Firm Dynamics with Downward Nominal Wage Rigidity *

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Abstract

We propose a framework that integrates (i) firm dynamics á la Hopenhayn and Rogerson (1993); (ii) wage posting with on-the-job search á la Burdett and Mortensen (1998); and (iii) downward nominal wage rigidity constraint in wage settings. We use the framework to quantify the efficiency losses from downward nominal wage rigidity by comparing with a counterfactual economy with flexible wage, which we show is constrained efficient.

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1 Introduction

Downward nominal wage rigidity (DNWR) is a well-documented feature of labor markets across countries and sectors, with important implications for wage setting, employment dynamics, and worker allocation (Grigsby, Hurst and Yildirmaz (2021), Elsby and Solon (2019)). Workers strongly resist wage cuts, and firms, anticipating this aversion, take it into account when setting wages. This rigidity affects how firms respond to economic fluctuations, potentially distorting employment outcomes and the efficient allocation of labor.

In this paper, we propose a framework that integrates firm dynamics á la Hopenhayn and Rogerson (1993) and wage posting with on-the-job search á la Burdett and Mortensen (1998) with downward nominal wage rigidity and use it to evaluate the missallocation effects of the later. Multi-worker firms subject to productivity shocks post contracts in the form of a promised utility as in Spear and Srivastava (1987) and the constraint that the nominal wage cannot decrease. Real wages decrease at the inflation rate unless firms decide to raise them and therefore inflation "grease the wheels" of the labor market (Tobin (1972)). On the other side of the market forward-looking workers search for jobs while employed and move to firms offering a higher utility.

We start by characterizing the equilibrium with flexible wages. Firms inherit some value of past utility promises, but we show that they can choose the current posted utility without any restriction. This is because the wage adjusts to satisfy the promise-keeping constraint of the workers. A worker with a given promised utility is compensated in two ways: she receives a wage and a promised utility in the future. If the firm wants to post high utility contracts, it lowers current wages. Conversely, if it posts low utility contracts, it has to increase wages today to compensate the workers. Nor do workers or firms care about wages per se, and only the promised utility is the relevant state, which can jump at any moment. Despite our focus on DNWR, we provide a tractable model of firm dynamics with wage posting tractable enough to be extended in several directions, which we leave for further research.

Once we introduce downward wage rigidity, changing the utility offered by firms has a cost and wages play a key role in defining firm policies. A firm that wants to increase the utility offered to new workers can't lower the wage to incumbents, and therefore they receive a net transfer. On the other end, a firm that wants to lower posted utility has to raise wages to compensate incumbent workers, but then is stuck at a higher than desired wage for a while. Eventually, inflation erodes the real wage and gets it back to the desired level for the firm. DNWR acts as an adjustment cost in posted utility, dampening its response, and also the response of wages.

We use the model to assess the misallocation effects of DNWR and how inflation alleviates them. With higher inflation, the nominal rigidity is less restrictive because real wages decrease at a faster rate. We calibrate to match the characteristics of the US labor market and perform several exercises. First, we test how much firms value wage rigidity in partial equilibrium. Taking the steady state distribution of offers, the value of a firm that is not subject to DNWR would increase by around 10%. Next, we compare how the equilibrium changes, in general equilibrium, when *all* firms can post flexible wages, the unmeployment rate goes from XX% to XX% and labor productivity rises XX%.

Finally, we quantify the general equilibrium effects of moving from a 2% inflation target to a 4% target. Unlike getting rid of DNWR, the inflation target is a parameter that the policy maker has control. A higher inflation rate increases employment by XX% and labor productivity by XX% due to an improved allocation of workers.

Having studied the properties of the steady state, we turn to the dynamics of the model. For that, we use the Sequente Space Jacobian method Auclert, Bardóczy, Rognlie and Straub (2021) to simulate the effect of inflation and interest rate shocks.

Literature Review Our model builds on the long-stranding literature on firm dyanmics pioneered by Hopenhayn and Rogerson (1993). Like Hopenhayn and Rogerson (1993), our model features firms operate decreasing returns to scale production function subject idiosincratic shocks to productivity, as well as the endogenous mass of firms through the free entry condition. Unlike Hopenhayn and Rogerson (1993), firms do not take wages as given in the presence of search frictions. Instead, firms offer long-term contracting trading off higher expenses with higher retention rates of workers. In this sense, our model has many features in common with recent literature that combines labor market frictions with firm dynamics (Bilal, Engbom, Mongey and Violante, 2022; Elsby and Gottfries, 2022; Schaal, 2017; McCrary, 2022). Our model differs not only in adding down nominal wage rigidity constraints but also in adopting Burdett and Mortensen (1998) wage setting protocol.

Finally, we relate to the wage rigidity literature. In the search and matching framework, Gertler, Trigari, Journal and February (2009) propose a Nash-bargaining problem with staggered actions a la Calvo. Blanchard and Galí (2010) also propose a search model with real wage rigidities. In both cases, only aggregate shocks are considered. On the other side, Ehrlich and Montes (2024) present a model with idiosicnratcic firm risk and wage rigidity, but with an reduced form labor supply curve and solve the problem in partial equilibrium.

2 Model

Time is continuous and the horizon is infinite. Until Section XXX, we focus on the steady state of the economy and drop time subscript.

Production Technology. There is a single homogenous goods. Workers can be either unemployed or employed by a firm. When workers are unemployed, they produce b units of goods at home. A firm with productivity z employing n workers produce $Y(n, z)$ units of goods. We assume $Y(n, z)$ is strictly increasing in n and z, concave in n, and is homogenous of degree one. For example, a Cobb-Douglas production $Y(n, z) = z^{1-\alpha}n^{\alpha}$ with $\alpha \in (0, 1)$ satisfies this property.

Firm Dempgraphics. Firm's productivity z evolves stochastically following geometric Brownian motion:

$$dz = \mu z dt + \sigma z dW, \tag{1}$$

where μ is the drift parameter, $\sigma \geq 0$ governs the volatility, and dW denotes the increment of the standard Brownian motion. Firm's employment n evolves beacuse of four reasons. Firt, workers exogenous separate into unemployment at rate $s > 0$. Second, employed workers receive outside offers at rate $\lambda^E \geq 0$ and may quit to move to the other firms. Third, firms can hire workers by posting vacancies subject to hiring costs. Hiring costs of a firm with employment n take the form $c(h)n$, where h denotes the hiring rate (number of hires divided by current employment). We adopt the iso-elastic hiring cost function:

$$c(h) = \bar{c}\frac{h^{1+\nu}}{1+\nu}, \tag{2}$$

where $\bar{c} > 0$. In what follows, we will restrict attention to the case with $\nu > 1$ for the reasons we will describe. In the baseline model, we assume there is no vacancy cost, following Coles and Mortensen (2016) and Elsby and Gottfries (2022). As a result, firms post vacancies as much as needed to acheive the deisired level of hiring. Finally, firms can always fire workers with no cost.

There is free entry. A potential firm can pay fixed cost c^e to become an operating firm. After paying the fixed cost, firms draw initial employment size, n_0, and initial productivity, z_0, from a joint distribution function $\psi(n_0, z_0)$. Firms exit at an exogenous rate $\varkappa > 0$. We abstract from endogenous exit. We denote the total mass of firms in the economy as m and the mass of entrants per unit of time as m_0.

Preferences and Worker Demographics. All agents discount the future with the real interest rate $r > 0$ and are risk-neutral. We normalize the total mass of workers to be one. Let $u \in [0, 1]$ denote the fraction of unemployed workers.

Matching. An unemployed worker has a search efficiency of 1 and meets a firm at rate $\lambda^U > 0$. An employed worker has a search efficiency $\zeta > 0$ and meets another firm at rate $\lambda^E = \zeta\lambda^U > 0$.

As in Coles and Mortensen (2016), there is not an explicit matching function and instead, each vacancy matches with a worker regardless of their employment status

Cotracts. Firms offer wage contracts to workers. As in the tradition of Burdett and Mortensen (1998), we impose equal-treatment constraint within a firm. That is, firms have to offer the same wage w to all the workers within a firm, and whenever firms fire, firms have to randomize which workers to fire. This assumption rules out outside offer matching.

Firms have offer state-contingent wage contracts to all the workers with full commitment. Workers cannot commit. As in Spear and Srivastava (1987), firm's past commitment can be summarized by the promised utility to the workers. Let $\mathcal{W}$ be worker's utility and U be unemployment value. In what follows, we write the contracts in terms of worker utility in excess of unemployment value:

$$W = \mathcal{W} - U. \tag{3}$$

At time t, given the past commitment W as well as other state variables (n, z, w), firms offer a combination of changes in real wage w_t, changes in promised utility dW_t, and exposure to firm-level productivity shocks, a_t that satisfy the promise keeping constraint:

$$rW_t dt = w_t dt - rU dt - (s + \varkappa)W_t dt + \lambda^E \int \max\{\tilde{W}_t - W_t, 0\} dF(\tilde{W}) dt + dW_t + a_t \sigma dZ. \tag{4}$$

The first term on the right-hand side captures the worker's flow utility from wage payments. The second term captures the possibility of moving into unemployment through exogenous separation or firm exits. The third term captures the possibility of moving to other firms, and $F(W)$ denotes the endogenous offer distribution of the firms in the economy. The last term captures the continuation value from changes in firm's promises.

In equation (4), we have implicitly imposed that firms offer state non-contingent contracts because W_t is not conditioned on the realization of firm productivity z_t. Given the risk-neutrality of workers and firms, we view this is a natural benchmark assumption. In fact, without an additional wage constraint that we describe below, this assumption is inconsequential for anything. We relax this assumption in Section XXX. The long-term contracts must satisfy the no-Ponzi condition:

$$\lim_{T \to \infty} e^{-(r+s+\varkappa)T} W_T = 0 \tag{5}$$

for all possible path of $\{W_t\}$.

Cruicially, we assume that firms have to respect the downward nominal wage rigidity constraint. Let π denote the steady state inflation rate. We assume that real wages that firms offer cannot fall more than the inflation rate:

$$dw \geq -\pi w dt. \tag{6}$$

We assume the central bank determines the steady state inflation rate π, and we are primalily interested in the comparative statics with respect to π.

2.1 Firm's Problem

Given the state variables (z, n, W, w), firm's problem is to choose hiring rates h, how many workers to fire, how much utility to promise dW, and how much wages to offer dw in order to maximize present discounted value of profits. Let $\tilde{J}(z, n, W, w)$ denote the firm's value function. The following describes the firm's Bellman equation. In what follows, we use short-hand notation of $\partial_x J$ to denote $\frac{\partial J}{\partial x}$ and $\partial^2_{xx} J$ to denote for $\frac{\partial^2 J}{\partial x^2}$. We also let U denote the value of unemployment.

Problem 1. *The firm's value function is homogenous in z, $\tilde{J}(z, n, W, w) = J(\hat{n}, W, w)z$, where $\hat{n} \equiv n/z$. The value function $J \equiv J(\hat{n}, W, w)$ (dependence ommited for brevity) solves the following Hamiltonian-Jacobi-Bellman quasi-variational inequality (HJB-QVI):*

$$\min \left\{ \rho J(\cdot) - \max_{h,a} \left\{ y(\hat{n}) - w\hat{n} - c(h)\hat{n} + \mathcal{L}(h, a)J(\cdot) \right\}, J(\cdot) - J^*(\hat{n}, W, w) \right\} = 0, \tag{7}$$

where $\rho \equiv r + \varkappa - \mu$ and

$$\begin{aligned} \mathcal{L}(h, a)J \equiv\ & \hat{n} \left(h - \mu - s - \lambda^E (1 - F(W)) \right) J_n(\cdot) \\ & + \left[rW - \left[w - rU - (\varkappa + s)W + \lambda^E \textstyle\int \max\{W' - W, 0\} dF(W') \right] \right] J_W(\cdot) \\ & - \pi w J_w(\cdot) + \frac{\sigma^2 \hat{n}^2}{2} J_{nn}(\cdot) + a^2 \frac{\sigma^2}{2} J_{WW}(\cdot) + a\sigma^2 \left[J_W(\cdot) - \hat{n} J_{\hat{n}W}(\cdot) \right] \end{aligned} \tag{8}$$

and $y(\hat{n}) \equiv Y(1, \hat{n})$ and $J^*(\hat{n}, W, w)$ is the value from firing, raising promised utility, or raising wages:

$$J^*(\hat{n}, W, w) \equiv \max_{\hat{n}^* \leq \hat{n}, W^*, w^* \geq w} J(\hat{n}^*, W^*, w^*) \tag{9}$$

$$\textit{s.t.} \quad W^* \tfrac{n^*}{n} = W. \tag{10}$$

Although the orignal problem features four state variables, Problem 1 shows that the firm's problem simplifies to three-state Bellman equation. This is due to the homogeneity assumptions in production technology and hiring technology.

The first term in the minimum operator of equation (27) describes the Bellman equation when both constraints (4) and (6) bind, and the firms do not fire. This is the situation where there is no jump in any of the state variables. The firm's flow profits consist of revenue minus wage costs and hiring costs. The evolution of firm value consists of changes in the employment-to-proudctivity ratio ($\hat{n}$), changes in promised utility, and changes in real wages. The firm chooses optimal hiring rates to maximize the firm value, which results in the first-order condition of the form

$$\partial_{\hat{n}} J(\hat{n}, W, w) = c'(h). \tag{11}$$

The above equation equates the marginal value of a hire to the marginal cost of a hire. Let $h(\hat{n}, W, w)$ denote the policy function for h.

The second term in the minimum operator captures the value from a jump in state variables, which we denote as $J^*(\hat{n}, W, w)$. The term ensures that the firm's value function $J(\hat{n}, W, w)$ is always weakly greater than $J^*(\hat{n}, W, w)$. The jump can happen for three reasons, as described in equation (9). First, firms can always fire workers, in which case, firms have to promise higher utility to non-fired workers to fulfill the commitment, as captured by the constraint (10). Second, firms are always free to raise the promised utility above the past commitment level. Finally, firms can always raise wages.

Since the entrants do not have any past commitment or past wages, they choose W and w at the unconstrained optimum:

$$J^0(\hat{n}) \equiv \max_{W \geq U, w} J(\hat{n}, W, w), \tag{12}$$

conditional on the realization of $\hat{n} \equiv n_0/z_0$. Let $W^0(\hat{n})$ and $w^0(\hat{n})$ be the policy functions of entrants. The free-entry condition is

$$\int J^0(n^0/z^0) d\Psi(z^0, n^0) = c^e. \tag{13}$$

2.2 Aggregation and Equilibrium Definition

Let $g(z, \hat{n}, W, w)$ denote the steady state mass of firms with productivty z, employment-to-productivity $\hat{n}$, promised utility W and wages w. It satisfies the Kolmogorov Forward equation described in Appendix A.3. Let $G(z, \hat{n}, W, w)$ be the cumulative mass.

Now we aggregate firm's problem to define equilibrium of the economy. As in Coles and Mortensen (2016), there is not an explicit matching function and instead each vacancy matches with a worker regardless of their employment status. This means that the hiring rate and the vacancy rate satisfy

$$h(\hat{n}, W, w) = (p^u + (1 - p^u)H(W))v(\hat{n}, W, w) \quad . \tag{14}$$

With probability $p^u \equiv \frac{u}{u+\zeta(1-u)}$ vacancies meet unemployed workers who accept the offer right away. With probability $1 - p_u$ the worker is employed and accepts it with probability $H(W)$, which is the distribution of W of employed workers, given by

$$H(W) = \int_{\tilde{W} \leq W} \hat{n} z dG(z, \hat{n}, \tilde{W}, w) \quad . \tag{15}$$

Since the entrants post vacancies as much as needed to achieve the initial level of employment, a entrant with employment n and productivity z posts

$$v^0(z, n) = \frac{n}{p^u + (1 - p^u)H(W^0(\hat{n}))} \tag{16}$$

vacancies. The aggregate vacancy posting is given by

$$V = \int v(\hat{n}, W, w)\hat{n} z dG(z, \hat{n}, W, w) + m^0 \int v^0(n_0, z_0) d\Psi(z_0, n_0), \tag{17}$$

where the first term is the vacancy creation of incumbent firms, and the second term is the va- cancy creation of entrants. The offer distribution $F(W)$ is the vacancy weighted distribution of promised utility offered by firms:

$$F(W) = \frac{1}{V}\left[\int_{\tilde{W} \leq W} v(\hat{n}, \tilde{W}, w) dG(z, \hat{n}, \tilde{W}, w) + \int_{W^0(n_0/z_0) \leq W} v^0(n_0/z_0) d\Psi(z_0, n_0)\right]. \tag{18}$$

The value of unemployment, U, solves

$$rU = b + \lambda^U \int W dF(W). \tag{19}$$

In equilibrium, $\lambda^U = p^u V$. The steady state unemployment satisfies

$$u = 1 - \int \hat{n} z dG(z, \hat{n}, W, w) \tag{20}$$

We define the equilibrium of this economy as follows. (i) Given $\{F(W), \lambda^E, U\}$, firms solve Problem 1. (ii) The steady state firm distribution $g(z, \hat{n}, W, w)$ solve Kolmogorov forward equation described in Appendix A.3. (iii) The offer distribution $F(W)$ satisfies (**??**)-(18). (iv) The unemployment value satisfies (19). (v) The meeting rates satisfy (**??**), (**??**)-(17), and (20).

3 Flexible Wage Equilibrium

We first characterize the equilibrium without downward nominal wage rigidity constraint (6). This serves two purposes. First, it provides a benchmark against the model with downward nomina wage rigidity constraint that we study below. Second, it might be on its own interest. To the best of our knowledge, we are the first to combine the cononical model of firm dynamics à la Hopenhayn and Rogerson (1993) and wage posting models with on-the-job search à la Burdett and Mortensen (1998).

In the absence of downward nominal wage rigidity constraint (6), firm's problem further simplifies as follows.

Problem 2. *In the absence of downward nominal wage rigidity constraint* (6)*, the firm's value functions take the form*

$$J(\hat{n}, W, w) = S(\hat{n}) - Wn, \tag{21}$$

where we call $S(\hat{n})$ the joint value. The joint value solves the following HJB equation:

$$\begin{aligned} \rho S(\hat{n}) = \max_{h\geq 0, W\geq 0} & \, y(\hat{n}) - c(h)\hat{n} - b\hat{n} - Wh\hat{n} \\ & - \lambda^U \int \tilde{W} dF(\tilde{W})\hat{n} + \lambda^E \int_{\tilde{W}\geq W} \tilde{W} dF(\tilde{W})\hat{n} \\ & + S_n(\hat{n})\hat{n}(h - \mu - s - \lambda^E(1 - F(W)) + S_{nn}(\hat{n})\frac{1}{2}(\sigma\hat{n})^2 \end{aligned} \tag{22}$$

for $\hat{n} \leq \hat{n}^*$, where $\hat{n}^*$ is the threshold above which firms fire satisfying $S'(\hat{n}^*) = 0$.

Several remarks are in order. First, the derivation relies on the fact that with flexible wages, the promise keeping constraint (4) always binds. If it were not binding, firms can always lower wages w to increase its profits, a contradiction. Second, now promised utility is no longer a state variable in a sense that current W does not influence firm's policy functions. With flexible wages, firms are free to choose any promise by moving wages (potentially with $w = \pm\infty$). The only relevant state variable is the employment-to-proudctivity ratio, which dictates the joint value of a firm and workers. This structure resembles those in Bilal et al. (2022) and McCrary (2022) under wage bargaining, as well as Elsby and Gottfries (2022) under exogenously specified wage rule. Our contribution here is to show the similar structure can be recovered with wage posting in the tradition of Burdett and Mortensen (1998).

The optimality condition with respect to the promised utility in the next instance, W', gives

$$\left(S_{\hat{n}}(\hat{n}) - W\right) \lambda^E f(W) = h. \tag{23}$$

The left-hand side is the marignal benefit of promising more utility. It raises the retention of the workers in proportion to $\lambda^E f(W')$, and each retained worker has a net value of $(S_{\hat{n}}(\hat{n}) - W')$. The right-hand side is the marginal cost. When firms offer more utility, it has to promise the same value to new hires without any pre-existing commitment. This comes from our assumption of the fairness constraint within a firm. The optimality condition with respect to hiring rate h is given by

$$S_{\hat{n}}(\hat{n}) - W' = \underbrace{\bar{c}h^{\nu}}_{c'(h)}, \tag{24}$$

which equates the marginal value of new hire with the marginal cost of hiring.

Equations (23) and (24) jointly lead to interesting implications of our model. Consider first a case where hiring rate is fixed by setting $\nu \to \infty$. In this case, it is clear that firm's profits are strictly supermodular in $S_{\hat{n}}$ and W', implying that W' is stcitly increasing in marginal joint value $S_{\hat{n}}$ — firms with higher marginal worker value offer higher promised utility. Since workers always move toward firms offering higher promised utility, workers always move toward firms with higher marginal value. This is a ubiquitous feature in the class of models that combine job-ladder and firm dynamics (see Bilal et al., 2022; Elsby and Gottfries, 2022; McCrary, 2022). One can show this conclusion extends as long as $\nu > 1$.

Exploiting the rank-preserving property, we have a tight analytical characterization of the flexible wage equilibrium, as the following proposition summarizes.

Proposition 1. *There exists an equilibrium with the following properties*

- *Workers move from one firm with $\hat{n}$ to another firm with $\hat{n}'$ if and only if $\hat{n} > \hat{n}'$. That is, job-ladder is strictly decreasing in $\hat{n}$.*

- *The utility each firm with $\hat{n}$ offer is given by*

$$\begin{aligned} W(\hat{n}) &= \int_{\tilde{n} \geq \hat{n}} \frac{\zeta \tilde{n} z}{1 - \int \hat{n}' z dg + \zeta \int_{\hat{n}' \geq \tilde{n}} \hat{n}' z dg} S_n(\tilde{n}) dg \\ &\equiv \mathbb{E}_n \Big[S_n(\tilde{n}) | \tilde{n} \geq \hat{n} \Big] \end{aligned} \tag{25}$$

- *Hiring, firing and entry satisfy*

$$h(\hat{n}) = c'^{-1}\Big(S_n(\hat{n}) - W(\hat{n})\Big), \quad S'(\hat{n}^*) = 0, \quad \int S(\hat{n}_0) z_0 d\psi = c^e \tag{26}$$

4 Efficiency of Flexible Wage Equilibrium

Is the flexible wage equilibrium efficient? This question is important to understand the consequences of downward nominal wage rigidity. If the flexible wage equilibrium is not efficient to begin with, it is entirely possible that the downard wage rigidity is welfare improving.

We define the constrained planner's problem as follows.

Problem 3. *The constrained planner solves*

$$\max_{\{g_t,h_t,m_t^0,\mathbb{I}_{t,zn,z'n'},dn_t\}} \int_0^\infty e^{-rt}\left\{\int (Y(z,n)-c(h_t(z,n))n)\,dg_t + (1-\textstyle\int n dg_t)b - m_t^0 c^e\right\} dt$$

subject to

$$\partial_t g_t(n,z) = \partial_n\left[dn_t(n,z)g_t(n,z)\right] + \partial_z[\mu z g_t(n,z)] + \frac{1}{2}\partial_{zz}[(\sigma z)^2 g_t(n,z)] - \varkappa g_t(n,z) + m_t^0\psi(z,n)$$

$$dn_t(z,n) = \left(h(z,n) - s - \zeta\int \mathbb{I}_{t,zn,z'n'}\frac{h(z',n')n'}{1-\int \tilde{n}dg_t(\tilde{z},\tilde{n}) + \zeta\int \mathbb{I}_{t,\tilde{z}\tilde{n},z'n'}\tilde{n}dg_t(\tilde{z},\tilde{n})}dg_t(z',n')\right)n$$

We say the economy is constrained efficient if the allocation solves the constrained planning problem.

Proposition 2. *The flexible wage equilibrium characterized in Proposition 1 is constrained efficient.*

5 Inflation and Misallocation under DNWR

5.1 Calibration

We assume the production function is Cobb-Douglas, $y(\hat{n}) = \hat{n}^\alpha$. The matching function is also Cobb-Douglas, $M(\tilde{u}, V) = \tilde{u}^\eta V^{1-\eta}$, where we have normalize the matching efficiency parameter to one. We asume the joint of distribution of entrants (z^0, n^0) is such that the initial employment is a point mass at $n = n^0$, and z^0 is drawn from Pareto distribution with scale parameter $\bar{z}^0$ and shape parameter γ^e. We calibrate our model so that one period in the model corresponds to a quarter.

Our calibration proceeds in two steps. In the first step, we externally assign subset of parameters. They are either normalizations or set by matching the data moments that can be computed without solving the model. In the second step, we calibrate the remaining parameters in order to target data moments that require solving the model.

First, we externally assign values for $(\rho, \alpha, \eta, \varkappa, \pi, \bar{c})$. The discount rate r is set to 1.75%, which implies the annual discount rate of 7%. The curvature in the production function is set to

$\alpha = 0.64$, a standard value in the litearture that also mirrors the estimate in Cooper, Haltiwanger and Willis (2015). The elasticity in the matching function is set to 0.5, based on the convention in the literature (Petrongolo and Pissarides, 2001). The exit rate is set to $\varkappa = 2.25\%$, which implies the annual exit rate of 9%, in line with the data counterpart reported in Business Dynamics Statistics 2019. We set the steady state inflation rate is $\pi = 0.5\%$, which implies the annual inflation rate of 2%, as in the data in 2019. The size of entrants is set to 5.4, which is the data counterpart reported in Business Dynamics Statistics in 2019. Finally, the hiring cost parameter, $\bar{c}$, is not separately identified from the productivity, z, as what matters for the labor market outcome is $z/\bar{c}$. For this reason, we normalize $\bar{c} = 1$.

Second, we choose the remaining nine parameters $(\mu, \sigma, \nu, c^e, s, \zeta, \bar{z}^0, \gamma^e, b)$ to target nine moments in the data. Although all parameters are jointly calibrated, we provide hueritic arguments for how each parameter is informed by each data moment. The productivity drift parameter, μ, governs the steady state firm size distribution. A higher value of μ implies that the economy is more likely to be dominated by a handful of large firms. The productivity variance parameter, σ, is informed by the volatility of firm growth. The hiring elasticity parameter ν

5.2 Computational Algorithm

We solve the equilibrium of the model numerically with the following algorithm.

1. Guess (U, λ^E).

2. Guess $F(W)$.

   - Solve HJB-QVI stated in Problem 1 to obtain policy functions.
   - Using the policy functions, solve the KFE in Appendix A.3.
   - Using (**??**)

6 Transition Dynamics

Along the transitiosn, the firm's value function $J_t(\hat{n}, W, w)$ solves the following HJB-QVI:

$$\min \left\{ \begin{array}{c} \rho J_t - \max_{h, \mu_{\hat{n}}, \mu_W} \left\{ y(\hat{n}) - w\hat{n} - c(h)\hat{n} + \mu_{\hat{n}} \partial_{\hat{n}} J_t + \frac{\sigma^2}{2} \hat{n}^2 \partial^2_{\hat{n}\hat{n}} J_t + \mu_W \partial_W J_t - \pi_t w \partial_w J_t \right\} - \partial_t J_t, \\ J_t - J_t^*(\hat{n}, W, w) \end{array} \right\} = 0 \tag{27}$$

Paramter	Value	Description	Target
A. Externally assigned parameters			
r	0.0125	Steady state interest rate	Annual 5% interest rate
α	0.85	Returns to scale	Standard
$\varkappa$	0.0225	Exit rate	Annual 9% exit rate
π	0.005	Inflation rate	Annual 2% inflation
$\bar{n}^0$	5.4	Entrants initial employment	BDS 2019
$\bar{c}$	100	Hiring cost level	Nomralization
ν	3.45	Hiring cost elasticity	Bilal et al. (2022)
B. Ineternally calibrated parameters			
μ	0.015	Productivity growth drift	Emp. share 500+ firms 50%
σ	0.18	Productivity growth variance	std($\Delta \ln(n)$) of 0.41
c^e	10.6	Entry cost	Average firm size 23
s	0.02	Exogenous separation rate	Monthly EU rate 2%
ζ	0.10	On-the-job search efficiency	Monthly EE rate 2%
$\bar{z}^0$	0.0038	Entrants initial productivity	Unemp. rate 5%
b	0.065	Flow value of leisure	Hiring cost = monthly wage

Table 1: Parameter Values

Moments	Target	Model
Firm size Pareto tail	1.05	
Average firm size	23	23
Standard deviation of firm growth		
Unemployment rate	0.05	0.05
EU rate		
EE rate		
Entrants productivity gap		
Profit share or Std($\log w$)		

Table 2: Targeted Moments

subject to

$$\mu_{\hat{n}} = \hat{n} \left(h - \mu - s - \lambda_t^E (1 - F_t(W)) \right) \tag{28}$$

$$\mu_W = \underbrace{rW - \left[w + (s + \varkappa) (U_t - W) + \lambda_t^E \int \max\{W' - W, 0\} dF_t(W') \right]}_{\equiv d\mathcal{W}_t(W,w)} \quad \text{for } W > U_t, \tag{29}$$

and $\mu_{Wt} = \max\{d\mathcal{W}_t(W, w), 0\}$ for $W = U_t$.

We solve the transition dynamics of our model by adopting Auclert et al. (2021) to our environment.

Outcome	DNWR	Flexible Wage	Change
A. Macro Aggregates			
GDP	0.233	0.237	+1.5%
Employment	0.95	0.949	-0.1%
TFP	0.245	0.249	+1.6%
Consumption	0.223	0.224	+1.0%
B. Labor Market Flows			
UE rate	0.41	0.47	+14.6%
EU rate	0.021	0.025	+19.0%
EE rate	0.07	0.08	+14.2%
C. Firm Dynamics			
Mass of firms	0.041	0.045	+8%
std($\ln \hat{n}$)	0.38	0.30	-16%

Table 3: Parameter Values

Moments	Target	Model
Firm size Pareto tail	1.05	
Average firm size	23	23
Standard deviation of firm growth		
Unemployment rate	0.05	0.05
EU rate		
EE rate		
Entrants productivity gap		
Profit share or Std($\log w$)		

Table 4: Targeted Moments

7 Extensions

7.1 Vacancy Costs

7.2 State-contingent Contracts

References

Achdou, Yves, Jiequn Han, Jean-Michel Lasry, Pierre-Louis Lions, and Benjamin Moll, "Income and wealth distribution in macroeconomics: A continuous-time approach," *The review of economic studies*, 2022, *89* (1), 45–86.

Auclert, Adrien, Bence Bardóczy, Matthew Rognlie, and Ludwig Straub, "Using the sequence-space Jacobian to solve and estimate heterogeneous-agent models," *Econometrica*, 2021, *89* (5), 2375–2408.

Bertucci, Charles, "Fokker-Planck equations of jumping particles and mean field games of impulse control," in "Annales de l'Institut Henri Poincaré C, Analyse non linéaire," Vol. 37 Elsevier 2020, pp. 1211–1244.

Bilal, Adrien, Niklas Engbom, Simon Mongey, and Giovanni L Violante, "Firm and worker dynamics in a frictional labor market," *Econometrica*, 2022, *90* (4), 1425–1462.

Blanchard, Olivier and Jordi Galí, "Labor markets and monetary policy: A new keynesian model with unemployment," *American Economic Journal: Macroeconomics*, 2010, *2*, 1–30.

Burdett, Kenneth and Dale T Mortensen, "Wage differentials, employer size, and unemployment," *International Economic Review*, 1998, pp. 257–273.

Coles, Melvyn G and Dale T Mortensen, "Equilibrium labor turnover, firm growth, and unemployment," *Econometrica*, 2016, *84* (1), 347–363.

Cooper, Russell, John Haltiwanger, and Jonathan L Willis, "Dynamics of labor demand: Evidence from plant-level observations and aggregate implications," *Research in Economics*, 2015, *69* (1), 37–50.

Ehrlich, Gabriel and Joshua Montes, "Wage Rigidity and Employment Outcomes: Evidence from Administrative Data," *American Economic Journal: Macroeconomics*, 2024, *16*, 147–206.

Elsby, Michael W L and Gary Solon, "How Prevalent Is Downward Rigidity in Nominal Wages? International Evidence from Payroll Records and Pay Slips," *The Journal of Economic Perspectives*, 2019, *33*, 185–201.

Elsby, Michael WL and Axel Gottfries, "Firm dynamics, on-the-job search, and labor market fluctuations," *The Review of Economic Studies*, 2022, *89* (3), 1370–1419.

Gertler, Mark, Antonella Trigari, Source Journal, and No February, "Unemployment Fluctuations with Staggered Nash Wage Bargaining Nash Wage Bargaining Mark Gertler Antonella Trigari," *Journal of Political Economy*, 2009, *117*, 38–86.

Grigsby, John, Erik Hurst, and Ahu Yildirmaz, "Aggregate Nominal Wage Adjustments: New Evidence from Administrative Payroll Data," *American Economic Review*, 2021, *111*, 428–471.

Hopenhayn, Hugo and Richard Rogerson, "Job turnover and policy evaluation: A general equilibrium analysis," *Journal of political Economy*, 1993, *101* (5), 915–938.

McCrary, Sean, "A Job Ladder Model of Firm, Worker, and Earnings Dynamics," *Worker, and Earnings Dynamics (November 4, 2022)*, 2022.

Petrongolo, Barbara and Christopher A Pissarides, "Looking into the black box: A survey of the matching function," *Journal of Economic literature*, 2001, *39* (2), 390–431.

Schaal, Edouard, "Uncertainty and unemployment," *Econometrica*, 2017, *85* (6), 1675–1721.

Spear, Stephen E and Sanjay Srivastava, "On repeated moral hazard with discounting," *The Review of Economic Studies*, 1987, *54* (4), 599–617.

Tobin, James, "Inflation and Unemployment," *American Economic Review*, 1972, *62*, 1–18.

Appendix

A Theory Appendix

A.1 Derivation of Problem 1

The firm's value function $\mathcal{J}(z,n,W,w)$ solves the following HJB-QVI:

$$\min\left\{(r+\varkappa)\mathcal{J}(\cdot)-\max_{h,a}\left\{Y(z,n)-wn-c(h)n+\mathcal{L}(h,a)\mathcal{J}(\cdot)\right\},\mathcal{J}(\cdot)-\mathcal{J}^*(\cdot)\right\}=0, \tag{30}$$

where

$$\begin{aligned}\mathcal{L}(h,a)J(\cdot)\equiv n\left(h-s-\lambda^E(1-F(W))\right)\mathcal{J}_n(\cdot)\\ +\left[rW-\left[w-rU-(\varkappa+s)W+\lambda^E\textstyle\int\max\{W'-W,0\}dF(W')\right]\right]\mathcal{J}_W(\cdot)\\ -\pi w\mathcal{J}_w(\cdot)+\mu z\mathcal{J}_z(\cdot)+\frac{\sigma^2z^2}{2}\mathcal{J}_{zz}(\cdot)+a^2\frac{\sigma^2}{2}\mathcal{J}_{WW}(\cdot)+a\sigma^2z\mathcal{J}_{zW}\end{aligned} \tag{31}$$

and

$$\mathcal{J}^*(z,n,W,w)\equiv\max_{z,n^*\le n,W^*,w^*\ge w}\mathcal{J}(z,n^*,W^*,w^*) \tag{32}$$

$$\text{s.t.}\quad W^*\tfrac{n^*}{n}=W. \tag{33}$$

We guess and verify that the value function takes the form

$$\mathcal{J}(z,n,W,w)=J(\hat{n},W,w)z, \tag{34}$$

where $\hat{n}\equiv n/z$. Taking dervatives of both sides,

$$\mathcal{J}_z=J-J_n\hat{n} \tag{35}$$

$$\begin{aligned}\mathcal{J}_{zz}&=-J_n\hat{n}\frac{1}{z}\hat{n}+J_{nn}\hat{n}^2\frac{1}{z}+J_n\hat{n}\frac{1}{z}\\&=J_{nn}\hat{n}^2\frac{1}{z}\end{aligned} \tag{36}$$

$$\mathcal{J}_n=J_nz \tag{37}$$

$$\mathcal{J}_W=J_Wz \tag{38}$$

$$\mathcal{J}_{WW}=J_{WW}z \tag{39}$$

$$\mathcal{J}_{Wz}=J_W-J_{nW}\hat{n} \tag{40}$$

$$\mathcal{J}_w=J_wz \tag{41}$$

Plugging (34)-(41) into (30)-(33), we verify our guess and obtain Problem 1.

A.2 Derivation of Problem 2

We guess and verify

$$J(\hat{n}, W, w) = S(\hat{n}) - W\hat{n} \tag{42}$$

in the absence of downard nominal wage rigidity constraint. Under the guess,

$$J_n = S_n - W \tag{43}$$
$$J_W = -\hat{n} \tag{44}$$
$$J_w = 0 \tag{45}$$
$$J_{nn} = S_n n \tag{46}$$
$$J_{WW} = 0 \tag{47}$$
$$J_{Wz} = -1; \tag{48}$$

Plugging these expressions into the first inside the minimum operator of Problem 1, we have

$$\rho J(\cdot) - W\hat{n} - [y(\hat{n}) - w\hat{n} - c(h)\hat{n} + \mathcal{L}(h,a)J] \tag{49}$$
$$= \rho S(\cdot) - (r + \varkappa - \mu)W\hat{n} - [y(\hat{n}) - w\hat{n} - c(h)\hat{n}\hat{n}] \tag{50}$$
$$- \hat{n}\left(h - \mu - s - \lambda^E(1 - F(W))\right) S_n(\cdot) \tag{51}$$
$$+ W\hat{n}\left(h - \mu - s - \lambda^E(1 - F(W))\right) \tag{52}$$
$$+ \left[rW - \left[w - rU - (\varkappa + s)W + \lambda^E \textstyle\int \max\{W' - W, 0\} dF(W')\right]\right]\hat{n} \tag{53}$$
$$- \frac{\sigma^2\hat{n}^2}{2} S_{nn}(\cdot) \tag{54}$$
$$= \rho S(\cdot) - [y(\hat{n}) - c(h)\hat{n} - rU\hat{n}] \tag{55}$$
$$- \hat{n}\left(h - \mu - s - \lambda^E(1 - F(W))\right) S_n(\cdot) \tag{56}$$
$$+ W\hat{n}\left(h - \lambda^E(1 - F(W))\right) - \lambda^E \textstyle\int \max\{W' - W, 0\} dF(W')\hat{n} \tag{57}$$
$$- \frac{\sigma^2\hat{n}^2}{2} S_{nn}(\cdot) \tag{58}$$
$$= \rho S(\cdot) - [y(\hat{n}) - c(h)\hat{n}] \tag{59}$$
$$- \hat{n}\left(h - \mu - s - \lambda^E(1 - F(W))\right) S_n(\cdot) \tag{60}$$
$$+ W\hat{n}h - \lambda^E \textstyle\int_{W' \geq W} W' dF(W')\hat{n} \tag{61}$$
$$- \frac{\sigma^2\hat{n}^2}{2} S_{nn}(\cdot) \tag{62}$$

The second term inside the minimum operator is now

$$\max_{\hat{n}^*} S(\hat{n}^*) - Wn, \tag{63}$$

which implies the boundary condition of

$$S_n(\hat{n}^*) = 0. \tag{64}$$

Therefore, the firm's HJB-QVI is now given by

$$\rho S(\cdot) = \max_{h\geq 0, W\geq 0} y(\hat{n}) - c(h)\hat{n} - W\hat{n}h + \lambda^E \int_{W'\geq W} W' dF(W')\hat{n} \tag{65}$$

$$+ \hat{n}\left(h - \mu - s - \lambda^E(1 - F(W))\right) S_n(\cdot) + \frac{\sigma^2 \hat{n}^2}{2} S_{nn}(\cdot) \tag{66}$$

with a boundary condition of (64), as desired.

A.3 Kolmogorov forward equation

Let $\mu_{\hat{n}}(\hat{n}, W, w)$ be the firm's policy functions that describe the drift of the employment-to-productivity ratio:

$$\mu_{\hat{n}}(\hat{n}, W, w) = \hat{n}\left(h(\hat{n}, W, w) - \mu - s - \lambda^E(1 - F(W))\right). \tag{67}$$

Let $\mu_W(W, w)$ denote the firm's policy function that describes the drift of the firm's promised utility absent a jump:

$$\mu_W(W, w) = rW - \left[w + (s + \varkappa)(U - W) + \lambda^E \int \max\{W' - W, 0\} dF(W')\right]. \tag{68}$$

Let $\hat{n}^*(\hat{n}, W, w)$ and $w^*(\hat{n}, W, w)$ be the policy functions that solve (9).

Using these notations, define the following infinitesimal generator $\mathcal{L}$ defined for arbitrary test function $f(z, \hat{n}, W, w)$

$$\begin{aligned}\mathcal{L}f(z, \hat{n}, W, w) = {} & \mu z \partial_z f(z, \hat{n}, W, w) + \mu_{\hat{n}}(\hat{n}, W, w)\partial_{\hat{n}} f(z, \hat{n}, W, w) \\ & + \mu_W(W, w)\partial_W f(z, \hat{n}, W, w) - \pi w \partial_w f(z, \hat{n}, W, w) \\ & + \frac{(\sigma\hat{n})^2}{2}\partial^2_{\hat{n}\hat{n}} f(z, \hat{n}, W, w) + \frac{(\sigma z)^2}{2}\partial^2_{zz} f(z, \hat{n}, W, w) - \sigma^2 \hat{n} z \partial^2_{\hat{n}z} f(z, \hat{n}, W, w) \\ & - \varkappa f(z, \hat{n}, W, w) \\ & + \Lambda(\hat{n}, W, w)\left[f(z, \hat{n}^*(\hat{n}, W, w), W^*(\hat{n}, W, w), w^*(\hat{n}, W, w)) - f(z, \hat{n}, W, w)\right],\end{aligned}$$

where

$$\Lambda(\hat{n}, W, w) = \begin{cases} 0 & \text{if } \hat{n} = \hat{n}^*(\hat{n}, W, w), W = W^*(\hat{n}, W, w) \text{ and } w = w^*(\hat{n}, W, w) \\ \infty & \text{otherwise} \end{cases} \tag{69}$$

The steady state distribution of firm distribution $g(z, \hat{n}, W, w)$ satisfies the following Kolmogorov forward equation

$$0 = \mathcal{L}^\dagger g(z, \hat{n}, W, w) + m^0 \psi(z, \hat{n}z)\delta(W^0(\hat{n}), w^0(\hat{n})), \tag{70}$$

where $\mathcal{L}^\dagger$ is the adjoint operator of $\mathcal{L}$, and $\delta(W^0, w^0)$ is the dirac function that takes one if $W = W^0$ and $w = w^0$. See Bertucci (2020) for the rigorous characterization of Kolmogorov forward equation with jump processes as a limit of finite Poisson process.

A.4 Characterization of Felxbile Wage Equilibrium

We look for an equilibrium in which $F(W)$ is smooth with support $[0, \bar{W}]$ and almost all firms offer W that is an interior optimum. In this equilibrium, the following first-order condition holds for almost all firms:

$$(S_n(\hat{n}) - W)\lambda^E f(W) = \tilde{h}(\hat{n}, W), \tag{71}$$

where $\tilde{h}(\hat{n}, W) \equiv c'^{-1}(S_n(\hat{n}) - W)$ together with the associated second-order condition:

$$(S_n(\hat{n}) - W)\lambda^E f'(W) - \lambda^E f(W) - \frac{\partial \tilde{h}(\hat{n}, W)}{\partial W} < 0. \tag{72}$$

Totally differentiating (71), we have

$$\left[S_{nn}(\hat{n})\lambda^E f(W) - \frac{\partial \tilde{h}(\hat{n}, W)}{\partial \hat{n}}\right] d\hat{n} + \left[-\lambda^E f(W) + (S_n(\hat{n}) - W)\lambda^E f(W) - \frac{\partial \tilde{h}(\hat{n}, W)}{\partial W}\right] dW = 0, \tag{73}$$

which results in

$$\frac{dW}{d\hat{n}} = S_{nn}(\hat{n}) \frac{\lambda^E f(W) - \frac{\tilde{h}(\hat{n},W)}{S_n(\hat{n})-W} \frac{\partial \ln \tilde{h}(\hat{n},W)}{\partial \ln(S_n(\hat{n})-W)}}{\left[-\lambda^E f(W) + (S_n(\hat{n}) - W)\lambda^E f(W) - \frac{\partial h(\hat{n},W)}{\partial W}\right]} \tag{74}$$

$$= S_{nn}\lambda^E f(W)(\hat{n}) \frac{1 - \frac{1}{\nu}}{\left[-\lambda^E f(W) + (S_n(\hat{n}) - W)\lambda^E f(W) - \frac{\partial \tilde{h}(\hat{n},W)}{\partial W}\right]}. \tag{75}$$

The standard argument implies that the value function is strictly concave, $S_{nn}(\hat{n}) < 0$, as $y(\hat{n})$ is strictly concave. This implies that

$$\frac{dW}{d\hat{n}} > 0 \tag{76}$$

under our assumption that

$$\nu > 1. \tag{77}$$

Since $\frac{dW}{d\hat{n}} > 0$, the job-ladder is inversely ranked by the employment-to-productivity ratio, $\hat{n}$. Let $\hat{F}(\hat{n})$ be the survival function of offer distribution, which is given by

$$\hat{F}(\hat{n}) \equiv \int_{\hat{n}' \geq \hat{n}} \frac{h(\hat{n}')\hat{n}'z}{(p^u + p^e H(\hat{n}'))} \frac{1}{V} dg, \tag{78}$$

where we used the fact that entrants iniitally sit at the bottom of the job-ladder. By the rank-preserving property,

$$\hat{F}(\hat{n}) = F(W(\hat{n})). \tag{79}$$

Differentiating both sides by $\hat{n}$,

$$\hat{F}'(\hat{n}) = f(W(\hat{n}))W'(\hat{n}), \tag{80}$$

which can be rewritten as

$$-\frac{h(\hat{n})\hat{n}zdg}{p^u + p^e H(\hat{n}')} \frac{1}{V} = f(W(\hat{n}))W'(\hat{n}). \tag{81}$$

Plugging (81) into the FOC: $(S_n(\hat{n}) - W(\hat{n}))\lambda^E f(W(\hat{n})) = h(\hat{n})$, we have

$$(S_n(\hat{n}) - W(\hat{n}))\lambda^E \frac{h(\hat{n})\hat{n}zdg}{V} = h(\hat{n})W'(\hat{n}), \tag{82}$$

which simplifies to

$$(S_n(\hat{n}) - W(\hat{n}))\frac{1}{u+\zeta(1-u)}\frac{\zeta \hat{n} z dg}{p^u + p^e H(\hat{n})} = W'(\hat{n}) \tag{83}$$

$$\Leftrightarrow \quad (S_n(\hat{n}) - W(\hat{n}))\frac{\zeta \hat{n} z dg}{u+\zeta \int_{\hat{n}'\geq \hat{n}} \hat{n}' z dg} = W'(\hat{n}). \tag{84}$$

Solving the above ODE with the boundary condition $W(\hat{n}^*) = 0$,

$$W(\hat{n}) = \int_{\hat{n}'\geq \hat{n}} \frac{\zeta \hat{n} z}{u+\zeta \int_{\hat{n}'\geq \hat{n}} \hat{n}' z dg} S_n(\hat{n}') dg \tag{85}$$

$$\equiv \mathbb{E}_n \left[S_n(\hat{n}') | \hat{n}' \geq \hat{n}\right], \tag{86}$$

as desired. Now we verify that the second-order conditions, (72) are satisfied. Since $\nu > 1$, $-\lambda^E f(W) - \frac{\partial \tilde{h}(\hat{n},W)}{\partial W} < 0$ and $S_n(\hat{n}) - W > 0$, it is sufficient to show $f'(W(\hat{n}) < 0$. Further differenciating (80),

$$F''(\hat{n}) = f'(W(\hat{n}))W'(\hat{n})^2 + f(W(\hat{n}))W''(\hat{n}) \tag{87}$$

Recall the first-order condition was

$$(S_n(\hat{n}) - W(\hat{n}))\lambda^E \hat{F}'(\hat{n}) = h(\hat{n})W'(\hat{n}). \tag{88}$$

Taking the derivative w.r.t. $\hat{n}$,

$$(S_{nn}(\hat{n}) - W'(\hat{n}))\lambda^E \hat{F}'(\hat{n}) + (S_n(\hat{n}) - W(\hat{n}))\lambda^E \hat{F}''(\hat{n}) = h(\hat{n})W''(\hat{n}) + h'(\hat{n})W'(\hat{n}). \tag{89}$$

Eliminating $W''(\hat{n})$ from (87) and (89), we have

$$F''(\hat{n}) = f'(W(\hat{n}))W'(\hat{n})^2 \tag{90}$$

$$+ f(W(\hat{n}))\frac{(S_{nn}(\hat{n}) - W'(\hat{n}))\lambda^E \hat{F}'(\hat{n}) + (S_n(\hat{n}) - W(\hat{n}))\lambda^E \hat{F}''(\hat{n}) - h'(\hat{n})W'(\hat{n})}{h(\hat{n})} \tag{91}$$

$$\Leftrightarrow \quad 0 = f'(W(\hat{n}))W'(\hat{n})^2 + f(W(\hat{n}))\frac{(S_{nn}(\hat{n}) - W'(\hat{n}))\lambda^E \hat{F}'(\hat{n}) - h'(\hat{n})W'(\hat{n})}{h(\hat{n})} \tag{92}$$

$$\Leftrightarrow \quad h(\hat{n})\frac{f'(W(\hat{n})}{f(W(\hat{n}))} = -\frac{(S_{nn}(\hat{n}) - W'(\hat{n}))\lambda^E \hat{F}'(\hat{n}) - h'(\hat{n})W'(\hat{n})}{(W'(\hat{n}))^2} \tag{93}$$

Using (71) and (72), we can rewrite the second-order coondition as

$$h(\hat{n})\frac{f'(W(\hat{n})}{f(W(\hat{n}))} - \lambda^E f(W) - \frac{\partial \tilde{h}(\hat{n}, W)}{\partial W} < 0. \tag{94}$$

Plugging (93) into the above expression,

$$\begin{aligned}
& - \frac{(S_{nn}(\hat{n}) - W'(\hat{n}))\lambda^E \hat{F}'(\hat{n}) - h'(\hat{n})W'(\hat{n})}{(W'(\hat{n}))^2} - \lambda^E f(W(\hat{n})) - \frac{\partial \tilde{h}(\hat{n}, W)}{\partial W} && (95)\\
=& - \frac{S_{nn}(\hat{n})\lambda^E \hat{F}'(\hat{n}) - h'(\hat{n})W'(\hat{n})}{(W'(\hat{n}))^2} - \frac{\partial \tilde{h}(\hat{n}, W)}{\partial W} && (96)\\
=& - \frac{S_{nn}(\hat{n})\lambda^E \hat{F}'(\hat{n})}{(W'(\hat{n}))^2} + \frac{1}{W'(\hat{n})}\frac{\partial \tilde{h}(\hat{n}, W)}{\partial \hat{n}} && (97)\\
=& - \frac{1}{W'(\hat{n})}\left[S_{nn}(\hat{n})\lambda^E f(W(\hat{n})) - \frac{\partial \tilde{h}(\hat{n}, W)}{\partial \hat{n}}\right] && (98)\\
=& - \frac{1}{W'(\hat{n})} S_{nn}(\hat{n})\lambda^E f(W(\hat{n}))\left[1 - \frac{1}{\nu}\right] && (99)\\
<& 0, && (100)
\end{aligned}$$

where the second line uses $F'(\hat{n}) = f(W(\hat{n}))W'(\hat{n})$, the third line uses $\frac{\partial \tilde{h}}{\partial \hat{n}} + \frac{\partial \tilde{h}}{\partial W}W'(\hat{n}) = h'(\hat{n})$, and the final follows from $\nu > 1$.

A.5 Tranformed Problem

Define $\hat{W} \equiv W - U$, then we can rewrite the firm's problem as follows. Define the offer distribution of $\hat{W}$ as $\hat{F}(\hat{W}) = F(\hat{W} + U)$.

Problem 4. *The firm's value function is homogenous in z,* $\tilde{J}(z, n, \hat{W}, w) = J(\hat{n}, \hat{W}, w)z$*, where* $\hat{n} \equiv n/z$*. The value function* $J \equiv J(\hat{n}, \hat{W}, w)$ *(dependence ommited for brevity) solves the following Hamiltonian-Jacobi-Bellman quasi-variational inequality (HJB-QVI):*

$$\min \left\{ \begin{array}{c} \rho J - \max_{h, \mu_{\hat{n}}, d\hat{W}} \left\{ y(\hat{n}) - w\hat{n} - c(h)\hat{n} + \mu_{\hat{n}}\partial_{\hat{n}} J + \frac{\sigma^2}{2}\hat{n}^2 \partial^2_{\hat{n}\hat{n}} J + \mu_W \partial_W J - \pi w \partial_w J \right\}, \\ J - J^*(\hat{n}, W, w) \end{array} \right\} = 0 \tag{101}$$

subject to

$$\mu_{\hat{n}} = \hat{n}\left(h - \mu - s - \lambda^E(1 - \hat{F}(\hat{W}))\right) \tag{102}$$

$$\mu_W = \underbrace{r\hat{W} - \left[w - rU - (s + \varkappa)\,\hat{W} + \lambda^E \int \max\{\hat{W}' - \hat{W}, 0\} d\hat{F}(\hat{W}')\right]}_{\equiv d\mathcal{W}(\hat{W}, w)} \quad \textit{for } \hat{W} > 0, \tag{103}$$

and $\mu_W = \max\{d\mathcal{W}(\hat{W}, w), 0\}$ *for* $\hat{W} = 0$*, where* $\rho \equiv r + \varkappa - \mu$*,* $y(\hat{n}) \equiv Y(1, \hat{n})$ *and* $J^*(\hat{n}, \hat{W}, w)$ *is the value from firing, raising promised utility, or raising wages:*

$$J^*(\hat{n}, \hat{W}, w) \equiv \max_{\hat{n}^* \leq \hat{n}, w^* \geq w, \hat{W}^*} J(\hat{n}^*, \hat{W}^*, w^*) \tag{104}$$

$$\textit{s.t.} \quad \hat{W}^* \tfrac{n^*}{n} \geq \hat{W}. \tag{105}$$

The value of unemployment can be rewritten as

$$rU = b + \lambda^U \int \hat{W}' d\hat{F}(\hat{W}'). \tag{106}$$

A.6 Normalization

The problem of the firm can be normalized up to a constant. Given a hiring cost function $c(h)$, unemployment value U and offer distribution $F(W)$, we can compute the value function $J(z, n, W, w; 1)$. Now, imagine another economy with hiring costs $\lambda c(h)$, unemployment value λU, and offer distribution that satisfies $\hat{F}(\lambda W) = F(W)$. That is, hiring costs, the value of unemployment, and the offer distribution increase by a factor of λ. Let $J(z, n, W, w; \lambda)$ be the value function of this other economy. Then, we have that

$$J\left(\lambda^{\frac{1}{1-\alpha}} z, n, \lambda W, \lambda w; \lambda\right) = \lambda J(z, n, W, w; 1) \tag{107}$$

$$\min \left\{ \begin{array}{c} \rho J - \max\limits_{h, \mu_n, \mu_W} \left\{ z^{1-\alpha} n^\alpha - wn - c(h)n + \mu_z \partial_z J + \frac{\sigma^2}{2} z^2 \partial^2_{zz} J + \mu_n \partial_n J + \mu_W \partial_W J - \pi w \partial_w J \right\}, \\ J - J^*(z, n, W, w) \end{array} \right\} = 0$$

$$\mu_n = n(h - s - \lambda^E(1 - F(W))) \tag{108}$$

$$\mu_W = rW - \left[w + (s + \varkappa)\,(U - W) + \lambda^E \int \max\{W' - W, 0\} dF(W')\right]. \tag{109}$$

Guess $J\left(\lambda^{\frac{1}{1-\alpha}}z, n, \lambda W, \lambda w\right) = \lambda J(z,n,W,w)$. This implies that

$$\partial_z J\left(\lambda^{\frac{1}{1-\alpha}}z, n, \lambda W, \lambda w\right)\lambda^{\frac{1}{1-\alpha}} = \lambda \partial_z J(z,n,W,w)$$
$$\partial_{zz} J\left(\lambda^{\frac{1}{1-\alpha}}z, n, \lambda W, \lambda w\right)\lambda^{\frac{2}{1-\alpha}} = \lambda \partial_{zz} J(z,n,W,w)$$
$$J_n\left(\lambda^{\frac{1}{1-\alpha}}z, n, \lambda W, \lambda w\right) = \lambda J_n(z,n,W,w)$$
$$J_W\left(\lambda^{\frac{1}{1-\alpha}}z, n, \lambda W, \lambda w\right) = J_W(z,n,W,w)$$
$$J_w\left(\lambda^{\frac{1}{1-\alpha}}z, n, \lambda W, \lambda w\right) = J_w(z,n,W,w)$$

Let $\tilde{J} \equiv J\left(\lambda^{\frac{1}{1-\alpha}}z, n, \lambda W, \lambda w\right)$

$$\mu_n = n(h - s - \lambda^E(1 - F(W)) \tag{110}$$
$$\mu_W = rW - \left[w + (s + \varkappa)(U - W) + \lambda^E \int \max\{W' - W, 0\} dF(W')\right]. \tag{111}$$

B Computation Appendix

We discretize the state space using non-uniform grids: $z_{i_z} \in [z_1, \dots, z_{N_z}]$, $\hat{n}_{i_{\hat{n}} \in [\hat{n}_1, \dots, \hat{n}_{N_{\hat{n}}}]}$, $W_{i_W} \in [W_1, \dots, W_{N_W}]$, and $w_{i_w} \in [w_1, \dots, w_{N_w}]$ and define J on that grid.

B.1 Solving the HJBQVI of the firm

In solving (), the firm continuously adjusts h but it has to solve an optimal stopping time problem and decide when to change w or W. To solve the HJB-QVI equation, first we solve a simpler problem, a HJBVI problem where J^* is fixed. The idea is to solve the HJBVI for a fixed J^*, then compute a new J^*, and iterate until convergence.

To avoid firms from backloading the payment of wages forever, we assume firms cannot promise more than $W_{N_W} \equiv \bar{W}$. This means that a firm at $\bar{W}$, if it has a wage that would imply $dW(\hat{n}, w, \bar{W}) \geq 0$, is forced to raise the wage to $\bar{w}$ satisfying $dW(\hat{n}, \bar{w}, \bar{W}) = 0$. Note that $\bar{w}$ does not depend on $\hat{n}$ because the law of motion of dW does not depend on employment or productivity. Thus, we want to impose that

$$J(\hat{n}, w, \bar{W}) = J(\hat{n}, \bar{w}, \bar{W}) \quad \text{for} \quad w \leq \bar{w} \tag{112}$$

To do so, we can build an M matrix that has ones in the diagonal everywhere except in states where $dW(\hat{n}, w, \bar{W}) \geq 0$. There it has a 1 to the column corresponding to the state $[\hat{n}, \bar{w}, \bar{W}]$

[1]. Given a general value function J, the transformed value function MJ ensures that (112) is satisfied.

Rows of $\tilde{A}$ and elements of $\tilde{b}$ satisfy

$$[\tilde{A}]_i = \begin{cases} [\rho I - A^n M]_i & \text{if} \quad J_i^{*,m} - J_i^n \geq A^n M J_i^n + M b_i^n - \rho J_i^n \\ [I]_i & \text{otherwise} \end{cases}$$

$$\tilde{b}_i^n = \begin{cases} M b_i^n & \text{if} \quad J_i^{*,m} - J_i^n \geq A^n M J_i^n + M b_i^n - \rho J_i^n \\ J_i^* & \text{otherwise} \end{cases}$$

We update the new value function to

$$J^{n+1} = M(\tilde{A} \backslash \tilde{b}) \tag{113}$$

This forces $J_i^{n+1} = J_i^{*,m}$ when the firm jumps. Once J^{n+1} is sufficiently close to J^n, we consider the HJBVI problem solved and we update $J^{*,m+1}$. We keep iterating on $J^{*,m+1}$ until $J^{*,m}$ and $J^{*,m+1}$ are sufficiently close

We solve the HJBQVI problem of the firm in the following way. We start with a guess of $J^{*,0} \to -\infty$, and solve the HJB of the firm assuming it never adjusts wages.

B.2 Computing the steady state distribution

The homogeneity of the partial equilibrium problem of the firm allowed us to work with only three states, $\hat{n}, W$ and w. This is not the case when we want to solve the KFE of the steady state distribution of firms. For that, we build M_{KFE}, which is a matrix with ones in the diagonal when the firm in state $[z, \hat{n}, W, w]$ does not jump, and if the firm wants to jump, it has a 0 on the diagonal and a 1 at the corresponding row of the destination state. That is, columns of M_{KFE} add up to one and $[M]_{ij} = 1$ if firm in state j adjust to state i. With that, given a distribution g, we have that $M_{KFE} g$ has 0 mass on the adjustment regions.

When the firm does not adjust, we define the matrix A_{KFE}. $[A_{KFE}]_{ij}$ is the rate at which firm in state i transitions to state j. Rows of A_{KFE} add up to $-\chi$, which is the exit rate of firms, have negative entries at the diagonal and non-negative entries elsewhere. Given a distribution g,

Then we can find the steady-state distribution by solving

$$g = (D_{KFE} + M_{KFE} A_{KFE} M_{KFE}) \backslash (-m_0 M_{KFE} g_{entry}) \quad . \tag{114}$$

D_{KFE} is a diagonal matrix where $[D_{KFE}]_{ii} = 1$ if i is in the adjustment region and 0 oth-

[1]We set $\bar{w}$ to the lowest wage level that ensures $dW(\hat{n}, \bar{w}, \bar{W}) \leq 0$

erwise. This avoids the system to have a singular matrix and force $g_i = 0$ when i is in the adjustment region.

As in Achdou, Han, Lasry, Lions and Moll (2022), we approximate the second derivative of an arbitrary function $f(x_i, \ldots)$ as

$$\frac{\partial^2}{\partial x^2} f(x_i, \ldots) \approx \frac{\Delta x_{-,i} f_{i+1} - (\Delta x_{+,i} + \Delta x_{-,i}) f_i + \Delta x_{+,i} f_{i-1}}{\frac{1}{2}(\Delta x_{+,i} + \Delta x_{-,i}) \Delta x_{+,i} \Delta x_{-,i}}, \tag{115}$$

where $f_i \equiv f(x_i, \ldots)$.

We approximate the cross partial derivative of an arbitrary function $f(x_i, y_j)$ with

$$\sigma_{xy} \frac{\partial^2}{\partial x \partial y} f(x_i, y_j) \approx \begin{cases} \frac{\sigma_{xy}}{2} \frac{\frac{\Delta x_{+,i}}{\Delta x_{+,i} + \Delta x_{-,i}} (f_{i+1,j+1} - f_{i+1,j}) + \frac{\Delta y_{+,j}}{\Delta y_{+,j} + \Delta y_{-,j}} (f_{i+1,j+1} - f_{i,j+1})}{\Delta x_{+,i} \Delta y_{+,i}} \\ \quad - \frac{\sigma_{xy}}{2} \frac{\frac{\Delta y_{-,j}}{\Delta y_{+,j} + \Delta y_{-,j}} (f_{i,j-1} - f_{i-1,j-1}) + \frac{\Delta x_{-,i}}{\Delta x_{+,i} + \Delta x_{-,i}} (f_{i-1,j} - f_{i-1,j-1})}{\Delta x_{-,i} \Delta y_{-,j}} & \text{if } \sigma_{xy} > 0 \\ \\ \frac{\sigma_{xy}}{2} \frac{\frac{\Delta x_{+,i}}{\Delta x_{+,i} + \Delta x_{-,i}} (f_{i+1,j} - f_{i+1,j-1}) - \frac{\Delta y_{-,i}}{\Delta y_{+,i} + \Delta y_{-,i}} (f_{i+1,j-1} - f_{i,j-1})}{\Delta x_{+,i} \Delta y_{-,i}} \\ - \frac{\sigma_{xy}}{2} \frac{\frac{\Delta x_{-,i}}{\Delta x_{+,i} + \Delta x_{-,i}} (f_{i-1,j+1} - f_{i-1,j}) - \frac{\Delta y_{+,i}}{\Delta y_{+,i} + \Delta y_{-,i}} (f_{i,j+1} - f_{i-1,j+1})}{\Delta x_{-,i} \Delta y_{+,i}} & \text{if } \sigma_{xy} < 0 \end{cases}, \tag{116}$$

where $f_{i,j} \equiv f(x_i, y_j)$.

B.3 Algorithm for Counterfactual Equilibrium

Guess (λ^E, U, m^0) to clear

- $\lambda^E = \zeta M(\tilde{u}, V)/\tilde{u}$
- $rU = b + \lambda^U \int \tilde{W} d\hat{F}(\tilde{W})$
- Free entry $\mathbb{E}J = c^e$