Undominated Mechanisms*

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Abstract

We study the design of mechanisms when the designer faces multiple plausible scenarios and is uncertain about the true scenario. A mechanism is dominated by another if the latter performs at least as well in all plausible scenarios and strictly better in at least one. A mechanism is undominated if no other feasible mechanism dominates it. We show how analyzing undominated mechanisms could be useful and illustrate the tractability of characterizing such mechanisms. This approach provides an alternative criterion for mechanism design under non-Bayesian uncertainty, complementing existing methods.

KEYWORDS: robust mechanism design, undominated mechanisms, maxmin approrach, regret minimization, second-price auction, random reserve price

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1 Introduction

Mechanism design has been one of the most successful areas within economic theory, significantly advancing our understanding of incentives under private information. It has led to numerous theoretical and methodological breakthroughs, while also greatly influencing the design and analysis of real-world mechanisms and institutions. However, the classical approach has notable limitations, as it typically imposes strong assumptions about the detailed knowledge of the designer about the economic environment. As a result, theoretical conclusions can be fragile; mechanisms optimized for specific assumptions may perform poorly when those assumptions are not met. The Wilson doctrine suggests that practical mechanisms should be designed without presuming that the designer has precise knowledge of the economic environment.

The literature on robust mechanism design aims to relax the strong assumptions about the designer's knowledge of the environment. This often involves incorporating non-Bayesian uncertainty about certain aspects of the economic setting, in line with the broader goal of relaxing the assumptions about the designer's knowledge. Typically, the designer is modeled as a maxmin decision maker who evaluates mechanisms based on their worst-case performance, where the worst case is taken over all scenarios that are perceived to be plausible. While this framework has made considerable advances, it has also drawn some criticisms. One common critique questions the need to model the designer as a maxmin decision maker. Another concern is that the robust optimality of certain mechanisms arises from extremely pessimistic scenarios within the uncertainty set. After all, in the presence of uncertainty, a mechanism's performance should be viewed as a vector of outcomes, reflecting its effectiveness across different potential scenarios.

To fix ideas and illustrate some of the motivations behind our analysis, consider a situation where the designer could only choose from two mechanisms: Γ_0 and Γ_1. Figure 1 depicts the performance of these mechanisms, where the x-axis represents scenarios that are perceived to be plausible and the y-axis measures the performance of these mechanisms. A maxmin decision maker would identify both Γ_0 and Γ_1 as optimal, since their worst-case performance over all plausible scenarios are the same. However, when examining the outcome vectors across all plausible scenarios, we observe that Γ_1 performs weakly better than Γ_0 in all scenarios, with strictly better performance in at least one. This provides a compelling argument in favor of Γ_1 over Γ_0, a situation we refer to as Γ_1 dominating Γ_0.

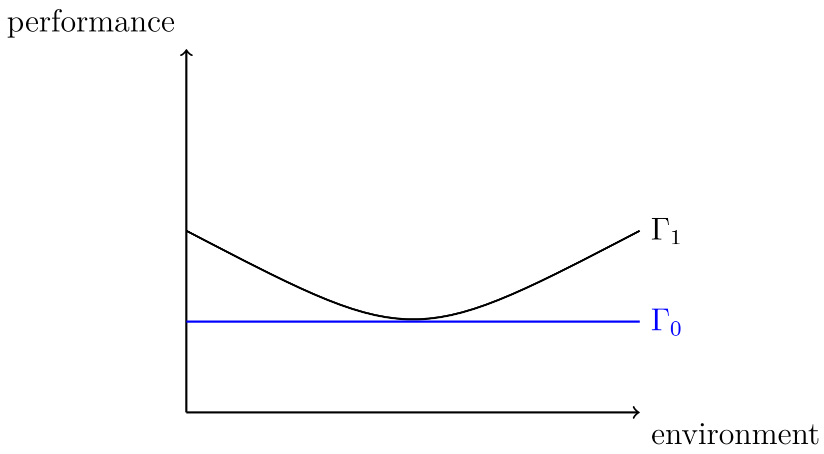

Figure 1: The performance of the two mechanisms across potential scenarios

In this paper, we consider undominated mechanism as an alternative notion in designing mechanisms when the designer has non-Bayesian uncertainty. Say that a mechanism is dominated by another mechanism if the performance of the latter mechanism is weakly better in all plausible scenarios and strictly better in at least one scenario. A mechanism is undominated if there is no other feasible mechanism that dominates it.

Example 1 below revisits the design of mechanisms within the correlation-robust framework (see He and Li (2022)) and demonstrates that the second-price auction with a random reserve price, which is shown to be maxmin optimal in their setting, is dominated.

Example 1 (Maxmin approach)**.** Consider an auctioneer who seeks to sell a single indivisible object to two bidders. Each bidder i holds private information about his value of the object, denoted v_i. The auctioneer knows that $v_i \sim U[0,1]$ for $i = 1, 2$, but has non-Bayesian uncertainty about the correlation structure between v_1 and v_2. Let Π denote the collection of all joint distributions that are consistent with the marginals. The auctioneer ranks mechanisms according to the revenue guarantee, that is, the worst-case expected revenue where the worst case is taken over all $\pi \in \Pi$.

He and Li (2022, Theorem 3) show that a second-price auction with a random reserve price generates the highest revenue guarantee among all standard dominant strategy mechanisms, where the random reserve price follows the distribution $G(r) = \frac{4}{3}r$ with support $[0, \frac{3}{4}]$. Let (q, t) denote this auction, $REV(v)$ the ex post revenue of this auction under the valuation profile v, and $REV(\pi)$ the expected revenue of this auction under the joint distribution π. In what follows, we explicitly construct another dominant strategy mechanism (q^A, t^A) such that

$$REV^A(\pi) \geq REV(\pi),\ \forall \pi \in \Pi \text{ and } REV^A(\tilde{\pi}) > REV(\tilde{\pi}) \text{ for some } \tilde{\pi} \in \Pi,$$

where $REV^A(\pi)$ denotes the expected revenue of the mechanism (q^A, t^A) under the joint distribution π. Also let $REV^A(v)$ denotes the ex post revenue of the mechanism (q^A, t^A) under the valuation profile v.

We construct the mechanism (q^A, t^A) by modifying the mechanism (q, t) for the valuation profiles v where $v_1 = v_2 \in [0, \frac{3}{4}]$. For these profiles, since the two bidders' values are the same, conditional on the object being allocated, the reserve price plays no role in determining the ex post revenue to the auctioneer. Thus, the auctioneer could potentially generate higher revenue by increasing the allocation probability for such profiles. The auctioneer, however, may not be able to arbitrarily increase the allocation probability, as that might distort the agents' incentive to report their values truthfully. The mechanism (q^A, t^A) exploits this observation by increasing the allocation probability for these valuation profiles while preserving the bidders' incentive constraints. Specifically, if both bidders have a value of $x \in [0, \frac{3}{4}]$, let

$$q_i^A(x, x) = \min\{G(x), \frac{1}{2}\} \ \textit{and}\ t_i^A(x, x) = x q_i^A(x, x),\ \forall i = 1, 2,$$

where the term $G(x)$ ensures that the modified mechanism remains a dominant strategy mechanism, and the term $\frac{1}{2}$ guarantees feasibility. For any other v, the allocation rule and the payment rule remain unchanged.

It is straightforward to verify that $q^A(\cdot)$ is ex post monotone and that (q^A, t^A) is a dominant strategy mechanism. Next, we compare the performance of the two mechanisms (q, t) and (q^A, t^A). If both bidders have a value of $x \in (0, \frac{3}{4})$, then

$$REV^A(v) = 2x \min\{G(x), \frac{1}{2}\} > xG(x) = REV(v).$$

Clearly, $REV^A(v) = REV(v)$ for any other v. It follows that $REV^A(\pi) \geq REV(\pi)$ for all correlation structures π consistent with the marginals, and there exist correlation structures $\tilde{\pi}$ consistent with marginals under which $REV^A(\tilde{\pi}) > REV(\tilde{\pi})$. Thus, (q^A, t^A) dominates (q, t) in the correlation-robust framework.

Example 1 highlights how the undominated mechanism approach could be useful in guiding the design of mechanisms. While certain mechanisms may be optimal according to a specific criterion, they could still be dominated. Though we have focused on the maxmin approach so far, this idea applies equally to other frameworks, such as the regret minimization approach. For instance, Zhang (2022a) examines an auctioneer who knows only the upper bound of each bidder's value without further distributional information. The author shows that a regret-minimizing auctioneer would select a second-price auction with a random reserve price. A similar analysis as in Example 1 also shows that this regret-minimizing auction is dominated. Naturally, if a mechanism is optimal according to a specific criterion but is dominated, the dominating mechanism must also be optimal under that criterion. Thus, the concept of undominated mechanisms can help a designer filter out some undesirable mechanisms when multiple mechanisms are optimal and thus cannot be distinguished under other approaches.

From a practical standpoint, being dominated is a compelling reason to reject a mechanism. By definition, if a mechanism is dominated, there exists some mechanism that performs weakly better in every plausible scenario and strictly better in at least one scenario. This makes it easy to provide a straightforward and convincing argument for avoiding dominated mechanisms, without needing to rely on extreme or highly unlikely scenarios. This offers a robust justification for not using such mechanisms, reinforcing the practicality and reliability of the undominated mechanism approach.

Evidently, the undominated mechanism approach could be adopted in various design settings and under various assumptions about the designer's knowledge about the environment. As a proof of concept, in this paper, we focus on the screening environment and the single-unit auction environments, assuming that the designer has minimal information about the environment. This is arguably the most standard design environment. While the assumed knowledge of the designer is rather extreme, this is a natural starting point. Given the information assumption, the definition of dominance relation between two mechanisms reduces to a dominance relation where the performance meausre is the ex post revenue to the auctioneer under each value profile.

Section 2 characterizes the set of undominated mechanisms in the screening environment. Some preliminary observations indicate that for a mechanism to be undominated, it must be that the lowest type has no rent, the allocation rule must be weakly increasing and right-continuous in the buyer's value, and there is full efficiency at the top. These observations imply that for a mechanism to be undominated, it must be a random posted-price mechanism. A more involved analysis show that the converse is also true: any random posted-price mechanism is undominated.

Section 3 studies undominated mechanisms in the single-unit auction setting. The analysis is more subtle in this environment given the fact that a higher ex post revenue to the auctioneer does not neessary mean that the payment from each bidder is higher. We focus on the class of second-price auctions with deterministic or random reserve prices in this environment. Theorem 2 shows that a deterministic reserve price is undominated, providing a undominance foundation for its use if it were optimal under other criterion. Rather surprisingly, Theorem 3 shows that any (non-degenerate) random reserve price is dominated. And we explicitly construct a mechanism that dominates it. This construction generalizes the logic in Example 1 to an arbitrary random reserve price. This is in sharp contrast with the screening environment, when there is a single buyer. Furthermore, Theorem 4 shows that the superior mechanism that we construct is itself undominated. Thus, we provide a concrete recommendation when the second-price auction with a random reserve price is optimal under other criterion.

Section 4 introduces an alternative notion of dominance which we call strong dominance. Loosely speaking, strong dominance is a more demanding notion as it requires the superior mechanism to generate a strictly better preformance over a set of scenarios with positive Lebesgue measure (while leading to no worse outcome for all scenarios). We highlight the contrasting effects of applying each notion, using the screening environment as a case study. Section 4 also discusses how variations in the designer's knowledge of the environment affect the set of undominated mechanisms.

Our findings imply that the set of undominated mechanisms is rather large. Section 5 concludes the paper with reflections on this issue and provides some directions for future research.

1.1 Related literature

This paper contributes to the theory of robust mechanism design that addresses the design of mechanisms when the designer has non-Bayesian uncertainty about the design environment; see Carroll (2019) for a recent survey. While typical papers establish the robust optimility of certain mechanisms using either the maxmin approach or the minimax approach, the approach of undominated mechanisms differs in that it directly compares the vector outcomes of the mechanisms without imposing any assumptions on the decision-making procedure of the designer.

In environments in which the designer has an estimate of the distribution of the agents' payoff types but does not have any reliable information about the agents' beliefs, Chung and Ely (2007) and Chen and Li (2018) show that under certain condition, the use of dominant strategy mechanisms has a maxmin foundation. Börgers (2017) shows that in these environments, all dominant strategy mechanisms are actually dominated, exploiting the possibility of side bets when the agents' beliefs are not derived from common priors. Li and Dworczak (2021) study dominance relations among mechanisms when the designer faces agents that are unsophisticated. Curello and Sinander (2024) study undominated and optimal mechanisms in a dynamic agency problem of incentivizing prompt disclosure of productive information. Mishra and Patil (2024) study undominated mechanisms for regulating a monopolist who privately observes the marginal cost of production.

Manelli and Vincent (2007) consider a multiple-good monopoly pricing problem. The authors show that if a mechanism is undominated, then the mechanism is Bayesian optimal for some distribution over values of the buyer.

Viewed as a theory of dominance among choices, our paper is related to the strict or weak dominance among strategies in games; see for example Bernheim (1984) and Pearce (1984). Cheng and Börgers (2023) provide a general theory of dominance among choices that includes strict and weak dominance among strategies in games, Blackwell dominance among experiments, and first or second-order stochastic dominance among lotteries as special cases.

2 Screening

In this section, we study undominated mechanisms in the screening environment. Section 2.1 presents the model, and Section 2.2 provides a complete characterization of undominated mechanisms in this environment.

2.1 Model

A seller (she) seeks to sell a single indivisible object to a potential buyer (he). The buyer has a value v, his willingness to pay for the object, belonging to a closed interval $V = [\underline{v}, 1]$ where $\underline{v} > 0$. The value v is private information to the buyer. The seller has only minimal information about the buyer's value of the object—the only information she has is that $v \in V$.

A mechanism consists of a set M of messages for the buyer, an allocation rule $q : M \to [0, 1]$, and a transfer rule $t : M \to \mathbb{R}$. The buyer will select a message from the set M, and based on the message m, the buyer gets the object with probability $q(m)$ and makes a transfer $t(m)$ to the seller. By the revelation principle, it is without loss to focus on direct mechanisms.

Definition 1. *A direct mechanism (q, t) is incentive compatible (IC) if*

$$vq(v) - t(v) \geq vq(v') - t(v') \text{ for all } v, v' \in V,$$

and is individually rational (IR) if

$$vq(v) - t(v) \geq 0 \text{ for all } v \in V.$$

Lemma 1 below is a standard result that characterizes incentive compatibility and individual rationality for the screening environment; see for example Börgers (2015, Section 2.2).

Lemma 1. *A direct mechanism (q, t) is IC if and only if*

- *$q(v)$ is weakly increasing in v, and*
- *for all $v \in V$, we have*

$$t(v) = vq(v) - \int_{\underline{v}}^{v} q(x)\, dx - \Big(\underline{v}q(\underline{v}) - t(\underline{v})\Big).$$

An IC direct mechanism is IR if and only if $\underline{v}q(\underline{v}) - t(\underline{v}) \geq 0$.

We focus on mechanisms that are undominated in the following sense.

Definition 2. *An IC and IR mechanism (q, t) is dominated by another IC and IR mechanism (q', t') if*

(1) $t'(v) \geq t(v)$ for all $v \in V$, and

(2) there exists some $\tilde{v} \in V$ such that $t'(\tilde{v}) > t(\tilde{v})$.

An IC and IR mechanism is undominated if there is no IC and IR mechanism that dominates it.

2.2 Undominated mechanisms

In this section, we fully characterize the class of undominated mechanisms in the screening environment. We first present several straightforward necessary conditions for a mechanism to be undominated.

First, consider any IC and IR mechanism (q, t) where the lowest possible type has positive rent, that is, $\underline{v}q(\underline{v}) - t(\underline{v}) > 0$. We construct a new mechanism (q', t') where for all $v \in V$,

$$q'(v) = q(v) \text{ and } t'(v) = t(v) + \big(\underline{v}q(\underline{v}) - t(\underline{v})\big) > t(v).$$

Obviously, the mechanism (q', t') is IC, IR, and dominates (q, t).

Observation 1. *If an IC and IR mechanism (q, t) is undominated, then*

$$\underline{v}q(\underline{v}) - t(\underline{v}) = 0.$$

Second, consider any IC and IR mechanism (q, t) where the allocation rule q is not right-continuous at some $\hat{v} \in V$, that is, $q(\hat{v}) < q(\hat{v}+)$.[1] We consider another mechanism (q', t') where

$$\begin{aligned}
q'(\hat{v}) &= q(\hat{v}+) > q(\hat{v}), \\
q'(v) &= q(v), \forall v \neq \hat{v}, \\
t'(v) &= vq'(v) - \int_{\underline{v}}^{v} q'(x)\, dx - \big(\underline{v}q(\underline{v}) - t(\underline{v})\big), \forall v \in V.
\end{aligned}$$

Here, we only modify the allocation probability for type $\hat{v}$ and keep the allocation probabilities for the other types unchanged. Clearly, $q'(v)$ is weakly increasing in v, and the mechanism (q', t') is IC and IR. Since $q'(\hat{v}) > q(\hat{v})$ and $q'(v) = q(v)$ for any

[1] $q(\hat{v}+) = \lim_{v \to \hat{v}^+} q(v)$ is the right-hand limit of $q(v)$ at $\hat{v}$.

$v \neq \hat{v}$, we have

$$
\begin{aligned}
t'(\hat{v}) &= \hat{v}q'(\hat{v}) - \int_{\underline{v}}^{\hat{v}} q'(x)\,dx - \big(\underline{v}q(\underline{v}) - t(\underline{v})\big) \\
&> \hat{v}q(\hat{v}) - \int_{\underline{v}}^{\hat{v}} q(x)\,dx - \big(\underline{v}q(\underline{v}) - t(\underline{v})\big) \\
&= t(\hat{v}).
\end{aligned}
$$

It is easy to verify that $t'(v) = t(v)$ for any $v \neq \hat{v}$. Therefore, (q, t) is dominated.

Observation 2. *If an IC and IR mechanism (q, t) is undominated, then the allocation rule $q(v)$ must be right-continuous.*

Third, consider any IC and IR mechanism (q, t) where the allocation is not efficient at the top, that is, $q(1) < 1$. We consider another mechanism (q', t') where

$$
\begin{aligned}
&q'(1) = 1 > q(1), \\
&q'(v) = q(v),\ \forall v \neq 1, \\
&t'(1) = t(1) + (1 - q(1)) > t(1), \\
&t'(v) = t(v),\ \forall v \neq 1.
\end{aligned}
$$

Here, we only modify the allocation probability for the highest possible type 1 and keep the allocation probabilities for the other types unchanged. It is straightforward to verify that (q', t') is IC, IR, and dominates (q, t).

Observation 3. *If an IC and IR mechanism (q, t) is undominated, then $q(1) = 1$.*

We summarize the above observations in the following lemma.

Lemma 2. *If an IC and IR mechanism (q, t) is undominated, then we must have*

- $\underline{v}q(\underline{v}) - t(\underline{v}) = 0$,
- *$q(v)$ is weakly increasing and right-continuous, and*
- $q(1) = 1$.

Say that a mechanism is a random posted-price mechanism if for some distribution $G \in \Delta[\underline{v}, 1]$ we have

$$
q(v) = G(v) \text{ and } t(v) = \int_{\underline{v}}^{v} p\,dG(p),\ \forall v \in V.
$$

In words, the seller posts a (random) price p that follows the distribution G and the buyer buys the object if and only if the buyer's value is weakly higher than the posted price. Consider any mechanism (q, t) that satisifies the conditions in Lemma 2. Since $\underline{v}q(\underline{v}) - t(\underline{v}) = 0$, $q(\cdot)$ is weakly increasing and right-continuous, and $q(1) = 1$, the mechanism (q, t) is the random posted-price mechanism where the posted price follows the distribution q. Lemma 2 states that if an IC and IR mechanism is undominated, then it must be a random posted-price mechanism. Thus, in our quest to characterize undominated mechanisms, it is without loss of generality to focus on random posted-price mechanisms.

Before we present our main result for the screening environment, we consider some special cases of random posted-price mechanisms, where the arguments are particularly easy to follow. We first consider the class of posted-price mechanisms with a deterministic price. The allocation rule and the payment scheme for such a mechanism with a deterministic price p are given as follows:

$$q(v) = \begin{cases} 0 & \text{if } v \in [\underline{v}, p), \\ 1 & \text{if } v \in [p, 1]. \end{cases} \quad \text{and} \quad t(v) = \begin{cases} 0 & \text{if } v \in [\underline{v}, p), \\ p & \text{if } v \in [p, 1]. \end{cases}$$

Proposition 1. *Any posted-price mechnaism with a deterministic price $p \in [\underline{v}, 1]$ is undominated.*

Proof. Fix any $p \in [\underline{v}, 1]$, and let (q, t) denote the posted-price mechanism with price p. Consider any IC and IR mechanism (q', t') such that $t'(v) \geq t(v)$ for all $v \in V$. In what follows, we show that $t'(v) = t(v)$ for all $v \in V$. This then implies that (q, t) is undominated.

Case 1. We first consider the case in which $v = p$.

Suppose that $t'(p) > t(p)$. Clearly, $t'(p) > t(p) = p$. Note that the seller is extracting more than the buyer's value of the object in the mechanism (q', t'). This is not possible without violating the IR constraint of type p. We have a contradiction. Thus, we must have $t'(p) = t(p) = p$. Furthermore, since in the mechanism (q', t') the seller is extracting the full surplus from type p, it must be that $q'(p) = 1$.

Case 2. Next, we consider the case in which $v > p$.

We claim that in the mechanism (q', t') the seller can never extract more than p from the buyer of any type. This is because, following our analysis in Case 1 above, the buyer of any type can always mimick type p, paying only p to get the object with probability 1. Thus, for any $v > p$,

$$t'(v) \leq p = t(v) \leq t'(v) \implies t'(v) = t(v).$$

Case 3. Last, we consider the case in which $v < p$.

Suppose that there exists some $\tilde{v} \in [\underline{v}, p)$ such that $t'(\tilde{v}) > t(\tilde{v})$. Clearly, $t'(\tilde{v}) > t(\tilde{v}) = 0$. It follows from the IR constraint of type $\tilde{v}$ that $q'(\tilde{v}) > 0$. It follows from the IC constraint of type p (mimicking type $\tilde{v}$) and the IR constraint of type $\tilde{v}$ that

$$pq'(p) - t'(p) \geq pq'(\tilde{v}) - t'(\tilde{v}) = (p - \tilde{v})q'(\tilde{v}) + \tilde{v}q'(\tilde{v}) - t'(\tilde{v}) > 0.$$

In words, type p has a strictly positive rent in the mechanism (q', t') because type p can always mimic the lower type $\tilde{v}$ that gets the good with positive probability. We arrive at a contradiction. This is because, as argued in Case 1 above, the seller extracts full surplus from type p and does not leave any rent for type p in the mechanism (q', t'). We conclude that $t'(v) = t(v) = 0$ for all $v < p$. $\square$

Next, we consider any random posted-price mechanism where $G \in \Delta V$ has finite support.

Proposition 2. *Any random posted-price mechanism where $G \in \Delta V$ has finite support is undominated.*

Proof. Fix any $G \in \Delta V$ with finite support, and denote the support of G by $\{p_i\}_{i=1}^K$, where $\underline{v} \leq p_1 < p_2 < \cdots < p_K \leq 1$. Denote the probability of the seller posting the price p_k by λ_k, $k = 1, 2, \ldots, K$. The allocation rule and the payment scheme for this posted-price mechanism are given as follows:

$$q^G(v) = \begin{cases} 0 & \text{if } v \in [\underline{v}, p_1), \\ \sum_{i=1}^k \lambda_i & \text{if } v \in [p_k, p_{k+1}),\ \forall k = 1, 2, \ldots, K-1, \\ 1 & \text{if } v \in [p_K, 1]. \end{cases}$$

$$t^G(v) = \begin{cases} 0 & \text{if } v \in [\underline{v}, p_1), \\ \sum_{i=1}^k \lambda_i p_i & \text{if } v \in [p_k, p_{k+1}),\ \forall k = 1, 2, \ldots, K-1, \\ \sum_{i=1}^K \lambda_i p_i & \text{if } v \in [p_K, 1]. \end{cases}$$

Consider any IC and IR mechanism (q, t) that satisfies $t(v) \geq t^G(v)$ for all $v \in V$. Step 1 shows that $q(v) \geq q^G(v)$ for all $v \in V$. This uses an induction argument. Step 2 shows that $q(v) = q^G(v)$ for all $v \in V$, which further implies that $t(v) = t^G(v)$ for all $v \in V$. Thus, there does not exist a type $\tilde{v}$ such that $t(\tilde{v}) > t^G(\tilde{v})$, and (q^G, t^G) is undominated.

Step 1. We show that $q(v) \geq q^G(v)$, $\forall v \in V$.

For any $v \in [\underline{v}, p_1)$, since $q^G(v) = 0$, we immediately have $q(v) \geq q^G(v)$.

For $v = p_1$, since $p_1 q(p_1) \geq t(p_1) \geq t^G(p_1) = \lambda_1 p_1$, we have $q(p_1) \geq \lambda_1$. Since $q(v)$ is weakly increasing and $q^G(v)$ is constant on $[p_1, p_2)$, we have $q(v) \geq q^G(v)$ for any $v \in [p_1, p_2)$.

Suppose that $q(v) \geq q^G(v)$ for all $v < p_k$ for some $k = 2, 3, \ldots, K-1$, we show that $q(v) \geq q^G(v)$ for all $v < p_{k+1}$. For $v = p_k$, we can show that both the payment and the buyer's rent in the mechanism (q, t) are weakly higher than those in the mechanism (q^G, t^G), which further implies that the allocation probability for type p_k must also be weakly higher in the mechanism (q, t). Formally, $t(p_k) \geq t^G(p_k)$, and

$$\begin{aligned} p_k q(p_k) - t(p_k) &= \Big(\underline{v} q(\underline{v}) - t(\underline{v})\Big) + \int_{\underline{v}}^{p_k} q(x)\, dx \\ &\geq \int_{\underline{v}}^{p_k} q(x)\, dx \\ &\geq \int_{\underline{v}}^{p_k} q^G(x)\, dx \\ &= p_k q^G(p_k) - t^G(p_k). \end{aligned}$$

It follows that

$$p_k q(p_k) \geq p_k q^G(p_k) \implies q(p_k) \geq q^G(p_k).$$

Since $q(v)$ is weakly increasing and $q^G(v)$ is constant on $[p_k, p_{k+1})$, we have $q(v) \geq q^G(v)$ for any $v < p_{k+1}$.

The induction analysis shows that $q(v) \geq q^G(v)$ for any $v < p_K$. Using similar arguments, one can further show that $q(p_K) \geq q^G(p_K)$ and $q(v) \geq q^G(v)$ for any $v \in V$.

Step 2. We show that $q(v) = q^G(v)$, $\forall v \in V$.

Suppose this is not true. Since $q(v) \geq q^G(v)$ for all $v \in V$, there exists some $v^* \in V$ such that $q(v^*) > q^G(v^*)$. Clearly, $v^* < p_K$. Let k^* be the smallest $k \in \{1, 2, \ldots, K\}$ such that $v^* < p_{k^*}$. Since $q(v)$ is weakly increasing and $q^G(v)$ is constant on $[v^*, p_{k^*})$, we have $q(v) > q^G(v)$ for any $v \in [v^*, p_{k^*})$. Thus,

$$\begin{aligned} p_K q(p_K) - t(p_K) &= \Big(\underline{v} q(\underline{v}) - t(\underline{v})\Big) + \int_{\underline{v}}^{p_K} q(x)\,dx \\ &\geq \int_{\underline{v}}^{p_K} q(x)\,dx \\ &> \int_{\underline{v}}^{p_K} q^G(x)\,dx \\ &= p_K q^G(p_K) - t^G(p_K). \end{aligned}$$

Since $t(p_K) \geq t^G(p_K)$, we have

$$p_K q(p_K) > p_K q^G(p_K) \implies q(p_K) > q^G(p_K) = 1.$$

We arrive at a contradiction. Thus, $q(v) = q^G(v)$ for all $v \in V$.

Since $q(v) = q^G(v)$ and $t(v) \geq t^G(v)$ for all $v \in V$, we must have $\underline{v} q(\underline{v}) - t(\underline{v}) = 0$ and $t(v) = t^G(v)$ for all $v \in V$. We conclude that the mechanism (q^G, t^G) is undominated. □

Theorem 1 below provides a complete characterization of undominated mechanisms. The proof is conceptually similar to that of Proposition 2, but involves some continuity arguments to handle all random posted-price mechanisms. The proof can be found in Appendix A.

Theorem 1. *An IC and IR mechanism is undominated if and only if it is a random posted-price mechanism.*

Remark 1. Proposition 1 shows that any posted-price mechanism with a deterministic price is undominated. Thus, a random posted-price mechanism is a randomization over undominated mechanisms. The readers might wonder whether the set of undominated mechanisms is convex. If this property holds, then the characterization of undominated mechanisms could be seen as a straightforward corollary of Proposition 1. Since Theorem 1 gives a complete characterization of undominated mechanisms, this result confirms that this is the case in the screening environment.[2] However, this property does not hold more generally. In the single-unit auction environment, we show that while the second-price auction with any deterministic reserve price is undominated, the second-price auction with any (non-degenerate) random reserve price is dominated.

[2]It is not clear to us how to show the convexity of the set of undominated mechanisms in the screening environment without first characterizing the set of undominated mechanisms.

Remark 2. In contract to Bayesian environments in which it is without loss of optimility to focus on posted-price mechanisms with a deterministic price, we show that random posted-price mechanisms are undominated when the seller has minimal information about the screening environment. This highlights the hedging benefits of randomization. With a single price p, the seller has a revenue of zero when the buyer's value is less than p. If the seller were to simply lower the deterministic price to some $p' < p$, she would miss the chance of extracting revenue from higher valuation types. A randomized price provides hedging benefits against these various scenarios.

3 Single-unit auction

In this section, we study undominated mechanisms in the single-unit auction setting. Here, we focus on the class of second-price auctions, an important class of auction formats that are both theoretically appealing and widely adopted in practice. Section 3.1 introduces the auction environment, Section 3.2 studies second-price auctions with a deterministic reserve price, and Section 3.3 studies second-price auctions with a (non-degenerate) random reserve price.

3.1 Model

A seller seeks to sell a single indivisible object. There are $n \geq 2$ bidders competing for the object. We denote by $N = \{1, 2, \ldots, n\}$ the set of bidders and i a generic bidder. Each bidder i holds private information about his valuation of the object, denoted v_i, belonging to a closed interval $V_i = [\underline{v}, 1]$ where $\underline{v} > 0$. The set of value profiles is $V = \times_{i \in N} V_i$ with a generic element v. We write v_{-i} for a value profile of bidder i's opponents, that is, $v_{-i} \in V_{-i} = \times_{j \neq i} V_j$. The only information the seller has is that $v_i \in V_i$ for all $i \in I$.

We focus on dominant strategy mechanisms. The revelation principle applies, and we restrict attention to direct mechanisms. A direct mechanism (q, t) consists of an allocation rule $q : V \to [0, 1]^n$ and a payment rule $t : V \to \mathbb{R}^n$. Each bidder will report a value v_i, and based on the resulting profile of reports v, bidder i receives the object with probability $q_i(v)$ and makes a transfer $t_i(v)$ to the seller. Let $REV(v) = \sum_{i \in N} t_i(v)$ for all $v \in V$.

Definition 3. *A direct mechanism* (q, t) *is dominant strategy incentive compatible (DIC) if* $\forall i \in N$, $\forall v \in V$, $\forall v_i' \in V_i$,

$$v_i q_i(v) - t_i(v) \geq v_i q(v_i', v_{-i}) - t(v_i', v_{-i}),$$

and is dominant strategy individually rational (DIR) if $\forall i \in N$, $\forall v \in V$,

$$v_i q_i(v) - t_i(v) \geq 0.$$

Lemma 3 below is a standard result that characterizes dominant strategy incentive compatibility and dominant strategy individual rationality for the single-unit auction environment; see for example Börgers (2015, Section 4.2).

Lemma 3. *A direct mechanism* (q, t) *is DIC if and only if* $\forall i \in N$, $\forall v_{-i} \in V_{-i}$,

- $q_i(v_i, v_{-i})$ *is weakly increasing in* v_i, *and*
- *for all* $v_i \in V_i$, *we have*

$$t_i(v_i, v_{-i}) = v_i q(v_i, v_{-i}) - \int_{\underline{v}}^{v_i} q_i(x, v_{-i}) dx - \Big(\underline{v} q_i(\underline{v}, v_{-i}) - t_i(\underline{v}, v_{-i})\Big).$$

A DIC direct mechanism is DIR if and only if $\forall i \in N$, $\forall v_{-i} \in V_{-i}$,

$$\underline{v} q_i(\underline{v}, v_{-i}) - t_i(\underline{v}, v_{-i}) \geq 0.$$

An important class of dominant strategy mechanisms is the class of second-price auctions with a deterministic or random reserve price. We present a formal definition here. In the second-price auction with reserve price r, each bidder i submits a bid $m_i \in \mathbb{R}_+$. Conditional on the submitted bids $m = (m_1, m_2, \ldots, m_n)$, bidder i's probability of winning the object $q_i(m)$ and the payment from bidder i to the auctioneer $t_i(m)$ are given as follows:

$$q_i(m) = \begin{cases} \frac{1}{|W(m)|} & \text{if } i \in W(m) \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad t_i(m) = \begin{cases} \frac{\max(m(2), r)}{|W(m)|} & \text{if } i \in W(m) \\ 0 & \text{otherwise} \end{cases}$$

where $W(m) = \{i \in N : m_i = m(1),\, m_i \geq r\}$.[3]

[3] For any set S, let $|S|$ denote its cardinality. For any real-valued vector $x \in \mathbb{R}^n$, we write $x(k)$ for the k-th largest element of the vector.

We are interested in the seller's revenue in the dominant strategy equilibrium in which each bidder submits a bid that is equal to her value of the object. For the second-price auction with reserve price r, let

$$REV(r, v) = \begin{cases} 0 & \text{if } v(1) < r; \\ r & \text{if } v(2) < r \leq v(1); \\ v(2) & \text{if } v(2) \geq r. \end{cases}$$

That is, we use $REV(r, v)$ to denote the seller's ex post revenue by using the deterministic reserve price r when the realized value profile is v. Let $\mathcal{G}$ denote the set of all cumulative distribution functions on the $[\underline{v}, 1]$ interval. For any random reserve price $G \in \mathcal{G}$, let

$$REV(G, v) = \int_{\underline{v}}^{1} REV(r, v)\, dG(r)$$

denote the seller's ex post revenue by using the random reserve price G when the realized value profile is v.

We study mechanisms that are undominated in the following sense.

Definition 4. *A DIC and DIR mechanism (q, t) is dominated by another DIC and DIR mechanism (q', t') if*

(1) $REV'(v) \geq REV(v)$ for all $v \in V$, and

(2) there exists some $\tilde{v} \in V$ such that $REV'(\tilde{v}) > REV(\tilde{v})$.

A DIC and DIR mechanism is undominated if there is no DIC and DIR mechanism that dominates it.

3.2 Second-price auctions with a deterministic reserve price

Theorem 2 below shows that any second-price auction with a deterministic reserve price is undominated. This result includes the second-price auction with no reserve price as a special case, since we allow the deterministic reserve price to be set at $\underline{v}$.

Theorem 2. *Any second-price auction with a deterministic reserve price $r \in [\underline{v}, 1]$ is undominated.*

Proof. Fix any $r \in [\underline{v}, 1]$. Let (q, t) denote the second-price auction with the reserve price r and $REV(v)$ the seller's ex post revenue at the value profile v. Fix any DIC and DIR mechanism (q', t') such that $REV'(v) \geq REV(v)$ for all $v \in V$. In what follows, we show that $REV'(v) = REV(v)$ for all $v \in V$. This then implies that (q, t) is undominated.

Case 1. Consider any $v \in V$ such that $v(1) = r$ or $v(1) = v(2) > r$.

For any such $v \in V$, the seller already extracts the entire surplus using the second-price auction with the reserve price r. Thus, the seller could not obtain a strictly higher ex post revenue for any such value profile without violating the DIR constraint of some bidder. Thus, we can conclude that $REV'(v) = REV(v)$ and the seller also extract the full surplus in the mechanism (q', t') for any $v \in V$ such that $v(1) = r$ or $v(1) = v(2) > r$.

Case 2. Consider any $v \in V$ such that $v(1) < r$.

Suppose that for some $v \in V$ such that $v(1) < r$, we have $REV'(v) > REV(v)$. Clearly, $REV'(v) > REV(v) = 0$. There necessarily exists some bidder $i \in N$ such that $t'_i(v) > 0$. It follows from the DIR constraint of bidder i that $q'_i(v) > 0$. Consider the following value profile where we increase the value of bidder i from v_i to r while keeping other bidders' value unchanged:

$$\hat{v} = (v_1, v_2, \ldots, v_{i-1}, r, v_{i+1}, \ldots, v_n).$$

Note that $\hat{v}$ constructed above is a type considered in Case 1. It follows from the DIC and DIR constraints of bidder i that

$$rq'_i(\hat{v}) - t'_i(\hat{v}) \geq rq'_i(v) - t'_i(v) = (r - v_i)q'_i(v) + v_i q'_i(v) - t'_i(v) > 0.$$

In words, bidder i has positive rent at the value profile $\hat{v}$. But this is impossible since we have already concluded from Case 1 above that the seller must extract the full surplus at the value profile $\hat{v}$ in the mechanism (q', t'). Thus, it must be that for any $v \in V$ such that $v(1) < r$, we have $REV'(v) = REV(v) = 0$.

Case 3. Finally, we consider any $v \in V$ with $v(1) > r$ and $v(1) > v(2)$.

Suppose that for some $v \in V$ such that $v(1) > r$ and $v(1) > v(2)$, we have $REV'(v) > REV(v)$. Let $z = \max \left\{ v(2), r \right\}$. Clearly, $REV'(v) > REV(v) = z$. Let i^1 denote the bidder with the highest value $v(1)$. Since $v(1) > v(2)$, i^1 is unique. Let i^2 denote one of the bidders with the second highest value $v(2)$.

In the mechanism (q,t), we have

$$q_{i^1}(v) = 1,\ REV(v) = t_{i^1}(v) = z.$$

For sufficiently small $\epsilon > 0$, consider the following value profile $\tilde{v}$ where

$$\tilde{v}_{i^1} = z + \epsilon,$$
$$\tilde{v}_i = v_i,\ \forall i \neq i^1.$$

In words, we decrease the value of bidder i^1 from v_{i^1} to $z+\epsilon$ without changing the values of the other bidders. By construction, bidder i^1 still has the highest value at the value profile $\tilde{v}$. Thus, in the original mechanism (q,t), we have $REV(\tilde{v}) = z$. It follows that $REV'(\tilde{v}) \geq REV(\tilde{v}) = z$.

We claim that, in the mechanism (q',t'), bidders other than bidder i^1 cannot get the good with any positive probability at the value profile $\tilde{v}$. Suppose to the contrary, for some bidder $j \neq i^1$, we have $q'_j(\tilde{v}) > 0$. We consider another value profile $\tilde{\tilde{v}}$ where we increase the value of bidder j from $\tilde{v}_j$ to $z+\epsilon$ while leaving the other bidders' values unchanged:

$$\tilde{\tilde{v}}_j = z + \epsilon,$$
$$\tilde{\tilde{v}}_i = \tilde{v}_i,\ \forall i \neq j.$$

Note that v and $\tilde{v}$ only differ in bidder i^1's value, and $\tilde{v}$ and $\tilde{\tilde{v}}$ only differ in bidder j's value. Furthermore, $\tilde{\tilde{v}}$ is a type considered in Case 1. Since $q'_j(\tilde{v}) > 0$, we can use the DIC and DIR constraints of bidder j to show that type $\tilde{\tilde{v}}_j$ has positive rent at the value profile $\tilde{\tilde{v}}$. But this contradicts with our finding in Case 1 that in the mechanism (q',t') the seller must extract the full surplus and leave no rent to the bidders at the value profile $\tilde{\tilde{v}}$. Thus, in the mechanism (q',t'), bidders other than bidder i^1 cannot get the good with any positive probability at the value profile $\tilde{v}$.

It follows that

$$z = REV(\tilde{v}) \leq REV'(\tilde{v}) = \sum_{i \in N} t'_i(\tilde{v}) \leq \sum_{i \in N} \tilde{v}_i q'_i(\tilde{v}) = (z+\epsilon) q'_{i^1}(\tilde{v}),$$

which further implies that

$$q'_{i^1}(\tilde{v}) \geq \frac{z}{z+\epsilon}.$$

Thus, we have

$$\begin{aligned} t'_{i^1}(v) &\leq v_{i^1} q'_{i^1}(v) - v_{i^1} q'_{i^1}(\tilde{v}) + t'_{i^1}(\tilde{v}) \\ &\leq v_{i^1} - v_{i^1} q'_{i^1}(\tilde{v}) + t'_{i^1}(\tilde{v}) \\ &\leq v_{i^1} - v_{i^1} \frac{z}{z+\epsilon} + (z+\epsilon) \\ &= v_{i^1} \frac{\epsilon}{z+\epsilon} + (z+\epsilon) \\ &\to z \end{aligned}$$

as $\epsilon \to 0$, where the first line follows from the DIC constraint of bidder i^1, the second line is due to the feasibility constraint $q'_{i^1}(v) \leq 1$, the third line follows from the lower bound on $q'_{i^1}(\hat{v})$ and the DIR constraint of bidder i^1. Thus, we have that $t'_{i^1}(v) \leq z$.

Using similar arguments as above, we can show that in the mechanism (q', t'), bidders other than bidder i^1 cannot get the good with any positive probability at the value profile v. Thus, we have

$$z = REV(v) \leq REV'(v) = \sum_{i \in N} t'_i(v) \leq t'_{i^1}(v) \leq z,$$

which implies that $REV'(v) = REV(v)$. □

3.3 Second-price auction with a random reserve price

Theorem 2 shows that any second-price auction with a deterministic reserve price is undominated. One natural conjecture is that if we randomize between two mechanisms that are undominated, the resulting mechanism remains undominated.[4] However, this property does not hold here. As we have seen in Example 1, a second-price auction with a random reserve price, while being maxmin optimal in the correlation-robust auction design framework, is dominated. In what follows, we generalize Example 1 and show that any second-price auction with a (non-degenerate) random reserve price is dominated, and we explicitly construct a superior mechanism that dominates it.

[4]This is true in the case of the screening environment; see Theorem 1

Theorem 3. *Any second-price auction with a (non-degenerate) random reserve price $G \in \mathcal{G}$ is dominated.*

Fix any (non-degenerate) random reserve price $G \in \mathcal{G}$. Let (q, t) denote the second-price auction with the random reserve price G. Let

$$\underline{r} = \min \operatorname{supp}(G), \quad \bar{r} = \max \operatorname{supp}(G).$$

Clearly, $G(v) = 1$ if and only if $v \geq \bar{r}$.

We explicitly construct a mechanism (q^A, t^A) by modifying the mechanism (q, t) for value profiles v with $v(1) = v(2) \in [\underline{r}, \bar{r})$, and proceed to show that (q^A, t^A) dominates (q, t). The intuition for modifying the allocation rule for these value profiles is as follows. For any v with $v(1) = v(2)$, conditional on the object being allocated, the reserve price plays no role in determining the ex post revenue to the seller. Thus, the seller could potentially generate higher revenue by increasing the allocation probability for such v. The seller, however, may not be able to arbitrarily increase the allocation probability, as that might distort the bidders' incentives to report their values truthfully. The mechanism (q^A, t^A) exploits this observation by increasing the allocation probability for these valuation profiles while preserving the bidders' incentive constraints. For any v, let

$$M(v) = \{i \in N \mid v_i = v(1)\}$$

denote the collection of bidders who have the highest value at the value profile v. For any v with $v(1) = v(2) \in [\underline{r}, \bar{r})$, let

$$q_i^A(v) = \begin{cases} \min \left\{G(v(1)), \frac{1}{|M(v)|}\right\} & \forall i \in M(v); \\ 0 & \forall i \notin M(v), \end{cases}$$

and let

$$t_i^A(v) = v_i \, q_i^A(v), \ \forall i \in N.$$

In words, (q^A, t^A) is identical to (q, t) except for $v \in V$ with $v(1) = v(2) \in [\underline{r}, \bar{r})$. For these value profiles, if $|M(v)|G(v(1)) \leq 1$, we increase the allocation probability for each bidder $i \in M(v)$ to $G(v(1))$. As such, for each $i \in M(v)$, $q_i^A(v_i, v_{-i})$ is right-continuous at the point $v_i = v(1)$, that is,

$$q_i^A(v(1), v_{-i}) = q_i(v(1)+, v_{-i}) = G(v(1)).$$

If we increase the allocation probability any further, the allocation would no longer be ex post monotone. When $|M(v)|G(v(1)) > 1$, due to the feasibility constraint, we could not increase the allocation probability for each bidder $i \in M$ to $G(v(1))$. Thus, $q_i^A(v) = \frac{1}{|M(v)|}$ for all $i \in M(v)$ in this case. The allocation rule is modified so that we increase the allocation probability as much as possible while satisfying the monotonicity constraint and the feasibility constraint. We say that (q^A, t^A) is an augmented mechanism of (q, t).

Proof of Theorem 3. Fix any (non-degenerate) random reserve price $G \in \mathcal{G}$. Let (q, t) denote the second-price auction with the random reserve price G. In what follows, we show that (q^A, t^A), the argumented mechanism of (q, t), dominates (q, t). For any $v \in V$ with $v(1) = v(2) \in [\underline{r}, \bar{r})$, we have

$$\begin{aligned}
REV^A(v) &= \sum_{i \in N} t_i^A(v) \\
&= \sum_{i \in N} v_i\, q_i^A(v) \\
&= \sum_{i \in M(v)} v_i\, q_i^A(v) \\
&= \sum_{i \in M(v)} v(1) \min \left\{ G(v(1)), \frac{1}{|M(v)|} \right\} \\
&= \min \left\{ |M(v)| \cdot v(1) G(v(1)), v(1) \right\} \\
&\geq v(1) G(v(1)) \\
&= REV(v),
\end{aligned}$$

where the inequality holds because $|M(v)| \geq 2$ and $G(v(1)) < 1$ for any v with $v(1) = v(2) \in [\underline{r}, \bar{r})$. Furthermore, the inequality is strict if $G(v(1)) > 0$. This trivially holds for any v with $v(1) = v(2) \in (\underline{r}, \bar{r})$.

Clearly, $REV^A(v) = REV(v)$ for any other v. Therefore, we conclude that (q^A, t^A) dominates (q, t). $\square$

Remark 3. The class of second-price auctions with (non-degenerate) random reserve prices is an important class of auction mechanisms, particularly in the literature of robust mechanism design. Among many others, He and Li (2022), Zhang (2022a), Zhang (2022b), and Che (2022) establish the optimality of second-price auctions with random reserve prices across various design environments. These papers model a seller who has various degrees of uncertainty about the environment and take either the maxmin approach or the minimax regret approach. Theorem 3 highlights the usefulness of the analysis of undominated mechanisms. Perhaps surprisingly, the identified optimal mechanisms according to the various criteria in these papers are, in fact, dominated. Our analysis suggests that the argmented mechanisms we propose dominate the previously identified optimal mechanisms. Therefore, they are also optimal in these studies. In the next section, we further show that the argumented mechanims that we construct are undominated.

3.4 Augmented mechanisms are undominated

Section 3.3 shows that for any second-price auction with a (non-degenerate) random reserve price, the corresponding augmented mechanism dominates it. The question remains: is the augmented mechanism itself dominated? This section addresses this question.

Theorem 4. *For any second-price auction with a (non-degenerate) random reserve price that has finite support, the corresponding augmented mechanism dominates it and is itself undominated.*

We illustrate the main idea of the proof with the following example where there are only two bidders and the reserve price only takes two possible values. The proof of Theorem 4 can be found in Appendix B.

Example 2. Suppose that there are two bidders. Let (q, t) denote the second-price auction with a random reserve price, which takes the value $\underline{r}$ with probability p and $\bar{r}$ with probability $1 - p$, for some fixed $p \in (0, 1)$. Formally, the reserve price is distribution according to the following distribution:

$$G(r) = \begin{cases} 0 & \text{if } r < \underline{r}; \\ p & \text{if } \underline{r} \leq r < \bar{r}; \\ 1 & \text{if } r \geq \bar{r}. \end{cases}$$

Let (q^A, t^A) be the corresponding augmented mechanism. Formally, (q^A, t^A) differs from (q, t) only for value profiles v where $v_1 = v_2 \in [\underline{r}, \bar{r})$. For any such v,

$$q_i^A(v) = \min\left\{p, \frac{1}{2}\right\} \text{ and } t_i^A(v) = v_i\, q_i^A(v),\ \forall i = 1, 2.$$

In what follows, we show that (q^A, t^A) is undominated.

Fix any DIC and DIR mechanism (q', t') such that $REV'(v) \geq REV^A(v)$ for all $v \in V$. We show that $REV'(v) = REV^A(v)$ for all $v \in V$. This then implies that (q^A, t^A) is undominated.

Case 1. Consider any v where $v(2) \geq \bar{r}$.

First consider any v with $v_1 = v_2 \geq \bar{r}$. For any such $v \in V$, the seller already extracts the entire surplus in the mechanism (q^A, t^A). Thus, the seller could not obtain a strictly higher ex post revenue for any such value profile without violating the DIR constraint of some bidder. Thus, we can conclude that $REV'(v) = REV^A(v)$ and the seller also extract the full surplus in the mechanism (q', t') for any $v \in V$ such that $v_1 = v_2 \geq \bar{r}$. In particular, at the value profile $(\bar{r}, \bar{r})$, both bidders have zero rent. This further implies that for any value profile where one bidder's value is $\bar{r}$ and the other bidder's value is less than $\bar{r}$, the bidder whose value is lower does not get the object with any positive probability. Using similar arguments as in the proof of Theorem 2, one can show that $REV'(v) = REV^A(v)$ for any v where $v(1) > v(2) \geq \bar{r}$.

Case 2. Consider any v where $v_1 = v_2 \in [\underline{r}, \bar{r})$.

For any such v, we have

$$v_1 q_1^A(v) + v_2 q_2^A(v) = REV^A(v) \leq REV'(v) \leq v_1 q_1'(v) + v_2 q_2'(v), \tag{1}$$

where the last inequality follows from the DIR constraints. Since $v_1 = v_2$, we have

$$q_1'(v) + q_2'(v) \geq q_1^A(v) + q_2^A(v). \tag{2}$$

We shall show that (2) must hold with equality. This, together with (1), shows that $REV^A(v) = REV'(v)$.

If $p \geq \frac{1}{2}$, then by construction, $q_1^A(v) + q_2^A(v) = 1$. Clearly, in this case, (2) holds with equality. If $p < \frac{1}{2}$, then by construction, $q_1^A(v) = q_2^A(v) = p$. Suppose to the contrary, (2) holds with strict inequality. Then, there necessarily exists some bidder i with $q_i'(v) > p$.

Consider the following profile $\hat{v}$:

$$\hat{v}_i = \bar{r} \geq \hat{v}_j = v_j \in [\underline{r}, \bar{r}).$$

From our analysis in Case 1, we know that $q'_j(\hat{v}) = 0$. Thus, $t'_j(\hat{v}) \leq 0$. It follows that

$$t'_i(\hat{v}) \geq REV'(\hat{v}) \geq REV^A(\hat{v}) = (1-p)\bar{r} + p\hat{v}_j. \tag{3}$$

It follows from the DIC and DIR constraints of bidder i that

$$\bar{r}q'_i(\hat{v}) - t'_i(\hat{v}) \geq \bar{r}q'_i(v) - t'_i(v) \geq (\bar{r} - v_i)q'_i(v) > (\bar{r} - v_i)p. \tag{4}$$

From (3) and (4), we obtain that

$$\bar{r}q'_i(\hat{v}) > \bar{r} \implies q'_i(\hat{v}) > 1.$$

We arrive at a contradiction. Therefore, if $p < \frac{1}{2}$, (2) also holds with equality. We have

$$v_1 q_1^A(v) + v_2 q_2^A(v) = REV^A(v) = REV'(v) = v_1 q'_1(v) + v_2 q'_2(v).$$

This further implies that both bidders have zero rent at v.

Case 3. Consider any v where $v(2) < \underline{r}$.

It follows from Case 2 that at the value profile $(\underline{r}, \underline{r})$, both bidders have zero rent. This implies that for any value profile v where one bidder's value is $\underline{r}$ and the other bidder's value is less than $\underline{r}$, the bidder whose value is lower does not get the object with any positive probability. Without loss of generality, suppose that bidder 2's value v_2 is less than $\underline{r}$. Consider the following two value profiles:

$$w^1 = (\underline{r}, v_2), \quad w^2 = (\bar{r}, v_2).$$

At w^1, we have $q'_2(w^1) = 0$ and

$$p\underline{r} = REV^A(w^1) \leq REV'(w^1) \leq t'_1(w^1) \leq \underline{r}q'_1(w^1) \implies q'_1(w^1) \geq p. \tag{5}$$

At w^2, we have $q'_2(w^2) = 0$ and

$$(1-p)\bar{r} + p\underline{r} = REV^A(w^2) \leq REV'(w^2) \leq t'_1(w^2). \tag{6}$$

Furthermore, for bidder 1 we have

$$\bar{r}q_1'(w^2) - t_1'(w^2) = \underline{r}q_1'(w^1) - t_1'(w^1) + \int_{\underline{r}}^{\bar{r}} q_1'(x, v_2)\ \mathrm{d}x \geq \int_{\underline{r}}^{\bar{r}} q_1'(x, v_2)\ \mathrm{d}x. \tag{7}$$

It follows from (5), (6), and (7) that

$$\begin{aligned}
\bar{r} &\geq \bar{r}q_1'(w^2) \\
&= \left(\bar{r}q_1'(w^2) - t_1'(w^2)\right) + t_1'(w^2) \\
&\geq \int_{\underline{r}}^{\bar{r}} q_1'(x, v_2)\ \mathrm{d}x + (1-p)\bar{r} + p\underline{r} \\
&\geq (\bar{r} - \underline{r})q_1'(w^1) + (1-p)\bar{r} + p\underline{r} \\
&\geq (\bar{r} - \underline{r})p + (1-p)\bar{r} + p\underline{r} \\
&= \bar{r}.
\end{aligned}$$

Therefore, the above inequalities must all hold with equality, and we have $q_1'(v_1, v_2) = p$ if $v_1 \in [\underline{r}, \bar{r})$, and $q_1'(\bar{r}, v_2) = 1$. Since $q_1'(v_1, v_2)$ is weakly increasing in v_1, we further have $q_1'(v_1, v_2) = 1$ if $v_1 \in [\bar{r}, 1]$.

Next, we show that $q_1'(v_1, v_2) = 0$ if $v_1 \in [\underline{v}, \underline{r}]$. Suppose not, there exists some $v_1 < \underline{r}$ such that $q_1'(v_1, v_2) > 0$. It follows from the DIC and DIR constraints of bidder 1 that $q_1'(w^1) > p$, contradicting with our finding in the previous paragraph. Thus $q_1'(v_1, v_2) = 0$ if $v_1 \in [\underline{v}, \underline{r}]$.

Since for any $v_2 < \underline{r}$, $q_1'(v_1, v_2) = q_1^A(v_1, v_2)$ for all v_1, we have $REV'(v) = REV^A(v)$ for any v where $v(2) < \underline{r}$.

Case 4. Consider any v with $v(1) > v(2) \in [\underline{r}, \bar{r})$.

Without loss of generality, we consider the case in which $v_1 > v_2 \in [\underline{r}, \bar{r})$. Consider the value profile $w = (w_1, w_2) = (v_2 + \epsilon, v_2)$ for some fixed $v_2 \in [\underline{r}, \bar{r})$, where $\epsilon > 0$ is sufficiently small. Since both bidders have zero rent at $(v_2 + \epsilon, v_2 + \epsilon)$, a value profile considered in Case 2, we have $q_2'(w) = 0$. Thus,

$$p\, v_2 = REV^A(w) \leq REV'(w) \leq t_1'(w) \leq (v_2 + \epsilon)q_1'(w),$$

which implies that

$$q_1'(w) \geq p\frac{v_2}{v_2+\epsilon} > 0.$$

Consider the profile $w' = (\bar{r}, v_2)$. Since both bidders have zero rent at $(\bar{r}, \bar{r})$, a value profile considered in Case 1, we have $q_2'(w') = 0$. Thus,

$$t_1'(w') \geq REV'(w') \geq REV^A(w') = p\, v_2 + (1-p)\,\bar{r}.$$

Furthermore, for bidder 1 we have

$$\bar{r}q_1'(w') - t_1'(w') = (v_2+\epsilon)q_1'(w) - t_1'(w) + \int_{v_2+\epsilon}^{\bar{r}} q_1'(x, v_2)\ \mathrm{d}x \geq \int_{v_2+\epsilon}^{\bar{r}} q_1'(x, v_2)\ \mathrm{d}x.$$

It follws from these inequalities that

$$\begin{aligned}
\bar{r} &\geq \bar{r}q_1'(w') \\
&= \Big(\bar{r}q_1'(w') - t_1'(w')\Big) + t_1'(w') \\
&\geq \int_{v_2+\epsilon}^{\bar{r}} q_1'(x, v_2)\ \mathrm{d}x + p\, v_2 + (1-p)\,\bar{r} \\
&\geq \Big(\bar{r} - (v_2+\epsilon)\Big)q_1'(w) + p\, v_2 + (1-p)\,\bar{r} \\
&\geq \Big(\bar{r} - (v_2+\epsilon)\Big)\frac{v_2}{v_2+\epsilon}p + p\, v_2 + (1-p)\,\bar{r} \\
&\to r,
\end{aligned}$$

as $\epsilon \to 0$. Therefore, the above inequalities must all hold with equalities, and we have $q_1'(v_1, v_2) = p$ if $v_1 \in [\underline{r}, \bar{r})$, and $q_1'(\bar{r}, v_2) = 1$. Since $q_1'(v_1, v_2)$ is weakly increasing in v_1, we further have $q_1'(v_1, v_2) = 1$ if $v_1 \in [\bar{r}, 1]$. Therefore, we have $REV'(v) = REV^A(v)$ for any v with $v(1) > v(2) \in [\underline{r}, \bar{r})$.

The above analysis show that $REV'(v) = REV^A(v)$ for all v. Therefore, (q^A, t^A) is undominated.

4 Discussion

Section 4.1 introduces an alternative notion of dominance and highlights the contrasting effects of applying each notion, using the screening environment as a case study. Section

4.2 discusses how variations in the designer's knowledge of the environment affect the set of undominated mechanisms.

4.1 Strong dominance

The notion of dominance we consider so far only requires that there exist at least one scenario under which the superior mechanism achieves strict improvement (while leading to no worse outcome for all scenarios). In this section, we briefly consider an alternative notion of dominance which requires that the superior mechanism strictly improve over a set of scenarios with positive Lebesgue measure. In this sense, this alternative notion of dominance is more demanding. We illustrate the distinction between the notions of dominance and strong dominance in the screening environment.

Definition 5. *An IC and IR mechanism (q, t) is strongly dominated by another IC and IR mechanism (q', t') if $t'(v) \geq t(v)$ for all $v \in V$, with strict inequality in a set of positive Lebesgue measure. An IC and IR mechanism is not strongly dominated if there is no IC and IR mechanism that strongly dominates it.*

At this point, the readers might wonder which of the two notions is preferable. Here, we clarify the distinctions between the two notions and discuss how the seller's information affects the choice of a more suitable notion. First, by definition, f a mechanism (q, t) is strongly dominated by another mechanism (q', t'), then it is dominated by (q', t'). Thus, the set of mechanisms that are not strongly dominated is a superset of the set of undominated mechanisms. Second, the choice between the two notions depends on the designer's knowledge about the environment. When the seller has only minimal information about the environment and only knows the bounds on the buyer's value, we favor the original notion of dominance. This is because, to justify using strong dominance, additional assumptions about the seller's knowledge of the environment are necessary—specifically, that the seller views any given scenario as having zero probability of being the true scenario. That said, the concept of strong dominance holds interest in its own right. One notable advantage of strong dominance is that it allows us to use duality arguments to construct mechanisms that are not strongly dominated, as we shall see shortly.

Lemma 4 presents the necessary conditions for a mechanism to be not strongly dominated.

Lemma 4. *If a mechanism (q, t) is not strongly dominated, then we must have*

- $\underline{v}q(\underline{v}) - t(\underline{v}) = 0$, *and*
- $q(1) = 1$.

Proof. The same arguments for the case of dominance implies that if a mechanism (q, t) is not strongly dominated, then $\underline{v}q(\underline{v}) - t(\underline{v}) = 0$; see Observation 1. In what follows, we show that if a mechanism (q, t) is not strongly dominated, then $q(1) = 1$.

Suppose to the contrary, $q(1) < 1$. We explicitly construct another mechanism (q', t') that strongly dominates (q, t). Let

$$\begin{aligned}
&q'(v) = q(v), \forall v \in [\underline{v}, 1 - \epsilon), \\
&q'(v) = q(v) + (1 - q(1)), \forall v \in [1 - \epsilon, 1], \\
&t'(v) = vq'(v) - \int_{\underline{v}}^{v} q'(x)dx, \forall v \in V
\end{aligned}$$

for some sufficiently small $\epsilon > 0$. In words, we shift upward the allocation probability on the interval $[1 - \epsilon, 1]$ and adjust the payments accordingly. Clearly, the mechanism (q', t') is IC and IR. Furthermore, it is easy to verify that $t'(v) = t(v)$ for all $v \in [\underline{v}, 1 - \epsilon)$ and

$$\begin{aligned}
t'(v) - t(v) &= v(q'(v) - q(v)) - \int_{\underline{v}}^{v} (q'(x) - q(x))dx \\
&= v(q'(v) - q(v)) - \int_{1-\epsilon}^{v} (q'(x) - q(x))dx \\
&= v(1 - q(1)) - \Big(v - (1 - \epsilon)\Big)(1 - q(1)) \\
&= (1 - \epsilon)(1 - q(1)) \\
&> 0,
\end{aligned}$$

for all $v \in [1 - \epsilon, 1]$. Thus, we can conclude that (q', t') strongly dominates (q, t). $\square$

We say that an IC and IR mechanism (q, t) is rationalizable with respect to some distribution $F \in \Delta(V)$ if the mechanism (q, t) generates the highest expected revenue with respect to the distribution F. In particular, we consider the following distribution that rationalizes a wide array of mechanisms:

$$F^*(v) = \begin{cases} 1 - \frac{\underline{v}}{v} & \text{if } v \in [\underline{v}, 1); \\ 1 & \text{if } v = 1. \end{cases}$$

Note that the distribution F^* rationalizes any mechanism (q, t) such that $\underline{v}q(\underline{v}) - t(\underline{v}) = 0$ and $q(1) = 1$.

Lemma 5. *If a mechanism (q, t) is rationalizable with respect to F^*, then (q, t) is not strongly dominated.*

Proof. Fix any mechanism (q, t) that is rationalizable with respect to F^*. Suppose to the contrary, (q, t) is strongly dominated by another mechanism (q', t'). By definition, we have $t'(v) \geq t(v)$ for all $v \in V$, with strict inequality in a set of positive Lebesgue measure. It follows that, again the distribution F^*, the expected revenue generated by the mechanism (q', t') is strictly higher than that by the mechanism (q, t). Thus, F^* cannot rationalize the use of the mechanism (q, t). We arrive at a contradiction. □

Theorem 1. *A mechanism (q, t) is not strongly dominated if and only if it satisfies $\underline{v}q(\underline{v}) - t(\underline{v}) = 0$ and $q(1) = 1$.*

Theorem 1 follows from Lemma 4 and Lemma 5.

4.2 The designer's knowledge

Our analysis suggests that the set of undominated mechanisms is rather large. There are two compounding factors driving this. First, the notion of dominance itself directly compares vector outcomes corresponding to the mechanisms without making any assumptions about the designer's decision-making procedure. This comparison leads to a partial order over mechanisms. Thus, to a certain extent, it is not surprising to find a wide array of undominated mechanisms. Second, the minimal information assumed in the two applications we consider amplifies this effect. In particular, we assume that the seller has minimal knowledge about the environment, knowing only the bounds of valuations but has no additional distributional information. Our results show that, the set of undominated mechanisms turns out to be large under this environment.

To understand the role of the designer's information, consider the other extreme case in which the designer has a well-defined Bayesian model. In this case, each mechanism's vector outcome simplifies to a scalar—the expected performance of the mechanism with respect to the Bayesian environment—allowing for a total ordering among mechanisms and typically resulting in a unique undominated mechanism.

In practice, the designer's knowledge is likely somewhere between these extremes. While the designer may not fully rely on a specific Bayesian model, the designer is likely to possess some information about the design environment. The literature of robust mechanism design has explored various instances of this possibility, modeling a designer who knows, for example, marginal distributions, moment conditions, or aspects of the payoff environment but not agents' beliefs. A reasonable conjecture is that more precise knowledge of the environment would enable the designer to eliminate more dominated mechanisms. We illustrate this point in Example 3, as a contrast to the minimal information assumption findings in Section 2.

Example 3. *A seller seeks to sell a single indivisible object to one risk-neutral buyer. The buyer has a private value $v \in [0,1]$ for the good. The seller knows that the buyer's value v follows some distribution from the following set of distributions:*

$$F_n(v) = v^n, n = 1, 2,$$

but has non-Bayesian uncertainty about the true distribution. In this setting, each mechanism is associated with two possible levels of expected revenue, $(p(1-p), p(1-p^2))$, one for each possible distribution. It is straightforward to check that any price $p < \frac{1}{2}$ is dominated by the price $\frac{1}{2}$ and any price $p > \frac{\sqrt{3}}{3}$ is dominated by the price $\frac{\sqrt{3}}{3}$.

5 Conclusion

This paper is devoted to the analysis of undominated mechanisms. While understanding the set of undominated mechanisms requires a distinct analytical approach from typical optimization problems (such as Bayesian optimization, maxmin, or minimax), we show that the analysis is still largely tractable.

The notion of dominance establishes only a partial ordering over mechanisms, as it relies on directly comparing their vector outcomes. As such, it should not come as a surprise that the set of undominated mechanisms is large, especially when dealing with a large uncertainty set. Consequently, in many cases, the undominated mechanism analysis might not always yield sharp recommendations for the designer. Indeed, our findings reveal that many mechanisms are undominated. Below, we offer some reflections on this approach and suggest some directions for future research.

First, as discussed in Section 4.2, the large set of undominated mechanisms is partly driven by the notion of dominance itself and by the large uncertain set we assume in this paper. In practice, the designer's knowledge likely falls between the extremes of minimal information and specific Bayesian models. Indeed, the literature of robust mechanism design has already explored various design environments with designers who have partial knowledge of the environment, often using maxmin or minimax regret approach. The analysis of undominated mechanisms in environments where the designer has additional information beyond the minimal information considered in the paper is a particularly fruitful direction for further research.

Second, even when the designer has minimal information about the environment, as studied in the current paper, we can already derive (rather surprising) results about specific mechanisms. For instance, a large number of papers advocate the use of second-price auctions with random reserve prices due to their desirable performance even in the adversarial scenario. It is hitherto unknown that this well-known auction format is actually dominated by the corresponding augmented mechanism. Being dominated is a compelling reason to reject a mechanism, involving a straightforward and convincing argument for avoiding dominated mechanisms, without the need to rely on extreme or highly unlikely scenarios. This offers a robust justification for not using such mechanisms, reinforcing the practicality and reliability of the undominated mechanism approach.

Third, analyzing undominated mechanisms allows us to avoid justifying the robust optimality of certain mechanisms based on specific, isolated scenarios, a common criticism raised on the maxmin or minimax regret approach. To make meaningful progress along this dimension, it seems necessary to examine the performance of any given mechanism across all plausible scenarios, which necessitates the comparison of vector outcomes.

Fourth, the analysis of undominated mechanisms provides a comprehensive understanding of a mechanism's performance across various scenarios and facilitates a deeper understanding of the trade-off across the performance of a mechanism against different scenarios. This aspect of mechanisms can go unnoticed when one takes the Bayesian approach, the maxmin appraoch, or the minimax regret approach, as typically these approaches involve the analysis against a particular type of scenario. For example, while a random posted-price mechanism is not the unique Bayesian optimal mechanism with respect to some distribution, its performance across different values provides hedging benefit when the seller has minimal information about the environment and is hence undominated.

Last, the set of undominated mechanisms can be a useful first step in a two-stage decision-making process. In the first stage, we identify the set of undominated mechanisms, ensuring that focusing on this set does not entail any loss of generality for the designer. In the second stage, this set can be further refined based on the designer's specific decision criteria, allowing for more concrete recommendations. In this way, the undominated mechanism approach complements other methods in mechanism design when there is uncertainty in the design environment.

Overall, the undominated mechanism approach enriches the toolkit for mechanism design under uncertainty, and complements existing methodologies to broaden the scope of mechanism design.

A Proof of Theorem 1

The only if direction follows immediately from Lemma 2. In what follows, we prove the if direction. Fix any random posted-price mechanism (q^G, t^G). Suppose to the contrary, there exists some IC and IR mechanism (q, t) that dominates (q^G, t^G). If $q(v)$ is not right-continuous or $\underline{v}q(\underline{v}) - t(\underline{v}) > 0$, then there exists another mechanism (q', t') such that

- $q'(v)$ is right-continuous and $\underline{v}q'(\underline{v}) - t'(\underline{v}) = 0$, and
- (q', t') dominates (q, t) and thus dominates (q^G, t^G).

Therefore, without loss of generality, we assume that for the mechanism (q, t) that dominates (q^G, t^G), $q(v)$ is right-continuous and $\underline{v}q(\underline{v}) - t(\underline{v}) = 0$.

The proof is conceptually similar to that of Proposition 2, but involves some continuity arguments to handle all random posted-price mechanisms. Step 1 shows that $q(v) \geq q^G(v)$ for all $v \in V$, and Step 2 shows that $q(v) = q^G(v)$ for all $v \in V$, which further implies that $t(v) = t^G(v)$ for all $v \in V$.

Step 1. We show that $q(v) \geq q^G(v)$, $\forall v \in V$.

Suppose to the contrary, there exists some $v^* \in V$ such that $q(v^*) < q^G(v^*)$. Let

$$d = q(v^*) - q^G(v^*) < 0 \text{ and } e = \inf\left\{q(v) - q^G(v) \,|\, v \in V\right\}.$$

Clearly, $e \leq d < 0$. Since $\underline{v}q(\underline{v}) = t(\underline{v}) \geq t^G(\underline{v}) = \underline{v}q^G(\underline{v})$, we have $q(\underline{v}) - q^G(\underline{v}) \geq 0$.

Since the function $q(v) - q^G(v)$ is right-continuous, there exists some $\epsilon > 0$ such that

$$q(x) - q^G(x) > \frac{d}{2},\ \forall x \in [\underline{v}, \underline{v} + \epsilon]. \tag{8}$$

Let $\delta = \min\Big\{ -\frac{d}{2}, -(\underline{v} + \frac{\epsilon}{2})e \Big\}$. Clearly, $\delta > 0$. By the definition of e, there exists $v^{**} \in V$ such that

$$q(v^{**}) - q^G(v^{**}) < e + \delta \leq d - \frac{d}{2} = \frac{d}{2}. \tag{9}$$

It follows from (8) and (9) that $v^{**} > \underline{v} + \epsilon$.

We then have

$$\begin{aligned}
\int_{\underline{v}}^{v^{**}} \Big(q(x) - q^G(x)\Big)\, dx &= \int_{\underline{v}}^{\underline{v}+\epsilon} \Big(q(x) - q^G(x)\Big)\, dx + \int_{\underline{v}+\epsilon}^{v^{**}} \Big(q(x) - q^G(x)\Big) dx \\
&> \epsilon\frac{d}{2} + \Big(v^{**} - (\underline{v} + \epsilon)\Big)e \\
&\geq \epsilon\frac{e}{2} + \Big(v^{**} - (\underline{v} + \epsilon)\Big)e \\
&= v^{**}e - (\underline{v} + \frac{\epsilon}{2})e \\
&\geq \Big(e - (\underline{v} + \frac{\epsilon}{2})e\Big)v^{**} \\
&\geq (e + \delta)v^{**} \\
&> \Big(q(v^{**}) - q^G(v^{**})\Big)v^{**},
\end{aligned}$$

where the second line follows from (8) and the definition of e, the second-to-last line follows from the definition of δ, and the last line follows from (9). However, by Lemma 1, the condition $t(v) \geq t^G(v)$ for all $v \in V$ implies that

$$v\Big(q(v) - q^G(v)\Big) \geq \int_{\underline{v}}^{v} \Big(q(x) - q^G(x)\Big)\, dx,\ \forall v \in V.$$

We arrive at a contradiction. Thus, we can conclude that $q(v) \geq q^G(v)$ for all $v \in V$.

Step 2. We show that $q(v) = q^G(v)$, $\forall v \in V$.

Suppose this is not true. Since $q(v) \geq q^G(v)$ for all $v \in V$, there exists some $v^{***} \in V$ such that $q(v^{***}) > q^G(v^{***})$. Obviously, $v^{***} < 1$. Since the function $q(\cdot) - q^G(\cdot)$ is right-continuous, there exists some $\eta \in (0, 1 - v^{***})$ such that

$$q(x) - q^G(x) > 0,\ \forall x \in [v^{***}, v^{***} + \eta]. \tag{10}$$

Since we know from Step 1 that $q(v) \geq q^G(v)$ for all $v \in V$, we have

$$\int_{\underline{v}}^{1} \Big(q(x) - q^G(x)\Big)\, dx > 0.$$

Since $t(1) \geq t^G(1)$, by Lemma 1, we have

$$1\Big(q(1) - q^G(1)\Big) \geq \int_{\underline{v}}^{1} \Big(q(x) - q^G(x)\Big)\, dx > 0 \implies q(1) > q^G(1) = 1.$$

We arrive at a contradiction. Thus, we can conclude that $q(v) = q^G(v)$ for all $v \in V$.

Since $q(v) = q^G(v)$ for all $v \in V$, by Lemma 1, we further have $t(v) = t^G(v)$ for all $v \in V$. This shows that (q^G, t^G) is undominated.

B Proof of Theorem 4

Fix any $G \in \Delta[\underline{v}, 1]$ with finite support, and denote the support of G by $\{r_i\}_{i=1}^K$ where $\underline{v} \leq r_1 < r_2 < \cdots < r_K \leq 1$. Denote the probability of the reserve price being r_k by λ_k, $k = 1, 2, \ldots, K$. Let (q, t) denote the second-price auction with a random reserve price G, and (q^A, t^A) the corresponding augmented mechanism.

Consider any DIC and DIR mechanism (q', t') that satisfies $REV'(v) \geq REV^A(v)$ for all $v \in V$. We show that $REV'(v) = REV^A(v)$ for all $v \in V$. This then implies that (q^A, t^A) is undominated.

Case 1. Consider any v with $v(2) \geq r_K$.

First consider any v with $v(1) = v(2) \geq r_K$. For any such $v \in V$, the seller already extracts the entire surplus in the mechanism (q^A, t^A). Thus, the seller could not obtain a strictly higher ex post revenue for any such value profile without violating the DIR constraint of some bidder. Thus, we can conclude that $REV'(v) = REV^A(v)$ and the seller also extract the full surplus in the mechanism (q', t') for any $v \in V$ such that $v(1) = v(2) \geq r_K$. Using similar arguments as in the proof of Theorem 2, one can show that $REV'(v) = REV^A(v)$ for any v where $v(1) > v(2) \geq r_K$.

Case 2. Consider any v with $v(1) = v(2) \in [r_1, r_K)$.

For any such v, we have

$$v(1) \sum_{i \in M} q_i^A(v) = REV^A(v) \leq REV'(v) \leq \sum_{i \in N} v_i q_i'(v) \leq v(1) \sum_{i \in N} q_i'(v), \tag{11}$$

where the second inequality follows from the DIR constraints. We have

$$\sum_{i \in M} q_i^A(v) \leq \sum_{i \in N} q_i'(v). \tag{12}$$

We shall show that (12) must hold with equality. This, together with (11), shows that $REV^A(v) = REV'(v)$.

If $|M(v)|\, G(v(1)) \geq 1$, then by construction, $\sum_{i \in M} q_i^A(v) = 1$. Clearly, in this case, (12) holds with equality. Thus, we only have to consider value profiles v such that $|M(v)|\, G(v(1)) < 1$. By construction, $q_i^A(v) = G(v(1))$ for all $i \in M$. Suppose to the contrary, (12) holds with strict inequality. Then, there necessarily exists some bidder i with $q_i'(v) > G(v(1))$.

Step 1.1. Consider any v with $v(1) = v(2) = \ldots = v(n) \in [r_{K-1}, r_K)$.

We can apply the same arguments as in Case 2 of Example 2 to show that $q_i'(v) = G(v(1))$ for all $i \in M$. Therefore, we have

$$\sum_{i \in N} q_i'(v) = \sum_{i \in M} q_i'(v) = \sum_{i \in M} q_i^A(v).$$

Consequently, $REV^A(v) = REV'(v)$ for any v with $v(1) = v(2) = \ldots = v(n) \in [r_{K-1}, r_K)$. Furthermore, none of the bidders has positive rent at any such v in the mechanism (q', t').

Step 1.2. Consider any v with $v(1) = v(2) \in [r_{K-1}, r_K)$.

We can apply the same arguments as in Case 2 of Example 2 to show that $q_i'(v) = G(v(1))$ for all $i \in M$. Using the conclusion from Step 1.1, we further have $q_i'(v) = 0$ for all $i \notin M$. Thus, we have

$$\sum_{i \in N} q_i'(v) = \sum_{i \in M} q_i'(v) = \sum_{i \in M} q_i^A(v),$$

and $REV^A(v) = REV'(v)$ for any v with $v(1) = v(2) \in [r_{K-1}, r_K)$. Furthermore, none of the bidders has positive rent at any such v in the mechanism (q', t').

Step 2. Suppose that $REV^A(v) = REV'(v)$ for any v with $v(1) = v(2) \in [r_k, r_K)$ and none of the bidders has positive rent at any such v in the mechanism (q', t'). Consider any v with $v(1) = v(2) \in [r_{k-1}, r_k)$.

For each bidder $i \in M$, for each value profile $(r_{k'}, v_{-i})$ where $k' = k, k+1, \ldots, K$, by the assumption of Step 2, we must have $q'_j(r_{k'}, v_{-i}) = 0$, for all $k' = k, k+1, \dots, K$ and for all $j \neq i$. Thus, we can apply the similar arguments as for (random) posted-price mechanism with finite support (see the proof of Proposition 2) to show that

$$q'_i(v) = q_i^A(v).$$

Consequently, $REV^A(v) = REV'(v)$ for any v with $v(1) = v(2) \in [r_{k-1}, r_K)$ and none of the bidders has positive rent at any such v in the mechanism (q', t').

The induction process shows that $REV^A(v) = REV'(v)$ for any v with $v(1) = v(2) \in [r_1, r_K)$ and none of the bidders has positive rent at any such v in the mechanism (q', t').

Similar arguments as in Case 3 and Case 4 of Example 2 show that $REV^A(v) = REV'(v)$ for all $v \in V$. Therefore, (q^A, t^A) is undominated.

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