NON-DISTORTIONARY BELIEF ELICITATION

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Abstract. A researcher wants to ask a decision-maker about a belief related to a choice the decision-maker made. When can the researcher provide incentives for the decision-maker to report her belief truthfully without distorting her choice? We identify necessary conditions and sufficient conditions for non-distortionary elicitation and fully characterize which questions can be incentivized in this way in three canonical classes of problems. For these questions, we construct simple variants of the classic Becker-DeGroot-Marschak mechanism that can be used to elicit beliefs.

1. Introduction

Researchers who conduct experiments in economics frequently ask subjects to make a choice facing some uncertainty and then wish to elicit subjects' beliefs related to their choice. Examples include:

(1) A decision-maker chooses an action with a payoff that depends on an unknown state of the world. The researcher asks what probability she assigns to her action being correct, i.e, maximizing the *ex post* payoff.
(2) Instead of asking about probabilities, the researcher asks how much the decision-maker would be willing to pay for the option to change their action after the state is realized.
(3) The decision-maker provides a guess of some quantity and is to receive a reward according to how good her guess is. The researcher asks her how likely she believes her guess is within some fixed amount x of the correct value.
(4) The decision-maker takes a test consisting of a number of true/false questions, and is to be rewarded for each correct answer. The researcher asks her about the probability that her score exceeds $y\%$.
(5) The test in the previous example has two parts. The researcher asks the decision-maker how much she believes her score improved from the first part to the second.

To ensure that the subjects' reported beliefs are reliable, researchers typically provide incentives that make truthful reporting optimal.[1] However, when the belief to be elicited is tied to an action choice, doing so could distort the incentives governing that choice. To take a simple example, suppose the subject must answer a multiple-choice question and then is asked the probability that they gave

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[1]See Healy and Leo (2024) for a discussion of incentivized vs. unincentivized belief elicitation.

the correct answer. Suppose moreover that the subject is rewarded at the belief elicitation stage with a payment that is increasing in the probability the subject assigns to the true event (namely, whether their answer was correct or not). Then a subject who is not confident about the correct answer but is confident that one of the answers is incorrect may be able to increase their overall expected payment by choosing the obviously incorrect answer and then reporting a high probability that it is not correct, thereby obtaining a high expected payoff at the belief elicitation stage.

Designing payments for belief elicitation that do not distort the incentives in the original problem allows the researcher to honestly tell the subject that she will maximize the payment she can expect to receive by choosing the action she believes is optimal in the decision problem and then reporting her belief truthfully.[2] These instructions should minimize any attempts by the subject to distort her behavior, even in cases where profitable distortions are not obvious.[3]

We introduce a model combining a general decision problem with a belief elicitation stage. The model allows us to consider a wide variety of belief elicitation questions: a question is described by a function $X(a,\theta)$ that, following an action choice a in the decision problem, asks the subject the expectation of $X(a,\theta)$ according to their subjective belief about the unknown state θ. We say that a question is *incentivizable* if there exists a payment scheme at the belief elicitation stage for which (i) truthfully reporting the expectation of $X(a,\theta)$ always maximizes the subject's expected payment, and (ii) the incentives in the decision problem are not distorted, meaning that for any belief the subject may have about θ, the optimal action in the decision problem remains optimal in the combined problem that includes the belief elicitation stage.

We first identify questions that are incentivizable regardless of the decision problem; we refer to these questions as being aligned with the utility $u(a,\theta)$ in the decision problem. Alignment allows for all questions of the form $X(a,\theta) = u(a,\theta) + d(\theta)$, as well as all affine transformations of such questions with parameters that may depend on the action a. Examples include: asking the subject about the payoff she expects to receive in the decision problem; and asking her willingness to pay to have her action replaced with an *ex post* optimal one.[4] On the other hand, the question that asks the subject about the probability that her choice is *ex post* optimal does not generally take this form (and indeed is, in many problems, not incentivizable). A researcher interested in eliciting a measure of cognitive

[2] One alternative approach would be not to inform subjects about the belief elicitation stage until after they have chosen an action. This approach is unlikely to be effective in experiments with repeated choices and more than one instance of belief elicitation. In any case, we would view this approach as a form of deception if the experimental design relies on the subject believing that they can maximize their earnings by treating the decision problem in isolation.

[3] Danz, Vesterlund, and Wilson (2022) find that instructions along these lines increase truthful reporting in a belief elicitation problem relative to explicit incentives.

[4] The latter question directly extends a simpler one used by Hu (2023) to elicit whether subjects are uncertain about the optimality of their choices.

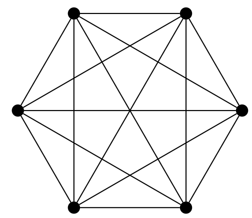
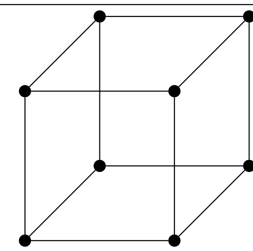
	Tree	Complete graph	Product structure
Adjacency graph			
Examples	Cognitive uncertainty (Enke and Graeber, 2023), one-dimensional supermodular problems	Multiple choice question	Random problem selection, multiple choice test
Representation	Piecewise aligned	Aligned	Product aligned

Figure 1.1. Summary of main results.

uncertainty (Enke and Graeber, 2023) may therefore do better to ask about the subject's expected payoff relative to the optimum rather than the probability that their action is optimal.

For questions that are aligned with the utility, we provide a simple construction of payments satisfying both of our incentivizability criteria. This construction is based on the classic Becker-DeGroot-Marschak method. One can first normalize $X(a, \theta)$ to lie in $[0, 1]$, and then elicit the value of $y \in [0, 1]$ at which the subject is indifferent between winning the prize with probability y and winning it with probability $X(a, \theta)$.[5] A similar construction applies to other questions that we show are incentivizable for some decision problems.

Whether and which other questions are incentivizable depends on the structure of the decision problem. A particularly important role is played by what we call the "adjacency graph." Two actions are adjacent if there is some belief at which they are both optimal and no other action is. We show that each adjacency places restrictions on how the questions following the adjacent actions are related to one another. Problems with more adjacencies therefore tend to involve stronger restrictions on which questions are incentivizable.

We fully characterize the set of incentivizable questions in three canonical classes of decision problems that differ in the structure of the adjacency graph: complete adjacency, adjacency trees, and product adjacency. (For complete and product adjacency, we also make some mild richness assumptions regarding linear independence of payoffs.) Figure 1.1 summarizes the results.

[5]A typical method for eliciting y is a multiple-price list that asks the subject to choose between probabilities y and $X(a, \theta)$ for various values of y, then choose one such choice at random for payment.

Complete adjacency graphs naturally arise in problems in which the decision-maker chooses an action to match an unknown state and receives a payoff based on whether or not they succeed, as in a multiple-choice question where the state corresponds to which answer is correct and the subject receives a payment for a correct answer. In these problems, only questions that are aligned with the utility (in the sense described above) are incentivizable.

Adjacency trees naturally arise in some problems with ordered states and actions that are monotone in beliefs. For example, if states and actions are real numbers and the subject incurs a quadratic loss based on the distance between her action and the state, then the adjacency graph forms a line.[6] Because there are so few adjacencies, this case is, in a sense, the most permissive in terms of which questions are incentivizable. In particular, alignment with the utility on the full set of actions is no longer necessary; it suffices for the question to be "piecewise aligned," meaning, in this case, that for each pair of adjacent actions it is aligned with the utility (but with the parameters governing the alignment possibly differing across pairs).

Product adjacencies arise when the decision problem comprises a number of separate tasks with complete adjacency graphs and the subject's expected reward is a sum of rewards across these tasks. For example, the subject may complete a multiple-choice test and receive a payment proportional to their score. More generally, the researcher may ask the subject about their aggregate choices in an experiment with a sequence of tasks, one of which is randomly chosen to be rewarded. The adjacency graph has a simple structure: two actions are adjacent if only if they differ on a single task. In this case, a question is incentivizable if and only if it is aligned with some weighted sum of the utilities in the various tasks (but not necessarily aligned with the overall utility in the decision problem). Thus, for example, a question that asks the subject about the likelihood that her score is above some fixed cutoff is not incentivizable, while a question that asks about the expected improvement in her score across two parts of the test is.

Two related features distinguish our approach from previous work on belief elicitation. First, the researcher asks the subject only to report a single number.[7] Second, the quantity of interest to the researcher—as described by the question $X(a, \theta)$—depends non-trivially on the subject's choice of action in the decision problem. In the absence of either of these features, any question $X(a, \theta)$ is incentivizable using standard methods. For example, if the researcher could ask the subject to report her entire belief, it would be enough to incentivize truthful reports and randomly reward the subject either for her choice in the decision problem or for her reported belief. From a practical perspective, however, this approach could be burdensome for subjects if there are more than a few states to report

[6]A typical example is when the decision problem is itself a belief elicitation problem—such as one of belief updating—with the states representing objective probabilities known to the researcher.

[7]We discuss in Section 8.4 how to extend our methods to multiple questions.

on; if the researcher is only interested in a one-dimensional statistic of the belief, asking about it directly could reduce noise in the reports.

We are not the first to observe that incentivized belief elicitation can distort other decisions. Chambers and Lambert (2021) and Healy and Leo (2024) discuss the possibility that a subject would purposefully fail a test to increase their payment from belief elicitation about their likelihood of passing. Möbius et al. (2022) describes how their belief elicitation mechanism is designed with the intention of preserving incentives in their main task. Blanco et al. (2010) find evidence that, in some problems, subjects who are paid for both a choice in a game and a reported belief take advantage of hedging opportunities, distorting either choice (or both). We implicitly assume that subjects are randomly paid either for the main task or the reported belief, eliminating such hedging opportunities.

The problem we study is motivated in part by Enke and Graeber (2023) and related and follow-up papers (e.g., Amelio, 2022; Arts, Ong, and Qiu, 2024; Xiang et al., 2021).[8] In each of these papers, subjects' cognitive uncertainty is elicited using unincentivized questions. Hu (2023) is the first paper we are aware of that provides strict incentives for subjects to reveal whether they are uncertain about their decision. His mechanism is essentially a simplified version of the Becker-DeGroot-Marschak mechanism we employ; in his, subjects make a binary choice of whether to pay a cost to have some chance that their action can be replaced with the optimal one.

Belief elicitation has been widely studied and used in both theory and experiments (see Schlag, Tremewan, and Van der Weele (2015), Charness, Gneezy, and Rasocha (2021), Haaland, Roth, and Wohlfart (2023), and Healy and Leo (2024) for surveys). The closest theoretical work to this proposal is that of Lambert, Pennock, and Shoham (2008) and Lambert (2019), which ask which properties of distributions can be elicited. Our model shares the feature that the belief elicitation question does not ask the decision-maker to report their entire belief. We sidestep their question of elicitability by restricting attention to questions that always correspond to elicitable properties, and we add the condition that the elicitation must not distort the decisions in the main decision problem.

In experiments, the binarized scoring rule of Hossain and Okui (2013) has become a popular choice for eliciting beliefs. Danz, Vesterlund, and Wilson (2022) find that subjects report more accurate beliefs when they are told that reporting truthfully will maximize the payment they can expect to receive than when the payments in the binarized scoring rule are described explicitly. In keeping with this finding, we would expect to see less distortion in behaviour in belief elicitation settings like ours if subjects are instructed that they have incentives to choose what they believe to be the optimum, which researchers can do only if the belief elicitation question is incentivizable.

Azrieli, Chambers, and Healy (2018) study incentives in a sequence of tasks and find that paying for a randomly selected problem is the only incentive-compatible mechanism when allowing for a general

[8] This work follows a longer history of using questions about confidence in decisions that is not expressed in probabilistic terms, going back at least to Butler and Loomes (2007).

class of preferences. In their model, the sequence of tasks is exogenously given, whereas in ours the belief elicitation task depends on the subjects' choice in the main task; random selection for payment is therefore not sufficient to ensure incentive compatibility.

2. Model

A decision-maker (DM) chooses an action and then faces a belief elicitation problem posed by a researcher that may depend on the action she chose.

A *decision problem* consists of a tuple (Θ, A, u), where Θ is a finite set of states of the world, A is a finite set of actions, and $u : A \longrightarrow \mathbb{R}^{\Theta}$ is a utility function specifying, for each action, the vector of payoffs across all states. We write $u(a;\theta)$ for the θ-coordinate of the vector $\mathsf{u}(a)$. For each belief $p \in \Delta(\Theta)$, let $A(p) = \arg\max_{a\in A} \sum_{\theta} p(\theta)u(a;\theta)$ denote the set of optimal actions at p. For simplicity, we assume that (i) there are no redundant actions, i.e., no a and a' such that $\mathsf{u}(a) = \mathsf{u}(a')$, and (ii) there are no dominated actions, i.e., for each a, there exists some p such that $A(p) = \{a\}$.

After choosing an action a, the DM faces a *question* about her belief described by a function $X : A \longrightarrow \mathbb{R}^{\Theta}$, with θ-coordinate $X(a;\theta)$. We interpret the question $\mathsf{X}(a)$ as asking the DM to report her subjective expected value $\mathbb{E}_p X(a;\cdot)$ given the action a that she chose in the first stage and her belief p. The dependence of X on a allows for the possibility that the researcher seeks information about the DM's beliefs that are related to the chosen action.

As the following examples illustrate, this formulation allows for considerable flexibility in what the DM is asked to report.

Example 1. The question "is the chosen action a correct *ex post*?" corresponds to

$$X(a;\theta) = \begin{cases} 1 & \text{if } a \in \arg\max_{b\in A} u(b;\theta) \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mathbb{E}_p X(a;\cdot)$ is the subjective probability of the chosen action being correct *ex post*.

Example 2. Given any action $a_0 \in A$, the question "is action a_0 correct *ex post* (regardless of what action was chosen)?" corresponds to

$$X(a;\theta) = \begin{cases} 1 & \text{if } a_0 \in \arg\max_{b\in A} u(b;\theta) \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mathbb{E}_p X(a;\cdot)$ is the subjective probability of action a_0 being correct *ex post*.

Example 3. The question "what is the regret from the chosen action?" corresponds to

$$X(a;\theta) = u(a;\theta) - \max_{b\in A} u(b;\theta).$$

Then $\mathbb{E}_p X(a;\cdot)$ is the subjective expected loss from the chosen action a relative to the *ex post* optimal action.

The DM announces a report $r \in \mathbb{R}$ and is given a reward that depends her report, her action a, and the realized state θ. Her overall payoff—including the payoff from the decision problem in the first stage—is given by a bounded function $V : \mathbb{R} \times A \times \Theta \longrightarrow [0,1]$ (which we normalize to the unit interval for convenience), the first argument of which is the DM's report r. We refer to V as an *elicitation method.*

For simplicity, we do not include the payoff $\mathsf{u}(a)$ explicitly in the elicitation method; adding it would not change anything as the payoff from the elicitation problem can be adjusted accordingly so as to give the same overall payoff. In the applications we have in mind, the researcher pays the DM for the decision problem with some fixed probability $\alpha \in (0,1)$ and for the belief elicitation problem with the remaining probability $1-\alpha$. The elicitation method V therefore takes the form $\alpha u(a;\theta) + (1-\alpha)V_0(r,a,\theta)$ for some V_0 and some $\alpha \in (0,1)$, where V_0 is the payoff from the belief elicitation mechanism. As is standard in the recent literature on belief elicitation, we implicitly view V_0 as the probability of winning a fixed prize to avoid any influence of risk preferences on the reported belief.

Definition 1. A question X is *incentivizable* if there exists an elicitation method V such that, for every $p \in \Delta(\Theta)$,

$$\arg\max_{r,a} \mathbb{E}_p V(r,a,\theta) \subseteq \left\{ \left(\mathbb{E}_p X(a;\theta), a\right) : a \in \arg\max_{b \in A} \mathbb{E}_p u(b;\cdot) \right\}.$$

Any V satisfying this condition *incentivizes X*.

Incentivizability combines two requirements of the elicitation method: first, the payoffs at the belief elicitation stage must not distort her action choices in the decision problem in the sense that any action a she optimally chooses in the overall problem with payoffs V is also optimal in the original decision problem with payoffs u; and second, the DM must have strict incentives to report her true subjective expectation of $\mathbb{E}_p X(a;\cdot)$ given her action choice a. For any question that cannot be incentivized, the researcher must either give up on truthful reporting of the DM's belief or on actions that accurately reflect the subject's belief in the decision problem. In particular, if the researcher is conducting an experiment with the DM as a subject, no matter how she designs the incentives at the belief elicitation stage, she cannot honestly tell the subjects that they will maximize their earnings by considering the decision problem in isolation and by reporting beliefs truthfully.

Our formulation imposes two substantive restrictions on the belief elicitation problem. First, the DM is asked to report only a single number rather than, say, a full probability distribution. In practice, collecting more complicated information about beliefs quickly becomes impractical in experiments. If, however, the full probability distribution could be elicited, our problem would reduce to a standard belief elicitation problem since there would be no need to make the question or incentives dependent on the action chosen in the decision problem. Second, the elicited belief is based on the expectation of some question $X(a;\cdot)$. While this formulation captures many relevant cases, in principle, the researcher may want to elicit other properties of the distribution for which our approach may not apply. We discuss in Section 8 what changes if either of these restrictions is relaxed.

3. Sufficient conditions

We begin by identifying simple sufficient conditions under which a question X is incentivizable. In the following section, we show that these conditions are also necessary in some natural applications.

Definition 2. A question X is *aligned with u on $B \subseteq A$* if there exist $\gamma : B \longrightarrow \mathbb{R}\setminus\{0\}$, $\kappa : B \longrightarrow \mathbb{R}$, and $\mathsf{d} \in \mathbb{R}^{\Theta}$ such that either

$$\mathsf{X}(a) \equiv_B \gamma(a)\left(\mathsf{u}(a) + \mathsf{d}\right) + \kappa(a)\mathsf{1}$$

or

$$\mathsf{X}(a) \equiv_B \gamma(a)\mathsf{d} + \kappa(a)\mathsf{1},$$

where $\mathsf{1} \in \mathbb{R}^{\Theta}$ is the vector of all ones and $\equiv_B$ denotes equality for all $a \in B$. We say that X is *non-trivially aligned with u on B* in the former case, and *trivially* so in the latter case. If $B = A$, we say simply that X is *aligned with u* (and similarly with the (non)-trivial qualifier).

Relative to the question $X = u$ that asks the DM about the expected utility from her chosen action, questions aligned with u allow for three changes. First, a vector d may be added to payoffs. Since d is independent of a, this change has no effect on the optimal action in the decision problem. Then, for each a, the question $\mathsf{X}(a)$ can be rescaled by a (non-zero) constant $\gamma(a)$ and translated by another constant $\kappa(a)$ uniformly across θ. These changes make each question $\mathsf{X}(a)$ essentially equivalent to the question $\mathsf{u}(a)+\mathsf{d}$ (in the sense that the expectation of $\mathsf{X}(a)$ can be computed from that of $\mathsf{u}(a)+\mathsf{d}$, and vice versa). The case of $\mathsf{X}(a) \equiv_B \gamma(a)\mathsf{d} + \kappa(a)\mathsf{1}$ can be viewed as a limit of these operations as d is scaled up and $\gamma(a)$ scaled down by the same constant, causing the $\mathsf{u}(a)$ term to vanish.

Note that for X aligned with u, the parameters γ, κ, and d are not uniquely determined in general.

Proposition 1. *If X is aligned with u, then it is incentivizable.*

Proofs omitted from the main text may be found in the appendix.

The proposition indicates that alignment with u is sufficient for incentivizability. The proof proceeds by construction using a standard Becker-DeGroot-Marschak (BDM) mechanism. The idea is to first renormalize each question using an affine transformation to make it of the form $\mathsf{X}(a) = \mathsf{u}(a) + \mathsf{d}$ or $\mathsf{X}(a) = \mathsf{d}$. Let $[L, M]$ be an interval containing every value of $X(a,\theta)$. After learning the DM's report, r, of her expectation of $\mathsf{X}(a)$, the researcher draws a number x uniformly from $[L, M]$. The DM receives a fixed prize with probability $(X(a,\theta) - L)/(M - L)$ if $r > x$, and a probability $(x - L)/(M - L)$ otherwise. By standard arguments, this mechanism provides strict incentives for the DM to truthfully report her expectation of $\mathsf{X}(a)$. Alignment of X with u ensures that the DM is also incentivized to choose an action that maximizes her expected payoff in the decision problem.[9]

For a researcher interested in eliciting a measure of the DM's confidence in her choice of action, an immediate implication of the proposition, captured in the following corollary, is that it is possible to elicit the DM's expected regret without distorting her decisions.

Corollary 1. *For any decision problem, the question about regret in Example 3 is incentivizable, as is any question that does not depend on the chosen action (such as that in Example 2).*

4. Necessary conditions

We now identify simple necessary conditions for questions to be incentivizable. Since such questions must not distort incentives for the action choice in the decision problem, it is natural to focus on beliefs where the DM is indifferent between two actions.

Say that two actions $a, b \in A$ are *adjacent* if there is a belief $p \in \Delta(\Theta)$ such that $A(p) = \{a, b\}$, that is, at belief p, a and b are both optimal and there is no other optimal action. Adjacency of actions a and b implies that there is a $(|\Theta| - 2)$-dimensional set of beliefs $p \in \Delta(\Theta)$ such that $a, b \in \arg\max_{c \in A} \mathbb{E}_p\, u\,(c; \cdot)$.[10]

For any vector $\mathsf{v} \in \mathbb{R}^\Theta$, let

$$\bar{\mathsf{v}} = \mathsf{v} - \frac{1}{|\Theta|} \sum_{\theta' \in \Theta} v(\theta')\mathsf{1}.$$

Thus $\bar{\mathsf{v}}$ is the projection of v onto the hyperplane of vectors whose coordinates sum to 0. Recall that two vectors $\mathsf{u}, \mathsf{v} \in \mathbb{R}^\Theta$ are *collinear* if there exists $\alpha \neq 0$ such that $\mathsf{v} = \alpha \mathsf{u}$.

Let $\Delta_a^b = \bar{\mathsf{u}}(b) - \bar{\mathsf{u}}(a)$ be the payoff difference vector.

Lemma 1 (Adjacency Lemma)**.** *Suppose that X is incentivizable. If actions a and b are adjacent, then there exist $\rho \in \mathbb{R}$ and $\sigma \in \mathbb{R}\setminus\{0\}$ such that*

$$\bar{\mathsf{X}}\,(b) = \rho \Delta_a^b + \sigma \bar{\mathsf{X}}\,(a)\,.$$

If $\bar{\mathsf{X}}(a)$ or $\bar{\mathsf{X}}(b)$ is collinear with Δ_a^b, then we can take $\rho \neq 0$.

[9]This mechanism can be used either to determine the combined payment from the decision problem together with the belief elicitation stage, or can be used with some fixed probability as the reward for the belief elicitation stage together with a reward of $u(a, \theta)$ with the remaining probability.

[10]A weaker version of adjacency would require only that there is some belief at which both actions are optimal (possibly along with some other action(s)). While one can prove a result analogous to the Adjacency Lemma below based on this weaker definition, we have not found the analogous result to be useful.

The Adjacency Lemma provides a key tool in testing whether a question in incentivizable: it identifies a restriction on the values of the question on each pair of adjacent actions a and b. An equivalent way to state this restriction is that the vector $\bar{\mathsf{X}}(b)$ must belong to the linear subspace spanned by the vectors $\bar{\mathsf{X}}(a)$ and Δ_a^b.

To understand the intuition behind this result, consider two adjacent actions, a and b. Suppose an elicitation method V incentivizes some X. Notice that, for any fixed r, the expected value $\mathbb{E}_p[V(r, a, \cdot)]$ is affine in p. Moreover, the set of beliefs at which a given r is optimal following action a consists of the intersection of the simplex with the hyperplane defined by $r = \mathbb{E}_p[X(a, \cdot)]$. If this set intersects with the set of beliefs at which both a and b are optimal, then the value obtained from action b followed by the optimal report must itself be affine within this intersection. This in turn implies that the optimal report $r' = \mathbb{E}_p[X(b, \cdot)]$ following action b must be constant along this intersection since the value for any fixed report is affine. In other words, for all beliefs such that the DM is indifferent between the two actions and the expected value of the question for action a is constant, the expected value for the other action must be constant as well. A standard linear algebra argument shows that vector $\mathsf{X}(b)$ must belong to the linear space spanned by vectors $\mathsf{u}(b) - \mathsf{u}(a)$, $\mathsf{X}(a)$, and $\mathtt{1}$ (the latter because we apply the linear condition to the space of beliefs). The result then follows from straightforward algebra.

Because the projection $\bar{\mathsf{v}}$ maps vectors v into a $|\Theta - 1|$-dimensional space, the Adjacency Lemma has no bite when $|\Theta| = 2$, and limited bite when $|\Theta| = 3$. (In the latter case, if $\bar{\mathsf{X}}(a)$ and Δ_a^b are linearly independent, the thesis of the Lemma holds trivially.) If there are only two states, any question $X(a, \theta)$ that is not constant in θ is equivalent to simply asking the DM to report her belief, which can be easily incentivized by adding a standard scoring rule to the utility $u(a, \theta)$. With three states, the set of beliefs at which two adjacent actions are optimal forms a line segment. Further restricting to a particular optimal report for one of the actions generically reduces the set to a single point. Matching the values along the line segment does not, therefore, impose restrictions on the question. Cycles of adjacencies can, however, imply substantive restrictions; we expand on this point in Section 8.

Example 4 (Second-order beliefs)**.** The following example is a slightly simplified version of a belief-updating experiment from Enke and Graeber (2023); the same conclusions apply to their original experiment.

The decision problem involves forecasting a binary event. The action set and state space $A = \Theta = \{0, 1/n, \ldots, 1\}$ consist of (discretized) probabilities that the event occurs, where $n \geq 3$. One can think of θ as the "true" probability given the available information, which is known to the researcher but about which the DM may be uncertain (for example because she has doubts about how to update her beliefs in light of the information she observes). This uncertainty is captured by her belief $p \in \Delta(\Theta)$.

The DM is rewarded more for forecasts that are closer to the state according to the payoff function $u(a;\theta) = -(a-\theta)^2$. The DM optimally chooses an action closest to $\mathbb{E}_p[\theta]$. The adjacency graph therefore forms a line: a_i and a_j are adjacent if and only if $|a_i - a_j| = 1/n$.

The researcher wishes to elicit the DM's confidence in her report by asking how likely she believes it is that her action is within x of the true value of θ for some fixed $x \in [0, 1/2]$. This question is described by

$$X(a,\theta) = \begin{cases} 1 & \text{if } |a-\theta| \leq x \\ 0 & \text{otherwise.} \end{cases}$$

To check whether X is incentivizable, we use Theorem 1.

Consider adjacent actions a and $b = a + 1/n$. For each θ,

$$u(b,\theta) - u(a,\theta) = -\left(a + \frac{1}{n} = \theta\right)^2 + (a-\theta)^2 = -\frac{1}{n}\left(2a - 2\theta + \frac{1}{n}\right).$$

The difference between $u(b,\theta) - u(a,\theta)$ and $\bar{u}(b,\theta) - \bar{u}(a,\theta)$ is the constant $(a-1)^2 - (a-1/n)^2$. Thus we obtain $\bar{u}(b,\theta) - \bar{u}(a,\theta) = 2a + 2\theta/n - 1$. In particular, the coordinates of any vector collinear with Δ_a^b must feature constant differences in θ. However, there is no σ for which $\bar{X}(b) - \sigma\bar{X}(a)$ satisfies this property. To see this, note that if $x < 1/n$, $\bar{X}(b) - \sigma\bar{X}(a)$ is constant across all $\theta \notin \{a,b\}$, and if $x \geq 1/n$, then its coordinates for $\theta = a$ and $\theta = b$ are equal. Therefore, there is no σ such that $\bar{X}(b) - \sigma\bar{X}(a)$ is collinear with Δ_a^b, and by Lemma 1, X is not incentivizable.

This result is not specific to the quadratic payoffs in the decision problem; the same conclusion applies if the payoff is replaced with any strictly proper scoring rule (one for which reporting an action close to the expectation of θ is optimal).

Faced with this result, what should the researcher do? One option is to use a difference measure of decision confidence that *is* incentivizable. For instance, according to Corollary 1, the expected regret question of Example 3 is incentivizable in every decision problem.

The Adjacency Lemma has an equivalent formulation using the language of alignment from the previous section.

Corollary 2. *If a question is incentivizable, then it is aligned with u on $\{a,b\}$ for every adjacent pair of actions a and b.*

Proof. Note first that X is non-trivially aligned with u on a set of actions B if and only if $\bar{\mathsf{X}}(a) \equiv_B \gamma(a)\bar{\mathsf{u}}(a) + \mathsf{d}$ for some $\gamma(a) \in \mathbb{R}$ and $\mathsf{d} \in \mathbb{R}^\Theta$, and is trivially aligned with u on B if and only if $\bar{\mathsf{X}}(a) \equiv_A \gamma(a)\mathsf{d}$ for some $\gamma(a) \in \mathbb{R}$ and $\mathsf{d} \in \mathbb{R}^\Theta$. If $\rho = 0$, then X is trivially aligned with u on $\{a,b\}$. If $\rho \neq 0$, then the alignment is nontrivial, with

$$\mathsf{d} = \frac{1}{\rho}\bar{\mathsf{X}}(b) - \bar{\mathsf{u}}(b) = \frac{\sigma}{\rho}\bar{\mathsf{X}}(a) - \bar{\mathsf{u}}(a),$$

$\gamma(a) = \rho/\sigma$, and $\gamma(b) = \rho$. □

This corollary highlights a potential gap between the necessary conditions from the Adjacency Lemma—namely, alignment on all adjacent pairs—and the sufficient condition from Proposition 1—namely, alignment on the full set of actions. In the rest of the paper, we show how to close this gap in three canonical classes of decision problems.

Our approach relies on the following two observation. If the decision problem has three actions a, b, and c such that $\{a, b\}$ and $\{b, c\}$ are both adjacent pairs, the restrictions implied by the Adjacency Lemma for these two pairs may interact with each other, leading to additional information about which questions are incentivizable. We therefore look to analyze the restrictions across all adjacent pairs simultaneously.

The *adjacency graph* is the undirected graph with vertices A and edges consisting of the adjacent pairs $\{a, b\}$. Note that, since there are no redundant or dominated actions, the adjacency graph is connected for every decision problem. A basic intuition across the next three sections is that the more edges there are in the adjacency graph, the more powerful are the restrictions imposed by the Adjacency Lemma.

5. Adjacency trees

We first consider the case in which the adjacency graph is a tree. Trees naturally arise in problems with ordered actions, as in Example 4. Example 5 below describes a simple decision problem in which the adjacency graph forms a star. In this case, as the following result shows, alignment with u on the full set of actions is not necessary for incentivizability.

Theorem 1. *Suppose the adjacency graph is a tree. Then X is incentivizable if and only if it is aligned with u on every pair of adjacent actions.*

That alignment with u is necessary for incentivizability is implied by Corollary 2. That it is sufficient follows from Proposition 2 below, which is a generalization of Proposition 1.

Say that action a is *splitting* if removing it from the adjacency graph makes the graph disconnected. If $B_0, B_1 \subset A$ are disjoint sets such that $B_0 \cup B_1 = A \setminus \{a\}$ and the adjacency graph contains no edges between B_0 and B_1, we say that *a splits the adjacency graph into B_0 and B_1*.[11]

A *splitting collection* $\{A_0, \ldots, A_k\}$ is a collection of subsets $A_i \subseteq A$ such that $\bigcup A_i = A$ and, for each i and j, either $A_i \cap A_j = \emptyset$ or $A_i \cap A_j = \{a\}$ for some splitting action a. If the adjacency graph is a tree, a splitting collection is formed by the set $\mathcal{A} = \{A_0, \ldots, A_k\}$ consisting of all adjacent pairs of actions; that is, $\tilde{A} \in \mathcal{A}$ if and only if $\tilde{A} = \{a, b\}$ for some adjacent actions a and b.

[11]Note that B_0 and B_1 need not be connected, and hence that the sets into which a splitting action splits the adjacency graph are not uniquely determined in general.

Example 5. Suppose the decision problem involves guessing the correct state, with the option of opting out and choosing a safe action. The action set is $A = \Theta \cup \{a_s\}$, with payoffs

$$u(a;\theta) = \begin{cases} 1 & \text{if } a = \theta, \\ s & \text{if } a = a_s, \\ 0 & \text{otherwise,} \end{cases}$$

where $s \in (0,1)$. If $s \geq 1/2$, then the adjacency graph is a star with action a_s in the centre. Letting $\{B_0, B_1\}$ be any partition of Θ, $\{B_0 \cup \{a_s\}, B_1 \cup \{a_s\}\}$ forms a splitting collection. Another splitting collection is given by the set of pairs $\{\theta, a_s\}$ for all $\theta \in \Theta$.

We say that X is *piecewise aligned with* u if it is aligned with u on each element of a splitting collection.

Proposition 2. *If X is piecewise aligned with u, then it is incentivizable.*

To understand the main idea of the proof of this result, consider a binary splitting collection with sets of actions A_0 and A_1 and splitting action a_0. If actions are restricted to either A_i, by the piecewise alignment assumption, the BDM construction of Proposition 1 can be used to incentivize X. This construction gives rise to a well-defined elicitation method on the full action set if the two methods agree on $A_0 \cap A_1 = \{a_0\}$. In the proof, we construct a positive affine transformation of one of the elicitation methods that ensures agreement on a_0.

We show that the new elicitation method incentivizes X on the union of the two sets. Indeed, by construction, given any belief p at which an action $a \in A_i$ is optimal in the decision problem, no other action in A_i leads to a higher expected payoff from the elicitation method. It remains to show that, similarly, no action $a' \in A_j$ (for $j \neq i$) does better than a at p. To do so, we prove that if p_0 and p_1 are beliefs such that, for each i, some action in A_i is optimal in the decision problem at belief p_i, then there exists a convex combination of p_0 and p_1 at which a_0 is optimal. Taking one of these beliefs to be p and the other to be a belief at which a' is optimal in the decision problem, it follows that there is a belief at which a_0 is optimal in the decision problem but not in the elicitation method, contrary to the way the method was constructed.

Proof of Theorem 1. The "only if" direction follows immediately from Corollary 2. For the "if" direction, as noted above, the set of adjacent pairs forms a splitting collection when the adjacency graph is a tree. As a result, alignment with u on adjacent pairs implies piecewise alignment with u, which in turn implies incentivizability by Proposition 2. □

6. Adjacency cycles

When the adjacency graph forms a tree, the incentivizable questions are those that are piecewise aligned with u; the relationship between $\bar{\mathsf{X}}(a)$ and $\bar{\mathsf{X}}(b)$ described in the Adjacency Lemma is both necessary and sufficient for incentivizability. Cycles in the adjacency graph impose additional restrictions: not only must the relationship in the Adjacency Lemma hold for adjacent actions, it must also be consistent all the way around each cycle.

A common setting in which adjacency cycles appear is when the adjacency graph is complete. This is the case, for instance, in Example 5 when $s < 1/2$.

The main result of this section shows that for many decision problems with complete adjacency graphs, a question is incentivizable if and only if it is aligned with u.

Theorem 2. *Suppose the adjacency graph is complete and $|A| \geq 4$. Suppose in addition that for any four distinct actions a, b_0, b_1, b_2, the set of vectors $\{\Delta_a^{b_i}\}_{i=0,1,2}$ is linearly independent. Then X is incentivizable if and only if it is aligned with u.*

In addition to completeness of the adjacency graph, this theorem relies on two assumptions: the set of actions must be sufficiently large and the payoffs from the actions must be sufficiently independent. The assumptions make the Adjacency Lemma particularly powerful. For example, as discussed in Section 4, the Adjacency Lemma only has any bite if there are at least three states of the world, which must be the case if the two assumptions of the theorem hold. These assumptions also exclude decision problems with larger state spaces that can effectively be reduced to problems with three states, such as those in which the payoff from each action is 0 in every state but the first three.

Example 6. Consider a decision problem with at least four states in which the DM is asked to guess the state and receives a reward for guessing correctly that may depend on which state is realized. Thus $A = \Theta$ and

$$u(a, \theta) = \begin{cases} r_\theta & \text{if } a = \theta \\ 0 & \text{otherwise,} \end{cases}$$

where $r_\theta > 0$ for all θ. The adjacency graph for this problem is complete: given any two states θ and θ', the actions θ and θ' are the only optimal actions at the belief that assigns probability $r_{\theta'}/(r_\theta + r_{\theta'})$ to state θ and all of the remaining probability to state θ'.

Suppose the researcher seeks to elicit the DM's belief about whether she correctly guessed the state, which is described by the question

$$X(a, \theta) = \begin{cases} 1 & \text{if } a = \theta \\ 0 & \text{otherwise.} \end{cases}$$

Taking $\gamma(a) = 1/r_a$ and $\kappa(a) = 0$ for each a and $\mathsf{d} = 0$, we see that X is aligned with u, and therefore can be incentivized using the BDM construction of Proposition 1.

Now consider the same question X in a different decision problem where the DM can also receive a smaller reward for a "close" guess. Let $\Theta = \{1, \ldots, n\}$ and

$$u(a, \theta) = \begin{cases} r_\theta & \text{if } a = \theta \\ r_\theta/2 & \text{if } |a - \theta| = 1 \\ 0 & \text{otherwise,} \end{cases}$$

where $\min_\theta r_\theta > \max_\theta r_\theta/2 > 0$. The adjacency graph for this problem is again complete. In this case, however, X is not aligned with u. Since the linear independence condition in Theorem 2 holds, X is not incentivizable.

To explain the main ideas in the proof of Theorem 2, we first need some additional terminology. An *adjacency cycle* is a tuple $C = (a_0, \ldots, a_n)$ such that $a_n = a_0$ and actions a_i and a_{i+1} are adjacent for each $i = 0, \ldots, n-1$. We say that n is the *length* of the cycle C. We abuse notation slightly by writing $a \in C$ to mean that $a = a_i$ for some i.

An adjacency cycle is *internally independent* if, for some $a \in C$, the set of vectors $\left\{\Delta_a^b : b \in C \setminus \{a\}\right\}$ is linearly independent. Let V_C be the linear space spanned by $\left\{\Delta_a^b : b \in C \setminus \{a\}\right\}$. The cycle C is internally independent if and only if $\dim V_C = n - 1$, where n is the length of the cycle. Notice that the space V_C and the linear independence of $\left\{\Delta_a^b : b \in C \setminus \{a\}\right\}$ do not depend on the choice of a, only on C itself.

For each action a_0 in an adjacency cycle, applying the Adjacency Lemma iteratively along adjacent pairs $\{a, b\}$ of actions in the cycle gives rise to an expression for some multiple of $\bar{\mathsf{X}}(a_0)$ as a linear combination of Δ_a^b terms. If this cycle is internally independent, then either $\bar{\mathsf{X}}(a_0) \in V_C \setminus \{\mathsf{0}\}$ or all of the coefficients in the linear combination must be equal to 0. In the latter case, iteratively applying the Adjacency Lemma again gives a linear relationship between any $\bar{\mathsf{X}}(c)$ (for c in the cycle) and $\bar{\mathsf{X}}(a_0)$ together with Δ_a^b terms. This relationship can be simplified using the fact that the coefficients in the previous linear combination are zero to show that X is aligned with u on C, leading to the following lemma.

Lemma 2. *Let C be an internally independent adjacency cycle. If X is incentivizable, then either X is aligned with u on C, or $\bar{\mathsf{X}}(a) \in V_C \setminus \{\mathsf{0}\}$ for all $a \in C$.*

While Lemma 2 identifies conditions that an incentivizable question must satisfy on a given internally independent cycle, it says nothing about more complicated adjacency graphs. The following lemma shows how alignment with u on multiple cycles or other subsets of actions can, under mild conditions, be combined to obtain alignment with u on their union.

Lemma 3. *Suppose X is aligned with u on sets of actions B and D. If there exist actions $a_0, a_1 \in B \cap D$ such that $\bar{\mathsf{X}}(a_0)$ and $\bar{\mathsf{X}}(a_1)$ are not collinear, then X is aligned with u on $B \cup D$.*

Alignment with u on B gives rise, for each action in B, to a system of equations that the parameters relating X and u must satisfy; naturally, the same is true for D. In the proof of this lemma, we show that if B and D share two actions for which the values of the questions are linearly independent, then the corresponding systems of equations must share the same solution for B as for D.

The main idea behind Theorem 2 is that, given any cycle C through an action a, Lemma 2 implies that any incentivizable X is either aligned with u on C or lies in $V_C \setminus \{\mathsf{0}\}$. The linear independence condition can be used to eliminate the latter possibility by considering multiple cycles through a; more precisely, there must be *some* cycle C on which X is aligned with u, as the intersection of the sets $V_{C'}$ across cycles C' containing a is $\{\mathsf{0}\}$. Varying a and applying Lemma 3 leads to alignment with u on the entire action set.

In Appendix D, we identify a more general class of problems in which alignment with u is both necessary and sufficient for incentivizability. This more general result applies when there is a rich enough structure of internally independent cycles in the adjacency graph, which can be the case even when the graph is not complete.

7. Product problems

In many experiments, subjects perform a sequence of tasks. These tasks may be identical or they may differ; the subject's payoff is a sum or weighted average payoff of the payoffs in the various tasks, as in the common experimental design in which one task is randomly selected for payment (see Charness, Gneezy, and Halladay (2016) and Azrieli, Chambers, and Healy (2018) and the references therein). In such cases, the researcher may be interested in eliciting beliefs related to the entire sequence of actions chosen by the subject. For example, a student may solve a test with multiple questions—with their payoff being equal to the score—and the researcher may want to ask the subject what she believes about her overall performance on the test. Alternatively, to gauge the impact of learning across repetitions of the same task, the researcher may want to elicit subjects' beliefs about their change performance between the beginning and the end of the experiment.

To formalize this idea, we define a *product problem* (Θ, A, u) to be a decision problem in which $\Theta = \times_i \Theta_i$, $A = \times_i A_i$, and

$$u\left(a, \theta\right) = \sum_i u_i\left(a_i, \theta_i\right)$$

for some sets $(\Theta_1, \ldots, \Theta_I)$ and $(A_1, \ldots, A_I)$, and some functions $u_i : A_i \times \Theta_i \longrightarrow \mathbb{R}$. As noted above, the additive separability of u captures commonly used incentives in which one choice is randomly selected for payment. We refer to each (Θ_i, A_i, u_i) as a *task*. We write a_{-i} for a profile

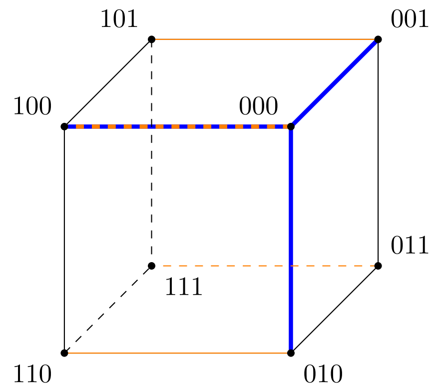


FIGURE 7.1. Adjacency graph for Example 7 with $\Omega = \{0, 1\}$ and $I = 3$.

$(a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_I)$ and $a_i a_{-i}$ for the profile whose ith coordinate is a_i and remaining coordinates are given by a_{-i}.

Example 7. The decision problem is a test consisting of $I \geq 3$ multiple choice questions. The state space and action space are given by $\Theta = A = \Omega^I$, where Ω is a finite set containing at least two elements describing the possible answers to any given question. Coordinate i corresponds to the ith question: θ_i is the correct answer to question i and a_i is the DM's answer to question i. The payoff in the decision problem is the score on the test:

$$u(a, \theta) = \sum_{i=1}^{I} \mathbb{1}\{\theta_i = a_i\}.$$

The DM has a belief $p \in \Delta(\Theta)$. The optimal choice of action in each task i is the most likely state according to the marginal distribution of p over Θ_i.

Note that we make no assumptions about the correlation among states across tasks: the DM can hold any belief about the joint distribution of $(\theta_1, \dots, \theta_I)$. In particular, the DM need not view the states as independent, nor must there be a fixed state across tasks.

Product problems share a structure that distinguishes them from the problems we have analyzed so far. Two actions $a, b \in A$ are adjacent only if they differ in exactly one task, that is, if there is some i such that $a_i \neq b_i$ and $a_{-i} = b_{-i}$. Conversely, if a_i and b_i are adjacent in task i, then the product actions $a_i a_{-i}$ and $b_i a_{-i}$ are adjacent for all a_{-i}. The adjacency graph of the product problem is typically neither complete (even if the adjacency graphs in each task are complete) nor a tree.

Figure 7.1 depicts one example. Note that while the collection of payoff difference vectors associated with edges exiting a single node—such as the blue edges Δ^{100}_{000}, Δ^{010}_{000}, and Δ^{001}_{000}—are linearly independent, no cycle is internally independent because parallel edges correspond to identical payoff difference vectors. Thus, for instance, the orange edges correspond to $\Delta^{100}_{000} = \Delta^{110}_{010} = \Delta^{101}_{001} = \Delta^{111}_{011}$.

We say that question X depends on task i *trivially* if, for each a_{-i}, the vectors $\bar{\mathsf{X}}(a_i a_{-i})$ are collinear for all a_i. The following result provides necessary and sufficient conditions for incentivizability in product problems.

Theorem 3. *Let $(\Theta_i, A_i, u_i)_{i=1}^{I}$ be a product problem with $I \geq 3$. Suppose that for each i, either A_i contains only two actions, or the adjacency graph for problem (Θ_i, A_i, u_i) is complete and the vectors $\left\{\Delta_{a_i}^{b_i}, \Delta_{a_i}^{c_i}\right\}$ are linearly independent for all $a_i, b_i, c_i \in A_i$. Suppose in addition that there are at least three tasks on which X does not depend on trivially. Then X is incentivizable if and only if there exist $v(a), \kappa(a) \in \mathbb{R}$ with $v(a) \neq 0$ for each $a \in A$, $\tau_i \in \mathbb{R}$ for each i, and $\mathsf{d} \in \mathbb{R}^{\Theta}$ such that*

$$X(a, \theta) = \kappa(a) + v(a)\left(d(\theta) + \sum_i \tau_i u_i(a_i, \theta_i)\right) \tag{7.1}$$

for all a and θ.

Theorem 3 requires that the question non-trivially depends on at least three tasks. If instead the question depends non-trivially on only one problem, it is straightforward to adapt the analysis from Section 6 to apply here. In the intermediate case in which the question depends non-trivially on exactly two tasks, we do not know whether the conclusion of Theorem 3 holds.

Along the same lines as Theorem 2, we assume that each task has a complete adjacency graph. However, relative to Theorem 2, the other requirements for each task are significantly weaker: there are no restrictions on the number of actions, and we require linear independence only of pairs of payoff difference vectors instead of triples.

The characterization of incentivizable questions in (7.1) is more permissive than that in Theorem 2. If τ_i is constant across i, the expression in (7.1) implies that X is aligned with u. However, in contrast to the case of a single problem with complete adjacency, there are many questions not aligned with u that are also incentivizable (namely, those for which τ_i varies across i). The additional freedom in the product problem results from a smaller number of cycles and a larger number of linear dependencies in the payoffs.

The following example illustrates the added flexibility afforded by (7.1).

Example 8 (Example 7 continued)**.** It is straightforward to verify that in the product problem of Example 7, if $|\Omega| > 2$, the adjacency graph for each problem i is complete and all pairs $\left\{\Delta_{a_i}^{b_i}, \Delta_{a_i}^{c_i}\right\}$ are linearly independent. Therefore, Theorem 3 applies to any question that depends non-trivially on at least three tasks.

Suppose the test consists of two parts: the first part comprises questions 1 through I_1, while the second part comprises questions $I_1 + 1$ through I. Consider the question that asks the DM about the expected improvement in her average score from the first part of the test to the second:

$$X(a,\theta) = \sum_{i=1}^{I} \zeta_i \mathbb{1}(a_i = \theta_i) \text{ where } \zeta_i = \begin{cases} -\frac{1}{I_1} & \text{if } i \leq I_1 \\ \frac{1}{I-I_1} & \text{if } i > I_1 \end{cases}$$

This question is of the form given in (7.1), and is therefore incentivizable by Theorem 3. At first glance, this result may be surprising as X seems to create opposing incentives in the two parts of the test. Nevertheless, X can be incentivized using a simple modification of the BDM elicitation mechanism from Proposition 1 that involves adding the payoff from the decision problem:

$$\begin{aligned} V(r,a,\theta) &= \int_0^r X(a,\theta)dx + \int_r^1 x dx - \frac{1}{2} + \sum_i u_i(a_i,\theta_i) \\ &= \sum_i (1 + r\zeta_i) u_i(a_i,\theta_i) - \frac{r^2}{2}. \end{aligned}$$

The two integral terms provide incentives for truthful reporting of r. On their own, since ζ_i is negative for some i, these terms distort the incentives in the original decision problem and change the optimal choice of a. Adding the final sum restores the correct incentives since $1 + r\zeta_i$ is positive for each i.

The next example illustrates the restrictiveness of (7.1).

Example 9 (Example 7 continued)**.** Let $x \in \{1, \dots, I\}$. The researcher would like to elicit the probability the DM assigns to receiving a score of at least x,[12] which corresponds to the question

$$X(a,\theta) = \begin{cases} 1 & \text{if } \sum_{i=1}^{I} \mathbb{1}\{\theta_i = a_i\} \geq x \\ 0 & \text{otherwise.} \end{cases}$$

We claim that X is not incentivizable. Since there is no problem on which X depends trivially, Theorem 3 applies. Thus it suffices to show that X is not of the form specified in (7.1). Suppose for contradiction that it is. Notice that there exist at least two distinct scores on the test that are either both below x or both at least x. It follows that, given any a and i, there exist θ and θ' such that $a_i = \theta_i \neq \theta'_i$, $\theta_j = \theta'_j$ for all $j \neq i$, and $X(a,\theta) = X(a,\theta')$. From (7.1), since $v(a) \neq 0$, it follows that $d(\theta) + \tau_i = d(\theta')$. Applying the same argument to the action a' defined by

$$a'_j = \begin{cases} \theta'_i & \text{if } j = i \\ a_j & \text{otherwise,} \end{cases}$$

we obtain $d(\theta') + \tau_i = d(\theta)$. Therefore, $\tau_i = 0$ for all i. Since $X(a,\theta) = X(a,\theta')$, it now follows that $d(\theta) = d(\theta')$. Letting a'' be any action that agrees with θ on exactly x coordinates, including coordinate i, we have $X(a'',\theta) = 1 \neq 0 = X(a'',\theta')$, implying that $d(\theta) \neq d(\theta')$, a contradiction.

[12] This question is similar to that of Möbius et al. (2022), who elicit the subject's belief that their score on an incentivized IQ test is above the median among the participants.

7.1. **Sketch of proof.** The proof of Theorem 3 can be found in Appendix E. Here, we describe the key ideas.

The proof that (7.1) is sufficient for incentivizability is relatively straightforward: the argument directly extends the construction in Example 8.

For the remainder of this section, we assume that question X is incentivizable. By the Adjacency Lemma (Lemma 1), for any pair of adjacent actions a and b, there are coefficients $x(a,b) \neq 0$ and $y(a,b)$ such that $\bar{\mathsf{X}}(a) = x(a,b)\bar{\mathsf{X}}(b) + y(a,b)\Delta_a^b$. For the purpose of this discussion, we assume that the coefficients $x(a,b)$ and $y(a,b)$ are uniquely defined.

The key step in the proof is to show that each cycle $a_0, \ldots, a_n = a_0$ of adjacent actions is *exact*, that is, that

$$x(a_0,a_1)x(a_1,a_2)\cdots x(a_{n-1},a_n) = 1.$$

We first explain the connection between exactness of all cycles and (7.1), and then explain why all cycles are exact.

Fix an arbitrary action a_0. The exactness of all cycles implies that we can define

$$v(a) = x(a_0,a_1)\cdots x(a_{n-1},a_n)$$

for any path $a_0, \ldots, a_n = a$ of adjacent actions, and the definition does not depend on the chosen path.[13] In particular, for any two adjacent actions, $v(a) = x(a,b)v(b)$. Letting $\bar{\mathsf{X}}^*(a) = \frac{1}{v(a)}\bar{\mathsf{X}}(a)$ and $y^*(a,b) = \frac{1}{v(a)}y(a,b)$, the Adjacency Lemma implies that, for any two adjacent actions a and b,

$$\bar{\mathsf{X}}^*(a) = \bar{\mathsf{X}}^*(b) + y^*(a,b)\Delta_a^b. \tag{7.2}$$

To illustrate how (7.2) implies (7.1), consider Example 7. Let a and b be adjacent actions, and recall that adjacency implies there must be some task i such that $a_{-i} = b_{-i}$. Notice that $\Delta_a^b = \Delta_a^{b_i a_{-i}} = \Delta_{a_i}^{b_i}$, where the last vector refers to the payoff difference $\bar{\mathsf{u}}_i(b_i) - \bar{\mathsf{u}}_i(a_i)$ in task i. We will argue that the coefficient $y^*(a,b)$ depends only on a_i and b_i, i.e., that $y^*(a,b) = y^*(a_i a'_{-i}, b_i a'_{-i})$ for all a'_{-i}. Indeed, for any two problems i,j, an application of (7.2) to two paths $a, b_i a_{-i}, b_i b_j a_{-ij}$ and $a, b_j a_{-j}, b_i b_j a_{-ij}$ yields

$$y^*(a,b_i a_{-i})\Delta_a^{b_i a_{-i}} + y^*(b_i a_{-i}, b_i b_j a_{-ij})\Delta_{b_i a_{-i}}^{b_i b_j a_{-ij}} = y^*(a,b_j a_{-j})\Delta_a^{b_j a_{-j}} + y^*(b_j a_{-j}, b_i b_j a_{-ij})\Delta_{b_j a_{-j}}^{b_i b_j a_{-ij}}.$$

Since $\Delta_a^{b_i a_{-i}} = \Delta_{b_j a_{-j}}^{b_i b_j a_{-ij}} = \Delta_{a_i}^{b_i}$ and $\Delta_a^{b_j a_{-j}} = \Delta_{b_i a_{-i}}^{b_i b_j a_{-ij}} = \Delta_{a_j}^{b_j}$, this last equation simplifies to

$$\left(y^*(a,b_i a_{-i}) - y^*(b_j a_{-j}, b_i b_j a_{-ij})\right)\Delta_{a_i}^{b_i} + \left(y^*(b_i a_{-i}, b_i b_j a_{-ij}) - y^*(a, b_j a_{-j})\right)\Delta_{a_j}^{b_j} = 0.$$

[13]The term "exact" is borrowed from differential geometry, where an exact vector field is a gradient of a scalar function.

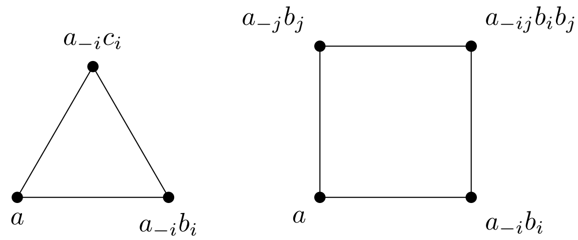


FIGURE 7.2. Two basic types of cycles in product problems.

Both coefficients must be equal to 0 since $\Delta_{a_i}^{b_i}$ and $\Delta_{a_j}^{b_j}$ are linearly independent; in particular,

$$y^*(a, b_i a_{-i}) = y^*(b_j a_{-j}, b_i b_j a_{-ij}).$$

Applying this equation repeatedly to change a_{-i} to any a'_{-i} one component at a time shows that $y^*(a, b_i a_{-i})$ depends only on a_i and b_i.

Now consider Example 7 with $\Omega = \{0, 1\}$. Let $y_i = y^*(0_i a_{-i}, 1_i a_{-i})$ for any a_{-i}. (By the previous observation, y_i is independent of the choice of a_{-i}.) Taking $b = \mathsf{0}$ in (7.2), we obtain

$$\begin{aligned}\bar{\mathsf{X}}^*(a) &= \bar{\mathsf{X}}^*(\mathsf{0}) + \sum_{i \text{ s.t. } a_i = 1} y_i \Delta_{0_i}^{1_i} \\ &= \bar{\mathsf{X}}^*(\mathsf{0}) - \sum_i y_i \bar{\mathsf{u}}_i(0_i) + \sum_i y_i \bar{\mathsf{u}}_i(a_i).\end{aligned}$$

Taking $d(\theta) = \bar{\mathsf{X}}^*(\mathsf{0}) - \sum_i y_i \bar{\mathsf{u}}_i(0_i)$, we obtain an expression of the form in (7.1).

This argument is not specific to Example 7: it extends directly to all product problems with binary actions in every task. Extending beyond binary actions requires more care and conditions on linear independence of payoffs within tasks. We leave the details to the formal proof.

We now return to the question of why all cycles are exact. The first step is to notice that it is sufficient to establish the exactness of two types of cycles, depicted in Figure 7.2. Cycles of the first type consist of three actions that differ only in choices within the same task. Those of the second type consist of four actions that differ in choices in two tasks. Under the assumptions of Theorem 3, one can show that all cycles can, in a sense, be decomposed into cycles of these two types whose exactness implies exactness of the original cycle.

In Example 7, only cycles of the second type exist. Consider the four-action cycle corresponding to the front face in Figure 7.1. A repeated application of the Adjacency Lemma (Lemma 1) along the cycle leads to

$$\bar{\mathsf{X}}(000) = x(000, 100)x(100, 110)x(110, 010)x(010, 000)\bar{\mathsf{X}}(000) + s_1 \Delta_{0_1}^{1_1} + s_2 \Delta_{0_2}^{1_2}$$

for some $s_1, s_2 \in \mathbb{R}$. If $\bar{\mathsf{X}}(000)$ is not in the subspace spanned by vectors $\Delta_{0_1}^{1_1}$ and $\Delta_{0_2}^{1_2}$, then the product of the x coefficients on the right-hand side must be equal to 1, meaning that the cycle is exact. Otherwise, by analyzing a number of cases, one can show that the cycles corresponding to the other five faces in Figure 7.1 are all exact. These five cycles can be combined in such a way that all edges not belonging to the front face "cancel out," thereby implying exactness of the original cycle.

8. Discussion

8.1. **Three states.** We have assumed throughout that there are at least four states. Our necessary conditions for incentivizability rely on this assumption insofar as it ensures that, for any two adjacent actions a and b, intersections of the set of beliefs at which the DM is indifferent between a and b with the level sets of $X(a, \cdot)$ or $X(b, \cdot)$ have dimension at least one.

If there are only two states, the problem becomes trivial as the DM's belief about the state can be elicited independent of the action, which is sufficient for the researcher to determine the expectation of any question X. With three states, eliciting the entire belief may still be a practical option as it requires asking for only two probabilities.

If the researcher wants to ask for only one number in a problem with three states, although our necessary conditions no longer apply, looking at adjacencies can nonetheless be useful for understanding incentivizability. Suppose the overall payoff is a weighted sum of the payoff from the decision problem and that from a scoring rule applied at the belief elicitation stage. Suppose moreover that the scoring rule depends only on the reported belief and the realized value of $X(a, \theta)$, and not on θ directly.[14] At any belief at which the DM is indifferent between two actions, the value of truthful reporting at the belief elicitation stage must be equal following these two actions. Depending on the structure of the decision problem, following these constant values along cycles of adjacent actions can imply restrictions on X; see Figure 8.1.

8.2. **Independent questions.** Our necessary conditions make use of independence assumptions on the payoffs in the decision problem. Similar results can be obtained if one replaces these assumptions with assumptions about independence of $\bar{X}$ across actions. For example, along the lines of Lemma 2, if X is incentivizable and the set of questions $\bar{\mathsf{X}}(a)$ is linearly independent for actions a in some cycle C of adjacencies, then one can show that X must be aligned with u on C. Lemma 3 can then be used to obtain necessary conditions on the full set of actions.

8.3. **Non-affine questions.** We have restricted attention to eliciting beliefs about the expectation of some function $\mathsf{X}(a)$. Lambert (2019) studies elicitation of "properties" of beliefs, where a property

[14]This is a natural restriction that we do not impose in our model; doing so would have no effect on our results. When there are three states, we do not know whether this restriction has any bite; we expect that the convexity of the value function would place restrictions on incentivizable questions even if the value can depend on θ directly.

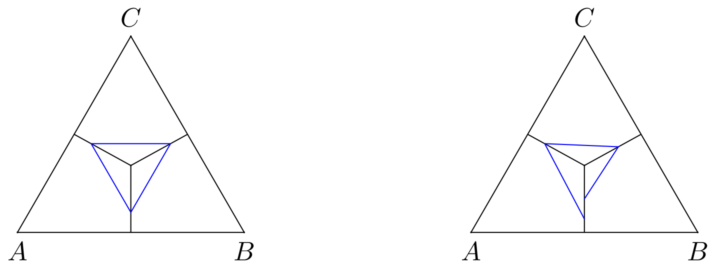


FIGURE 8.1. Problems with three states. Each triangle depicts the simplex of beliefs. The black line segments illustrate the partition of the simplex according to which action is optimal. Blue line segments represent sets of beliefs on which $\mathsf{X}(a^*)$ is constant for the optimal action a^*. In the left triangle, because the blue lines form a triangle, adjacency considerations do not rule out incentivizability of X. In the right triangle, X is not incentivizable if the payment for elicitation depends only on the value of X and the reported belief.

corresponds to a partition of the simplex. He characterizes which properties are incentivizable, i.e., for which ones there exists a scoring rule incentivizing truthful reporting when the DM is asked only about the property associated with her belief. A question $\mathsf{X}(a)$ in our framework corresponds to a property that partitions the simplex into parallel hyperplanes (unless the question is trivial, in which case there is a single property for the entire simplex). This formulation captures many properties of interest and ensures incentivizability in the absence of an additional decision problem, or if the question is independent of the action choice. There are, however, properties—such as the median of some X—which may be of interest that are incentivizable in Lambert's context but to which our results do not apply. Nonetheless, we expect our general approach of focusing on adjacencies between actions to be useful for such non-affine questions.

8.4. **Multi-dimensional question.** In our model, we assume that the researcher can only ask a single question. Here, we show how our methods can be extended to multiple questions. For simplicity, we focus on the case of two questions; the logic extends directly to more than two questions.

We say that questions $X, Y : A \longrightarrow \mathbb{R}^\Theta$ are *jointly incentivizable* if there exists an elicitation method $V : \mathbb{R}^2 \times A \times \Theta \longrightarrow [0,1]$ such that, for every $p \in \Delta(\Theta)$,

$$\arg\max_{a,r,s} \mathbb{E}_p V(r,s,a,\theta) \subseteq \left\{ \left(\mathbb{E}_p X(a;\theta), \mathbb{E}_p Y(a;\theta), a\right) : a \in \arg\max_{b \in A} \mathbb{E}_p u(b;\cdot) \right\}.$$

If X and Y are both aligned with u, it is straightforward to extend Proposition 1 to show that they are jointly incentivizable.

For necessary conditions, our key result—Lemma 1—extends as follows.

Lemma 4. *Suppose that X and Y are jointly incentivizable. If actions a and b are adjacent, then there are ρ_X, ρ_Y and σ_x^y for $x, y = X, Y$, not all equal to 0, such that*

$$\bar{\mathsf{X}}(b) = \rho_X (\bar{\mathsf{u}}(b) - \bar{\mathsf{u}}(a)) + \sigma_X^X \bar{\mathsf{X}}(a) + \sigma_X^Y \bar{\mathsf{Y}}(a)$$

$$\text{and} \qquad \bar{\mathsf{Y}}(b) = \rho_Y (\bar{\mathsf{u}}(b) - \bar{\mathsf{u}}(a)) + \sigma_Y^X \bar{\mathsf{X}}(a) + \sigma_Y^Y \bar{\mathsf{Y}}(a).$$

If $\bar{\mathsf{X}}(a)$ or $\bar{\mathsf{X}}(b)$ is collinear with $\bar{\mathsf{u}}(b) - \bar{\mathsf{u}}(a)$, then we can take $\rho \neq 0$.

The proof, which we omit, follows the same reasoning as that of Lemma 1. We leave to future research the details of how to use this lemma to identify precise conditions for joint incentivizability.

8.5. **Robust scoring rules.** We have assumed throughout that the researcher knows the utility function in the decision problem. This assumption is reasonable in many lab experiments, but questionable in other settings—such as field experiments—in which a researcher may want to elicit beliefs. Suppose instead that the researcher has in mind a set of possible utility functions, and requires that questions be incentivizable for every utility function in that set. For questions that (as in our model) are independent of the utility function, this requirement is typically very demanding. For instance, under the conditions of Theorem 2, X must be aligned with every u. Notice, however, that some questions are naturally formulated in a way that depends on the utility function; the expected ex post regret of Example 3 is one such question. Allowing X to depend on u accommodates these questions. Our results can then be applied separately for each utility function in the set considered by the researcher to determine whether a question is incentivizable.

Appendix A. Proofs for Section 3

Lemma 5. *For any $\mathsf{d} \in \mathbb{R}^\Theta$, $\mathsf{X}(a) = \mathsf{u}(a) + \mathsf{d}$ and $\mathsf{X}(a) = \mathsf{d}$ are incentivizable.*

Proof. Let $L, M \in \mathbb{R}$ be such that, for all a and θ, $L < u(a;\theta) + d(\theta) < M$ and $L < d(\theta) < M$. If $\mathsf{X}(a) = \mathsf{u}(a) + \mathsf{d}$, let

$$V(r, a, \theta) = \int_L^r X(a;\theta)\, dx + \int_r^M x dx - \frac{M^2}{2} = (u(a;\theta) + d(\theta))(r - L) - \frac{1}{2} r^2,$$

where $d(\theta)$ is the θ-coordinate of d. For this V, simple calculations show that the optimal choice of r given a is $\mathbb{E}_p X(a;\theta)$. Since the optimal r is greater than L, the optimal choice of action a is the same as in the decision problem with utility u. A similar argument applies if $\mathsf{X}(a) = \mathsf{d}$, in which case we let $V(r, a, \theta) = u(a;\theta) + d(\theta)(r - L) - \frac{1}{2} r^2$. □

Proof of Proposition 1. Suppose V incentivizes a question X. Let $\mathsf{Y}(a) = \gamma(a)\mathsf{X}(a) + \kappa(a)\mathsf{1}$ for some $\gamma, \kappa : A \longrightarrow \mathbb{R}$. Letting $W(r, a, \theta) = V\left(\frac{1}{\gamma(a)}(r - \kappa(a)), a, \theta\right)$, it is straightforward to verify that W incentivizes Y. The result now follows from Lemma 5. □

Appendix B. Proofs for Section 4

Proof of Lemma 1. Let V be an elicitation method that incentivizes X. Consider two beliefs p_0 and p_1 such that (i) actions a and b are both optimal in the decision problem, i.e., $a, b \in \arg\max_{a'} \mathbb{E}_{p_k}[u(a', \cdot)]$ for $k = 0, 1$), and (ii) the question $\mathsf{X}(a)$ attains the same value at p_0 and p_1, i.e., $\mathbb{E}_{p_0}[X(a, \cdot)] = \mathbb{E}_{p_1}[X(a, \cdot)]$.Letting $r = \mathbb{E}_{p_0}[X(a, \cdot)]$, it follows that (a, r) is optimal given V at all $p_\alpha = \alpha p_1 + (1-\alpha)\, p_0$ for $\alpha \in [0, 1]$. The optimal expected payoff $\mathbb{E}_{p_\alpha}[V(r, a, \cdot)]$ is therefore linear in α. Since b is also an optimal action at each p_α, the optimal expected payoff $\max_s \mathbb{E}_{p_\alpha}[V(s, b, \cdot)]$ must be linear in α as well. Therefore, there exists some r' such that, for each α, $r' \in \arg\max_s \mathbb{E}_{p_\alpha}[V(s, b, \cdot)]$. In particular, $\mathbb{E}_{p_k}[X(b, \cdot)] = r'$ for each $k = 0, 1$.

Fixing p_0, notice that the preceding argument applies to all $p_1 \in \mathbb{R}^\Theta$ satisfying the following orthogonality conditions:

(1) $p_1 - p_0 \perp \mathsf{1}$, ensuring that p_1 is a well-defined belief;
(2) $p_1 - p_0 \perp \mathsf{u}(a) - \mathsf{u}(b)$, ensuring that the DM is indifferent between a and b at p_1; and
(3) $p_1 - p_0 \perp \mathsf{X}(a)$, ensuring that $\mathsf{X}(a)$ attains the same value at p_0 as at p_1.

For any such p_1, the preceding argument implies that $p_1 - p_0 \perp \mathsf{X}(b)$.

By a standard linear algebra argument, it follows that

$$\mathsf{X}(b) \in \operatorname{span}(\mathbf{1}, \mathsf{u}(a) - \mathsf{u}(b), \mathsf{X}(a))\,.$$

Noting that, for any vector v, $\bar{\mathsf{v}}$ differs from v by a scalar multiple of $\mathsf{1}$, and that $\mathsf{v} \perp \mathsf{1}$, we obtain

$$\bar{\mathsf{X}}(b) = \rho\,(\bar{\mathsf{u}}(a) - \bar{\mathsf{u}}(b)) + \sigma \bar{\mathsf{X}}(a)$$

for some $\rho, \sigma \in \mathbb{R}$.

If $\bar{\mathsf{X}}(b)$ is not collinear with $\bar{\mathsf{u}}(b) - \bar{\mathsf{u}}(a)$, then we must have $\sigma \neq 0$, as needed. Otherwise, switching the roles of a and b in the preceding argument yields that $\bar{\mathsf{X}}(a)$ is also collinear with $\bar{\mathsf{u}}(b) - \bar{\mathsf{u}}(a)$, and therefore one can take ρ and σ to be nonzero. □

Appendix C. Proofs for Section 5

Lemma 6. *Suppose action a splits the adjacency graph into B_0 and B_1. Then, for any pair of beliefs p_0 and p_1 such that, for each i, some action $b_i \in B_i$ is optimal at p_i in the decision problem, there is a convex combination $p = \alpha p_1 + (1-\alpha)\, p_0$ such that a is optimal at p.*

Proof. Suppose not. Then all actions that are optimal at convex combinations of p_0 and p_1 must be either from B_0 or B_1. Hence, for some such convex combination p', the set of optimal actions contains at least one element from each of B_0 and B_1. Then in any neighbourhood of p', there must be a belief at which there is exactly one member of each B_i that is optimal, contradicting the fact that no action in B_0 is adjacent to any action in B_1. □

Proof of Proposition 2. We first derive explicit formulas for an elicitation method that incentivizes a question X aligned with u. When X is non-trivially aligned with u, using the BDM construction from the proof of Proposition 1 gives

$$\begin{aligned}
V^{BDM}\left(r,a,\theta;\gamma,\kappa,\mathsf{d}\right) &= \left(u\left(a;\theta\right)+d\left(\theta\right)\right)\left(\frac{1}{\gamma(a)}\left(r-\kappa(a)\right)-L\right)-\frac{1}{2}\left(\frac{1}{\gamma(a)}\left(r-\kappa(a)\right)\right)^2 \\
&= \frac{1}{\gamma(a)^2}\left[\left(X\left(a;\theta\right)-\kappa(a)\right)\left(r-\kappa(a)-\gamma(a)L\right)-\frac{1}{2}\left(r-\kappa(a)\right)^2\right] \\
&= \frac{1}{\gamma(a)^2}\left(X\left(a;\theta\right)-\frac{1}{2}r-\frac{1}{2}\kappa(a)\right)\left(r-\kappa(a)\right)-\frac{1}{\gamma(a)}(X(a;\theta)-\kappa(a))L.
\end{aligned} \tag{C.1}$$

Similarly, if X is trivially aligned with u, adding $u(a;\theta)$ to either of the last two lines gives V^{BDM}.

When X is non-trivially aligned with u, the expected payoff of a DM who chooses action a and then chooses r optimally is equal to

$$\begin{aligned}
&\max_r\left[\mathbb{E}_p\left[u\left(a;\cdot\right)+d\left(\cdot\right)\right]\left(\frac{1}{\gamma(a)}\left(r-\kappa(a)\right)-L\right)-\frac{1}{2}\left(\frac{1}{\gamma(a)}\left(r-\kappa(a)\right)\right)^2\right] \\
&\qquad=\max_x\left[\mathbb{E}_p\left[u\left(a;\cdot\right)+d\left(\cdot\right)\right]\left(x-L\right)-\frac{1}{2}x^2\right] \\
&\qquad=\frac{1}{2}\left(\mathbb{E}_p\left[u\left(a;\cdot\right)+d\left(\cdot\right)\right]\right)^2-L\,\mathbb{E}_p\left[u\left(a;\cdot\right)+d\left(\cdot\right)\right].
\end{aligned} \tag{C.2}$$

Similarly, the expected payoff is equal to $\mathbb{E}_p[u(a;\cdot)]+\left(\mathbb{E}_p[d(\cdot)]\right)^2/2-L\,\mathbb{E}_p[d(\cdot)]$ if X is trivially aligned with u.

To keep the notation simple, we present the argument only for the case in which the splitting collection contains two elements, A_0 and A_1, with splitting action a_0. Extending the argument to the general case is straightforward.

We construct an elicitation method on each A_i, then show that they agree on a_0 and therefore give rise to well-defined elicitation method on the full action set, A.

First suppose X is non-trivially aligned with u on each A_i. For each $i=0,1$, let $(\gamma_i,\kappa_i,\mathsf{d}_i)$ be such that $\mathsf{X}(a)\equiv_{A_i}\gamma_i(a)(\mathsf{u}(a)+\mathsf{d})+\kappa(a)\mathsf{1}$. Define the elicitation methods

$$\begin{aligned}
V_i\left(r,a,\theta;w_i,\omega_i\right) = &\left(\frac{\gamma_i(a_0)}{\gamma_i(a)}\right)^2\left(X\left(a;\theta\right)-\frac{1}{2}r-\frac{1}{2}\kappa_i(a)\right)\left(r-\kappa_i(a)\right) \\
&+w_i\left(\theta\right)-\omega_i-\frac{\gamma_i(a_0)^2}{\gamma_i(a)}(X(a;\theta)-\kappa_i(a))L,
\end{aligned} \tag{C.3}$$

where $w_i\in\mathbb{R}^\Theta$ and $\omega_i\in\mathbb{R}$. Note that this expression differs from the expression for V^{BDM} in (C.1) only by multiplication by a positive constant and addition of a function that depends only on the state. As neither of these changes affects the optimal choices of a and r, it follows from the argument in the proof of Proposition 1 that V_i incentivizes X on A_i (i.e., in the decision problem with actions restricted to A_i).

Notice that

$$\begin{aligned}
&V_1\big(r, a_0, \theta; w_1, \omega_1\big) - V_0\left(r, a_0, \theta; w_0, \omega_0\right) \\
&\quad = \left(X\left(a_0;\theta\right) - \frac{1}{2}r - \frac{1}{2}\kappa_1(a_0)\right)\left(r - \kappa_1(a_0)\right) - \left(X\left(a_0;\theta\right) - \frac{1}{2}r - \frac{1}{2}\kappa_0(a_0)\right)\left(r - \kappa_0(a_0)\right) \\
&\qquad + \left(w_1\left(\theta\right) - w_0\left(\theta\right)\right) - \left(\omega_1 - \omega_0\right) - \gamma_1(a_0)(X(a_0;\theta) - \kappa_1(a_0))L + \gamma_0(a_0)(X(a_0;\theta) - \kappa_0(a_0))L \\
&\quad = X\left(a_0;\theta\right)\left(\kappa_0(a_0) - \kappa_1(a_0) + \gamma_0(a_0) - \gamma_1(a_0)\right) + \frac{1}{2}\left(\left(\kappa_1(a_0)\right)^2 - \left(\kappa_0(a_0)\right)^2\right) \\
&\qquad\qquad + \left(w_1\left(\theta\right) - w_0\left(\theta\right)\right) - \left(\omega_1 - \omega_0\right) + (\gamma_1(a_0)\kappa_1(a_0) - \gamma_0(a_0)\kappa_0(a_0))L.
\end{aligned}$$

Given any w_0 and ω_0, let

$$w_1(\theta) = w_0(\theta) - X\left(a_0,\theta\right)\left(\kappa_0(a_0) - \kappa_1(a_0) + \gamma_0(a_0) - \gamma_1(a_0)\right)$$

for each θ, and

$$\omega_1 = \frac{1}{2}\left(\kappa_1(a_0)^2 - \kappa_0(a_0)^2\right) + \omega_0 + (\gamma_1(a_0)\kappa_1(a_0) - \gamma_0(a_0)\kappa_0(a_0))L.$$

Then $V_1\big(r, a_0, \theta; w_1, \omega_1\big) = V_0\left(r, a_0, \theta; w_0, \omega_0\right)$ for all r and θ.[15]

Let

$$V\left(r, a, \theta\right) = V_i\left(r, a, \theta\right) \text{ for each } a \in A_i.$$

Because V_0 and V_1 agree for action a_0, V is well defined.

In case X is trivially aligned with u on some A_i, we add $\gamma_i(a_0)^2 u(a;\theta)$ to the expression for $V_i(r, a, \theta; w_i, \omega_i)$ in (C.3) and adjust the definition of $w_1(\theta)$ accordingly to include any such additional $u(a_0;\theta)$ terms.

To verify that V incentivizes X, it suffices to show that at any belief p at which no action in A_i is optimal in the decision problem, $\arg\max_a \max_r \mathbb{E}_p[V(r, a, \cdot)] \subseteq A_j$, where $j \neq i$. Without loss of generality, take $i = 1$ and $j = 0$, and let p_0 denote such a belief.

Note first that, by (C.2), the expected value from choosing an action $a \in A_i$ followed by an optimal choice of r is equal to

$$\text{(C.4)} \quad \max_r \mathbb{E}_p[V_i(r, a, \cdot)] = \gamma_i(a_0)^2 \frac{1}{2}\left(\mathbb{E}_p[u\left(a, \cdot\right)] + \mathbb{E}_p[d\left(\cdot\right)]\right)^2 + \mathbb{E}_p[w_i(\cdot)] - \omega_i - L\gamma_i(a_0)^2\, \mathbb{E}_p\left[u\left(a; \cdot\right) + d\left(\cdot\right)\right].$$

It follows that the expected value from an action $b \in A_i$ is at least as large as that from an action $a \in A_i$ if and only if

$$\frac{1}{2}\left(\mathbb{E}_p[u(b;\cdot)] + \mathbb{E}_p[d(\cdot)]\right)^2 - L\,\mathbb{E}_p[u(b;\cdot)] \geq \frac{1}{2}\left(\mathbb{E}_p[u(a;\cdot)] + \mathbb{E}_p[d(\cdot)]\right)^2 - L\,\mathbb{E}_p[u(a;\cdot)],$$

[15] In the general case with splitting collection $\{A_1, \ldots, A_k\}$, one can recursively define each w_{i+1} and ω_{i+1} given w_i and ω_i in an analogous fashion.

which holds if and only if $\mathbb{E}_p[u(b;\cdot)] \geq \mathbb{E}_p[u(a;\cdot)]$ since the function $(x+y)^2/2 - Lx$ is increasing in x for $x+y > L$ and L satisfies $\mathbb{E}_p[u(a';\cdot)] + \mathbb{E}_p[d(\cdot)] > L$ for all actions a'.

Suppose for contradiction that there is some $b \in A_1 \setminus \{a_0\}$ such that $b \in \arg\max_a \max_r \mathbb{E}_{p_0} V(r,a,\cdot)$. It follows that

$$\max_r \mathbb{E}_{p_0} V_1(r,b,\cdot) = \max_r \mathbb{E}_{p_0} V(r,b,\cdot) \geq \max_r \mathbb{E}_{p_0} V(r,a_0,\cdot) = \max_r \mathbb{E}_{p_0} V_1(r,a_0,\cdot).$$

By the observation in the preceding paragraph,

$$\mathbb{E}_{p_0} u\left(b,\cdot\right) \geq \mathbb{E}_{p_0} u\left(a_0,\cdot\right).$$

Let p_1 be a belief at which b is strictly optimal, i.e., $\{b\} = \arg\max_a \mathbb{E}_{p_1}[u\left(a,\cdot\right)]$. By Lemma 6, action a_0 must be optimal at some convex combination $p = \alpha p_1 + (1-\alpha)\, p_0$. At the same time, the above inequalities imply that

$$\begin{aligned}
\mathbb{E}_p[u\left(a_0,\cdot\right)] &= \alpha\, \mathbb{E}_{p_1}[u\left(a_0,\cdot\right)] + (1-\alpha)\, \mathbb{E}_{p_0}[u\left(a_0,\cdot\right)] \\
&< \alpha\, \mathbb{E}_{p_1}[u\left(b,\cdot\right)] + (1-\alpha)\, \mathbb{E}_{p_0}[u\left(b,\cdot\right)] \\
&= \mathbb{E}_p[u\left(b,\cdot\right)],
\end{aligned}$$

contradicting the optimality of a_0 at p. □

Appendix D. Proofs for Section 6 and extension of Theorem 2

Proof of Lemma 2. Without loss of generality, let $a_0 \in C$ be such that either $\bar{\mathsf{X}}(a_0) = 0$ or $\bar{\mathsf{X}}(a_0) \notin V_C$. (If no such a_0 exists, the lemma holds trivially.)

By Lemma 1, for each $i = 1, \ldots, n$, we have $\bar{\mathsf{X}}\left(a_i\right) = \rho_i \Delta_{a_{i-1}}^{a_i} + \sigma_i \bar{\mathsf{X}}\left(a_{i-1}\right)$ for some $\sigma_i \neq 0$ and some ρ_i. Iterating these equations gives

$$\bar{\mathsf{X}}\left(a_0\right) = \bar{\mathsf{X}}\left(a_n\right) = \sum_{i=1}^{n} \Gamma_i \rho_i \Delta_{a_{i-1}}^{a_i} + \Gamma_0 \bar{\mathsf{X}}\left(a_0\right),$$

where $\Gamma_i = \sigma_n \cdots \sigma_{i+1}$ and $\Gamma_n = 1$. Because $\Delta_{a_0}^{a_1} + \cdots + \Delta_{a_{n-1}}^{a_n} = 0$, we get

$$\sum_{i=1}^{n-1} \left(\Gamma_i \rho_i - \rho_n\right) \Delta_{a_{i-1}}^{a_i} + \left(\Gamma_0 - 1\right) \bar{\mathsf{X}}\left(a_0\right) = 0.$$

Since either $\bar{\mathsf{X}}(a_0) = 0$ or $\bar{\mathsf{X}}(a_0) \notin V_C$, it follows that

$$\sum_{i=1}^{n-1} \left(\Gamma_i \rho_i - \rho_n\right) \Delta_{a_{i-1}}^{a_i} = 0.$$

Internal independence implies that $\Gamma_i \rho_i = \rho_n$ for each $i = 1, \ldots, n-1$.

If $\rho_n = 0$, then $\rho_i = 0$ for each i since $\Gamma_i \neq 0$. In this case, all $\bar{\mathsf{X}}(a_i)$ are collinear, which implies that X is trivially aligned with u on C.

For the case of $\rho_n \neq 0$, first note that, by the same iteration as above, for each k,

$$\begin{aligned}\bar{\mathsf{X}}(a_k) &= \sum_{i=1}^{k-1} \rho_i \frac{\Gamma_i}{\Gamma_k} \Delta^{a_i}_{a_{i-1}} + \frac{\Gamma_0}{\Gamma_k} \bar{\mathsf{X}}(a_0) \\ &= \frac{\rho_n}{\Gamma_k} \sum_{i=1}^{k-1} \Delta^{a_i}_{a_{i-1}} + \frac{\Gamma_0}{\Gamma_k} \bar{\mathsf{X}}(a_0) \\ &= \frac{\rho_n}{\Gamma_k} (\bar{\mathsf{u}}(a_k) - \bar{\mathsf{u}}(a_0)) + \frac{\Gamma_0}{\Gamma_k} \bar{\mathsf{X}}(a_0).\end{aligned}$$

Letting $\gamma(a_k) = \rho_k = \rho_n / \Gamma_k \neq 0$ and $\mathsf{d} = -\bar{\mathsf{u}}\,(a_0) + \rho_n^{-1} \Gamma_0 \bar{\mathsf{X}}\,(a_0)$, we have $\bar{\mathsf{X}}(a_k) = \gamma(a_k)\,(\bar{\mathsf{u}}(a_k) + \mathsf{d}))$ for all k, and thus X is (non-trivially) aligned with u on C. $\square$

Proof of Lemma 3. Alignment with u on B implies that there exist $\gamma_0^B, \gamma_1^B \in \mathbb{R}$ and $\mathsf{d}^B \in \mathbb{R}^\Theta$ such that

$$\gamma_0^B \bar{\mathsf{X}}(a_0) - \bar{\mathsf{u}}(a_0) = \gamma_1^B \bar{\mathsf{X}}(a_1) - \bar{\mathsf{u}}(a_1) = \mathsf{d}^B.$$

Similarly, for D,

$$\gamma_0^D \bar{\mathsf{X}}(a_0) - \bar{\mathsf{u}}(a_0) = \gamma_1^D \bar{\mathsf{X}}(a_1) - \bar{\mathsf{u}}(a_1) = \mathsf{d}^D.$$

Rearranging these equations gives

$$\begin{aligned} &\gamma_0^B \bar{\mathsf{X}}(a_0) - \gamma_1^B \bar{\mathsf{X}}(a_1) = \bar{\mathsf{u}}(a_0) - \bar{\mathsf{u}}(a_1) \\ \text{and} \qquad &\gamma_0^D \bar{\mathsf{X}}(a_0) - \gamma_1^D \bar{\mathsf{X}}(a_1) = \bar{\mathsf{u}}(a_0) - \bar{\mathsf{u}}(a_1).\end{aligned}$$

Subtracting one equation from the other leads to

$$\left(\gamma_0^B - \gamma_0^D\right) \bar{\mathsf{X}}(a_0) - \left(\gamma_1^B - \gamma_1^D\right) \bar{\mathsf{X}}(a_1) = 0.$$

Since $\bar{\mathsf{X}}(a_0)$ and $\bar{\mathsf{X}}(a_1)$ are linearly independent, it must be that $\gamma_k^B = \gamma_k^D$ for each $k = 0, 1$. This in turn implies that $\mathsf{d}^B = \mathsf{d}^D = \mathsf{d}$.

All that remains is to show that for any other action $a \in B \cup D$, the corresponding parameters γ_a^B and γ_a^D are equal. Since $\bar{\mathsf{X}}(a)$ cannot be collinear with both $\bar{\mathsf{X}}(a_0)$ and $\bar{\mathsf{X}}(a_1)$, we can repeat the argument replacing one of a_0 or a_1 with a to obtain $\gamma_a^B = \gamma_a^D$. $\square$

D.1. **General necessary conditions.** A set of actions $B \subseteq A$ is *cycle-rich* if it contains at least four elements and, for any proper subset $B' \subset B$ with at least three elements, there exists $a \in B \setminus B'$ such that

$$\bigcap \{V_C : C \text{ is internally independent}, a \in C, \text{ and } |C \cap B'| \geq 2\} = \{\mathsf{0}\}.$$

The intersection above goes over all internally independent cycles that contain a and at least two elements of B'.

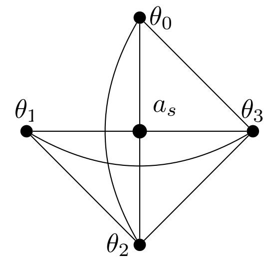


FIGURE D.1. Adjacency graph for Example 10.

Example 10. Consider a variant of Example 6 in which there is an additional safe action. Thus $A = \Theta \cup \{a_s\}$ with

$$u(a,\theta) = \begin{cases} r_\theta & \text{if } a = \theta \\ s & \text{if } a = a_s \\ 0 & \text{otherwise.} \end{cases}$$

Suppose in addition that $\Theta = \{\theta_0, \theta_1, \theta_2, \theta_3\}$, with $r_\theta = 1/2$ for $\theta = \theta_0, \theta_1$ and $r_\theta = 1$ for $\theta = \theta_2, \theta_3$. Let $s = 3/10$. Then the adjacency graph—which is depicted in Figure D.1—is incomplete since actions θ_0 and θ_1 are not adjacent, but A is cycle-rich.

Theorem 4. *Suppose $B \subseteq A$ is cycle-rich. If X is incentivizable, then it is aligned with u on B.*

Proof. We begin with the following observation.

Lemma 7. *If X is non-trivially aligned with u on B, then for any $a, b \in B$, if $\bar{\mathsf{X}}(a)$ and $\bar{\mathsf{X}}(b)$ are collinear, they are also collinear with Δ_a^b.*

Proof. Let $a, b \in B$ be such that $\bar{\mathsf{X}}(a)$ and $\bar{\mathsf{X}}(b)$ are collinear and let $\gamma(\cdot) \neq 0$ and d be such that $\bar{\mathsf{X}}(a) = \gamma(a)\,(\bar{\mathsf{u}}(a) + \mathsf{d})$ and $\bar{\mathsf{X}}(b) = \gamma(b)\,(\bar{\mathsf{u}}(b) + \mathsf{d})$. By the collinearity assumption, there exists $\alpha \neq 0$ such that

$$\bar{\mathsf{u}}(a) + \mathsf{d} = \alpha\,(\bar{\mathsf{u}}(b) + \mathsf{d})\,.$$

Because $\bar{\mathsf{u}}(a) \neq \bar{\mathsf{u}}(b)$, it must be that $\alpha \neq 1$. It follows that $\mathsf{d} = \frac{1}{1-\alpha}\,(\alpha\bar{\mathsf{u}}(b) - \bar{\mathsf{u}}(a))$ and

$$\frac{1}{\gamma(a)}\bar{\mathsf{X}}(a) = \bar{\mathsf{u}}(a) + \frac{1}{1-\alpha}\,(\alpha\bar{\mathsf{u}}(b) - \bar{\mathsf{u}}(a)) = \frac{\alpha}{1-\alpha}\,(\bar{\mathsf{u}}(b) - \bar{\mathsf{u}}(a))\,,$$

as needed. □

Cycle-richness implies that there is some $a \in B$ and a collection of internally independent cycles $\tilde{C}$ with $a \in \tilde{C} \subseteq B$ for which the intersection of the spaces $V_{\tilde{C}}$ is $\{\mathsf{0}\}$. Thus either $\bar{\mathsf{X}}(a) = \mathsf{0}$ or $\bar{\mathsf{X}}(a) \notin V_C$ for some such cycle C. By Lemma 2, X is aligned with u on C.

Let $B' \subseteq B$ be a subset of B of maximal cardinality on which X is aligned with u. By the above argument, B' has at least three elements. Suppose for contradiction that $B' \neq B$. By the same argument as in the preceding paragraph, cycle-richness implies that there exists $a \in B \setminus B'$ and a cycle C containing a such that $|C \cap B'| \geq 2$ and either $\bar{\mathsf{X}}(a) = \mathsf{0}$ or $\bar{\mathsf{X}}(a) \notin V_C$. By Lemma 2, X is aligned with u on C.

If there exists a pair of distinct actions $a_0, a_1 \in C \cap B'$ such that $\bar{\mathsf{X}}(a_0)$ and $\bar{\mathsf{X}}(a_1)$ are not collinear, Lemma 3 implies that X is aligned with u on $C \cup B'$, contradicting the maximality of B'.

From now on, suppose that $a_0, a_1 \in C \cap B'$ are distinct actions such that $\bar{\mathsf{X}}(a_0)$ and $\bar{\mathsf{X}}(a_1)$ are collinear.

If $\bar{\mathsf{X}}(a_0)$ and $\bar{\mathsf{X}}(a_1)$ are not collinear with $\Delta_{a_0}^{a_1}$, then Lemma 7 implies that the alignment with u on C and that on B' must both be trivial, which further implies that, for all $b \in C \cup B'$, $\bar{\mathsf{X}}(b)$ is collinear with $\bar{\mathsf{X}}(a_0)$. Thus, X is trivially aligned with u on $C \cup B'$, contradicting the maximality of B'.

If $\bar{\mathsf{X}}(a_0)$ and $\bar{\mathsf{X}}(a_1)$ are collinear with $\Delta_{a_0}^{a_1}$, then, by Lemma 1, $\bar{\mathsf{X}}(a) \in \operatorname{span}(\Delta_a^{a_0}, \Delta_{a_0}^{a_1}) = V_C$. The choice of cycle C implies that $\bar{\mathsf{X}}(a) = 0$. Another application of Lemma 1 shows that $\bar{\mathsf{X}}(a_0)$ is collinear with $\Delta_a^{a_0}$. But the latter contradicts collinearity with $\Delta_{a_0}^{a_1}$ due to the independence assumptions. The contradiction finishes the proof of the Theorem. □

Proof of Theorem 2. It suffices to show that the set of all actions is cycle-rich; the result then follows from Theorem 4.

Take any proper subset $B \subset A$ with at least three actions $b_0, b_1, b_2 \in B$ and let $a \in A \setminus B$. Consider cycles $C_i = B \setminus \{b_i\} \cup \{a\}$. Then, $V_{C_i} = \operatorname{span}\{\Delta_a^{b_j} : j \neq i\}$. The independence assumption implies that $\bigcap_i V_{C_i} = \{0\}$. □

It is straightforward to extend Theorem 4 to problems in which there is a splitting collection $\{A_0, \ldots, A_k\}$ such that, for each l, either A_l is cycle-rich or it contains exactly two elements. In that case, only questions piecewise aligned with u are incentivizable.

Appendix E. Proof of Theorem 3

This Appendix is divided into the following subsections. Section E.1 shows that each incentivizable question in the product problem can be decomposed into linearly independent vectors that correspond to different tasks. We use this decomposition together with Lemma 1 to derive restrictions on questions for adjacent actions. Subsections E.2 to E.5 show that all adjacency cycles are exact. Section E.3 develops useful tools, and Sections E.4 and E.5 deal with different classes of cycles. Section E.6 concludes the proof.

E.1. **Decomposition.** Assume throughout that X is incentivizable.

For each i and each $t_i \in \Theta_i$, let $\mathsf{e}_i(t_i) \in \mathbb{R}^\Theta$ be the vector such that, for each $\theta \in \Theta$, $\mathsf{e}_i(\theta|t_i) = \mathbf{1}\{\theta_i = t_i\}$. Let $E_i = \operatorname{span}\{\mathsf{e}_i(t_i) : t_i \in \Theta_i\}$ be the linear subspace spanned by such vectors. Notice that we can interpret

$$\Delta_{a_i}^{b_i} = \bar{\mathsf{u}}_i(a_i) - \bar{\mathsf{u}}_i(b_i) = \sum_{t_i} \left(\bar{\mathsf{u}}_i(t_i|a_i) - \bar{\mathsf{u}}_i(t_i|b_i)\right) \mathsf{e}_i(t_i)$$

for each i, a_i, and b_i as a vector in the subspace E_i. Let E_0 be a complementary space to the sum of E_1 through E_I. Note that the subspaces E_i are linearly independent.

For each a, $\bar{\mathsf{X}}(a)$ admits a unique decomposition

$$\bar{\mathsf{X}}(a) = \sum_{i=0}^{I} \mathsf{w}_i(a) \tag{E.1}$$

with $\mathsf{w}_i(a) \in E_i$ for all i. Note that the vectors $\mathsf{w}_0(a), \mathsf{w}_1(a), \ldots, \mathsf{w}_I(a)$ are linearly independent.

Take any a, b, and i, such that $a_{-i} = b_{-i}$. By Lemma 1, there are $x(a,b) \neq 0$ and $y(a,b)$ such that $\bar{\mathsf{X}}(a) = x(a,b)\bar{\mathsf{X}}(b) - y(a,b)\Delta_b^a$. Using the above decomposition, we have

$$\begin{aligned} 0 = &\left[\mathsf{w}_i(a) - x(a,b)\mathsf{w}_i(b) - y(a,b)\Delta_{b_i}^{a_i}\right] \\ &+ \sum_{j\neq i} \left[\mathsf{w}_j(a) - x(a,b)\mathsf{w}_j(b)\right] + \left[\mathsf{w}_0(a) - x(a,b)\mathsf{w}_0(b)\right]. \end{aligned}$$

The proof of Lemma 1 shows that $x(a,b)$ is uniquely defined if and only if $\bar{\mathsf{X}}(a) \neq 0$ and $\bar{\mathsf{X}}(a)$ is not collinear with $\Delta_{b_i}^{a_i}$. When this is not the case, we say that the transition (a,b) is *free*. The values of $x(a,b)$ for free transitions are carefully chosen below. Our choice always satisfies $x(a,b)x(b,a) = 1$ (which always holds for non-free transitions). Let $x(a,a) = 1$.

Because all vectors in square brackets form a linearly independent system, all these vectors must be equal to 0:

$$\mathsf{w}_i(a) - x(a,b)\mathsf{w}_i(b) = y(a,b)\Delta_{b_i}^{a_i}, \tag{E.2}$$

$$\mathsf{w}_j(a) - x(a,b)\mathsf{w}_j(b) = 0 \text{ for each } j \neq i, \tag{E.3}$$

$$\mathsf{w}_0(a) - x(a,b)\mathsf{w}_0(b) = 0. \tag{E.4}$$

Lemma 8. *For each a, there exist $\gamma_i(a) \neq 0$ and vectors $\mathsf{w}_i^*(a_i)$ for each $i = 1, \ldots, I$ and $\gamma_0(a) \neq 0$ and a vector w_0^* such that*

$$\mathsf{w}_i(a) = \gamma_i(a)\mathsf{w}_i^*(a_i) \text{ for each } i, \text{ and } \mathsf{w}_0(a) = \gamma_0(a)\mathsf{w}_0^*.$$

Proof. For the first claim, fix an action a^*. For each a_i, let $\mathsf{w}_i^*(a_i) = \mathsf{w}_i(a_{-i}^* a_i)$. For each a, fix an arbitrary a path of adjacent actions $a^0 = a_{-i}^* a_i, \ldots, a^n = a$ such that for each $l < n$, $a_i^l = a_i$. A repeated application of (E.3) shows that

$$\mathsf{w}_i(a) = x(a^n, a^{n-1}) \cdots x(a^1, a^0)\mathsf{w}_i^*(a_i).$$

Let $\gamma_i(a) = x(a^n, a^{n-1}) \cdots x(a^1, a^0)$.

For the second claim, fix $\mathsf{w}_0^* = \mathsf{w}_0(a^*)$ and repeatedly apply (E.4). □

This result says that, for a fixed a_i, the vectors $\mathsf{w}_i(a_{-i}a_i)$ for a_{-i} are either all collinear or all equal to 0.

Lemma 9. *For each i, one of the following is true:*

(a) $\mathsf{w}_i^*(a_i) = 0$ *for each* a_i,
(b) *there exists* $a_i^0 \in A_i$ *such that* $\mathsf{w}_i^*(a_i^0) = 0$ *and* $\mathsf{w}_i^*(a_i) \neq 0$ *for each* $a_i \neq a_i^0$,
(c) $\mathsf{w}_i^*(a_i) \neq 0$ *for each* a_i.

If question X does not depend trivially on problem i, then there is an action a_i such that $\mathsf{w}_i^*(a_i) \neq 0$.

Proof. Suppose that there are three different actions a_i, b_i, and c_i such that $\mathsf{w}^*(a_i) = \mathsf{w}^*(b_i) = \mathsf{0}$ and $\mathsf{w}^*(c_i) \neq 0$. Equation (E.2) together with Lemma 8 implies that $\mathsf{w}_i(c_i)$ is simultaneously collinear with $\Delta_{c_i}^{a_i}$ and $\Delta_{c_i}^{b_i}$. But the latter is impossible given the independence assumption.

For the last claim, suppose that $\mathsf{w}^*(a_i) = 0$ for each a_i. Equation (E.2) implies that $y(a,b) = 0$ for each a and b such that $a_{-i} = b_{-i}$. But then, for each a_{-i}, the vectors $\bar{\mathsf{X}}(a_{-i}a_i)$ and $\bar{\mathsf{X}}(a_{-i}b_i) = x(a_{-i}b_i, a_{-i}a_i)\bar{\mathsf{X}}(a_{-i}a_i)$ are collinear. Hence, X depends on problem i trivially. □

Because X depends non-trivially on at least three problems, Lemma 9 implies that at least three problems satisfy (b) or (c).

E.2. **Exact cycles.** Say that an adjacency cycle $a^0, \ldots, a^n = a^0$ is *exact* if

$$x(a^0, a^1)x(a^1, a^2) \cdots x(a^{n-1}, a^n) = 1.$$

The goal of this subsection as well as subsections E.3 to E.5 is to prove the following result.

Lemma 10. *The values $x(a,b)$ for free transitions (a,b) can be chosen so that (i) $x(a,b)x(b,a) = 1$ for all adjacent a and b, and (ii) every adjacency cycle is exact.*

We begin with the following observation.

Lemma 11. *Suppose that* $\mathsf{w}_0^* \neq 0$*. Then, all adjacency cycles are exact.*

Proof. The result follows from a repeated application of equality (E.4). □

From now on, we assume that $\mathsf{w}_0^* = 0$.

E.3. **Tools.** The two results in this section develop tools that we use in the subsequent analysis.

The first tool allows us to replace the question of whether a cycle is exact with an analogous question about related cycles. For each path $c = (a^0, \ldots, a^{n_c})$ and each transition (a,b) between two adjacent actions, define

$$m_a^b(c) = \#\left\{l < n_c : a^l = a, a^{l+1} = b\right\} - \#\left\{l < n_c : a^l = b, a^{l+1} = a\right\}.$$

Lemma 12. *Suppose that, for some adjacency cycle c, there exists a collection D of exact adjacency cycles such that for each adjacent pair a, b, $m_a^b(c) = \sum_{d \in D} m_a^b(d)$. Then cycle c is exact.*

Proof. Let $\prec$ be an arbitrary strict order on the set of actions A. Then,

$$\prod_{l<n_c} x(a^{c,l}, a^{c,l+1}) = \prod_{a \prec b} (x(a,b))^{m_a^b(c)} = \prod_{a \prec b} (x(a,b))^{\sum_{d \in D} m_a^b(d)}$$
$$= \prod_{d \in D} \prod_{a \prec b} (x(a,b))^{m_a^b(d)} = \prod_{d \in D} \prod_{l<n_d} x(a^{d,l}, a^{d,l+1}) = 1,$$

where the last equality comes from the fact that cycles in D are exact. □

The second result shows that it is enough to consider a particular category of "small" cycles.

Lemma 13. *Each adjacency cycle is exact if and only if the following cycles are exact:*

(1) (a, b, a) *for any two adjacent actions a and b,*
(2) (a, b, c, a) *for any three actions a, b, and c such that $a_{-i} = b_{-i} = c_{-i}$,*
(3) $(a, a_{-i}b_i, a_{-ij}b_ib_j, a_{-j}b_j, a)$ *for any action a, any $i \neq j$, and any b_i and b_j.*

Proof. Take any "large" cycle of adjacent actions $a = a^0, \ldots, a^n = a$ and let i_l be such that, for each $l < n$, $a^l_{-i_l} = a^{l+1}_{-i_l}$. For future reference, notice that if the action in some problem i is ever changed, then it must be changed at least twice: if $i_l = i$ for some l, then there is some $l' \neq l$ such that $i_{l'} = i$. We use the "small" cycles to re-order and reduce the "large" cycle without changing the value of the product of associated the x terms:

- if $i_l > i_{l+1}$ for $l < n-1$ we use the "small" cycle of type (3) to switch the order of the two problems, i.e., replace the cycle fragment $\ldots, a^l, a^{l+1}, a^{l+2}, \ldots$, where $a^{l+1} = a^l_{-i_l} a^{l+2}_{i_l}$, with $\ldots, a^l, a^l_{-i_{l+1}} a^{l+2}_{i_{l+1}}, a^{l+2}, \ldots$;
- if $i_l = i_{l+1}$, including $i_{n-1} = i_0$, we use either type (1) or type (2) to reduce the "large" cycle, i.e, replace the cycle fragment $\ldots, a^l, a^{l+1}, a^{l+2}, \ldots$ with $\ldots, a^l, a^{l+2}, \ldots$ in the case of a type (2) cycle or with $\ldots, a^l, \ldots$ in the case of a type (1) cycle.

Consider a process in which one of the above operations is applied until it cannot be applied anymore. Because the operations either reduce the size of the cycle or they re-order problems in an increasing direction, the process never reverts and it will eventually stop. If the process stops at a single-element cycle a, then, because $x(a, a) = 1$, the original cycle must be exact.

Otherwise, the process stops with a non-trivial cycle $a = a^0, \ldots, a^m = a$ for some $2 \leq m \leq n$. Then it must be that $i_l < i_l + 1$ for each $l < m$. But this contradicts the earlier observation that, in an adjacency cycle, if i appears at least once, it must appear at least twice. □

E.4. **Cycles without free transitions.** Here, we consider the exactness of small cycles without free transitions. We refer to i from the definition of type (2) cycles as the *relevant problem* for this cycle; similarly, we refer to i and j as the relevant problems for type (3) cycles. We say that a type (2) or type (3) cycle $\left(a^0, \ldots, a^n = a^0\right)$ is *grounded* if there exists some k such that k is not a relevant problem and $\mathsf{w}^*_k(a^0_k) \neq 0$.

Lemma 14. *Any grounded type (2) or type (3) cycle is exact.*

Proof. Suppose that $\mathsf{w}^*_k(a^0_k) \neq 0$ for some irrelevant problem k. Then $\mathsf{w}_k(a^i) \neq 0$ for each action a^i in the cycle. A repeated application of (E.3) shows that for each $l \leq n$,

$$\mathsf{w}_k(a^l) = \mathsf{w}_k(a)x(a^0, a^1)\cdots x(a^{l-1}, a^l).$$

The result follows from the fact that $a^n = a^0$. □

Lemma 15. *Suppose that for some a and $i \neq j$ and any b_i and b_j, $(a, a_{-i}b_i, a_{-ij}b_ib_j, a_{-j}b_j, a)$ is a type (3) "small" cycle such that either (i) $\mathsf{w}^*_i(a_i) \neq 0$ and $\mathsf{w}^*_i(a_i)$ is not collinear with $\Delta^{b_i}_{a_i}$, or (ii) $\mathsf{w}^*_j(a_j) \neq 0$ and $\mathsf{w}^*_j(a_j)$ is not collinear with $\Delta^{b_j}_{a_j}$. Then the cycle is exact.*

Proof. Using E.2 and E.3, we get

$$\begin{aligned}
x(a_{-i}b_i, a_{-ij}b_ib_j) &\left(x(a, a_{-i}b_i)\mathsf{w}_i(a) + x(a, a_{-i}b_i)\Delta^{b_i}_{a_i}\right) \\
&= x(a_{-i}b_i, a_{-ij}b_ib_j)\mathsf{w}_i(a_{-i}b_i) \\
&= \mathsf{w}_i(a_{-ij}b_ib_j) \\
&= x(a_{-j}b_j, a_{-ij}b_ib_j)\mathsf{w}_i(a_{-j}b_j) + y(a_{-j}b_j, a_{-ij}b_ib_j)\Delta^{b_i}_{a_i} \\
&= x(a_{-j}b_j, a_{-ij}b_ib_j)x(a, a_{-j}b_j)\mathsf{w}_i(a) + y(a_{-j}b_j, a_{-ij}b_ib_j)\Delta^{b_i}_{a_i}.
\end{aligned}$$

Suppose without loss of generality that $\mathsf{w}^*_i(a_i) \neq 0$ and $\mathsf{w}^*_i(a_i)$ is not collinear with $\Delta^{b_i}_{a_i}$, which implies that $\mathsf{w}_i(a) \neq 0$ and $\mathsf{w}_i(a)$ is not collinear with $\Delta^{b_i}_{a_i}$. The first and the last line of the above sequence of equalities yield

$$x(a_{-i}b_i, a_{-ij}b_ib_j)x(a, a_{-i}b_i) = x(a_{-j}b_j, a_{-ij}b_ib_j)x(a, a_{-j}b_j).$$

Hence, the cycle is exact. □

Lemma 16. *Suppose that a type (2) or type (3) "small" cycle is such that $\mathsf{w}^*_i(a_i) \neq 0$ for each action a in the cycle and each relevant problem i. Then it is exact.*

Proof. If the cycle is grounded, the result follows from Lemma 14. Accordingly, suppose henceforth that the cycle is not grounded. Then, because of Lemma 9, and due to the assumption that at least three different problems are nontrivial, there exist non-relevant k and action $b_k \neq a_k$ such that $\mathsf{w}^*_k(b_k) \neq 0$.

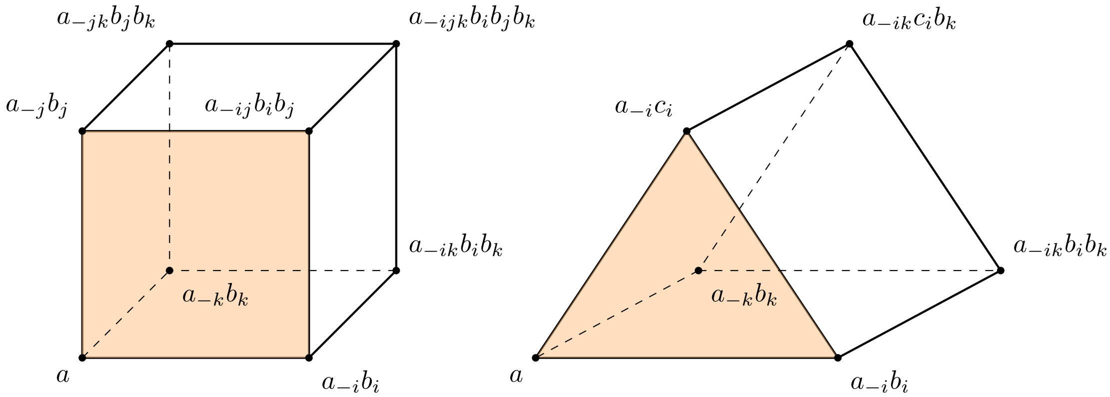


FIGURE E.1. Small cycles of type (3) and type (2)

Suppose that the original cycle $(a, a_{-i}b_i, a_{-ij}b_ib_j, a_{-j}b_j, a)$ has type (3). The cycle corresponds to the orange face on the left-side of Figure E.1. Consider the type (3) cycles that are associated with all five remaining faces of the cube:

- $a, a_{-i}b_i, a_{-ik}b_ib_k, a_{-k}b_k, a$;
- $a_{-i}b_i, a_{-ij}b_ib_j, a_{-ijk}b_ib_jb_k, a_{-ik}b_ib_k, a_{-i}b_i$;
- $a_{-j}b_j, a_{-jk}b_jb_k, a_{-ijk}b_ib_jb_k, a_{-ij}b_ib_j, a_{-j}b_j$;
- $a, a_{-k}b_k, a_{-jk}b_jb_k, a_{-j}b_j, a$;
- $a_{-k}b_k, a_{-ik}b_ib_k, a_{-ijk}b_ib_jb_k, a_{-jk}b_jb_k, a_{-k}b_k$.

The first cycle corresponds to the bottom face, the second to the right face, the third to the top face, the fourth to the left face, and the fifth to the back face. All five cycles are grounded, and hence exact by Lemma 14. Moreover, the conditions of Lemma 12 are satisfied. Therefore, the original cycle is exact as well.

An analogous argument applies to type (2) cycles (see the right panel of Figure E.1). □

Lemma 17. *If a type (2) or (3) cycle has no free transitions, then it is exact.*

Proof. By Lemmas 14 and 16, it is enough assume that the cycle is not grounded and $\mathsf{w}_i^*(a_i) = 0$ for some relevant i and action a in the cycle. It follows that $w_k^*(a_i) \neq 0$ if and only if $k = j$, where j is the other relevant problem of the cycle.

In such a case, if the cycle were of type (2), all transitions to action a would be free.

Suppose the cycle is of type (3). Let b be the action in the cycle such that $a_{-j} = b_{-j}$. Then

$$\bar{\mathsf{X}}(a) = \gamma_j(a)\mathsf{w}_j^*(a_j)$$
$$\text{and} \qquad \bar{\mathsf{X}}(b) = \gamma_j(b)\mathsf{w}_j^*(b_j) = x(a,b)\gamma_j(a)\mathsf{w}_j^*(a_j) + y(a,b)\Delta_{a_j}^{b_j}.$$

Because the transition (a, b) is not free, it must be that $\mathsf{w}_j^*(a_j) \neq 0$ and $\mathsf{w}_j^*(a_j)$ is not collinear with $\Delta_{a_j}^{b_j}$. Lemma 15 therefore implies that the cycle is exact.

□

E.5. **Cycles with free transitions.** Next, we consider cycles with free transitions.

Notice first that, if a transition between adjacent actions a and b such that $a_{-i} = b_{-i}$ is free, then it must be that $\mathsf{w}_j^*(a_j) = 0$ for each $j \neq i$. Indeed, if, for some $\alpha \neq 0$,

$$\bar{\mathsf{X}}(a) = \alpha \bar{\mathsf{X}}(b) = x(a, b)\bar{\mathsf{X}}(b) + y(a, b)\Delta_{a_i}^{b_i},$$

then $\bar{\mathsf{X}}(b)$, and hence $\bar{\mathsf{X}}(a)$, must be collinear with $\Delta_{a_i}^{b_i}$. Because of the linear independence of $\Delta_{a_i}^{b_i}$ and $\mathsf{w}_j(a)$ for each $j \neq i$, it must be that $\mathsf{w}_j(a) = 0$, and hence $\mathsf{w}_j^*(a_j) = 0$. We refer to this property as the *test* for freeness of the transition (which provides necessary conditions). It follows that, if $j \neq i$ is non-trivial, then $a_j = a_j^0$.

In what follows, we consider two cases:

I: There exists a single non-trivial i such that $\mathsf{w}_i^*(a_i) \neq 0$ for all a_i. Assume without loss of generality that $i = I$. In this case, all free transitions must be between adjacent actions a and b such that $a_{-I} = b_{-I}$. Moreover, it must be that $w_I^*(a_I)$ and $w_I^*(b_I)$ are collinear with $\Delta_{a_I}^{b_I}$.

Assume without loss of generality that problem 1 is non-trivial and fix action $b_1 \neq a_1$ so that $\mathsf{w}^*(b_1) \neq 0$. Let

$$x(a, b) = x(a, a_{-1}b_1)x(a_{-1}b_1, a_{-1I}b_1b_I)x(a_{-1I}b_1b_I, b).$$

The above definition implies that the cycle $(a, a_{-1}b_1, a_{-1I}b_1b_I, b, a)$ is exact. This cycle corresponds to the red face(s) in Figure E.5. The orange edge corresponds to the free transition. Notice that each red face cycle contains only one free transition. This is because of the test: all other transitions of the cycle either keep fixed the action in problem I or the action b_1, and those actions are associated with non-zero $\mathsf{w}_\cdot^*(\cdot)$ vectors.

There are three types of cycles that contain free transitions, which are depicted as orange faces in Figure E.5.

The top left panel corresponds to the cycle $(a, a_{-i}b_i, a_{-iI}b_ib_I, a_{-I}b_I, a)$ when problem i is non-trivial. In this case, $b_i \neq a_i^0$, and, by Lemma 9, $\mathsf{w}_i^*(b_i) \neq 0$. The test implies that none of the other transitions in the orange cycle are free: either action b_i or action a_I is fixed. Analogously, an application of the test shows that none of the other cycles (the uncolored faces) is free: one of the actions a_I, b_i, or b_1 is fixed. Proceeding as in the proof of Lemma 16, we see that this cycle is exact.

The top right panel corresponds to a situation when problem i is trivial. In this case, the transition $(a_{-i}b_i, a_{-iI}b_ib_I)$ is free and $x\,(a_{-i}b_i, a_{-iI}b_ib_I)$ can be chosen to make the cycle on the bottom face exact. None of the other transitions are free. Because the red face and the

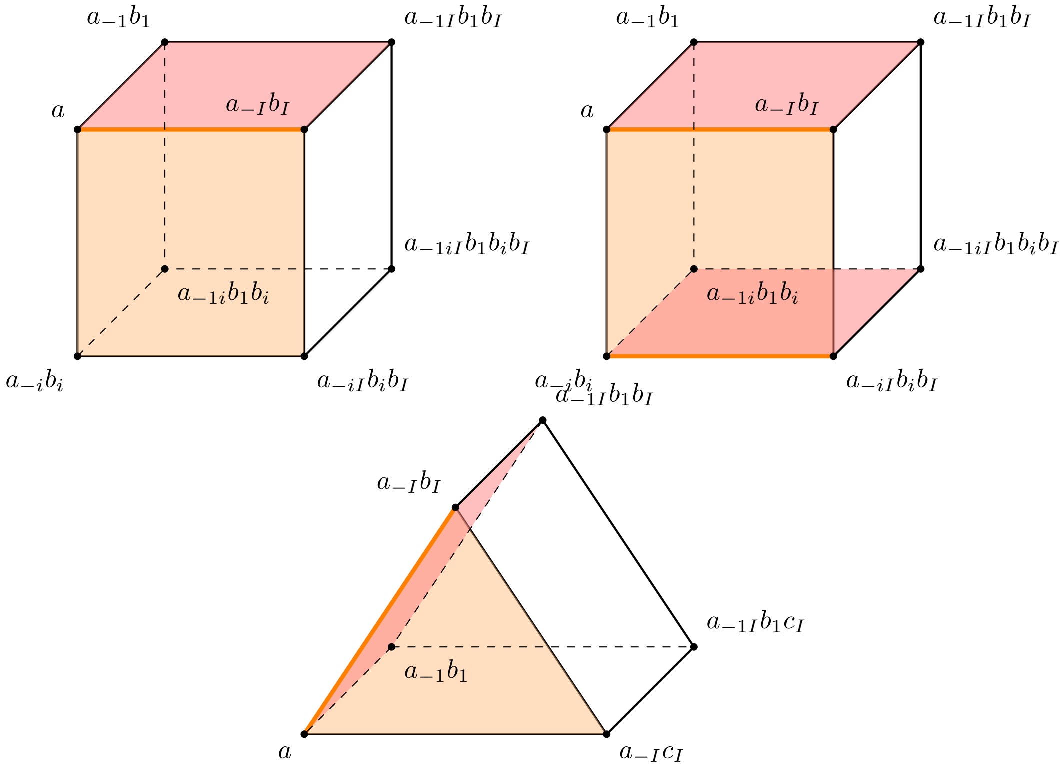


FIGURE E.2. Cycles with free transitions in case I.

uncolored cycles are exact, the above argument implies that the orange face cycle is exact as well.

The bottom panel corresponds to the orange cycle $(a, a_{-I}b_I, a_{-I}c_I, a)$. The other transitions of the orange cycle are not free (otherwise, $w^*(a_I)$ would be collinear with $\Delta_{a_I}^{c_I}$, which would violate linear independence of the latter vector with $\Delta_{a_I}^{b_I}$). All other transitions fix one of the actions: a_I, b_I, c_I, or b_1. Hence, due to the test, none of the remaining transitions are free. The claim follows from the same reasoning as in Lemma 16.

II: For all non-trivial i, there exists a unique a_i^0 such that $\mathsf{w}_i^*(a_i^0) = 0$. Let a^0 be the product problem action that consists of actions a_i^0. Assume without loss of generality that problem $i = 1$ is non-trivial. Fix action $a_1^* \neq a_1^0$.

In this case, a transition is free if and only if it takes the form $(a_{-i}^0 a_i, a^0)$ for some i. Indeed, the above observation implies that if transition (a, b) is free, then $a_{-i} = a_{-i}^0$. Furthermore, if neither $a = a^0$ nor $b = a^0$, then both $w_i^*(a_i)$ and $w_i^*(b_i)$ must be collinear with $\Delta_{a_i}^{b_i}$. But, together with the linear independence assumption, this implies that $w_i^*(a_i)$ is not collinear with $\Delta_{a_i}^{a_i^0}$, which contradicts the fact that $x(a, a^0)\gamma_i(a)w_i^*(a_i) + y(a, a^0)\Delta_{a_i}^{a_i^0} = 0$.

For each $b_1 \neq a_1^*$, each $i \neq 1$, and each b_i, let

$$x(a_{-1}^0 a_1^*, a^0) = 1,$$
$$x(a_{-1}^0 b_1, a^0) = x(a_{-1}^0 b_1, a_{-1}^0 a_1^*),$$
$$\text{and} \qquad x(a_{-i}^0 b_i, a^0) = x(a_{-i}^0 b_i, a_{-1i}^0 a_1^* b_i) x(a_{-1i}^0 a_1^* b_i, a_{-1}^0 a_1^*).$$

By the above definition, the cycles $(a^0, a_{-1}^0 b_1, a_{-1}^0 a_1^*, a^0)$ and $(a^0, a_{-i}^0 b_i, a_{-i1}^0 a_1^* b_i, a_{-1}^0 a_1^*, a^0)$ are exact. These cycles correspond to the red faces in Figure E.5.

There are three types of cycles that contain free transitions in this case other than the cycles listed above.

First, consider a cycle $(a^0, a_{-i}^0 b_i, a_{-ij}^0 b_i b_j, a_{-j}^0 b_j, a^0)$ for $i \neq 1$. This cycle is depicted in orange on the left panel of Figure E.5. The two cycles depicted in red are exact due to the choice of the x coefficients. Finally, all of the other cycles are exact because they do not contain free transitions.

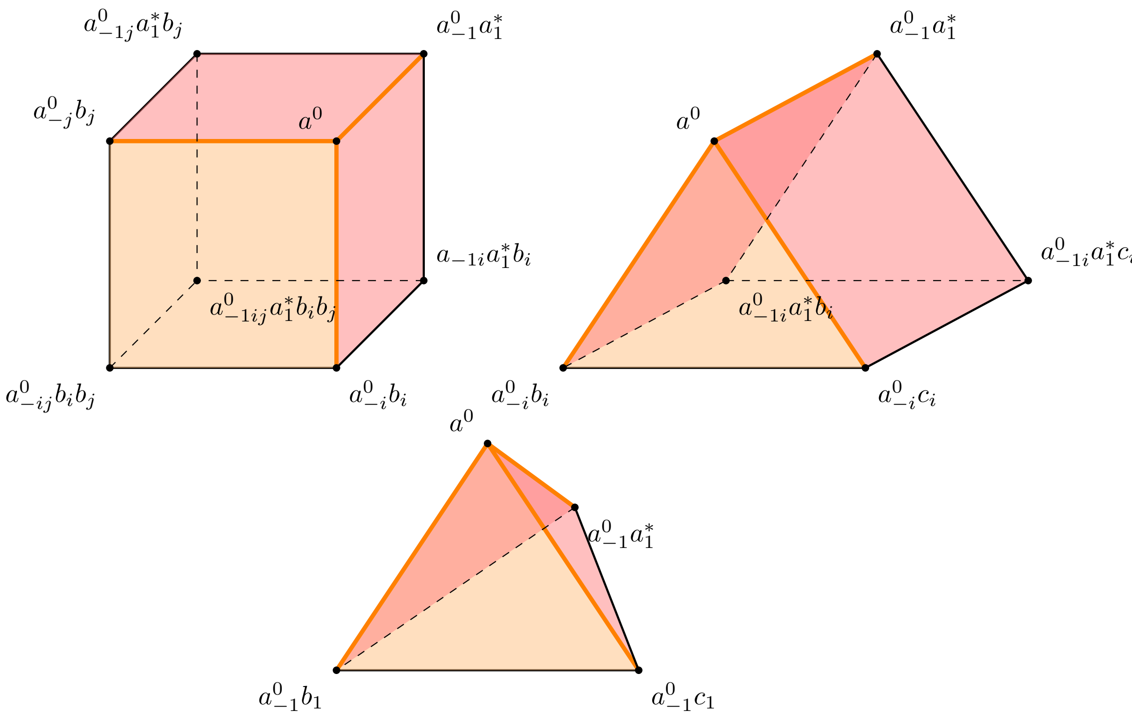


FIGURE E.3. Cycles with free transitions in case II.

Second, consider a cycle $(a^0, a_{-i}^0 b_i, a_{-i}^0 c_i, a^0)$ for $i \neq 1$. This cycle corresponds to front wall depicted in orange on the right panel of Figure E.5. The two cycles depicted in red are exact due to the choice of the x coefficients. All of the other cycles (corresponding to the back and bottom walls) are exact because they do not contain free transitions.

Third, consider a cycle $(a^0, a^0_{-1}b_1, a^0_{-1}c_1, a^0)$. This cycle corresponds to the front wall depicted in orange on the bottom panel of Figure E.5. The two cycles depicted in red (corresponding to the left and right walls) are exact due to the choice of the x coefficients. The remaining cycle (corresponding to the bottom wall) is exact because it does not contain any free transitions.

Proceeding as in the proof of Lemma 16, we see that each of the considered cycles is exact.

Together with Lemma 17, this concludes the proof of Lemma 10.

E.6. **Conclusion of the proof of Theorem 3.** Recall that $\mathsf{w}_i(a) = \gamma_i(a)\mathsf{w}^*_i(a_i)$. The next result delivers additional information about the function $\gamma_i(\cdot)$.

Lemma 18. *There exist $\gamma : A \longrightarrow \mathbb{R}$, $\gamma^*_i : A_i \longrightarrow \mathbb{R}$, and $\gamma^*_0 \in \mathbb{R}$ such that, for each a, $\gamma_i(a) = \gamma(a)\gamma^*_i(a_i)$ for each $i = 1, \ldots, I$, and $\gamma_0(a) = \gamma(a)\gamma^*_0$. In addition, $x(a, b) = \frac{\gamma(a)}{\gamma(b)}$ for any two adjacent actions a and b.*

Proof. Fix an action a^*. For each a, find a path of adjacent actions $a^* = a^0, \ldots, a^m = a$. Define

$$\gamma(a) = x(a^m, a^{m-1}) \cdots x(a^1, a^0).$$

Lemma 10 implies that $\gamma(a)$ is well defined in that its definition does not depend on the choice of path from a^* to a. Moreover, for any two adjacent actions a and b, if $a^* = a^0, \ldots, a^m = a$ is an adjacency path from a^* to a, then $a^* = a^0, \ldots, a^m, b$ is an adjacency path from a^* to b, and

$$\gamma(b) = x(b, a)x(a^m, a^{m-1}) \cdots x(a^1, a^0) = x(b, a)\gamma(a).$$

Let $\gamma^*_i(a_i) = \gamma_i(a^*_{-i}a_i)/\gamma(a^*_{-i}a_i)$. The claim for $i > 0$ follows from the fact that, for each a, if $\mathsf{w}^*_i(a_i) \neq 0$, then

$$\mathsf{w}_i(a) = \frac{\gamma_i(a)}{\gamma_i(a^*_{-i}a_i)}\mathsf{w}_i(a^*_{-i}a_i) = \frac{\gamma(a)}{\gamma(a^*_{-i}a_i)}\mathsf{w}_i(a^*_{-i}a_i).$$

A similar argument establishes the claim for $i = 0$ (see also the proof of Lemma 8). □

Lemma 19. *There exist $y^*_i \in \mathbb{R}$ and $\mathsf{d}_i \in E_i$ such that*

$$\gamma^*_i(a_i)\mathsf{w}^*_i(a_i) = y^*_i\bar{\mathsf{u}}_i(a_i) + \mathsf{d}_i \textit{ for any } a_i.$$

Proof. By (E.2), for any actions a and b such that $a_{-i} = b_{-i}$,

$$\begin{aligned} \gamma_i^*(a_i)\mathsf{w}_i^*(a_i) - \gamma_i^*(b_i)\mathsf{w}_i^*(b_i) &= \frac{1}{\gamma(a)}\mathsf{w}_i(a) - \frac{1}{\gamma(b)}\mathsf{w}_i(b) \\ &= \frac{1}{\gamma(a)}\left(\mathsf{w}_i(a) - x(a,b)\mathsf{w}_i(b)\right) \\ &= \frac{1}{\gamma(a)}y(a,b)\Delta_{b_i}^{a_i}. \end{aligned} \tag{E.5}$$

Because the left-hand side and $\Delta_{b_i}^{a_i}$ do not depend on a_{-i}, neither does $y(a,b)/\gamma(a)$. Let $y^*(a_i, b_i) = y(a,b)/\gamma(a)$.

If problem i has only two actions, it is easy to see that $y_i^*(a_i, b_i) = y_i^*(b_i, a_i) =: y_i^*$. The claim follows.

If problem i has at least three actions, take a, b, and c such that $a_{-i} = b_{-i} = c_{-i}$ and a_i, b_i, and c_i are distinct. Applying the above equation to pairs (a,b), (b,c), and (c,a) yields

$$\begin{aligned} y_i^*(a_i,c_i)\Delta_{b_i}^{a_i} + y_i^*(a_i,c_i)\Delta_{c_i}^{b_i} &= y_i^*(a_i,c_i)\Delta_{c_i}^{a_i} \\ &= \gamma_i^*(a_i)\mathsf{w}_i^*(a_i) - \gamma_i^*(c_i)\mathsf{w}_i^*(c_i) \\ &= \gamma_i^*(a_i)\mathsf{w}_i^*(a_i) - \gamma_i^*(b_i)\mathsf{w}_i^*(b_i) + \gamma_i^*(b_i)\mathsf{w}_i^*(b_i) - \gamma_i^*(c_i)\mathsf{w}_i^*(c_i) \\ &= y_i^*(a_i,b_i)\Delta_{b_i}^{a_i} + y_i^*(b_i,c_i)\Delta_{c_i}^{b_i}. \end{aligned}$$

The independence assumption implies that $y_i^*(a_i, b_i) = y_i^*(b_i, c_i)$. Because the claim holds for arbitrary and distinct actions, there must be y_i^* such that for all a_i and b_i, we have $y_i^*(a_i, b_i) = y_i^*$.

Finally, fix a_i^* and take $\mathsf{d}_i = \gamma_i^*(a_i)\mathsf{w}_i^*(a_i) - y_i^*\bar{\mathsf{u}}_i(a_i^*)$. The claim follows from equation (E.5). □

Substituting the observations from the two lemmas back into (E.1), we obtain

$$\begin{aligned} \bar{X}(a) &= \sum_{i=0}^{n} \mathsf{w}_i(a) \\ &= \gamma(a)\left(\sum_{i=1}^{n} \gamma_i^*(a_i)\mathsf{w}_i^*(a_i) + \gamma_0\mathsf{w}_0^*\right) \\ &= \gamma(a)\left(\sum_{i=1}^{n} y_i^*\bar{\mathsf{u}}_i(a_i) + \left[\sum_{i=1}^{n}\mathsf{d}_i + \gamma_0\mathsf{w}_0^*\right]\right). \end{aligned}$$

Let d be the vector in the square brackets. The result follows.

E.7. **Converse.** We have shown that (7.1) is necessary for incentivizability. All that remains is to show that it is sufficient.

Notice that if $\tau_i > 0$ for all i, the product problem is equivalent (in terms of optimal choices) to a problem with payoffs

$$u(a,\theta) = \sum_i \tau_i u_i(a_i, \theta_i).$$

In this latter problem, any X satisfying (7.1) is aligned with u, and is therefore incentivizable. This in turn implies that X is incentivizable in the original problem.

If $\tau_i \leq 0$ for some i, note that, by Proposition 1, to show that X of the form described in (7.1) is incentivizable, it suffices to show that

$$X(a, \theta) = d(\theta) + \sum_i \tau_i u_i(a_i, \theta_i)$$

is, where we may assume $|\tau_i| < 1$ for all i. Letting

$$\begin{aligned} V(r, a, \theta) &= \int_0^r X(a, \theta) dx + \int_r^1 x dx - \frac{1}{2} + \sum_i u_i(a_i, \theta_i) \\ &= r d(\theta) + \sum_i (1 + r\tau_i) u_i(a_i, \theta_i) - \frac{r^2}{2}, \end{aligned}$$

the optimal choice of a is the same as in the original problem since $1 + r\tau_i > 0$ for all i and r, and the optimal choice of r is $\mathbb{E}_p[X(a, \theta)]$, as needed.

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