Econ 897 (math camp). Part I

Exam

August 19

You should work on the exam independently. The exam is closed book, so you are not allowed to use any material. Questions about **this** part are not allowed. If you believe something is ambiguous, clearly identify it in your answer, explain the interpretation you assigned (it should be reasonable) and proceed accordingly. This part is supposed to take one hour. **Please write your answers on the sheets in the space provided.**

Each question in each problem has the same weight (3 points) and there are 18 questions. However **the maximal score you can get on this part is** 40 (so your final score is minimum of 40 and what you've actually scored). **That means you don't have to answer all questions to get full score**. You can also use results from previous questions even if you have not solved them. Good luck!

Problem 1 (15 points)**.** Agree or disagree with each of this following statements. Briefly explain your answer.

(a) Set of all finite sequences of 0 and 1 has the same cardinality as $\mathbb{N}$.

(b) If X is a complete metric space, (x_n) is a Cauchy sequence in it and $f\colon X \to Y$ is a continuous function from X to some topological space Y, then $f(x_n)$ converges in Y.

(c) Set of all rays $(a, +\infty)$ $(a \in \mathbb{R})$ is a base of some topology on $\mathbb{R}$ and this topology is different from the one generated by Euclidean metric.

(d) Subset of topological space can be connected and not path-connected at the same time.

(e) Correspondence $\phi\colon \mathbb{R}_+ \rightrightarrows \mathbb{R}_+$ defined by $\phi(x) = (0, x)$ is upper hemicontinuous at 5.

Problem 2 (24 points)**.** Let's define a vector space X over $\mathbb{R}$ where $X \subset \mathbb{R}^\infty$ is a set of all sequences $x = (x_0, x_1, \ldots)$ such that $\sum_{i=0}^{\infty} {x_i}^2 < \infty$. In other words, sequences x_n such that the derivative sequence

$$y_n = \sum_{i=0}^{n} {x_i}^2$$

converges.

Hint. Any geometric sequence $x_n = ar^n$ belongs to X if $|r| < 1$.

(a) Is it possible for $x \in X$ to have $x_n \to a$ for some $a > 0$?

(b) Show that is indeed a vector space over $\mathbb{R}$ (i. e. it is vector subspace of $\mathbb{R}^\infty$).

Hint. First you may need to show that $(x + y)^2 \leqslant 2(x^2 + y^2)$.

(c) Show that

$$\|x\|_2 = \sum_{i=0}^{\infty} {x_i}^2$$

is a well-defined norm on X.

(d) Let's denote a metric generated from this norm as d_2:

$$d_2(x, y) = \|x - y\|_2.$$

Is the metric space (X, d_2) bounded?

(e) Is X (with the same metric) compact?

(f) Consider the set Y of all sequences that are zero from some point:

$$Y = \{(x_0, x_1, \ldots) \mid \exists n \in \mathbb{N} \text{ s.t.} \forall i \geqslant n \ x_i = 0\}.$$

It is a subset of X. Is it open, closed or neither (in topology induced by metric d_2)?

(g) Consider now the following subset Z of X:

$$Z = \{(x_0, x_1, \ldots) \in X \mid \forall i \in \mathbb{N} \ x_i \in [-1, 1]\}.$$

Is Z open, closed or neither (in topology induced by metric d_2))?

(h) Is Z compact?

Hint. Is Z sequentially compact?

Problem 3 (15 points)**.** Consider the following correspondence $\phi : [0, 10] \rightrightarrows [0, 10] \times [0, 10]$ (all spaces in this question have standard topology induced by Euclidean metric):

$$\phi(x) = \begin{cases} \{(5,5)\}, & \text{if } 0 \leqslant x \leqslant 2; \\ \{(y,z)\colon \max(|y-5|, |z-5|) \leqslant \frac{5}{2}(x-2)\}, & \text{if } 2 < x \leqslant 4; \\ \{(0,0), (0,10), (10,0), (10,10)\}, & \text{if } 4 < x \leqslant 6; \\ 0 \times [0,10], & \text{if } 6 < x \leqslant 8; \\ ([0,10] \cap \mathbb{R} \setminus \mathbb{Q}) \times [0,10], & \text{if } 8 < x \leqslant 10. \end{cases}$$

Hint. $[0, 10] \cap \mathbb{R} \setminus \mathbb{Q}$ is the set of all irrational points on the interval $[0, 10]$.

(a) For each point in $[0, 10]$ show whether ϕ is lower hemicontinuous at it.

(b) For each point in $[0, 10]$ show whether ϕ is upper hemicontinuous at it.

(c) For each point in $[0, 10]$ show whether $\phi(x)$ is closed, open or neither.

(d) For each point in $[0, 10]$ show whether $\phi(x)$ is compact. Is $\phi([0, 10])$ compact?

(e) Find all points such that $\phi(x) = (x, x)$. Do they form a closed set?

Upenn Math Camp Part II

Final

August 19, 2024

If you are not able to answer an item of a question you can continue to the next assuming that the previous item holds.

1. **(25 points)** In Leontief model, we study the following linear equations:

$$A\mathbf{x} + \mathbf{d} = \mathbf{x}$$

We want to know the solution structure $\mathbf{x}$ for the non-negative $n \times n$ matrix A and non-negative vector $\mathbf{d}$.

Define a non-negative matrix A is productive if there exists $\mathbf{x} \gg \mathbf{0}$ such that $\mathbf{x} \gg A\mathbf{x}$. In this exercise, you will show that for any $\mathbf{d} > \mathbf{0}$, there exists a unique non-negative solution if A is productive.[1]

(a) **(5 points)** Let $\mathbf{x}^1$ and $\mathbf{x}^2$ be two vectors such that $\mathbf{x}^1 \geq \mathbf{x}^2$ and let A be a non-negative matrix. Show that $A\mathbf{x}^1 \geq A\mathbf{x}^2$.

(b) **(5 points)** If A is a productive matrix, show that all elements of the matrix A^s converge to 0 as $s \to \infty$. [Hint: show that $\exists \lambda < 1$ s.t. $\lambda^s x > A^s x$]

(c) **(5 points)** Show that if A is productive matrix and if

$$\mathbf{x} \geq A\mathbf{x}$$

for some $\mathbf{x}$ then

$$\mathbf{x} \geq \mathbf{0}.$$

(d) **(5 points)** Show that if A is a productive matrix then $(I - A)$ full rank.

(e) **(5 points)** Show that given any non-negative $\mathbf{d}$, the system

$$(I - A)\mathbf{x} = \mathbf{d}$$

has a unique non-negative solution if the matrix A is productive.

[1] To make the discussion complete, the only if direction also holds and is straightforward.

2. **(10 points)** A firm currently produces a good according to the production:

$$f(k,l) = Ak^{\alpha}l^{\beta}$$

The factors of production k and l are hired at market prices r and w, respectively. The firm chooses the amounts of labor and capital hired that maximize profits, which are given by the first order conditions:

$$\alpha PAk^{\alpha-1}l^{\beta} = r, \qquad \beta PAk^{\alpha}l^{\beta-1} = w$$

1. Compute the hessian of f.
2. Use the Implicit Function Theorem to find expressions for the derivatives of k and l with respect to the prices P, w and r? State clearly any assumptions we have to make.

3. **(5 points)** Let $f : \mathbb{R}^n_+ \to \mathbb{R}$ such that $f(x) = \frac{h(x)}{g(x)}$ and $h(x), g(x) > 0$ for all $x \in \mathbb{R}^n_+$. Assume h is concave and g is convex. Show that f is quasiconcave.

Math Camp Part III Final, August 19, 2024

Throughout, unless otherwise indicated, $\mathbb{R}$ is equipped with the standard Lebesgue measure λ defined on the Borel σ-algebra $\mathcal{B}$.

Question 1: True/False

State which of the following is True or False. If it is true, explain why. If it is false, give a counterexample. Points will be awarded only for the explanation.

(a) The arbitrary intersection of measurable sets is measurable.

(b) If a sequence of functions $f_n \to f$ point-wise, then $f_n \to_\mu f$.

(c) If a sequence of random variables X_n converges to X in probability, then it converges to X almost surely.

(d) There exists a measure on $(\mathbb{R}, \mathcal{B})$ that assigns positive measure to the set $\{0\}$.

(e) Let $(Y, \mathcal{A})$ be a measurable space. Any function $f : Y \to \mathbb{R}$ such that $f(Y) \subseteq \{c_1, ..., c_n\}$ for $n < \infty$ must be an $\mathcal{A}$-simple function.

Question 2: Optimization

Consider the maximization problem

$$\max f(x, y) := xy$$

subject to the constraint

$$g_1(x, y) := x^2 + y^2 \leq 1$$
$$g_2(x, y) := y \geq 0$$

(a) Write out (including all conditions on the multipliers), and solve, the first order conditions for a maximizer. In general, what cases would we need to consider? Can we reduce the number of cases to check?

(b) State the second order necessary and sufficient conditions for f to have a maximum at a point (x, y) in the constraint set. There is no need to compute anything here.

(c) Explain a scenario where conditions on f could be used to simplify/bypass checking the second order conditions. Explain also an alternative to the second order conditions in order to verify that a point is a maximizer.

Question 3: Lebesgue Integration

Let λ be the standard Lebesgue measure. Consider the sequence of functions

$$f_n(x) := e^{-nx^2}$$

(a) Show that $f_n \to f \equiv 0$ point-wise almost everywhere. On which set of points does it not converge?

(b) State the Dominated Convergence theorem.

(c) Give a definition of the integral of a function $f : \mathbb{R} \to [0, \infty)$. If you need subsidiary definitions, give them.

(d) Apply DCT to show that

$$\lim_{n\to\infty} \int f_n d\lambda = 0$$

Hint: You may use that $\int_{-\infty}^{\infty} e^{-x^2} = \sqrt{\pi}$

(e) Compute $||f_n||_p$. Do we have that $f_n \to_{\mathcal{L}_p} 0$?

Now consider the sequence of functions

$$g_n(x) := e^{-(x-n)^2}$$

(f) Show that $g_n \to g \equiv 0$ point-wise almost everywhere.

(g) Explain how DCT fails in showing that $\lim_{n\to\infty} \int g_n \neq 0$.