this depends only on how $\succcurlyeq$ ranks streams without risk. Consider a lottery that returns with equal chances constant streams a or d; and a lottery that returns with equal chances constant streams b or c. If all risk aversion is included in the curvature of u, these two lotteries must be indifferent since $u(a) + u(d) = u(b) + u(c)$. With additional risk aversion, since $a > b > c > d$ or $a < b < c < d$, the individual should prefer the lottery between b and c, in which the utility spread is smaller.

We first link Residual Risk Aversion to the KM representation, showing that the curvature of ϕ is related to Residual Risk Aversion similarly to how the curvature of the Bernoulli utility function is related to risk aversion in Expected Utility:

Proposition 3. *Suppose $\succcurlyeq$ admits a KM representation (ϕ, D, u). Then, $\succcurlyeq$ displays Residual Risk Aversion/Seeking/Neutrality if and only if ϕ is concave/convex/affine.*

It follows that Residual Risk Neutrality characterizes EDU given Assumptions 1 and 2, and that EZ allows for Residual Risk Aversion:

Observation 3 (EDU is characterized by Residual Risk Neutrality)**.** Suppose $\succcurlyeq$ satisfies Assumptions 1 and 2. Then, it admits an EDU representation if and only if it displays Residual Risk Neutrality.

[16] Similar to Proposition 3 and Observation 5 below, Traeger also gives a functional characterization of attitudes towards intertemporal risk aversion in his framework.

Observation 4 (EZ preferences)**.** Suppose $\succcurlyeq$ admits a representation as in (2). Then $\succcurlyeq$ displays Residual Risk Aversion/Neutrality/Seeking if and only if $\alpha \geq / = / \leq \frac{1}{\psi}$.

Recall that the CRRA-CES version of EZ displays a preference for early (late/neutrality towards) resolution of uncertainty if $\alpha > (< / =)\frac{1}{\psi}$. Therefore, in this model, $\succcurlyeq$ displays Residual Risk Aversion (Seeking/Neutrality) if and only if, in the space of temporal lotteries, there is a preference for early (a preference for late/neutrality towards) resolution of uncertainty.

We now introduce a comparative notion.

Definition 3. Consider two relations $\succcurlyeq_1$ and $\succcurlyeq_2$ over Δ. We say that $\succcurlyeq_1$ has more Residual Risk Aversion than $\succcurlyeq_2$ if they coincide on degenerate lotteries and if, for all $a > b > c > d$,

$$\frac{1}{2}(b, b, \dots) + \frac{1}{2}(c, c, \dots) \succcurlyeq_2 \frac{1}{2}(a, a, \dots) + \frac{1}{2}(d, d, \dots)$$

implies

$$\frac{1}{2}(b, b, \dots) + \frac{1}{2}(c, c, \dots) \succcurlyeq_1 \frac{1}{2}(a, a, \dots) + \frac{1}{2}(d, d, \dots).$$

This comparative notion parallels standard ones for risk and ambiguity (Epstein, 1999; Ghirardato and Marinacci, 2002). It has an immediate counterpart in KM representations.

Observation 5. Let $\succcurlyeq_1$ and $\succcurlyeq_2$ be two preferences with KM representations (ϕ_1, u, D) and (ϕ_2, u, D). Then, $\succcurlyeq_1$ has more Residual Risk Aversion than $\succcurlyeq_2$ if and only if there exist a strictly increasing and concave function $f : \mathbb{R} \to \mathbb{R}$ s.t. $\phi_1 = f \circ \phi_2$.

4.3 General bounds by Stochastic Impatience

We now present our general result. As usual, we say that ϕ is more concave/convex than g if $\phi = f \circ g$ for some concave/convex f.

Proposition 4. *Let $C = [\underline{x}, \overline{x}]$, $\succcurlyeq$ be a preference relation over Δ that satisfies Assumptions 1 and 2, and (ϕ, u, D) be a KM representation.*

(i) If $\phi(v)$ is weakly less concave than $\ln(v - u(\underline{x}))$ for all $v \in u(C)$, then $\succcurlyeq$ satisfies Stochastic Impatience;

(ii) There exist $v_1, v_2 \in u(C)$ such that, if $\phi(v)$ is strictly more concave than $\ln(v - u(\underline{x}))$ for all $v \in [v_1, v_2]$, then $\succcurlyeq$ violates Stochastic Impatience;

(iii) There exists another preference relation $\succcurlyeq'$ over Δ that satisfies Assumptions 1 and 2, has more Residual Risk Aversion than $\succcurlyeq$, and violates Stochastic Impatience;

(iv) Let $\succcurlyeq'$ be another preference relation over Δ that satisfies Assumptions 1 and 2 and has more Residual Risk Aversion than $\succcurlyeq$. Then:

(a) if $\succcurlyeq$ violates Stochastic Impatience, so does $\succcurlyeq'$;

(b) if $\succcurlyeq'$ satisfies Stochastic Impatience, so does $\succcurlyeq$.

Proposition 4 shows that Stochastic Impatience restricts how high risk aversion can be given EIS both in terms of the KM representation (*i* and *ii*) and behaviorally (*iii* and *iv*). The intuition is the same as in EZ: Stochastic Impatience implies that the individual should prefer the lottery with a higher average but also higher spread in discounted utilities. Under EDU such spread does not matter. Under KM, however, concavity of ϕ induces the agent to dislike such spread. Proposition 4 shows that the 'breaking-point' where the aversion to spread overcomes the preference for higher average is when the curvature of ϕ is that of the log.

Note that *(i)* and *(ii)* together are not an equivalence statement: *(ii)* requires ϕ to be more concave on an entire range $[v_1, v_2]$—which depends on u and D—while the negation of *(i)* requires it to hold at a (neighborhood of a) single point. This is due to the discreteness of time intervals: in Appendix A we consider a continuous time version of KM and show that, in that case, Stochastic Impatience holds if and only if $\phi(v)$ is globally less concave than $\ln(v - u(\underline{x}))$.

5 Discussion

We have shown that within a broad class of models, Stochastic Impatience restricts how high risk aversion can be relative to EIS, ruling out the parameters used in most applications of EZ and HS. This result has two implications. First, it provides a simple way to test common parametrizations of existing models: they imply that Stochastic Impatience must be violated. Second, it implies that widespread approaches violate Stochastic Impatience. To preserve Stochastic Impatience, one needs to either consider a class of models where our results do not hold, or find different ways to match empirical patters while keeping lower risk aversion. We conclude with a discussion of these two possibilities.

Beyond our assumptions. Our results assume Expected Utility under risk and Discounted Utility without risk (with generic discounting). Do they hold more broadly?

There are two natural ways to extend beyond Expected Utility. First, one can adopt models of non-Expected Utility developed in the atemporal environment. In Appendix B we consider a very broad class that includes models with probability weighting and Disappointment Aversion.[17] We show that (in a continuous time setting) these models always violate Stochastic Impatience whenever they exhibit First-Order Risk Aversion (Segal and Spivak, 1990)—as is the case in most specifications that use them.

Alternatively, one can consider models that maintain Expected Utility within each period, but violate it across periods. While this avenue has received little attention in the literature on non-Expected Utility, there are models that fit into this category. One example is the Dynamic Ordinal Certainty Equivalent model (Selden, 1978; Selden and Stux, 1978), where the individual first calculates the per-period certainty equivalents using one utility function, and then calculates their discounted value using a different utility. This model satisfies Stochastic Impatience while allowing for a separation of time and risk preferences (Dejarnette et al., 2020; Selden and Wei, 2019). A discussion of the appeal of this model is beyond our scope, although some papers pointed to concerns with dynamic consistency and notions of monotonicity (Epstein and Zin, 1989; Chew and Epstein, 1990; Bommier et al., 2017).

Our other assumption is Discounted Utility without risk. Going beyond it requires dropping additive separability, as in models of habit formation or memorable consumption. Our results qualitatively extend, in the sense that for any such model Stochastic Impatience imposes a bound on how high risk aversion can be for a fixed EIS.[18] However, this bound depends on the specifics of the preferences considered.

Beyond high risk aversion. Models in finance and macroeconomics often require high risk aversion to capture the individuals' unwillingness to take financial or similar risks. An alternative approach is to consider other features that have the same effect—e.g., ambiguity aversion/robustness concerns, incorrect beliefs about stock returns, or

[17] Applications of these models have been suggested, starting in the original paper of Epstein and Zin (1989). See Backus et al. (2004) and references therein.

[18] Even weakening Assumption 1 while maintaining Assumption 2, a concave enough ϕ makes the value of any lottery be arbitrarily close to the value of its worst outcome, thus generating a violation of Stochastic Impatience.

rational inattention. If these aspects are relevant but omitted from the model, risk aversion may be overestimated.

For example, if some of the equity premium is due to ambiguity aversion, incorporating it may allow for much lower coefficients of risk aversion (Barillas et al., 2009). This would reduce the preference for early resolution of uncertainty to more realistic levels (Epstein et al., 2014) and allow for Stochastic Impatience: since the latter is based on objective lotteries, it is unaffected by ambiguity aversion. In general, any feature that reduces the individual's willingness to undertake financial risk without modifying her attitude towards objective lotteries, as discussed in the surveys of Backus et al. (2004), Epstein and Schneider (2010), and Hansen and Sargent (2014), could provide a way to reconcile the empirical fit of the model with more moderate levels of risk aversion and—as we show in this paper—also with Stochastic Impatience.

Appendices

A Continuous time and additional results

We now consider a continuous time formulation of the model from Section 4. Let $C = [\underline{x}, \overline{x}] \subset \mathbb{R}_+$ denote the space of per-period consumption. The set of dates is now $T = [0, \bar{t}]$, where $\bar{t} > 0$ may be $+\infty$. For each $\delta > 0$, (c, t, x, δ) denotes the consumption stream that returns $c \in C$ for $\tilde{t} \in T \backslash [t, t+\delta)$ and $x \in C$ for $\tilde{t} \in [t, t+\delta) \cap T$. Note that $(c, t, x, \bar{t} - t)$ is the stream that returns c until date t and x from t onward.

We consider preferences $\succcurlyeq$ over lotteries over streams that can be represented by

$$V(p) = \mathbb{E}_p\Big[\phi\Big(\frac{\int_0^{\bar{t}} D(t)u(x(t))dt}{\int_0^{\bar{t}} D(t)dt}\Big)\Big],$$

for lotteries p such that the integrals are well defined, where $u : C \to \mathbb{R}_+$ is continuous and strictly increasing, $D : T \to [0,1]$ is continuous and strictly decreasing, and $\phi : [u(\underline{x}), u(\overline{x})] \to \mathbb{R}$ is strictly increasing.

Recall that Stochastic Impatience posits that individuals prefer to associate higher prizes to earlier dates. The analogous notion in continuous time is as follows:

Definition 4 (Stochastic Impatience'). The relation $\succcurlyeq$ satisfies Stochastic Impatience' if for any $t_1, t_2 \in \mathbb{R}_+$ with $t_1 < t_2 < \bar{t}$ and any $c, x_1, x_2 \in C$ with $x_1 > x_2 > c$

$$\frac{1}{2}(c, t_1, x_1, \bar{t} - t_1) + \frac{1}{2}(c, t_2, x_2, \bar{t} - t_2) \succcurlyeq \frac{1}{2}(c, t_2, x_1, \bar{t} - t_2) + \frac{1}{2}(c, t_1, x_2, \bar{t} - t_1). \quad (6)$$

In the definition above, increases in consumption are permanent. We can also consider a stronger version that allows for prizes with a finite duration.

Definition 5 (Strong Stochastic Impatience). The relation $\succcurlyeq$ satisfies Strong Stochastic Impatience if for any $t_1, t_2 \in \mathbb{R}_+$ with $t_1 < t_2 < \bar{t}$, any $\delta > 0$, and any $c, x_1, x_2 \in C$ with $x_1 > x_2 > c$

$$\frac{1}{2}(c, t_1, x_1, \delta) + \frac{1}{2}(c, t_2, x_2, \delta) \succcurlyeq \frac{1}{2}(c, t_2, x_1, \delta) + \frac{1}{2}(c, t_1, x_2, \delta). \quad (7)$$

We now show that, with continuous time, the two versions of Stochastic Impatience are equivalent and correspond to a bound to the concavity of ϕ.

Proposition 5. *Suppose time is continuous and let (ϕ, u, D) be a KM representation of $\succcurlyeq$. The following statements are equivalent:*

1. *The relation $\succcurlyeq$ satisfies Stochastic Impatience';*

2. *The relation $\succcurlyeq$ satisfies Strong Stochastic Impatience;*

3. *The function $\phi(v)$ is weakly less concave than* $\ln(v - u(\underline{x}))$.

Note that this result is stronger than the one obtained in the first two parts of Proposition 4 in discrete time: here one concavity condition is both necessary and sufficient for Stochastic Impatience.[19]

B Beyond Expected Utility

In this section we show that the tension between Stochastic Impatience and the separation of time and risk preferences does not rely on Expected Utility.

We extend beyond Expected Utility by assuming that preferences are at least *locally bilinear* at $\frac{1}{2}$. This generalization includes as special cases popular models such as those of probability weighting (Rank-Dependent Utility, Quiggin 1982, and Cumulative Prospect Theory, Tversky and Kahneman 1992) and Disappointment Aversion (Gul, 1991).[20] In general, bilinearity holds if there is an increasing onto function $\pi : [0, 1] \to [0, 1]$, and a function f that evaluates (arbitrary) prizes, such that, for any x, y such that $f(x) > f(y)$, the prospect that yields x with probability λ and y otherwise is evaluated by $\pi(\lambda) f(x) + [1 - \pi(\lambda)] f(y)$. Since our goal is to be as general as possible, we only require preferences to be bilinear for equally likely binary lotteries ($\lambda = \frac{1}{2}$)—the *local* bilinear model (Dean and Ortoleva, 2017).[21]Applying it to our setting, we obtain the following generalization of the KM model using the continuous time setup of Appendix A.

Definition 6. We say that $\succcurlyeq$ admits a *local bilinear KM* representation if there exist strictly increasing and continuous $u : C \to \mathbb{R}_+$, a strictly decreasing $D : T \to [0, 1]$, a strictly increasing and differentiable $\phi : u(C) \to \mathbb{R}$, and $\pi(\frac{1}{2}) \in (0, 1)$, such that

[19] In discrete time, Stochastic Impatience also does not imply Strong Stochastic Impatience.

[20] It also allows for generalizations of Rank-Dependent Expected Utility, e.g., the minimum from a set of probability distortions (Dean and Ortoleva, 2017). On the other hand, it does not encompass all models of risk preferences (e.g., it does not encompass Cautious Expected Utility, Cerreia-Vioglio et al. 2015).

[21] This is a local specification of the bilinear (or biseparable) model of Ghirardato and Marinacci (2001) for objective risk. Here, preferences are not restricted to be bilinear in general, but only that there is some bilinear representation for 50/50 lotteries.

for all $\mathbf{x}, \mathbf{y} \in \mathcal{X}$, $p = \frac{1}{2}\mathbf{x} + \frac{1}{2}\mathbf{y}$ with $\int_0^{\bar{t}} D(t)u(x(t))dt \geq \int_0^{\bar{t}} D(t)u(y(t))dt$ is evaluated according to:

$$V(p) = \pi\Big(\frac{1}{2}\Big)\phi\Big(\frac{\int_0^{\bar{t}} D(t)u(x(t))dt}{\int_0^{\bar{t}} D(t)dt}\Big) + \Big[1 - \pi\Big(\frac{1}{2}\Big)\Big]\phi\Big(\frac{\int_0^{\bar{t}} D(t)u(y(t))dt}{\int_0^{\bar{t}} D(t)dt}\Big).$$

It is easy to see that in a local bilinear KM representation Residual Risk Aversion can be achieved either by adding curvature to ϕ, as in the KM representation, or by adding non-Expected Utility and First-Order Risk Aversion (Segal and Spivak, 1990) by positing that $\pi(\frac{1}{2}) < \frac{1}{2}$, i.e., underweighting the best outcome.

Proposition 6. *Let $\succcurlyeq$ be a preference relation over Δ that admits a local bilinear KM representation (u, D, ϕ, π). If $\pi(\frac{1}{2}) < \frac{1}{2}$, then $\succcurlyeq$ violates Stochastic Impatience'.*

The result above shows that, in continuous time, even if we go beyond Expected Utility by looking at the broad class of local bilinear models, displaying First-Order Risk Aversion *always* leads to violations of Stochastic Impatience, independently of the shape of ϕ. Intuitively, this derives from the fact that First-Order Risk Aversion implies extreme amounts of risk aversion in a neighborhood around certainty; and we have already seen how Stochastic Impatience is violated once risk aversion towards discounted utilities is high enough.

C Proofs

C.1 Proof of Observation 1

Using the representation, Stochastic Impatience holds if and only if

$$\frac{D(t_1)u(x_1) + D(t_2)u(x_2)}{2} + \sum_{t \notin \{t_1, t_2\}} D(t)u(c) \geq \frac{D(t_1)u(x_2) + D(t_2)u(x_1)}{2} + \sum_{t \notin \{t_1, t_2\}} D(t)u(c)$$

for all all $t_1 < t_2$, c, and $x_1 > x_2$. Rearrange this expression to obtain:

$$[D(t_1) - D(t_2)]\,[u(x_1) - u(x_2)] \geq 0.$$

Since u is strictly increasing, this inequality holds if and only if $D(t_1) \geq D(t_2)$. ■

C.2 Proof of Proposition 1

Let $\rho \equiv \frac{1}{\psi}$ denote the inverse of EIS. For notational simplicity, we will work with ρ instead of ψ.

Since preferences are dynamically consistent, it suffices to look at lotteries in which the earliest payment is made in period 1. Consider a lottery that, with equal probability, either starts paying an increment of x in period 1 or starts paying an increment of y in period t: $\frac{1}{2}(c,1,c+x)+\frac{1}{2}(c,t,c+y)$.

Note that $\succcurlyeq$ satisfies Stochastic Impatience (SI) if:

$$\frac{1}{2}(c,1,c+x)+\frac{1}{2}(c,t,c+y) \succcurlyeq \frac{1}{2}(c,1,c+y)+\frac{1}{2}(c,t,c+x)$$

for any $c>0$, any $x>y$, and any $t \in \{2,3,...\}$. The proof will be given through a series of lemmas.

Lemma 1. *The value of lottery* $\frac{1}{2}(c,1,c+x)+\frac{1}{2}(c,t,c+y)$ *is*

$$\left\{(1-\beta)\,c^{1-\rho}+\beta\left[\frac{(c+x)^{1-\alpha}+\left\{c^{1-\rho}+\beta^{t-1}\left[(c+y)^{1-\rho}-c^{1-\rho}\right]\right\}^{\frac{1-\alpha}{1-\rho}}}{2}\right]^{\frac{1-\rho}{1-\alpha}}\right\}^{\frac{1}{1-\rho}}.$$

Proof. Recall that in EZ, lotteries are evaluated using the recursion:

$$V_t=\left\{(1-\beta)\,x(t)^{1-\rho}+\beta\left[E_t\left(V_{t+1}^{1-\alpha}\right)\right]^{\frac{1-\rho}{1-\alpha}}\right\}^{\frac{1}{1-\rho}}. \tag{8}$$

With 50% chance, the individual gets $c+x$ in all future periods, giving a continuation value of $V_1=c+x$. With 50% chance, consumption equals c up to period $t-1$, after which the individual gets $c+y$. Therefore, $V_t=c+y$. Proceeding backwards, we obtain:

$$V_{t-n}=\left\{(1-\beta)\,c^{1-\rho}\left(1+\beta+...+\beta^{n-1}\right)+\beta^{n}\,(c+y)^{1-\rho}\right\}^{\frac{1}{1-\rho}},$$

for any $n=1,...,t-1$. In particular,

$$\begin{aligned} V_1 &= \left\{(1-\beta)\,c^{1-\rho}\,(1+\beta+...+\beta^{t-2})+\beta^{t-1}\,(c+y)^{1-\rho}\right\}^{\frac{1}{1-\rho}} \\ &= \left\{c^{1-\rho}+\beta^{t-1}\left[(c+y)^{1-\rho}-c^{1-\rho}\right]\right\}^{\frac{1}{1-\rho}} \end{aligned},$$

where the second line uses the fact that $1+\beta+...+\beta^{t-2}=\frac{1-\beta^{t-1}}{1-\beta}$.

Moving to period 0, we obtain:

$$V_0 = \left\{ (1-\beta)\, c^{1-\rho} + \beta \left[\frac{(c+x)^{1-\alpha} + \left\{ c^{1-\rho} + \beta^{t-1} \left[(c+y)^{1-\rho} - c^{1-\rho} \right] \right\}^{\frac{1-\alpha}{1-\rho}}}{2} \right]^{\frac{1-\rho}{1-\alpha}} \right\}^{\frac{1}{1-\rho}}.$$

$\square$

By the homotheticity of EZ preferences, we can, without loss of generality, take a background consumption of $c = 1$. Therefore, Stochastic Impatience holds if and only if, for all $z_H > z_L > 1$ and all $t \in \{2, 3, ...\}$, we have

$$\begin{aligned} &\left\{ 1 - \beta + \beta \left[\frac{z_H^{1-\alpha} + \left[1 + \beta^{t-1} \left(z_L^{1-\rho} - 1 \right) \right]^{\frac{1-\alpha}{1-\rho}}}{2} \right]^{\frac{1-\rho}{1-\alpha}} \right\}^{\frac{1}{1-\rho}} \\ \geq &\left\{ 1 - \beta + \beta \left[\frac{z_L^{1-\alpha} + \left[1 + \beta^{t-1} \left(z_H^{1-\rho} - 1 \right) \right]^{\frac{1-\alpha}{1-\rho}}}{2} \right]^{\frac{1-\rho}{1-\alpha}} \right\}^{\frac{1}{1-\rho}} . \end{aligned} \tag{9}$$

Lemma 2. *Stochastic Impatience holds if and only if*

$$z^{\rho-\alpha} \geq \left[1 + \beta^{t-1} \left(z^{1-\rho} - 1 \right) \right]^{\frac{\rho-\alpha}{1-\rho}} \beta^{t-1} \tag{10}$$

for all $z \geq 1$ and all $t = \{2, 3, ...\}$.

Proof. Let $\Phi : [1, +\infty) \to \mathbb{R}$ be given by

$$\Phi(z) \equiv z^{1-\alpha} - \left[1 + \beta^{t-1} \left(z^{1-\rho} - 1 \right) \right]^{\frac{1-\alpha}{1-\rho}} .$$

The proof has two parts. In the first part, we show that Stochastic Impatience holds if and only if:

- $\Phi'(z) \geq 0$ for all $z > 1$ if $\alpha < 1$.
- $\Phi'(z) \leq 0$ for all $z > 1$ if $\alpha > 1$.

To establish this result, we rearrange inequality (9) in each of 4 possible cases.

Case 1: $\alpha, \rho < 1$.

$$\begin{aligned} & z_H^{1-\alpha} + \left[1 + \beta^{t-1} \left(z_L^{1-\rho} - 1 \right) \right]^{\frac{1-\alpha}{1-\rho}} \geq z_L^{1-\alpha} + \left[1 + \beta^{t-1} \left(z_H^{1-\rho} - 1 \right) \right]^{\frac{1-\alpha}{1-\rho}} \\ \iff & z_H^{1-\alpha} - \left[1 + \beta^{t-1} \left(z_H^{1-\rho} - 1 \right) \right]^{\frac{1-\alpha}{1-\rho}} \geq z_L^{1-\alpha} - \left[1 + \beta^{t-1} \left(z_L^{1-\rho} - 1 \right) \right]^{\frac{1-\alpha}{1-\rho}} . \end{aligned}$$

Case 2: $\alpha, \rho > 1$.

$$z_H^{1-\alpha} + \left[1 + \beta^{t-1}\left(z_L^{1-\rho} - 1\right)\right]^{\frac{1-\alpha}{1-\rho}} \leq z_L^{1-\alpha} + \left[1 + \beta^{t-1}\left(z_H^{1-\rho} - 1\right)\right]^{\frac{1-\alpha}{1-\rho}}$$

$$\iff z_H^{1-\alpha} - \left[1 + \beta^{t-1}\left(z_H^{1-\rho} - 1\right)\right]^{\frac{1-\alpha}{1-\rho}} \leq z_L^{1-\alpha} - \left[1 + \beta^{t-1}\left(z_L^{1-\rho} - 1\right)\right]^{\frac{1-\alpha}{1-\rho}}.$$

Case 3: $\alpha > 1 > \rho$.

$$z_H^{1-\alpha} - \left[1 + \beta^{t-1}\left(z_H^{1-\rho} - 1\right)\right]^{\frac{1-\alpha}{1-\rho}} \leq z_L^{1-\alpha} - \left[1 + \beta^{t-1}\left(z_L^{1-\rho} - 1\right)\right]^{\frac{1-\alpha}{1-\rho}}.$$

Case 4: $\rho > 1 > \alpha$.

$$z_H^{1-\alpha} - \left[1 + \beta^{t-1}\left(z_H^{1-\rho} - 1\right)\right]^{\frac{1-\alpha}{1-\rho}} \geq z_L^{1-\alpha} - \left[1 + \beta^{t-1}\left(z_L^{1-\rho} - 1\right)\right]^{\frac{1-\alpha}{1-\rho}}.$$

In the second part, we differentiate Φ to obtain:

$$\Phi'(z) = (1-\alpha)\left\{z^{-\alpha} - \left[1 + \beta^{t-1}\left(z^{1-\rho} - 1\right)\right]^{\frac{\rho-\alpha}{1-\rho}} \beta^{t-1} z^{-\rho}\right\}.$$

Since, by the previous result, Stochastic Impatience holds if $\Phi'(z) \geq 0$ when $\alpha < 1$ and $\Phi'(z) \leq 0$ if $\alpha > 1$, it follows that the term inside brackets must be weakly positive for all $z \geq 1$:

$$z^{-\alpha} \geq \left[1 + \beta^{t-1}\left(z^{1-\rho} - 1\right)\right]^{\frac{\rho-\alpha}{1-\rho}} \beta^{t-1} z^{-\rho}.$$

Multiplying both sides by $z^{\rho} > 0$, we obtain (10). □

We now use Lemma 2 to determine when Stochastic Impatience holds.

Lemma 3. *Stochastic Impatience if and only if either (i)* $\alpha \leq \frac{1}{\psi}$*, or (ii)* $\alpha < 1$.

Proof. The proof considers each of the six possible cases. We start with the two cases under which Stochastic Impatience fails:

Case 1. $\alpha > 1 > \rho$**.** To show that Stochastic Impatience fails in this case, note that after some algebraic manipulations, we can rewrite (10) as:

$$\frac{1 - \beta^{t-1}}{z^{1-\rho}} \geq \beta^{(t-1)\frac{1-\rho}{\alpha-\rho}} - \beta^{t-1}$$

for all $z \geq 1$. Note that the LHS converges to zero as $z \nearrow +\infty$. Moreover, the LHS is bounded away from zero since

$$\beta^{(t-1)\frac{1-\rho}{\alpha-\rho}} > \beta^{t-1} \iff (t-1)\frac{1-\rho}{\alpha-\rho} < t-1 \iff 1 < \alpha.$$

Therefore, there exists $\bar{z}$ such that (10) fails for all $z > \bar{z}$, showing that Stochastic Impatience fails.

Case 2. $\alpha > \rho > 1$. To show that Stochastic Impatience fails, rearrange (10) as

$$\beta^{\frac{(t-1)(\rho-1)}{\alpha-\rho}} \geq \left(1-\beta^{t-1}\right) z^{\rho-1} + \beta^{t-1}.$$

Note that, as $z \searrow 1$, the RHS converges to 1, whereas the LHS is always strictly less than 1 (since $\frac{(t-1)(\rho-1)}{\alpha-\rho} > 0$). Therefore, this inequality fails for z close to 1, showing that Stochastic Impatience fails.

We now turn to the cases where Stochastic Impatience holds:

Case 3. $1 > \alpha > \rho$. Rearranging (10), we find that Stochastic Impatience holds if and only if

$$\left[1 + \beta^{t-1}\left(z^{1-\rho} - 1\right)\right]^{\frac{\alpha-\rho}{1-\rho}} \geq z^{\alpha-\rho}\beta^{t-1}$$

for all $z > 1$. Rearrange this inequality as

$$\frac{1-\beta^{t-1}}{z^{1-\rho}} + \beta^{t-1} \geq \beta^{\frac{(t-1)(1-\rho)}{\alpha-\rho}}.$$

Because LHS is decreasing in z and converges to β^{t-1} as $z \nearrow +\infty$, this condition holds for all $z > 1$ if and only if

$$\beta^{t-1} \geq \beta^{(t-1)\left(\frac{1-\rho}{\alpha-\rho}\right)},$$

which is true since $\rho < \alpha \leq 1$. Thus, Stochastic Impatience holds in this case.

Case 4. $1 < \alpha < \rho$. Rewrite (10) as

$$z^{\rho-1} \cdot \left(1-\beta^{t-1}\right) + \beta^{t-1} \geq \beta^{\frac{(t-1)(\rho-1)}{\rho-\alpha}}.$$

Since the LHS is increasing in z, this inequality holds for all $z > 1$ if and only if it holds for $z = 1$:

$$1 \geq \beta^{\frac{(t-1)(\rho-1)}{\rho-\alpha}},$$

which is true because $\frac{(t-1)(\rho-1)}{\rho-\alpha} > 0$. Thus, Stochastic Impatience also holds in this case.

Case 5. $\alpha < \rho < 1$. Rewrite (10) as

$$\frac{1}{\beta^{\frac{(t-1)(1-\rho)}{\rho-\alpha}}} \geq \frac{1-\beta^{t-1}}{z^{1-\rho}} + \beta^{t-1}.$$

Since the RHS is decreasing in z, this condition holds for all $z > 1$ if and only if it holds for $z = 1$:

$$\frac{1}{\beta^{\frac{(t-1)(1-\rho)}{\rho-\alpha}}} \geq 1 - \beta^{t-1} + \beta^{t-1} \iff 1 \geq \beta^{\frac{(t-1)(1-\rho)}{\rho-\alpha}},$$

which is true since $\frac{(t-1)(1-\rho)}{\rho-\alpha} \geq 0$ under the parameters above.

Case 6. $\alpha < 1 < \rho$. Notice that (10) can be simplified as

$$z^{\rho-1}\left(1-\beta^{t-1}\right) + \beta^{t-1} \geq \beta^{(t-1)\left(\frac{\rho-1}{\rho-\alpha}\right)}.$$

Since the LHS is increasing in z, this condition holds for all $z > 1$ if and only if it holds for $z = 1$:

$$1 \geq \beta^{(t-1)\left(\frac{\rho-1}{\rho-\alpha}\right)},$$

which is true since $(t-1)\left(\frac{\rho-1}{\rho-\alpha}\right) > 0$. □

This concludes the proof of Proposition 1. ■

C.3 Proof of Proposition 2

The utility of the lottery $\frac{1}{2}(c, 1, c+x) + \frac{1}{2}(c, t, c+y)$ is

$$-\frac{\exp\left(-k\frac{\beta}{1-\beta}u(c+x)\right) + \exp\left(-k\cdot\left(\frac{\beta-\beta^t}{1-\beta}u(c) + \frac{\beta^t}{1-\beta}u(c+y)\right)\right)}{2}.$$

Therefore, Stochastic Impatience fails if and only if there exist $x > y > c$ and $t \in \{2, 3, ...\}$ such that

$$-\frac{\exp\left(-k\frac{\beta}{1-\beta}u(c+y)\right) + \exp\left(-k\cdot\left(u(c)\frac{\beta-\beta^t}{1-\beta} + u(c+x)\frac{\beta^t}{1-\beta}\right)\right)}{2}$$
$$> -\frac{\exp\left(-k\frac{\beta}{1-\beta}u(c+x)\right) + \exp\left(-k\cdot\left(u(c)\frac{\beta-\beta^t}{1-\beta} + u(c+y)\frac{\beta^t}{1-\beta}\right)\right)}{2}$$

Rearrange this inequality as:

$$\exp\left(-k\frac{\beta}{1-\beta}u(c+y)\right) - \exp\left(-k\cdot\left(u(c)\frac{\beta-\beta^t}{1-\beta} + u(c+y)\frac{\beta^t}{1-\beta}\right)\right)$$
$$< \exp\left(-k\frac{\beta}{1-\beta}u(c+x)\right) - \exp\left(-k\cdot\left(u(c)\frac{\beta-\beta^t}{1-\beta} + u(c+x)\frac{\beta^t}{1-\beta}\right)\right).$$

For simplicity, we will assume that u is a differentiable function, although it is immediate to generalize the argument for when it is not. Then, Stochastic Impatience fails if and only if there exist $x > c$ and $t \in \{2, 3, ...\}$ such that

$$\frac{d}{dx}\left[\exp\left(-k\frac{\beta}{1-\beta}u\left(c+x\right)\right) - \exp\left(-k\cdot\left(u\left(c\right)\frac{\beta-\beta^t}{1-\beta} + u\left(c+x\right)\frac{\beta^t}{1-\beta}\right)\right)\right] > 0$$

Evaluating the derivative, the previous inequality becomes

$$\exp\left(-k\frac{\beta}{1-\beta}u\left(c+x\right)\right) < \beta^{t-1}\exp\left(-k\cdot\left(u\left(c\right)\frac{\beta-\beta^t}{1-\beta} + u\left(c+x\right)\frac{\beta^t}{1-\beta}\right)\right).$$

Since both sides are positive, we can take logs to obtain:

$$-k\frac{\beta}{1-\beta}u\left(c+x\right) < \left(t-1\right)\ln\beta - k\cdot\left(u\left(c\right)\frac{\beta-\beta^t}{1-\beta} + u\left(c+x\right)\frac{\beta^t}{1-\beta}\right),$$

which can be rearranged as:

$$u\left(c+x\right) - u\left(c\right) > \frac{\left(t-1\right)\left(1-\beta\right)}{\beta-\beta^t}\cdot\frac{-\ln\beta}{k}. \tag{11}$$

Therefore, Stochastic Impatience fails if and only if there exist $x > c$ and $t = \{2, 3, ...\}$ such that (11) holds. For $t = 2$, condition (11) becomes

$$u\left(c+x\right) - u\left(c\right) > \frac{-\ln\beta}{\beta\cdot k}.$$

To complete the proof, we verify that this inequality holds for some t if and only if it holds for $t = 2$, that is:

$$\frac{-\ln\beta}{\beta\cdot k} \leq \frac{\left(t-1\right)\left(1-\beta\right)}{\beta-\beta^t}\cdot\frac{-\ln\beta}{k}$$

for all $t > 2$. To see this, rearrange the inequality above as

$$\left(t-1\right)\left(1-\beta\right) - 1 + \beta^{t-1} \geq 0. \tag{12}$$

At $\beta = 1$, both sides equal zero. The derivative of the expression on the RHS with respect to β is:

$$-\left(t-1\right) + \left(t-1\right)\beta^{t-2} = -\left(t-1\right)\left(1-\beta^{t-2}\right),$$

which is strictly negative for all $\beta \in [0, 1)$ and all $t > 2$. Thus, (12) holds for all $\beta \in [0, 1]$, concluding the proof. ■

C.4 Proof of Observation 2

For necessity, note that, when restricted to degenerate streams, the representation is a monotone transformation of $\sum_T D(t)u(x(t))$, so preferences must satisfy Assumption 1. Moreover, since risky lotteries are evaluated by taking expectations, preferences satisfy Assumption 2 as in Expected Utility Theory.

For sufficiency, by Assumption 1, there exist a strictly increasing and continuous $u : [\underline{x}, \overline{x}] \to \mathbb{R}$ and a strictly decreasing $D : T \to [0, 1]$ such that $\succcurlyeq$ restricted to $\mathcal{X}$ is represented by

$$F^*(\mathbf{x}) := \sum_{t \in T} D(t)u(x(t)).$$

Applying a positive transformation, the same preference is also represented by

$$F(\mathbf{x}) := \frac{1}{\sum_T D(t)} \sum_T D(t)u(x(t)),$$

Note that $F(\mathcal{X}) = u(C)$. By Assumption 2, there exists $U : \mathcal{X} \to \mathbb{R}$ such that $\succcurlyeq$ is represented by

$$V(p) := \mathbb{E}_p\big[U\big].$$

It follows that U and F represent the same preferences over $\mathcal{X}$, i.e., for all $\mathbf{x}, \mathbf{y} \in \mathcal{X}$,

$$U(\mathbf{x}) \geq U(\mathbf{y}) \Leftrightarrow \mathbf{x} \succcurlyeq \mathbf{y} \Leftrightarrow F(\mathbf{x}) \geq F(\mathbf{y}). \tag{13}$$

Therefore, there must exist an increasing $\phi : u(C) \to \mathbb{R}$ such that $U = \phi \circ F$.

We claim that ϕ must be strictly increasing. Suppose not. Then, there are $a, b \in u(C)$ with $a > b$ and $\phi(a) = \phi(b)$. Consider the streams $\mathbf{x}$ and $\mathbf{y}$ that return $u^{-1}(a)$ and $u^{-1}(b)$ each period, respectively. Since $a > b$ we must have $F(\mathbf{x}) = a > b = F(\mathbf{y})$. At the same time, since $\phi(a) = \phi(b)$, we have $U(\mathbf{x}) = \phi(F(\mathbf{x})) = \phi(a) = \phi(b) = \phi(F(\mathbf{y})) = U(\mathbf{y})$, violating (13).

The uniqueness claims follow from the same arguments as in the Expected Utility Theorem. ■

C.5 Proof of Proposition 3

The proof will be presented in three lemmas.

Lemma 4. *Preferences are Residual Risk Averse (Seeking) if for all* $v_1, v_2 \in [u(\underline{x}), u(\overline{x})]$,

$$\phi\left(\gamma v_1 + (1 - \gamma)\, v_2\right) + \phi\left(\gamma v_2 + (1 - \gamma)\, v_1\right) \geq (\leq) \phi\left(v_1\right) + \phi\left(v_2\right) \tag{14}$$

where $\gamma \equiv \frac{D(1)}{D(1)+D(2)} \in (\frac{1}{2}, 1)$.

Proof. Note that by Definition 2 and the KM representation, preferences display Residual Risk Aversion whenever:

$$\phi\left(\frac{D(1)u(a)+D(2)u(d)+\sum_{t\notin\{1,2\}}D(t)u(x)}{\sum_T D(t)}\right) = \phi\left(\frac{[D(1)+D(2)]\,u(b)+\sum_{t\notin\{1,2\}}D(t)u(x)}{\sum_T D(t)}\right)$$

and

$$\phi\left(\frac{D(1)u(d)+D(2)u(a)+\sum_{t\notin\{1,2\}}D(t)u(x)}{\sum_T D(t)}\right) = \phi\left(\frac{[D(1)+D(2)]\,u(c)+\sum_{t\notin\{1,2\}}D(t)u(x)}{\sum_T D(t)}\right)$$

imply

$$\frac{\phi\left(u(b)\right)+\phi\left(u(c)\right)}{2} \geq \frac{\phi\left(u(a)\right)+\phi\left(u(d)\right)}{2}.$$

Since ϕ is strictly increasing, the first two equations can be simplified as:

$$u(b)=\frac{D(1)u(a)+D(2)u(d)}{D(1)+D(2)} \text{ and } u(c)=\frac{D(1)u(d)+D(2)u(a)}{D(1)+D(2)}.$$

Therefore, Residual Risk Aversion holds if and only if, for all a and all d,

$$\phi\left(\frac{D(1)u(a)+D(2)u(d)}{D(1)+D(2)}\right)+\phi\left(\frac{D(1)u(d)+D(2)u(a)}{D(1)+D(2)}\right) \geq \phi\left(u(a)\right)+\phi\left(u(d)\right). \quad (15)$$

Letting $\gamma \equiv \frac{D(1)}{D(1)+D(2)}$, $v_1 \equiv u(a)$, and $v_2 \equiv u(d)$ concludes the proof. □

Lemma 5. *Let (ϕ, u, D) be a KM representation of $\succcurlyeq$.*

- *If ϕ is discontinuous at any point $v \neq u\left(\underline{x}\right)$, then $\succcurlyeq$ is not Residual Risk Averse.*
- *If ϕ is discontinuous at any point $v \neq u\left(\overline{x}\right)$, then $\succcurlyeq$ is not Residual Risk Seeking.*

Proof. Suppose ϕ is discontinuous at $v > u\left(\underline{x}\right)$. Let $\{h_n\} \searrow v$ be a non-increasing sequence that converges to v and $\{l_n\} \nearrow v$ be a non-decreasing sequence that converges to v. Let

$$\phi_+ := \lim_{n\to\infty}\phi(h_n) > \lim_{n\to\infty}\phi(l_n) = \phi_-.$$

For each n, let $u_{a_n} := h_n$ and $u_{d_n} := \frac{l_n - \gamma h_n}{1-\gamma}$. Note that

$$u_{d_n} < \gamma u_{d_n} + (1-\gamma)\, u_{a_n} < \gamma u_{a_n} + (1-\gamma)\, u_{d_n} = l_n < v < h_n = u_{a_n}.$$

Since ϕ is bounded (by $\phi(u(\underline{x})$ and $\phi(u(\overline{x}))$, we can assume that the sequences $\{\phi(\gamma u_{a_n} + (1-\gamma)\, u_{d_n})\}$, $\{\phi(\gamma u_{d_n} + (1-\gamma)\, u_{a_n})\}$, $\{\phi(u_{a_n})\}$, and $\{\phi(u_{d_n})\}$ are convergent (taking a subsequence if necessary). Therefore,

$$\lim_{n\to\infty} \phi(u_{d_n}) = \lim_{n\to\infty} \phi(\gamma u_{d_n} + (1-\gamma)\, u_{a_n}) = \lim_{n\to\infty} \phi(\gamma u_{a_n} + (1-\gamma)\, u_{d_n}) = \phi_-,$$

and

$$\lim_{n\to\infty} \phi(u_{a_n}) = \phi_+ > \phi_-.$$

Therefore, there exists $\bar{n}$ such that for all $n > \bar{n}$,

$$\phi(\gamma u_{a_n} + (1-\gamma)\, u_{d_n}) + \phi(\gamma u_{d_n} + (1-\gamma)\, u_{a_n}) < \phi(u_{a_n}) + \phi(u_{d_n}),$$

which, by (14), shows that preferences are not Residual Risk Averse.

Next, suppose ϕ is discontinuous at $v < u(\overline{x})$. Let $\{h_m\} \searrow v$ be a non-increasing sequence that converges to v, let $\{l_m\} \nearrow v$ be an increasing sequence that converges to v. As before, let

$$\phi_+ := \lim_{m\to\infty} \phi(h_m) > \lim_{m\to\infty} \phi(l_m) = \phi_-,$$

where the limits exist by the Monotone Convergence Theorem.

For each m, take $u_{d_m} := l_m$ and take $u_{a_m} = \frac{h_m - \gamma l_m}{1-\gamma}$. Note that

$$u_{a_m} > \gamma u_{a_m} + (1-\gamma)\, u_{d_m} > \gamma u_{d_m} + (1-\gamma)\, u_{a_m} = h_m > x > l_m = u_{d_m}.$$

As before (taking a subsequence if necessary), we have

$$\lim_{m\to\infty} \phi(u_{a_m}) = \lim_{m\to\infty} \phi(\gamma u_{a_m} + (1-\gamma)\, u_{d_m}) = \lim_{m\to\infty} \phi(\gamma u_{d_m} + (1-\gamma)\, u_{a_m}) = \phi_+,$$

and

$$\lim_{m\to\infty} \phi(u_{d_m}) = \phi_- < \phi_+.$$

Thus, there exists $\bar{m}$ such that for all $m > \bar{m}$,

$$\phi(\gamma u_{a_m} + (1-\gamma)\, u_{d_m}) + \phi(\gamma u_{d_m} + (1-\gamma)\, u_{a_m}) > \phi(u_{a_m}) + \phi(u_{d_m}),$$

showing that preferences are not Residual Risk Seeking. $\square$

Lemma 6. *Let (ϕ, u, D) be a KM representation of $\succcurlyeq$.*

- *$\succcurlyeq$ is Residual Risk Averse if and only if ϕ is concave.*
- *$\succcurlyeq$ is Residual Risk Seeking if and only if ϕ is convex.*

Proof. To establish sufficiency, suppose, without loss of generality, that $v > w$, so that

$$v > \gamma v + (1 - \gamma)w > \gamma w + (1 - \gamma)v > w.$$

It follows from the definition of concavity (convexity) and inequality (14) that preferences are Residual Risk Averse (Seeking) if ϕ is concave (convex). We now establish necessity.

Suppose preferences are Residual Risk Averse. By the Lemma 5, ϕ must be continuous at any point $v > u(\underline{x})$. We need to show that ϕ is concave. Suppose not. Then, there exist $v, w \in [u(\underline{x}), u(\overline{x})]$ with $v > w$ and $\lambda \in (0, 1)$ such that

$$\lambda\phi(v) + (1 - \lambda)\,\phi(w) > \phi\left(\lambda v + (1 - \lambda)\,w\right). \tag{16}$$

Let $F : [0, 1] \to \mathbb{R}$ given by

$$F(\tilde{\lambda}) \equiv \phi\left(\tilde{\lambda}v + \left(1 - \tilde{\lambda}\right)w\right) - \left[\tilde{\lambda}\phi(v) + \left(1 - \tilde{\lambda}\right)\phi(w)\right],$$

and note that $F(\lambda) < 0$, while $F(1) = F(0) = 0$. Since ϕ can only be discontinuous at $u(\underline{x})$, $F(\tilde{\lambda})$ is continuous at all $\tilde{\lambda} > 0$. It is continuous at $\tilde{\lambda} = 0$ if either $w > u(\underline{x})$ or if ϕ is continuous at $u(\underline{x})$.

Let

$$L \equiv \left\{\tilde{\lambda} \in [0, \lambda] : \; F(\tilde{\lambda}) \leq 0\right\} \text{ and } H \equiv \left\{\tilde{\lambda} \in [\lambda, 1] : \; F(\tilde{\lambda}) \geq 0\right\}.$$

Let

$$l \equiv \sup L \text{ and } h \equiv \inf H.$$

Because $F(\tilde{\lambda})$ is continuous at all $\tilde{\lambda} > 0$ and $F(\lambda) < 0$, it follows that $l < \lambda < h$. Moreover, it follows from the definitions of the supremum and infimum that

$$F\left(\tilde{\lambda}\right) < 0 \quad \forall\tilde{\lambda} \in (l, h).$$

We claim that $F(l) = 0$. There are two cases to consider. If F is continuous at 0, then L is a compact and non-empty set ($0 \in L$), which implies that $F(l) = 0$. Suppose, instead, that F is discontinuous at 0, which can only happen if $w = u(\underline{x})$ and ϕ is discontinuous at $u(\underline{x})$. Because ϕ is increasing, the discontinuity must correspond to an upwards jump: $\phi(u(\underline{x})) < \lim_{z \searrow u(\underline{x})} \phi(z) =: \phi(u(\underline{x})_+)$. Since

$$\begin{aligned}\lim_{\tilde{\lambda} \searrow 0} F(\tilde{\lambda}) = \lim_{\tilde{\lambda} \searrow 0} \left\{ \phi\left(\tilde{\lambda} v + \left(1 - \tilde{\lambda}\right) w\right) - \left[\tilde{\lambda}\phi(v) + \left(1 - \tilde{\lambda}\right)\phi(w)\right]\right\} \\ = \phi\left(u(\underline{x})_+\right) - \phi(u(\underline{x})) > 0,\end{aligned}$$

and F is continuous at any $\tilde{\lambda} > 0$, there exists $\bar{\lambda} > 0$ such that $F(\tilde{\lambda}) > 0$ for all $\tilde{\lambda} \in (0, \bar{\lambda})$. Hence,again by continuity of F for $\tilde{\lambda} > 0$, $l \geq \bar{\lambda}$ by the definition of supremum. Therefore,

$$l \equiv \sup L = \sup \left\{ \tilde{\lambda} \in [\bar{\lambda}, \lambda] : \ F(\tilde{\lambda}) \leq 0 \right\}.$$

Because $\left\{ \tilde{\lambda} \in [\bar{\lambda}, \lambda] : \ F(\tilde{\lambda}) \leq 0 \right\}$ is compact ($F(\tilde{\lambda})$ is continuous for all $\tilde{\lambda} > 0$) and non-empty (λ belongs to it), it again follows that $F(l) = 0$.

Next, we show that $F(h) = 0$. Because $F(\lambda) < 0$ and $F(0) = 0$, (16) implies that $\lambda > 0$. Therefore, $F(\tilde{\lambda})$ is continuous at $[\alpha, 1]$, implying that H is a compact set. Because it is also non-empty ($1 \in H$), we must have $F(h) = 0$.

Substituting the definition of F, we have shown:

$$l\phi(v) + (1 - l)\,\phi(w) = \phi\left(lv + (1 - l)\,w\right), \tag{17}$$

$$h\phi(v) + (1 - h)\,\phi(w) = \phi\left(hv + (1 - h)\,w\right), \tag{18}$$

and

$$\tilde{\lambda}\phi(v) + \left(1 - \tilde{\lambda}\right)\phi(w) > \phi\left(\tilde{\lambda} v + \left(1 - \tilde{\lambda}\right) w\right) \tag{19}$$

for all $\tilde{\lambda} \in (l, h)$.

Let $w' \equiv lv + (1 - l)w$ and $v' \equiv hv + (1 - h)w$, so that $w < w' < v' < v$. Note that, for all $\lambda \in (0, 1)$, we have

$$\begin{aligned} \lambda w' + (1 - \lambda)\, v' &= \lambda\left[lv + (1 - l)w\right] + (1 - \lambda)\left[hv + (1 - h)w\right] \\ &= \left[\lambda l + (1 - \lambda)\, h\right] v + \left\{1 - \left[\lambda l + (1 - \lambda)\, h\right]\right\} w \end{aligned}. \tag{20}$$

Since $\lambda l + (1 - \lambda)\, h \in (l, h)$, we have

$$\begin{aligned} \phi\left(\lambda w' + (1 - \lambda)\, v'\right) &= \phi\left(\left[\lambda l + (1 - \lambda)\, h\right] v + \left\{1 - \left[\lambda l + (1 - \lambda)\, h\right]\right\} w\right) \\ &< \left[\lambda l + (1 - \lambda)\, h\right]\phi(v) + \left\{1 - \left[\lambda l + (1 - \lambda)\, h\right]\right\}\phi\left(w\right) \\ &= \lambda\left[l\phi(v) + (1 - l)\phi(w)\right] + (1 - \lambda)\left[h\phi(v) + (1 - h)\phi(w)\right] \\ &= \lambda\phi\left(lv + (1 - l)\, w\right) + (1 - \lambda)\,\phi\left(hv + (1 - h)\, w\right) \\ &= \lambda\phi\left(w'\right) + (1 - \lambda)\,\phi\left(v'\right) \end{aligned}$$

for all $\lambda \in (0,1)$, where the first line uses (20), the second line uses equation (19), the third line follows from algebraic manipulations, the fourth line uses (17) and (18), and the last line substitutes the definitions of v' and w'. Since this inequality holds for all $\lambda \in (0,1)$, in particular, it must hold for γ and $1-\gamma$:

$$\phi\left(\gamma w' + (1-\gamma)v'\right) < \gamma\phi\left(w'\right) + (1-\gamma)\phi\left(v'\right)$$

and

$$\phi\left(\gamma v' + (1-\gamma)w'\right) < \gamma\phi\left(v'\right) + (1-\gamma)\phi\left(w'\right).$$

Combining these two inequalities, gives

$$\phi\left(\gamma w' + (1-\gamma)v'\right) + \phi\left(\gamma v' + (1-\gamma)w'\right) < \phi\left(w'\right) + \phi\left(v'\right),$$

showing that Residual Risk Aversion fails. The proof for Residual Risk Seeking is analogous. □

This concludes the proof of Proposition 3. ■

C.6 Proof of Proposition 4

Before presenting the proof, it is helpful to rewrite Stochastic Impatience in terms of the KM model. Stochastic Impatience holds if and only if, for all $x, y, c \in C$ with $x > y > c$ and all $t_1, t_2 \in T$ with $t_1 < t_2$,

$$\begin{aligned}
&\phi\left(\frac{\sum_{\tilde{t}<t_1} D(\tilde{t})u(c) + \sum_{\tilde{t}\geq t_1} D(\tilde{t})u(x)}{\sum_{\tilde{t}=1}^{\bar{t}} D(\tilde{t})}\right) + \phi\left(\frac{\sum_{\tilde{t}<t_2} D(\tilde{t})u(c) + \sum_{\tilde{t}\geq t_2} D(\tilde{t})u(y)}{\sum_{\tilde{t}=1}^{\bar{t}} D(\tilde{t})}\right)\\
\geq\ &\phi\left(\frac{\sum_{\tilde{t}<t_1} D(\tilde{t})u(c) + \sum_{\tilde{t}\geq t_1} D(\tilde{t})u(y)}{\sum_{\tilde{t}=1}^{\bar{t}} D(\tilde{t})}\right) + \phi\left(\frac{\sum_{\tilde{t}<t_2} D(\tilde{t})u(c) + \sum_{\tilde{t}\geq t_2} D(\tilde{t})u(x)}{\sum_{\tilde{t}=1}^{\bar{t}} D(\tilde{t})}\right)
\end{aligned}$$

Let $d(t) \equiv \frac{\sum_{\tilde{t}\geq t} D(\tilde{t})}{\sum_{\tilde{t}=1}^{\bar{t}} D(\tilde{t})} \in (0,1)$. Then, the previous condition becomes

$$\phi\left(d(t_1)u(x) + (1-d(t_1)\,u(c))\right) + \phi\left(d(t_2)u(y) + (1-d(t_2))\,u(c)\right)$$

$$\geq \phi\left(d(t_1)u(y) + (1-d(t_1))\,u(c)\right) + \phi\left(d(t_2)u(x) + (1-d(t_2))\,u(c)\right)$$

for all $u(x), u(y), u(c) \in [u(\underline{x}), u(\overline{x})]$ with $u(x) > u(y) > u(c)$ and all $d(t_1), d(t_2) \in \left\{\frac{\sum_{\tilde{t}\geq t}^{\bar{t}} D(\tilde{t})}{\sum_{\tilde{t}=1}^{\bar{t}} D(\tilde{t})}\right\}_{t\in T}$ with $d(t_1) > d(t_2)$.

Proof of Part (i). Let $\phi = g \circ \ln$ for some increasing and convex function g. It suffices to show that

$$g\left(\ln\left(\lambda u(x) + (1-\lambda)\,u(c)\right)\right) + g\left(\ln\left(\gamma u(y) + (1-\gamma)\,u(c)\right)\right) \\ \geq g\left(\ln\left(\lambda u(y) + (1-\lambda)\,u(c)\right)\right) + g\left(\ln\left(\gamma u(x) + (1-\gamma)\,u(c)\right)\right) \quad (21)$$

for all $u(x) > u(y) > u(c) \geq u(\underline{x})$ and all $0 < \gamma < \lambda \leq 1$. Let

$$z \equiv \ln\left[\frac{\left(\lambda u(y) + (1-\lambda)\,u(c)\right)\left(\gamma u(x) + (1-\gamma)\,u(c)\right)}{\lambda u(x) + (1-\lambda)\,u(c)}\right],$$

so that:

$$\ln\left(\lambda u(x) + (1-\lambda)\,u(c)\right) + z \\ = \ln\left(\lambda u(y) + (1-\lambda)\,u(c)\right) + \ln\left(\gamma u(x) + (1-\gamma)\,u(c)\right). \quad (22)$$

Because $\lambda u(x) + (1-\lambda)\,u(c) > \max\left\{\lambda u(y) + (1-\lambda)\,u(c), \gamma u(x) + (1-\gamma)\,u(c)\right\}$, the equation above implies

$$z < \min\left\{\ln\left(\lambda u(y) + (1-\lambda)\,u(c)\right),\ \ln\left(\gamma u(x) + (1-\gamma)\,u(c)\right)\right\}. \quad (23)$$

Combining (22) and (23) with the fact that g is convex, we have

$$g\left(\ln\left(\lambda u(x) + (1-\lambda)\,u(c)\right)\right) + g\left(z\right) \\ \geq g\left(\ln\left(\lambda u(y) + (1-\lambda)\,u(c)\right)\right) + g\left(\ln\left(\gamma u(x) + (1-\gamma)\,u(c)\right)\right). \quad (24)$$

It can also be shown that

$$z < \ln\left(\gamma u(y) + (1-\gamma)\,u(c)\right).$$[22]

Then, since g is increasing, replacing $g\left(z\right)$ by $g\left(\ln\left(\gamma u(y) + (1-\gamma)\,u(c)\right)\right)$ in (24) implies (21).

Proof of Part (ii). Pick any $y \in (\underline{x}, \bar{x})$, any $t_2 > 1$, and recall that $d(t) \equiv \frac{\sum_{\tau=t}^{\bar{t}} D(\tau)}{\sum_{\tau=1}^{\bar{t}} D(\tau)} \in (0,1]$. Define $v_1 = u\left(\underline{x}\right) + d(t_2)\left(u\left(y\right) - u\left(\underline{x}\right)\right)$ and $v_2 = u\left(\bar{x}\right)$. Suppose $\phi(z)$ is more concave than $\ln(z - u(\underline{x}))$ in $[v_1, v_2]$. Then, there exists g strictly

[22]To see this, observe that the ratio $\frac{\lambda u(x)+(1-\lambda)u(c)}{\lambda u(y)+(1-\lambda)u(c)}$ is strictly increasing in λ. Therefore, if we replace z by $\ln\left(\gamma u(y) + (1-\gamma)\,u(c)\right)$ in (22), we will have that the left hand side is strictly greater than the right hand side.

concave in this range such that $\phi(z) = g(\ln(z - u(\underline{x}))$. Note that we must have

$$\ln\left(d(1)\left[u(\bar{x}) - u(\underline{x})\right]\right) + \ln\left(d(t_2)\left[u(y) - u(\underline{x})\right]\right) \\ = \ln\left(d(1)\left[u(y) - u(\underline{x})\right]\right) + \ln\left(d(t_2)\left[u(\bar{x}) - u(\underline{x})\right]\right).$$

Since $u(y) < u(\bar{x})$, $d(1) > d(t_2)$, and g is stricty concave, it follows that:

$$g\left(\ln\left(d(1)\left[u(\bar{x}) - u(\underline{x})\right]\right)\right) + g\left(\ln\left(d(t_2)\left[u(y) - u(\underline{x})\right]\right)\right) \\ < g\left(\ln\left(d(1)\left[u(y) - u(\underline{x})\right]\right)\right) + g\left(\ln\left(d(t_2)\left[u(\bar{x}) - u(\underline{x})\right]\right)\right),$$

so that

$$\phi\left(d(1)\left[u(\bar{x}) - u(\underline{x})\right] + u(\underline{x})\right) + \phi\left(d(t_2)\left[u(y) - u(\underline{x})\right] + u(\underline{x})\right) \\ < \phi\left(d(1)\left[u(y) - u(\underline{x})\right] + u(\underline{x})\right) + \phi\left(d(t_2)\left[u(\bar{x}) - u(\underline{x})\right] + u(\underline{x})\right).$$

Substituting back from the definition of $d(\cdot)$ and noting that $d(1) = 1$, we obtain:

$$\phi\left(u(\bar{x})\right) + \phi\left(\frac{\sum_{t=t_2}^{\bar{t}} D(t)u(y) + \sum_{t=1}^{t_2} D(t)u(\underline{x})}{\sum_{t=1}^{\bar{t}} D(t)}\right) \\ < \phi\left(u(y)\right) + \phi\left(\frac{\sum_{t=t_2}^{\bar{t}} D(t)u(\bar{x}) + \sum_{t=1}^{t_2} D(t)u(\underline{x})}{\sum_{t=1}^{\bar{t}} D(t)}\right).$$

Hence $\frac{1}{2}\bar{x} + \frac{1}{2}(\underline{x}, t_2, y) \prec \frac{1}{2}y + \frac{1}{2}(\underline{x}, t_2, \bar{x})$, violating Stochastic Impatience.

Proof of part (iii). Consider a preference relation $\succcurlyeq$ with KM representation (ϕ, u, D). Define ϕ^c as the concave envelope of ϕ. By definition, ϕ^c is both concave and more concave than ϕ.

Now consider $\phi'(v) = \phi^c\big(g(\ln(v - u(\underline{x})))\big)$ for some concave $g : \mathbb{R} \to \mathbb{R}$. Consider the preference $\succcurlyeq'$ induced by the KM representation (ϕ', u, D). Because both ϕ^c and g are concave, $\phi'(v)$ is more concave than $\ln(v - u(\underline{x}))$: by Proposition 4, $\succcurlyeq'$ violates Stochastic Impatience. Because $\ln(v - u(\underline{x}))$ and $g(v)$ are concave, ϕ' is more concave than ϕ^c, and thus of ϕ. By Observation 5, $\succcurlyeq'$ has more Residual Risk Aversion than $\succcurlyeq$.

Proof of part (iv). The statement Follows from part (i) and Observation 5. ■

C.7 Proof of Observations 3, 4, and 5

Observation 3 is due to Proposition 3 and the fact that KM coincides with EDU if and only if ϕ is affine. Observation 4 follows from Proposition 3 and the KM representation of EZ given in Example 4. Observation 5 follows directly from Proposition 3. ■

C.8 Proof of Proposition 5

The proof of Proposition 5 will be presented though two lemmas. We first show that items 1 and 3 are equivalent to each other:

Lemma 7. *Stochastic Impatience' holds if and only if* ϕ *is less concave than* $\ln(v - u(\underline{x}))$.

Proof. The fact that Stochastic Impatience' holds if ϕ is less concave than $\ln(v-u(\underline{x}))$ follows by the same exact argument as in the discrete time case (see proof of part (i) of Proposition 4). We now show that Stochastic Impatience' fails if ϕ is not less concave than $\ln(v - u(\underline{x}))$.

For notational simplicity, let $h(x) \equiv u(x) - u(\underline{x})$. Let $\phi(v) = g \circ \ln$ for some increasing function $g : h(C) \to \mathbb{R}$. Suppose g is not convex. Then, there exist $H > L$ and $\epsilon > 0$ such that $L, H, (L + \epsilon), (H + \epsilon)$ are in the interior of $\ln(h(C))$ and

$$g(H + \epsilon) - g(H) < g(L + \epsilon) - g(L). \tag{25}$$

Let $y \equiv h^{-1}(\exp(H))$, $x \equiv h^{-1}(\exp(\epsilon + H))$, and $\lambda \equiv \exp(L - H) \in (0, 1)$, so that

$$H = \ln(h(y)), \ H + \epsilon = \ln(h(x)), \text{ and } L = \ln(\lambda h(y)).$$

Note that $x > y > \underline{x}$ (since H and L are in the interior of C). Moreover, a straightforward algebraic manipulation yields

$$L + \epsilon = \ln(\lambda h(y)) + \ln(h(x)) - \ln(h(y)) = \ln(\lambda h(x)).$$

Therefore, (25) becomes

$$g(\ln(h(x))) - g(\ln(h(y))) < g(\ln(\lambda h(x)) - g(\ln(\lambda h(y))),$$

which can be rearranged as:

$$g(\ln(u(x)-u(\underline{x})))+g(\ln(\lambda(u(y)-u(\underline{x})))) < g(\ln(u(y)-u(\underline{x})))+g(\ln(\lambda(u(x)-u(\underline{x})))),$$

violating Stochastic Impatience' with $c = \underline{x}$, $t_1 = 0$, and $t_2 \in (0, \bar{t})$ such that $\lambda = \frac{\int_{t_2}^{\bar{t}} D(t)dt}{\int_0^{\bar{t}} D(t)dt}$. $\square$

Lemma 7 established that $1 \iff 3$. Since Strong Stochastic Impatience becomes Stochastic Impatience' if we take $\delta = +\infty$, it follows that $2 \implies 1$. To conclude the proof, it suffices to show that $3 \implies 2$.

Lemma 8. *Suppose ϕ is less concave than $\ln(v - u(\underline{x}))$. Then, preferences satisfy Strong Stochastic Impatience.*

Proof. Use the representation to obtain the value of $\frac{1}{2}(c, t_1, x, \delta) + \frac{1}{2}(c, t_2, y, \delta)$:

$$\frac{\phi\left(\lambda_{t_1}^{\delta} u(c+x) + \left(1 - \lambda_{t_1}^{\delta}\right) u(c)\right) + \phi\left(\gamma_{t_2}^{\delta} u(c+y) + \left(1 - \gamma_{t_2}^{\delta}\right) u(c)\right)}{2},$$

where $\lambda_{t_1}^{\delta} \equiv \frac{\int_{t \in [t_1, t_1+\delta] \cap T} D(t)dt}{\int_0^{\bar{t}} D(t)dt} > \gamma_{t_2}^{\delta} \equiv \frac{\int_{t \in [t_2, t_2+\delta] \cap T} D(\tilde{t})d\tilde{t}}{\int_0^{\bar{t}} D(\tilde{t})dt}$.

It suffices to show that if ϕ is less concave than $\ln(v - u(\underline{x}))$, then

$$\begin{aligned} &\phi\left(\lambda u(c+x) + (1-\lambda) u(c)\right) + \phi\left(\gamma u(c+y) + (1-\gamma) u(c)\right) \\ &\qquad \geq \phi\left(\lambda u(c+y) + (1-\lambda) u(c)\right) + \phi\left(\gamma u(c+x) + (1-\gamma) u(c)\right) \end{aligned} \tag{26}$$

for all $0 < \gamma < \lambda < 1$, $c < y < x$. Let $\phi(v) = g(\ln(v - u(\underline{x})))$ for some increasing and weakly convex g, and $u_x \equiv u(c+x) - u(\underline{x}) > u_y \equiv u(c+y) - u(\underline{x}) > u_c \equiv u(c) - u(\underline{x}) \geq 0$. Then equation (26) becomes:

$$\begin{aligned} &g\left(\ln\left(\lambda u_x + (1-\lambda) u_c\right)\right) + g\left(\ln\left(\gamma u_y + (1-\gamma) u_c\right)\right) \\ &\qquad \geq g\left(\ln\left(\lambda u_y + (1-\lambda) u_c\right)\right) + g\left(\ln\left(\gamma u_x + (1-\gamma) u_c\right)\right). \end{aligned} \tag{27}$$

We now show that the points on the LHS of (27) have both a higher mean and a higher spread than the points on the LHS. It will then follow from the fact that g is increasing and weakly convex that the inequality holds. To see that the points on the LHS have a higher mean, note that

$$\begin{aligned} \ln\left(\lambda u_x + (1-\lambda) u_c\right) + \ln\left(\gamma u_y + (1-\gamma) u_c\right) &\geq \ln\left(\lambda u_y + (1-\lambda) u_c\right) + \ln\left(\gamma u_x + (1-\gamma) u_c\right) \\ \iff \left[\lambda(1-\gamma) - (1-\lambda)\gamma\right] u_c u_x &\geq \left[(1-\gamma)\lambda - (1-\lambda)\gamma\right] u_c u_y \\ \iff (\lambda - \gamma) u_c (u_x - u_y) &\geq 0, \end{aligned}$$

which is true since $\lambda > \gamma$, $u_x > u_y$, and $u_c \geq 0$. To see that the points on the LHS have a higher spread, note that

$$\begin{aligned} \ln\left(\lambda u_x + (1-\lambda) u_c\right) &> \max\left\{\ln\left(\lambda u_y + (1-\lambda) u_c\right),\ \ln\left(\gamma u_x + (1-\gamma) u_c\right)\right\} \\ &> \ln\left(\gamma u_y + (1-\gamma) u_c\right). \end{aligned}$$

Thus, condition (27) holds. $\square$

This conclues the proof of Proposition 5. $\blacksquare$

C.9 Proof of Proposition 6

Take $c = \underline{x}$ and, for simplicity, assume that $u(\underline{x}) = 0$.[23] Let $x_1 \in \text{int}\,(C)$ be such that $\phi'(u(x_1)) > 0$ (which exists because ϕ is differentiable and strictly increasing). Fix $t_1 = 0$ and $t_2 > 0$ and let x_2 be such that:

$$u\left(x_2\right) = u\left(x_1\right) \frac{\int_{t_2}^{\bar{t}} D(t)dt}{\int_0^{\bar{t}} D(t)dt}. \tag{28}$$

Note that $x_2 \in \text{int}\,(C)$ because $0 < u\left(x_2\right) < u\left(x_1\right)$. Thus, by construction, $\left(c, t_2, x_1\right) \sim \left(c, t_1, x_2\right)$.

We will show that, if t_2 is close enough to 0, then

$$\begin{aligned} &\pi\left(\frac{1}{2}\right)\phi\left(u(x_1)\right) + (1 - \pi\left(\frac{1}{2}\right))\phi\left(u(x_2)\frac{\int_{t_2}^{\bar{t}} D(t)dt}{\int_0^{\bar{t}} D(t)dt}\right) \\ &\qquad = \pi\left(\frac{1}{2}\right)\phi\left(u(x_1)\right) + (1 - \pi\left(\frac{1}{2}\right))\phi\left(u\left(x_1\right)\left(\frac{\int_{t_2}^{\bar{t}} D(t)dt}{\int_0^{\bar{t}} D(t)dt}\right)^2\right) \\ &\qquad\qquad < \phi\left(u(x_1)\frac{\int_{t_2}^{\bar{t}} D(t)dt}{\int_0^{\bar{t}} D(t)dt}\right) \end{aligned}$$

where the equality above uses (28). First note that both sides equal $\phi(u(x_1))$ for $t_2 = 0$. We now show that the LHS falls faster than RHS when we increase t_2 slightly, generating a violation of Stochastic Impatience. By Leibniz's rule we have

$$\frac{\partial}{\partial t_2}\phi\left(u(x_1)\frac{\int_{t_2}^{\bar{t}} D(t)dt}{\int_0^{\bar{t}} D(t)dt}\right)_{|t_2=0} = -\phi'\left(u(x_1)\right)\frac{D(0)}{\int_0^{\bar{t}} D(t)dt}$$

whereas

$$\begin{aligned} &\frac{\partial}{\partial t_2}\left[\pi\left(\frac{1}{2}\right)\phi\left(u(x_1)\right) + (1 - \pi\left(\frac{1}{2}\right))\phi\left(u\left(x_1\right)\left(\frac{\int_{t_2}^{\bar{t}} D(t)dt}{\int_0^{\bar{t}} D(t)dt}\right)^2\right)\right]_{|t_2=0} \\ &\qquad\qquad = -(1 - \pi\left(\frac{1}{2}\right))\phi'\left(u\left(x_1\right)\right)\frac{2D(0)}{\int_0^{\bar{t}} D(t)dt} \end{aligned}$$

[23] Note that this is without loss as u can always be normalized as long as ϕ is appropriately adjusted to maintain the total utility value unchanged.

So we want to show that

$$\frac{-\phi'\left(u(x_1)\right)D(0)}{\int_0^{\bar{t}} D(t)dt} > -\left[1-\pi\left(\frac{1}{2}\right)\right]\phi'\left(u\left(x_1\right)\right)\frac{2D(0)}{\int_0^{\bar{t}} D(t)dt}$$

Since $\frac{\phi'(u(x_1))D(0)}{\int_0^{\bar{t}} D(t)dt} > 0$, this is true if and only if $\pi(\frac{1}{2}) < \frac{1}{2}$. ■

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