A Theory of Labor Markets with Inefficient Turnover*

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Abstract

We develop a theory of labor markets with four features: search frictions, worker productivity shocks, wage rigidity, and two-sided lack of commitment. Inefficient job separations occur in the form of endogenous quits and layoffs that are unilaterally initiated whenever a worker's wage-to-productivity ratio moves outside an inaction region. We derive sufficient statistics for the labor market response to aggregate shocks based on the distribution of workers' wage-to-productivity ratios. These statistics depend on the incidence of inefficient job separations and are linked to readily available microdata on wage changes and worker flows between jobs.

Keywords: Wage Rigidity, Directed Search, Limited Commitment, Job Separations, Inflation

JEL Classification: E31, E52, J64

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1 Introduction

The classical idea that inflation "greases the wheels of the labor market" (Keynes, 1936; Tobin, 1972) forms the bedrock of many theories of macroeconomic fluctuations: After the onset of a recession, nominal wage rigidities lead to inefficiently high wages and depressed labor demand, which creates a role for inflation to restore the economy's optimal employment by reducing real wages. While models in the Keynesian tradition (e.g., Erceg *et al.*, 2000) take seriously the proposition that frictions prevent the efficient adjustment of wages, they are usually silent on the micromechanics of the labor market, including the determinants of *turnover*—i.e., which jobs are saved, destroyed, and created. Conversely, models in the search-theoretic tradition (e.g., Mortensen and Pissarides, 1994) yield rich predictions for the distribution of wages and employment flows in the labor market but abstract from inefficient turnover due to the simplifying assumption that contracts can be continuously and costlessly renegotiated.

Motivated by empirical evidence linking wage rigidity to the employment sensitivity to aggregate shocks, bilaterally inefficient job separations, and real effects of monetary policy in the labor market, we explore the implications of inefficient turnover in a labor market model with four features. First, job search is frictional. Second, workers are subject to idiosyncratic productivity shocks. Third, wages are rigid in between staggered renegotiations. Fourth, neither workers nor firms can commit to staying in a match. In our environment, all four features are necessary to generate inefficient turnover. At the center stage of our model are endogenous quits and layoffs that are unilaterally initiated whenever a worker's wage-to-productivity ratio moves outside an inaction region, giving rise to inefficient job separations.[1] In turn, inefficiencies on the separation margin feed back into job creation through workers' and firms' search decisions. In summary, our contribution is to analytically characterize inefficient turnover and to derive *sufficient statistics* for the economy's response to aggregate shocks in an equilibrium labor market model.

We first study a stationary environment in continuous time. The labor market is populated by a unit mass of heterogeneous workers and an endogenous mass of homogeneous firms. Workers' incomes depend on their employment state and idiosyncratic productivity, which follows a Brownian motion in logarithms. Unemployed workers and idle firms direct their search across

[1] Coase (1960) explicitly pointed out that realistic transaction costs may prevent bilaterally efficient outcomes since *"it is necessary to discover who it is that one wishes to deal with, to inform people that one wishes to deal and on what terms, to conduct negotiations leading up to a bargain, to draw up the contract, to undertake the inspection needed to make sure that the terms of the contract are being observed"* and that *"these operations are often extremely costly, sufficiently costly at any rate to prevent many transactions that would be carried out in a world in which the pricing system worked without cost."*

submarkets indexed by their wage rate and productivity, as in Shimer (1996) and Moen (1997). Worker-firm matches are subject to two contractual frictions. First, wages are rigid in between staggered renegotiations à la Calvo (1983).[2] Second, neither workers nor firms can commit to staying in a match, which can be endogenously dissolved in the form of unilateral quits and layoffs.

Once matched, a worker and a firm play a *nonzero-sum stochastic differential game with stopping times* (Bensoussan and Friedman, 1977). Their interaction forms a *game* due to their strategic choices of their own *stopping times*, defining when to unilaterally separate from the match. The game is *stochastic* and *differential* because worker productivity evolves according to a Brownian motion. It is *nonzero-sum* because the equilibrium match surplus is positive. To characterize the solution to this problem, we leverage powerful tools based on *variational inequalities* (Lions and Stampacchia, 1967).

While workers and firms engage in complex forward-looking behavior, we show that their decisions depend only on a single state variable: the wage-to-productivity ratio. A match is dissolved when this ratio falls outside an inaction region with two thresholds. On one side, workers quit when their wage-to-productivity ratio falls beneath a lower threshold. On the other side, firms lay off workers whose wage-to-productivity ratio exceeds an upper threshold. Endogenous job separations due to quits and layoffs are unilateral in the sense that they occur voluntarily in the eyes of one party, even if they are involuntary in the eyes of the other party (McLaughlin, 1991).

Our analysis yields three main results. First, we prove the existence and uniqueness of a *block-recursive equilibrium* (BRE). This result requires substantially different methods than those in the seminal work of Menzio and Shi (2010, 2011), which we extend to a continuous-time setting with two-sided lack of commitment. Second, we provide a novel characterization of match surplus, entry wages, and job-separation under inefficient turnover by linking them to the expected discounted duration of a match, which here—unlike in models with flexible wages or full commitment—distinctly depends on rent sharing between a worker and a firm. Third, we demonstrate that two-sided lack of commitment has implications for labor markets that are profoundly different from prominent models of product pricing and investment. Unlike in models of inaction (e.g., Barro, 1972; Bernanke, 1983), workers' and firms' ability to unilaterally separate bounds the option value of a match, even as the volatility of productivity shocks grows unboundedly. Compared to Sheshinski and Weiss (1977), the quit threshold and entry wage in our environment are less

[2] We follow a long tradition of modeling wage rigidity through staggered renegotiations, as in Erceg *et al.* (2000). While our model abstracts from their specific microfoundations, we think of such rigidities as capturing a list of reasons surveyed by Bewley (1999), including transaction costs, wage norms, fairness concerns, or information asymmetries. On the relationship between time- and state-dependent pricing models, see Alvarez *et al.* (2016a,b) and Auclert *et al.* (2023).

responsive to expected productivity growth and trend inflation.

Having characterized the stationary economy, we then introduce aggregate shocks. To this end, we assume that incumbents' wages are nominally rigid while there are fluctuations in *aggregate revenue productivity* (TFPR)—i.e., either *aggregate physical productivity* (TFPQ) or the price level. Such aggregate shocks shift incumbents' TFPR-adjusted wages, leading to movements in the rate of endogenous job separations in the form of quits and layoffs. Under flexible entry wages, the wage that unemployed workers search for responds to the aggregate shock. Motivated by the allocative role of new-hire wages (Pissarides, 2009) and the limited cyclicality of reservation wages (Koenig *et al.*, 2023), we also study rigid entry wages, under which unemployed workers search for the same nominal wage schedule as before the aggregate shock, thereby changing firms' vacancy posting incentives. In this environment, inflation can "grease the wheels of the labor market" by affecting both job-separation and job-finding rates.

To study the effects of an economy-wide TFPR shock on aggregate employment, we analyze the economy's *cumulative impulse response* (CIR), defined as the area under an *impulse response function* (IRF). To this end, we extend the seminal work of Alvarez *et al.* (2016a) on sufficient statistics in the product market to a labor market context. Under flexible entry wages, the CIR of aggregate employment is fully described by three data moments: the job-finding rate, the variance of workers' wage changes across jobs, and a measure of the skewness of wage changes across jobs. That skewness appears in the sufficient statistic is a novel result. Intuitively, wage changes between jobs reflect workers' wage-to-productivity ratios, the skewness of which reflects the relative mass of workers near the quit versus layoff thresholds.

Under rigid entry wages, the CIR of aggregate employment additionally depends on the job-finding rate's elasticity with respect to the aggregate shock, which itself is a function of the share of inefficient job separations. Intuitively, an increase in TFPR incentivizes firms to post more vacancies but the magnitude of this effect is decreasing in the share of inefficient job separations: Firms choose when to lay off workers but do not control workers' quit decisions, which limit a firm's expected returns from vacancies.

While our theory highlights the relevant mechanisms at play in labor markets with inefficient turnover, we also lay the foundation for quantifying these mechanisms. To this end, we show how to recover the distribution of unobserved wage-to-productivity ratios and the productivity process parameters from conventional labor market microdata on wage changes between job spells.

Related Literature. Relative to the existing literature, we make two contributions. Our first contribution is to develop an equilibrium framework with inefficient turnover, in which nominal fluctuations affect both job-finding and job-separation rates through the split of match surplus. This approach sets us apart from the two traditions. On one hand, models in the Keynesian tradition have highlighted wage rigidity as the key friction for quantitative models to generate a realistic transmission of shocks (e.g., Christiano *et al.*, 2005). We add to this literature an equilibrium model of endogenous job creation and destruction with inefficient turnover due to wage rigidity, which naturally connects to labor market microdata on quits and layoffs (Graves *et al.*, 2023).

On the other hand, models in the search-theoretic tradition have studied the role of wage rigidity in amplifying unemployment fluctuations, following Shimer (2005a). These models restrict attention to bilaterally efficient contracts, as in Hall (2003) and Elsby *et al.* (2023), where costless wage renegotiations prevent the dissolution of matches with positive surplus. Similarly, the wage-setting protocols assumed by Hall (2005), Hall and Milgrom (2008), and Moscarini and Postel-Vinay (2023) yield only efficient job separations. In related work by Gertler and Trigari (2009) and Gertler *et al.* (2020, 2022), inefficient job separations arise from wage rigidity and productivity shocks in theory but are ignored in practice.

All aforementioned models steer clear of the Barro (1977) critique of inefficient outcomes under long-term contracts. Models in this tradition have produced many important insights. At the same time, there is mounting empirical evidence linking wage rigidity to the employment sensitivity to aggregate shocks,[3] bilaterally inefficient job separations,[4] and real effects of monetary policy in the labor market.[5] Our work represents a stark departure from this tradition in that we explicitly model inefficient turnover as a result of search frictions, productivity shocks, wage rigidity, and two-sided lack of commitment. In doing so, our theory yields a novel characterization of job creation, job separation, and wage determination under inefficient turnover in steady state and over the business cycle. In this sense, our work connects with a theoretical literature's conclusion that "*regrettable layoffs when demand is weak and regrettable quits when demand is strong are the outcome of practical limitations on contracts*" (p. 255 of Hall and Lazear, 1984). Recent work along these lines

[3] Schmieder and von Wachter (2010) document that workers with higher wages and more rigid wages face increased layoff risk in the U.S. Kaur (2019) finds that wage rigidity distorts employment in the presence of labor demand shocks in India. Ehrlich and Montes (2024) show that wage rigidity increases layoffs but decreases quits and hiring in Germany.

[4] Jäger *et al.* (2022) provide quasi-experimental evidence of inefficient job separations following changes in unemployment insurance (UI) policies. Furthermore, many UI recipients would accept significant wage cuts in lieu of being laid off (Davis and Krolikowski, 2023), yet employers do not consider pay cuts a substitute for layoffs (Bertheau *et al.*, 2023).

[5] Olivei and Tenreyro (2007, 2010) show that staggered wage contracts transmit monetary policy to output in the U.S. and other countries. Coglianese *et al.* (2023) link wage rigidity to monetary policy-induced unemployment in Sweden. Faia and Pezone (2023) find greater employment sensitivity to monetary policy at firms with more rigid wages in Italy.

includes Mueller (2017) who calibrates a model with inefficient separations due to wage rigidity, Carlsson and Westermark (2022) who develop a model of inefficient layoffs, and Heathcote and Cai (2023) who study the implications of inefficient quits for optimal UI design. More broadly, our theory opens up the door to a new research agenda studying the propagation of aggregate shocks in frictional labor markets subject to wage rigidity.

Our second contribution is methodological in nature and adds to two prominent literatures. Relative to the search-theoretic literature, we introduce the powerful tools of nonzero-sum stochastic differential games with stopping times (Bensoussan and Friedman, 1977). Such continuous-time methods are well suited to our environment because they offer three distinct benefits. First, they allow us to prove the existence and uniqueness of a BRE under inefficient turnover. Second, they yield convenient properties of value functions (e.g., continuity) and policy functions (e.g., connectedness), allowing us to study equilibrium conditions using variational inequalities. Third, they allow us to derive sharp comparative statics (e.g., anticipatory and option value effects). The foundational work of Menzio and Shi (2010) studies BRE in a discrete-time model under efficient turnover. We complement their work by leveraging new tools to characterize BRE in a continuous-time model under two-sided lack of commitment.

There are important differences between our analysis and the product pricing literature.[6] Specifically, we introduce new methods to extend the sufficient statistic approach to an environment with no commitment on behalf of two strategically interacting parties (i.e., workers and firms) and endogenous transitions between discrete states (i.e., employment and unemployment) in our labor market setting. A notable contribution of Alvarez *et al.* (2016a) is to link the CIR of output to the ratio of the kurtosis and frequency of price changes in a large class of product pricing models. We complement their important insights by deriving the novel result that the CIR of employment in our labor market context is proportional to a measure of skewness of wage changes.

Outline. The rest of the paper is organized as follows. Section 2 characterizes inefficient turnover in the labor market. Section 3 derives sufficient statistics for the economy's response to aggregate shocks. Section 4 connects the model to labor market microdata. Finally, Section 5 concludes.

[6] See, for example, Alvarez *et al.* (2021) and Baley and Blanco (2021, 2022).

2 A Model of Labor Markets with Inefficient Turnover

In this section, we develop a model of inefficient turnover arising from the combination of search frictions, idiosyncratic productivity shocks, wage rigidity, and two-sided lack of commitment.

2.1 Environment

Time is continuous and indexed by t. A unit mass of heterogeneous workers and an endogenous mass of homogeneous firms meet in a frictional labor market.

Preferences. Both workers and firms discount the future at rate $r > 0$. Firms maximize profits. Workers have risk-neutral preferences over consumption streams $\{C_t\}_{t=0}^{¥}$ given by $\mathbb{E}\left[\int_0^{¥} e^{-rt} C_t \, \mathrm{d}t\right]$.

Technology. A worker's flow income depends on their productivity Z_t and their employment state E_t, which can be either employed (h) or unemployed (u). Employed workers produce $Y_t = Z_t$ and consume their wage W_t. Unemployed workers consume $\tilde{B} Z_t$ from home production, with $\tilde{B} \in (0, 1)$. Henceforth, lower-case letters denote the logarithm of variables in uppercase.

Stochastic Process. Workers' idiosyncratic productivities follow a Brownian motion in logarithms, $\mathrm{d}z_t = g \, \mathrm{d}t + s \, \mathrm{d}\mathcal{W}_t^z$, with drift g, volatility s, and a Wiener process $\mathcal{W}_t^z$ that is iid across workers.

Search Frictions. Unemployed workers and idle firms direct their search across segmented submarkets indexed by worker productivity z and the wage w. In each submarket $(z; w)$, firms post vacancies at flow cost $\tilde{K} e^z$ for $\tilde{K} > 0$. Given $\mathcal{U}(z; w)$ unemployed workers and $\mathcal{V}(z; w)$ vacancies, a Cobb-Douglas matching function with constant returns to scale produces $m(z; w) = \mathcal{U}(z; w)^a \mathcal{V}(z; w)^{1-a}$ matches, where a is the elasticity of matches with respect to the unemployed. Given market tightness $q(z; w) := \mathcal{V}(z; w)/\mathcal{U}(z; w)$, workers' job-finding rate is $f(q(z; w)) = m(z; w)/\mathcal{U}(z; w) = q(z; w)^{1-a}$ and firms' job-filling rate is $q(q(z; w)) = m(z; w)/\mathcal{V}(z; w) = q(z; w)^{-a}$. Existing matches can end for any of three reasons: they can be exogenously dissolved at Poisson rate d, or they can be endogenously and unilaterally dissolved by either the worker or the firm.

Wage Determination. While wages are competitively set at match formation, they are intermittently rigid thereafter, with staggered wage renegotiations occurring at rate $d^r \geq 0$ and following a

Nash bargaining protocol with worker weight a. We present the limiting case with $d^r = 0$ in the main text. All results extend to the case of $d^r > 0$, shown in Online Appendix II.3.[7]

Agents' Choices. An unemployed worker's choice of submarket $(z; w)$ is associated with a job-finding rate $f(q(z; w))$. Exogenous separations occur at rate d, inducing a stopping time t^d. Given the wage w, a matched worker chooses a continuation productivity set $\mathcal{Z}^h(w)$, inducing the worker's stopping time $t^h(z; w) = \inf\{t \geq 0 : z_t \in \mathcal{Z}^h(w)^c, z_0 = z\}$, where $X^c := \mathbb{R} \backslash X$. Similarly, given w, a matched firm chooses a continuation productivity set $\mathcal{Z}^j(w)$, inducing the firm's stopping time $t^j(z; w) = \inf\{t \geq 0 : z_t \in \mathcal{Z}^j(w)^c, z_0 = z\}$. Naturally, agents' stopping times must be measurable with respect to their productivity history. Given the worker's and the firm's continuation sets and the exogenous separation hazard, the match duration is the first stopping time in $\vec{t}^m = (t^h, t^j, t^d)$, denoted $t^m = \min\{t^h, t^j, t^d\}$.

2.2 Block-Recursive Equilibrium

A BRE can be described in two steps.[8] In the first step, we describe the optimal search behavior of unmatched workers and firms. Let $u(z)$ be the value of an unemployed worker under the optimal search policy given productivity z. Let $q(z; w)$ denote market tightness in submarket $(z; w)$. Let $h(z; w)$ and $j(z; w)$ be the equilibrium values of an employed worker and a filled job. The problem of an unemployed worker is characterized by the Hamilton-Jacobi-Bellman (HJB) equation,

$$ru(z) = \tilde{B}e^z + g\frac{\P u(z)}{\P z} + \frac{s^2}{2}\frac{\P^2 u(z)}{\P z^2} + \max_w f(q(z; w))[h(z; w) - u(z)], \tag{1}$$

with optimal search strategy $w^*(z)$. Equation (1) states that unemployed workers' flow value is that of an asset with a return equal to the sum of flow dividends (i.e., home production) and expected capital gains (i.e., productivity fluctuations and finding a job). Free entry requires that

$$\min\left\{\tilde{K}e^z - q(q(z; w))j(z; w), q(z; w)\right\} = 0. \tag{2}$$

Equation (2) dictates zero profits in open submarkets and nonpositive profits in closed ones.

In the second step, which is the novel focus of this paper, we describe the strategic interaction

[7]We treat wage rigidity as technological in nature, similar to adjustment costs in product pricing (Barro, 1972) and investment (Cooper and Haltiwanger, 2006).

[8]BRE objects do not depend on the distribution of productivities, wages, and employment states, allowing us to omit it from all notation, as in Menzio and Shi (2010).

that forms part of the *game* between a matched worker-firm pair, which has three features. First, payoffs are *nonzero-sum*, since the match flow value, e^z, exceeds the flow value of separating, $\tilde{B}e^z$. Second, agents' payoffs are *stochastic* and *differential*, since worker productivity z follows a Brownian motion. Third, agents' strategies consist of when to unilaterally separate from the match; i.e., the *stopping times* implied by their continuation sets $\mathcal{Z}^h(w)$ and $\mathcal{Z}^j(w)$. Thus, the interaction between a worker-firm pair can be formulated as a nonzero-sum stochastic differential game with stopping times (Bensoussan and Friedman, 1977). The application of these mathematical methods in a labor market context is different from existing work and a key contribution of this paper.

Value Functions. As long as one agent stays in the match with state z, the other agent chooses whether to stay in the match or to separate, reflecting the two-sided lack of commitment. Thus, we use *variational inequalities* to characterize the values of both agents. The HJB equation of a worker employed at wage w with productivity z inside the firm's optimal continuation set $\mathcal{Z}^{j*}(w)$ is

$$rh(z;w) = \max\left\{e^w + g\frac{\P h(z;w)}{\P z} + \frac{s^2}{2}\frac{\P^2 h(z;w)}{\P z^2} + d\left[u(z) - h(z;w)\right], ru(z)\right\}. \tag{3}$$

Equation (3) reflects the employed worker's choice between staying matched and quitting the firm. The flow value of staying is that of an asset for which the return is the sum of flow dividends (i.e., the wage) and expected capital gains (i.e., productivity fluctuations and separation). The flow value of quitting the firm is simply that of unemployment. The variational inequality in equation (3) satisfies $h(\cdot;w) \in \mathbb{C}^1(\mathcal{Z}^{j*}(w)) \cap \mathbb{C}(\mathbb{R})$. That is, the value of the employed worker is continuously once-differentiable inside the firm's optimal continuation set and continuous everywhere. These continuity and differentiability conditions correspond to the *value matching* and *smooth pasting* conditions of the worker's value function under their own best response. Importantly, a smooth pasting condition characterizes the optimal boundary of the worker's continuation region.

Analogously, the HJB equation of a firm employing a worker at wage w with productivity z inside the worker's optimal continuation set $\mathcal{Z}^{h*}(w)$ is

$$rj(z;w) = \max\left\{e^z - e^w + g\frac{\P j(z;w)}{\P z} + \frac{s^2}{2}\frac{\P^2 j(z;w)}{\P z^2} - dj(z;w), 0\right\}. \tag{4}$$

Equation (4) reflects the firm's choice between staying matched and laying off the worker. The flow value of staying is that of an asset for which the return is the sum of flow dividends (i.e., profits) and expected capital gains (i.e., productivity fluctuations and separation). The flow value of laying off the worker is simply that of being idle. The variational inequality in equation (4) satisfies $j(\cdot; w) \in \mathbb{C}^1(\mathcal{Z}^{h*}(w)) \cap \mathbb{C}(\mathbb{R})$. That is, the value of the matched firm is continuously once-differentiable inside the worker's optimal continuation set and continuous everywhere. Again, a smooth pasting condition characterizes the optimal boundary of the firm's continuation region.

If either agent dissolves the match, then the other agent receives their outside option value. Therefore, the worker's and the firm's values of a match with productivity z and wage w satisfy:

$$h(z; w) = u(z) \quad \forall z \in (\mathcal{Z}^{j*}(w))^c, \tag{5}$$

$$j(z; w) = 0 \qquad \forall z \in (\mathcal{Z}^{h*}(w))^c. \tag{6}$$

Equations (5)–(6) define each agent's payoff outside the other agent's continuation set. Value-matching conditions imply the continuity of each agent's value function at the boundaries of the other agent's continuation set. However, smooth pasting conditions *do not* apply to either agent's value at the boundary of the other agent's continuation set because the HJB equations (3)–(4) do not hold when an agent has no optimization problem to solve, which happens outside the other agent's continuation set.[9] For the same reason, we do not require value functions to be differentiable in the entire domain, but only in the part where an agent has a choice between staying matched or not.

Continuation Sets. Two sets of conditions characterize agents' optimal continuation sets. First, agents optimally choose to continue whenever

$$h(z; w) > u(z), \tag{7}$$

$$j(z; w) > 0. \tag{8}$$

Second, to resolve any ambiguity in the strategic choice of an indifferent party, we focus on the socially (weakly) preferable outcome by invoking an equilibrium refinement. Specifically, we assume that agents choose to continue whenever staying in the match is a weakly dominant strategy. For *any* policy of the worker, the firm strictly prefers to continue the match if flow profits are strictly positive—i.e., $e^z - e^w > 0$—because the firm always has the option of firing the worker in the future. Therefore, the firm's optimal continuation set is

$$\mathcal{Z}^{j*}(w) := int\,\{z : j(z; w) > 0 \text{ or } e^z - e^w > 0\}\,. \tag{9}$$

[9]For example, for $z \in (\mathcal{Z}^{h*}(w))^c$, $0 = r j(z; w) < \max\{e^z - e^w + g \P j(z; w)/\P z + (s^2/2)\P^2 j(z; w)/\P z^2 - d j(z; w)\,, 0\}$.

Analogously, the worker's optimal continuation set includes all productivity levels for which the sum of the current wage and the discounted capital gains from unemployment is positive:

$$\mathcal{Z}^{h*}(w) := int\left\{z : h(z;w) > u(z) \text{ or } 0 < e^{w} - ru(z) + g\frac{¶u(z)}{¶z} + \frac{s^2}{2}\frac{¶^2u(z)}{¶z^2}\right\}. \tag{10}$$

Intuitively, the unemployed worker's HJB equation (1) implies $0 < e^{w} - ru(z) + g(¶u(z)/¶z) + (s^2/2)¶^2u(z)/¶z^2$ if and only if $\tilde{B}e^{z} + \max_{w'} f(q(z;w'))[h(z;w') - u(z)] < e^{w}$. Thus, continuing strictly dominates quitting precisely when the wage strictly exceeds the flow opportunity cost.

Figure 1 illustrates the equilibrium values and optimal policies of a worker-firm match. The firm's continuation set is $\mathcal{Z}^{j*}(w) = (z^{-}(w), ¥)$, which contains productivities for which the firm makes strictly positive flow profits—i.e., $z > \tilde{z}^{-}(w) := w$—as well as productivities for which the worker and the firm continue despite negative flow profits due to a positive and large enough continuation value—i.e., $z \in (z^{-}(w), \tilde{z}^{-}(w))$. Analogously, the worker's continuation set is $\mathcal{Z}^{h*}(w) = (-¥, z^{+}(w))$, which contains productivities for which the worker's wage strictly exceeds the flow opportunity cost—i.e., $z < \tilde{z}^{+}(w)$, where $\tilde{z}^{+}$ satisfies $0 = e^{w} - ru(\tilde{z}^{+}) + g¶u(\tilde{z}^{+})/¶z + (s^2/2)¶^2u(\tilde{z}^{+})/¶z^2$—as well as productivities for which the worker and the firm continue despite the worker's negative net flow value due to a positive and large enough continuation value—i.e., $z \in (\tilde{z}^{+}(w), z^{+}(w))$. The existence and uniqueness of a threshold characterizing each agent's separation policy are not assumptions but results formally derived below.

A Markov perfect equilibrium of this game is a fixed point between agents' best-response mappings involving continuation productivity levels z, given wage w.[10] To address the trivial multiplicity of equilibria, our equilibrium definition implicitly imposes weakly dominant strategies.

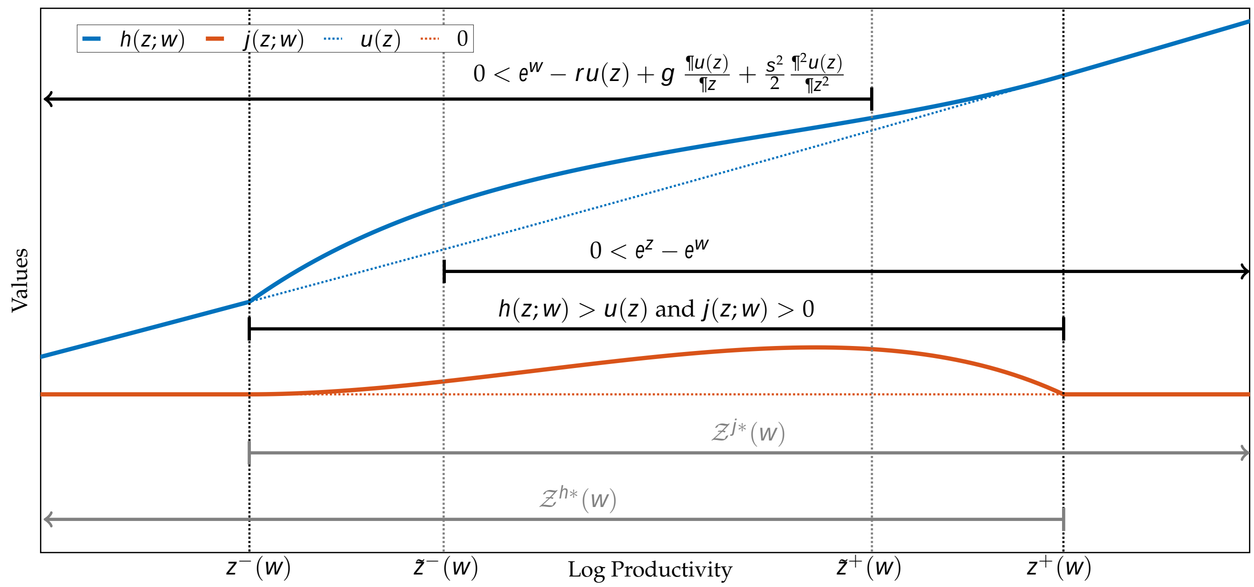
Definition 1. *A BRE consists of a set of value functions $\{u(z), h(z;w), j(z;w)\}$, a market tightness function $q(z;w)$, the matched worker's and the matched firm's continuation sets $\{\mathcal{Z}^{h*}(w), \mathcal{Z}^{j*}(w)\}$, and the unemployed worker's search strategy function $w^{*}(z)$ s.t.:*

1. *Given $h(z;w)$ and $q(z;w)$, $u(z)$ solves (1) with optimal search strategy $w^{*}(z)$.*
2. *Given $j(z;w)$, market tightness $q(z;w)$ satisfies the free-entry condition (2).*
3. *Given $u(z)$ and $\mathcal{Z}^{j*}(z)$, $h(z;w) \in \mathbb{C}^1(\mathcal{Z}^{j*}(w)) \cap \mathbb{C}(\mathbb{R})$ solves (3) and (5). Given $\mathcal{Z}^{h*}(z)$, $j(z;w) \in \mathbb{C}^1(\mathcal{Z}^{h*}(w)) \cap \mathbb{C}(\mathbb{R})$ solves (4) and (6).*

[10] The Markovian nature of the equilibrium reflects two-sided lack of commitment. Online Appendix I.3 derives the recursive equilibrium in continuous time from its discrete-time counterpart.

4. *Given $u(z)$, the continuation set of the firm, $\mathcal{Z}^{j*}(z)$, is (9) and that of the worker, $\mathcal{Z}^{h*}(z)$, is (10).*

FIGURE 1. EQUILIBRIUM VALUES AND OPTIMAL POLICIES



Notes: The figure plots the value functions of workers (blue lines) and firms (red lines) for a given log wage w as a function of log productivity z. Solid lines show the values in the match, which are $h(z;w)$ for the worker and $j(z;w)$ for the firm. Dashed lines show the values outside of a match, which are $u(z)$ for the worker and 0 for the firm. The equilibrium continuation sets of the worker and the firm are $\mathcal{Z}^{h*}(w) = (-¥, z^{+}(w))$ and $\mathcal{Z}^{j*}(w) = (z^{-}(w), ¥)$, respectively. The worker has positive net flow payoff for any productivity level $z < \tilde{z}^{+}(w)$, where $\tilde{z}^{+}$ satisfies $0 = e^{w} - ru(\tilde{z}^{+}) + g¶u(\tilde{z}^{+})/¶z + (s^2/2)¶^2 u(\tilde{z}^{+})/¶z^2$. The firm makes strictly positive flow profits for any productivity level $z > \tilde{z}^{-}(w) := w$. *Source:* Model simulations.

Part 1 of Definition 1 requires unemployed workers' search strategies to be optimal. Part 2 imposes free entry. The remaining parts describe agents' best responses in two steps. Given the other agent's optimal continuation set, Part 3 describes the value function under the optimal continuation policy. Given these value functions, Part 4 describes the optimal continuation sets.

Equilibrium Refinement. Our equilibrium definition incorporates an equilibrium refinement based on weakly dominant strategies. For illustration, suppose time is discrete, a period lasts $\mathrm{d}t$, and the match will end in the following period with certainty. If continuation is optimal today in expectation of match separation next period, which is the worst possible outcome from the next period onward, then continuation must be optimal under any possible outcome from next period onward. Table 1 lists the payoffs in the period game. Suppose that productivity z is such that flow payoffs in the match exceed flow payoffs from the outside options for both the worker and the firm—i.e., $(e^{z} - e^{w})\,\mathrm{d}t > 0$ and $e^{w}\,\mathrm{d}t + \mathbb{E}_{z'}[e^{-r\,\mathrm{d}t}u(z')|z] > u(z)$. Then, there are two equilibria: one in which both agents choose to separate and one in which both players decide to continue. However, the first equilibrium does not survive the *iterated elimination of weakly dominated strategies* since, independent of what the other agent does, it is weakly better to continue. As $\mathrm{d}t \to 0$, we recover the continuation sets in equations (9)–(10), which incorporate a restriction to weakly dominant strategies in continuous time. That is, $(e^z - e^w)\,\mathrm{d}t > 0$ and $e^w\,\mathrm{d}t + \mathbb{E}_{z'}[e^{-r\,\mathrm{d}t}u(z')|z] > u(z)$ imply $e^z - e^w > 0$ and $0 < e^w - ru(z) + g\partial u(z)/\partial z + (s^2/2)\partial^2 u(z)/\partial z^2$ as $\mathrm{d}t \to 0$.

TABLE 1. ILLUSTRATING THE EQUILIBRIUM REFINEMENT USING PAYOFFS IN THE PERIOD GAME

	Worker separates	**Worker continues**
Firm separates	$(0, u(z))$	$(0, u(z))$
Firm continues	$(0, u(z))$	$((e^z - e^w)\,\mathrm{d}t, e^w\,\mathrm{d}t + \mathbb{E}_{z'}[e^{-r\,\mathrm{d}t}u(z')|z])$

Notes: This table shows the payoffs in a discrete-time approximation of the game played between a worker and a firm under the assumption that in the next period, match separat.

Inefficient Turnover. The flow benefit of a match, net of its opportunity cost, is given by $e^z - (\tilde{B}e^z + \max_w f(q(z;w))[h(z;w) - u(z)]) > 0$, reflecting the positive social value of a match. Given that wages are *allocative* in the sense that match duration depends on their level, given wage rigidity. For this reason, *inefficient job separations* occur whenever a match is endogenously dissolved by either the worker or the firm. The lack of commitment is reflected in the equilibrium definition: Endogenous separations are optimal at each point of the state space for at least one of the agents. Importantly, inefficiencies on the job separation margin also imply *inefficient job match creation* due to their effects on unemployed workers' search decisions through $h(z;w)$ and on firms' vacancy posting decisions through $j(z;w)$. Thus, *inefficient turnover* manifests itself in the form of endogenous quits versus layoffs, in contrast to standard labor market theories (e.g., Mortensen and Pissarides, 1994) in which the two events are not separately defined and separations occur when match surplus is exhausted.

2.3 Equilibrium Characterization

To understand the dependence of equilibrium objects on state variables, we recast the model in terms of a reduced state space. It turns out that the relevant state variable for both workers and firms is the log-wage-to-productivity ratio, $\hat{w} := w - z$. We can express agents' values and policies as functions of the scalar $\hat{w}$ instead of the duplet $(z;w)$. To simplify notation, we define the transformed drift $\hat{g} := g + s^2$ and the transformed discount factor $\hat{r} := r - g - s^2/2$. The following Lemma characterizes the equilibrium.

Lemma 1. *Suppose that the set* $(u(z), h(z;w), j(z;w), q(z;w))$ *satisfies the equilibrium conditions* (1)–(6), *given the continuation sets* $\mathcal{Z}^{h*}(w)$ *and* $\mathcal{Z}^{j*}(w)$ *defined in* (9)–(10) *and search policy* $w^*(z)$*. Then,*

$$(\hat{U}\,,\,\hat{J}(w-z)\,,\,\hat{W}(w-z)\,,\,\hat{q}(w-z)) = \left(\frac{u(z)}{e^z}\,,\,\frac{j(z;w)}{e^z}\,,\,\frac{h(z;w)-u(z)}{e^z}\,,\,q(z;w)\right)$$

equivalently characterizes the equilibrium if the following conditions are satisfied:

1. *Given* $\hat{W}(\hat{w})$ *and* $q(\hat{w})$, $\hat{U}$ *satisfies*

$$\hat{r}\hat{U} = \tilde{B} + \max_{\hat{w}} f(\hat{q}(\hat{w}))\hat{W}(\hat{w}), \tag{11}$$

where the optimal choice of submarket for an unemployed worker to search in is $\hat{w}^* = w^*(z) - z$.

2. *Given* $\hat{J}(\hat{w})$*, free entry is satisfied:* $\min\left\{\tilde{K} - q(\hat{q}(\hat{w}))\hat{J}(\hat{w})\,,\,\hat{q}(\hat{w})\right\} = 0$.

3. *Given* $\hat{\mathcal{Z}}^{h*} := int\left\{\hat{w} : \hat{W}(\hat{w}) > 0 \text{ or } e^{\hat{w}} > \hat{r}\hat{U}\right\}$ *and* $\hat{\mathcal{Z}}^{j*} := int\left\{\hat{w} : \hat{J}(\hat{w}) > 0 \text{ or } e^{\hat{w}} < 1\right\}$*, the transformed value functions* $\hat{W}(\hat{w})$ *and* $\hat{J}(\hat{w})$ *satisfy the variational inequalities*

$$\hat{r}\hat{W}(\hat{w}) = \begin{cases} \max\left\{e^{\hat{w}} - \hat{r}\hat{U} - \hat{g}\hat{W}'(\hat{w}) + \frac{s^2}{2}\hat{W}''(\hat{w}) - d\hat{W}(\hat{w})\,,\,0\right\} & \forall \hat{w} \in \hat{\mathcal{Z}}^{j*}, \\ 0 & \forall \hat{w} \in (\hat{\mathcal{Z}}^{j*})^c, \end{cases} \tag{12}$$

$$\hat{r}\hat{J}(\hat{w}) = \begin{cases} \max\left\{1 - e^{\hat{w}} - \hat{g}\hat{J}'(\hat{w}) + \frac{s^2}{2}\hat{J}''(\hat{w}) - d\hat{J}(\hat{w})\,,\,0\right\} & \forall \hat{w} \in \hat{\mathcal{Z}}^{h*}, \\ 0 & \forall \hat{w} \in (\hat{\mathcal{Z}}^{h*})^c, \end{cases} \tag{13}$$

with $\hat{W} \in \mathbb{C}^1(\hat{\mathcal{Z}}^{j*}) \cap \mathbb{C}(\mathbb{R})$ *and* $\hat{J} \in \mathbb{C}^1(\hat{\mathcal{Z}}^{h*}) \cap \mathbb{C}(\mathbb{R})$*. Finally, the optimal stopping times are given by* $t^{h*} = \inf\{t \geq 0 : \hat{w}_t \in (\hat{\mathcal{Z}}^{h*})^c, w_0 = \hat{w}^*\}$ *and* $t^{j*} = \inf\{t \geq 0 : \hat{w}_t \in (\hat{\mathcal{Z}}^{j*})^c, w_0 = \hat{w}^*\}$.

Proof. See Appendix A.1. □

The equilibrium conditions in Lemma 1 are transformed versions of those of the original problem stated above. Part 1 gives the value of unemployment under the optimal search strategy in equation (11). Part 2 states the transformed free-entry condition. Part 3 describes a nontrivial equilibrium, with equations (12)–(13) referencing agents' optimal continuation regions such that workers' wages are above the flow value of unemployment whenever $e^{\hat{w}} > \hat{r}\hat{U}$ and firms' flow profits are positive whenever $e^{\hat{w}} < 1$.

Next, we state a key result on equilibrium existence and uniqueness.

Proposition 1. *There exists a unique BRE.*

Proof. See Appendix A.2. □

Although equilibrium existence and uniqueness are important properties of models of directed search, in our context they do not follow from previous work. Standard arguments in discrete time with only exogenous job separations involve Schauder's fixed-point theorem (e.g., Menzio and Shi, 2010; Schaal, 2017), which critically relies on two conditions: continuity in the value functions and continuity in the mapping between value functions that characterize the BRE. These standard arguments no longer apply to the above-referenced models in discrete time after the inclusion of endogenous separations, nor do they carry over to our continuous-time setup.

Instead, we leverage quasi-variational inequalities to prove the existence and uniqueness of a nontrivial equilibrium in our model. The proof proceeds in three steps. In the first step, we represent the equilibrium conditions (12)–(13) in terms of quasi-variational inequalities (cf. Antman, 1983). In the second step, we use the existence and uniqueness results in Lions and Stampacchia (1967) to show the existence of the agents' best response functions and their associated value functions. In the third step, we define a functional equation $Q(\cdot)$ that maps the worker's value function to itself using both agents' best response functional equations. Thus, proving the existence of a unique nontrivial Nash equilibrium becomes equivalent to finding a fixed point $\hat{W}^*$ such that $Q(\hat{W}^*) = \hat{W}^*$. To this end, we show that the operator $Q(\cdot)$ is monotonic, thus allowing us to establish the existence of the fixed point by invoking the Birkhoff-Tartar theorem (Aubin, 2007), which applies under relatively weak regularity conditions. Finally, we show that the operator $Q(\cdot)$ satisfies a type of concavity property, which allows us to establish the uniqueness of the fixed point. This uniqueness result is nontrivial given the complementarity in agents' continuation decisions based on strategic worker-firm interactions within a match. Importantly, our continuous-time setup also allows us to leverage properties of the employed worker's and the firm's value functions—e.g., continuity with respect to $\hat{U}$—which are necessary to find a unique equilibrium of this economy.

At an intuitive level, the result follows from two observations. First, taking the value of unemployment as given, the firm chooses a layoff productivity threshold $z^-(w)$ and the worker chooses a quit productivity threshold $z^+(w)$. If the firm decides to delay a layoff (i.e., a lower $z^-(w)$), then the worker's best response is also to delay a quit (i.e., higher $z^+(w)$). But the response is less than one-for-one, because the benefit from delaying separation materializes in the future. In the extreme, as one agent commits to staying in the match, the other party still has an incentive to separate in some future states. Thus, each agent's best-response threshold is decreasing and concave in the other agent's threshold. Second, if the future value of unemployment is higher, then workers will quit sooner (i.e., lower $z^+(w)$) and the match value is lower. This in turn reduces the current value of an unemployed worker. Thus, current and future unemployment values are "strategic substitutes," pushing toward the uniqueness of equilibrium.

Next, we characterize properties of the BRE. Recalling the definition of the transformed state variable $\hat{w} := w - z$, we postulate that there exist optimal policies $\hat{w}^- < \hat{w}^* < \hat{w}^+$, where $\hat{w}^-$ is the worker's optimal job-separation threshold, $\hat{w}^*$ is the optimal search strategy at match formation, and $\hat{w}^+$ is the firm's optimal job separation threshold. We define the transformed surplus of the match as $\hat{S}(\hat{w}) := \hat{J}(\hat{w}) + \hat{W}(\hat{w})$ and the worker's share of the transformed surplus as $h(\hat{w}) := \hat{W}(\hat{w})/\hat{S}(\hat{w})$.

Proposition 2. *The BRE has the following properties:*

1. *The joint match surplus satisfies*

$$\hat{S}(\hat{w}) = (1 - \hat{r}\hat{U})\mathcal{T}(\hat{w}, \hat{r}), \tag{14}$$

where $\tilde{B} < \hat{r}\hat{U} < 1$ and the expected discounted match duration is given by

$$\mathcal{T}(\hat{w}, \hat{r}) := \mathbb{E}\left[\int_0^{t^{m*}} e^{-\hat{r}t}\,\mathrm{d}t \middle| \hat{w}_0 = \hat{w}\right]. \tag{15}$$

2. *The competitive entry wage, $\hat{w}^* = \arg\max_{\hat{w}} f(\hat{q}(\hat{w}))\hat{W}(\hat{w})$, exists and is unique. Moreover, it solves*

$$\hat{w}^* = \arg\max_{\hat{w}} \left\{\hat{W}(\hat{w})^a \hat{J}(\hat{w})^{1-a}\right\} = \arg\max_{\hat{w}} \left\{h(\hat{w})^a (1 - h(\hat{w}))^{1-a}\mathcal{T}(\hat{w}, \hat{r})\right\}, \tag{16}$$

with the unique solution characterized by the following optimality condition:

$$\underbrace{h'(\hat{w}^*)\left(\frac{a}{h(\hat{w}^*)} - \frac{1-a}{1-h(\hat{w}^*)}\right)}_{\text{share channel}} = -\underbrace{\frac{\mathcal{T}'_{\hat{w}}(\hat{w}^*, \hat{r})}{\mathcal{T}(\hat{w}^*, \hat{r})}}_{\text{surplus channel}}. \tag{17}$$

3. *The equilibrium job-finding rate $f(\hat{q}(\hat{w}^*))$ and the flow opportunity cost of employment $\hat{r}\hat{U}$ are*

$$f(\hat{q}(\hat{w}^*)) = \left[(1 - h(\hat{w}^*))(1 - \hat{r}\hat{U})\mathcal{T}(\hat{w}^*, \hat{r})/\tilde{K}\right]^{\frac{1-a}{a}}, \tag{18}$$

$$\hat{r}\hat{U} = \tilde{B} + \left(\tilde{K}^{a-1}\left(1 - h(\hat{w}^*)\right)^{1-a} h(\hat{w}^*)^a \left(1 - \hat{r}\hat{U}\right)\mathcal{T}(\hat{w}^*, \hat{r})\right)^{\frac{1}{a}}. \tag{19}$$

4. *If $g \neq 0$ or $s \neq 0$, then each agent's continuation set is connected and that of the game is bounded:*

$$\hat{\mathcal{Z}}^{h*} = (\hat{w}^-, ¥) \quad \textit{and} \quad \hat{\mathcal{Z}}^{j*} = (-¥, \hat{w}^+), \tag{20}$$

where $-¥ < \hat{w}^- \leq \log(\hat{r}\hat{U}) < 0 \leq \hat{w}^+ < ¥$. Workers' and firms' value functions satisfy the following smooth pasting conditions: $\hat{W}'(\hat{w}^-) = \hat{J}'(\hat{w}^+) = 0$.

Proof. See Appendix A.3. □

Starting with Part 1 of Proposition 2, equation (14) states that the match surplus equals the product of the transformed flow surplus $1 - \hat{r}\hat{U}$ and the expected discounted match duration $\mathcal{T}(\hat{w}, \hat{r})$ defined in equation (15), which depends on the wage $\hat{w}$ and the width of the match's continuation set $(\hat{w}^-, \hat{w}^+)$. Also, the flow opportunity cost of employment $\hat{r}\hat{U}$ is bounded between 1 (i.e., the transformed value of flow output in the match) and $\tilde{B}$ (i.e., the transformed value of home production). Since $1 > \hat{r}\hat{U}$, the joint match surplus is always strictly positive, so that all endogenous job separations are inefficient.

Equations (16)–(17) of Part 2 show that the optimal entry wage $\hat{w}^*$ balances a *share channel* and a *surplus channel*. Unemployed workers search for wages that are competitively set *as if* they were a Nash bargaining solution with worker weight a, thereby satisfying the Hosios (1990) condition. This result derives from free entry, which implies that workers' job-finding rate is proportional to the firm's value. A larger entry wage increases the worker's surplus share by $h'(\hat{w}^*)a/h(\hat{w}^*)$ but reduces the job-finding probability by $h'(\hat{w}^*)(1-a)/(1-h(\hat{w}^*))$. This trade-off is reflected in the share channel and standard in models of directed search (e.g., Shimer, 1996; Moen, 1997).

In addition, the novel *surplus channel* captures the dependence of expected match duration on the wage set at match formation. The higher (lower) the entry wage, the sooner the firm (worker) will dissolve the match in expectation. Only if $\mathcal{T}'_{\hat{w}}(\hat{w}^*, \hat{r}) = 0$, then the worker's surplus share is $h(\hat{w}^*) = a$, as in bilaterally efficient models. These considerations are unique to our environment.

Part 3 states workers' job-finding rate (18) and the flow opportunity cost of employment (19) as functions of the worker's surplus share and the expected discounted match duration.

Part 4 shows that the continuation set of the worker and that of the firm in (20) follow threshold rules in the log-wage-to-productivity ratio $\hat{w}$. Workers do not quit as long as $\hat{w} > \hat{w}^-$, while firms refrain from firing the worker as long as $\hat{w} < \hat{w}^+$. Thus, the continuation set for the match is given by $\hat{\mathcal{Z}}^{h*} \cap \hat{\mathcal{Z}}^{j*} = (\hat{w}^-, \hat{w}^+)$. These thresholds satisfy $\hat{w}^- \leq \log(\hat{r}\hat{U})$ and $\hat{w}^+ \geq 0$, reflecting both parties' willingness to accept flow payoffs below that from their respective outside option. Finally, the smooth pasting conditions apply at the worker's quit threshold $\hat{w}^-$ and at the firm's firing threshold $\hat{w}^+$, reflecting the optimality of agents' continuation thresholds.

Finally, it is worth highlighting that the optimal entry wage (see Part 2) will be set at an optimal distance from both separation thresholds (see Part 4). To convey the intuition, consider a wage-to-productivity ratio $\hat{w}$ close to the quit threshold $\hat{w}^-$. The worker's and firm's value functions are increasing for $\hat{w}$ sufficiently close to $\hat{w}^-$ since both values are zero when $\hat{w} < \hat{w}^-$ and positive when $\hat{w} > \hat{w}^-$.[11] Therefore, around the quit threshold, raising wages is Pareto improving, as it results in a higher flow payoff for the worker and at the same time a lower quit probability, which extends the expected match duration and increases the firm's value. Following a symmetric argument, lowering wages is Pareto improving near the layoff threshold.

2.4 Understanding the Economic Mechanisms

Static Considerations. We first consider equilibrium policies under fixed productivity.

Proposition 3. *If $g = s = 0$, then optimal policies are given by*

$$(\hat{w}^-, \hat{w}^*, \hat{w}^+) = \log(\hat{r}\hat{U}, a + (1-a)\hat{r}\hat{U}, 1),$$

with $h(\hat{w}^*) = a$ and $\mathcal{T}(\hat{w}^*, \hat{r}) = 1/(\hat{r}+d)$, and no smooth pasting conditions apply.

Proof. See Appendix A.4. □

Note that $\hat{w}^- < \hat{w}^* < \hat{w}^+$ and $\hat{w} = \hat{w}^*$ for the match duration, so there are no endogenous job separations absent productivity fluctuations. From this, we see that lack of commitment and wage rigidity by themselves do not generate inefficient job separations. Absent productivity fluctuations, agents' behavior is bilaterally efficient, in that it maximizes the joint match surplus.

In addition to the static forces outlined above, two dynamic considerations guide workers' and firms' choices: the *option value effect* and the *anticipatory effect*.

Dynamic Consideration I: The Option Value Effect. To understand the option value due to productivity fluctuations, we temporarily abstract from the drift in worker productivity.

Proposition 4. *If $\hat{g} = 0$ and $a = 1/2$, then, to a first-order approximation of flow payoffs around the entry wage, $\hat{w}^* = \log\left((1+\hat{r}\hat{U})/2\right)$ and job-separation thresholds satisfy $\hat{w}^{\pm} = \hat{w}^* \pm h(j, \mathsf{F})$ for some function $h(j, \mathsf{F})$ with $j := \sqrt{2(\hat{r}+d)}/s$ and $\mathsf{F} := (1-\hat{r}\hat{U})/(1+\hat{r}\hat{U})$. The following properties apply:*

[11] Figure I1 in Online Appendix I.1 plots workers' and firms' values as functions of $\hat{w}$.

1. *$h(j, \mathsf{F})$ is decreasing in j and increasing in F.*

2. *$\lim_{j \to 0} h(j, \mathsf{F}) = 3\mathsf{F}$ and $\lim_{j \to ¥} h(j, \mathsf{F}) = \mathsf{F}$.*

3. *$j h(j, \mathsf{F})$ is increasing in j.*

The equilibrium surplus share is $h(\hat{w}^*) = a = 1/2$ and the expected discounted match duration,

$$\mathcal{T}(\hat{w}^*, \hat{r}) = \frac{1 - 2\left(e^{j h(j, \mathsf{F})} + e^{-j h(j, \mathsf{F})}\right)^{-1}}{\hat{r} + d}, \tag{21}$$

is increasing in j and F and satisfies $\mathcal{T}'_{\hat{w}}(\hat{w}^*, \hat{r}) = 0$.

Proof. See Appendix A.4. □

Proposition 4 demonstrates that idiosyncratic volatility, by itself, does not affect the split of the match surplus between the worker and the firm. Such an economy is symmetric around the entry wage, which implies $\mathcal{T}'_{\hat{w}}(\hat{w}^*, r) = 0$ and $h(\hat{w}^*) = a$. Thus, a larger $\hat{w}^*$ reduces the match duration by increasing the likelihood of a layoff but increases the match duration by reducing the likelihood of a quit. Weighing both forces, $\mathcal{T}(\cdot, r)$ is maximized at $\hat{w}^* = (1 + \hat{r}\hat{U})/2$ and $h(\hat{w}^*) = 1/2$.

This result provides a tight characterization of the worker's and the firm's optimal policy functions, which yield the continuation region of the match $(\hat{w}^-, \hat{w}^+)$ being symmetrically centered around the optimal entry wage $\hat{w}^*$. Second, the width of the continuation region is increasing in volatility s and decreasing in $\hat{r}\hat{U}$ (Part 1). The width of the inaction region increases with s due to the option value effect: Though the worker's productivity might fall below the wage, the firm is willing to wait before firing the worker because productivity may increase in the future. The width of the inaction region decreases with $\hat{r}\hat{U}$, a higher value of which decreases match surplus and makes it more costly to wait.

The option value effect naturally arises in models of inaction. However, our model features a departure from canonical models of inaction (e.g., Barro, 1972; Bernanke, 1983). In those models, the width of the continuation region typically grows unboundedly with the volatility s. Instead, in our model, the width of the continuation region has an upper bound (Part 2). To see the intuition behind this result, consider the problem of a firm that finds itself in a match with negative flow profits—the worker case is exactly analogous. The marginal benefit from remaining in a currently unprofitable match is that, with some probability in the future, productivity increases enough to make the match profitable by rendering the wage-to-productivity ratio less than unity. At the same time, inaction on the part of the firm is risky: Productivity may increase by a large enough amount for the worker to choose to quit. Given the two job-separation thresholds, as the volatility goes to infinity, the probability of remaining in the profitable part of the inaction region approaches zero. Thus, the two-sided lack of commitment imposes an upper bound on the option value associated with remaining in a match.

The inefficiency due to the lack of commitment also manifests itself in the expected duration of the match in (21). Since the separation thresholds, indexed by $h(j, F)$, remain bounded as $s \to$ ¥, the expected match duration decreases as the volatility of productivity shocks increases (Part 3).

Dynamic Consideration II: The Anticipatory Effect. To understand the anticipatory effect due to the productivity drift, we temporarily abstract from volatility in worker productivity (i.e., $s = 0$) and focus on the case with weakly positive drift (i.e., $\hat{g} \geq 0$), with other cases being analogous.

Proposition 5. *If $s = 0$ and $\hat{g} \geq 0$, then the quit threshold is $\hat{w}^- = \log\left(\hat{r}\hat{U}\right)$ and*

$$w^* = \hat{w}^- + \tilde{T}\left(\frac{a + (1-a)\hat{r}\hat{U}}{\hat{r}\hat{U}}, \frac{\hat{r}+d}{\hat{g}}, \frac{(1-a)(1-\hat{r}\hat{U})}{\hat{r}\hat{U}}\right),$$

where $\tilde{T}(\cdot)$, defined in equation (A.32) of Appendix A.4, is increasing in its first argument and decreasing in its second argument. Moreover:

1. *As $\hat{g} \to 0$, then $\left(\tilde{T}(\cdot), T(\hat{w}^*, \hat{r}), h(\hat{w}^*)\right) \to \left(\log\left(\frac{a+(1-a)\hat{r}\hat{U}}{\hat{r}\hat{U}}\right), \frac{1}{\hat{r}+d}, a\right)$.*
2. *As $\hat{g} \to$ ¥, then $\left(\tilde{T}(\cdot), T(\hat{w}^*, \hat{r}), h(\hat{w}^*)\right) \to \left(\tilde{T}^{limit}, 0, h^{limit}\right)$, where $\tilde{T}^{limit}$ and h^{limit} satisfy*

$$\frac{a + (1-a)\hat{r}\hat{U}}{\hat{r}\hat{U}} = \frac{e^{\tilde{T}^{limit}} - 1 - \frac{(1-a)(1-\hat{r}\hat{U})}{\hat{r}\hat{U}}\left(1 - \frac{\tilde{T}^{limit}}{e^{\tilde{T}^{limit}}-1}\right)}{\tilde{T}^{limit}},$$

$$h^{limit} = a + \frac{1-a}{\tilde{T}^{limit}} \frac{(1-\hat{r}\hat{U})h^{limit}}{h^{limit} + \hat{r}\hat{U}(1-h^{limit})}. \tag{22}$$

Proof. See Appendix A.4. □

When productivity grows at a constant rate, the job-separation threshold $\hat{w}^-$ equals the static opportunity cost of employment since workers benefit from remaining matched up to that point and workers have no incentive to delay separation beyond that point. The fact that $\hat{w}^-$ is insensitive to the drift differs from the canonical result in Sheshinski and Weiss (1977) who studied the problem of a firm setting prices subject to menu costs with positive trend inflation. Their main result is that, in order to economize on menu costs associated with price changes, firms both decrease the lower threshold of the inaction region for real prices and increase the nominal reset price in response to higher trend inflation. Here, the quit threshold $\hat{w}^{-}$ is independent of the drift due to limited commitment—the firm has no control over worker quits. From Proposition 5, the entry wage $\hat{w}^{*}$ is increasing in both the weighted sum of opportunity costs ($a + (1-a)\hat{r}\hat{U}$) and the drift ($\hat{g}$). We refer to the latter as the anticipatory effect: Workers anticipate higher future productivity and modify their search strategy accordingly. The following two cases illustrate this point by exploring two limiting behaviors of the anticipatory effect.

As $\hat{g} \rightarrow 0$ (Part 1), the equilibrium entry wage $\hat{w}^{*}$ is the same as in the case without drift; thus, $h(\hat{w}^{*}) = a$. As the drift increases, workers optimally search for a job with a higher entry wage. Therefore, the average wage in the economy increases above the weighted sum of opportunity costs; recall that $\hat{w}^{-}$ remains fixed. This results from the worker internalizing the trade-off whereby a higher wage implies (i) a reduced job-finding rate and (ii) a lower frequency of inefficient job separations and, thus, a longer expected match duration. As $\hat{g} \rightarrow$ ¥ (Part 2), the entry wage w^{*} becomes unresponsive to the drift because the job-finding rate becomes so small that it dominates the trade-off. Thus, the effect of the drift on the entry wage is bounded, in contrast to the reset price in Sheshinski and Weiss (1977). Finally, as seen in (22), the anticipatory effect gives workers a higher surplus share when $\hat{g} \rightarrow$ ¥ compared to $\hat{g} \rightarrow 0$.

Workers' lack of commitment gives them the option to quit, which implies the invariance of $\hat{w}^{-}$ to $\hat{g}$ and a decreased value of searching for a job. To see this, suppose a worker commits to some $\hat{w}^{-}$ as $d \rightarrow 0$. Then, the worker chooses a single instrument, namely the entry wage w^{*} to balance two objectives. On one hand, the worker chooses w^{*} to steer the rate of inefficient separations, which occur at a tenure of $(w^{*} - \hat{w}^{-})/\hat{g}$, as captured by the surplus channel. On the other hand, the worker chooses w^{*} close to the weighted sum of opportunity costs, as captured by the share channel. Since these objectives are conflicting, lack of commitment distorts both the expected match duration and job-finding rates in equilibrium.

2.5 Discussion of Model Assumptions

For expositional clarity, we imposed certain assumptions that are not essential for our theory of labor markets with inefficient turnover: (i) homotheticity of the home production technology and vacancy costs; (ii) no on-the-job search; and (iii) time dependence of wage setting.

Regarding (i), shocks to worker productivity Z_t affect agents' choices through the relative flow value of employment (W_t/Z_t), home production ($\tilde{B}Z_t$), and vacancy costs ($\tilde{K}Z_t$). In order to focus on the novel first margin, we abstract from the other two by assuming that home production and vacancy costs are homothetic in worker productivity. This assumption implies that all workers face the same job-finding rate and entry wage per efficiency unit and also rules out any efficient endogenous job separations. As a result, it allows us to focus on our economic mechanisms within—rather than between—worker types. It is straight-forward to relax these homotheticity assumptions in numerical simulations.

Regarding (ii), workers can reset their wages by undergoing a costly unemployment spell, similar to models with costly on-the-job search. Even allowing for on-the-job search, inefficient separations into unemployment would occur for analogous reasons. Qualitatively, on-the-job search would widen the inaction region since now employment yields an option value of receiving outside employment offers. However, a fully specified model of on-the-job search under wage rigidity would need to take a stance on the wage renegotiation protocol. Blanco and Drenik (2023) take a step in this direction.

Finally, regarding (iii), time-dependent wage setting à la Calvo (1983) is common in macroeconomic modeling (e.g., Erceg *et al.*, 2000) . Wages have been empirically documented to be reset at certain intervals (Taylor, 1979), synchronized within firms (Grigsby *et al.*, 2021), and subject to staggered institutional contracts (Adamopoulou *et al.*, 2022). While necessarily parsimonious, the current model of wage setting is motivated by these empirical regularities. Our assumptions allow for sharp analytical results, the essence of which we expect to carry over to alternative models of state-dependent wage setting that require numerical solution methods (cf. Alvarez *et al.*, 2016a,b; Auclert *et al.*, 2023).[12]

3 Aggregate Shocks in Labor Markets with Inefficient Turnover

How does inefficient turnover affect the transmission of aggregate shocks in the labor market? To answer this question, we extend our model to encompass shocks to aggregate productivity and monetary policy.

[12]E.g., Alvarez *et al.* (2016a) conclude that "for small aggregate shocks the [multiproduct pricing] models behave similarly irrespective of the nature of the sticky price friction" (p. 2850).

3.1 An Economy with Aggregate Shocks

To characterize the labor market response to a broad set of aggregate shocks, we modify the baseline model by introducing shocks to economy-wide TFPR, defined as $TFPR_t := A_t P_t$, where A_t denotes aggregate productivity and P_t denotes the aggregate price level. We assume that the logarithm of TFPR follows a Brownian motion with drift c and volatility z:

$$\mathrm{d}\log TFPR_t = c\,\mathrm{d}t + z\,\mathrm{d}\mathcal{W}_t^{\mathrm{TFP}},$$

where $\mathcal{W}_t^{\mathrm{TFP}}$ is a Wiener process. Studying shocks to TFPR has two benefits. On one hand, it allows us to study shocks to aggregate productivity, which are the predominant source of exogenous fluctuations studied in the quantitative macro-labor literature (Shimer, 2005a; Hall, 2005). On the other hand, it allows us to study shocks to the aggregate price level, which is endogenously determined in a monetary economy. We provide two alternative microfoundations for the aggregate price level when monetary policy is conducted either via money supply (Online Appendix II.1) or an interest rate-based Taylor rule (Online Appendix II.2). In both models, monetary policy moves the aggregate price level P_t and thus $TFPR_t$.[13] We assume that the vacancy posting cost $\tilde{K}Z_t$ and the value of home production $\tilde{B}Z_t$ are linear in TFPR. This assumption arises naturally when the TFPR shock is due to price movements as long as $\tilde{K}Z_t$ and $\tilde{B}Z_t$ are denominated in real terms. Under the interpretation of the TFPR shock being driven by productivity, this assumption can be justified by appealing to recruiting expenses incurred in the process of workers operating a recruiting technology (cf. Shimer, 2010).

The introduction of aggregate shocks requires minor adjustments to our framework. Aggregate shocks do not change the analysis, beyond altering the stochastic process for productivity and introducing dynamics in the aggregate state. Given fluctuations in TFPR, the relevant state variable becomes the *real wage-to-productivity ratio* $\hat{w} := w - z - \log TFPR$, which equals the worker's nominal wage w minus *worker productivity* $z + \log TFPR$. All policies $(\hat{w}^+, \hat{w}^*, \hat{w}^-)$ are then expressed in TFPR-adjusted terms. In addition, it will be useful to keep track of the negative of the cumulative shocks to $z + \log TFPR$ since the beginning of a spell of employment or unemployment, denoted

[13] By studying the labor market effects of monetary policy through the aggregate price level, we abstract from other important monetary policy channels (e.g., Hall, 2017; Kehoe *et al.*, 2019). Our goal is to highlight how monetary policy "greases the wheels of the labor market" (Tobin, 1972) by redistributing surplus between workers and firms.

$\mathsf{D}z := \hat{w} - \hat{w}^*$, which evolves as

$$\mathrm{d}\mathsf{D}z = -(g+c)\,\mathrm{d}t + s\,\mathrm{d}\mathcal{W}_t^z + z\,\mathrm{d}\mathcal{W}_t^{\mathrm{TFP}}.$$

Let $G_h(\mathsf{D}z)$ and $g^h(\mathsf{D}z)$ denote the cumulative distribution function (CDF) and probability density function (PDF), respectively, of cumulative worker productivity shocks within a spell in steady state. This distribution's support is given by $[-\mathsf{D}^-, \mathsf{D}^+]$, where $\mathsf{D}^- := \hat{w}^* - \hat{w}^-$ and $\mathsf{D}^+ := \hat{w}^+ - \hat{w}^*$. For any $k \in \mathbb{N}$, we define this distribution's k^{th} moment as $\mathbb{E}_h(\mathsf{D}z^k) := \int_{\mathsf{D}z} \mathsf{D}z^k \,\mathrm{d}G_h(\mathsf{D}z)$.

Our model implies a set of observable steady-state statistics. First, agents transition from employment to unemployment at rate s, from unemployment to employment at rate $f(\hat{q}(\hat{w}^*))$, and total employment is $\mathcal{E}$. Second, the model implies a distribution of log nominal wage changes between consecutive job spells $\mathsf{D}w$ and distributions of employment durations t^m and unemployment durations t^u.[14] We use subscript $\mathcal{D}$ to denote moments of these distributions observed in the microdata—e.g., $\mathbb{E}_{\mathcal{D}}[\cdot]$ and $\mathbb{V}ar_{\mathcal{D}}[\cdot]$ denote the mean and the variance of this distribution, respectively. These moments will be useful to define sufficient statistics for the effects of aggregate shocks on labor market outcomes.

3.2 Sufficient Statistics for Aggregate Employment and Real Wages

Starting from the steady state without aggregate shocks, we consider a small, unanticipated shock $z > 0$ to TFPR at time $t = 0$, so that $\log(TFPR_0) = \lim_{t\uparrow 0} \log(TFPR_t) + z$. We are interested in the economy's CIR of aggregate employment and TFPR-adjusted wages to such an aggregate shock.[15]

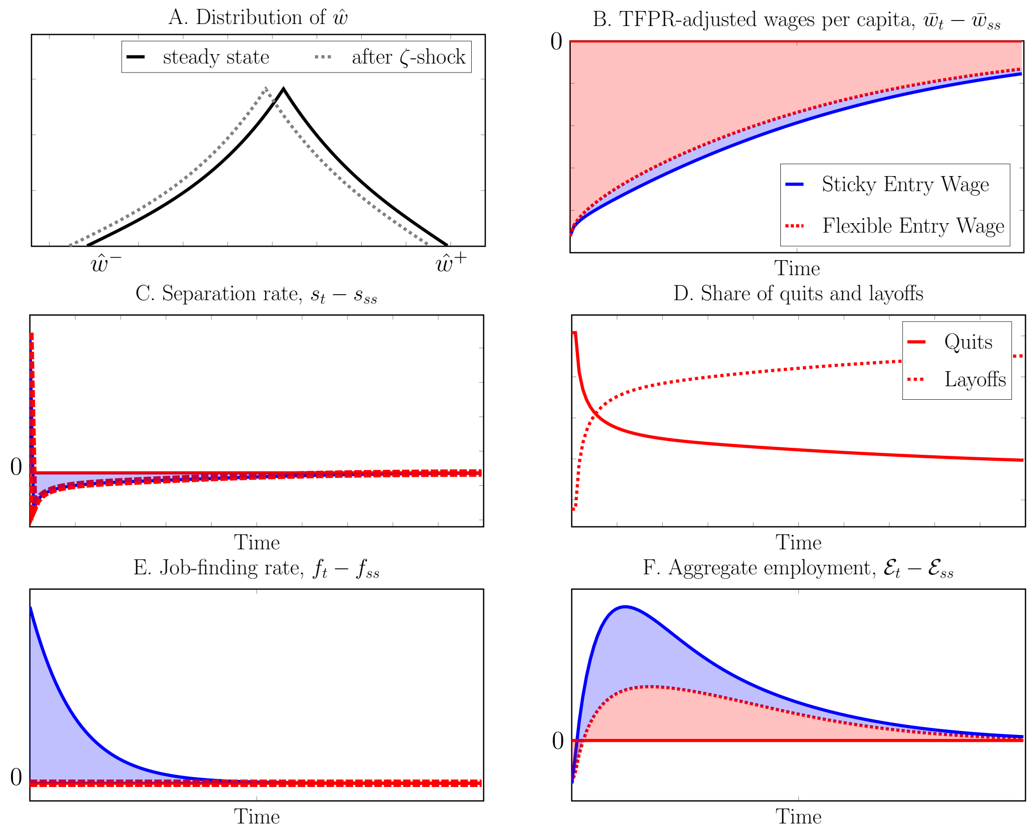
An Illustration. Figure 2 shows the evolution of key variables after an unanticipated one-off increase in TFPR—i.e., the price level or productivity. The distribution of real wage-to-productivity ratios $\hat{w}$ shifts to the left (Panel A), resulting in lower TFPR-adjusted wages per capita $\overline{w}_t := \int_0^1 \mathbb{1}[E_{it} = h] w_{it}\,\mathrm{d}i$ (Panel B), movements in the job-separation rate s_t (Panel C), the shares of quits and layoffs (Panel D), the job-finding rate f_t (Panel E), and aggregate employment $\mathcal{E}_t$ (Panel F).

While employed workers' wages are rigid, we allow for two polar cases guiding the wages of new matches, which are commonly considered a key determinant of the job-finding rate (Pissarides,

[14]While worker wages and productivities do not have a stationary distribution in *levels*, the distribution of wage *changes* across jobs is stationary. Although not necessary for our purposes, the former could be rendered stationary by assuming, for example, that workers permanently leave the labor force at a constant hazard rate.

[15]By the certainty equivalence principle, the IRF following an aggregate shock from the steady state with steady-state policies is equivalent to the solution based on a first-order perturbation of the model with business cycle fluctuations.

FIGURE 2. IMPULSE RESPONSE FUNCTIONS OF LABOR MARKET VARIABLES



Notes: Panel A shows the distribution of real wage-to-productivity ratios $\hat{w} := w_{it} - z_{it} - \log TFPR_t$ in steady state and after a TFPR shock of size z. Panels B–F show the IRFs of the average log TFPR-adjusted per-capita wage $\overline{w}_t$, the job-separation rate s_t, the shares of quits and layoffs, the job-finding rate f_t, and aggregate employment $\mathcal{E}_t$, respectively.
Source: Model simulations.

2009). In the first case of *flexible entry wages*, we assume that unemployed workers adjust their search behavior to the new TFPR level, so $\hat{w}^*$ remains at its steady-state level. Consequently, firms' TFPR-adjusted value of hiring is unaffected, so job-filling and job-finding rates remain unchanged (dashed line in Panel E). The only effect of the TFPR shock is to shift $\hat{w}$ in the inaction region, which affects the time path of endogenous job separations in the form of quits and layoffs (Panel D). Thus, employment dynamics under flexible entry wages are driven only by job-separation rates.

In the second case of *sticky entry wages*, we assume that unemployed workers are unaware of the shock realization at $t = 0$ and learn about it only after becoming employed. Given this lack of information, unemployed workers do not adjust their search behavior to the higher TFPR and keep searching for jobs that pay the old steady-state nominal wage-to-productivity ratio, which is $\hat{w}^* - z$ in real terms. Once they find a job, workers' search strategies incorporate their knowledge about the shock and search for jobs that pay the steady-state real wage $\hat{w}^*$. Since firms know about the shock realization, the job-finding rate is affected by the free-entry condition. Consequently, temporarily lower entry wages induce firms to post more vacancies and the job-finding rate increases (solid line in Panel E). In summary, employment dynamics under sticky entry wages are driven by both job-separation and job-finding rates.

The case of sticky entry wages is motivated by the empirical evidence that new-hire wages evolve similarly to incumbent workers within a firm at business cycle frequencies (Grigsby *et al.*, 2021) and that wages for new hires rarely change between successive vacancies at the same job (Hazell and Taska, 2022). Microfounding this assumption is beyond the scope of this paper.[16]

Impulse Responses. Our goal is to characterize the effects of a TFPR shock on aggregate employment $\mathcal{E}$. To this end, we define $IRF_x(z,t) := x_t - x_{ss}$ as the value of variable x at time t relative to its steady-state value x_{ss}, following an unanticipated one-off TFPR shock z at time 0. Following Alvarez *et al.* (2016a), we define the CIR of variable x to a TFPR shock z as

$$CIR_x(z) = \int_0^{¥} IRF_x(z,t)\,\mathrm{d}t,$$

which is simply the area under the IRF for all $t > 0$. The CIR summarizes in a single scalar the full path—i.e., the on-impact response and dynamics—of the labor market response to the TFPR shock. Therefore, the CIR can be interpreted as a *TFPR multiplier*. To illustrate the logic behind the CIR, suppose that there are no nominal rigidities so that the nominal wages of both newly hired and incumbent workers respond one-for-one to the shock. In this case, $IRF_x(z,t) = 0$ for all t and thus $CIR_x(z) = 0$ for $x \in \{\mathcal{E}, \overline{w}\}$, which reflects the fact that given our assumptions there are no consequences of TFPR shocks. With nominal rigidities, a TFPR shock affects both employment and wages, the magnitude of which is given by the CIR.

Next, we relate the economy's CIR to conventional labor market microdata. A key insight is that the CIR can be characterized only in terms of cross-sectional steady-state moments. Intuitively, changes in a worker's idiosyncratic productivity and changes in TFPR symmetrically affect the log-real-wage-to-productivity ratio $W_{it}/(Z_{it}TFPR_t)$, so the response of a match to idiosyncratic

[16]Since the steady-state entry wage is constrained efficient, any perturbation around that level has a second-order welfare effect on workers. Thus, the assumption of sticky entry wages could be replaced by any first-order cost of entry wage adjustments arising from imperfect knowledge about aggregate shocks, as in models of sticky information (Alvarez *et al.*, 2021), rational inattention (Maćkowiak and Wiederholt, 2009), and level-k thinking (Farhi and Werning, 2019). For notable models of rigid entry wages, see Fukui (2020) and Menzio (2022).

worker productivity changes in steady state can inform the aggregate effects of shocks to TFPR.

For ease of exposition, we assume $g + c = 0$ for the remainder of the main text. However, all results and their proofs in Appendix B refer to the general case with $g + c \gtreqless 0$. At the end of this section, we discuss the differences with the general case.

CIR of Employment with Flexible Entry Wages. To facilitate the exposition, we first present the case with flexible entry wages. Proposition 6 characterizes the CIR up to a first order.[17]

Proposition 6. *Up to first order, the CIR of employment under flexible entry wages is*

$$\frac{CIR_{\mathcal{E}}(z)}{z} = -(1-\mathcal{E}_{ss})\frac{\mathbb{E}_h[\mathrm{D}z]}{s^2} + o(z) \tag{23}$$

$$= \underbrace{\frac{1}{f(\hat{q}(\hat{w}^*))}}_{\text{avg. unemployment duration}} \times \underbrace{\frac{1}{\mathbb{V}ar_{\mathcal{D}}[\mathrm{D}w]}}_{\text{inverse dispersion}} \times \underbrace{\frac{1}{3}\mathbb{E}_{\mathcal{D}}\left[\mathrm{D}w\frac{\mathrm{D}w^2}{\mathbb{E}_{\mathcal{D}}[\mathrm{D}w^2]}\right]}_{\text{asymmetries}} + o(z). \tag{24}$$

Proof. See Appendix B.1. □

Let us begin by inspecting the result in equation (23) of Proposition 6, which expresses the CIR in terms of model objects. To build intuition, we consider two cases in which aggregate employment has a zero response to a TFPR shock. In the first case, all job separations are exogenous, so the IRF of the job-separation rate identically equals zero. In the second case, all job separations are endogenous but the mass of workers quitting exactly equals the mass of workers saved from layoffs along the entire IRF. In both cases, equation (23) features $\mathbb{E}_h[\mathrm{D}z] = 0$. As a third case, consider an economy with $\mathbb{E}_h[\mathrm{D}z] < 0$. Such an economy features a larger share of layoffs than quits, so a shock-induced reduction in TFPR-adjusted wages reduces the separation rate and increases total employment. Finally, the CIR is scaled by the steady-state unemployment rate, $1 - \mathcal{E}_{ss}$, which is informative of the steady-state job-finding rate $f(\hat{q}(\hat{w}^*))$ and thus the speed of (re-)matching.

Next, we inspect the result in equation (24), which expresses the CIR in terms of a sufficient statistic that depends only on the observed distributions of wage changes across jobs and unemployment duration. This sufficient statistic is composed of three terms: (i) the average unemployment duration; (ii) the inverse of the dispersion of wage changes; and (iii) a measure of the asymmetries of the wage change distribution. Note that these moments summarize the entire distribution of workers over the inaction band, not just the mass of workers at the separation thresholds. Each of the three terms in the CIR plays an intuitive role. First, the steady-state unemployment rate

[17] That is, $CIR_x(z) = CIR_x(0) + (CIR_x)'(0)z + o(z^2)$, where $CIR_x(0) = 0$.

scales the aggregate employment response. Second, a larger dispersion of wage changes indicates a wider inaction region or matches that are more resilient to shocks, which is inversely related to the share of endogenous separations and responsiveness of aggregate employment to a given impulse. Third, the measure of asymmetries reflects the relative distances of the separation thresholds $\hat{w}^-$ and $\hat{w}^+$ from the entry wage $\hat{w}^*$ and thus the relative incidence of quits versus layoffs. For example, consider a distribution of nominal wage changes that is positively skewed—i.e., featuring a large mass of workers who experience small wage cuts due to a relatively high layoff risk. In this example, a positive shock to TFPR reduces the relative cost of wages, leading firms to reduce layoffs and thereby increasing aggregate employment.

Proposition 6 also shed new light on the conventional wisdom whereby fluctuations in the job separation rate are not the primary driver of aggregate employment dynamics (e.g., Shimer, 2005b). In the context of a TFPR shock, equation (23) points to conditions under which aggregate employment fluctuations due to endogenous job separations are either small or large.[18] Moreover, it allows us to verify those conditions in the data. Given the conventional wisdom, one might be tempted to conclude that sticky wages cannot lead to significant inefficiencies at the micro and macro level. However, equation (23) shows that the CIR of aggregate employment can be small despite the presence of inefficient separations at the micro level. Thus, time-series data on aggregate job separations cannot be used to assess the incidence of inefficient turnover. Instead, in order to do so, labor market microdata is needed.

CIR of Employment with Sticky Entry Wages. Having characterized the aggregate employment response under flexible entry wages, Proposition 7 describes the case of sticky entry wages.

Proposition 7. *Up to first order, the CIR of employment under sticky entry wages is*

$$\frac{CIR_{\mathcal{E}}(z)}{z} = -(1-\mathcal{E}_{ss})\left[\frac{\mathbb{E}_h[\mathrm{D}z]}{s^2} - \frac{1}{f(\hat{q}(\hat{w}^*))+s}\left[\underbrace{\frac{1-a}{a}\left[\frac{h'(\hat{w}^*)}{(1-h(\hat{w}^*))} - \frac{\mathcal{T}'_{\hat{w}}(\hat{w}^*,\hat{r})}{\mathcal{T}(\hat{w}^*,\hat{r})}\right]}_{\text{job-finding effect}} - \underbrace{\frac{\mathcal{T}'_{\hat{w}}(\hat{w}^*,0)}{\mathcal{T}(\hat{w}^*,0)}}_{\substack{\text{new-hires'}\\ \text{separation}\\ \text{effect}}}\right]\right] + o(z) \tag{25}$$

$$= -(1-\mathcal{E}_{ss})\left[\frac{\mathbb{E}_h[\mathrm{D}z]}{s^2} - \frac{1}{f(\hat{q}(\hat{w}^*))+s}\left[\frac{h'(\hat{w}^*)}{h(\hat{w}^*)} + \frac{\mathcal{T}'_{\hat{w}}(\hat{w}^*,\hat{r})}{\mathcal{T}(\hat{w}^*,\hat{r})} - \frac{\mathcal{T}'_{\hat{w}}(\hat{w}^*,0)}{\mathcal{T}(\hat{w}^*,0)}\right]\right] + o(z). \tag{26}$$

[18] For example, all else equal, the rate of inefficient job separations is more responsive to TFPR shocks for larger TFPR trends c. Alternatively, following a sequence of negative productivity shocks, an inflationary shock reduces the incidence of inefficient job separations due to firings (see Blanco *et al.*, 2022b, for empirical evidence consistent with this theoretical result).

Proof. See Appendix B.2. □

Focusing first on equation (25) of Proposition 7, the first term in brackets reflects the same forces at play in the CIR under flexible entry wages. The remaining terms in brackets capture two new mechanisms at play when entry wages are sticky. First, the *job-finding effect* captures the fact that lower TFPR-adjusted entry wages increase the firm's surplus share (i.e., $h'(\hat{w}^*)/(1-h(\hat{w}^*))$) but also could affect the expected match duration (i.e., $\mathcal{T}'_{\hat{w}}(\hat{w}^*,\hat{r})/\ \mathcal{T}(\hat{w}^*,\hat{r})$) and the match surplus, both of which shape firms' incentives to post vacancies. Second, the *new hires' separation effect* captures the fact that lower TFPR-adjusted entry wages directly affect the separation rate of initially unemployed workers (i.e., $\mathcal{T}'_{\hat{w}}(\hat{w}^*,0)/\ \mathcal{T}(\hat{w}^*,0)$).

Next, we move to equation (26), which comes from combining (25) with the optimality condition for $\hat{w}^*$ in (17). This step's goal is to take advantage of the fact that workers internalize the effect of entry wages on net job creation. To shed light on the two key elasticities appearing in equation (26), we first show that $\mathcal{T}'_{\hat{w}}(\hat{w}^*,\hat{r})/\mathcal{T}(\hat{w}^*,\hat{r}) - \mathcal{T}'_{\hat{w}}(\hat{w}^*,0)/\mathcal{T}(\hat{w}^*,0) \approx 0$. While this property trivially holds when $\hat{r} \downarrow 0$, the following lemma shows that the elasticity of the expected match duration to the entry wage is independent of the discount factor $\hat{r}$ up to second order.

Lemma 2. *Up to a second-order approximation of the match duration $\mathcal{T}(\hat{w},\hat{r})$ around $\hat{w} = \hat{w}^*$ and for all $\hat{r}$, we have $\mathcal{T}'_{\hat{w}}(\hat{w}^*,\hat{r})/\mathcal{T}(\hat{w}^*,\hat{r}) = (\mathsf{D}^+ - \mathsf{D}^-)/(\mathsf{D}^+\mathsf{D}^-)$.*

Proof. See Appendix B.3. □

Lemma 2 shows that the elasticity of match duration is a function of the quit and layoff thresholds expressed in terms of cumulative shocks to worker productivity, D^- and D^+. Thus, the key sufficient statistic for the effect of lower entry wages on job creation in equation (26) is $h'(\hat{w}^*)/h(\hat{w}^*)$. From this, one may be inclined to conclude that the prevalence of inefficient separations cannot be an important determinant of aggregate job creation. However, we find that this is not generally the case. The following result shows this by characterizing the elasticity of the worker's share to changes in the entry wage.

Proposition 8. *The rent-sharing elasticity $h'(\hat{w}^*)/h(\hat{w}^*)$ satisfies the following properties:*

1. *If $\mathsf{D}^-, \mathsf{D}^+ \to$ ¥, then*

$$\frac{h'(\hat{w}^*)}{h(\hat{w}^*)} = \frac{a + (1-a)\hat{r}\hat{U}}{a(1-\hat{r}\hat{U})}. \tag{27}$$

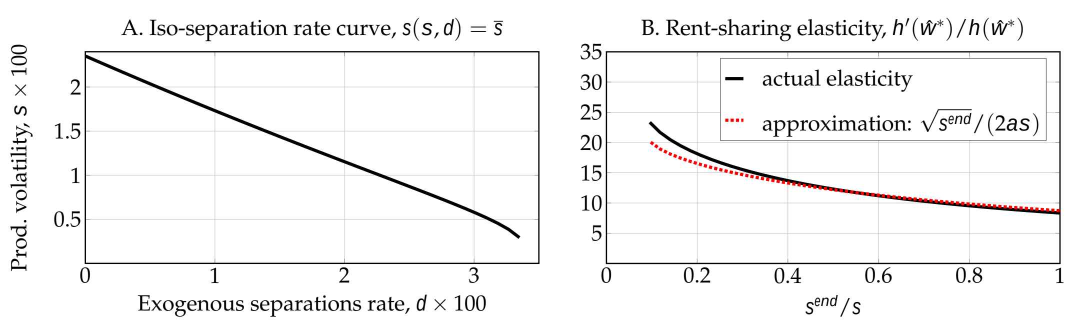
2. *If* $\mathsf{D}^- = \mathsf{D}^+$ *and* D^+ *is small enough, then*

$$\frac{h'(\hat{w}^*)}{h(\hat{w}^*)} = \frac{\sqrt{s^{end}}}{2as}. \tag{28}$$

Proof. See Appendix B.4. □

Proposition 8 characterizes the rent-sharing elasticity $h'(\hat{w}^*)/h(\hat{w}^*)$ under two polar cases, namely as the inaction region grows infinitely wide (Part 1) and for a symmetric and narrow enough inaction region (Part 2). The two results are best explained with the aid of Figure 3, which we construct in two steps. First, we set $d = 0$ and calibrate the model to match the U.S. economy's job-finding rate $\overline{f}$ and separation rate $\overline{s}$ with a replacement ratio $\tilde{B}$ of 0.29. We purposely choose a so that $\mathsf{D}^+ = \mathsf{D}^-$ and thus $\mathcal{T}'_{\hat{w}}(\hat{w}^*, \hat{r}) = 0$. Second, for different levels of the exogenous separation rate d, we find the productivity volatility s as a function of d that keeps the total separation rate constant. The objective of this exercise is to vary the fraction of endogenous job separations s^{end}/s from 0 to 100 percent while keeping the opportunity cost $\hat{r}\hat{U}$ and the total separation rate fixed by construction. Panel A of the figure shows combinations of d and s that constitute the "iso-separation rate curve" defined by $s(d, s) = \overline{s}$, while Panel B plots the rent-sharing elasticity $h'(\hat{w}^*)/h(\hat{w}^*)$ as a function of the share of endogenous job separations s^{end}/s.

FIGURE 3. ISO-SEPARATION RATE CURVE AND THE ELASTICITY OF RENT SHARING



Notes: Panel A shows the iso-separation rate curve defined by $s(d, s) = \overline{s}$. Panel B shows the rent-sharing elasticity as a function of the share of endogenous separations (black solid line) and compares it to an approximation of the rent-sharing elasticity given by $\sqrt{s^{end}}/(2as)$ based on equation (28). Note that the productivity volatility s is a function of d derived from the iso-separation rate curve. The parameter values for $d = 0$ are $(g + c, s, r, a, \tilde{K}, d, \tilde{B}) = (0, 0.0235, 0.0048, 0.452, 1.87, 0, 0.29)$. The steady-state targets for this calibration are $(f(q(\hat{w}^*)), s) = (0.55, 0.034)$ with $\mathsf{D}^+ = \mathsf{D}^-$. *Source:* Model simulations.

Consider the limiting case as $d \to \overline{s}$ (i.e., $s^{end}/s \to 0$), so that all separations are exogenous, as in Part 1 of Proposition 8. Then, a marginal increase in the entry wage increases workers' surplus share according to equation (27), reflecting the well-known result that, absent inefficient turnover, the rent-sharing elasticity is inversely proportional to the flow surplus $1 - \hat{r}\hat{U}$ (Shimer, 2005a). As the share of inefficient separations (i.e., s^{end}/s) increases in Panel B of Figure 3, the rent-sharing elasticity (black solid line) decreases due to a novel mechanism in our framework with sticky entry wages. A higher entry wage increases the layoff probability and decreases the quit probability. By construction, the expected duration of the match does not change, so match surplus is constant. As workers make optimal quit decisions, a marginally lower quit probability leaves their value unchanged due to an envelope condition (i.e., $\hat{W}'(\hat{w}^-) = 0$). But a marginal increase in the layoff probability reduces the worker's value, since the firm makes layoff decisions. Therefore, the rent-sharing elasticity decreases in the share of endogenous job separations, which the following section shows how to measure using conventional labor market microdata.

Model Extension: Staggered Wage Renegotiations. In Online Appendix II.3, we extend our model to feature staggered wage renegotiations, which we assume to follow a Nash bargaining protocol with worker weight a and to occur at rate $d^r \geq 0$ à la Calvo (1983). This allows us to convexify between models of rigid and flexible wages. Under this generalization, we derive all main results, including the CIR of aggregate employment.[19]

4 Mapping the Model to Labor Market Microdata

A key feature of the previous section's sufficient statistics is that they can be expressed in terms of measurable labor market moments. In this section, we show how to connect our model to the data in two steps. First, we use labor market microdata on wage changes between jobs to recover the unobserved distribution of cumulative productivity changes. Second, we link the distribution of cumulative productivity changes to the prevalence of inefficient job separations. For simplicity, we focus on the case with $g + c = 0$. Appendix C presents the general case.

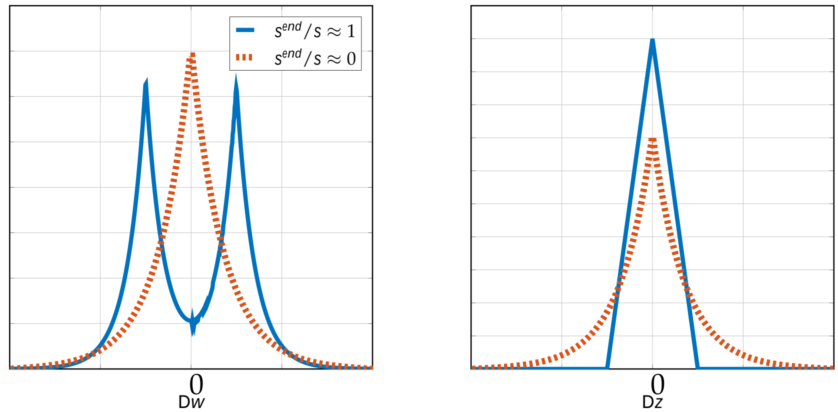
Intuitively, how are observed wage changes between jobs informative about the prevalence of inefficient job separations? Figure 4 illustrates their distribution, $l^w(\mathsf{D}w)$ (Panel A), and that of cumulative productivity shocks in employment, $g^h(\mathsf{D}z)$ (Panel B), for each of two extreme cases.

In the first case, when most job separations are endogenous (solid blue line), then most separated workers experienced cumulative productivity shocks in employment of either $-\mathsf{D}^-$ or D^+. As a result, the distribution of wage changes of laid-off workers is concentrated around $-\mathsf{D}^-$, while

[19] As an additional extension of interest, we can model worker- and firm-specific costs of unilateral separations, which allows us to convexify between models of full and no commitment. Results are available upon request.

FIGURE 4. DISTRIBUTIONS OF WAGE CHANGES AND CUMULATIVE SHOCKS

A. Wage changes between jobs, $\mathsf{D}w$ B. Cumulative productivity shocks in employment, $\mathsf{D}z$



Notes: The figure plots the distribution of wage changes between jobs $l^w(\mathsf{D}w)$ and the distribution of cumulative productivity shocks in employment $g^h(\mathsf{D}z)$ for two calibrations. In the first calibration, we set $(\mathsf{D}^-, \mathsf{D}^+, g + c, s, d, f(q(\hat{w}^*))) = (0.05, 0.05, 0, 0.02, 0, 0.5)$ so that $s^{end}/s \approx 1$ (blue solid line). In the second calibration, we set $(\mathsf{D}^-, \mathsf{D}^+, g + c, s, d, f(q(\hat{w}^*))) = (0.2, 0.2, 0, 0.1, 0.04, 0.5)$ so that $s^{end}/s \approx 0$ (red dashed line). *Source:* Model simulations.

that of workers who quit is concentrated around D^+. This results in a bimodal distribution of wage changes between jobs, with dispersion around the two modes caused by productivity shocks during unemployment.

In the second case, when most job separations are exogenous (dashed red line), then most separated workers experienced cumulative productivity shocks in employment close to zero—i.e., away from the two endogenous separation thresholds. With a constant job-finding probability during unemployment, the distribution of wage changes between jobs mimics the distribution of cumulative productivity shocks in employment, which is symmetric and single peaked at zero.

More generally, we provide equilibrium conditions characterizing the steady-state distributions of cumulative productivity shocks $g^h(\mathsf{D}z)$ and $g^u(\mathsf{D}z)$ in Appendix C.1. The following result shows how to recover the distribution of cumulative productivity shocks in employment, $g^h(\mathsf{D}z)$.

Proposition 9. *Given the volatility of workers' productivity shocks,*

$$s^2 = \frac{\mathbb{E}_{\mathcal{D}}[(\mathsf{D}w)^2]}{\mathbb{E}_{\mathcal{D}}[t^m + t^u]}, \tag{29}$$

the distribution of workers' cumulative productivity shocks can be expressed as

$$g^h(\mathsf{D}z) = s\mathcal{E}\left[\int_{-\mathsf{D}^-}^{\mathsf{D}z} \frac{2(\mathsf{D}z - y)}{s^2}\bar{g}^h(y)\,\mathrm{d}y + \overline{G}^h(-\mathsf{D}^-)\frac{2(\mathsf{D}z + \mathsf{D}^-)}{s^2}\right], \tag{30}$$

where

$$\overline{G}^h(\text{D}z) = \frac{s^2}{2f(\hat{q}(\hat{w}^*))}\frac{\text{d}l^w(-\text{D}z)}{\text{d}z} - [1 - L^w(-\text{D}z)] \tag{31}$$

is the distribution of $\text{D}z$ *conditional on a job separation, and* $L^w(\text{D}w)$ *is the CDF corresponding to the PDF of wage changes between jobs,* $l^w(\text{D}w)$.

Proof. See Appendix C.2. □

Equation (29) of Proposition 9 states that the volatility of productivity s equals the dispersion of wage changes between jobs, $\mathbb{E}_{\mathcal{D}}[(\text{D}w)^2]$, divided by the average time between two consecutive jobs' starting times, $\mathbb{E}_{\mathcal{D}}[t^m + t^u]$. Next, in order to recover the distribution of $\text{D}z$ conditional on a job separation, we exploit h-to-u and u-to-h worker flows. Consider a worker who at time t_0 starts a job with wage w_{t_0}, at time $t_0 + t^m$ separates, and at time $t_0 + t^m + t^u$ finds a new job with wage $w_{t_0+t^m+t^u}$. This worker's wage change between jobs is given by

$$\begin{aligned}
\text{D}w &= w_{t_0+t^m+t^u} - w_{t_0}, && (32)\\
&= \underbrace{(w_{t_0+t^m+t^u} - z_{t_0+t^m+t^u})}_{=\hat{w}^*} - \underbrace{(w_{t_0} - z_{t_0})}_{=\hat{w}^*} + \underbrace{z_{t_0+t^m+t^u} - z_{t_0}}_{=\text{D}z \text{ after } h\text{-}u\text{-}h \text{ transition}}, && (33)\\
&= \underbrace{\hat{w}^* - \hat{w}^*}_{=0} + \underbrace{z_{t_0+t^m} - z_{t_0}}_{\text{D}z|h\text{-}u \text{ transition starting from } z_{t_0}} + \underbrace{z_{t_0+t^m+t^u} - z_{t_0+t^m}}_{\text{D}z|u\text{-}h \text{ transition starting from } z_{t_0+t^m}}. && (34)
\end{aligned}$$

Equation (32) gives the definition of $\text{D}w$. Next, equation (33) adds and subtracts $z_{t_0+t^h+t^u} - z_{t_0}$ before grouping terms into the wage-to-productivity ratio in the old job, the wage-to-productivity ratio in the new job, and the cumulative productivity shocks between the starting dates of the two jobs. Then, equation (34) adds and subtracts $z_{t_0+t^m}$ before applying the definition of $\hat{w}^*$ and that of $\text{D}z$. In summary, the wage change across jobs equals the sum of three random variables: (i) the difference of entry wage-to-productivity ratios across jobs, which is identically zero; (ii) $\text{D}z$ conditional on a job separation starting from z_{t_0}; and (iii) $\text{D}z$ conditional on finding a new job, which is independent of productivity z_t for $t \in (t_0 + t^m, t_0 + t^m + t^u)$. Exploiting this independence, we can use data on $\text{D}w$ to infer the distribution of the second term, which is given by (31). Finally, the distribution of cumulative productivity shocks in (30) can be derived from (31) by exploiting ergodicity—i.e., the cross-sectional distribution of cumulative shocks can be deduced from the distribution of shocks experienced during completed job spells.

Proposition 10. *The share of inefficient job separations is given by*

$$\frac{s^{end}}{s} = \frac{\frac{s^2}{2\mathcal{E}}\left[\lim_{\Delta z \downarrow -\Delta^-}(g^h)'(\Delta z) - \lim_{\Delta z \uparrow \Delta^+}(g^h)'(\Delta z)\right]}{s}. \tag{35}$$

Proof. The proof follows from the conditions in Appendix C.1. □

Equation (35) in Proposition 10 expresses the share of inefficient job separations, s^{end}/s, in terms of the distribution of cumulative productivity shocks, $g^h(\Delta z)$, recovered in Proposition 9 above.

Discussion of Assumptions. We conclude with a discussion of some assumptions underlying the mapping between key model objects and the data, which could be relaxed: (i) the threshold nature of policies; (ii) the absence of other sources of wage changes; (iii) specifics of the productivity process; and (iv) the absence of alternative sources of heterogeneity.

Regarding (i), that the job-separation rate is δ for $\Delta z_t \in (-\Delta^-, \Delta^+)$ and ∞ for $\Delta z_t \in \{-\Delta^-, \Delta^+\}$ is not crucial and can be replaced with a general job-separation hazard, as in Alvarez *et al.* (2021).

Regarding (ii), we have ignored other sources of wage changes, such as those due to job-to-job moves. This assumption could be relaxed following the methodology in Baley and Blanco (2022).

Regarding (iii), our assumption of a particular stochastic process for Δz_t can be empirically tested and generalized, as in Baley and Blanco (2021). For example, it would be straightforward to let the parameters of the productivity process depend on a worker's employment state. What is critical is that the data contain enough information to recover productivity changes during unemployment. In our model, lack of selection in job finding and the inferred productivity process in employment together yield this result.

Finally, regarding (iv), we abstract from firm and match productivity shocks. This simplification is motivated by empirical evidence suggesting that worker heterogeneity explains the largest share of wage dynamics (Guiso *et al.*, 2005; Friedrich *et al.*, 2024; Engbom *et al.*, 2023). Additionally, a benefit of focusing on worker heterogeneity is that it allows our model to parsimoniously speak to both worker quits and firm layoffs—both of which are empirically salient (Elsby *et al.*, 2010). Conversely, a model with only firm- or match-specific heterogeneity would predict no worker quits because workers' flow value of employment and flow value of nonemployment would both be constant within a wage segment, even in the presence of staggered renegotiations. Adding other sources of heterogeneity would require different data (e.g., linked employer-employee records), different model ingredients (e.g., a multi-worker firm wage-setting protocol), and a different identification strategy (e.g., exploiting synchrony in coworker outcomes). Future work could calibrate richer models with multiple dimensions of empirically disciplined heterogeneity.

5 Conclusion

There is mounting empirical evidence that not all job separations can be rationalized using bilaterally efficient models. To understand the sources and consequences of inefficient turnover, we develop a theory of labor markets with four features: search frictions, productivity fluctuations, wage rigidity due to staggered renegotiations, and two-sided lack of commitment to remaining in a match. A defining feature of our theory is the distinction between quits and layoffs as two separate equilibrium outcomes following a voluntary-involuntary interpretation. Inefficient turnover manifests itself not only in job separations but also in job creation and wage determination. We first characterize the unique BRE of this model. We then derive sufficient statistics for the labor market response to aggregate shocks based on conventional labor market microdata on wage changes between jobs.

While the parsimony of this framework is useful in delineating several novel theoretical insights, an empirically grounded quantification may require several extensions. Adding nonhomotheticities (e.g., in home production), alternative sources of heterogeneity (e.g., match productivity shocks), and additional quit motives (e.g., on-the-job search) could yield efficient endogenous separations. Blanco and Drenik (2023) take a step in this direction. Furthermore, adding asymmetric renegotiation costs would add a state-dependent motive for wage adjustments and allow the model to match asymmetries in the empirical distribution of wage changes (e.g., Blanco *et al.*, 2022a). We expect that many of our insights will carry over to such richer environments. Incorporating these and other features into a unified framework with empirical discipline will allow future work to assess the implications of inefficient turnover in the labor market for issues including monetary policy (e.g., state dependent employment effects), fiscal policy (e.g., UI), and labor market regulations (e.g., severance pay).

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A Theory of Labor Markets with Inefficient Turnover

Appendix

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Contents of the Appendix

A Proofs for Section 2: A Model of Labor Markets with Inefficient Turnover

Notation. We use the following mathematical notation throughout the Appendix and Online Appendix.

1. $H^l(\mathbb{R})$: Sobolev space; i.e., $H^l(\mathbb{R}) \subset L^2(\mathbb{R})$ and its weak derivatives up to order l have a finite L^p norm.

2. Characteristic operator $\mathcal{A}$: Given a function $f : \mathbb{R} \to \mathbb{R}$ and a diffusion process $\{x_t\}$, the characteristic operator of X is given by $\mathcal{A}f = \lim_{U \downarrow x} \frac{\mathbb{E}[f(X_{t_U}|x_0=x] - f(x)]}{\mathbb{E}[t_U|x_0=x]}$.

3. Let $u, v : \mathbb{R} \to \mathbb{R}$, $(u, v) = \int_{\mathbb{R}} u(x)v(x)\,\mathrm{d}x$ denotes the inner product in the Hilbert space $L^2(\mathbb{R})$ with the Lebesgue measure, and $||u|| = \left(\int u(x)^2\,\mathrm{d}x\right)^{1/2}$.

4. $a(u, v)$ is a bilinear continuous form. We say $a(u, v)$ is coercive if $a(u, u) \geq a||u||^2$.

5. We use $a \wedge b$ to denote the minimum between a and b. We also use the notation $[x]^+ = max\{0, x\}$.

Some Useful and Known Results. Our mathematical arguments will make extensive use of the following useful and known results.

Proposition A.1. *Let $\mathcal{A}$ be the characteristic operator of $\{X_t\}$ with $X_t \in \mathbb{R}^n$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a twice differentiable function with compact (i.e., bounded and closed in $\mathbb{R}$) support, $support(f) = \{x : f(x) \neq 0\}$. If t is a stopping time with $\mathbb{E}_x[t] <$ ¥, then*

$$\mathbb{E}_x[f(x_t)] = f(x) + \mathbb{E}_x\left[\int_0^t \mathcal{A}f(X_t)\,\mathrm{d}t\right]. \tag{A.1}$$

Moreover, if t is the first exit time of a bounded set, then (A.1) holds for any twice differentiable function.

Proof. This is Dynkin's formula, the proof of which can be found in Øksendal (2007). □

Proposition A.2. *Let x_t be a strong Markov process, t be a stopping time measurable with the filtration generated by x_t, and t^d an exponential random variable independent of t. Then*

$$\mathbb{E}\left[\int_0^{t\wedge t^d} e^{-rt} f(x_t)\,\mathrm{d}t + e^{-r(t\wedge t^d)} g(x_{t\wedge t^d})\middle|\, x_0 = x\right] = \mathbb{E}\left[\int_0^t e^{-(r+d)t}[f(x_t) + dg(x_t)]\,\mathrm{d}t + e^{-(r+d)t} g(x_t)\middle|\, x_0 = x\right].$$

Proposition A.3. *Let V be a Hilbert space and H a closed convex set. Assume that $a(u,v)$ with $u,v \in V$ is a coercive bilinear continuous form. Then, there exists a unique solution to $a(u, v-u) \geq (f, v-u), \forall v \in H, u \in H$, where f belongs to the dual of V.*

Proof. See Lions and Stampacchia (1967). □

Proposition A.4. *Let $(V,(\cdot,\cdot))$ be a Hilbert space and $V_+ \subset V$ a closed convex cone satisfying $V_+ = \{x \in V \text{ such that } (x,y) \geq 0\ \forall y \in V_+\}$. We say that $x \geq y$ according to the vector ordering $\geq$ if and only if $x - y \in V_+$ with $x, y \in V$. Let $T : V \to V$ be an increasing map from V into itself. Suppose that there exists a $\underline{x}, \overline{x} \in V$ with $\underline{x} \leq \overline{x}$, $\underline{x} \leq T(\underline{x})$, $T(\overline{x}) \leq \overline{x}$. Then, the subset of fixed points x^* of T satisfying $\underline{x} \leq x^* \leq \overline{x}$ is nonempty and has a larger and smallest element.*

Proof. See the proof of Proposition 2 of Chapter 15 on page 539 of Aubin (2007). □

We will use Propositions A.3 and A.4 in the proof of Proposition 1. Proposition A.3 is used to show the existence of the best response function and its associated value function for each agent. Notice that we are solving the differential equations associated with the HJB equations using a quasi-variational approach—i.e., we are after the weak solution of the differential equation. Proposition A.4 is our main tool to show the existence of the nontrivial Nash Equilibrium. Notice that while we impose monotonicity from the order generated with the positive cone, we do not impose that the set V is a complete lattice. Thus, we are not invoking an order-theoretical approach to showing the existence of a fixed point. The reason is that the completeness property (i.e., all subsets of V have both a supremum and an infimum) is hard to satisfy in the space of functions. The best example of V and V_+ are $L^2(\mathbb{R})$—integrable functions using the Lebesgue measure—and the nonnegative function subset of this Hilbert space.

A.1 Proof of Lemma 1

The equilibrium conditions are:

$$
\begin{aligned}
ru(z) &= \tilde{B}e^z + g\frac{¶u(z)}{¶z} + \frac{s^2}{2}\frac{¶^2u(z)}{¶z^2} + \max_w f(q(z;w))[h(z;w) - u(z)], \qquad \text{(A.2)}\\
0 &= \min\left\{\tilde{K}e^z - q(q(z;w))j(z;w), q(z;w)\right\},\\
rh(z;w) &= \begin{cases} \max\left\{e^w + g\frac{¶h(z;w)}{¶z} + \frac{s^2}{2}\frac{¶^2h(z;w)}{¶z^2} + d\left[u(z) - h(z;w)\right],\ ru(z)\right\} & \forall z \in \mathcal{Z}^{j*}(w),\\ ru(z) & \forall z \in (\mathcal{Z}^{j*}(w))^c,\end{cases}
\end{aligned}
$$

$$
rj(z;w) = \begin{cases} \max\left\{e^z - e^w + g\frac{\partial j(z;w)}{\partial z} + \frac{s^2}{2}\frac{\partial^2 j(z;w)}{\partial z^2} - dj(z;w)\,,\, 0\right\} & \forall z \in \mathcal{Z}^{h*}(w), \\ 0 & \forall z \in (\mathcal{Z}^{h*})(w)^c, \end{cases}
$$

$$
\mathcal{Z}^{j*}(w) = int\left\{z \in \mathbb{R} : j(z;w) > 0 \text{ or } e^z - e^w > 0\right\}, \tag{A.3}
$$

$$
\mathcal{Z}^{h*}(w) = int\left\{z \in \mathbb{R} : h(z;w) > u(z) \text{ or } 0 < e^w - ru(z) + g\frac{\partial u(z)}{\partial z} + \frac{s^2}{2}\frac{\partial^2 u(z)}{\partial z^2}\right\}, \tag{A.4}
$$

$$
j(\cdot;w) \in \mathbb{C}^1(\mathcal{Z}^{h*}(w)) \cap \mathbb{C}(\mathbb{R}),\;\; h(\cdot;w) \in \mathbb{C}^1(\mathcal{Z}^{j*}(w)) \cap \mathbb{C}(\mathbb{R}). \tag{A.5}
$$

The equilibrium conditions in the normalized state space $\hat{w}$ are:

$$
\hat{r}\hat{U} = \tilde{B} + \max_{\hat{w}} f(\hat{q}(\hat{w}))\hat{W}(\hat{w}), \tag{A.6}
$$

$$
0 = \min\left\{\tilde{K} - q(\hat{q}(\hat{w}))\hat{J}(\hat{w}), \hat{q}(\hat{w})\right\},
$$

$$
\hat{r}\hat{W}(\hat{w}) = \begin{cases} \max\{0, e^{\hat{w}} - \hat{r}\hat{U} - \hat{g}\frac{\partial \hat{W}(\hat{w})}{\partial \hat{w}} + \frac{s^2}{2}\frac{\partial^2 \hat{W}(\hat{w})}{\partial \hat{w}^2} - d\hat{W}(\hat{w})\} & \forall \hat{w} \in \hat{\mathcal{Z}}^{j*} \\ 0 & \forall \hat{w} \in (\hat{\mathcal{Z}}^{j*})^c \end{cases}
$$

$$
\hat{r}\hat{J}(\hat{w}) = \begin{cases} \max\{0, 1 - e^{\hat{w}} - \hat{g}\frac{\partial \hat{J}(\hat{w})}{\partial \hat{w}} + \frac{s^2}{2}\frac{\partial^2 \hat{J}(\hat{w})}{\partial \hat{w}^2} - d\hat{J}(\hat{w})\} & \forall \hat{w} \in \hat{\mathcal{Z}}^{h*} \\ 0 & \forall \hat{w} \in (\hat{\mathcal{Z}}^{h*})^c \end{cases}
$$

$$
\hat{\mathcal{Z}}^{h*} := int\left\{\hat{w} \in \mathbb{R} : \hat{W}(\hat{w}) > 0 \text{ or } (e^{\hat{w}} - \hat{r}\hat{U}) > 0\right\}, \tag{A.7}
$$

$$
\hat{\mathcal{Z}}^{j*} := int\left\{\hat{w} \in \mathbb{R} : \hat{J}(\hat{w}) > 0 \text{ or } \left(1 - e^{\hat{w}}\right) > 0\right\}, \tag{A.8}
$$

$$
\hat{J} \in \mathbb{C}^1(\hat{\mathcal{Z}}^{h*}) \cap \mathbb{C}(\mathbb{R}),\;\; \hat{W} \in \mathbb{C}^1(\hat{\mathcal{Z}}^{j*}) \cap \mathbb{C}(\mathbb{R}), \tag{A.9}
$$

where $\hat{w} = w - z$, $\hat{r} = r - g - s^2/2$ and $\hat{g} = g + s^2$.

Lemma 1. *Assume that values $(u(z), h(z;w), j(z;w), q(z;w))$ and policies $(w^*(z), \mathcal{Z}^{j*}(w), \mathcal{Z}^{h*}(w))$ are a recursive equilibrium—i.e., they satisfy conditions (A.2)–(A.5)—, then*

$$
\left(\hat{U}, \hat{J}(w - z), \hat{W}(w - z), \hat{q}(w - z), \hat{w}^*\right) = \left(\frac{u(z)}{e^z}, \frac{j(z;w)}{e^z}, \frac{h(z;w) - u(z)}{e^z}, q(z;w), w^*(z) - z\right).
$$

satisfy (A.6)–(A.9) with continuation sets $\hat{\mathcal{Z}}^{h*}$ and $\hat{\mathcal{Z}}^{j*}$ given by (A.7)–(A.8). Moreover, if $\left(\hat{U}, \hat{J}(\hat{w}), \hat{W}(\hat{w}), \hat{q}(\hat{w})\right)$ and policies $(\hat{w}^*, \hat{\mathcal{Z}}^{j*}, \hat{\mathcal{Z}}^{h*})$ satisfy (A.6)-(A.9), then

$$
(u(z), j(z;w), h(z;w), q(z;w), w^*(z)) = \left(\hat{U}e^z, \hat{J}(w - z)e^z, (\hat{W}(w - z) + \hat{U})e^z, \hat{q}(w - z), \hat{w}^* + z\right)
$$

satisfy (A.2)–(A.5) with continuation sets $\mathcal{Z}^{h*}(w)$ and $\mathcal{Z}^{j*}(w)$ given by (A.3)–(A.4).

Proof. We use a guess-and-verify strategy for each equilibrium condition.

□

A.2 Proof of Proposition 1

Proposition 1. *Let $\hat{W}(\hat{w}), \hat{J}(\hat{w}), \hat{q}(\hat{w})$ be bounded functions with compact support. Then, there exists a unique solution to*

$$\hat{r}\hat{U} = \tilde{B} + \max_{\hat{w}} f(\hat{q}(\hat{w}))\hat{W}(\hat{w}),$$

$$0 = \min\left\{\tilde{K} - q(\hat{q}(\hat{w}))\hat{J}(\hat{w}), \hat{q}(\hat{w})\right\},$$

$$\hat{W}(\hat{w}) \geq 0, \tag{A.10}$$

$$\hat{J}(\hat{w}) \geq 0, \tag{A.11}$$

$$if\ \hat{w} \in (\hat{\mathcal{Z}}^h)^c \Rightarrow \hat{J}(\hat{w}) = 0, \tag{A.12}$$

$$if\ \hat{w} \in (\hat{\mathcal{Z}}^j)^c \Rightarrow \hat{W}(\hat{w}) = 0, \tag{A.13}$$

$$0 = \max\{-r\hat{W}(\hat{w}), \hat{\mathcal{A}}\hat{W}(\hat{w}) + e^{\hat{w}} - \hat{r}\hat{U}\},\ \forall \hat{w} \in \hat{\mathcal{Z}}^j, \hat{W} \in \mathbb{C}^1(\hat{\mathcal{Z}}^j) \cap \mathbb{C}(\mathbb{R}) \tag{A.14}$$

$$0 = \max\{-r\hat{J}(\hat{w}), \hat{\mathcal{A}}\hat{J}(\hat{w}) + 1 - e^{\hat{w}}\},\ \forall \hat{w} \in \hat{\mathcal{Z}}^h, \hat{J} \in \mathbb{C}^1(\hat{\mathcal{Z}}^h) \cap \mathbb{C}(\mathbb{R}) \tag{A.15}$$

$$\hat{\mathcal{Z}}^h := int\left\{\hat{w} \in \mathbb{R} : \hat{W}(\hat{w}) > 0\ or\ (e^{\hat{w}} - \hat{r}\hat{U}) > 0\right\}, \tag{A.16}$$

$$\hat{\mathcal{Z}}^j := int\left\{\hat{w} \in \mathbb{R} : \hat{J}(\hat{w}) > 0\ or\ \left(1 - e^{\hat{w}}\right) > 0\right\}, \tag{A.17}$$

$$\hat{\mathcal{A}}(v) := -(\hat{r} + d)v - \hat{g}\frac{\P v(\hat{w})}{\P \hat{w}} + \frac{s^2}{2}\frac{\P^2 v(\hat{w})}{\P \hat{w}^2},$$

Before going to the proof, observe that conditions (A.10)–(A.11) are implied by conditions (A.12)–(A.15) and, therefore, they are redundant. Nevertheless, they will help with the proof of existence.

The proof uses results from a branch of mathematics that most economists may not be familiar with. For this reason, before presenting the proof, we provide some intuition about the steps we show below. In a nutshell, there are two steps in the proof. First, we need to show that, for a given value of unemployment $\hat{U}$, there is a unique nontrivial Nash equilibrium of the game played by the matched worker-firm pair. To understand the intuition behind this step, define $\hat{w}^+(\hat{w}^-; r\hat{U})$ as the best response function of the firm in terms of its layoff threshold, and $\hat{w}^-(\hat{w}^+; r\hat{U})$ as the best response function of the worker in terms of her quit threshold. It is easy to show that optimal policies are given by wage-to-productivity thresholds. $\hat{w}^+(\hat{w}^-; r\hat{U})$ is the solution to the differential equation

$$(\hat{r}+d)\hat{J}(\hat{w}) = 1 - e^{\hat{w}} - \hat{g}\hat{J}'(\hat{w}) + \frac{s^2}{2}\hat{J}''(\hat{w}), \qquad \forall w \in (\hat{w}^-, \hat{w}^+)$$

with border conditions $\hat{J}(\hat{w}^+) = \hat{J}(\hat{w}^-) = \hat{J}'(\hat{w}^+) = 0$. Notice that the smooth pasting condition $\hat{J}'(\hat{w}^+) = 0$ determines $\hat{w}^+$. In the same way, $\hat{w}^-(\hat{w}^+; r\hat{U})$ is the solution to the differential equation

$$(\hat{r}+d)\hat{W}(\hat{w}) = e^{\hat{w}} - \hat{r}\hat{U} - \hat{g}\hat{W}'(\hat{w}) + \frac{s^2}{2}\hat{W}''(\hat{w}), \qquad \forall w \in (\hat{w}^-, \hat{w}^+)$$

with border conditions $\hat{W}(\hat{w}^+) = \hat{W}(\hat{w}^-) = \hat{W}'(\hat{w}^-) = 0$, where $\hat{W}'(\hat{w}^-) = 0$ determines $\hat{w}^-$. Let $\hat{W}(\hat{w}; r\hat{U})$ and $\hat{J}(\hat{w}; r\hat{U})$ be the values associated with the nontrivial equilibrium policies.

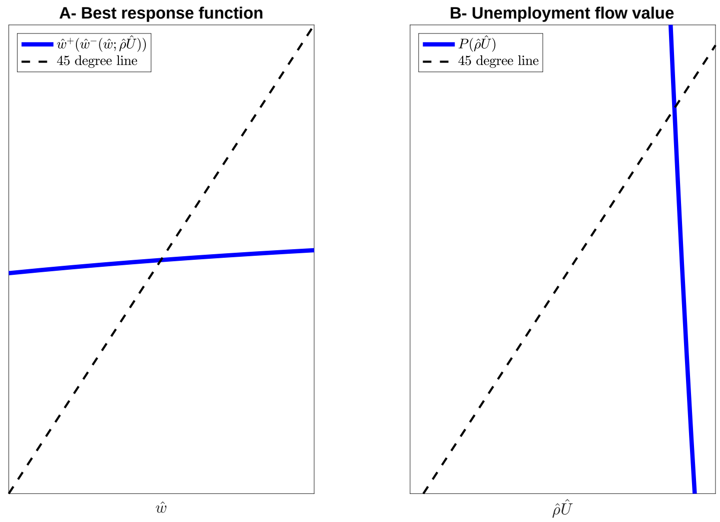
Second, we need to find the equilibrium value of unemployment. This value satisfies

$$\mathbb{P}(\hat{r}\hat{U}) = \tilde{B} + \max_{\hat{w}} \frac{1}{\tilde{K}^{1/a}} \hat{J}(\hat{w}; \hat{r}\hat{U})^{\frac{1-a}{a}} \hat{W}(\hat{w}; \hat{r}\hat{U}).$$

Panel A of Figure A1 shows the composition of $\mathbb{Q}(\hat{w}) := \hat{w}^+(\hat{w}^-(\hat{w}; \hat{r}\hat{U}))$ and Figure A1-Panel B shows $\mathbb{P}(\hat{r}\hat{U})$. As we can see in the figure, the composition of the best response functions satisfies two properties: (i) monotonicity (i.e., $\mathbb{Q}'(\hat{w}) > 0$) and (ii) concavity (i.e., $\mathbb{Q}''(\hat{w}) < 0$). Intuitively, the monotonicity property arises from the fact that if one agent prefers to stay in the match for longer, then the incentives for the other agent to stay in the match are larger; thus, the other agent also prefers to stay longer. Concavity arises from the fact that there is a decreasing value of delaying the separation. As the figure clearly shows, a unique nontrivial Nash Equilibrium exists under these two properties. Equipped with the values from the nontrivial Nash Equilibrium as a function of $\hat{U}$, we can then characterize the decision problem of the unemployed worker. The mapping $\mathbb{P}(\hat{r}\hat{U})$ satisfies three properties: (i) $\mathbb{P}(\tilde{B}) > \tilde{B}$ with $\mathbb{P}(1) = \tilde{B}$, (ii) it is continuous and (iii) it is decreasing. Intuitively, if the flow value of unemployment is equal to $\tilde{B}$, then the surplus of the match is positive, and the unemployed worker obtains a positive continuation value from searching for a job. If, instead, the flow value of unemployment equals the value of (normalized) output, then the surplus is zero, and the unemployed worker does not benefit from finding a job. Also, the larger the unemployment value, the lower the value of the match, and, therefore, the value of searching for a job. As the figure clearly shows, a unique equilibrium exists under these three properties of $\mathbb{P}(\hat{r}\hat{U})$.

<u>Proof.</u> We divide the proof into four steps. Step 1 shows the existence of a nontrivial Nash equilibrium for a given $\hat{U}$. In this step, we show the existence of a solution to conditions (A.10) to

FIGURE A1. INTUITION



Notes: The figure illustrates the properties of the policy and value functions. Panel A shows the composition of $\mathbb{Q}(\hat{w}) := \hat{w}^+(\hat{w}^-(\hat{w};\hat{\rho}\hat{U}))$ and the 45 degree line. The nontrivial Nash Equilibrium is given by the intersection between these two lines. Panel B shows the composition of the individual best response functions and the fixed point in the equilibrium $\mathbb{P}(\hat{\rho}\hat{U})$.

(A.17). To simplify the exposure, we divide step 1 into three propositions. Proposition A.5 shows the equivalence between the equilibrium conditions and the *quasi-variational inequalities* (i.e., a generalization of variational inequalities to the case when the feasible set is a function of the state variables), which is required to apply known fixed-point theorems. Proposition A.6 shows the existence and uniqueness of the agents' best responses. Proposition A.7 shows the existence of equilibrium by invoking Proposition A.4 (Birkhoff-Tartar's fixed-point theorem). Observe that we restrict the functions $\hat{W}(\hat{w})$ and $\hat{J}(\hat{w})$ to have bounded support. This restriction is without loss of generality since it is a result of Proposition 2—i.e., the match's continuation region is bounded.

Step 2 shows the uniqueness of the solution to conditions (A.10) to (A.17). We divide this proof into two propositions. Proposition A.8 shows that the operator defined in step 1 is strong order concave. Using concavity and techniques in the spirit of Marinacci and Montrucchio (2019) applied to our own problem, we show uniqueness in Proposition A.9.

Step 3 shows that value functions are continuous and decreasing. We divide this step into two propositions. First, we show in Proposition A.10 that the value associated with the worker's "best response" is continuous and decreasing in $\hat{U}$. Proposition A.11 shows these properties for the nontrivial Nash equilibrium. Finally, step 4 proves the uniqueness of the equilibrium by showing the existence of the unique fixed point in the unemployed worker's value $\hat{U}$.

Step 1. We begin by defining a continuous bilinear form in a more general space of functions. The objective here is to find the weak solution of the nontrivial Nash equilibrium. Since the bilinear form uses the first derivative, we work in $H_0^1(\mathbb{R})$—i.e., the Sobolev space of order 1 with bounded support. Let $V := H_0^1(\mathbb{R})$ be a Hilbert space and define the bilinear continuous form $a : V \times V \to \mathbb{R}$

$$a(v_1, v_2) := \frac{s^2}{2} \int_{\mathbb{R}} \frac{\mathrm{d}v_1}{\mathrm{d}\hat{w}} \frac{\mathrm{d}v_2}{\mathrm{d}\hat{w}} \,\mathrm{d}\hat{w} + \hat{g} \int_{\mathbb{R}} \frac{\mathrm{d}v_1}{\mathrm{d}\hat{w}} v_2(\hat{w}) \,\mathrm{d}\hat{w} + (\hat{r} + d) \int_{\mathbb{R}} v_1(\hat{w}) v_2(\hat{w}) d\hat{w}.$$

Notice that $a(v_1, v_2)$ is a bilinear form since it satisfies two properties for all $v_1, v_2, v_3 \in H_0^1(\mathbb{R})$: (i) $a(v_1 + v_3, v_2) = a(v_1, v_2) + a(v_3, v_2)$ and $a(v_1, v_2 + v_3) = a(v_1, v_2) + a(v_1, v_3)$; and (ii) $a(v_1 a, v_2) = aa(v_1, v_2)$ and $a(v_1, v_2 a) = aa(v_1, v_2)$ with $a \in \mathbb{R}$. To show these properties notice that the derivative and the integral of functions are linear operators. Thus,

$$\begin{aligned}
a(v_1 + v_3, v_2) &= \frac{s^2}{2} \int_{\mathbb{R}} \frac{\mathrm{d}(v_1 + v_3)}{\mathrm{d}\hat{w}} \frac{\mathrm{d}v_2}{\mathrm{d}\hat{w}} \,\mathrm{d}\hat{w} + \hat{g} \int_{\mathbb{R}} \frac{\mathrm{d}(v_1 + v_3)}{\mathrm{d}\hat{w}} v_2(\hat{w}) \,\mathrm{d}\hat{w} + (\hat{r} + d) \int_{\mathbb{R}} (v_1(\hat{w}) + v_3(\hat{w})) v_2(\hat{w}) d\hat{w} \\
&= \frac{s^2}{2} \left(\int_{\mathbb{R}} \frac{\mathrm{d}v_1}{\mathrm{d}\hat{w}} \frac{\mathrm{d}v_2}{\mathrm{d}\hat{w}} \,\mathrm{d}\hat{w} + \int_{\mathbb{R}} \frac{\mathrm{d}v_3}{\mathrm{d}\hat{w}} \frac{\mathrm{d}v_2}{\mathrm{d}\hat{w}} \,\mathrm{d}\hat{w} \right) + \hat{g} \left(\int_{\mathbb{R}} \frac{\mathrm{d}v_1}{\mathrm{d}\hat{w}} v_2(\hat{w}) \,\mathrm{d}\hat{w} + \int_{\mathbb{R}} \frac{\mathrm{d}v_3}{\mathrm{d}\hat{w}} v_2(\hat{w}) \,\mathrm{d}\hat{w} \right) \\
&\quad + (\hat{r} + d) \left(\int_{\mathbb{R}} v_1(\hat{w}) v_2(\hat{w}) d\hat{w} + \int_{\mathbb{R}} v_3 w(\hat{w}) v_2(\hat{w}) d\hat{w} \right) = a(v_1, v_2) + a(v_3, v_2). \\
a(v_1 a, v_2) &= \frac{s^2}{2} \int_{\mathbb{R}} \frac{\mathrm{d}(a v_1)}{\mathrm{d}\hat{w}} \frac{\mathrm{d}v_2}{\mathrm{d}\hat{w}} \,\mathrm{d}\hat{w} + \hat{g} \int_{\mathbb{R}} \frac{\mathrm{d}(a v_1)}{\mathrm{d}\hat{w}} v_2(\hat{w}) \,\mathrm{d}\hat{w} + (\hat{r} + d) \int_{\mathbb{R}} (a v_1(\hat{w})) v_2(\hat{w}) d\hat{w} = aa(v_1, v_2).
\end{aligned}$$

The proof for $a(v_1, v_2 + v_3) = a(v_1, v_2) + a(v_1, v_3)$ and $a(v_1, v_2 a) = aa(v_1, v_2)$ are similar. To show it is continuous, we need to show that $a(v_1, v_2) = a||v_1||||v_2||, \ a \in \mathbb{R}$. It is easy to verify that the bilinear form is continuous using the inner product of the Soloveb space, the Cauchy-Schwarz inequality, and compact support.

Now, we define the boundary conditions imposed by the other agent. Define $K^h(\hat{J})$ and $K^j(\hat{W})$ as

$$\begin{aligned}
K^h(\hat{J}) &:= \left\{ \hat{W} \in V : \hat{W}(\hat{w}) \geq 0 \,\&\, if \ \hat{J}(\hat{w}) = 0 \ and \ \hat{w} \geq 0 \Rightarrow \hat{W}(\hat{w}) = 0 \right\}, \\
K^j(\hat{W}) &:= \left\{ \hat{J} \in V : \hat{J}(\hat{w}) \geq 0 \,\&\, if \ \hat{W}(\hat{w}) = 0 \ and \ \hat{w} \leq \log(\hat{r}\hat{U}) \Rightarrow \hat{J}(\hat{w}) = 0 \right\}.
\end{aligned}$$

From now on, we look for solutions satisfying the variational approach within these sets.

Proposition A.5. *Assume $\hat{W}(\hat{w}) \in \mathbb{C}^1(\hat{\mathcal{Z}}^j) \cap \mathbb{C}(\mathbb{R})$ and $\hat{J}(\hat{w}) \in \mathbb{C}^1(\hat{\mathcal{Z}}^h) \cap \mathbb{C}(\mathbb{R})$ bounded with compact support, where $\hat{\mathcal{Z}}^h$ and $\hat{\mathcal{Z}}^j$ are constructed with $\hat{W}$ and $\hat{J}$ following* (A.16) *and* (A.17)*. Then, $\hat{W}(\hat{w})$ and*

$\hat{J}(\hat{w})$ solve

$$
\begin{aligned}
&\hat{W} \in K^h(\hat{J}), \;\; \hat{J} \in K^j(\hat{W}) \\
&a(\hat{J}, v - \hat{J}) \geq \int_{\mathbb{R}} \left(1 - e^{\hat{w}}\right)\left(v - \hat{J}\right) d\hat{w}, \;\; \forall\, v \in K^j(\hat{W}) && \text{(A.18)} \\
&a(\hat{W}, v - \hat{W}) \geq \int_{\mathbb{R}} \left(e^{\hat{w}} - \hat{r}\hat{U}\right)\left(v - \hat{W}\right) d\hat{w}, \;\; \forall\, v \in K^h(\hat{J}). && \text{(A.19)}
\end{aligned}
$$

iff. $\hat{W}(\hat{w})$ and $\hat{J}(\hat{w})$ solve (A.10), (A.11), (A.12), (A.13), (A.14), *and* (A.15).

Before going to the proof, it is worth making some remarks. First, conditions (A.18) and (A.19) provide a weak solution to the differential equations and not a classical solution. For the same reason, we did not define the sets $K^h(\hat{J})$ and $K^j(\hat{W})$ in terms of conditions holding almost everywhere. We come back to this issue below.

Proof of Step 1—Proposition A.5. We verify conditions (A.10), (A.11), (A.12), (A.13), (A.14), and (A.15) focusing on the firm (the worker's conditions are verified following similar steps). It is easy to show the converse.

Conditions (A.10) and (A.11) are satisfied. Since $\hat{J} \in K^j(\hat{W})$, we have $\hat{J}(\hat{w}) \geq 0$.

Conditions (A.12) and (A.13) are satisfied. Define $\hat{\mathcal{Z}}^h$ with $\hat{W}$. Then, $(\hat{\mathcal{Z}}^h)^c = cl\{\hat{w} \in \mathbb{R} : \hat{W}(\hat{w}) \leq 0 \text{ and } (e^{\hat{w}} - \hat{r}\hat{U}) \leq 0\}$. Since $\hat{W}(\hat{w}) \geq 0$, we have $(\hat{\mathcal{Z}}^h)^c = cl\{\hat{w} \in \mathbb{R} : \hat{W}(\hat{w}) = 0 \text{ and } \hat{w} \leq \log(\hat{r}\hat{U})\}$. Since $\hat{J} \in K^j(\hat{W})$, if $\hat{w} \in (\hat{\mathcal{Z}}^h)^c$, then $\hat{J}(\hat{w}) = 0$.

Conditions (A.14) and (A.15) are satisfied. Take any $v \in K^j(\hat{W})$. Then, if $\hat{w} \in (\hat{\mathcal{Z}}^h)^c$, we have $\hat{J}(\hat{w}) = v(\hat{w}) = 0$. Therefore, we have that, for every $v, \hat{J} \in K^j(\hat{W})$,

$$
\begin{aligned}
&a(\hat{J}, v - \hat{J}) \geq \int_{\mathbb{R}} (1 - e^{\hat{w}})\,(v - \hat{J}) \iff \\
&\underbrace{\frac{s^2}{2}\int_{(\hat{\mathcal{Z}}^h)^c} \frac{d\hat{J}(\hat{w})}{d\hat{w}} \frac{d(v(\hat{w}) - \hat{J}(\hat{w}))}{d\hat{w}}\, d\hat{w} + \hat{g}\int_{(\hat{\mathcal{Z}}^h)^c} \frac{d\hat{J}(\hat{w})}{d\hat{w}} (v(\hat{w}) - \hat{J}(\hat{w}))\, d\hat{w} + (\hat{r} + d)\int_{(\hat{\mathcal{Z}}^h)^c} \hat{J}(\hat{w})(v(\hat{w}) - \hat{J}(\hat{w}))\, d\hat{w}}_{=0} + \\
&\frac{s^2}{2}\int_{\hat{\mathcal{Z}}^h} \frac{d\hat{J}(\hat{w})}{d\hat{w}} \frac{d(v(\hat{w}) - \hat{J}(\hat{w}))}{d\hat{w}}\, d\hat{w} + \hat{g}\int_{\hat{\mathcal{Z}}^h} \frac{d\hat{J}(\hat{w})}{d\hat{w}} (v(\hat{w}) - \hat{J}(\hat{w}))\, d\hat{w} + (\hat{r} + d)\int_{\hat{\mathcal{Z}}^h} \hat{J}(\hat{w})(v(\hat{w}) - \hat{J}(\hat{w}))\, d\hat{w} \geq \\
&\int_{\hat{\mathcal{Z}}^h} (1 - e^{\hat{w}})\,(v(\hat{w}) - \hat{J}(\hat{w}))\, d\hat{w} + \underbrace{\int_{(\hat{\mathcal{Z}}^h)^c} (1 - e^{\hat{w}})\,(v(\hat{w}) - \hat{J}(\hat{w}))\, d\hat{w}}_{=0} \iff \\
&\frac{s^2}{2}\int_{\hat{\mathcal{Z}}^h} \frac{d\hat{J}(\hat{w})}{d\hat{w}} \frac{d(v(\hat{w}) - \hat{J}(\hat{w}))}{d\hat{w}}\, d\hat{w} + \hat{g}\int_{\hat{\mathcal{Z}}^h} \frac{d\hat{J}(\hat{w})}{d\hat{w}} (v(\hat{w}) - \hat{J}(\hat{w}))\, d\hat{w} + (\hat{r} + d)\int_{\hat{\mathcal{Z}}^h} \hat{J}(\hat{w})(v(\hat{w}) - \hat{J}(\hat{w}))\, d\hat{w} \geq \\
&\int_{\hat{\mathcal{Z}}^h} (1 - e^{\hat{w}})\,(v(\hat{w}) - \hat{J}(\hat{w}))\, d\hat{w}.
\end{aligned}
$$

Using integration by parts, we obtain

$$\frac{s^2}{2}\int_{\hat{\mathcal{Z}}^h}\frac{\mathrm{d}\hat{J}(\hat{w})}{\mathrm{d}\hat{w}}\frac{\mathrm{d}(v(\hat{w})-\hat{J}(\hat{w}))}{\mathrm{d}\hat{w}}\,\mathrm{d}\hat{w} =^{(1)} \underbrace{\frac{s^2}{2}\frac{\mathrm{d}\hat{J}(\hat{w})}{\mathrm{d}\hat{w}}(v(\hat{w})-\hat{J}(\hat{w}))\Big|_{\hat{w}\in\P\in\hat{\mathcal{Z}}^h}}_{=0} - \frac{s^2}{2}\int_{\hat{\mathcal{Z}}^h}\frac{d^2\hat{J}(\hat{w})}{\mathrm{d}\hat{w}}(v(\hat{w})-\hat{J}(\hat{w}))\,\mathrm{d}\hat{w}.$$

In (1), there could be two cases for the first term. The first case is a finite limit of integration (i.e., $\hat{\mathcal{Z}}^h$ is bounded). In this case, we use the continuity of the functions and the fact that if $\hat{w}\to\P\hat{\mathcal{Z}}^h$ ($\hat{\mathcal{Z}}^h$ is open), then $\hat{w}\to(\hat{\mathcal{Z}}^h)^c$ and, therefore, $\hat{J}(\hat{w})=v(\hat{w})=0$. The second case is an infinite limit of integration. In this case, the assumption of bounded support implies $\hat{J}(\hat{w})=0$ for sufficiently large or small $\hat{w}$, thus $\hat{J}'(\hat{w})=0$. In conclusion, $\int_{\hat{\mathcal{Z}}^h}\left(\hat{\mathcal{A}}\hat{J}(\hat{w})+(1-e^{\hat{w}})\right)(v(\hat{w})-\hat{J}(\hat{w}))\,\mathrm{d}\hat{w}\le 0$.

Before continuing, we remark that the previous equality holds for all $v(\hat{w})\in K^j(\hat{W})$. Let $\mathcal{O}$ be an open ball in $\hat{\mathcal{Z}}^h$ that covers an arbitrary point $\hat{w}\in\hat{\mathcal{Z}}^h$. Then, we can find a family of smooth functions indexed by n with $o_{\hat{w}}(n)\in[0,1]$, s.t. $o_{\hat{w}}(n)=0$ outside $\hat{\mathcal{Z}}^h$, $o_{\hat{w}}(n)\to 1$ in $\mathcal{O}$, and $o_{\hat{w}}(n)\to 0$ outside $\mathcal{O}$. Since $\hat{J}(\hat{w})+o_{\hat{w}}(n)\ge 0$, $\hat{J}(\hat{w})+o_{\hat{w}}(n)\in K^j(\hat{W})$ and

$$\int_{\mathcal{O}}\left(\hat{\mathcal{A}}\hat{J}(\hat{w})+(1-e^{\hat{w}})\right)o_{\hat{w}}(n)\,\mathrm{d}\hat{w}+\int_{\hat{\mathcal{Z}}^h/\mathcal{O}}\left(\hat{\mathcal{A}}\hat{J}(\hat{w})+(1-e^{\hat{w}})\right)o_{\hat{w}}(n)\,\mathrm{d}\hat{w}\le 0.$$

Taking the limit $n\to ¥$, we have that $\int_{\mathcal{O}}\left(\hat{\mathcal{A}}\hat{J}(\hat{w})+(1-e^{\hat{w}})\right)\mathrm{d}\hat{w}\le 0$. Since $\mathcal{O}$ is arbitrary, $\hat{\mathcal{A}}\hat{J}(\hat{w})+1-e^{\hat{w}}\le 0$ a.e. in $\hat{\mathcal{Z}}^h$. Since $\hat{J}(\hat{w})\in\mathbb{C}^1(\hat{\mathcal{Z}}^h)$, then $\hat{\mathcal{A}}\hat{J}(\hat{w})+1-e^{\hat{w}}\le 0$ for all $\hat{w}$ whenever the second derivative is defined. To obtain the other inequality, consider $\hat{J}(\hat{w})(1-o_{\hat{w}}(n))+0o_{\hat{w}}(n)\in K^j(\hat{W})$ and we have

$$-\int_{\mathcal{O}}\left(\hat{\mathcal{A}}\hat{J}(\hat{w})+(1-e^{\hat{w}})\right)\hat{J}(\hat{w})o_{\hat{w}}(n)\,\mathrm{d}\hat{w}-\int_{\hat{\mathcal{Z}}^h/\mathcal{O}}\left(\hat{\mathcal{A}}\hat{J}(\hat{w})+(1-e^{\hat{w}})\right)\hat{J}(\hat{w})o_{\hat{w}}(n)\,\mathrm{d}\hat{w}\le 0$$

Taking the limit $n\to ¥$, we have that $\int_{\mathcal{O}}\left(\hat{\mathcal{A}}\hat{J}(\hat{w})+(1-e^{\hat{w}})\right)(-\hat{J}(\hat{w}))\,\mathrm{d}\hat{w}\le 0$ almost everywhere. Since $\hat{J}(\hat{w})\in\mathbb{C}^1(\hat{\mathcal{Z}}^h)$, we have that for all $\hat{w}\in\hat{\mathcal{Z}}^h$

$$\left(\hat{\mathcal{A}}\hat{J}(\hat{w})+(1-e^{\hat{w}})\right)(-\hat{J}(\hat{w}))\le 0.$$

Since $\hat{J}(\hat{w})\ge 0$ and $\left(\hat{\mathcal{A}}\hat{J}(\hat{w})+(1-e^{\hat{w}})\right)\le 0$, we have that $\left(\hat{\mathcal{A}}\hat{J}(\hat{w})+(1-e^{\hat{w}})\right)(-\hat{J}(\hat{w}))\ge 0$. Thus, $\left(\hat{\mathcal{A}}\hat{J}(\hat{w})+1-e^{\hat{w}}\right)(-\hat{J}(\hat{w}))=0$ or written more compactly $0=\max\{-\hat{J}(\hat{w}),\hat{\mathcal{A}}\hat{J}(\hat{w})+1-e^{\hat{w}}\}$, $\forall\hat{w}\in\hat{\mathcal{Z}}^h$, with $\hat{J}(\hat{w})\in\mathbb{C}^1(\hat{\mathcal{Z}}^h)\cap\mathbb{C}(\mathbb{R})$. □

Proposition A.6. *Define the value functions that are obtained from the best responses as* $BR^h: H^1(\mathbb{R})\to$

$H^1(\mathbb{R})$ and $BR^j : H_0^1(\mathbb{R}) \to H_0^1(\mathbb{R})$ such that

$$BR^h(\hat{J}) = \{\hat{W} \in H^1(\mathbb{R}) : a(\hat{W}, v - \hat{W}) \geq (e^{\hat{w}} - \hat{r}\hat{U}, v - \hat{W}),\ \forall\, v \in K^h(\hat{J}),\ \hat{W} \in K^h(\hat{J})\},$$
$$BR^j(\hat{W}) = \{\hat{J} \in H^1(\mathbb{R}) : a(\hat{J}, v - \hat{J}) \geq (1 - e^{\hat{w}}, v - \hat{J}),\ \forall\, v \in K^j(\hat{W}),\ \hat{J} \in K^j(\hat{W})\}.$$

Then, $BR^h(\hat{J})$ and $BR^j(\hat{W})$ exist and are unique.

Proof of Step 1—Proposition A.6. Here, we show that the value functions that are obtained from the best responses are well-defined. For this, we need to verify the conditions in Proposition A.3. Basically, we need to show that $K^j(\hat{W})$ is closed and convex, and that $a(\cdot,\cdot)$ is coercive.

$\mathbf{K^j(\hat{W})}$ is closed and convex. First, we show that $K^j(\hat{W})$ is closed. Take a sequence $\hat{J}^n \in K^j(\hat{W})$ s.t. $\hat{J}^n$ converges to some $\hat{J}^*$. Since $\hat{J}^n \in K^j(\hat{W})$,

$$\hat{J}^n(\hat{w}) \geq 0, \text{if } \hat{W}(\hat{w}) = 0 \textit{ and } \hat{w} \leq \log(\hat{r}\hat{U}),\ \text{then}\ \ \hat{J}^n(\hat{w}) = 0$$

for all n and all $\hat{w}$. Taking the limit in the real numbers,

$$\hat{J}^*(\hat{w}) \geq 0, \text{if } \hat{W}(\hat{w}) = 0 \textit{ and } \hat{w} \leq \log(\hat{r}\hat{U}),\ \text{then}\ \ \hat{J}^*(\hat{w}) = 0$$

where we use the fixed domain in the second limit. Thus, $K^j(\hat{W})$ is closed.

To show that $K^j(\hat{W})$ is convex, take $\hat{J}^1, \hat{J}^2 \in K^j(\hat{W})$, then

$$\hat{J}^1(\hat{w}) \geq 0, \text{if } \hat{W}(\hat{w}) = 0 \textit{ and } \hat{w} \leq \log(\hat{r}\hat{U}),\ \text{then}\ \ \hat{J}^1(\hat{w}) = 0,$$
$$\hat{J}^2(\hat{w}) \geq 0, \text{if } \hat{W}(\hat{w}) = 0 \textit{ and } \hat{w} \leq \log(\hat{r}\hat{U}),\ \text{then}\ \ \hat{J}^2(\hat{w}) = 0,$$

all $\hat{w}$. Taking the convex combination with $l \in [0,1]$

$$l\hat{J}^1 + (1-l)\hat{J}^2 \geq 0, \text{if } \hat{W}(\hat{w}) = 0 \textit{ and } \hat{w} \geq 0,\ \text{then}\ \ l\hat{J}^1 + (1-l)\hat{J}^2 = 0.$$

Thus, $K^j(\hat{W})$ is convex.

$\mathbf{a(u,v)}$ is coercive. Operating over the bilinear operator

$$a(v,v) = \frac{s^2}{2}\int_{\mathbb{R}} \frac{dv(\hat{w})}{d\hat{w}}\frac{dv(\hat{w})}{d\hat{w}}\, d\hat{w} + \hat{g}\int_{\mathbb{R}} \frac{dv(\hat{w})}{d\hat{w}} v(\hat{w})\, d\hat{w} + (\hat{r} + d)\int_{\mathbb{R}} v(\hat{w})^2 d\hat{w}$$

$$
\begin{aligned}
&=^{(1)} \frac{s^2}{2}\underbrace{\int_{\mathbb{R}}\left(\frac{\mathrm{d}v(\hat{w})}{\mathrm{d}\hat{w}}\right)^2 \mathrm{d}\hat{w}}_{\geq 0} + \hat{g}\underbrace{v(\hat{w})^2|_{-¥}^{¥}}_{=0} + (\hat{r}+d)\int_{\mathbb{R}} v(\hat{w})^2 d\hat{w} \\
&\geq^{(2)} (\hat{r}+d)\int_{\mathbb{R}} v(\hat{w})^2 d\hat{w} = (\hat{r}+d)||v||^2
\end{aligned}
$$

Step (1) integrates $\int_{\mathbb{R}} \frac{\mathrm{d}v(\hat{w})}{\mathrm{d}\hat{w}} v(\hat{w})\,\mathrm{d}\hat{w} = \frac{1}{2}v(\hat{w})^2|_{-¥}^{¥}$ and uses compact support. Step (2) uses the nonnegativity of the squared derivative term.

With the properties verified, we can apply Proposition A.3. Thus, the best response exists, and it is unique. □

Proposition A.7. *Define $Q(\hat{W}) = (BR^h \circ BR^j)(\hat{W})$, then there exists a fixed point $Q(\hat{W}^*) = \hat{W}^*$ and $\hat{J}^* = BR^j(\hat{W}^*)$. The set of fixed points is bounded above and below by*

$$
\begin{aligned}
0 \leq \underline{\hat{W}} \leq \hat{W}^* \leq \overline{\hat{W}}, \\
0 \leq \underline{\hat{J}} \leq \hat{J}^* \leq \overline{\hat{J}},
\end{aligned}
$$

where

$$
\begin{aligned}
a(\underline{\hat{W}}, v - \underline{\hat{W}}) &\geq (e^{\hat{w}} - \hat{r}\hat{U}, \underline{\hat{W}}),\ \forall\, v \in K^{small}, \underline{\hat{W}} \in K^{small}, \\
a(\underline{\hat{J}}, v - \underline{\hat{J}}) &\geq (1 - e^{\hat{w}}, \underline{\hat{J}}),\ \forall\, v \in K^{small}, \underline{\hat{J}} \in K^{small}, \\
a(\overline{\hat{W}}, v - \overline{\hat{W}}) &\geq (e^{\hat{w}} - \hat{r}\hat{U}, \overline{\hat{W}}),\ \forall\, v \in K^{big}, \overline{\hat{W}} \in K^{big}, \\
a(\overline{\hat{J}}, v - \overline{\hat{J}}) &\geq (1 - e^{\hat{w}}, \overline{\hat{J}}),\ \forall\, v \in K^{big}, \overline{\hat{J}} \in K^{big},
\end{aligned}
$$

with

$$
\begin{aligned}
K^{small} &:= \left\{v \in V : v(\hat{w}) \geq 0 \,\&\, if\ \ \hat{w} \notin (\log(\hat{r}\hat{U}), 0) \Rightarrow v(\hat{w}) = 0\right\}, \\
K^{big} &:= \{v \in V : v(\hat{w}) \geq 0\},
\end{aligned}
$$

with a maximum and minimum element.

Proof of Step 1—Proposition A.7. The first step consists in showing that the function $Q(W)$ is monotonically increasing—i.e., if $\hat{W}_1 \geq \hat{W}_2$, then $Q(\hat{W}_1) \geq Q(\hat{W}_2)$. To show this result, first, we need to prove that $K^j(\hat{W})$ is increasing—i.e., if $\hat{W}_1 \geq \hat{W}_2$, then $K^j(\hat{W}_2) \subset K^j(\hat{W}_1)$. Take $\hat{J}_2 \in K^j(\hat{W}_2)$, then

$$\hat{J}_2 \geq 0, \ \& \ if \ \hat{W}_2(\hat{w}) = 0 \ and \ \hat{w} \leq \log(\hat{r}\hat{U}) \Rightarrow \hat{J}_2(\hat{w}) = 0.$$

Since $\hat{W}_2(\hat{w}) \geq 0$, we have

$$\hat{J}_2 \geq 0, \ \& \ \hat{J}_2(\hat{w}) = 0 \ \forall \hat{w} \in \{\hat{w} : \hat{W}_2(\hat{w}) \leq 0 \ \& \ \hat{w} \leq \log(\hat{r}\hat{U})\}.$$

Now, we show that $\{\hat{w} : \hat{W}_1(\hat{w}) \leq 0 \ \& \ \hat{w} \leq \log(\hat{r}\hat{U})\} \subset \{\hat{w} : \hat{W}_2(\hat{w}) \leq 0 \ \& \ \hat{w} \leq \log(\hat{r}\hat{U})\}$. Take $\hat{w} \in \{\hat{w} : \hat{W}_1(\hat{w}) \leq 0 \ \& \ \hat{w} \leq \log(\hat{r}\hat{U})\}$. Then, $\hat{W}_1(\hat{w}) \leq 0$ and, since $\hat{W}_1(\hat{w}) \geq \hat{W}_2(\hat{w})$, we have that $\hat{W}_2(\hat{w}) \leq 0$. Since $\{\hat{w} : \hat{W}_1(\hat{w}) \leq 0 \ \& \ \hat{w} \leq \log(\hat{r}\hat{U})\} \subset \{\hat{w} : \hat{W}_2(\hat{w}) \leq 0 \ \& \ \hat{w} \leq \log(\hat{r}\hat{U})\}$, the previous condition holds for the larger set, so it will also hold for the smaller set

$$\hat{J}_2 \geq 0, \ \& \ \hat{J}_2(\hat{w}) = 0, \forall \hat{w} \in \{\hat{w} : \hat{W}_1(\hat{w}) \leq 0 \ \& \ \hat{w} \leq \log(\hat{r}\hat{U})\}.$$

Thus, $\hat{J}_2 \in K^j(W_1)$ and $K^j(\hat{W}_2) \subset K^j(W_1)$.

Now, let $\hat{W}_1 \geq \hat{W}_2$. We need to show that $\hat{J}_1 = BR^j(\hat{W}_1) \geq BR^j(\hat{W}_2) = \hat{J}_2$. Since $K^j(\hat{W})$ is increasing—i.e., $K^j(\hat{W}_2) \subset K^j(\hat{W}_1)$—$\hat{J}_1, \hat{J}_2 \in K^j(\hat{W}_1)$ and the envelope $\max\{\hat{J}_1, \hat{J}_2\} \in K^j(\hat{W}_1)$. Now, we show that $\min\{\hat{J}_1, \hat{J}_2\} \in K^j(\hat{W}_2)$. Since $\hat{J}_1, \hat{J}_2 \geq 0$, we have that $\min\{\hat{J}_1, \hat{J}_2\} \geq 0$. Moreover, take a $\hat{w}$ s.t. $\hat{W}_2(\hat{w}) \leq 0$ and $\hat{w} \leq \log(\hat{r}\hat{U})$, then $0 = \hat{J}_2 = \min\{\hat{J}_2, \hat{J}_1\}$. Thus, $\min\{\hat{J}_1, \hat{J}_2\} \in K^j(\hat{W}_2)$. In conclusion, we can use $\max\{\hat{J}_1, \hat{J}_2\}$ as a test function for $K^j(\hat{W}_1)$ and $\min\{\hat{J}_1, \hat{J}_2\}$ as a test function for $K^j(\hat{W}_2)$:

$$\min\{\hat{J}_1, \hat{J}_2\} = \hat{J}_2 - \max\{\hat{J}_2 - \hat{J}_1, 0\} \text{ for test function for } K^j(\hat{W}_2)$$
$$\max\{\hat{J}_1, \hat{J}_2\} = \hat{J}_1 + \max\{\hat{J}_2 - \hat{J}_1, 0\} \text{ for test function for } K^j(\hat{W}_1)$$

Using the quasi-variational inequality

$$a(\hat{J}_2, -\max\{\hat{J}_2 - \hat{J}_1, 0\}) \geq (1 - e^{\hat{w}}, -\max\{\hat{J}_2 - \hat{J}_1, 0\})$$
$$a(\hat{J}_1, \max\{\hat{J}_2 - \hat{J}_1, 0\}) \geq (1 - e^{\hat{w}}, \max\{\hat{J}_2 - \hat{J}_1, 0\}).$$

Thus, since $a(\cdot, \cdot)$ is a bilinear form

$$-a(\hat{J}_2, \max\{\hat{J}_2 - \hat{J}_1, 0\}) \geq -(1 - e^{\hat{w}}, \max\{\hat{J}_2 - \hat{J}_1, 0\})$$

$$a(\hat{J}_1, \max\{\hat{J}_2 - \hat{J}_1, 0\}) \geq (1 - e^{\hat{w}}, \max\{\hat{J}_2 - \hat{J}_1, 0\}).$$

Summing these two equalities, we obtain

$$a(\hat{J}_1, \max\{\hat{J}_2 - \hat{J}_1, 0\}) - a(\hat{J}_2, \max\{\hat{J}_2 - \hat{J}_1, 0\}) \geq 0$$

or equivalently,

$$a(\hat{J}_2, \max\{\hat{J}_2 - \hat{J}_1, 0\}) - a(\hat{J}_1, \max\{\hat{J}_2 - \hat{J}_1, 0\}) \leq 0.$$

Next, we show that the previous inequality implies $a(\max\{\hat{J}_2 - \hat{J}_1, 0\}, \max\{\hat{J}_2 - \hat{J}_1, 0\}) \leq 0$. Define the set $\mathbb{X} = \{x : \hat{J}_2 > \hat{J}_1\}$. Then,

$$\begin{aligned}
& a(\hat{J}_2, \max\{\hat{J}_2 - \hat{J}_1, 0\}) - a(\hat{J}_1, \max\{\hat{J}_2 - \hat{J}_1, 0\}) \\
& = \frac{s^2}{2}\left(\int_{\mathbb{X}} \frac{\mathrm{d}\hat{J}_2(\hat{w})}{\mathrm{d}\hat{w}} \frac{\mathrm{d}(\hat{J}_2 - \hat{J}_1)}{\mathrm{d}\hat{w}}\,\mathrm{d}\hat{w} - \int_{\mathbb{X}} \frac{\mathrm{d}\hat{J}_1(\hat{w})}{\mathrm{d}\hat{w}} \frac{\mathrm{d}(\hat{J}_2 - \hat{J}_1)}{\mathrm{d}\hat{w}}\,\mathrm{d}\hat{w} + \int_{\mathbb{R}/\mathbb{X}} 0\,\mathrm{d}x\right) \\
& \cdots + \hat{g}\left(\int_{\mathbb{X}} \frac{\mathrm{d}\hat{J}_2(\hat{w})}{\mathrm{d}\hat{w}}(\hat{J}_2 - \hat{J}_1)\,\mathrm{d}\hat{w} - \int_{\mathbb{X}} \frac{\mathrm{d}\hat{J}_1(\hat{w})}{\mathrm{d}\hat{w}}(\hat{J}_2 - \hat{J}_1)\,\mathrm{d}\hat{w} + \int_{\mathbb{R}/\mathbb{X}} 0\,\mathrm{d}x\right) \\
& \cdots + (\hat{r} + d)\left(\int_{\mathbb{X}} \hat{J}_2(\hat{J}_2 - \hat{J}_1)\,\mathrm{d}\hat{w} - \int_{\mathbb{X}} \hat{J}_1(\hat{J}_2 - \hat{J}_1)\,\mathrm{d}\hat{w} + \int_{\mathbb{R}/\mathbb{X}} 0\,\mathrm{d}\hat{w}\right) \\
& = \frac{s^2}{2}\int_{\mathbb{X}}\left(\frac{\mathrm{d}(\hat{J}_2 - \hat{J}_1)}{\mathrm{d}\hat{w}}\right)^2 \mathrm{d}\hat{w} + \hat{g}\int_{\mathbb{X}} \frac{\mathrm{d}(\hat{J}_2(\hat{w}) - \hat{J}_1)}{\mathrm{d}\hat{w}}(\hat{J}_2 - \hat{J}_1)\,\mathrm{d}\hat{w} + (\hat{r} + d)\left(\int_{\mathbb{X}} (\hat{J}_2 - \hat{J}_1)^2\,\mathrm{d}\hat{w}\right) \\
& = a(\max\{\hat{J}_2 - \hat{J}_1, 0\}, \max\{\hat{J}_2 - \hat{J}_1, 0\}).
\end{aligned}$$

Since $a(\cdot,\cdot)$ is a coercive bilinear form, $0 \geq a(\max\{\hat{J}_2 - \hat{J}_1, 0\}, \max\{\hat{J}_2 - \hat{J}_1, 0\}) \geq K||\max\{\hat{J}_2 - \hat{J}_1, 0\}||^2$. Thus, $\hat{J}_1 \geq \hat{J}_2$ a.e., and by continuity $\hat{J}_1 \geq \hat{J}_2$ for all $\hat{w}$. Applying similar arguments to $BR^h(\hat{J})$, we have that $\hat{W}_1 \geq \hat{W}_2$ implies $Q(\hat{W}_1) \geq Q(\hat{W}_2)$, so by Proposition A.4 (Birkhoff-Tartar's fixed-point theorem), there exists a fixed point. Moreover, the set of fixed points has a maximum and a minimum—i.e., $\{\hat{W} \in H_0^1(\mathbb{R}) : \hat{W} = Q(\hat{W})\}$ has a $\hat{W}^{\min}$ and $\hat{W}^{\max}$ s.t. $\hat{W}^{\min} \leq \hat{W}^* \leq \hat{W}^{\max}$ for all $\hat{W}^* \in \{\hat{W} \in H_0^1(\mathbb{R}) : \hat{W} = Q(\hat{W})\}$.

Observe that since the flow payoff function and the coefficient of the characteristic operator are infinitely differentiable and the continuation set is bounded, by Theorems 3 and 6 of Chapter 6 of Evans (2022), we have that $\hat{W}, \hat{J}$ are infinite differentiable in the continuation set of the game and differentiable in the continuation set of the other agent.

To find the upper and lower bound, observe that we can write the nontrivial Nash equilibrium policies as $\hat{J}^*(w) = \max_{\{t^j \in \mathcal{T} : t^j \leq t^{h*}\}} \mathbb{E}\left[\int_0^{t^j} e^{-(\hat{r}+d)t}(1 - e^{\hat{w}_t})\,\mathrm{d}t | \hat{w}_0 = \hat{w}\right]$. Since ¥ $> t^{h*} \geq$

$t_{(\log(\hat{r}\hat{U},0))}$,[20] we have

$$0 \leq \underline{\hat{J}} = \max_{\{t^j \in \mathcal{T}: t^j \leq t_{(\log(\hat{r}\hat{U},0))}\}} \mathbb{E}\left[\int_0^{t^j} e^{-(\hat{r}+d)t}(1-e^{\hat{w}_t})\,\mathrm{d}t | \hat{w}_0 = \hat{w}\right]$$
$$\leq \max_{\{t^j \in \mathcal{T}: t^j \leq t^{h*}\}} \mathbb{E}\left[\int_0^{t^j} e^{-(\hat{r}+d)t}(1-e^{\hat{w}_t})\,\mathrm{d}t | \hat{w}_0 = \hat{w}\right] = \hat{J}^*(w)$$
$$\leq \max_{\{t^j \in \mathcal{T}\}} \mathbb{E}\left[\int_0^{t^j} e^{-(\hat{r}+d)t}(1-e^{\hat{w}_t})\,\mathrm{d}t | \hat{w}_0 = \hat{w}\right] = \overline{\hat{J}}.$$

□

Step 2. This step proves the uniqueness of the fixed point. The first proposition shows that $Q : H_0^1(\mathbb{R}) \to H_0^1(\mathbb{R})$ is concave. Since the Q operator is only defined for nonnegative functions, we assume that the domain is restricted to nonnegative functions without loss of generality. Since the game's continuation region is bounded, flow payoffs are bounded. Therefore, the equilibrium value functions are also bounded. Thus, without loss of generality, we restrict the $Q : \mathcal{A} \to \mathcal{A}$ operator in $\mathcal{A} = \{v \in H_0^1(\mathbb{R}) : v(\hat{w}) \in [0, \bar{v}], \forall \hat{w}\}$. Observe that $\mathcal{A}$ is order convex—i.e., if $a, b \in \mathcal{A}$ with $a \leq c \leq b$, then $c \in \mathcal{A}$. Define the operator $a : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, where $a(\hat{W}', \hat{W}'') = a(\hat{w})\hat{W}'(\hat{w}) + (1 - a(\hat{w}))\hat{W}''(\hat{w})$, with $a(\hat{w}) \in [0, 1]$.

Proposition A.8. *$Q : \mathcal{A} \to \mathcal{A}$ is strongly order concave—i.e., $Q(a(\hat{W}', \hat{W}'')) \geq a(Q(\hat{W}'), Q(\hat{W}''))$ for all $\hat{W}' \leq \hat{W}''$.*

Proof of Step 2—Proposition A.8. Take $\hat{W}' \leq \hat{W}''$. The proof has three arguments. First, we show that $K^j(a(\hat{W}', \hat{W}'')) = K^j(\hat{W}'')$. With this result, we show that the $BR^j(a(\hat{W}', \hat{W}'')) \geq a(BR^j(\hat{W}'), BR^j(\hat{W}''))$. Finally, we show that $Q(a(\hat{W}', \hat{W}'')) \geq a(Q(\hat{W}'), Q(\hat{W}''))$.

To see that $K^j(a(\hat{W}', \hat{W}'')) = K^j(\hat{W}'')$, observe that since $a(\hat{W}', \hat{W}'') \leq \hat{W}''$ and $K^j(\cdot)$ is increasing, we have $K^j(a(\hat{W}', \hat{W}'')) \subset K^j(\hat{W}'')$. Now, we show that $K^j(\hat{W}'') \subset K^j(a(\hat{W}', \hat{W}''))$. For any $\hat{J} \in K^j(\hat{W}'')$,

$$\hat{J} \geq 0, \ \& \ if \ \hat{W}''(\hat{w}) = 0 \text{ and } \hat{w} \leq \log(\hat{r}\hat{U}) \Rightarrow \hat{J}(\hat{w}) = 0.$$

If $\hat{W}''(\hat{w}) = 0$, then $\hat{W}''(\hat{w}) \geq \hat{W}'(\hat{w}) = 0$, which is then also true for any convex combination.

[20] $t_{(\log(\hat{r}\hat{U},0))} := \inf\{t \geq 0 : \hat{w}_t \notin (\log(\hat{r}\hat{U}, 0))\}$.

Thus, $a(\hat{W}', \hat{W}'') \le \hat{W}'' = 0$ and

$$\hat{J} \ge 0, \quad \text{and} \quad (a(\hat{W}', \hat{W}'') = 0 \wedge \hat{w} \le \log(\hat{r}\hat{U})) \Rightarrow \hat{J}(\hat{w}) = 0.$$

In conclusion, $\hat{J} \in K^j(a(\hat{W}', \hat{W}''))$ and $K^j(\hat{W}'') \subset K^j(a(\hat{W}', \hat{W}''))$. Therefore, $K^j(a(\hat{W}', \hat{W}'')) = K^j(\hat{W}'')$.

Since the constraint set—i.e., $\hat{W}$ and any test function v in $K^j(\cdot)$—is the same for $a(\hat{W}', \hat{W}'')$ and $\hat{W}''$:

$$\begin{aligned} BR^j(a(\hat{W}', \hat{W}'')) &= BR^j(\hat{W}''), \\ &= a(BR^j(\hat{W}''), BR^j(\hat{W}'')), \\ &\ge a(BR^j(\hat{W}'), BR^j(\hat{W}'')), \end{aligned}$$

where the last inequality uses monotonicity of $BR^j(\hat{W})$. A similar property holds for $BR^h(\hat{J})$. In conclusion, $BR^j(\hat{W})$ and $BR^h(\hat{J})$ are increasing and strongly order concave. Using this result, for $\hat{W}' \le \hat{W}''$:

$$\begin{aligned} Q(a(\hat{W}', \hat{W}'')) &= BR^h(BR^j(a(\hat{W}', \hat{W}''))) \\ &\ge^{(1)} BR^h(a(BR^j(\hat{W}'), BR^j(\hat{W}''))) \\ &\ge^{(2)} a(BR^h(BR^j(\hat{W}')), BR^h(BR^j(\hat{W}''))) \\ &= a(Q(\hat{W}'), Q(\hat{W}'')). \end{aligned}$$

Step (1) uses the monotonicity of $BR^h(\hat{J})$ and the strong order concavity of $BR^j(\hat{W})$. Step (2) uses the strong order concavity of $BR^h(\hat{J})$. □

Proposition A.9. *$Q : \mathcal{A} \to A$ has a unique fixed point.*

Proof of Step 2—Proposition A.9. We have shown that $Q(\hat{W})$ is monotone and order concave defined in an order convex set. Now, we prove the result by contradiction. Let $\underline{\hat{W}}$ be the minimum fixed point and let $\hat{W}^*$ be another fixed point with $\hat{W}^* > \underline{\hat{W}}$ (here, $>$ stand for $\hat{W}^*(\hat{w}) \ge \underline{\hat{W}}(\hat{w})$ for $\hat{w}$ and with stricly inequality for some $\hat{w}$). Then, we can write $\underline{\hat{W}} = a^*(0, \hat{W}^*)$ for some $a^*(\hat{w})$ function, where zero is the lower bound in the domain. Importantly, it is easy to see that $a^*(\hat{w}) \in [0, 1]$ for all

$\hat{w} \in (\log(\hat{r}\hat{U}, 0))$ and open interval for some $\hat{w}$. Thus,

$$\underline{\hat{W}} =^{(1)} Q(\underline{\hat{W}}) =^{(2)} Q(a^*(0, \hat{W}^*)) \geq^{(3)} a^*(Q(0), Q(\hat{W}^*)) =^{(4)} a^*(Q(0), \hat{W}^*) >^{(5)} a^*(0, \hat{W}^*) =^{(6)} \underline{\hat{W}}$$

Step (1) uses the fact that $\underline{\hat{W}}$ is a fixed point and step (2) uses the fact that $\underline{\hat{W}} = a^*(0, \hat{W}^*)$. Step (3) uses the strong order concavity of Q. Step (4) uses the fact that $\hat{W}^*$ is a fixed point. Step (5) uses that $Q(0) > 0$ for all $\hat{w} \in (\log(\hat{r}\hat{U}), 0)$. Since it cannot be that $\underline{\hat{W}} > \underline{\hat{W}}$, we have a contradiction. □

Step 3. Let $\hat{W}^*(\hat{w}; \hat{r}\hat{U})$ and $\hat{J}^*(\hat{w}; \hat{r}\hat{U})$ be the value functions from the unique nontrivial Nash equilibrium. We now show that they are continuous and decreasing in $\hat{U}$.

Proposition A.10. *Fix $\hat{J}$. Let $\hat{W}(\hat{w}; \hat{r}\hat{U}) = BR^h(\hat{J}; \hat{r}\hat{U})$ be the solution of*

$$a(\hat{W}, v - \hat{W}) \geq (1 - \hat{r}\hat{U}, v - \hat{W}),\ \forall v \in K^h(\hat{J}),\ \hat{W} \in K^h(\hat{J})$$

Then, $\hat{W}(\hat{w}; \hat{r}\hat{U})$ is continuous and decreasing in $\hat{r}\hat{U}$.

Proof of Step 3—Proposition A.10. First, we prove continuity. Take $\hat{U}_1$ and $\hat{U}_2$ and define $\hat{W}_1 = BR^h(\hat{J}; \hat{r}\hat{U}_1)$ and $\hat{W}_2 = BR^h(\hat{J}; \hat{r}\hat{U}_2)$. Then,

$$a(\hat{W}_1, v - \hat{W}_1) \geq (1 - \hat{r}\hat{U}_1, v - \hat{W}_1), \tag{A.20}$$

$$a(\hat{W}_2, v - \hat{W}_2) \geq (1 - \hat{r}\hat{U}_2, v - \hat{W}_2). \tag{A.21}$$

Let $\hat{W}_2$ be the test function for (A.20) and let $\hat{W}_1$ be the test function for (A.21). Summing both equations

$$a(\hat{W}_1, \hat{W}_2 - \hat{W}_1) + a(\hat{W}_2, \hat{W}_1 - \hat{W}_2) \geq (1 - \hat{r}\hat{U}_1, \hat{W}_2 - \hat{W}_1) + (1 - \hat{r}\hat{U}_2, \hat{W}_1 - \hat{W}_2)$$

or equivalently

$$a(\hat{W}_1 - \hat{W}_2, \hat{W}_2 - \hat{W}_1) \geq (\hat{r}(\hat{U}_2 - \hat{U}_1), \hat{W}_2 - \hat{W}_1).$$

Multiplying by -1 on both sides and under the observation that $(\hat{r}(\hat{U}_2 - \hat{U}_1), \hat{W}_2 - \hat{W}_1) = \hat{r}(\hat{U}_2 - \hat{U}_1)(1, \hat{W}_2 - \hat{W}_1)$, we obtain

$$a(\hat{W}_2 - \hat{W}_1, \hat{W}_2 - \hat{W}_1) \leq \hat{r}(\hat{U}_1 - \hat{U}_2)(1, \hat{W}_2 - \hat{W}_1).$$

Given that the operator is coercive and that

$$(1, \hat{W}_2 - \hat{W}_1) = \int_{\mathbb{R}} \left(\hat{W}(\hat{w}; \hat{r}\hat{U}_2) - \hat{W}(\hat{w}; \hat{r}\hat{U}_1)\right) \mathrm{d}\hat{w} \leq \left(\int_{\mathbb{R}} \left(\hat{W}(\hat{w}; \hat{r}\hat{U}_2) - \hat{W}(\hat{w}; \hat{r}\hat{U}_1)\right)^2 \mathrm{d}\hat{w}\right)^{1/2},$$

we have

$$b||\hat{W}_2 - \hat{W}_1||^2 \leq a(\hat{W}_2 - \hat{W}_1, \hat{W}_2 - \hat{W}_1) \leq \hat{r}(\hat{U}_1 - \hat{U}_2)(1, \hat{W}_2 - \hat{W}_1) \leq \hat{r}|\hat{U}_1 - \hat{U}_2|||\hat{W}_2 - \hat{W}_1||$$

for some $b > 0$. Thus, $||\hat{W}_2 - \hat{W}_1|| \leq \frac{\hat{r}}{b}|\hat{U}_1 - \hat{U}_2|$. With this inequality, we can verify the continuity of $\hat{W}(\hat{w}; \hat{r}\hat{U})$. Let $e > 0$ and choose $|\hat{U}_1 - \hat{U}_2| < e\frac{b}{\hat{r}}$. Then, $||\hat{W}_2 - \hat{W}_1|| < e$. Thus, $\hat{W}(\hat{w}; \hat{r}\hat{U})$ is continuous.

Now, we prove that $\hat{W}(\hat{w}; \hat{r}\hat{U})$ is decreasing in the second argument. Let $\hat{U}_1 > \hat{U}_2$ and define $\hat{W}_1 = BR^h(\hat{J}; \hat{r}\hat{U}_1)$ and $\hat{W}_2 = BR^h(\hat{J}; \hat{r}\hat{U}_2)$. Observe that $\hat{W}_1, \hat{W}_2 \in K^h(\hat{J})$. Thus, $\min\{\hat{W}_1, \hat{W}_2\}$ and $\max\{\hat{W}_1, \hat{W}_2\} \in K^h(\hat{J})$. Therefore, we can use $\min\{\hat{W}_1, \hat{W}_2\} = \hat{W}_1 - \max\{\hat{W}_1 - \hat{W}_2, 0\}$ as a test function with $\hat{U}_1$ and $\max\{\hat{W}_1, \hat{W}_2\} = \hat{W}_2 + \max\{\hat{W}_1 - \hat{W}_2, 0\}$ as a test function with $\hat{U}_2$. Therefore,

$$\begin{aligned} -a(\hat{W}_1, \max\{\hat{W}_1 - \hat{W}_2, 0\}) &\geq -(1 - \hat{r}\hat{U}_1, \max\{\hat{W}_1 - \hat{W}_2, 0\}), \\ a(\hat{W}_2, \max\{\hat{W}_1 - \hat{W}_2, 0\}) &\geq (1 - \hat{r}\hat{U}_2, \max\{\hat{W}_1 - \hat{W}_2, 0\}). \end{aligned}$$

Adding both inequalities, we obtain

$$a(\hat{W}_2 - \hat{W}_1, \max\{\hat{W}_1 - \hat{W}_2, 0\}) \geq \hat{r}\left(\hat{U}_1 - \hat{U}_2\right)(1, \max\{\hat{W}_1 - \hat{W}_2, 0\}).$$

Multiplying by -1 and under the observation that $a(\hat{W}^1 - \hat{W}^2, \max\{\hat{W}^1 - \hat{W}^2, 0\}) = a(\max\{\hat{W}^1 - \hat{W}^2, 0\}, \max\{\hat{W}^1 - \hat{W}^2, 0\}) \geq b||\max\{\hat{W}^1 - \hat{W}^2, 0\}||^2$ for some $b > 0$, we have

$$||\max\{\hat{W}_1 - \hat{W}_2, 0\}||^2 \leq \frac{\hat{r}}{b}\left(\hat{U}_2 - \hat{U}_1\right)(1, \max\{\hat{W}_1 - \hat{W}_2, 0\}).$$

Since $\hat{U}_1 > \hat{U}_2$, we have that $\hat{U}_2 - \hat{U}_1 < 0$. Assume, by contradiction, that $\hat{W}_1 > \hat{W}_2$, then $(1, \max\{\hat{W}_1 - \hat{W}_2, 0\}) > 0$. Operating,

$$0 < ||\max\{\hat{W}_1 - \hat{W}_2, 0\}||^2 \leq \frac{\hat{r}}{b}\left(\hat{U}_2 - \hat{U}_1\right)(1, \max\{\hat{W}_1 - \hat{W}_2, 0\}) < 0.$$

Thus, we have a contradiction. In conclusion, $\hat{W}(\hat{w}; \hat{r}\hat{U})$ is decreasing in the second argument.

Observe that $\hat{J}(\hat{w}) = BR^j(\hat{W})$ is independent of $\hat{\rho}\hat{U}$. ☐

Proposition A.11. *Let $\hat{W}^*(\hat{w};\hat{\rho}\hat{U})$ be the nontrivial Nash Equilibrium, then it is continuous and decreasing in the second argument.*

Proof of Step 3—Proposition A.11. First, we show that the value function in the nontrivial Nash equilibrium is decreasing in $\hat{U}$. If $\hat{U}_1 > \hat{U}_2$, we have, by the previous step, that $Q(\hat{W},\hat{\rho}\hat{U}_1) \leq Q(\hat{W},\hat{\rho}\hat{U}_2)$. Define recursively $Q^n(\hat{W},\hat{\rho}\hat{U}_1) = Q \circ Q^{n-1}(\hat{W},\hat{\rho}\hat{U}_1)$. By monotonicity, $Q^n(\hat{W},\hat{\rho}\hat{U}_1) \leq Q^n(\hat{W},\hat{\rho}\hat{U}_2)$ holds for all n. By Theorem 18 of Marinacci and Montrucchio (2019), $Q^n(\hat{W},\hat{\rho}\hat{U}_1) \to \hat{W}^*(\hat{w};\hat{\rho}\hat{U}_1)$ and $Q^n(\hat{W},\hat{\rho}\hat{U}_2) \to \hat{W}^*(\hat{w};\hat{\rho}\hat{U}_2)$. Thus, $\hat{W}^*(\hat{w};\hat{\rho}\hat{U}_1) \leq \hat{W}^*(\hat{w};\hat{\rho}\hat{U}_2)$. In conclusion, the nontrivial Nash equilibrium is decreasing in $\hat{U}$.

Now, we show continuity. Take $\hat{U}_n \uparrow \hat{U}^*$ (resp. $\hat{U}_n \downarrow \hat{U}^*$). Then, it is easy to see that $\hat{W}^*(\hat{w};\hat{\rho}\hat{U}^n)$ is monotonic, and by completeness, it is easy to see that $\hat{W}^*(\hat{w};\hat{\rho}\hat{U}^n)$ is a convergent series. Thus, $\hat{W}^*(\hat{w};\hat{\rho}\hat{U})$ is continuous in the second element. ☐

Step 4. We now show the existence of the unique fixed point in $\hat{\rho}\hat{U}$. Using the free entry condition, we can define the value of the unemployed worker as

$$P(\hat{\rho}\hat{U}) := \tilde{B} + \max_{\hat{w}} \frac{1}{\tilde{K}^{1/a}} \hat{J}(\hat{w};\hat{\rho}\hat{U})^{\frac{1-a}{a}} \hat{W}(\hat{w};\hat{\rho}\hat{U}).$$

We now show two propositions: (i) we show relevant properties of $P(\hat{\rho}\hat{U})$, (ii) we use these properties to show the existence of a unique fixed point $P(\hat{\rho}\hat{U}^*) = \hat{\rho}\hat{U}^*$.

Proposition A.12. *The following properties hold for $P(\hat{\rho}\hat{U})$:*

- *$P(\hat{\rho}\hat{U})$ exists and is unique.*
- *$P(\hat{\rho}\hat{U})$ is continuous.*
- *$P : [\tilde{B},\bar{P}] \to [\tilde{B},\bar{P}]$ and it is decreasing.*

Proof of Step 4—Proposition A.12. From Proposition 2, $\hat{\mathcal{Z}}^h \cap \hat{\mathcal{Z}}^j$ is bounded, so

$$\max_{\hat{w}} \frac{1}{\tilde{K}^{1/a}} \hat{J}(\hat{w};\hat{\rho}\hat{U})^{\frac{1-a}{a}} \hat{W}(\hat{w};\hat{\rho}\hat{U}) = \max_{\hat{w}\in cl\{\mathcal{Z}^j\cap\mathcal{Z}^h\}} \frac{1}{\tilde{K}^{1/a}} \hat{J}(\hat{w};\hat{\rho}\hat{U})^{\frac{1-a}{a}} \hat{W}(\hat{w};\hat{\rho}\hat{U}).$$

Since $\hat{J}(\cdot;\hat{\rho}\hat{U})$ and $\hat{W}(\cdot;\hat{\rho}\hat{U})$ are continuous and the optimization is over a compact support, by the extreme value theorem there exists a maximum, which is unique.

Since $\hat{J}(\hat{w}; \hat{r}\hat{U})$ and $\hat{W}(\hat{w}; \hat{r}\hat{U})$ are continuous in both arguments, by the maximum theorem, the maximal value is continuous.

Let $\hat{w}^*(\hat{r}\hat{U})$ be the solution to the optimization problem. Then, if $\hat{U} < \hat{U}'$,

$$\frac{1}{\tilde{K}^{1/a}} \hat{J}(\hat{w}^*(\hat{r}\hat{U}); \hat{r}\hat{U})^{\frac{1-a}{a}} \hat{W}(\hat{w}^*(\hat{r}\hat{U}); \hat{r}\hat{U}) \geq^{(1)} \frac{1}{\tilde{K}^{1/a}} \hat{J}(\hat{w}^*(\hat{r}\hat{U}'); \hat{r}\hat{U})^{\frac{1-a}{a}} \hat{W}(\hat{w}^*(\hat{r}\hat{U}'), \hat{r}\hat{U})$$
$$\geq^{(2)} \frac{1}{\tilde{K}^{1/a}} \hat{J}(\hat{w}^*(\hat{r}\hat{U}'), \hat{r}\hat{U}')^{\frac{1-a}{a}} \hat{W}(\hat{w}^*(\hat{r}\hat{U}'), \hat{r}\hat{U}').$$

Step (1) uses the optimality of $\hat{w}^*(\hat{r}\hat{U})$ and step (2) uses the fact that $\hat{J}$ and $\hat{W}$ are decreasing in the second argument. Thus, $P(\hat{r}\hat{U})$ is decreasing. By Proposition 2, we have that $\bar{P} =: P(\tilde{B}) > \tilde{B}$. Since $P(\hat{r}\hat{U}) \geq \tilde{B}$ ($\hat{J}(\cdot)$ and $\hat{W}(\cdot)$ are nonnegative), we have that $P : [\tilde{B}, \bar{P}] \to [\tilde{B}, \bar{P}]$. □

Proposition A.13. *$P(\hat{r}\hat{U})$ has a unique fixed point.*

Proof of Step 4—Proposition A.13. The existence of the fixed point follows directly from Brouwer's fixed point theorem. To show uniqueness, observe that if there were two fixed points $\hat{U}_1 < \hat{U}_2$, by definition, we would have that $P(\hat{r}\hat{U}_1) = \hat{r}\hat{U}_1 < \hat{r}\hat{U}_2 = P(\hat{r}\hat{U}_2)$ and $P(\hat{r}\hat{U})$ would be strictly increasing. By Step 4 of Proposition A.12, this is a contradiction. □

A.3 Proof of Proposition 2

Proof. We prove each equilibrium property separately.

1. Using the recursive definition of the value function, we have

$$\hat{W}(\hat{w}) = \mathbb{E}\left[\int_0^{t^{m*}} e^{-\hat{r}t}(e^{\hat{w}_t} - \hat{r}\hat{U})\, dt | \hat{w}_0 = \hat{w}\right]$$
$$\hat{J}(\hat{w}) = \mathbb{E}\left[\int_0^{t^{m*}} e^{-\hat{r}t}(1 - e^{\hat{w}_t})\, dt | \hat{w}_0 = \hat{w}\right]$$

where t^{m*} is the nontrivial Nash equilibrium of the game between the firm and the worker. Summing up the previous two equations, we have

$$\hat{S}(\hat{w}) := \hat{W}(\hat{w}) + \hat{J}(\hat{w}) = \mathbb{E}_{\hat{w}}\left[\int_0^{t^{m*}} e^{-\hat{r}t}(1 - \hat{r}\hat{U})\, dt\right] = (1 - \hat{r}\hat{U})\mathcal{T}(\hat{w}, \hat{r}).$$

Now, we show that $1 > \hat{r}\hat{U} > \tilde{B}$ by contradiction. Assume that $\hat{r}\hat{U} \leq \tilde{B} < 1$. Using the free entry condition and worker optimality, we have that $\hat{q}(\hat{w}) \geq 0$ and $\hat{W}(\hat{w}) \geq 0$ for all $\hat{w}$; thus, the product is also nonnegative at $\hat{w}^*$ and

$$\hat{r}\hat{U} = \tilde{B} + \max_{\hat{w}} f(\hat{q}(\hat{w}))\hat{W}(\hat{w}) \geq \tilde{B} \quad \Longrightarrow \quad \hat{r}U \geq \tilde{B},$$

So, we have that $\hat{r}\hat{U} = \tilde{B} < 1$. Then, we have that $\max_{\hat{w}} f(\hat{q}(\hat{w}))\hat{W}(\hat{w}) = 0$ and, therefore, $f(\hat{q}(\hat{w}))\hat{W}(\hat{w}) = 0 \; \forall \hat{w}$. By weakly dominated strategies, we have that $(\log(\hat{r}\hat{U}), 0) = (\log(\tilde{B}), 0) \subset \mathcal{Z}^j \cap \mathcal{Z}^h$. Thus, for any $\hat{w} \in (\log(\tilde{B}), 0)$, we have that $(\hat{J}(\hat{w}), \hat{W}(\hat{w})) > (0, 0)$ and using the free entry condition $f(\hat{q}(\hat{w}))\hat{W}(\hat{w}) > 0$. Thus, a contradiction. Assume instead that $\hat{r}\hat{U} \geq 1$. Then, $\mathcal{T}(\hat{w}^*, \hat{r}) = 0$ for all $\hat{w}$ since $\hat{S}(\hat{w})$ is nonnegative and $0 = \hat{S}(\hat{w}) \geq (\hat{J}(\hat{w}), \hat{W}(\hat{w})) \geq 0 \; \forall \hat{w}$ and $\max_{\hat{w}} f(\hat{q}(\hat{w}))\hat{W}(\hat{w}) = 0$ with the free entry condition. With these argument, we have that $\hat{r}\hat{U} = \tilde{B} + \max_{\hat{w}} f(\hat{q}(\hat{w}))\hat{W}(\hat{w}) = \tilde{B} < 1$, and we have the contradiction.

2. To show this, we first show that $\hat{J}(\hat{w}) > 0$ for all $\hat{w} \in (\log(\hat{r}\hat{U}), 0)$. Let

$$t_{(w^-,0)} = \inf_t \{t : \hat{w}_t \notin (\log(\hat{r}\hat{U}), 0)\}.$$

By optimality of the firm,

$$\hat{J}(\hat{w}) = \mathbb{E}_{\hat{w}}\left[\int_0^{t^{m*}} e^{-\hat{r}t}(1 - e^{\hat{w}_t})\,\mathrm{d}t\right] \geq \mathbb{E}_{\hat{w}}\left[\int_0^{\min\{t_{(\log(\hat{r}\hat{U}),0)}, t^{m*}\}} e^{-\hat{r}t}(1 - e^{\hat{w}_t})\,\mathrm{d}t\right] > 0.$$

Thus, there is an open set around the optimally chosen starting wage $\hat{w}$ that lies entirely within the continuation region s.t. $\hat{J}(\hat{w}) > 0$, $\hat{q}(\hat{w}) > 0$, and $\hat{J}(\hat{w}) - \hat{K}\hat{q}(\hat{w})^a = 0$. Therefore,

$$\arg\max_{\hat{w}} \{f(\hat{q}(\hat{w}))\hat{W}(\hat{w})\} = \arg\max_{\hat{w}} \left\{\left(\frac{\hat{J}(\hat{w})}{\hat{K}}\right)^{\frac{1-a}{a}} \hat{W}(\hat{w})\right\} = \arg\max_{\hat{w}} \left\{\hat{J}(\hat{w})^{1-a}\hat{W}(\hat{w})^a\right\}.$$

Since $\hat{W}(\hat{w}) = h(\hat{w})\hat{S}(\hat{w})$ and $\hat{J}(\hat{w}) = (1 - h(\hat{w}))\hat{S}(\hat{w})$ and $\hat{S}(\hat{w}) = (1 - \hat{r}\hat{U})\mathcal{T}(\hat{w}, \hat{r})$,

$$\arg\max_{\hat{w}} \{f(\hat{q}(\hat{w}))\hat{W}(\hat{w})\} = \arg\max_{\hat{w}} \left\{\hat{J}(\hat{w})^{1-a}\hat{W}(\hat{w})^a\right\} = \arg\max_{\hat{w}} \left\{(1 - h(\hat{w}))^{1-a} h(\hat{w})^a \mathcal{T}(\hat{w}, \hat{r})\right\}.$$

Taking first-order conditions, $h'(\hat{w}^*)\left(\frac{a}{h(\hat{w}^*)} - \frac{1-a}{1-h(\hat{w}^*)}\right) = -\frac{\mathcal{T}'_{\hat{w}}(\hat{w}^*, \hat{r})}{\mathcal{T}(\hat{w}^*, \hat{r})}$. We now show the following claim: There exists a unique solution to

$$\max_{\hat{w}} \hat{W}(\hat{w})^a \hat{J}(\hat{w})^{1-a}. \tag{A.22}$$

We divide this proof into 4 steps.

• **The following result holds:**

$$\arg\max_{\hat{w}} \hat{W}(\hat{w})^{a}\hat{J}(\hat{w})^{1-a} = \arg\max_{\hat{w}\in[\hat{w}^{-},\hat{w}^{+}]} a\log\left(\hat{W}(\hat{w})\right) + (1-a)\log\left(\hat{J}(\hat{w})\right).$$

As we show below, for $s^2 > 0$ we have $-¥ < \hat{w}^{-} < \log(\hat{r}\hat{U}) < 0 < \hat{w}^{+} < ¥$. Now, we show that there exists a $\hat{w} \in (\hat{w}^{-}, \hat{w}^{+})$ such that $\hat{W}(\hat{w}) > 0$ and $\hat{J}(\hat{w}) > 0$ by constradiction. Assume the opposite inequalities hold. Then, since the values satisfy $\hat{W}(\hat{w}) \geq 0$ and $\hat{J}(\hat{w}) \geq 0$, it must be the case that $\hat{W}(\hat{w}) = \hat{J}(\hat{w}) = 0$. Replacing these equalities into the definition of the values, we obtain

$$(r+d)\hat{W}(\hat{w}) = e^{\hat{w}} - \hat{r}\hat{U}, \quad (r+d)\hat{J}(\hat{w}) = 1 - e^{\hat{w}}, \tag{A.23}$$

which results in a contradiction since the values that satisfy (A.23) are positive for any $\hat{w} \in (\log(\hat{r}\hat{U}), 0)$. Thus, we can restrict the domain of $\hat{w}$ to $[\hat{w}^{-}, \hat{w}^{+}]$ in problem (A.22).

• **Problem (A.22) attains a maximum.** This result follows from the Weierstrass Theorem since the set $[\hat{w}^{-}, \hat{w}^{+}]$ is compact and the objective function is the composition and sum of two continuous value functions.

• **The functions $\hat{J}(\hat{w})$ and $\hat{W}(\hat{w})$ have unique global maxima $\hat{w}^{*j} < \hat{w}^{*h}$.** We will show that $\hat{w}^{*j} = \arg\max_{\hat{w}} \hat{J}(\hat{w})$ is unique. The proof for $\hat{W}(\hat{w})$ is similar. Assume, by contradiction, that there exist at least two global maxima at $\hat{w}^{*j} < \hat{w}^{**j}$ (from the argument above, we conclude that these maxima cannot occur at the boundary of the game's continuation set). Without loss of generality, assume they are consecutive. The HJB equation within the game's continuation set is given by $(\hat{r}+d)\hat{J}(\hat{w}) = 1 - e^{\hat{w}} - \hat{g}\hat{J}'(\hat{w}) + \frac{s^2}{2}\hat{J}''(\hat{w})$. Since the function is smooth, at the two optima, we have

$$(\hat{r}+d)\hat{J}(\hat{w}^{*j}) + e^{\hat{w}^{*j}} - 1 = \frac{s^2}{2}\hat{J}''(\hat{w}^{*j}),$$
$$(\hat{r}+d)\hat{J}(\hat{w}^{**j}) + e^{\hat{w}^{**j}} - 1 = \frac{s^2}{2}\hat{J}''(\hat{w}^{**j}),$$

with $\hat{J}(\hat{w}) \leq \hat{J}(\hat{w}^{*j})$ for all $\hat{w} \in (\hat{w}^{*j}, \hat{w}^{**j})$. There are two cases to consider. First, $\hat{J}(\hat{w}) = \hat{J}(\hat{w}^{*j})$ for all $\hat{w} \in (\hat{w}^{*j}, \hat{w}^{**j})$. Here, we have a contradiction since $\hat{J}(\hat{w})$ is constant in the interval, thus $\hat{J}'(\hat{w}) = \hat{J}''(\hat{w}) = 0$ for all $\hat{w} \in (\hat{w}^{*j}, \hat{w}^{**j})$ and $(\hat{r}+d)\hat{J}(\hat{w}^{*j}) + e^{\hat{w}} - 1 = 0$, $\forall \hat{w} \in (\hat{w}^{*j}, \hat{w}^{**j})$, which is not constant. Next, assume that the function is not constant. Then, since $\hat{J}(\hat{w})$ is continuous and the set $[\hat{w}^{*j}, \hat{w}^{**j}]$ is compact, the function has a minimum at some $\hat{w}^{\min j} < \hat{w}^{**j}$ satisfying $\hat{J}(\hat{w}^{\min j}) < \hat{J}(\hat{w}^{**j})$ and $e^{\hat{w}^{\min j}} - 1 < e^{\hat{w}^{**j}} - 1$. By definition of minimum, $\hat{J}''(\hat{w}^{\min j}) \geq 0$. Therefore, combining the previous inequalities, we have

$$0 \leq \frac{s^2}{2}\hat{J}''(\hat{w}^{\min j}) = (\hat{r}+d)\hat{J}(\hat{w}^{\min j}) + e^{\hat{w}^{\min j}} - 1 < (\hat{r}+d)\hat{J}(\hat{w}^{**j}) + e^{\hat{w}^{**j}} - 1 = \frac{s^2}{2}\hat{J}''(\hat{w}^{**j}).$$

Since the function is concave near a maximum, we have a contradiction. We can follow similar steps to rule multiple local maxima. Finally, it is easy to show that $\hat{w}^{*j} < \hat{w}^{*h}$.

• **There exists a unique** $\arg\max_{\hat{w}\in[\hat{w}^{*j},\hat{w}^{*h}]} a\log\left(\hat{W}(\hat{w})\right) + (1-a)\log\left(\hat{J}(\hat{w})\right)$**.** We first show that $\hat{W}(\hat{w})$ is strictly log-concave $\forall \hat{w} \in (\hat{w}^-, \hat{w}^{*h})$. The proof that shows that $\hat{J}(\hat{w})$ is log-concave is similar. Applying L'Hôpital's rule, we have that $\lim_{\hat{w}\downarrow\hat{w}^-} \frac{\hat{W}'(\hat{w})}{\hat{W}(\hat{w})} = \lim_{\hat{w}\downarrow\hat{w}^-} \frac{\hat{W}''(\hat{w})}{\hat{W}'(\hat{w})}$. Recall that $(d+\hat{r})\hat{W}(\hat{w}) = e^{\hat{w}} - \hat{r}\hat{U} - \hat{g}\hat{W}'(\hat{w}) + \frac{s^2}{2}\hat{W}''(\hat{w})$. Taking the limit $\hat{w}\downarrow\hat{w}^-$ and using the border conditions $\hat{W}(\hat{w}^-) = \hat{W}'(\hat{w}^-) = 0$, we have that $0 < \hat{r}\hat{U} - e^{\hat{w}^-} = \frac{s^2}{2}\hat{W}''(\hat{w})$. Therefore, $\lim_{\hat{w}\downarrow\hat{w}^-} \frac{\hat{W}''(\hat{w})}{\hat{W}'(\hat{w})} = ¥$. It is easy to check that $\frac{\hat{W}'(\hat{w})}{\hat{W}(\hat{w})}$ has a vertical asymptote when $\hat{w}\downarrow\hat{w}^-$ and, therefore, it must be decreasing near $\hat{w}^-$ from the right. Let $\underline{\hat{w}}$ be a wage-to-productivity ratio close to $\hat{w}^-$ such that $\frac{\hat{W}'(\underline{\hat{w}})}{\hat{W}(\underline{\hat{w}})} > 0$ and $\left(\frac{\hat{W}'(\underline{\hat{w}})}{\hat{W}(\underline{\hat{w}})}\right)' < 0$. Since $\hat{W}(\hat{w})$ has a single maximum $\hat{w}^{*h}$, $\frac{\hat{W}'(\hat{w})}{\hat{W}(\hat{w})}$ is positive for all $\hat{w} \in [\underline{\hat{w}}, \hat{w}^{*h})$ and $\frac{\hat{W}'(\hat{w})}{\hat{W}(\hat{w})} = 0$ when evaluated at $\hat{w} = \hat{w}^{*h}$. Now, we show that $\frac{\hat{W}'(\hat{w})}{\hat{W}(\hat{w})}$ is decreasing for all $\hat{w} \in [\underline{\hat{w}}, \hat{w}^{*h})$. Using the worker's HJB equation and the corresponding smooth-pasting and value-matching conditions, we have

$$\hat{W}'(\hat{w}) = \frac{2}{s^2}\int_{\hat{w}^-}^{\hat{w}} \left[(d+\hat{r})\hat{W}(x) - (e^x - \hat{r}\hat{U})\right] \mathrm{d}x + \frac{2\hat{g}}{s^2}\hat{W}(\hat{w}).$$

Dividing both sides by $\hat{W}(\hat{w})$,

$$\frac{\hat{W}'(\hat{w})}{\hat{W}(\hat{w})} = \frac{2}{s^2}\frac{\int_{\hat{w}^-}^{\hat{w}} \left[(d+\hat{r})\hat{W}(x) - (e^x - \hat{r}\hat{U})\right] \mathrm{d}x}{\hat{W}(\hat{w})} + \frac{2\hat{g}}{s^2}.$$

Taking the derivative w.r.t. $\hat{w}$, we obtain

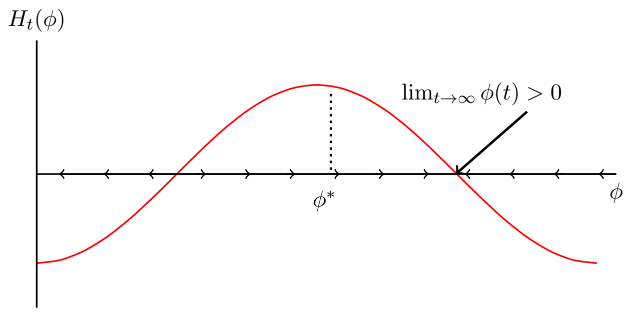
$$\left(\frac{\hat{W}'(\hat{w})}{\hat{W}(\hat{w})}\right)' = \frac{2}{s^2}\left[(\hat{r}+d) - \frac{(e^{\hat{w}} - \hat{r}\hat{U})}{\hat{W}(\hat{w})}\right] + \frac{2\hat{g}}{s^2}\frac{\hat{W}'(\hat{w})}{\hat{W}(\hat{w})} - \left(\frac{\hat{W}'(\hat{w})}{\hat{W}(\hat{w})}\right)^2.$$

Define the following function $f(\hat{w} - \underline{\hat{w}}) \equiv \frac{\hat{W}'(\hat{w})}{\hat{W}(\hat{w})}$. Then,

$$f'(\hat{w} - \underline{\hat{w}}) = \frac{2}{s^2}\left[(\hat{r}+d) - \frac{(e^{\hat{w}} - \hat{r}\hat{U})}{\hat{W}(\hat{w})}\right] + \frac{2\hat{g}}{s^2} f(\hat{w} - \underline{\hat{w}}) - f(\hat{w} - \underline{\hat{w}})^2.$$

Given this, the goal is to show $f'(\hat{w} - \underline{\hat{w}}) < 0$. Let $t \equiv \hat{w} - \underline{\hat{w}}$, then $f'(t) = \frac{2}{s^2}\left[(\hat{r}+d) - \frac{(e^{t+\underline{\hat{w}}} - \hat{r}\hat{U})}{\hat{W}(t+\underline{\hat{w}})}\right] +$

FIGURE A2. PHASE LINE FOR $f(t)$



$\frac{2\hat{g}}{s^2} f(t) - f(t)^2$. Next, we define $F(t) \equiv \frac{2}{s^2}\left[(\hat{r}+d) - \frac{(e^{t+\underline{\hat{w}}} - \hat{r}\hat{U})}{\hat{W}(t+\underline{\hat{w}})}\right]$. Thus, we have that the derivative of the log of the worker's value function satisfies the Ricatti equation $f'(t) = F(t) + \frac{2\hat{g}}{s^2} f(t) - f(t)^2$, with initial condition $f(0) > 0$, $f'(0) < 0$. Define $T = \hat{w}^{*h} - \underline{\hat{w}}$, then $f(T) = 0$ (which follows from $\hat{w}^{*h}$ being an interior maximum). Now, we show that $f'(t) < 0$ for all $t \in (0, T)$. Assume that this is not the case and there exists a $t^* \in (0, T)$ s.t. $f'(t^*) \geq 0$. Without loss of generality, let t^* be inside the first interval s.t. $f'(t) \geq 0$. Then, if we plot $H_t(f) := F(t) + \frac{2\hat{g}}{s^2} f(t) - f(t)^2$, there exists a $(t^*, f^*) > (0, 0)$ s.t. $H_{t^*}(f^*) \geq 0$. From Figure A2, since $f'(0) < 0$ with $f(0)$ arbitrary large, we can see that $\lim_{t\to ¥} f(t) \geq f^* > 0$ and, therefore, $f(T) > 0$, which contradicts the terminal condition $f(T) = 0$. Thus, $f'(t) < 0$ for all $t \in (0, T)$ and $\log(\hat{W}(\hat{w}))$ is a concave function $\forall \hat{w} \in (\hat{w}^{*j}, \hat{w}^{*h})$.

Since $\log\left(\hat{W}(\hat{w})\right)$ and $\log\left(\hat{J}(\hat{w})\right)$ are strictly concave $\forall \hat{w} \in [\hat{w}^{*j}, \hat{w}^{*h}]$ and the sum of strictly concave functions is strictly concave, $\arg\max_{\hat{w}\in[\hat{w}^{*j},\hat{w}^{*h}]} a \log\left(\hat{W}(\hat{w})\right) + (1-a)\log\left(\hat{J}(\hat{w})\right)$ exists and is unique.

3. This step follows directly from workers' and firms' optimality conditions.

4. To show that $\hat{\mathcal{Z}}^h$ and $\hat{\mathcal{Z}}^j$ are connected, assume they are not. Without loss of generality, assume that $\hat{\mathcal{Z}}^h = \{\hat{w} : \hat{w} > \hat{w}^-\} \cup (a, b)$ with $a < b < w^-$. Then, since $\hat{w}^- \leq \hat{r}\hat{U}$, it must be the case that for all $\hat{w} \in (a, b)$, we have $(e^{\hat{w}} - \hat{r}\hat{U}) < 0$ for all $\hat{w} \in (a, b)$, and $\hat{W}(\hat{w}) = \mathbb{E}_{\hat{w}}\left[\int_0^{t_{\hat{\mathcal{Z}}^h\cap\hat{\mathcal{Z}}^j}} e^{-(\hat{r}+d)t}(e^{w_t} - \hat{r}\hat{U})\,\mathrm{d}t\right] < 0$ for all $\hat{w} \in (a, b)$ due to continuity of Brownian motions. Since $\hat{W}(\hat{w}) \geq 0$, we have a contradiction. A similar argument holds for the firm's continuation set.

We prove that $-¥ < \hat{w}^-$ by contradiction. Assume that $-¥ = \hat{w}^-$, then

$$\hat{W}(\hat{w}, \hat{w}^+) := \mathbb{E}\left[\int_0^{t_{(-¥,\hat{w}^+)}\wedge t^d} e^{-\hat{r}t}\left(e^{\hat{w}_t} - \hat{r}\hat{U}\right)\mathrm{d}t|\hat{w}_0 = \hat{w}\right].$$

Then, since $\hat{r}\hat{U} < e^{\hat{w}^+}$, it is easy to show

$$\begin{aligned}
\hat{W}(\hat{w}, \hat{w}^+) &= \mathbb{E}\left[\int_0^{t_{(-¥,\hat{w}^+)} \wedge t^d} e^{-\hat{r}t}\left(e^{\hat{w}_t} - \hat{r}\hat{U}\right) dt | \hat{w}_0 = \hat{w}\right] \\
&\leq \mathbb{E}\left[\int_0^{¥} e^{-(\hat{r}+d)t}\left(e^{\hat{w}_t} - \hat{r}\hat{U}\right) dt | \hat{w}_0 = \hat{w}\right] \\
&= \frac{e^w}{\hat{r} + d + \hat{g} - s^2/2} - \frac{\hat{r}\hat{U}}{\hat{r} + d} \\
&= \frac{e^w}{r + d} - \frac{\hat{r}\hat{U}}{r - g - s^2/2 + d}
\end{aligned}$$

Thus, there exists a small enough $\hat{w}$ s.t. $\hat{W}(\hat{w}, \hat{w}^+) < 0$, a contradiction. A similar argument holds for the firm's separation threshold. The smooth pasting conditions are necessary and sufficient for optimality (see Brekke and Øksendal, 1990).

□

A.4 Proof of Propositions 3, 4, and 5

Define $\hat{\mathcal{Z}} = (\hat{w}^-, \hat{w}^+)$. From Proposition 1, when $\hat{g} > 0$ or $s > 0$, we can work with the HJB conditions

$$(\hat{r} + d)\hat{W}(\hat{w}) = e^{\hat{w}} - \hat{r}\hat{U} - \hat{g}\hat{W}'(\hat{w}) + \frac{s^2}{2}\hat{W}''(\hat{w}) \quad \forall \hat{w} \in \hat{\mathcal{Z}} \tag{A.24}$$

$$(\hat{r} + d)\hat{J}(\hat{w}) = 1 - e^{\hat{w}} - \hat{g}\hat{J}'(\hat{w}) + \frac{s^2}{2}\hat{J}''(\hat{w}) \quad \forall \hat{w} \in \hat{\mathcal{Z}} \tag{A.25}$$

$$\hat{r}\hat{U} = \tilde{B} + \tilde{K}^{1-a^{-1}}\hat{J}(\hat{w}^*)^{\frac{1-a}{a}}\hat{W}(\hat{w}^*)$$

$$(1-a)\frac{\mathrm{d}\log \hat{J}(\hat{w}^*)}{\mathrm{d}\hat{w}} = -a\frac{\mathrm{d}\log \hat{W}(\hat{w}^*)}{\mathrm{d}\hat{w}},$$

with the value-matching conditions $\hat{W}(\hat{w}^-) = \hat{J}(\hat{w}^-) = \hat{W}(\hat{w}^+) = \hat{J}(\hat{w}^+) = 0$ and smooth-pasting conditions $\hat{W}'(\hat{w}^-) = \hat{J}'(\hat{w}^+) = 0$.

Proof of Proposition 3. If $\hat{g} = s = 0$, conditions (A.24) and (A.25) imply $\hat{W}(\hat{w}) = \frac{e^{\hat{w}} - \hat{r}\hat{U}}{\hat{r}+d}$ and $\hat{J}(\hat{w}) = \frac{1-e^{\hat{w}}}{\hat{r}+d}$. The variation inequalities imply

$$(\hat{r} + d)\hat{W}(\hat{w}) = \max\{0, e^{\hat{w}} - \hat{r}\hat{U}\}, \ \forall \hat{w} \in \mathbb{R},$$

$$(\hat{r} + d)\hat{J}(\hat{w}) = \max\{0, 1 - e^{\hat{w}}\}, \quad \forall \hat{w} \in \mathbb{R}.$$

Thus, $\hat{W}(\hat{w}^-) = \hat{J}(\hat{w}^+) = 0$, $\hat{w}^+ = 0$ and $\hat{w}^- = \log\left(\hat{r}\hat{U}\right)$. Since

$$\mathcal{T}(\hat{w}, \hat{r}) = \begin{cases} (\hat{r}+d)^{-1} & \text{if } \hat{w} \in [\hat{w}^-, \hat{w}^+] \\ 0 & \text{otherwise,} \end{cases}$$

and $\mathcal{T}_{\hat{w}}(\hat{w}^*, \hat{r}) = 0$, we have that the worker's share of the surplus $h(\hat{w}^*) = a$. □

Proof of Proposition 4. Let us guess and verify the following solution $\hat{w}^* = \log\left(\frac{1+\hat{r}\hat{U}}{2}\right)$ and $\hat{w}^- = \hat{w}^* - h$ and $\hat{w}^+ = \hat{w}^* + h$ for a given h. Using a Taylor approximation of the flow profits around $\hat{w}^*$

$$e^{\hat{w}} - \hat{r}\hat{U} \approx e^{\hat{w}^*}(1 + (\hat{w} - \hat{w}^*)) - \hat{r}\hat{U} = \frac{1 - \hat{r}\hat{U}}{2} + e^{\hat{w}^*}(\hat{w} - \hat{w}^*),$$
$$1 - e^{\hat{w}} \approx 1 - e^{\hat{w}^*}(1 + (\hat{w} - \hat{w}^*)) = \frac{1 - \hat{r}\hat{U}}{2} - e^{\hat{w}^*}(\hat{w} - \hat{w}^*).$$

We can write the optimality conditions as

$$(\hat{r} + d)\hat{W}(\hat{w}) = \frac{1 - \hat{r}\hat{U}}{2} + e^{\hat{w}^*}(\hat{w} - \hat{w}^*) + \frac{s^2}{2}\hat{W}''(\hat{w}), \quad \forall \hat{w} \in (w^* - h, w^* + h)$$
$$(\hat{r} + d)\hat{J}(\hat{w}) = \frac{1 - \hat{r}\hat{U}}{2} - e^{\hat{w}^*}(\hat{w} - \hat{w}^*) + \frac{s^2}{2}\hat{J}''(\hat{w}), \quad \forall \hat{w} \in (w^* - h, w^* + h)$$

with the border conditions $\hat{W}(\hat{w}^* - h) = \hat{J}(\hat{w}^* - h) = \hat{W}(\hat{w}^* + h) = \hat{J}(\hat{w}^* + h) = 0$ and $\hat{W}'(\hat{w}^* - h) = \hat{J}'(\hat{w}^* + h) = 0$. Now, we show that we can transform $J(x) = \frac{\hat{J}(x+\hat{w}^*) - \frac{1-\hat{r}\hat{U}}{2(\hat{r}+d)}}{e^{\hat{w}^*}}$. A similar argument applies to the value function of the worker. Making the following transformation $J(x) = \frac{\hat{J}(x+\hat{w}^*) - \frac{1-\hat{r}\hat{U}}{2(\hat{r}+d)}}{e^{\hat{w}^*}}$, and using (A.25)

$$(\hat{r} + d)J(x) = -x + \frac{s^2}{2}J''(x).$$

Thus,

$$(\hat{r} + d)W(x) = x + \frac{s^2}{2}W''(x), \ \ (\hat{r} + d)J(x) = -x + \frac{s^2}{2}J''(x) \quad \forall x \in (-h, h)$$

Defining $F = \frac{\frac{1-\hat{r}\hat{U}}{2}}{e^{\hat{w}^*}} = \frac{1-\hat{r}\hat{U}}{1+\hat{r}\hat{U}} > 0$, it is easy to show that $W(h) = J(h) = W(-h) = J(-h) = -\frac{F}{\hat{r}+d}$ and $W'(-h) = J'(h) = 0$. Thus, $W(x) = J(-x)$. Given that this problem is symmetric, we verify the guess of symmetry of the Ss bands and $\frac{1}{2}W'(0) = -\frac{1}{2}J'(-0)$. The latter property implies that w^* satisfies the proposed Nash bargaining solution.

Now, we show that $h = w(j)\mathsf{F}$ with $j = \sqrt{2(\acute{r}+d)}/s$. Note that $W(x) = J(-x)$. Thus, we can only focus on $W(x)$ using the smooth pasting condition evaluated at $-h$. The solution to this system of differential equations is given by $W(x) = Ae^{jx} + Be^{-jx} + \frac{x}{\acute{r}+d}$ with border conditions $W(h) = W(-h) = -\frac{\mathsf{F}}{\acute{r}+d}$ and $W'(-h) = 0$, where $j = \sqrt{2(\acute{r}+d)}/s$. Writing the value-matching conditions

$$Ae^{jh} + Be^{-jh} + \frac{h}{\acute{r}+d} = -\frac{\mathsf{F}}{\acute{r}+d}; \quad Ae^{-jh} + Be^{jh} - \frac{h}{\acute{r}+d} = -\frac{\mathsf{F}}{\acute{r}+d}$$

Solving for A and B,

$$A = -\frac{1}{\acute{r}+d}\frac{e^{-jh}(-\mathsf{F}+h) + e^{jh}(h+\mathsf{F})}{e^{2jh} - e^{-2jh}}; \quad B = \frac{1}{\acute{r}+d}\frac{e^{jh}(-\mathsf{F}+h) + e^{-jh}(h+\mathsf{F})}{e^{2jh} - e^{-2jh}}$$

Therefore,

$$W(x) = -\frac{1}{\acute{r}+d}\frac{e^{-jh}(-\mathsf{F}+h) + e^{jh}(h+\mathsf{F})}{e^{2jh} - e^{-2jh}}e^{jx} + \frac{1}{\acute{r}+d}\frac{e^{jh}(-\mathsf{F}+h) + e^{-jh}(h+\mathsf{F})}{e^{2jh} - e^{-2jh}}e^{-jx} + \frac{x}{\acute{r}+d}$$

Taking the derivative, evaluating in $x = -h$ and imposing $W'(-h) = 0$, we obtain

$$-\mathsf{F}(e^{-2jh} + e^{2jh} - 2) = \frac{1}{j}(e^{2jh} - e^{-2jh}) - \frac{1}{2j}2jh\left(e^{2jh} + e^{-2jh} + 2\right). \tag{A.26}$$

It would be useful to express equation (A.26) using $\sinh(x) = \frac{e^x - e^{-x}}{2}$ and $\cosh(x) = \frac{e^x + e^{-x}}{2}$. Using the hyperbolic functions,

$$-\mathsf{F}2j\,(\cosh(2jh) - 1) = 2\sinh(2jh) - j2h\,(\cosh(2jh) + 1).$$

Next, we change variables with $q \equiv 2jh$ and define q as the implicit solution of

$$-2\mathsf{F}j\,(\cosh(q) - 1) + x\,(\cosh(q) + 1) = 2\sinh(q).$$

Thus, $h = \frac{q(2\mathsf{F}j)}{2j}$. Let $b = 2\mathsf{F}j > 0$, then we can express the function $x(\cdot)$ as the solution of $b = -\frac{2\sinh(q(b)) - q(b)(\cosh(q(b))+1)}{(\cosh(q(b))-1)}$. Notice that if we define $f(q) = -\frac{2\sinh(q) - q(\cosh(q)+1)}{(\cosh(q)-1)}$, the following properties about $f(q)$ hold:

1. $\lim_{q\to 0} f(q) = 0$ and $\lim_{q\to ¥} f(q) = ¥$.

2. $f(q)$ is increasing and convex, with $\lim_{q\to 0} f'(q) = 1/3$ and $\lim_{q\to ¥} f'(q) = 1$.

3. $\frac{\mathrm{d}\log(f(q))}{\mathrm{d}\log(q)} > 1.$

Given these properties, we can write $h(j, \mathsf{F}) = \frac{f^{-1}(2j\mathsf{F})}{2j}$ and show the following properties of $h(j, \mathsf{F})$

1. $h(j, \mathsf{F})$ is increasing in F: Since $f^{-1}(\cdot)$ is increasing, we have the result.

2. $h(j, \mathsf{F})$ is decreasing in j: Taking the derivative of $h(j, \mathsf{F}) = \frac{f^{-1}(2j\mathsf{F})}{2j}$ with respect to j and operating

$$\begin{aligned}\frac{¶h(j,\mathsf{F})}{¶j} &= \left.\frac{\mathrm{d}f^{-1}(q)}{\mathrm{d}q}\right|_{q=2j\mathsf{F}} \frac{2\mathsf{F}}{2j} - \frac{f^{-1}(2j\mathsf{F})}{2j^2} = \frac{f^{-1}(2j\mathsf{F})}{2j^2}\left[\left.\frac{\mathrm{d}f^{-1}(q)}{\mathrm{d}q}\right|_{q=2j\mathsf{F}} \frac{2j\mathsf{F}}{f^{-1}(2j\mathsf{F})} - 1\right] \\ &= \frac{f^{-1}(2j\mathsf{F})}{2j^2}\left[\left.\frac{\mathrm{d}\log(q)}{\mathrm{d}\log(f(q))}\right|_{x=2j\mathsf{F}} \frac{2j\mathsf{F}}{f^{-1}(2j\mathsf{F})} - 1\right] < 0.\end{aligned}$$

3. $\lim_{j\downarrow 0} h(j, \mathsf{F}) = 3\mathsf{F}$ and $\lim_{j\to ¥} h(j, \mathsf{F}) = \mathsf{F}$: Applying L'Hopital's rule and using properties of the derivative of the inverse,

$$\lim_{j\to ¥} h(j,\mathsf{F}) = \lim_{j\to ¥}\frac{f^{-1}(2j\mathsf{F})}{2j} = \lim_{j\to ¥}\frac{1}{f'(2j\mathsf{F})}\mathsf{F} = \mathsf{F}$$
$$\lim_{j\downarrow 0} h(j,\mathsf{F}) = \lim_{j\downarrow 0}\frac{f^{-1}(2j\mathsf{F})}{2j} = \lim_{j\downarrow 0}\frac{1}{f'(2j\mathsf{F})}\mathsf{F} = 3\mathsf{F}$$

4. $h(j, \mathsf{F}) = w(2j\mathsf{F})\mathsf{F}$: Define $w(z) = \frac{f^{-1}(z)}{z}$, then it is easy to see that $h(j, \mathsf{F}) = w(2j\mathsf{F})\mathsf{F}$. Moreover, from property 2 and 3, $w(z)$ is decreasing with $\lim_{z\downarrow 0} w(z) = 3$ and $\lim_{z\to ¥} w(z) = 1$. Moreover, it is easy to show with similar arguments that $w(2j\mathsf{F})\mathsf{F}$ is increasing in F and $w(2j\mathsf{F})j$ is increasing in j.

Now, we can compute $h(\hat{w}^*)$ and $\mathcal{T}(\hat{w}^*, \hat{r})$. Note that we can define $T(x) = \mathcal{T}(x + \hat{w}^*, \hat{r})$, which solves $(\hat{r} + d)T(x) = 1 + \frac{s^2}{2}T''(x)$, with $T(\pm h(j, \mathsf{F})) = 0$. The solution to this differential equation is given by $T(x) = \frac{1 - \frac{e^{jx}+e^{-jx}}{e^{jh}+e^{-jh}}}{\hat{r}+d}$. Thus, $T'(0) = 0$ and $h(\hat{w}^*) = a$. Finally, using the property that $sech(x) = \frac{2}{e^x+e^{-x}}$, we have $\mathcal{T}(\hat{w}^*, \hat{r}) = \frac{1 - sech(jw(2j\mathsf{F})\mathsf{F})}{\hat{r}+d}$. □

Proof of Proposition 5. Now, we take the limit $s \downarrow 0$. The equilibrium conditions in this case are

$$(\hat{r} + d)\hat{W}(\hat{w}) = e^{\hat{w}} - \hat{r}\hat{U} - \hat{g}\hat{W}'(\hat{w}) \quad \forall \hat{w} \in \hat{\mathcal{Z}}^j \cap \mathcal{Z}^h$$
$$(\hat{r} + d)\hat{J}(\hat{w}) = 1 - e^{\hat{w}} - \hat{g}\hat{J}'(\hat{w}) \quad \forall \hat{w} \in \hat{\mathcal{Z}}^j \cap \mathcal{Z}^h$$
$$(1 - a)\frac{\mathrm{d}\log \hat{J}(\hat{w}^*)}{\mathrm{d}\hat{w}} = -a\frac{\mathrm{d}\log \hat{W}(\hat{w}^*)}{\mathrm{d}\hat{w}}$$

with the value matching and smooth pasting conditions $\hat{W}(\hat{w}^-) = \hat{J}(\hat{w}^-) = \hat{W}(\hat{w}^+) = \hat{J}(\hat{w}^+) = 0$ and $\hat{W}'(\hat{w}^-) = \hat{J}'(\hat{w}^+) = 0$. Without idiosyncratic shocks and $g > 0$ the upper Ss band is not active. Thus, we discard the optimality condition for $\hat{w}^+$. In this case, the stopping time is a deterministic function; hence, it is easier to work with the sequential formulation.

$$\hat{W}(\hat{w}) = \max_T \int_0^T e^{-(\hat{r}+d)s} \left(e^{\hat{w}-\hat{g}s} - \hat{r}\hat{U}\right) ds \tag{A.27}$$

$$\hat{J}(\hat{w}) = \int_0^{T(\hat{w})} e^{-(\hat{r}+d)s} \left(1 - e^{\hat{w}-\hat{g}s}\right) ds. \tag{A.28}$$

In equation (A.28), $T(\hat{w})$ is the optimal policy of the worker. Taking the first order conditions with respect to $T(\hat{w})$, $e^{\hat{w}-\hat{g}T(\hat{w})} = \hat{r}\hat{U}$. Solving this equation, $T(\hat{w}) = \frac{\hat{w}-\log(\hat{r}\hat{U})}{\hat{g}}$. Thus, if $\hat{w} = \hat{w}^*$, we have that $\hat{w}^- = \hat{w}^* - \hat{g}T(\hat{w}^*)$ satisfies $\hat{w}^- = \log(\hat{r}\hat{U})$. Taking the derivatives of $\hat{W}(\hat{w})$ and $\hat{J}(\hat{w})$, and using the envelope condition for $\hat{W}'(\hat{w})$, we have

$$\hat{W}'(\hat{w}) = \int_0^{T(w)} e^{-(\hat{r}+d)s} \left(e^{\hat{w}-\hat{g}s}\right) ds, \tag{A.29}$$

$$\hat{J}'(\hat{w}) = -\int_0^{T(w)} e^{-(\hat{r}+d)s} \left(e^{\hat{w}-\hat{g}s}\right) ds + e^{-(\hat{r}+d)T(\hat{w})} \left(1 - e^{\hat{w}-\hat{g}T(\hat{w})}\right) \underbrace{T'(\hat{w})}_{=1/\hat{g}}. \tag{A.30}$$

From equations (A.29) and (A.30), we get the Nash bargaining solution

$$-a\frac{\int_0^{T^*} e^{-(\hat{r}+d)s} \left(e^{\hat{w}^*-\hat{g}s}\right) ds}{\int_0^{T^*} e^{-(\hat{r}+d)s} \left(e^{\hat{w}^*-\hat{g}s} - \hat{r}\hat{U}\right) ds} = (1-a)\frac{\left[-\int_0^{T^*} e^{-(\hat{r}+d)s} \left(e^{\hat{w}^*-\hat{g}s}\right) ds + e^{-(\hat{r}+d)T^*} \frac{(1-\hat{r}\hat{U})}{\hat{g}}\right]}{\int_0^{T^*} e^{-(\hat{r}+d)s} \left(1 - e^{\hat{w}^*-\hat{g}s}\right) ds} \tag{A.31}$$

Define $\mathrm{W}(a, T^*) := \frac{1-e^{-aT^*}}{a}$. Operating,

$$a\int_0^{T^*} e^{-(\hat{r}+d)s} \left(1 - e^{\hat{w}^*-\hat{g}s}\right) ds = (1-a)\int_0^{T^*} e^{-(\hat{r}+d)s} \left(e^{\hat{w}^*-\hat{g}s} - \hat{r}\hat{U}\right) ds \left[1 - \frac{e^{-(\hat{r}+d)T^*}(1-\hat{r}\hat{U})}{\hat{g}\int_0^{T^*} e^{-(\hat{r}+d)s}\left(e^{\hat{w}^*-\hat{g}s}\right) ds}\right] \iff$$

$$(a + (1-a)\hat{r}\hat{U})\,\mathrm{W}(\hat{r}+d, T^*) = e^{\hat{w}^*}\mathrm{W}(\hat{r}+d+\hat{g}, T^*) - \frac{(1-a)e^{-(\hat{r}+d)T^*}(1-\hat{r}\hat{U})}{\hat{g}}\left[1 - \hat{r}\hat{U}\frac{\mathrm{W}(\hat{r}+d, T^*)}{e^{\hat{w}^*}\mathrm{W}(\hat{r}+d+\hat{g}, T^*)}\right]$$

Define $\tilde{T} = \hat{g}T^*$ and $\mathrm{W}(a, T^*) := \frac{1-e^{-aT^*}}{a} = \hat{g}^{-1}\mathrm{W}(\frac{a}{\hat{g}}, \tilde{T})$. Then, the policy $(T^*, \hat{w}^*)$ solves $e^{\hat{w}^*-\tilde{T}} = \hat{r}\hat{U}$ and

$$(a + (1-a)\hat{r}\hat{U})\,\hat{g}^{-1}\mathrm{W}\left(\frac{\hat{r}+d}{\hat{g}}, \tilde{T}\right) = e^{\hat{w}^*}\hat{g}^{-1}\mathrm{W}\left(\frac{\hat{r}+d}{\hat{g}} + 1, \tilde{T}\right) - \frac{(1-a)e^{-\frac{\hat{r}+d}{\hat{g}}\tilde{T}}(1-\hat{r}\hat{U})}{\hat{g}}\left[1 - \hat{r}\hat{U}\frac{\mathrm{W}\left(\frac{\hat{r}+d}{\hat{g}}, \tilde{T}\right)}{e^{\hat{w}^*}\mathrm{W}\left(\frac{\hat{r}+d}{\hat{g}} + 1, \tilde{T}\right)}\right].$$

Therefore, the optimal stopping is given by

$$\frac{a+(1-a)\hat{r}\hat{U}}{\hat{r}\hat{U}} = e^{\tilde{T}}\frac{W\left(\frac{\hat{r}+d}{\hat{g}}+1,\tilde{T}\right)}{W\left(\frac{\hat{r}+d}{\hat{g}},\tilde{T}\right)} - \frac{(1-a)(1-\hat{r}\hat{U})\left[1-\frac{\hat{r}+d}{\hat{g}}W\left(\frac{\hat{r}+d}{\hat{g}},\tilde{T}\right)\right]}{\hat{r}\hat{U}W\left(\frac{\hat{r}+d}{\hat{g}},\tilde{T}\right)}\left[1-\frac{W\left(\frac{\hat{r}+d}{\hat{g}},\tilde{T}\right)}{e^{\tilde{T}}W\left(\frac{\hat{r}+d}{\hat{g}}+1,\tilde{T}\right)}\right] \tag{A.32}$$

Now, we show the properties satisfied by $\tilde{T}\left(\frac{a+(1-a)\hat{r}\hat{U}}{\hat{r}\hat{U}}, \frac{\hat{r}+d}{\hat{g}}, \frac{(1-a)(1-\hat{r}\hat{U})}{\hat{r}\hat{U}}\right)$. Let us define the function

$$f(a,b,c) := e^{a}\frac{1-e^{-(1+b)a}}{1-e^{-ba}}\frac{b}{b+1} - cb\frac{e^{-ba}}{1-e^{-ba}}\left[1-\frac{b+1}{b}\frac{1-e^{-ba}}{e^{a}-e^{-ba}}\right].$$

Observe that with this function:

$$\frac{a+(1-a)\hat{r}\hat{U}}{\hat{r}\hat{U}} = f\left(\tilde{T}\left(\frac{a+(1-a)\hat{r}\hat{U}}{\hat{r}\hat{U}}, \frac{\hat{r}+d}{\hat{g}}, \frac{(1-a)(1-\hat{r}\hat{U})}{\hat{r}\hat{U}}\right), \frac{\hat{r}+d}{\hat{g}}, \frac{(1-a)(1-\hat{r}\hat{U})}{\hat{r}\hat{U}}\right).$$

The following properties are easy to show:

1. $f(a,b,c)$ is increasing in a.

2. If $a,c>0, b\to \text{¥}$, then $f(a,b,c)\to e^{a}$: To see this property, taking the limit

$$\lim_{a>0,b\to\text{¥},c\text{µ}b}\left[e^{a}\frac{1-e^{-(1+b)a}}{1-e^{-ba}}\frac{b}{b+1} - cb\frac{e^{-ba}}{1-e^{-ba}}\left[1-\frac{b+1}{b}\frac{1-e^{-ba}}{e^{a}-e^{-ba}}\right]\right]$$
$$= e^{a}\underbrace{\lim_{a>0,b\to\text{¥}}\frac{1-e^{-(1+b)a}}{1-e^{-ba}}}_{=1}\underbrace{\lim_{a>0,b\to\text{¥}}\frac{b}{b+1}}_{=1} - \underbrace{\lim_{a>0,b\to\text{¥}}cb\frac{e^{-ba}}{1-e^{-ba}}}_{=0}\left[1-\underbrace{\lim_{b\to\text{¥}}\frac{b+1}{b}}_{=1}\underbrace{\lim_{a>0,b\to\text{¥}}\frac{1-e^{-ba}}{e^{a}-e^{-ba}}}_{=e^{-a}}\right] = e^{a}.$$

3. If $a,c>0$ and $b\to 0$ then $f(a,b,c)\to\frac{e^{a}-1-c\left(1-\frac{a}{e^{a}-1}\right)}{a}$: To see this property, taking the limit

$$\lim_{a>0,b\to 0}\left[e^{a}\frac{1-e^{-(1+b)a}}{1-e^{-ba}}\frac{b}{b+1} - cb\frac{e^{-ba}}{1-e^{-ba}}\left[1-\frac{b+1}{b}\frac{1-e^{-ba}}{e^{a}-e^{-ba}}\right]\right]$$
$$= e^{a}(1-e^{-a})\underbrace{\lim_{a>0,b\to 0}\frac{b}{1-e^{-ba}}}_{=1/a} - c\underbrace{\lim_{a>0,b\to 0}\frac{b}{1-e^{-ba}}}_{=1/a}\left[1-\frac{1}{e^{a}-1}\underbrace{\lim_{b\to\text{¥}}\frac{1-e^{-ba}}{b}}_{=a}\right] = \frac{e^{a}-1-c\left(1-\frac{a}{e^{a}-1}\right)}{a}.$$

4. $e^{a}\geq f(a,b,c)\geq\frac{e^{a}-1-c\left(1-\frac{a}{e^{a}-1}\right)}{a}$ where the upper bound is reached when $b\to\text{¥}$ and the lower bound when $b\downarrow 0$.

5. Duration of the match: It is easy to show that $\mathcal{T}(\hat{w}^{*},\hat{r}) = \frac{1-e^{-\frac{\hat{r}+d}{\hat{g}}\tilde{T}(\cdot)}}{\hat{r}+d}$.

6. The worker's share is given by

$$h(\hat{w}^*) = \frac{e^{\hat{g}T^*(\cdot)+\log(\hat{r}\hat{U})} \int_0^{T^*} e^{-(\hat{r}+d+\hat{g})t}\,\mathrm{d}t - \hat{r}\hat{U}\int_0^{T^*} e^{-(\hat{r}+d)t}\,\mathrm{d}t}{(1-\hat{r}\hat{U})\int_0^{T^*} e^{-(\hat{r}+d)t}\,\mathrm{d}t} = \frac{e^{\tilde{T}(\cdot)} \frac{1-e^{-\left(1+\frac{\hat{r}+d}{\hat{g}}\right)\tilde{T}(\cdot)}}{1-e^{-\frac{\hat{r}+d}{\hat{g}}\tilde{T}(\cdot)}} \frac{\hat{r}+d}{\hat{r}+d+\hat{g}} - 1}{1-\hat{r}\hat{U}}\hat{r}\hat{U} \quad \text{(A.33)}$$

With these properties, we can characterize the equilibrium policies:

1. $\tilde{T}\left(\frac{a+(1-a)\hat{r}\hat{U}}{\hat{r}\hat{U}}, \frac{\hat{r}+d}{\hat{g}}, \frac{(1-a)(1-\hat{r}\hat{U})}{\hat{g}\hat{r}\hat{U}}\right)$ is increasing in the first argument.
2. If $\hat{g} \to 0$, then $\frac{\hat{r}+d}{\hat{g}} \to ¥$ and $\lim_{(\hat{r}+d)/\hat{g}\to ¥} \tilde{T}(\cdot) = \log\left(\frac{a+(1-a)\hat{r}\hat{U}}{\hat{r}\hat{U}}\right)$. The expected discounted duration in the limit is equal to $\lim_{\hat{g}\to 0} \mathcal{T}(\hat{w}^*, \hat{r}) = \frac{1}{\hat{r}+d}$. The worker's share in the limit is equal to

$$h(\hat{w}^*) = \frac{e^{\tilde{T}(\cdot)} \frac{1-e^{-\left(1+\frac{\hat{r}+d}{\hat{g}}\right)\tilde{T}(\cdot)}}{1-e^{-\frac{\hat{r}+d}{\hat{g}}\tilde{T}(\cdot)}} \frac{\hat{r}+d}{\hat{r}+d+\hat{g}} - 1}{1-\hat{r}\hat{U}}\hat{r}\hat{U} = \frac{e^{\tilde{T}(\cdot)}-1}{1-\hat{r}\hat{U}}\hat{r}\hat{U} = \frac{\frac{a+(1-a)\hat{r}\hat{U}}{\hat{r}\hat{U}}-1}{1-\hat{r}\hat{U}}\hat{r}\hat{U} = a$$

3. If $\hat{g} \to ¥$, then $\frac{\hat{r}+d}{\hat{g}} \to 0$, which provides the same $\tilde{T}(\cdot)$ as $\hat{r}+d \to 0$. As we have shown before, under this limit, $\tilde{T}(\cdot)$ converges to the implicit solution given by

$$\frac{a+(1-a)\hat{r}\hat{U}}{\hat{r}\hat{U}} = \frac{e^{\tilde{T}(\cdot)} - 1 - \frac{(1-a)(1-\hat{r}\hat{U})}{\hat{r}\hat{U}}\left(1 - \frac{\tilde{T}(\cdot)}{e^{\tilde{T}(\cdot)}-1}\right)}{\tilde{T}(\cdot)}.$$

Given the convergence, we now show the limit for $h(\hat{w}^*)$ since clearly $\mathcal{T}(\hat{w}^*, r) \to 0$. Let us depart from equation (A.31)

$$-a\frac{\int_0^{T^*} e^{-(\hat{r}+d)s}\left(e^{\hat{w}^*-\hat{g}s}\right)\mathrm{d}s}{\int_0^{T^*} e^{-(\hat{r}+d)s}\left(e^{\hat{w}^*-\hat{g}s}-\hat{r}\hat{U}\right)\mathrm{d}s} = (1-a)\frac{\left[-\int_0^{T^*} e^{-(\hat{r}+d)s}\left(e^{\hat{w}^*-\hat{g}s}\right)\mathrm{d}s + e^{-(\hat{r}+d)T^*}\frac{(1-\hat{r}\hat{U})}{\hat{g}}\right]}{\int_0^{T^*} e^{-(\hat{r}+d)s}\left(1-e^{\hat{w}^*-\hat{g}s}\right)\mathrm{d}s}$$

Taking the limit as $\hat{r}+d \to 0$

$$a\int_0^{T^*}(1-e^{w_t})\,\mathrm{d}t = (1-a)\int_0^{T^*}\left(e^{w_t}-\hat{r}\hat{U}\right)\mathrm{d}t - \frac{(1-a)(1-\hat{r}\hat{U})}{\hat{g}}\frac{\int_0^{T^*}\left(e^{w_t}-\hat{r}\hat{U}\right)\mathrm{d}t}{\int_0^{T^*} e^{w_t}\,\mathrm{d}t}.$$

Operating and using the occupancy measure

$$a + (1-a)\hat{r}\hat{U} + \frac{(1-a)(1-\hat{r}\hat{U})}{\hat{g}T^*}\frac{\frac{\int_0^{T^*} e^{w_t}\,\mathrm{d}t}{T^*} - \hat{r}\hat{U}}{\frac{\int_0^{T^*} e^{w_t}\,\mathrm{d}t}{T^*}} = \frac{\int_0^{T^*} e^{w_t}\,\mathrm{d}t}{T^*}$$

It is easy to check that

$$a + (1-a)\hat{r}\hat{U} + \frac{1-a}{\hat{g}T^*}\frac{\mathbb{E}[e^{\hat{w}}] - \hat{r}\hat{U}}{\mathbb{E}[e^{\hat{w}}]}(1-\hat{r}\hat{U}) = \mathbb{E}[e^{\hat{w}}].$$

From (A.33), since $\hat{r} + d \to 0$, we have that $h(\hat{w}^*) = \frac{\mathbb{E}[e^{\hat{w}}] - \hat{r}\hat{U}}{1-\hat{r}\hat{U}}$. Combining these steps yields the desired result.

□

B Proofs for Section 3: Aggregate Shocks in Labor Markets with Inefficient Turnover

B.1 Proof of Proposition 6: CIR of Employment with Flexible Entry Wage

We divide the proof of Proposition 6 into three propositions. Let $g^h(\mathsf{D}z)$ and $g^u(\mathsf{D}z)$ be the distributions of $\mathsf{D}z$ for employed and unemployed workers, respectively. The support of $g^h(\mathsf{D}z)$ is $[-\mathsf{D}^-, \mathsf{D}^+]$, where $\mathsf{D}^- := \hat{w}^* - \hat{w}^-$ and $\mathsf{D}^+ := \hat{w}^+ - \hat{w}^*$. Denote by $\mathbb{E}_h[\cdot]$ and $\mathbb{E}_u[\cdot]$ the expectation operators under $g^h(\mathsf{D}z)$ and $g^u(\mathsf{D}z)$, respectively.

Proposition B.1. *Given steady-state policies $(\hat{w}^-, \hat{w}^*, \hat{w}^+)$ and distributions $(g^h(\mathsf{D}z), g^u(\mathsf{D}z))$, the CIR is given by*

$$CIR_{\mathcal{E}}(z) = \int_{-¥}^{¥} m_{\mathcal{E},h}(\mathsf{D}z) g^h(\mathsf{D}z + z)\,\mathrm{d}\mathsf{D}z + \int_{-¥}^{¥} m_{\mathcal{E},u}(\mathsf{D}z, z) g^u(\mathsf{D}z + z)\,\mathrm{d}\mathsf{D}z,$$

where the value functions $m_{\mathcal{E},h}(\mathsf{D}z)$ and $m_{\mathcal{E},u}(\mathsf{D}z, z)$ are defined as:

$$m_{\mathcal{E},h}(\mathsf{D}z) = \mathbb{E}\left[\int_0^{t^m} (1-\mathcal{E}_{ss})\,\mathrm{d}t + m_{\mathcal{E},u}(0,0) \,\middle|\, \mathsf{D}z_0 = \mathsf{D}z\right], \tag{B.1}$$

$$m_{\mathcal{E},u}(\mathsf{D}z, z) = \mathbb{E}\left[\int_0^{t^u(z)} (-\mathcal{E}_{ss})\,\mathrm{d}t + m_{\mathcal{E},h}(-z) \,\middle|\, \mathsf{D}z_0 = \mathsf{D}z\right]. \tag{B.2}$$

$$0 = \int_{-¥}^{¥} m_{\mathcal{E},h}(\mathsf{D}z) g^h(\mathsf{D}z)\,\mathrm{d}\mathsf{D}z + \int_{-¥}^{¥} m_{\mathcal{E},u}(\mathsf{D}z, 0) g^u(\mathsf{D}z)\,\mathrm{d}\mathsf{D}z.$$

with $t^u(z)$ being distributed according to a Poisson process with arrival rate $f(\hat{q}(\hat{w}^* - z))$.

Proof. We define the cumulative impulse response of aggregate employment to an aggregate TFPR shock as

$$\text{CIR}_{\mathcal{E}}(z) = \int_0^{¥} \int_{-¥}^{¥} \Big(g^h(\mathsf{D}z, z, t) - g^h(\mathsf{D}z) \Big)\, \mathrm{d}\mathsf{D}z\, \mathrm{d}t.$$

Note that $\mathcal{E}_t = \int_{-¥}^{¥} g^h(\mathsf{D}z, z, t)\, \mathrm{d}\mathsf{D}z$ is a function of z since aggregate shocks affect net flows into employment. The proof proceeds in three steps. Step 1 rewrites the CIR as the integral over time of two value functions, one for employed and unemployed workers, up to a finite time $\mathcal{T}$. Step 2 expresses the CIR as $\mathcal{T} \to ¥$. Step 3 uses the equivalence of the combined Dirichlet-Poisson problem (i.e., the mapping between the sequential problem and the corresponding HJB equations and boundary conditions).

Step 1. Here, we follow a recursive representation for the CIR. The CIR satisfies

$$\text{CIR}_{\mathcal{E}}(z) = \int_{-¥}^{¥} \lim_{\mathcal{T}\to ¥} \Big[m_{\mathcal{E},h}(\mathsf{D}z, \mathcal{T}) g^h(\mathsf{D}z + z) + m_{\mathcal{E},u}(\mathsf{D}z, \mathcal{T}) g^u(\mathsf{D}z + z) \Big]\, \mathrm{d}\mathsf{D}z,$$

where we defined

$$m_{\mathcal{E},h}(\mathsf{D}z_0, \mathcal{T}) := \int_0^{\mathcal{T}} \left[\int_{-¥}^{¥} \Big[(1 - \mathcal{E}_{ss})\, g^h(\mathsf{D}z, t|\mathsf{D}z_0, h) + (-\mathcal{E}_{ss}) g^u(\mathsf{D}z, t|\mathsf{D}z_0, h) \Big]\, \mathrm{d}\mathsf{D}z\, \mathrm{d}t \right],$$

$$m_{\mathcal{E},u}(\mathsf{D}z_0, z, \mathcal{T}) := \int_0^{\mathcal{T}} \left[\int_{-¥}^{¥} \Big[(1 - \mathcal{E}_{ss})\, g^h(\mathsf{D}z, z, t|\mathsf{D}z_0, u) + (-\mathcal{E}_{ss}) g^u(\mathsf{D}z, z, t|\mathsf{D}z_0, u) \Big]\, \mathrm{d}\mathsf{D}z\, \mathrm{d}t \right].$$

Proof of Step 1. Following Baley and Blanco (2022), it can be shown that

$$\begin{aligned} \text{CIR}_{\mathcal{E}}(z) &= \int_0^{¥} \int_{-¥}^{¥} \Big(g^h(\mathsf{D}z, z, t) - g^h(\mathsf{D}z) \Big)\, \mathrm{d}\mathsf{D}z\, \mathrm{d}t \\ &= \int_{-¥}^{¥} \lim_{\mathcal{T}\to ¥} m_{\mathcal{E},h}(\mathsf{D}z, \mathcal{T}) g^h(\mathsf{D}z + z)\, \mathrm{d}\mathsf{D}z + \int_{-¥}^{¥} \lim_{\mathcal{T}\to ¥} m_{\mathcal{E},u}(\mathsf{D}z, z, \mathcal{T}) g^u(\mathsf{D}z + z)\, \mathrm{d}\mathsf{D}z \end{aligned} \tag{B.3}$$

where we define

$$m_{\mathcal{E},h}(\mathsf{D}z_0, \mathcal{T}) \equiv \int_0^{\mathcal{T}} \left[\int_{-¥}^{¥} \Big[(1 - \mathcal{E}_{ss})\, g^h(\mathsf{D}z, t|\mathsf{D}z_0, h) + (-\mathcal{E}_{ss}) g^u(\mathsf{D}z, t|\mathsf{D}z_0, h) \Big]\, \mathrm{d}\mathsf{D}z\, \mathrm{d}t \right]$$

$$m_{\mathcal{E},u}(\mathsf{D}z_0, z, \mathcal{T}) \equiv \int_0^{\mathcal{T}} \left[\int_{-¥}^{¥} \Big[(1 - \mathcal{E}_{ss})\, g^h(\mathsf{D}z, z, t|\mathsf{D}z_0, u) + (-\mathcal{E}_{ss}) g^u(\mathsf{D}z, z, t|\mathsf{D}z_0, u) \Big]\, \mathrm{d}\mathsf{D}z\, \mathrm{d}t \right].$$

Step 2. The CIR satisfies

$$\text{CIR}_{\mathcal{E}}(z) = \int_{-¥}^{¥} m_{\mathcal{E},h}(\mathsf{D}z) g^h(\mathsf{D}z + z)\, \mathrm{d}\mathsf{D}z + \int_{-¥}^{¥} m_{\mathcal{E},u}(\mathsf{D}z, z) g^u(\mathsf{D}z + z)\, \mathrm{d}\mathsf{D}z$$

and the value functions $m_{\mathcal{E},h}(\mathsf{D}z_0)$ and $m_{\mathcal{E},u}(\mathsf{D}z_0, z)$ satisfy the following HJB and border conditions:

$$0 = 1 - \mathcal{E}_{ss} - (g + c) \frac{\mathrm{d}m_{\mathcal{E},h}(\mathsf{D}z)}{\mathrm{d}\mathsf{D}z} + \frac{s^2}{2} \frac{d^2 m_{\mathcal{E},h}(\mathsf{D}z)}{\mathrm{d}\mathsf{D}z^2} + d(m_{\mathcal{E},u}(0,0) - m_{\mathcal{E},h}(\mathsf{D}z)), \tag{B.4}$$

$$0 = -\mathcal{E}_{ss} - (g + c)\frac{dm_{\mathcal{E},u}(\mathrm{D}z, z)}{d\mathrm{D}z} + \frac{s^2}{2}\frac{d^2 m_{\mathcal{E},u}(\mathrm{D}z, z)}{d\mathrm{D}z^2} + f(\hat{q}(\hat{w}^* - z))(m_{\mathcal{E},h}(-z) - m_{\mathcal{E},u}(\mathrm{D}z, z)) \tag{B.5}$$

$$0 = m_{\mathcal{E},u}(0, 0) - m_{\mathcal{E},h}(\mathrm{D}z), \text{ for all } \mathrm{D}z \notin (-\mathrm{D}^-, \mathrm{D}^+) \tag{B.6}$$

$$0 = \lim_{\mathrm{D}z \to -¥} \frac{dm_{\mathcal{E},u}(\mathrm{D}z, z)}{d\mathrm{D}z} = \lim_{\mathrm{D}z \to ¥} \frac{dm_{\mathcal{E},u}(\mathrm{D}z, z)}{d\mathrm{D}z} \tag{B.7}$$

$$0 = \int_{-¥}^{¥} m_{\mathcal{E},h}(\mathrm{D}z) g^h(\mathrm{D}z)\, d\mathrm{D}z + \int_{-¥}^{¥} m_{\mathcal{E},u}(\mathrm{D}z, 0) g^u(\mathrm{D}z)\, d\mathrm{D}z. \tag{B.8}$$

Proof of Step 2. We divide this proof into steps *a*–*c*.

a. We show that $\lim_{\mathcal{T} \to ¥} m_{\mathcal{E},h}(\mathrm{D}z, \mathcal{T}) = m_{\mathcal{E},h}(\mathrm{D}z)$ and $\lim_{\mathcal{T} \to ¥} m_{\mathcal{E},u}(\mathrm{D}z, z, \mathcal{T}) = m_{\mathcal{E},u}(\mathrm{D}z, z)$: This property holds due to the convergence of the distribution of $\mathrm{D}z$ over time to its ergodic distribution for any initial condition (Stokey, 1989).

b. To show that $0 = \int_{-¥}^{¥} m_{\mathcal{E},h}(\mathrm{D}z, \mathcal{T}) g^h(\mathrm{D}z)\, d\mathrm{D}z + \int_{-¥}^{¥} m_{\mathcal{E},u}(\mathrm{D}z, 0, \mathcal{T}) g^u(\mathrm{D}z)\, d\mathrm{D}z$ and that $0 = \int_{-\mathrm{D}^-}^{\mathrm{D}^+} m_{\mathcal{E},h}(\mathrm{D}z) g^h(\mathrm{D}z)\, d\mathrm{D}z + \int_{-\mathrm{D}^-}^{\mathrm{D}^+} m_{\mathcal{E},u}(\mathrm{D}z, 0) g^u(\mathrm{D}z)\, d\mathrm{D}z$, see Baley and Blanco (2022).

c. We show that the CIR satisfies (B.3) with $m_{\mathcal{E},h}(\mathrm{D}z_0)$ and $m_{\mathcal{E},u}(\mathrm{D}z_0, z)$ satisfying (B.4)–(B.8): Writing the HJB for $m_{\mathcal{E},h}(\mathrm{D}z_0, \mathcal{T})$ and $m_{\mathcal{E},u}(\mathrm{D}z_0, z, \mathcal{T})$, we have that

$$\begin{aligned}
0 &= 1 - \mathcal{E}_{ss} - \frac{dm_{\mathcal{E},h}(\mathrm{D}z, \mathcal{T})}{d\mathcal{T}} - (g + c)\frac{dm_{\mathcal{E},h}(\mathrm{D}z, \mathcal{T})}{d\mathrm{D}z} + \frac{s^2}{2}\frac{d^2 m_{\mathcal{E},h}(\mathrm{D}z, \mathcal{T})}{d\mathrm{D}z^2} \\
&\quad + d(m_{\mathcal{E},u}(0, 0, \mathcal{T}) - m_{\mathcal{E},h}(\mathrm{D}z, \mathcal{T})), \\
0 &= -\mathcal{E}_{ss} - \frac{dm_{\mathcal{E},u}(\mathrm{D}z, z, \mathcal{T})}{d\mathcal{T}} - (g + c)\frac{dm_{\mathcal{E},u}(\mathrm{D}z, z, \mathcal{T})}{d\mathrm{D}z} + \frac{s^2}{2}\frac{d^2 m_{\mathcal{E},u}(\mathrm{D}z, z, \mathcal{T})}{d\mathrm{D}z^2} \\
&\quad + f(\hat{q}(\hat{w}^* - z))(m_{\mathcal{E},h}(-z, \mathcal{T}) - m_{\mathcal{E},u}(\mathrm{D}z, z, \mathcal{T})) \\
0 &= m_{\mathcal{E},u}(0, 0, \mathcal{T}) - m_{\mathcal{E},h}(\mathrm{D}z, \mathcal{T}), \text{ for all } \mathrm{D}z \notin (-\mathrm{D}^-, \mathrm{D}^+) \\
0 &= \lim_{\mathrm{D}z \to -¥} \frac{dm_{\mathcal{E},u}(\mathrm{D}z, z, \mathcal{T})}{d\mathrm{D}z} = \lim_{\mathrm{D}z \to ¥} \frac{dm_{\mathcal{E},u}(\mathrm{D}z, z, \mathcal{T})}{d\mathrm{D}z} \\
0 &= \int_{-\mathrm{D}^-}^{\mathrm{D}^+} m_{\mathcal{E},h}(\mathrm{D}z, \mathcal{T}) g^h(\mathrm{D}z)\, d\mathrm{D}z + \int_{-\mathrm{D}^-}^{\mathrm{D}^+} m_{\mathcal{E},u}(\mathrm{D}z, 0, \mathcal{T}) g^u(\mathrm{D}z)\, d\mathrm{D}z.
\end{aligned}$$

The border condition for $m_{\mathcal{E},u}(\mathrm{D}z, z, \mathcal{T})$ is implied from the fact that the job-finding rate $f(\hat{q}(\hat{w}^*))$ is independent of $\mathrm{D}z$, so the function $m_{\mathcal{E},u}(\mathrm{D}z, z, \mathcal{T})$ is constant in the entire domain. Taking the limit $\mathcal{T} \to ¥$ and using point-wise convergence of $m_{\mathcal{E},h}(\mathrm{D}z_0, \mathcal{T})$ and $m_{\mathcal{E},u}(\mathrm{D}z_0, z, \mathcal{T})$, we have the result.

Step 3. The solutions of the differential equations (B.4)–(B.7) satisfy (B.1) and (B.2).

Proof of Step 3. This is just an application of Øksendal (2007), Chapter 9.

□

Before starting the next step of the proof, we summarize the conditions that characterize the distributions of $\mathsf{D}z$.

Steady-State Cross-Sectional Distribution $\mathsf{D}z$. Below we describe the Kolmogorov Forward Equations (KFE) for $g^h(\mathsf{D}z)$ and $g^u(\mathsf{D}z)$.

$$dg^h(\mathsf{D}z) = (g+c)(g^h)'(\mathsf{D}z) + \frac{s^2}{2}(g^h)''(\mathsf{D}z)\ \forall \mathsf{D}z \in (-\mathsf{D}^-, \mathsf{D}^+)/\{0\} \tag{B.9}$$

$$f(\hat{q}(\hat{w}^*))g^u(\mathsf{D}z) = (g+c)(g^u)'(\mathsf{D}z) + \frac{s^2}{2}(g^u)''(\mathsf{D}z)\ \forall \mathsf{D}z \in (-¥, ¥)/\{0\} \tag{B.10}$$

$$g^h(\mathsf{D}z) = 0, \text{ for all } \mathsf{D}z \notin (-\mathsf{D}^-, \mathsf{D}^+) \tag{B.11}$$

$$\lim_{\mathsf{D}z \to -¥} g^u(\mathsf{D}z) = \lim_{\mathsf{D}z \to ¥} g^u(\mathsf{D}z) = 0. \tag{B.12}$$

$$1 = \int_{-¥}^{¥} g^u(\mathsf{D}z)\, \mathrm{d}\mathsf{D}z + \int_{-\mathsf{D}^-}^{\mathsf{D}^+} g^h(\mathsf{D}z)\, \mathrm{d}\mathsf{D}z, \tag{B.13}$$

$$f(\hat{q}(\hat{w}^*))(1-\mathcal{E}) = d\mathcal{E} + \frac{s^2}{2}\left[\lim_{\mathsf{D}z \downarrow -\mathsf{D}^-}(g^h)'(\mathsf{D}z) - \lim_{\mathsf{D}z \uparrow \mathsf{D}^+}(g^h)'(\mathsf{D}z)\right], \tag{B.14}$$

$$g^h(\mathsf{D}z), g^u(\mathsf{D}z) \in \mathbb{C},\ g^u(\mathsf{D}z) \in \mathbb{C}^2((-¥, ¥)/\{0\}),\ g^h(\mathsf{D}z) \in \mathbb{C}^2((-\mathsf{D}^-, \mathsf{D}^+)/\{0\})$$

Proposition B.2. *Assume flexible entry wages. Up to first order, the CIR of employment is given by:*

$$\frac{CIR_{\mathcal{E}}(z)}{z} = -(1-\mathcal{E}_{ss})\frac{(g+c)\mathbb{E}_h[a] + \mathbb{E}_h[\mathsf{D}z]}{s^2} + o(z).$$

Proof. The proof proceeds in three steps. Step 1 computes the value function for an unemployed worker $m_{\mathcal{E},u}(\mathsf{D}z)$ (when entry wages are flexible, the job-finding rate and this value function are independent of the shock z, so we omit this argument). Step 2 computes the value for the employed worker at $\mathsf{D}z = 0$—i.e., $m_{\mathcal{E},h}(0)$. Step 3 characterizes the CIR as a function of steady-state aggregate variables and moments.

Step 1. The CIR is given by

$$\mathrm{CIR}_{\mathcal{E}}(z) = \int_{-¥}^{¥} m_{\mathcal{E},h}(\mathsf{D}z) g^h(\mathsf{D}z + z)\, \mathrm{d}\mathsf{D}z + \left(-\frac{\mathcal{E}_{ss}}{f(\hat{q}(\hat{w}^*))} + m_{\mathcal{E},h}(0)\right)(1-\mathcal{E}_{ss}),$$

with

$$0 = 1 - \mathcal{E}_{ss} - (g+c)\frac{\mathrm{d}m_{\mathcal{E},h}(\mathsf{D}z)}{\mathrm{d}\mathsf{D}z} + \frac{s^2}{2}\frac{d^2 m_{\mathcal{E},h}(\mathsf{D}z)}{\mathrm{d}\mathsf{D}z^2} + d\left(-\frac{\mathcal{E}_{ss}}{f(\hat{q}(\hat{w}^*))} + m_{\mathcal{E},h}(0) - m_{\mathcal{E},h}(\mathsf{D}z)\right),$$

$$
\begin{aligned}
0 &= -\frac{\mathcal{E}_{ss}}{f(\hat{q}(\hat{w}^*))} + m_{\mathcal{E},h}(0) - m_{\mathcal{E},h}(\mathsf{Dz}), \text{ for all } \mathsf{Dz} \notin (-\mathsf{D}^-, \mathsf{D}^+) \\
0 &= \int_{-\yen}^{\yen} m_{\mathcal{E},h}(\mathsf{Dz}) g^h(\mathsf{Dz})\, \mathrm{d}\mathsf{Dz} + \left(-\frac{\mathcal{E}_{ss}}{f(\hat{q}(\hat{w}^*))} + m_{\mathcal{E},h}(0)\right)(1-\mathcal{E}_{ss}).
\end{aligned} \tag{B.15}
$$

Proof of Step 1. To show this result, note that the solution to (B.5) and (B.7) is $m_{\mathcal{E},u}(\mathsf{Dz}) = m_{\mathcal{E},u}(0)$, for all Dz. Thus,

$$
0 = -\mathcal{E}_{ss} + f(\hat{q}(\hat{w}^*))(m_{\mathcal{E},h}(0) - m_{\mathcal{E},u}(0)) \iff m_{\mathcal{E},u}(0) = -\frac{\mathcal{E}_{ss}}{f(\hat{q}(\hat{w}^*))} + m_{\mathcal{E},h}(0). \tag{B.16}
$$

Replacing (B.16) into the CIR, we have the result.

Step 2. We show that $m_{\mathcal{E},h}(0) = \frac{\mathcal{E}_{ss}}{f(\hat{q}(\hat{w}^*))} - (1-\mathcal{E}_{ss})\mathbb{E}_h[a]$, where $\mathbb{E}_h[a]$ is the cross-sectional expected age of the match or the worker's tenure at the current match.

Proof of Step 2. Observe that $m_{\mathcal{E},h}(\mathsf{Dz})$ satisfies the following recursive representation

$$
m_{\mathcal{E},h}(\mathsf{Dz}) = \mathbb{E}\left[\int_0^{t^m} (1-\mathcal{E}_{ss})\, \mathrm{d}t + \left(-\frac{\mathcal{E}_{ss}}{f(\hat{q}(\hat{w}^*))} + m_{\mathcal{E},h}(0)\right) \middle| \mathsf{Dz}_0 = \mathsf{Dz}\right]. \tag{B.17}
$$

Define the following auxiliary function

$$
\mathsf{Y}(\mathsf{Dz}|j) = \mathbb{E}\left[\int_0^{t^m} e^{jt}(1-\mathcal{E}_{ss})\, \mathrm{d}t + e^{jt^m}\left(-\frac{\mathcal{E}_{ss}}{f(\hat{q}(\hat{w}^*))} + m_{\mathcal{E},h}(0)\right) \middle| \mathsf{Dz}_0 = \mathsf{Dz}\right]. \tag{B.18}
$$

and note that $\mathsf{Y}(\mathsf{Dz}|0) = m_{\mathcal{E},h}(\mathsf{Dz})$. The auxiliary function $\mathsf{Y}(\mathsf{Dz}|j)$ satisfies the following HJB and border conditions:

$$
\begin{aligned}
-jY(\mathsf{Dz}|j) + d\left(Y(\mathsf{Dz}|j) - \left(-\frac{\mathcal{E}_{ss}}{f(\hat{q}(\hat{w}^*))} + m_{\mathcal{E},h}(0)\right)\right) &= (1-\mathcal{E}_{ss}) - (g+c)\frac{\P Y(\mathsf{Dz}|j)}{\P \mathsf{Dz}} + \frac{s^2}{2}\frac{\P^2 Y(\mathsf{Dz}|j)}{\P \mathsf{Dz}^2} \\
Y(\mathsf{Dz}, j) &= \left(-\frac{\mathcal{E}_{ss}}{f(\hat{q}(\hat{w}^*))} + m_{\mathcal{E},h}(0)\right) \text{ for all } \mathsf{Dz} \notin (-\mathsf{D}^-, \mathsf{D}^+).
\end{aligned} \tag{B.19}
$$

Taking the derivative with respect to j in (B.19), we have that

$$
\begin{aligned}
(d-j)\frac{\P Y(\mathsf{Dz}|j)}{\P j} - Y(\mathsf{Dz}|j) &= -(g+c)\frac{\P^2 Y(\mathsf{Dz}, j)}{\P \mathsf{Dz} \P j} + \frac{s^2}{2}\frac{\P^3 Y(\mathsf{Dz}|j)}{\P \mathsf{Dz}^2 \P j}, \\
\frac{\P Y(\mathsf{Dz}|j)}{\P j} &= 0 \text{ for all } \mathsf{Dz} \notin (-\mathsf{D}^-, \mathsf{D}^+).
\end{aligned}
$$

Using the Schwarz theorem to exchange partial derivatives, evaluating at $j = 0$, and using $Y(\mathsf{Dz}|0) = m_{\mathcal{E},h}(\mathsf{Dz})$, we obtain

$$
d\frac{\P Y(\mathsf{Dz}|0)}{\P j} - m_{\mathcal{E},h}(\mathsf{Dz}) = -(g+c)\frac{\P}{\P \mathsf{Dz}}\left(\frac{\P Y(\mathsf{Dz}|0)}{\P j}\right) + \frac{s^2}{2}\frac{\P^2}{\P \mathsf{Dz}^2}\left(\frac{\P Y(\mathsf{Dz}|0)}{\P j}\right), \tag{B.20}
$$

$$\frac{\P \mathsf{Y}(-\mathsf{D}^-|0)}{\P j} = \frac{\P \mathsf{Y}(\mathsf{D}^+|0)}{\P j} = 0. \tag{B.21}$$

Equations (B.20) and (B.21) correspond to the HJB and border conditions of the function $\frac{\P \mathsf{Y}(\mathsf{Dz}|0)}{\P j} = \mathbb{E}\left[\int_0^{t^m} m_{\mathcal{E},h}(\mathsf{Dz}_t)\,\mathrm{d}t \middle| \mathsf{Dz}_0 = \mathsf{Dz}\right]$. Evaluating $\frac{\P \mathsf{Y}(\mathsf{Dz}|0)}{\P j}$ at $\mathsf{Dz} = 0$, using the occupancy measure and result (B.15), we write the previous equation as:

$$\begin{aligned} \frac{\P \mathsf{Y}(0|0)}{\P j} &= \mathbb{E}\left[\int_0^{t^m} m_{\mathcal{E},h}(\mathsf{Dz}_t)\,\mathrm{d}t \middle| \mathsf{Dz}_0 = 0\right] = \mathbb{E}_{\mathcal{D}}[t^m] \frac{\int_{-¥}^{¥} m_{\mathcal{E},h}(\mathsf{Dz}) g^h(\mathsf{Dz})\,\mathrm{d}\mathsf{Dz}}{\mathcal{E}_{ss}} \\ &= \mathbb{E}_{\mathcal{D}}[t^m] \left(\frac{\mathcal{E}_{ss}}{f(\hat{q}(\hat{w}^*))} - m_{\mathcal{E},h}(0)\right) \frac{(1-\mathcal{E}_{ss})}{\mathcal{E}_{ss}}, \end{aligned} \tag{B.22}$$

where $\mathbb{E}_{\mathcal{D}}[t^m]$ is the mean duration of completed employment spells (the subscript highlights that the moment can be easily computed from the data). From (B.18), we also have that

$$\begin{aligned} \frac{\P \mathsf{Y}(0|0)}{\P j} &= \mathbb{E}\left[\int_0^{t^m} s\,(1-\mathcal{E}_{ss})\,\mathrm{d}s + t^m \left(-\frac{\mathcal{E}_{ss}}{f(\hat{q}(\hat{w}^*))} + m_{\mathcal{E},h}(0)\right) \middle| \mathsf{Dz}_0 = 0\right] \\ &= \mathbb{E}_{\mathcal{D}}\left[t^m\right] \left[(1-\mathcal{E}_{ss}) \frac{\mathbb{E}_h[a]}{\mathcal{E}_{ss}} + \left(-\frac{\mathcal{E}_{ss}}{f(\hat{q}(\hat{w}^*))} + m_{\mathcal{E},h}(0)\right)\right], \end{aligned} \tag{B.23}$$

Combining (B.22) and (B.23), and solving for $m_{\mathcal{E},h}(0)$ we obtain $m_{\mathcal{E},h}(0) = \frac{\mathcal{E}_{ss}}{f(\hat{q}(\hat{w}^*))} - (1-\mathcal{E}_{ss})\mathbb{E}_h[a]$.

Step 3. Up to a first-order approximation, the CIR is given by:

$$\mathrm{CIR}_{\mathcal{E}}(z) = -(1-\mathcal{E}_{ss}) \frac{(g+c)\mathbb{E}_h[a] + \mathbb{E}_h[\mathsf{Dz}]}{s^2} z + o(z^2).$$

Proof of Step 3. To help the reader, we summarize below the conditions used in this step of the proof.

$$\mathrm{CIR}_{\mathcal{E}}(z) = \int_{-¥}^{¥} m_{\mathcal{E},h}(\mathsf{Dz}) g^h(\mathsf{Dz}+z)\,\mathrm{d}\mathsf{Dz} + \left(-\frac{\mathcal{E}_{ss}}{f(\hat{q}(\hat{w}^*))} + m_{\mathcal{E},h}(0)\right)(1-\mathcal{E}_{ss}) \tag{B.24}$$

with

$$dm_{\mathcal{E},h}(\mathsf{Dz}) = 1 - \mathcal{E}_{ss} - (g+c)\frac{\mathrm{d}m_{\mathcal{E},h}(\mathsf{Dz})}{\mathrm{d}\mathsf{Dz}} + \frac{s^2}{2}\frac{d^2 m_{\mathcal{E},h}(\mathsf{Dz})}{\mathrm{d}\mathsf{Dz}^2} + dm_{\mathcal{E},u}(0), \tag{B.25}$$

$$m_{\mathcal{E},u}(0) = m_{\mathcal{E},h}(\mathsf{Dz}) \text{ for all } \mathsf{Dz} \notin (-\mathsf{D}^-, \mathsf{D}^+)$$

$$0 = \int_{-¥}^{¥} m_{\mathcal{E},h}(\mathsf{Dz}) g^h(\mathsf{Dz})\,\mathrm{d}\mathsf{Dz} + m_{\mathcal{E},u}(0)(1-\mathcal{E}_{ss}). \tag{B.26}$$

1. **Zeroth Order:** If $z = 0$, condition (B.26) implies

$$\text{CIR}_{\mathcal{E}}(0) = \int_{-¥}^{¥} m_{\mathcal{E},h}(\mathsf{D}z) g^h(\mathsf{D}z)\, \mathrm{d}\mathsf{D}z + \left(-\frac{\mathcal{E}_{ss}}{f(\hat{q}(\hat{w}^*))} + m_{\mathcal{E},h}(0)\right)(1 - \mathcal{E}_{ss}) = 0.$$

2. **First Order:** Taking the derivative of (B.24) we obtain $\text{CIR}'_{\mathcal{E}}(z) = \int_{-¥}^{¥} m_{\mathcal{E},h}(\mathsf{D}z)(g^h)'(\mathsf{D}z + z)\, \mathrm{d}\mathsf{D}z$, which evaluated at $z = 0$ becomes $\text{CIR}'_{\mathcal{E}}(0) = \int_{-\mathsf{D}^-}^{\mathsf{D}^+} m_{\mathcal{E},h}(\mathsf{D}z)(g^h)'(\mathsf{D}z)\, \mathrm{d}\mathsf{D}z$. Using condition (B.9) to replace $d = \frac{(g+c)(g^h)'(\mathsf{D}z) + \frac{s^2}{2}(g^h)''(\mathsf{D}z)}{g^h(\mathsf{D}z)}$ into equation (B.25), we obtain

$$\frac{(g+c)(g^h)'(\mathsf{D}z) + \frac{s^2}{2}(g^h)''(\mathsf{D}z)}{g^h(\mathsf{D}z)} m_{\mathcal{E},h}(\mathsf{D}z) = 1 - \mathcal{E}_{ss} - (g+c)m'_{\mathcal{E},h}(\mathsf{D}z) + \frac{s^2}{2} m''_{\mathcal{E},h}(\mathsf{D}z) + \frac{(g+c)g'(\mathsf{D}z) + \frac{s^2}{2} g''(\mathsf{D}z)}{g(\mathsf{D}z)} m_{\mathcal{E},u}(0).$$

Multiplying both sides by $g^h(\mathsf{D}z)\mathsf{D}z$ and integrating between $-\mathsf{D}^-$ and D^+,

$$0 = (1 - \mathcal{E}_{ss})\mathbb{E}_h[\mathsf{D}z] - (g+c)T_1 + \frac{s^2}{2} T_2 + m_{\mathcal{E},u}(0) T_3 \tag{B.27}$$
$$T_1 = \int_{-\mathsf{D}^-}^{\mathsf{D}^+} \mathsf{D}z \left[m'_{\mathcal{E},h}(\mathsf{D}z) g^h(\mathsf{D}z) + m_{\mathcal{E},h}(\mathsf{D}z)(g^h)'(\mathsf{D}z)\right] \mathrm{d}\mathsf{D}z$$
$$T_2 = \int_{-\mathsf{D}^-}^{\mathsf{D}^+} \mathsf{D}z \left[m''_{\mathcal{E},h}(\mathsf{D}z) g^h(\mathsf{D}z) - m_{\mathcal{E},h}(\mathsf{D}z)(g^h)''(\mathsf{D}z)\right] \mathrm{d}\mathsf{D}z$$
$$T_3 = \int_{-\mathsf{D}^-}^{\mathsf{D}^+} \mathsf{D}z \left((g+c)(g^h)'(\mathsf{D}z) + \frac{s^2}{2}(g^h)''(\mathsf{D}z)\right) \mathrm{d}\mathsf{D}z.$$

Next, we operate on the terms T_1, T_2, and T_3. The term T_1 is equal to

$$\begin{aligned} T_1 &= \int_{-\mathsf{D}^-}^{\mathsf{D}^+} \mathsf{D}z \left[m'_{\mathcal{E},h}(\mathsf{D}z) g^h(\mathsf{D}z) + m_{\mathcal{E},h}(\mathsf{D}z)(g^h)'(\mathsf{D}z)\right] \mathrm{d}\mathsf{D}z \\ &= m_{\mathcal{E},u}(0)(1 - \mathcal{E}_{ss}). \end{aligned} \tag{B.28}$$

The term T_2 satisfies

$$\begin{aligned} T_2 &= \int_{-\mathsf{D}^-}^{\mathsf{D}^+} \mathsf{D}z \left[m''_{\mathcal{E},h}(\mathsf{D}z) g^h(\mathsf{D}z) - m_{\mathcal{E},h}(\mathsf{D}z)(g^h)''(\mathsf{D}z)\right] \mathrm{d}\mathsf{D}z \\ &= -m_{\mathcal{E},u}(0)\ \mathsf{D}z(g^h)'(\mathsf{D}z)\Big|_{\mathsf{D}^-}^{\mathsf{D}^+} + 2\int_{\mathsf{D}^-}^{\mathsf{D}^+} m_{\mathcal{E},h}(\mathsf{D}z) g'(\mathsf{D}z)\, \mathrm{d}\mathsf{D}z. \end{aligned} \tag{B.29}$$

Finally, the term T_3 is equal to

$$T_3 = \int_{-\mathsf{D}-}^{\mathsf{D}^+} \mathsf{D}z \left((g+c)(g^h)'(\mathsf{D}z) + \frac{s^2}{2}(g^h)''(\mathsf{D}z)\right) \mathrm{d}\mathsf{D}z \tag{B.30}$$

$$= -(g+c)\mathcal{E}_{ss} + \frac{s^2}{2}\left[\mathsf{D}z(g^h)'(\mathsf{D}z)\Big|_{\mathsf{D}^-}^{\mathsf{D}^+}\right]$$

Combining results (B.27), (B.28), (B.29), (B.30) and those in Step 2, we obtain

$$0 = (1-\mathcal{E}_{ss})\mathbb{E}_h[\mathsf{D}z] - (g+c)T_1 + \frac{s^2}{2}T_2 + m_{\mathcal{E},u}(0)T_3$$
$$= (1-\mathcal{E}_{ss})\mathbb{E}_h[\mathsf{D}z] - (g+c)m_{\mathcal{E},u}(0) + s^2\int_{-\mathsf{D}^-}^{\mathsf{D}^+} m_{\mathcal{E},h}(\mathsf{D}z)(g^h)'(\mathsf{D}z)\,\mathrm{d}\mathsf{D}z,$$

which implies $\int_{-\mathsf{D}^-}^{\mathsf{D}^+} m_{\mathcal{E},h}(\mathsf{D}z)(g^h)'(\mathsf{D}z)\,\mathrm{d}\mathsf{D}z = -(1-\mathcal{E}_{ss})\frac{[(g+c)\mathbb{E}_h[a]+\mathbb{E}_h[\mathsf{D}z]]}{s^2}$.

□

Proposition B.3. *If $(g+c)=0$, up to first order, the $CIR_{\mathcal{E}}(z)$ can be expressed in terms of data moments as follows:*

$$\frac{CIR_{\mathcal{E}}(z)}{z} = \underbrace{\frac{1}{f(\hat{q}(\hat{w}^*))}}_{\text{avg. u dur.}}\underbrace{\frac{1}{\mathbb{V}ar_{\mathcal{D}}[\mathsf{D}w]}}_{\text{dispersion}}\left[\underbrace{\frac{1}{3}\mathbb{E}_{\mathcal{D}}\left[\mathsf{D}w\frac{\mathsf{D}w^2}{\mathbb{E}_{\mathcal{D}}\left[\mathsf{D}w^2\right]}\right]}_{\text{asymmetries}}\right] + o(z).$$

Proof. The goal is to express the sufficient statistics of the CIR, $\mathbb{E}_h[a]$ and $\mathbb{E}_h[\mathsf{D}z]$, in terms of moments of the distribution of $\mathsf{D}w$ and (t^u, t^m). We focus in the case of $(g+c)\neq 0$ and then we use the assumption $(g+c)=0$. Let $\tilde{x}\equiv x/\mathbb{E}_{\mathcal{D}}[x]$ denote random variable x relative to its mean in the data.

Proposition III.3 expresses moments of the wage distribution as a linear combination of moments of the distribution of productivity changes among completed employment and unemployment spells:

$$\mathbb{E}_{\mathcal{D}}[\mathsf{D}w] = -\left[\bar{\mathbb{E}}_u[\mathsf{D}z] + \bar{\mathbb{E}}_h[\mathsf{D}z]\right]$$
$$\mathbb{E}_{\mathcal{D}}\left[\mathsf{D}w^2\right] = \left[\bar{\mathbb{E}}_u\left[\mathsf{D}z^2\right] + 2\bar{\mathbb{E}}_h[\mathsf{D}z]\,\bar{\mathbb{E}}_u[\mathsf{D}z] + \bar{\mathbb{E}}_h\left[\mathsf{D}z^2\right]\right]$$
$$\mathbb{E}_{\mathcal{D}}\left[\mathsf{D}w^3\right] = -\left[\bar{\mathbb{E}}_u\left[\mathsf{D}z^3\right] + 3\bar{\mathbb{E}}_h[\mathsf{D}z]\,\bar{\mathbb{E}}_u\left[\mathsf{D}z^2\right] + 3\bar{\mathbb{E}}_h\left[\mathsf{D}z^2\right]\bar{\mathbb{E}}_u[\mathsf{D}z] + \bar{\mathbb{E}}_h\left[\mathsf{D}z^3\right]\right],$$

where $\bar{\mathbb{E}}_h[\cdot]$ and $\bar{\mathbb{E}}_u[\cdot]$ denote the expectation operators under the distributions $\bar{g}^h(\mathsf{D}z)$ and $\bar{g}^u(\mathsf{D}z)$, respectively. Using results from the same Proposition, we can express the moments of productivity changes for completed unemployment spells in terms of model parameters:

$$\bar{\mathbb{E}}_u[\mathsf{D}z] = \frac{\left(\mathcal{L}_2^{-1}-\mathcal{L}_2\right)}{\mathcal{L}_1},\; \bar{\mathbb{E}}_u\left[\mathsf{D}z^2\right] = \frac{2\left(\mathcal{L}_2^{-2}+\mathcal{L}_2^{2}-1\right)}{\mathcal{L}_1^2},\; \bar{\mathbb{E}}_u\left[\mathsf{D}z^3\right] = \frac{6\left(-\mathcal{L}_2^3+\mathcal{L}_2-\mathcal{L}_2^{-1}+\mathcal{L}_2^{-3}\right)}{\mathcal{L}_1^3},$$

where

$$\mathcal{L}_1 = \sqrt{\frac{2f(\hat{q}(\hat{w}^*))}{s^2}},\ \mathcal{L}_2 = \sqrt{\frac{(g+c)+\sqrt{(g+c)^2+2s^2 f(\hat{q}(\hat{w}^*))}}{-(g+c)+\sqrt{(g+c)^2+2s^2 f(\hat{q}(\hat{w}^*))}}}.$$

From these two sets of equations, we solve for the moments of productivity changes for completed employment spells and obtain

$$\begin{aligned}
\bar{\mathbb{E}}_h[\mathsf{D}z] &= -\left(\frac{\left(\mathcal{L}_2^{-1}-\mathcal{L}_2\right)}{\mathcal{L}_1}\right) - \mathbb{E}_{\mathcal{D}}[\mathsf{D}w] \\
\bar{\mathbb{E}}_h\left[\mathsf{D}z^2\right] &= \mathbb{E}_{\mathcal{D}}\left[\mathsf{D}w^2\right] + 2\mathbb{E}_{\mathcal{D}}[\mathsf{D}w]\left(\frac{\left(\mathcal{L}_2^{-1}-\mathcal{L}_2\right)}{\mathcal{L}_1}\right) - \frac{2}{\mathcal{L}_1^2} \\
\bar{\mathbb{E}}_h\left[\mathsf{D}z^3\right] &= -\mathbb{E}_{\mathcal{D}}\left[\mathsf{D}w^3\right] - 3\mathbb{E}_{\mathcal{D}}\left[\mathsf{D}w^2\right]\left(\frac{\left(\mathcal{L}_2^{-1}-\mathcal{L}_2\right)}{\mathcal{L}_1}\right) + \frac{6}{\mathcal{L}_1^2}\mathbb{E}_{\mathcal{D}}[\mathsf{D}w].
\end{aligned}$$

Assuming $(g+c)=0$, to obtain $\mathbb{E}_h[\mathsf{D}z]$, we evaluate (III.7) at $m=1$, use the fact that $\mathcal{L}_2=1$, $\mathbb{E}_{\mathcal{D}}[\mathsf{D}w]=0$ and $\frac{\mathbb{E}_{\mathcal{D}}[t^u]}{\mathbb{E}_{\mathcal{D}}[t]}=\mathcal{E}_{ss}$, and substitute s^2 from Lemma C.1: $\mathbb{E}_h[\mathsf{D}z] = -\frac{\mathbb{E}_{\mathcal{D}}\left[\mathsf{D}w^3\right]}{3\mathbb{E}_{\mathcal{D}}\left[\mathsf{D}w^2\right]}$. Finally, replace this expression into (23):

$$\begin{aligned}
\frac{CIR_{\mathcal{E}}(z)}{z} &= -(1-\mathcal{E}_{ss})\frac{\mathbb{E}_h[\mathsf{D}z]}{s^2} = (1-\mathcal{E}_{ss})\frac{\frac{\mathbb{E}_{\mathcal{D}}\left[\mathsf{D}w^3\right]}{3\mathbb{E}_{\mathcal{D}}\left[\mathsf{D}w^2\right]}}{\frac{\mathbb{E}_{\mathcal{D}}\left[\mathsf{D}w^2\right]}{\mathbb{E}_{\mathcal{D}}[t]}} \\
&= \frac{1}{f(\hat{q}(\hat{w}^*))}\frac{\mathbb{E}_{\mathcal{D}}\left[\mathsf{D}w^3\right]}{3\mathbb{E}_{\mathcal{D}}\left[\mathsf{D}w^2\right]^2} = \frac{1}{f(\hat{q}(\hat{w}^*))}\frac{1}{\mathbb{V}ar_{\mathcal{D}}\left[\mathsf{D}w^2\right]}\frac{1}{3}\mathbb{E}_{\mathcal{D}}\left[\mathsf{D}w\frac{\mathsf{D}w^2}{\mathbb{E}_{\mathcal{D}}\left[\mathsf{D}w^2\right]}\right].
\end{aligned}$$

□

B.2 Proof of Proposition 7: CIR of Employment with Sticky Entry Wage

Proposition 7. *Assume sticky entry wages. Up to first order, the CIR of employment is given by:*

$$\frac{CIR_{\mathcal{E}}(z)}{z} = (1-\mathcal{E}_{ss})\left[-\frac{[g\mathbb{E}_h[a]+\mathbb{E}_h[\mathsf{D}z]]}{s^2} + \frac{1}{f(\hat{q}(\hat{w}^*))+s}\left[\frac{h'(\hat{w}^*)}{h(\hat{w}^*)} + \frac{\mathcal{T}'_{\hat{w}}(\hat{w}^*,\hat{r})}{\mathcal{T}(\hat{w}^*,\hat{r})} - \frac{\mathcal{T}'_{\hat{w}}(\hat{w}^*,0)}{\mathcal{T}(\hat{w}^*,0)}\right]\right] + o(z). \tag{B.31}$$

Proof. We divide the proof in two steps. Step 1 characterizes $m_{\mathcal{E},u}(\mathsf{D}z,z)$. Steps 2 uses the equilibrium conditions to show (B.31). The starting point is the CIR for employment, which is given by

$$\text{CIR}_{\mathcal{E}}(z) = \int_{-¥}^{¥} m_{\mathcal{E},h}(\mathsf{D}z) g^h(\mathsf{D}z + z)\, \mathrm{d}\mathsf{D}z + \int_{-¥}^{¥} m_{\mathcal{E},u}(\mathsf{D}z, z) g^u(\mathsf{D}z + z)\, \mathrm{d}\mathsf{D}z, \tag{B.32}$$

with

$$0 = 1 - \mathcal{E}_{ss} - g\frac{\mathrm{d}m_{\mathcal{E},h}(\mathsf{D}z)}{\mathrm{d}\mathsf{D}z} + \frac{s^2}{2}\frac{d^2 m_{\mathcal{E},h}(\mathsf{D}z)}{\mathrm{d}\mathsf{D}z^2} + d(m_{\mathcal{E},u}(0,0) - m_{\mathcal{E},h}(\mathsf{D}z)), \text{ for all } \mathsf{D}z \in (-\mathsf{D}^-, \mathsf{D}^+) \tag{B.33}$$

$$0 = -\mathcal{E}_{ss} - g\frac{\mathrm{d}m_{\mathcal{E},u}(\mathsf{D}z, z)}{\mathrm{d}\mathsf{D}z} + \frac{s^2}{2}\frac{d^2 m_{\mathcal{E},u}(\mathsf{D}z, z)}{\mathrm{d}\mathsf{D}z^2} + f(\hat{q}(\hat{w}^* - z))(m_{\mathcal{E},h}(-z) - m_{\mathcal{E},u}(\mathsf{D}z, z)) \tag{B.34}$$

$$0 = m_{\mathcal{E},u}(0,0) - m_{\mathcal{E},h}(\mathsf{D}z), \text{ for all } \mathsf{D}z \notin (-\mathsf{D}^-, \mathsf{D}^+) \tag{B.35}$$

$$0 = \lim_{\mathsf{D}z \to -¥} \frac{\mathrm{d}m_{\mathcal{E},u}(\mathsf{D}z, z)}{\mathrm{d}\mathsf{D}z} = \lim_{\mathsf{D}z \to ¥} \frac{\mathrm{d}m_{\mathcal{E},u}(\mathsf{D}z, z)}{\mathrm{d}\mathsf{D}z} \tag{B.36}$$

$$0 = \int_{-¥}^{¥} m_{\mathcal{E},h}(\mathsf{D}z) g^h(\mathsf{D}z)\, \mathrm{d}\mathsf{D}z + \int_{-¥}^{¥} m_{\mathcal{E},u}(\mathsf{D}z) g^u(\mathsf{D}z)\, \mathrm{d}\mathsf{D}z \tag{B.37}$$

The key differences between the CIR with flexible wages and the CIR with sticky wages are found in the HJB equation at the moment of the shock. With sticky entry wages, the job-finding probability is given by $f(\hat{q}(\hat{w}^* - z))$, since now the TFPR-adjusted entry wage is lower. As a consequence, we need to evaluate $m_{\mathcal{E},h}(\mathsf{D}z)$ at $\mathsf{D}z = -z$ because conditional on finding a job, the TFPR-adjusted entry wage is lower. Observe that following the first job separation, the aggregate TFPR shock is fully absorbed (see the term $m_{\mathcal{E},u}(0,0)$ in equation (B.33)).

Step 1. The value function $m_{\mathcal{E},u}(\mathsf{D}z, z)$ is independent of $\mathsf{D}z$ and satisfies $m_{\mathcal{E},u}(\mathsf{D}z, z) = -\frac{\mathcal{E}_{ss}}{f(\hat{q}(\hat{w}^* - z))} + m_{\mathcal{E},h}(-z)$.

Proof of Step 1. We guess and verify that $m_{\mathcal{E},u}(\mathsf{D}z, z) = m_{\mathcal{E},u}(0, z)$ for all $\mathsf{D}z$. From the equilibrium conditions (B.34) and (B.36),

$$0 = -\mathcal{E}_{ss} + f(\hat{q}(\hat{w}^* - z))(m_{\mathcal{E},h}(-z) - m_{\mathcal{E},u}(0, z)).$$

Thus, $m_{\mathcal{E},u}(0, z) = m_{\mathcal{E},u}(\mathsf{D}z, z) = -\frac{\mathcal{E}_{ss}}{f(\hat{q}(\hat{w}^* - z))} + m_{\mathcal{E},h}(-z)$.

Step 2. Up to a first-order approximation, the CIR is given by:

$$\text{CIR}_{\mathcal{E}}(z) = -(1 - \mathcal{E}_{ss})\frac{(g + c)\mathbb{E}_h[a] + \mathbb{E}_h[\mathsf{D}z]}{s^2} z + \frac{(1 - \mathcal{E}_{ss})}{f(\hat{q}(\hat{w}^*)) + s}\left(\frac{h'(\hat{w}^*)}{h(\hat{w}^*)} + \frac{\mathcal{T}'(\hat{w}^*, \hat{r})}{\mathcal{T}(\hat{w}^*, \hat{r})} - \frac{\mathcal{T}'(\hat{w}^*, 0)}{\mathcal{T}(\hat{w}^*, 0)}\right) z + o(z^2).$$

Proof of Step 2. From Step 1, we have that

$$\mathrm{CIR}'_{\mathcal{E}}(0) = \int_{-\yen}^{\yen} m_{\mathcal{E},h}(\mathsf{D}z)(g^h)'(\mathsf{D}z)\,\mathrm{d}\mathsf{D}z + \left(-\frac{\mathcal{E}_{ss} f_{\hat{w}}(\hat{q}(\hat{w}^*))}{f(\hat{q}(\hat{w}^*))^2} - m'_{\mathcal{E},h}(0) \right)(1-\mathcal{E}_{ss}).$$

Since $\int_{-\yen}^{\yen} m_{\mathcal{E},h}(\mathsf{D}z)(g^h)'(\mathsf{D}z)\,\mathrm{d}\mathsf{D}z$ satisfies the same system of functional equations as the CIR with flexible entry wages characterized in Appendix B.1,

$$\int_{-\yen}^{\yen} m_{\mathcal{E},h}(\mathsf{D}z)(g^h)'(\mathsf{D}z)\,\mathrm{d}\mathsf{D}z = -(1-\mathcal{E}_{ss})\frac{g\mathbb{E}_h[a] + \mathbb{E}_h[\mathsf{D}z]}{s^2}. \tag{B.38}$$

Observe that we can write

$$\begin{aligned} m_{\mathcal{E},h}(\mathsf{D}z) &= \mathbb{E}\left[\int_0^{t^m} (1-\mathcal{E}_{ss})\,\mathrm{d}t + m_{\mathcal{E},u}(\mathsf{D}z, 0) \middle| \mathsf{D}z_0 = \mathsf{D}z \right], \\ &= (1-\mathcal{E}_{ss})\mathcal{T}(\hat{w}^* + \mathsf{D}z, 0) - \frac{\mathcal{E}_{ss}}{f(\hat{q}(\hat{w}^*))} + m_{\mathcal{E},h}(0). \end{aligned}$$

Taking the derivative with respect to $\mathsf{D}z$, evaluating it at $\mathsf{D}z = 0$, and using $s = 1/\mathcal{T}(\hat{w}^*, 0)$ from the Renewal Principle, we have that

$$m'_{\mathcal{E},h}(0) = (1-\mathcal{E}_{ss})\mathcal{T}'_{\hat{w}}(\hat{w}^*, 0) = \frac{s}{f(\hat{q}(\hat{w}^*)) + s}\mathcal{T}'_{\hat{w}}(\hat{w}^*, 0) = \frac{1}{f(\hat{q}(\hat{w}^*)) + s}\frac{\mathcal{T}'(\hat{w}^*, 0)}{\mathcal{T}(\hat{w}^*, 0)}. \tag{B.39}$$

From the free entry condition $f(\hat{q}(\hat{w}^*)) = \left(\frac{\hat{J}(\hat{w}^*)}{\tilde{K}}\right)^{\frac{1-a}{a}}$, and the definition $(1-h(\hat{w}^*)) = \hat{J}(\hat{w}^*)/\hat{S}(\hat{w}^*)$, we can compute the elasticity of the job finding rate with respect to the entry wage:

$$\frac{f_{\hat{w}}(\hat{q}(\hat{w}^*))}{f(\hat{q}(\hat{w}^*))} = \frac{\frac{1-a}{a}\left(\frac{\hat{J}(\hat{w}^*)}{\tilde{K}}\right)^{\frac{1-a}{a}-1}\frac{\hat{J}'(\hat{w}^*)}{\tilde{K}}}{\left(\frac{\hat{J}(\hat{w}^*)}{\tilde{K}}\right)^{\frac{1-a}{a}}} = \frac{1-a}{a}\frac{\hat{J}'(\hat{w}^*)}{\hat{J}(\hat{w}^*)} = \frac{1-a}{a}\left[-\frac{h'(\hat{w}^*)}{(1-h(\hat{w}^*))} + \frac{\mathcal{T}'(\hat{w}^*, \hat{r})}{\mathcal{T}(\hat{w}^*, \hat{r})}\right].$$

Finally, combining this result with the fact that $\mathcal{E}_{ss} = \frac{f(\hat{q}(\hat{w}^*))}{f(\hat{q}(\hat{w}^*))+s}$, $s = \frac{1}{\mathcal{T}(\hat{w}^*,0)}$, $h'(\hat{w}^*)\left(\frac{a}{h(\hat{w}^*)} - \frac{1-a}{1-h(\hat{w}^*)}\right) = -\frac{\mathcal{T}'(\hat{w}^*,\hat{r})}{\mathcal{T}(\hat{w}^*,\hat{r})}$, and operating, we obtain

$$\begin{aligned} -\frac{\mathcal{E}_{ss} f_{\hat{w}}(\hat{q}(\hat{w}^*))}{f(\hat{q}(\hat{w}^*))^2} &= \frac{1}{f(\hat{q}(\hat{w}^*)) + s}\left[-\frac{1-a}{a}\left[-\frac{h'(\hat{w}^*)}{(1-h(\hat{w}^*))} + \frac{\mathcal{T}'(\hat{w}^*, \hat{r})}{\mathcal{T}(\hat{w}^*, \hat{r})}\right]\right] \\ &= \frac{1}{f(\hat{q}(\hat{w}^*)) + s}\left[\frac{h'(\hat{w}^*)}{h(\hat{w}^*)} + \frac{\mathcal{T}'(\hat{w}^*, \hat{r})}{\mathcal{T}(\hat{w}^*, \hat{r})}\right]. \end{aligned} \tag{B.40}$$

Combining results in equations (B.38), (B.39), and (B.40), we obtain the desired result:

$$\mathrm{CIR}'_{\mathcal{E}}(0) = -(1-\mathcal{E}_{ss})\frac{[(g+c)\mathbb{E}_h[a]+\mathbb{E}_h[\mathsf{D}z]]}{s^2} + \frac{1-\mathcal{E}_{ss}}{f(\hat{q}(\hat{w}^*))+s}\left[\frac{h'(\hat{w}^*)}{h(\hat{w}^*)} + \frac{\mathcal{T}'(\hat{w}^*,\hat{r})}{\mathcal{T}(\hat{w}^*,\hat{r})} - \frac{\mathcal{T}'(\hat{w}^*,0)}{\mathcal{T}(\hat{w}^*,0)}\right].$$

□

B.3 Proof of Lemma 2

The following proposition provides a characterization of $\mathcal{T}'_{\hat{w}}(\hat{w}^*,\hat{r})/\mathcal{T}(\hat{w}^*,\hat{r})$ stated in Lemma 2.

Proposition B.4. *Up to a second-order approximation of $\mathcal{T}(\hat{w},\hat{r})$ around $\hat{w}=\hat{w}^*$,*

$$\frac{\mathcal{T}'_{\hat{w}}(\hat{w}^*,\hat{r})}{\mathcal{T}(\hat{w}^*,\hat{r})} = \frac{\mathsf{D}^+ - \mathsf{D}^-}{\mathsf{D}^+\mathsf{D}^-}.$$

Proof. To show this property, it is useful to change the state variable in $\mathcal{T}(\hat{w},\hat{r})$ from $\hat{w}$ to $\mathsf{D}z$. Define $\tilde{\mathcal{T}}(\mathsf{D}z,\hat{r}) := \mathcal{T}(\hat{w}^*+\mathsf{D}z,\hat{r})$. Then, applying Itô's Lemma, we obtain

$$d\tilde{\mathcal{T}}(\mathsf{D}z,\hat{r}) = 1 - \hat{r}\tilde{\mathcal{T}}(\mathsf{D}z,\hat{r}) - (g+c)\tilde{\mathcal{T}}'_{\mathsf{D}z}(\mathsf{D}z,\hat{r}) + \frac{s^2}{2}\tilde{\mathcal{T}}''_{\mathsf{D}z^2}(\mathsf{D}z,\hat{r})\ \forall\ \mathsf{D}z \in (-\mathsf{D}^-,\mathsf{D}^+), \tag{B.41}$$

$$\tilde{\mathcal{T}}(\mathsf{D}z,\hat{r}) = 0\ \forall\ \mathsf{D}z \notin (-\mathsf{D}^-,\mathsf{D}^+). \tag{B.42}$$

Let $(g+c)\neq 0$ and $\mathsf{D}^+ \neq \mathsf{D}^-$. In this case, we proceed with a second-order Taylor approximation of $\tilde{\mathcal{T}}(\mathsf{D}z,\hat{r})$ around $\mathsf{D}z=0$,

$$\tilde{\mathcal{T}}(\mathsf{D}z,\hat{r}) = \tilde{\mathcal{T}}(0,\hat{r}) + \tilde{\mathcal{T}}'_{\mathsf{D}z}(0,\hat{r})\mathsf{D}z + \frac{1}{2}\tilde{\mathcal{T}}''_{\mathsf{D}z^2}(0,\hat{r})\mathsf{D}z^2 + O(\mathsf{D}z^3).$$

From the border conditions in (B.42), we obtain (we omit the term $O(\mathsf{D}z^3)$ to save on notation)

$$\begin{aligned} \tilde{\mathcal{T}}(0,\hat{r}) + \tilde{\mathcal{T}}'_{\mathsf{D}z}(0,\hat{r})\mathsf{D}^+ + \frac{1}{2}\tilde{\mathcal{T}}''_{\mathsf{D}z^2}(0,\hat{r})(\mathsf{D}^+)^2 &= 0, \\ \tilde{\mathcal{T}}(0,\hat{r}) + \tilde{\mathcal{T}}'_{\mathsf{D}z}(0,\hat{r})(-\mathsf{D}^-) + \frac{1}{2}\tilde{\mathcal{T}}''_{\mathsf{D}z^2}(0,\hat{r})(\mathsf{D}^-)^2 &= 0. \end{aligned} \tag{B.43}$$

Taking the difference

$$\tilde{\mathcal{T}}'_{\mathsf{D}z}(0,\hat{r})(\mathsf{D}^+ + \mathsf{D}^-) = -\frac{1}{2}\tilde{\mathcal{T}}''_{\mathsf{D}z^2}(0,\hat{r})\left((\mathsf{D}^+)^2 - (\mathsf{D}^-)^2\right) \iff \tilde{\mathcal{T}}'_{\mathsf{D}z}(0,\hat{r}) = -\frac{1}{2}\tilde{\mathcal{T}}''_{\mathsf{D}z^2}(0,\hat{r})\left(\mathsf{D}^+ - \mathsf{D}^-\right).$$

Replacing this last equation into the HJB equation in (B.41) evaluated at $\mathsf{D}z=0$ and into (B.43), we obtain

$$\tilde{\mathcal{T}}(0,\hat{r}) = \frac{1+\left(\frac{s^2+(g+c)(\mathsf{D}^+-\mathsf{D}^-)}{2}\right)\tilde{\mathcal{T}}''_{\mathsf{D}z^2}(0,\hat{r})}{\hat{r}+d}$$
$$\tilde{\mathcal{T}}(0,\hat{r}) = -\frac{1}{2}\tilde{\mathcal{T}}''_{\mathsf{D}z^2}(0,\hat{r})\left((\mathsf{D}^+)^2-\mathsf{D}^+\left(\mathsf{D}^+-\mathsf{D}^-\right)\right).$$

Combining these equations and solving for $\tilde{\mathcal{T}}(0,\hat{r})$ and $\tilde{\mathcal{T}}'_{\mathsf{D}z}(0,\hat{r})$, we have $\frac{\tilde{\mathcal{T}}'_{\mathsf{D}z}(0,\hat{r})}{\tilde{\mathcal{T}}(0,\hat{r})} = \frac{\mathcal{T}'_{\hat{w}^*}(\hat{w}^*,\hat{r})}{\mathcal{T}(\hat{w}^*,\hat{r})} = \frac{\mathsf{D}^+-\mathsf{D}^-}{\mathsf{D}^+\mathsf{D}^-}$.

□

B.4 Proof of Proposition 8

We divide Proposition 8 into two propositions. Proposition B.5 "rescales the speed of time" to provide a recursive representation of $h(\hat{w})$.

Proposition B.5. *Define*

$$t^{end} = \inf\{t \geq 0 : \mathsf{G}_t \notin (\hat{w}^-, \hat{w}^+)\}$$

where $(\hat{w}^-, \hat{w}^+)$ is a Nash equilibrium. Then, the worker's share $h(\hat{w})$ satisfies the following Bellman equation

$$h(\hat{w}) = \mathbb{E}\left[\int_0^{t^{end}} e^{-(\hat{r}+d)t}(\hat{r}+d)\frac{e^{\mathsf{G}_t}-\hat{r}\hat{U}}{1-\hat{r}\hat{U}}\,\mathrm{d}t + e^{-(\hat{r}+d)t^{end}}\mathbb{1}[\mathsf{D}z_{t^{end}} = \mathsf{D}^+]|\mathsf{G}_0 = \hat{w}\right]$$

with

$$\mathrm{d}\mathsf{G}_t = (\hat{r}+d)(-\hat{g}\mathcal{T}(\mathsf{G}_t,\hat{r}) + s^2\mathcal{T}'_{\hat{w}}(\mathsf{G}_t,\hat{r}))\,\mathrm{d}t + s\sqrt{\mathcal{T}(\mathsf{G}_t,\hat{r})(\hat{r}+d)}\,\mathrm{d}\mathcal{W}^z_t.$$

Proof. The HJB equations for the worker's value and the surplus of the match are

$$(\hat{r}+d)\hat{W}(\hat{w}) = e^{\hat{w}} - \hat{r}\hat{U} - \hat{g}\hat{W}'(\hat{w}) + \frac{s^2}{2}\hat{W}''(\hat{w}) \quad \forall \hat{w} \in (\hat{w}^-, \hat{w}^+)$$
$$(\hat{r}+d)\hat{S}(\hat{w}) = 1 - \hat{r}\hat{U} - \hat{g}\hat{S}'(\hat{w}) + \frac{s^2}{2}\hat{S}''(\hat{w}) \quad \forall \hat{w} \in (\hat{w}^-, \hat{w}^+),$$

respectively. Replacing the definition of the worker's share $h(\hat{w}) = \hat{W}(\hat{w})/\hat{S}(\hat{w})$ into the worker's value function, we obtain

$$(\hat{r}+d)(h(\hat{w})\hat{S}(\hat{w})) = e^{\hat{w}} - \hat{r}\hat{U} - \hat{g}\left(h(\hat{w})\hat{S}'(\hat{w}) + h'(\hat{w})\hat{S}(\hat{w})\right) + \frac{s^2}{2}\left(h(\hat{w})\hat{S}''(\hat{w}) + 2h'(\hat{w})\hat{S}'(\hat{w}) + h''(\hat{w})\hat{S}(\hat{w})\right) \forall \hat{w} \in (\hat{w}^-, \hat{w}^+).$$

Using the HJB equation of the surplus to replace $(\hat{r}+d)\hat{S}(\hat{w})$ on the left hand side,

$$(1-\hat{r}\hat{U})h(\hat{w}) = e^{\hat{w}} - \hat{r}\hat{U} + h'(\hat{w})(-\hat{g}\hat{S}(\hat{w}) + s^2\hat{S}'(\hat{w})) + h''(\hat{w})\frac{s^2}{2}\hat{S}(\hat{w}) \quad \forall \hat{w} \in (\hat{w}^-, \hat{w}^+).$$

Since $\hat{S}(\hat{w}) = (1-\hat{r}\hat{U})\mathcal{T}(\hat{w},\hat{r})$, multiplying by $(\hat{r}+d)$, we arrive at

$$(\hat{r}+d)h(\hat{w}) = (\hat{r}+d)\frac{e^{\hat{w}}-\hat{r}\hat{U}}{1-\hat{r}\hat{U}} + h'(\hat{w})(\hat{r}+d)(-\hat{g}\mathcal{T}(\hat{w},\hat{r}) + s^2\mathcal{T}'_{\hat{w}}(\hat{w},\hat{r})) + h''(\hat{w})\frac{s^2}{2}(\hat{r}+d)\mathcal{T}(\hat{w},\hat{r}) \quad \forall \hat{w} \in (\hat{w}^-, \hat{w}^+).$$

Finally, recall the value-matching conditions $\hat{W}(\hat{w}^-) = \hat{J}(\hat{w}^-) = \hat{W}(\hat{w}^+) = \hat{J}(\hat{w}^+) = 0$, and the smooth pasting conditions $\hat{W}'(-D^-) = \hat{J}'(D^+) = 0$. The L'Hôpital's rule implies

$$\lim_{\hat{w}\downarrow\hat{w}^-} h(\hat{w}) = \lim_{\hat{w}\downarrow\hat{w}^-} \frac{\hat{W}(\hat{w})}{\hat{S}(\hat{w})} = \lim_{\hat{w}\downarrow\hat{w}^-} \frac{\hat{W}'(\hat{w})}{\hat{J}'(\hat{w})} = 0$$
$$\lim_{\hat{w}\uparrow\hat{w}^+} h(\hat{w}) = \lim_{\hat{w}\uparrow\hat{w}^+} \frac{\hat{W}(\hat{w})}{\hat{S}(\hat{w})} = \lim_{\hat{w}\uparrow\hat{w}^+} \frac{\hat{W}'(\hat{w})}{\hat{W}'(\hat{w})} = 1,$$

which are the boundary values for the worker's share at the separation triggers.

Finally, the equivalence of the combined Dirichlet-Poisson problem (i.e., the mapping from the corresponding HJB equations and boundary conditions of $h(\hat{w})$ to the sequential formulation) gives us the following Bellman equation

$$h(\hat{w}) = \mathbb{E}\left[\int_0^{t^{end}} e^{-(\hat{r}+d)t}(\hat{r}+d)\frac{e^{G_t}-\hat{r}\hat{U}}{1-\hat{r}\hat{U}}\,dt + e^{-(\hat{r}+d)t^{end}}\mathbb{1}[Dz_{t^{end}} = D^+] | G_0 = \hat{w}\right],$$

where $t^{end} = \inf\{t \geq 0 : G_t \notin (\hat{w}^-, \hat{w}^+)\}$ and

$$dG_t = (\hat{r}+d)(-\hat{g}\mathcal{T}(G_t,\hat{r}) + s^2\mathcal{T}'_{\hat{w}}(G_t,\hat{r}))\,dt + s\sqrt{\mathcal{T}(G_t,\hat{r})(\hat{r}+d)}\,d\mathcal{W}^z_t.$$

□

Proof of Proposition 8. Below, we prove each property.

1. If $D^+, D^- \to ¥$, then $\mathcal{T}(\hat{w},\hat{r}) = \int_0^{¥} e^{-(\hat{r}+d)t}\,dt = \frac{1}{\hat{r}+d}$. The optimality condition for $\hat{w}^*$ implies

$$0 = -\frac{\mathcal{T}'_{\hat{w}}(\hat{w}^*,\hat{r})}{\mathcal{T}(\hat{w}^*,\hat{r})} = h'(\hat{w}^*)\left(\frac{a}{h(\hat{w}^*)} - \frac{1-a}{1-h(\hat{w}^*)}\right) \iff a = h(\hat{w}^*).$$

Therefore, by the definition of $h(\hat{w})$,

$$a = h(\hat{w}^*) = \frac{\mathbb{E}\left[\int_0^{t^m} e^{-\hat{r}t+\hat{w}_t}\,\mathrm{d}t|\hat{w}_0 = \hat{w}^*\right] - \hat{r}\hat{U}\mathcal{T}(\hat{w},\hat{r})}{(1-\hat{r}\hat{U})\mathcal{T}(\hat{w},\hat{r})}$$

$$\iff \left[a + (1-a)\hat{r}\hat{U}\right]\mathcal{T}(\hat{w},\hat{r}) = \mathbb{E}\left[\int_0^{t^m} e^{-\hat{r}t+\hat{w}_t}\,\mathrm{d}t|\hat{w}_0 = \hat{w}^*\right].$$

Since $\mathcal{T}(\hat{w},\hat{r})$ is constant, the HJB equation of the worker's share $h(\hat{w})$ is given by

$$(\hat{r}+d)h(\hat{w}) = (\hat{r}+d)\frac{e^{\hat{w}} - \hat{r}\hat{U}}{1-\hat{r}\hat{U}} - \hat{g}h'(\hat{w}) + h''(\hat{w})\frac{s^2}{2} \;\; \forall \hat{w} \in (-¥, ¥). \tag{B.44}$$

Taking the derivative of (B.44) with respect to $\hat{w}$ yields

$$(\hat{r}+d)h'(\hat{w}) = (\hat{r}+d)\frac{e^{\hat{w}}}{1-\hat{r}\hat{U}} - \hat{g}h''(\hat{w}) + h'''(\hat{w})\frac{s^2}{2} \;\; \forall \hat{w} \in (-¥, ¥).$$

This expression corresponds to the HJB of the function $h'(\hat{w})$, which can be expressed as

$$h'(\hat{w}^*) = (\hat{r}+d)\frac{\mathbb{E}\left[\int_0^{t^m} e^{-\hat{r}t+\hat{w}_t}\,\mathrm{d}t|\hat{w}_0 = \hat{w}^*\right]}{1-\hat{r}\hat{U}}$$

Combining all these results, we finally obtain

$$\frac{h'(\hat{w}^*)}{h(\hat{w}^*)} = \frac{h'(\hat{w}^*)}{a} = (\hat{r}+d)\frac{\mathbb{E}\left[\int_0^{t^m} e^{-\hat{r}t+\hat{w}_t}\,\mathrm{d}t|\hat{w}_0 = \hat{w}^*\right]}{a(1-\hat{r}\hat{U})} = (\hat{r}+d)\frac{\left[a+(1-a)\hat{r}\hat{U}\right]\mathcal{T}(\hat{w},\hat{r})}{a(1-\hat{r}\hat{U})} = \frac{\left[a+(1-a)\hat{r}\hat{U}\right]}{a(1-\hat{r}\hat{U})}.$$

2. If $g + c = 0$ and $\mathsf{D}^+ = \mathsf{D}^-$, then $\mathcal{T}'_{\hat{w}}(\hat{w}^*,\hat{r}) = 0$ and $h(\hat{w}^*) = a$ (see the proof of Proposition B.4, item *a*). If $(\mathsf{D}^+ + \mathsf{D}^-)$ is small enough, then we can use a second-order approximation of $h(\hat{w})$ around $\hat{w} = \hat{w}^*$ to characterize $h'(\hat{w}^*)$ *only* using the border conditions. The approximation is given by

$$h(\hat{w}) = h(\hat{w}^*) + h'(\hat{w}^*)(\hat{w}-\hat{w}^*) + \frac{1}{2}h''(\hat{w}^*)(\hat{w}-\hat{w}^*)^2 + O((\hat{w}-\hat{w}^*)^3).$$

Evaluating this expression at $\hat{w}^-$ and $\hat{w}^+$, and omitting any terms of the order $O((\hat{w}-\hat{w}^*)^3)$, we obtain

$$h(\hat{w}^*) + h'(\hat{w}^*)(\hat{w}^- - \hat{w}^*) + \frac{1}{2}h''(\hat{w}^*)(\hat{w}^- - \hat{w}^*)^2 = 0,$$

$$h(\hat{w}^*) + h'(\hat{w}^*)(\hat{w}^+ - \hat{w}^*) + \frac{1}{2}h''(\hat{w}^*)(\hat{w}^+ - \hat{w}^*)^2 = 1,$$

respectively. The difference between both equations is given by $h'(\hat{w}^*) = \frac{1}{D^+ + D^-}$. From the proof of Proposition B.4 item b, we know that $\tilde{\mathcal{T}}(0,0) = 1/s = 1/(d + (s/D^+)^2) \Rightarrow s^{end} = (s/D^+)^2$. Replacing this result in the previous equation, we obtain, $\frac{h'(\hat{w}^*)}{h(\hat{w}^*)} = \frac{1}{a}\frac{1}{D^+ + D^-} = \frac{\sqrt{s^{end}}}{2as}$.

□

C Proofs for Section 4: Mapping the Model to Labor Market Microdata

C.1 Characterizing $g^h(Dz)$ and $g^u(Dz)$

The equilibrium policies $(\hat{w}^-, \hat{w}^*, \hat{w}^+)$ together with the stochastic process guiding Dz and the exogenous job separation rate determine the equilibrium distributions of cumulative productivity shocks $g^h(Dz)$ and $g^u(Dz)$. Due to the law of motion for Dz being independent of the worker's employment state, the KFEs for employed and unemployed workers are

$$dg^h(Dz) = (g + c)(g^h)'(Dz) + \frac{s^2}{2}(g^h)''(Dz) \quad \forall Dz \in (-D^-, D^+)\backslash\{0\}, \tag{C.1}$$

$$f(\hat{q}(\hat{w}^*))g^u(Dz) = (g + c)(g^u)'(Dz) + \frac{s^2}{2}(g^u)''(Dz) \quad \forall Dz \in \mathbb{R}\backslash\{0\}. \tag{C.2}$$

Since the entry state for a newly employed or unemployed worker is $Dz = 0$, the KFEs (C.1)–(C.2) do not hold at this point, but $g^h(\cdot)$ and $g^u(\cdot)$ must be continuous there.

The boundary conditions impose a zero measure of workers at the borders of the support, so that $g^h(-D^-) = g^h(D^+) = \lim_{Dz \to -¥} g^u(Dz) = \lim_{Dz \to ¥} g^u(Dz) = 0$. These distributions must also be consistent with (i) a unit measure of workers and (ii) a flow balance equation for steady-state employment:

$$1 = \int_{-¥}^{¥} g^u(Dz)\, dDz + \int_{-D^-}^{D^+} g^h(Dz)\, dDz, \tag{C.3}$$

$$\underbrace{f(\hat{q}(\hat{w}^*))(1 - \mathcal{E})}_{u\text{-to-}h \text{ flows}} = \underbrace{d\mathcal{E} + \frac{s^2}{2}\left[\lim_{Dz \downarrow -D^-}(g^h)'(Dz) - \lim_{Dz \uparrow D^+}(g^h)'(Dz)\right]}_{h\text{-to-}u \text{ flows}}. \tag{C.4}$$

In equation (C.3), the unit measure of workers is composed of $\int_{-¥}^{¥} g^u(Dz)\, dDz = 1 - \mathcal{E}$ unemployed and $\int_{-D^-}^{D^+} g^h(Dz)\, dDz = \mathcal{E}$ employed workers. In equation (C.4), the mass of u-to$h$ flows is

$f(\hat{q}(\hat{w}^*))(1-\mathcal{E})$, while the mass of h-to-u flows is $d\mathcal{E}+\frac{s^2}{2}[\lim_{\text{D}z\downarrow -\text{D}^-}(g^h)'(\text{D}z)-\lim_{\text{D}z\uparrow \text{D}^+}(g^h)'(\text{D}z)]$— i.e., the sum of exogenous and endogenous job separations.

To summarize, equations (C.1)–(C.4), together with the continuity of $g^u(\text{D}z)$ and $g^h(\text{D}z)$ at $\text{D}z=0$, constitute the equilibrium conditions for the steady-state distributions of cumulative productivity shocks.

C.2 Proof of Proposition 9

We divide the proof of Proposition 9 into three steps. Proposition C.1 recovers the parameters of the stochastic process of the wage-to-revenue productivity ratio $\text{D}z$. Proposition C.2 recovers the distribution of cumulative productivity shocks conditional on job transitions $\bar{G}^h(\text{D}z)$. Finally, Proposition C.3 recovers the distribution of $\text{D}z$ among employed workers $G^h(\text{D}z)$.

Proposition C.1. *Let $t := t^m + t^u$. The drift $(g+c)$ and volatility s of the stochastic process guiding cumulative productivity shocks can be recovered from the data with*

$$g+c=\frac{\mathbb{E}_{\mathcal{D}}[\text{D}w]}{\mathbb{E}_{\mathcal{D}}[t]},\quad s^2=\frac{\mathbb{E}_{\mathcal{D}}[(\text{D}w-(g+c)t)^2]}{\mathbb{E}_{\mathcal{D}}[t]}$$

Proof. From the law of motion $\text{d}z_t=(g+c)\,\text{d}t+s\,\text{d}\mathcal{W}_t^z$ and the fact that $w_{t_0}-z_{t_0}=\hat{w}^*$, we have

$$\text{D}w=-\text{D}z_t=(g+c)t+s\mathcal{W}_t^z. \tag{C.5}$$

Drift: Taking expectation on both sides conditional on a h-to-u-to-h transition, we have that $s\mathbb{E}[\mathcal{W}_t^z]=\mathbb{E}_{\mathcal{D}}[\text{D}w]-(g+c)\mathbb{E}_{\mathcal{D}}[t]$. Since $\mathcal{W}_t^z$ is a martingale, by Doob's Optional Stopping Theorem (OST) $\mathcal{W}_t^z$ is also a martingale, and $E[\mathcal{W}_t^z]=\mathbb{E}[\mathcal{W}_0^z]=0$, thus yielding the desired result.

Idiosyncratic volatility: Let us define $Y_t=(\text{D}z_t+(g+c)t)^2$. We apply Itô's Lemma to Y_t and obtain

$$\text{d}Y_t=2\,(\text{D}z_t+(g+c)t)\,(\text{d}\text{D}z_t+(g+c)\,\text{d}t)+\frac{1}{2}2(\text{d}\text{D}z_t)^2=2s\,(\text{D}z_t+(g+c)t)\,\text{d}\mathcal{W}_t^z+s^2\,\text{d}t$$

Integrating the previous equation between 0 and t and using condition (C.5), we obtain

$$(\text{D}w-(g+c)t)^2=2s\int_0^t(\text{D}z_t+(g+c)t)\,\text{d}\mathcal{W}_t^z+s^2t.$$

Since $\int_0^t(\text{D}z_t+(g+c)t)\,\text{d}\mathcal{W}_t^z$ is a martingale, by the OST, $\int_0^t(\text{D}z_t+(g+c)t)\,\text{d}\mathcal{W}_t^z$ is a martingale and $\mathbb{E}[\int_0^t (\mathsf{D}z_t + (g+c)t)\, \mathrm{d}\mathcal{W}_t^z] = 0$, thus yielding the desired result. □

Proposition C.2. *The cumulative distribution of* $\mathsf{D}z$ *conditional on a job separation is given by*

$$\bar{G}^h(\mathsf{D}z) = \frac{s^2}{2f(\hat{q}(\hat{w}^*))}\frac{\mathrm{d}l^w(-\mathsf{D}z)}{\mathrm{d}z} - \frac{(g+c)}{f(\hat{q}(\hat{w}^*))} l^w(-\mathsf{D}z) - [1 - L^w(-\mathsf{D}z)]\,. \qquad \text{(C.6)}$$

where $L^w(\mathsf{D}w)$ *denotes the CDF corresponding to the marginal distribution* $l^w(\mathsf{D}w)$.

Proof. The objective in this proof is to use the nondifferentiability of the distribution of $\bar{g}_s(\mathsf{D}z)$ for $s = \{h, u\}$ at $\mathsf{D}z = 0$ to express the distribution of $\mathsf{D}z$ conditional on a separation. Observe that

$$\begin{aligned} L^w(a) &= Pr^{l^w}(\mathsf{D}w \leq a) \\ &=^{(1)} Pr^{\bar{G}^h, \bar{G}^u}(-(\mathsf{D}z^h + \mathsf{D}z^u) \leq a) \\ &=^{(2)} Pr^{\bar{G}^h, \bar{G}^u}(\mathsf{D}z^h + \mathsf{D}z^u \geq -a) \\ &=^{(3)} 1 - Pr(\mathsf{D}z^h + \mathsf{D}z^u \leq -a) \\ &=^{(4)} 1 - \int_{-¥}^{¥} \bar{G}^h(-a-y)\bar{g}^u(y)\,\mathrm{d}y. \end{aligned}$$

Step (1) uses the definition of $\mathsf{D}w$. Steps (2)–(4) use independence of $\bar{G}^h(\cdot)$ and $\bar{g}^u(\cdot)$. It can be shown that

$$\bar{g}^u(\mathsf{D}z) = \mathcal{G}_u \begin{cases} e^{b_2(f(\hat{q}(\hat{w}^*)))\mathsf{D}z} & \text{if } \mathsf{D}z \in (-¥, 0] \\ e^{b_1(f(\hat{q}(\hat{w}^*)))\mathsf{D}z} & \text{if } \mathsf{D}z \in [0, ¥) \end{cases}, L^w(\mathsf{D}w) = 1 - C_1(\mathsf{D}w) - C_2(\mathsf{D}w), \qquad \text{(C.7)}$$

where

$$C_1(\mathsf{D}w) = \mathcal{G}_u \int_0^{¥} \bar{G}^h(-\mathsf{D}w - u)e^{b_1(f(\hat{q}(\hat{w}^*)))u}\,\mathrm{d}u, \quad C_2(\mathsf{D}w) = \mathcal{G}_u \int_{-¥}^{0} \bar{G}^h(-\mathsf{D}w - u)e^{b_2(f(\hat{q}(\hat{w}^*)))u}\,\mathrm{d}u.$$

Departing from $L^w(\mathsf{D}w) = 1 - \int_{-¥}^{¥} \bar{G}^h(-\mathsf{D}w - y)\bar{g}^u(y)\,\mathrm{d}y$ and doing the change of variable $x = -\mathsf{D}w - y$,

$$L^w(\mathsf{D}w) = 1 - \int_{-¥}^{¥} \bar{G}^h(x)\bar{g}^u(-\mathsf{D}w - x)\,\mathrm{d}x.$$

Taking the derivative on both sides with respect to $\mathsf{D}w$, we obtain

$$l^w(\mathsf{D}w) = \int_{-¥}^{¥} \bar{G}^h(x)\,(\bar{g}^u)'\,(-\mathsf{D}w - x)\,\mathrm{d}x.$$

Reverting the change of variables and using the fact that $\bar{g}^u(-\mathrm{D}w - x)$ is nondifferentiable at 0, we obtain

$$I^w(\mathrm{D}w) = b_1(f(\hat{q}(\hat{w}^*)))C_1(\mathrm{D}w) + b_2(f(\hat{q}(\hat{w}^*)))C_2(\mathrm{D}w).$$

Thus,

$$I^w(\mathrm{D}w) = b_1(f(\hat{q}(\hat{w}^*)))C_1(\mathrm{D}w) + b_2(f(\hat{q}(\hat{w}^*)))C_2(\mathrm{D}w). \tag{C.8}$$

To obtain the last condition, observe that

$$C_1(\mathrm{D}w) = \mathcal{G}_u \int_{-¥}^{-\mathrm{D}w} \bar{G}^h(y) e^{b_1(f(\hat{q}(\hat{w}^*)))(-\mathrm{D}w-y)}\,\mathrm{d}y, C_2(\mathrm{D}w) \quad = \mathcal{G}_u \int_{-\mathrm{D}w}^{¥} \bar{G}^h(y) e^{b_2(f(\hat{q}(\hat{w}^*)))(-\mathrm{D}w-y)}\,\mathrm{d}y.$$

Taking the derivative with respect to $\mathrm{D}w$ and using the Leibniz rule, we obtain

$$C_1'(\mathrm{D}w) = -\mathcal{G}_u \bar{G}^h(-\mathrm{D}w) - b_1(f(\hat{q}(\hat{w}^*)))C_1(\mathrm{D}w), \tag{C.9}$$

$$C_2'(\mathrm{D}w) = \mathcal{G}_u \bar{G}^h(-\mathrm{D}w) - b_2(f(\hat{q}(\hat{w}^*)))C_2(\mathrm{D}w). \tag{C.10}$$

Taking derivative of (C.8),

$$(I^w)'(\mathrm{D}w) = b_1(f(\hat{q}(\hat{w}^*)))C_1'(\mathrm{D}w) + b_2(f(\hat{q}(\hat{w}^*)))C_2'(\mathrm{D}w)$$

and using conditions (C.9) and (C.10),

$$\begin{aligned}(I^w)'(\mathrm{D}w) = {} & \bar{G}^h(-\mathrm{D}w)\mathcal{G}_u[b_2(f(\hat{q}(\hat{w}^*))) - b_1(f(\hat{q}(\hat{w}^*)))] \\ & - b_1(f(\hat{q}(\hat{w}^*)))^2 C_1(\mathrm{D}w) - b_2(f(\hat{q}(\hat{w}^*)))^2 C_2(\mathrm{D}w).\end{aligned} \tag{C.11}$$

Equations (C.7), (C.8), and (C.11) give a system of three functional equations with three unknowns:

$$\begin{aligned}1 - L^w(\mathrm{D}w) &= C_1(\mathrm{D}w) + C_2(\mathrm{D}w), \\ I^w(\mathrm{D}w) &= b_1(f(\hat{q}(\hat{w}^*)))C_1(\mathrm{D}w) + b_2(f(\hat{q}(\hat{w}^*)))C_2(\mathrm{D}w), \\ (I^w)'(\mathrm{D}w) &= \bar{G}^h(-\mathrm{D}w)\mathcal{G}_u[b_2(f(\hat{q}(\hat{w}^*))) - b_1(f(\hat{q}(\hat{w}^*)))] \\ &\quad - b_1(f(\hat{q}(\hat{w}^*)))^2 C_1(\mathrm{D}w) - b_2(f(\hat{q}(\hat{w}^*)))^2 C_2(\mathrm{D}w).\end{aligned}$$

Operating on the system of functional equations,

$$(l^w)'(\mathrm{D}w) + \left[b_2(f(\hat{q}(\hat{w}^*))) + b_1(f(\hat{q}(\hat{w}^*)))\right] l^w(\mathrm{D}w) + b_1(f(\hat{q}(\hat{w}^*)))b_2(f(\hat{q}(\hat{w}^*)))\left[1 - L^w(\mathrm{D}w)\right]$$

$$= \bar{G}^h(-\mathrm{D}w)\mathcal{G}_u[b_2(f(\hat{q}(\hat{w}^*))) - b_1(f(\hat{q}(\hat{w}^*)))],$$

with

$$\begin{aligned}
\mathcal{G}_u &= \left(b_2(f(\hat{q}(\hat{w}^*)))^{-1} - b_1(f(\hat{q}(\hat{w}^*)))^{-1}\right)^{-1} \\
b_1(f(\hat{q}(\hat{w}^*))) &= \frac{-(g+c) - \sqrt{(g+c)^2 + 2s^2 f(\hat{q}(\hat{w}^*))}}{s^2}, \\
b_2(f(\hat{q}(\hat{w}^*))) &= \frac{-(g+c) + \sqrt{(g+c)^2 + 2s^2 f(\hat{q}(\hat{w}^*))}}{s^2}.
\end{aligned}$$

Then,

$$\begin{aligned}
\mathcal{G}_u[b_2(f(\hat{q}(\hat{w}^*))) - b_1(f(\hat{q}(\hat{w}^*)))] &= \frac{2f(\hat{q}(\hat{w}^*))}{s^2} \\
\frac{b_2(f(\hat{q}(\hat{w}^*))) + b_1(f(\hat{q}(\hat{w}^*)))}{\mathcal{G}_u[b_2(f(\hat{q}(\hat{w}^*))) - b_1(f(\hat{q}(\hat{w}^*)))]} &= -\frac{(g+c)}{f(\hat{q}(\hat{w}^*))}. \\
\frac{b_1(f(\hat{q}(\hat{w}^*)))b_2(f(\hat{q}(\hat{w}^*)))}{\mathcal{G}_u[b_2(f(\hat{q}(\hat{w}^*))) - b_1(f(\hat{q}(\hat{w}^*)))]} &= -1.
\end{aligned}$$

Therefore, the differential equation is given by (C.6).

□

Proposition C.3. *If $(g+c) = 0$, the distribution of cumulative productivity shocks $g^h(\mathrm{D}z)$ is given by*

$$g^h(\mathrm{D}z) = s\mathcal{E}\left[\int_{-\mathrm{D}^-}^{\mathrm{D}z} \frac{2(\mathrm{D}z - y)}{s^2}\bar{g}^h(y)\,\mathrm{d}y + \bar{G}^h(-\mathrm{D}^-)\frac{2(\mathrm{D}z + \mathrm{D}^-)}{s^2}\right].$$

If $(g+c) \neq 0$, the distribution of cumulative productivity shocks $g^h(\mathrm{D}z)$ is given by

$$g^h(\mathrm{D}z) = \frac{s\mathcal{E}}{(g+c)}\left[\int_{-\mathrm{D}^-}^{\mathrm{D}z}\left(1 - e^{\frac{2(g+c)}{s^2}(y - \mathrm{D}z)}\right)\bar{g}^h(y)\,\mathrm{d}y + \bar{G}^h(-\mathrm{D}^-)\left[1 - e^{-\frac{2(g+c)}{s^2}(\mathrm{D}z + \mathrm{D}^-)}\right]\right].$$

Proof. During employment, the distribution of cumulative productivity shocks satisfies the following KFE and the boundary conditions

$$
\begin{aligned}
dg^h(\mathsf{D}z) &= (g+c)(g^h)'(\mathsf{D}z) + \frac{\mathsf{s}^2}{2}(g^h)''(\mathsf{D}z) \quad \forall \mathsf{D}z \in (-\mathsf{D}^-, \mathsf{D}^+) \setminus \{0\}, \\
g^h(-\mathsf{D}^-) &= g^h(\mathsf{D}^+) = 0, G^h(\mathsf{D}^+) = \mathcal{E}, \\
g^h(\mathsf{D}z) &\in \mathbb{C}.
\end{aligned}
$$

The distribution of cumulative productivity shocks conditional on a job separation satisfies

$$
\bar{G}^h(\mathsf{D}z) = \begin{cases} 1 & \text{if } \mathsf{D}z \in [\mathsf{D}_+, ¥) \\ \frac{1}{s\mathcal{E}}\left[\frac{\mathsf{s}^2}{2}\lim_{\mathsf{D}z\downarrow -\mathsf{D}^-}(g^h)'(\mathsf{D}z) + d\int_{-\mathsf{D}^-}^{\mathsf{D}z} g^h(x)\,\mathrm{d}x\right] & \text{if } \mathsf{D}z \in [-\mathsf{D}^-, \mathsf{D}^+) \\ 0 & \text{if } \mathsf{D}z \in (-¥, -\mathsf{D}^-). \end{cases}
$$

Combining these two conditions, we obtain

$$
\begin{aligned}
s\mathcal{E}\bar{g}^h(\mathsf{D}z) &= (g+c)(g^h)'(\mathsf{D}z) + \frac{\mathsf{s}^2}{2}(g^h)''(\mathsf{D}z) \quad \forall \mathsf{D}z \in (-\mathsf{D}^-, \mathsf{D}^+) \setminus \{0\} \\
g^h(-\mathsf{D}^-) &= g^h(\mathsf{D}^+) = 0, G^h(\mathsf{D}^+) = \mathcal{E}.
\end{aligned}
$$

Multiplying both sides of the first equation by $e^{\frac{2(g+c)}{s^2}\mathsf{D}z}$ we get

$$
s\mathcal{E}e^{\frac{2(g+c)}{s^2}\mathsf{D}z}\bar{g}^h(\mathsf{D}z) = \frac{\mathsf{s}^2}{2}\frac{\mathrm{d}(e^{\frac{2(g+c)}{s^2}\mathsf{D}z}(g^h)'(\mathsf{D}z))}{\mathrm{d}\mathsf{D}z}.
$$

Integrating both sides from $-\mathsf{D}^-$ to $\mathsf{D}z$, we obtain

$$
\begin{aligned}
s\mathcal{E}\int_{-\mathsf{D}^-}^{\mathsf{D}z} e^{\frac{2(g+c)}{s^2}x}\bar{g}^h(x)\,\mathrm{d}x &= \frac{\mathsf{s}^2}{2}\left[e^{\frac{2(g+c)}{s^2}\mathsf{D}z}(g^h)'(\mathsf{D}z) - \lim_{x\downarrow -\mathsf{D}^-} e^{\frac{2(g+c)}{s^2}x}(g^h)'(x)\right], \\
&= \frac{\mathsf{s}^2}{2}e^{\frac{2(g+c)}{s^2}\mathsf{D}z}(g^h)'(\mathsf{D}z) - s\mathcal{E}e^{-\frac{2(g+c)}{s^2}\mathsf{D}^-}\bar{G}^h(-\mathsf{D}^-),
\end{aligned}
$$

where the last equation uses the value of $\bar{G}^h(\mathsf{D}z)$ evaluated at $\mathsf{D}z = -\mathsf{D}^-$. Solving for $(g^h)'(\mathsf{D}z)$, integrating from $-\mathsf{D}^-$ to $\mathsf{D}z$ and taking the limit as $(g+c) \downarrow 0$, we get

$$
g^h(\mathsf{D}z) = s\mathcal{E}\left[\int_{-\mathsf{D}^-}^{\mathsf{D}z}\frac{2(\mathsf{D}z - y)}{\mathsf{s}^2}\bar{g}^h(y)\,\mathrm{d}y + \bar{G}^h(-\mathsf{D}^-)\frac{2(\mathsf{D}z + \mathsf{D}^-)}{\mathsf{s}^2}\right].
$$

□

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A Theory of Labor Markets with Inefficient Turnover

Online Appendix—Not for Publication

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Contents of the Online Appendix

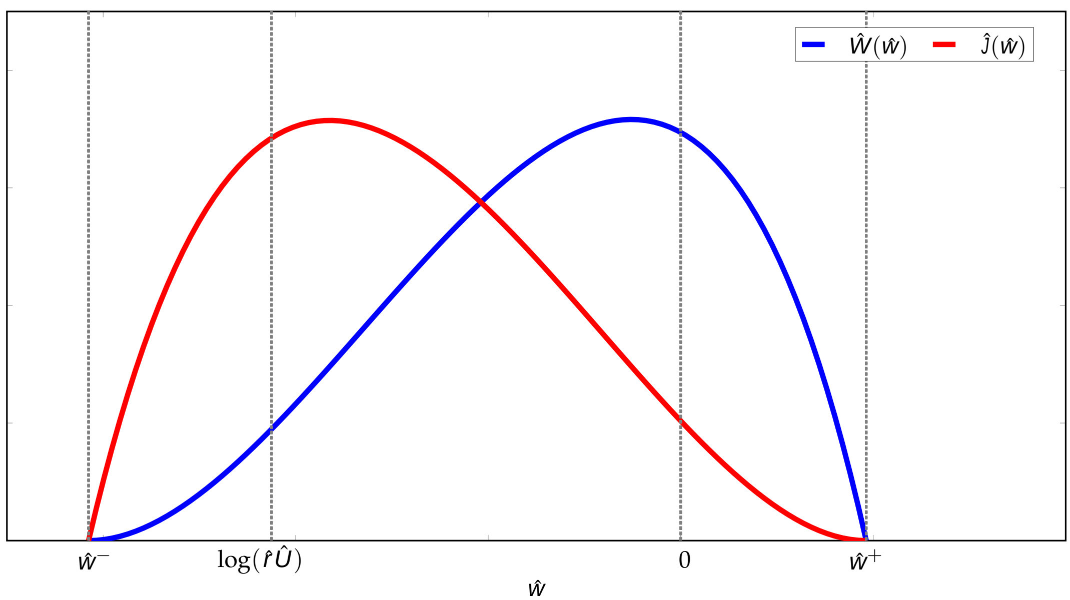
I Additional Results for Section 2: A Model of Labor Markets with Inefficient Turnover

This section presents additional results extending the analysis in Section 2.

I.1 Equilibrium Value Functions

The following figure shows the equilibrium values and continuation sets of the worker (blue solid line) and the firm (red solid line) in our baseline model.

FIGURE I1. EQUILIBRIUM VALUE FUNCTIONS AND CONTINUATION SETS



Notes: The figure plots the equilibrium value functions of the firm (i.e., $\hat{J}(\hat{w})$) and the employed worker (i.e., $\hat{W}(\hat{w})$) as a function of $\hat{w} = w - z$. The blue and red solid lines show the value function of the employed worker and the value function of the firm, respectively. The dotted vertical lines mark the boundaries of the firms' continuation set (−¥, $\hat{w}^+$) and the worker's continuation set ($\hat{w}^-$, ¥).

I.2 Sequential and Recursive Formulation of the Model

Here, we present the sequential formulation of the problem and show the one-to-one equivalence to our recursive formulation.

Environment. The environment—i.e., preferences, technology, shocks, and frictions—is the same as in Section 2. To focus on the novel component of the paper and to simplify the notation, we assume that a recursive representation holds across employment and unemployment spells.

An unemployed worker's choice of submarket $(z; w)$ is associated with a job-finding rate $f(z; w)$, which induces a stochastic job offer arrival time t^u. The value of an unemployed worker with productivity z is

$$U(z) = \max_{\{w_t\}_{t=0}^{t^u}} \mathbb{E}_0 \left[\int_0^{t^u} e^{-rt} \tilde{B} e^{z_t} \, dt + e^{-rt^u} H(z_{t_u}; w_{t^u}; \vec{t}^m(w_{t^u}, z_{t^u})) \right]. \quad \text{(I.1)}$$

That is, an unemployed worker searches for a job in submarket $(z_t; w_t)$ at time $t \leq t^u$ until becoming employed at wage w_{t^u} and receiving the value of employment $H(z_{t_u}; w_{t^u}; \vec{t}^m(w_{t^u}, z_{t^u}))$ at time t^u.

Given the (fixed) wage w and current productivity z, a matched worker chooses when to quit, which induces a stopping time t^h. Based on the same (w, z) pair, a matched firm chooses when to lay off the worker, which induces a stopping time t^j. Given the choices by workers and firms in addition to the exogenous stopping time t^d, the actual match duration is the minimum stopping time in the vector $\vec{t}^m = (t^h, t^j, t^d)$, denoted $t^m = \min\{t^h, t^j, t^d\}$. Given a vector of stopping times $\vec{t}^m$, the value of a worker employed at wage w with productivity z is

$$H(z; w, \vec{t}^m) = \mathbb{E}_0 \left[\int_0^{t^m} e^{-rt} e^{w} \, dt + e^{-rt^m} U(z_{t^m}) \right]. \quad \text{(I.2)}$$

That is, an employed worker consumes a constant wage w until time t^m when she either endogenously or exogenously transitions to unemployment. Similarly, given a vector of stopping times $\vec{t}^m$, the value of a firm matched with a worker with wage w and productivity z is

$$J(z; w, \vec{t}^m) = \mathbb{E}_0 \left[\int_0^{t^m} e^{-rt} \left[e^{z_t} - e^{w} \right] dt \right]. \quad \text{(I.3)}$$

That is, the match produces e^{z_t}, of which e^w is paid to the worker until it gets dissolved at time t^m.

Free Entry. In choosing the number of vacancies to post in each submarket, firms trade off the expected benefit—i.e., the product of the filling rate $q(q(z; w))$ and the value of a filled job $J(z; w, \vec{t}^m(z; w))$—with the flow cost $\tilde{K} e^z$ of posting a vacancy. In each submarket, firms post vacancies up to the point at which the marginal vacancy posting cost exceeds its expected benefits. Thus, free entry requires that

$$\min \left\{ \tilde{K} e^z - q(q(z; w)) J(z; w, \vec{t}^m(z; w)), q(z; w) \right\} = 0, \quad \text{(I.4)}$$

for all $(z; w)$.

__**Equilibrium Definition.**__ We are now ready to define an equilibrium. Let $\mathcal{T}$ be the set of all stopping times for a given match. Given the state $(z; w)$, staying in the match is a *weakly dominant strategy* for the worker if there exists a stopping time $t^{h\star}(z; w) \in \mathcal{T}$ such that $\Pr(t^{h\star}(z; w) > 0) = 1$ and

$$H(z; w, t^{h\star}(z; w), t^j, t^d) \geq H(z; w, t^h, t^j, t^d), \quad \forall t^h, t^j \in \mathcal{T},$$

with strict inequality for some t^j. Similarly, given $(z; w)$, staying in the match is a weakly dominant strategy for the firm if there exists a stopping time $t^{j\star}(z; w) \in \mathcal{T}$ such that $\Pr(t^{j\star}(z; w) > 0) = 1$ and

$$J(z; w, t^h, t^{j\star}(z; w), t^d) \geq J(z; w, t^h, t^j, t^d), \quad \forall t^h, t^j \in \mathcal{T},$$

with strict inequality for some t^h.

Definition I.2. *An equilibrium consists of a set of value functions $\{H(z; w, \vec{t}^m), J(z; w, \vec{t}^m), U(z)\}$, a market tightness function $q(z; w)$, and policy functions $\left\{t^{h*}(z; w), t^{j*}(z; w), w^*(z_t)\right\}$, such that:*

1. *Given $H(z; w, \vec{t}^{m*}(z; w))$, $U(z)$, and $q(z; w)$, the search strategy $\{w^*(z_t)\}_{t=0}^{t^{u*}}$ solves equation (I.1).*

2. *Given $J(z; w, \vec{t}^{m*}(z; w))$, market tightness $q(z; w)$ solves the free-entry condition (I.4).*

3. *Given $U(z)$, $(t^{h*}(z; w), t^{j*}(z; w))$ is a nontrivial Nash equilibrium with stopping times (t^h, t^j) that satisfy*

$$H(z; w, t^{h*}(z; w), t^{j*}(z; w), t^d) \geq H(z; w, t^h, t^{j*}(z; w), t^d), \quad \forall (z; w)$$
$$J(z; w, t^{h*}(z; w), t^{j*}(z; w), t^d) \geq J(z; w, t^{h*}(z; w), t^j, t^d), \quad \forall (z; w)$$

   and $Pr(t^{h*}(z; w) > 0) = 1$ (resp. $Pr(t^{j*}(z; w) > 0) = 1$) whenever staying in the match is a weakly dominant strategy for the worker (resp. the firm) given the state $(z; w)$.

__**Recursive Equilibrium Conditions.**__ Define the recursive equilibrium conditions:

$$ru(z) = \tilde{B}e^z + g\frac{\P u(z)}{\P z} + \frac{s^2}{2}\frac{\P^2 u(z)}{\P z^2} + \max_w f(q(z; w))[h(z; w) - u(z)],\ \forall z \in \mathbb{R} \tag{I.5}$$

$$0 = \min\left\{\tilde{K}e^z - q(q(z; w))j(z; w), q(z; w)\right\},\ \forall (z; w) \in \mathbb{R}^2$$

$$z \in (\mathcal{Z}^j(w))^c \Rightarrow h(z; w) = u(z), \tag{I.6}$$

$$z \in (\mathcal{Z}^h(w))^c \Rightarrow j(z; w) = 0, \tag{I.7}$$

$$0 = \max\{u(z) - h(z; w), \mathcal{A}^h h(z; w) + e^w\},\ \forall z \in \mathcal{Z}^j(w), h(\cdot; w) \in \mathbb{C}^1(\mathcal{Z}^j(w)) \cap \mathbb{C}(\mathbb{R}), \tag{I.8}$$

$$0 = \max\{-j(z;w), \mathcal{A}^j j(z;w) + e^z - e^w\},\ \forall z \in \mathcal{Z}^h(w), j(\cdot;w) \in \mathbb{C}^1(\mathcal{Z}^h(w)) \cap \mathbb{C}(\mathbb{R}), \quad \text{(I.9)}$$

$$\mathcal{Z}^h(w) := int\left\{z \in \mathbb{R} : h(z;w) > u(z) \text{ or } \mathcal{A}^h u(z) + e^w > 0\right\}, \quad \text{(I.10)}$$

$$\mathcal{Z}^j(w) := int\left\{z \in \mathbb{R} : j(z;w) > 0 \text{ or } e^z - e^w > 0\right\}, \quad \text{(I.11)}$$

where we define the characteristic operator for any function $v(z)$ for the firm and the worker as

$$\mathcal{A}^h(v(z)) := -rv + d(u(z) - v(z)) + g\frac{¶v(z)}{¶z} + \frac{s^2}{2}\frac{¶^2 v(z)}{¶z^2}$$
$$\mathcal{A}^j(v(z)) := -rv + d(0 - v(z)) + g\frac{¶v(z)}{¶z} + \frac{s^2}{2}\frac{¶^2 v(z)}{¶z^2}.$$

Lemma I.1. *The policy functions* $\{t^{h*}, t^{j*}, w^*(z)\}$ *and the value functions* $\{U(z), H(z;w,\tau^m), J(z;w,\tau^m)\}$ *given by* (I.1), (I.2) *and* (I.3) *and the market tightness function* $q(z;w)$ *form a BRE iff.* $\{u(z), h(z;w), j(z;w)\}$ *satisfy equations* (I.5)–(I.11) *and*

$$u(z) = U(z),$$
$$h(z;w) = H(z;w, t^{h*}(z;w), t^{j*}(z;w), t^d),$$
$$j(z;w) = J(z;w, t^{h*}(z;w), t^{j*}(z;w), t^d).$$

To simplify the exposition, we divide the proof into a sequence of steps.

Proposition I.1. *Let* $x := (z;w)$*. If there exist two functions* $h(z;w)$ *and* $j(z;w)$ *satisfying* (I.6), (I.7), (I.8) *and* (I.9) *given the continuation sets* (I.10) *and* (I.11)*, then*

$$t^{h*}(x) = \inf\left\{t \geq 0 : z_t \notin \mathcal{Z}^h(w)\right\},$$
$$t^{j*}(x) = \inf\left\{t \geq 0 : z_t \notin \mathcal{Z}^j(w)\right\}$$

form a nontrivial Nash equilibrium and

$$h(z;w) = H(x, t^{h*}(x), t^{j*}(x), t^d), \quad j(z;w) = J(x, t^{h*}(x), t^{j*}(x), t^d).$$

Moreover, if $(t^{h*}(x), t^{j*}(x))$ *is a nontrivial Nash equilibrium, then* $h(z;w)$ *and* $j(z;w)$ *satisfy* (I.6) *to* (I.9).

Proof. **Variational inequalities as sufficient conditions for Nash Equilibrium.** First, we prove that if $h(z;w)$ and $j(z;w)$ satisfy (I.6) to (I.9), then

$$h(z;w) = H(x, t^{h*}(x), t^{j*}(x), t^{d}) \geq H(x, t^{h}(x), t^{j*}(x), t^{d})$$

for any $t^h \in \mathcal{T}$. The proof of the statement

$$j(z;w) = J(x, t^{h*}(x), t^{j*}(x), t^{d}) \geq J(x, t^{h*}(x), t^{j}(x), t^{d}),$$

for any $t^j \in \mathcal{T}$, follows the same arguments.

Step 1: Here, we show that $h(z;w) \geq H(x, t^h(x), t^{j*}(x), t^d)$. Let t^h be any stopping time (not necessarily the optimal). Without loss of generality, we restrict the attention to $t^h \leq t_{(-¥,a)}$, where $t_{(-¥,a)} = \inf\{t > 0 : z_t \notin (-¥, a)\}$. Intuitively, it is never optimal for the worker to stay in the job at wage w when productivity is sufficiently large. Let $U_k \subset \mathbb{R}$ be an increasing sequence of bounded sets s.t. $\cup_{k=1}^{¥} U_k = \mathbb{R}$. Let $t_k = \inf\{t > 0 : z_t \notin U_k\}$. Since each U_k is bounded, we do not need to assume compact support of the function to apply Proposition A.1. Applying Dynkin's Lemma to the stopping time $t_k^h = t^h \wedge t^{j*} \wedge t^d \wedge t_k$,

$$\mathbb{E}[e^{-r t_k^h} h(x_{t_k^h}) | z_0 = z] = h(z;w) + \mathbb{E}\left[\int_0^{t_k^h} \mathcal{A}^h h(z_t;w)\, dt | z_0 = z\right].$$

Using condition (I.8), we have that $h(z;w) \geq u(z)$ for all $z \in \mathcal{Z}^j(w)$. Moreover, $h(z;w) = u(z)$ for all $z \in (\mathcal{Z}^j(w))^c$. Therefore, $h(z;w) \geq u(z)$ for all $z \in \mathbb{R}$. Thus, we have that $\mathbb{E}[e^{-r t_k^h} h(z_{t_k^h};w)|z_0 = z] \geq \mathbb{E}[e^{-r t_k^h} u(z_{t_k^h})|z_0 = z]$. Thus,

$$\mathbb{E}[e^{-r t_k^h} u(z_{t_k^h})|z_0 = z] - \mathbb{E}\left[\int_0^{t_k^h} \mathcal{A}^h h(z_t;w)\, dt | z_0 = z\right] \leq h(z;w).$$

From condition (I.8) , we have $\mathcal{A}^h h(z;w) + e^w \leq 0$ for all z. Thus,

$$\mathbb{E}\left[\int_0^{t_k^h} e^{-rt} e^w\, dt | z_0 = z\right] \leq -\mathbb{E}\left[\int_0^{t_k^h} \mathcal{A}^h h(z;w)\, dt | z_0 = z\right].$$

Using this result

$$\mathbb{E}\left[e^{-r t_k^h} u(z_{t_k^h}) + \int_0^{t_k^h} e^{-rt} e^w\, dt | z_0 = z\right] \leq h(z;w)$$

Now, we take the limit $k \to ¥$. It is easy to see that $\int_0^{t^h \wedge t^{j*} \wedge t^d \wedge t_k} e^{-rt+w}\, dt \leq \frac{1}{r} e^w$ a.e., so using the dominated convergence theorem $\lim_{k\to ¥} \mathbb{E}\left[\int_0^{t^h\wedge t^{j*}\wedge t^d\wedge t_k} e^{-rt+w}\,\mathrm{d}t|z_0 = z\right] = \mathbb{E}\left[\int_0^{t^h\wedge t^{j*}\wedge t^d} e^{-rt+w}\,\mathrm{d}t|z_0 = z\right]$.

As we show below, $u(z)$ µ e^z and since $e^{z_t} \leq e^a$ for all $t \leq t^h \leq t_{(-¥,a)}$, we have that $0 \leq e^{-rt}u(z_t) \leq e^a$. Applying the monotone convergence theorem, we have that

$$\lim_{k\to ¥} \mathbb{E}\left[e^{-r(t^h\wedge t^{j*}\wedge t^d\wedge t_k)}u(z_{t^h\wedge t^{j*}\wedge t^d\wedge t_k})|z_0 = z\right] = \mathbb{E}\left[e^{-r(t^h\wedge t^{j*}\wedge t^d)}u(z_{t^h\wedge t^{j*}\wedge t^d})|z_0 = z\right].$$

Therefore, taking the limit $k \to ¥$, we finally obtain

$$h(z;w) \geq H(x, t^h(x), t^{j*}(x), t^d).$$

Step 2: Now, we show that $h(z;w) = H(x, t^{h*}(x), t^{j*}(x), t^d)$. Applying Proposition A.1 to the stopping time $t_k^{h*} = t^{h*} \wedge t^{j*} \wedge t_k \wedge t^d$ we obtain

$$\mathbb{E}[e^{-rt_k^{h*}}h(z_{t_k^{h*}};w)|z_0 = z] = h(z;w) + \mathbb{E}\left[\int_0^{t_k^{h*}} \mathcal{A}^h h(z_t;w)\,\mathrm{d}t|z_0 = z\right].$$

For all $t < t_k^{h*}$, we have that $u(z) < h(z;w)$. Therefore, by (I.8), $\mathcal{A}^h h(z;w) + e^w = 0$ for all z. Thus,

$$\mathbb{E}\left[e^{-rt_k^{h*}}h(z_{t_k^{h*}};w) + \int_0^{t_k^{h*}} e^{-rt}e^w\,\mathrm{d}t|z_0 = z\right] = h(z;w).$$

Taking the limit $k \to ¥$ and following similar arguments as above, we obtain

$$\mathbb{E}\left[e^{-r(t^{h*}\wedge t^{j*}\wedge t^d)}h(z_{t^{h*}\wedge t^{j*}\wedge t^d};w) + \int_0^{t^{h*}\wedge t^{j*}\wedge t^d} e^{-rt}e^w\,\mathrm{d}t|z_0 = z\right] = h(z;w).$$

which, given Proposition A.2, is equivalent to

$$\mathbb{E}\left[e^{-(r+d)(t^{h*}\wedge t^{j*})}h(z_{t^{h*}\wedge t^{j*}};w) + \int_0^{t^{h*}\wedge t^{j*}} e^{-(r+d)t}(du(z_t) + e^w)\,\mathrm{d}t|z_0 = z\right] = h(z;w).$$

Since $z_{t^{h*}\wedge t^{j*}} \in ¶(\mathcal{Z}^h(w^*(z)) \cap \mathcal{Z}^j(w^*(z)))$ and $h(\cdot;w)$ is continuous, we have that

$$\mathbb{E}\left[e^{-(r+d)(t^{h*}\wedge t^{j*})}u(z_{t^{h*}\wedge t^{j*}};w) + \int_0^{t^{h*}\wedge t^{j*}} e^{-(r+d)t}(du(z_t) + e^w)\,\mathrm{d}t|z_0 = z\right] = h(z;w).$$

and $h(z;w) = H(x, t^{h*}(x), t^{j*}(x), t^d)$.

Variational inequalities as sufficient conditions for Nontrivial Nash Equilibrium. This part of the proof is constructive. Define $\mathbb{WD}^h(w) = \{z \in \mathbb{R} : 0 < e^w + \mathcal{A}^h u(z)\}$ and $t^\star(z;w) = \inf\{t \geq 0 : z_t \notin \mathbb{WD}^h(w),\ z_0 = z\}$. Now, we check that this set is where continuation is a weakly dominating strategy. Applying Dynkin's Lemma (and using similar arguments as before), for any stopping time t we obtain

$$\mathbb{E}\left[e^{-rt} u(z_t)|z_0 = z\right] = u(z) + \mathbb{E}\left[\int_0^{t(z;w)} \mathcal{A}^h u(z_t)\, \mathrm{d}t | z_0 = z\right].$$

Using $t(z;w) = \min\left\{t^\star(z;w), t^j(z;w), t^d(z;w)\right\}$,

$$\begin{aligned} u(z) &= \mathbb{E}\left[e^{-rt(z;w)} u(z_{t(z;w)})|z_0 = z\right] - \mathbb{E}\left[\int_0^{t(z;w)} \mathcal{A}^h u(z_t)\, \mathrm{d}t | z_0 = z\right] \\ &\leq \mathbb{E}\left[e^{-rt(z;w)} u(z_{t(z;w)})|z_0 = z\right] + \mathbb{E}\left[\int_0^{t(z;w)} e^{-rt+w}\, \mathrm{d}t | z_0 = z\right]. \end{aligned}$$

with strict inequality, if $Pr(t(z;w) > 0) = 1$. Thus, staying in the match weakly dominates dissolving the match.

Variational inequalities as necessary conditions. Now, we prove that if $t^{h*}(x)$ and $t^{j*}(x)$ is a nontrivial Nash equilibrium, then $h(z;w),\ j(z;w)$ satisfy (I.6) to (I.11). Notice that under the assumption that t^j and t^h are characterized by continuation sets, we can focus on these sets to prove conditions (I.6) to (I.11). By definition, we have that

$$h(z;w) = \max_{t^h} \mathbb{E}\left[\int_0^{t^h \wedge t^{j*} \wedge t^d} e^{-rt+w}\, \mathrm{d}t + e^{-r(t^h \wedge t^{j*} \wedge t^d)} u(z_{t^h \wedge t^{j*} \wedge t^d}; w)\, \mathrm{d}t | z_0 = z\right]. \tag{I.12}$$

- Condition (I.6): If $z \in (\mathcal{Z}^j(w))^c$, then $t^{j*}(x) = 0$ and $Pr[\min\{t^{h*}(x), t^{j*}(x), t^d(x)\} \leq t^{j*}(x)] = 1$, and $h(z;w) = u(z)$. A similar argument holds for the firm.

- Condition (I.8): Observe that this condition is the best response of the worker, given that the firm continues. See Øksendal (2007) and Brekke and Øksendal (1990) for a discussion of the necessity of the smooth pasting condition.

- Condition (I.10): For this part, we will assume that u is $\mathbb{C}^2$ and the set of productivities for which $e^w + \mathcal{A}^h u(z) = 0$ has measure zero (we show this property in Lemma 1). To show this, we need to characterize the continuation set in the Nash equilibrium that survives the iterated elimination of weakly dominated strategies. First, from the problem (I.12), if

$\Pr(t^{j*}(x) > 0) = 1$, then $\Pr(t^{h*}(x) > 0) = 1$ iff.

$$z \in int\left\{z \in \mathbb{R} : h(z;w) > u(z)\right\}.$$

Next, we proceed by contradiction. Assume that in the state $(z;w)$ staying in the match weakly dominates leaving and

$$0 < e^{w} + \mathcal{A}^{h}u(z). \tag{I.13}$$

Notice that here we are ignoring the case $e^{w} + \mathcal{A}^{h}u(z) = 0$ since it has measure 0. If $u(z) \in \mathbb{C}^2$, define an open set U, containing the chosen $(z;w)$, where $e^{w} + \mathcal{A}^{h}u(z) > 0$ and take any stopping time t_U. Then, applying Dynkin's Lemma (and using similar arguments as in Step 1), we obtain

$$\mathbb{E}\left[e^{-r\,t_U}u(z_{t_U})|z_0 = z\right] = u(z) + \mathbb{E}\left[\int_0^{t_U} \mathcal{A}u(z_t)\,\mathrm{d}t|z_0 = z\right].$$

Using the inequality in (I.13),

$$\begin{aligned} u(z) &= \mathbb{E}\left[e^{-r\,t_U}u(z_t)|z_0 = z\right] - \mathbb{E}\left[\int_0^{t_U} \mathcal{A}u(z_t)\,\mathrm{d}t|z_0 = z\right] \\ &> \mathbb{E}\left[e^{-r\,t_U}u(z_{t_U})|z_0 = z\right] + \mathbb{E}\left[\int_0^{t_U} e^{-rt+w}\,\mathrm{d}t|z_0 = z\right]. \end{aligned}$$

Thus, staying in the match is dominated for t^U, arriving at a contradiction.

□

Proposition I.2. *Define*

$$w^{*}(z) = \arg\max_{w} f(q(z;w))(h(z;w) - u(z)).$$

and $t^{u*} = \inf\{t \geq 0 : \mathrm{D}N_t^{f(q(z_t;w^{*}(z_t)))} = 1\}$ *where* $N_t^{f(q(z_t;w^{*}(z_t)))}$ *is a Poisson counter with arrival rate* $f(q(z_t;w^{*}(z_t)))$*. The function* $u(z)$ *satisfies* $u(z) \in \mathbb{C}^2(\mathbb{R})$ *and* (I.5) *iff.*

$$u(z) = \max_{\{w_t\}_{t=0}^{t^u}} \mathbb{E}\left[\int_0^{t^u} e^{-rt}B(z_t)\,\mathrm{d}t + e^{-r\,t^u}h(z_{t^u};w)\right].$$

Proof. The proof is the standard optimality conditions in the HJB (see Øksendal, 2007). □

Lemma I.1. *Assume $u(z)$, $h(z;w)$, $j(z;w)$, $q(z;w)$ satisfy (I.5)—(I.9) given the continuation sets (I.10) and (I.11). Then $\{t^{h*}, t^{j*}, \{w_t^*\}_{t=0}^{t^u}\}$ constructed with*

$$t^{h*}(x) = \inf\left\{t \geq 0 : z_t \notin \mathcal{Z}^h(w)\right\}$$
$$t^{j*}(x) = \inf\left\{t \geq 0 : z_t \notin \mathcal{Z}^j(w)\right\}$$
$$w^*(z) = \arg\max_w f(q(z;w))(h(z;w) - u(z)).$$

is a BRE with

$$h(z;w) = H(x, t^{h*}(x), t^{j*}(x), t^d),$$
$$j(z;w) = J(x, t^{h*}(x), t^{j*}(x), t^d),$$
$$u(z) = U(z).$$

If $\{H(z;w,\vec{t}^m), J(z;w,\vec{t}^m), U(z)\}$, market tightness $q(z;w)$, and policy functions $\{t^{h*}(z;w), t^{j*}(z;w), w^*(z_t)\}$ is a BRE with

$$h(z;w) = H(x, t^{h*}(x), t^{j*}(x), t^d),$$
$$j(z;w) = J(x, t^{h*}(x), t^{j*}(x), t^d),$$
$$u(z) = U(z).$$

then $u(z)$, $h(z;w)$, $j(z;w)$, $q(z;w)$ satisfy (I.5)—(I.9) given the continuation sets (I.10) and (I.11).

Proof. The proof is a combination of Propositions I.1 and I.2. □

I.3 Derivation of Recursive Equilibrium from Discrete Time

This section presents the discrete-time counterpart of the model described in Section 2 in time intervals Δt; i.e., $t = 0, \Delta t, 2\Delta t, \ldots$. We use the equilibrium concept of a Markov perfect equilibrium. We follow Stokey (2008) to construct a discrete-time approximation of the worker's idiosyncratic productivity:

$$z'_\Delta = \begin{cases} z_\Delta + \sigma\sqrt{\Delta t} & \text{with probability } \frac{1+\frac{g}{\sigma}\sqrt{\Delta t}}{2} \\ z_\Delta - \sigma\sqrt{\Delta t} & \text{with probability } \frac{1-\frac{g}{\sigma}\sqrt{\Delta t}}{2} \end{cases}. \tag{I.14}$$

Observe that the process is locally consistent with $dz_t = g\,dt + s\,d\mathcal{W}_t^z$ (see Kushner and Dupuis, 2001).

Given the discrete-time nature of the problem, the timing within the period is as follows. At the beginning of the period t, workers' idiosyncratic productivity shocks are realized. Then, the labor market opens: exogenous and endogenous separations and new matches are realized. Finally, production takes place, and agents receive their payoffs. We define all the value functions after the realization of the idiosyncratic shocks and before the labor market opens.

Value functions. The value of an unemployed worker $u_{\mathrm{D}t}(z)$ is

$$u_{\mathrm{D}t}(z) = \max_{w} \Big\{ e^{-f(q_{\mathrm{D}t}(z;w))\mathrm{D}t} \Big[\tilde{B}e^z \mathrm{D}t + e^{-r\mathrm{D}t}\mathbb{E}_{z'} \left[u_{\mathrm{D}t}(z') \middle| z \right] \Big] \\ + \Big[1 - e^{-f(q_{\mathrm{D}t}(z;w))\mathrm{D}t} \Big] \Big[e^w \mathrm{D}t + e^{-r\mathrm{D}t}\mathbb{E}_{z'} \left[h_{\mathrm{D}t}(z';w) \middle| z \right] \Big] \Big\}. \tag{I.15}$$

Here, $\tilde{B}e^z\mathrm{D}t$ is the flow income from unemployment, $1 - e^{-\mathrm{D}tf(q(z;w))}$ is the probability of finding a job with flow income $e^w\mathrm{D}t$ and continuation value $\mathbb{E}_{z'}\left[h_{\mathrm{D}t}(z';w)\middle| z\right]$, $e^{-r\mathrm{D}t}$ is the discount factor and z' is a random variable with law of motion (I.14). We use the notation $w^*_{\mathrm{D}t}(z)$ to denote the optimal search policy of an unemployed worker.

The vacancy cost for a period $\mathrm{D}t$ is $\tilde{K}e^z\mathrm{D}t$ and the expected return is $[1 - e^{-q(q(z;w))\mathrm{D}t}]j_{\mathrm{D}t}(z;w)$. The free entry condition is given by

$$\min\{\tilde{K}e^z\mathrm{D}t - \Big[1 - e^{-q(q_{\mathrm{D}t}(z;w))\mathrm{D}t}\Big] j_{\mathrm{D}t}(z;w), q_{\mathrm{D}t}(z;w)\} = 0. \tag{I.16}$$

Thus, if the cost of posting vacancies is larger than the expected value of finding a worker—i.e., $\tilde{K}e^z\mathrm{D}t - \Big[1 - e^{-q(q_{\mathrm{D}t}(z;w))\mathrm{D}t}\Big] j_{\mathrm{D}t}(z;w) > 0$—then $q_{\mathrm{D}t}(z;w) = 0$. Similarly, if the submarket $(z;w)$ is open, then the free entry condition holds with equality $\tilde{K}e^z\mathrm{D}t = \Big[1 - e^{-q(q_{\mathrm{D}t}(z;w))\mathrm{D}t}\Big] j_{\mathrm{D}t}(z;w)$.

Let $\mathbb{I}^h_{\mathrm{D}t}(z;w) \in \{0,1\}$ be an indicator variable equal to one when the worker chooses to continue in the match and 0 if the worker chooses to quit. Similarly, based on the $(z;w)$ pair, a matched firm chooses to lay a worker off when $\mathbb{I}^j_{\mathrm{D}t}(z;w) = 0$ and to continue in the match when $\mathbb{I}^j(z;w) = 1$.

Given firm policy $\mathbb{I}^j(z;w)$, the value function of a worker with productivity z employed at wage

w is

$$h^{\mathbb{I}^j}_{\mathsf{D}t}(z;w) = \begin{cases} \max\left\{e^{-d\mathsf{D}t}\left[e^{w}\mathsf{D}t + e^{-r\mathsf{D}t}\mathbb{E}_{z'}\left[h^{\mathbb{I}^j}_{\mathsf{D}t}(z';w)|z\right]\right] + (1-e^{-d\mathsf{D}t})u_{\mathsf{D}t}(z), u_{\mathsf{D}t}(z)\right\}, & \mathbb{I}^j_{\mathsf{D}t}(z;w) = 1 \\ u_{\mathsf{D}t}(z), & \mathbb{I}^j_{\mathsf{D}t}(z;w) = 0. \end{cases} \tag{I.17}$$

If the firm chooses not to lay the worker off, then the employed worker chooses between quitting her job or not while consuming the constant wage w. The notation makes it clear that the fixed point in (I.17) depends on the firm's policy function. We define $h_{\mathsf{D}t}(z;w) := h^{\mathbb{I}^{j*}_{\mathsf{D}t}}_{\mathsf{D}t}(z;w)$, where $\mathbb{I}^{j*}_{\mathsf{D}t}$ denotes the firm's *optimal* policy function.

Similarly, given a worker's policy $\mathbb{I}^h_{\mathsf{D}t}(z;w)$, the value of a firm matched with a worker with wage w and productivity z is

$$j^{\mathbb{I}^h_{\mathsf{D}t}}_{\mathsf{D}t}(z;w) = \begin{cases} \max\left\{e^{-d\mathsf{D}t}\left[\left(e^{z}-e^{w}\right)\mathsf{D}t + e^{-r\mathsf{D}t}\mathbb{E}_{z'}\left[j^{\mathbb{I}^h_{\mathsf{D}t}}_{\mathsf{D}t}(z';w)|z\right]\right], 0\right\} & \text{if } \mathbb{I}^h_{\mathsf{D}t}(z;w) = 1 \\ 0 & \text{if } \mathbb{I}^h_{\mathsf{D}t}(z;w) = 0 \end{cases} \tag{I.18}$$

We define $j_{\mathsf{D}t}(z;w) := h^{\mathbb{I}^{h*}_{\mathsf{D}t}}_{\mathsf{D}t}(z;w)$, where $\mathbb{I}^{h*}_{\mathsf{D}t}$ denotes the worker's *optimal* policy function.

We are ready to define a Markov Perfect equilibrium with the additional refinement that continuation in the match needs to be a *weakly* dominant strategy.

Definition I.3. *A Markov Perfect equilibrium is a set* $\{h_{\mathsf{D}t}(z;w), j_{\mathsf{D}t}(z;w), u_{\mathsf{D}t}(z), q_{\mathsf{D}t}(z;w)\}$ *of value functions and market tightness together with policy functions* $\{\mathbb{I}^{h*}_{\mathsf{D}t}, \mathbb{I}^{j*}_{\mathsf{D}t}, w^*_{\mathsf{D}t}(z)\}$ *such that:*

(i) Given $h_{\mathsf{D}t}(z;w)$ *and* $q_{\mathsf{D}t}(z;w)$, $u_{\mathsf{D}t}(z)$ *satisfies the value function* (I.15) *with optimal policy function* $w^*_{\mathsf{D}t}(z)$.

(ii) Given $j_{\mathsf{D}t}(z;w)$, *the market tightness* $q_{\mathsf{D}t}(z;w)$ *satisfies* (I.16).

(iii) Given $u(z)$ *and* $\mathbb{I}^{j*}_{\mathsf{D}t}$, $h_{\mathsf{D}t}(z;w) = h^{\mathbb{I}^{j*}_{\mathsf{D}t}}_{\mathsf{D}t}(z;w)$ *satisfies the value function* (I.17) *with optimal policy* $\mathbb{I}^{h*}_{\mathsf{D}t}(z;w)$. *Moreover, if for any function* $\mathbb{I}^j_{\mathsf{D}t}$, *the value function in* (I.17) *given by* $h^{\mathbb{I}^j}_{\mathsf{D}t}(z;w)$ *satisfies*

$$e^{-d\mathsf{D}t}\left[e^{w}\mathsf{D}t + e^{-r\mathsf{D}t}\mathbb{E}_{z'}\left[h^{\mathbb{I}^j}_{\mathsf{D}t}(z';w)|z\right]\right] + (1-e^{-d\mathsf{D}t})u_{\mathsf{D}t}(z) \geq u_{\mathsf{D}t}(z)$$

with strict inequality for some $\mathbb{I}^j_{\mathsf{D}t}$, *then* $\mathbb{I}^{h*}_{\mathsf{D}t}(z;w) = 1$.

(iv) Given $\mathbb{I}^{h*}_{\mathsf{D}t}$, $j_{\mathsf{D}t}(z;w) = j^{\mathbb{I}^{h*}_{\mathsf{D}t}}_{\mathsf{D}t}(z;w)$ *satisfies the value function* (I.18) *with optimal policy* $\mathbb{I}^{j*}_{\mathsf{D}t}(z;w)$.

Moreover, if for any function $\mathbb{I}^h_{\mathrm{D}t}$, the value function in (I.17) given by $j^{\mathbb{I}^h}_{\mathrm{D}t}(z;w)$ satisfies

$$e^{-d\mathrm{D}t}\left[(e^z - e^w)\,\mathrm{D}t + e^{-r\mathrm{D}t}\mathbb{E}_{z'}\left[j^{\mathbb{I}^h_{\mathrm{D}t}}_{\mathrm{D}t}(z';w)|z\right]\right] \geq 0$$

with strict inequality for some $\mathbb{I}^h_{\mathrm{D}t}$, then $\mathbb{I}^{j*}_{\mathrm{D}t}(z;w) = 1$.

A comparison with the main text's recursive equilibrium is helpful. First, in the main text, we use the *optimal* continuation set of each agent to define the equilibrium's best response. This is the reason why the value functions were not indexed by the continuation set of the other agent. Second, unmatched workers and firms internalize the outcome of the nontrivial Nash Equilibrium through $h_{\mathrm{D}t}(z;w)$ and $j_{\mathrm{D}t}(z;w)$, respectively. Third, the Nash equilibrium part of the definition imposes that the worker's optimal quit strategy is the best response to the firm's layoff policy and vice versa. Fourth, the refinement based on weakly dominating continuation strategies is applied in two steps. In the first step, we solve the decision problem of an agent for a given continuation policy of the other agent. In the second step, we verify that continuing in the match weakly dominates leaving it for all continuation policies of the other agent—not necessarily the optimal one.

We now proceed to derive the equilibrium conditions when $\mathrm{D}t \downarrow 0$. Define the following limits

$$u(z) = \lim_{\mathrm{D}t\downarrow 0} u_{\mathrm{D}t}(z), \quad h(z;w) = \lim_{\mathrm{D}t\downarrow 0} h_{\mathrm{D}t}(z;w),$$
$$j(z;w) = \lim_{\mathrm{D}t\downarrow 0} j_{\mathrm{D}t}(z;w),\ q(z;w) = \lim_{\mathrm{D}t\downarrow 0} q_{\mathrm{D}t}(z;w).$$

Below, we use the fact that for any function $u_{\mathrm{D}t}(z)$ the following two properties hold:

$$\lim_{\mathrm{D}t\downarrow 0} \frac{\mathbb{E}_{z'}\left[u_{\mathrm{D}t}(z')|z\right] - u(z)}{\mathrm{D}t} = \mathcal{A}^z u,$$

where $\mathcal{A}^z$ is the characteristic operator of $\mathrm{d}z_t = g\,\mathrm{d}t + s\,\mathrm{d}\mathcal{W}^z_t$, and

$$\lim_{\mathrm{D}t\downarrow 0} \mathbb{E}_{z'}[u_{\mathrm{D}t}(z')|z] = u(z).$$

Similar properties apply to $h_{\mathrm{D}t}(z;w)$ and $j_{\mathrm{D}t}(z;w)$. For details regarding the convergence of the limit when $\mathrm{D}t \downarrow 0$, see Chapters 9 and 10 of Kushner and Dupuis (2001).

Unemployed worker's HJB equation. Using the fact that $e^{-r\mathrm{D}t} = 1 - r\mathrm{D}t + o(\mathrm{D}t^2)$ and $e^{-f(q(z;w))\mathrm{D}t} = 1 - f(q(z;w))\mathrm{D}t + o(\mathrm{D}t^2)$, from (I.15) we have that $0 = \tilde{B}e^z + \frac{\mathbb{E}_{z'}[u_{\mathrm{D}t}(z')|z] - u_{\mathrm{D}t}(z)}{\mathrm{D}t} - r\mathbb{E}_{z'}\left[u_{\mathrm{D}t}(z')|\,z\right] + \max_w f(q_{\mathrm{D}t}(z;w))\mathbb{E}_{z'}\left[(h_{\mathrm{D}t}(z';w) - u_{\mathrm{D}t}(z'))|\,z\right] + o(\mathrm{D}t)$. Using the fact that $\lim_{\mathrm{D}t\downarrow 0} \frac{\mathbb{E}_{z'}[u_{\mathrm{D}t}(z')|z] - u_{\mathrm{D}t}(z)}{\mathrm{D}t} =$

$g\frac{¶u(z)}{¶z} + \frac{s^2}{2}\frac{¶^2u(z)}{¶z^2}$, $\lim_{Dt\downarrow 0}\mathbb{E}_{z'}\left[u_{Dt}(z')\right|z] = u(z)$, and $\lim_{Dt\downarrow 0}\mathbb{E}_{z'}\left[h_{Dt}(z';w)\right|z] = h(z;w)$, we have that $ru(z) = \tilde{B}e^z + g\frac{¶u(z)}{¶z} + \frac{s^2}{2}\frac{¶^2u(z)}{¶z^2} + \max_w f(q(z;w))\left(h(z;w) - u(z)\right)$.

Free entry condition. For free entry in (I.16), notice that $[1 - e^{-q(q_{Dt}(z;w))Dt}] = q(q_{Dt}(z;w))Dt + o(Dt^2)$. Thus, taking the limit, we obtain $\min\left\{\tilde{K}e^z - q(q(z;w))j(z;w), q(z;w)\right\} = 0$.

Nontrivial Nash Equilibrium. First, assume that $\mathbb{I}^{j*}_{Dt}(z;w) = 0$. Then, $h^{\mathbb{I}^{j*}_{Dt}}_{Dt}(z;w) = h_{Dt}(z;w) = u_{Dt}(z)$. Taking the limit, $\mathbb{I}^{j*}(z;w) = 0$, then $h(z;w) = u(z)$.

If $\mathbb{I}^{j*}_{Dt}(z;w) = 1$ and $I^{h*}_{Dt}(z;w) = 1$, then

$$h_{Dt}(z;w) = e^{-dDt}\left[e^w Dt + e^{-rDt}\mathbb{E}_{z'}\left[h_{Dt}(z';w)|z\right]\right] + (1 - e^{-dDt})u_{Dt}(z)$$

and $h_{Dt}(z;w) \geq u_{Dt}(z)$. Or equivalently,

$$0 = e^w + \frac{\mathbb{E}_{z'}\left[h_{Dt}(z';w)\right|z] - h_{Dt}(z;w)}{Dt} - r\mathbb{E}_{z'}\left[h_{Dt}(z';w)\right|z] + d\left[(u_{Dt}(z) - h_{Dt}(z;w))\right] + o(Dt)$$

and $h_{Dt}(z;w) \geq u_{Dt}(z)$. Taking the limit, if $\mathbb{I}^{j*}(z;w) = 1$ and $\mathbb{I}^{h*}(z;w) = 1$, then

$$\begin{aligned} rh(z;w) &= e^w + g\frac{¶h(z;w)}{¶z} + \frac{s^2}{2}\frac{¶^2h(z;w)}{¶z^2} + d\left(u(z) - h(z;w)\right), \\ h(z;w) &\geq u(z). \end{aligned}$$

If $\mathbb{I}^{j*}_{Dt}(z;w) = 1$ and $\mathbb{I}^{h*}_{Dt}(z;w) = 0$, then

$$h_{Dt}(z;w) \geq e^{-dDt}\left[e^w Dt + e^{-rDt}\mathbb{E}_{z'}\left[h_{Dt}(z';w)|z\right]\right] + (1 - e^{-dDt})u_{Dt}(z)$$

and

$$h_{Dt}(z;w) = u_{Dt}(z).$$

In the limit, $\mathbb{I}^{j*}(z;w) = 1$ and $\mathbb{I}^{h*}(z;w) = 0$,

$$\begin{aligned} rh(z;w) &\geq e^w + g\frac{¶h(z;w)}{¶z} + \frac{s^2}{2}\frac{¶^2h(z;w)}{¶z^2} + d\left(u(z) - h(z;w)\right), \\ h(z;w) &= u(z). \end{aligned}$$

Therefore, we can summarize the worker's optimality condition as

$$rh(z;w) = \max\left\{e^w + g\frac{¶h(z;w)}{¶z} + \frac{s^2}{2}\frac{¶^2h(z;w)}{¶z^2} + d\left(u(z) - h(z;w)\right), ru(z)\right\} \text{ if } \mathbb{I}^{j*}(z;w) = 1.$$

Applying the same argument to the firm's problem, we have that

$$rj(z;w) = \max\left\{e^z - e^w + g\frac{\P j(z;w)}{\P z} + \frac{s^2}{2}\frac{\P^2 j(z;w)}{\P z^2} + d\left(-j(z;w)\right), 0\right\} \text{ if } \mathbb{I}^{h*}(z;w) = 1$$

Finally, we characterize agents' continuation sets. We show that the worker's continuation set is

$$\mathcal{Z}^h_{\mathrm{D}t}(w) = \{z : h_{\mathrm{D}t}(z;w) > u_{\mathrm{D}t}(z) \text{ or } e^w \mathrm{D}t + e^{-r\mathrm{D}t}\mathbb{E}_{z'}\left[u_{\mathrm{D}t}(z') - u_{\mathrm{D}t}(z)|z\right] > 0\}.$$

Clearly, the worker will continue in the match if $h_{\mathrm{D}t}(z;w) > u_{\mathrm{D}t}(z)$. We now derive the equilibrium condition for continuation to be a weakly dominating strategy at $(z;w)$. Let us start from the definition of a weakly dominating strategy: Continuing in the match weakly dominates separating when the state is $(z;w)$ if, for all firm's policies $\mathbb{I}^j_{\mathrm{D}t}$, we have that

$$e^{-d\mathrm{D}t}\left[e^w\mathrm{D}t + e^{-r\mathrm{D}t}\mathbb{E}_{z'}\left[h^{\mathbb{I}^j}_{\mathrm{D}t}(z';w)|z\right]\right] + (1 - e^{-d\mathrm{D}t})u_{\mathrm{D}t}(z) \geq u_{\mathrm{D}t}(z),$$

with strict inequality for at least one policy $\mathbb{I}^j_{\mathrm{D}t}$. Operating

$$e^{-d\mathrm{D}t}\left[e^w\mathrm{D}t + e^{-r\mathrm{D}t}\mathbb{E}_{z'}\left[h^{\mathbb{I}^j}_{\mathrm{D}t}(z';w) - u(z)|z\right]\right] \geq 0.$$

Since this holds for all $\mathbb{I}^j_{\mathrm{D}t}(z;w)$ at $(z;w)$, it also holds for the infimum of the firm's policy function. Thus,

$$e^{-d\mathrm{D}t}\left[e^w\mathrm{D}t + e^{-r\mathrm{D}t}\mathbb{E}_{z'}\left[\inf_{\mathbb{I}^j} h^{\mathbb{I}^j}_{\mathrm{D}t}(z';w) - u_{\mathrm{D}t}(z)|z\right]\right] \geq 0$$

Since worker's optimality imposes that $h^{\mathbb{I}^j}_{\mathrm{D}t}(z;w) \geq u_{\mathrm{D}t}(z)$, with equality when $\mathbb{I}^j(z;w) = 0$, we have that $\inf_{\mathbb{I}^j} h^{\mathbb{I}^j}_{\mathrm{D}t}(z;w) = u_{\mathrm{D}t}(z)$ and

$$e^w\mathrm{D}t + e^{-r\mathrm{D}t}\mathbb{E}_{z'}\left[u_{\mathrm{D}t}(z') - u_{\mathrm{D}t}(z)|z\right] \geq 0.$$

Define the productivity set

$$\mathbb{WD}^h_{\mathrm{D}t}(w) = \left\{z : e^w\mathrm{D}t + e^{-r\mathrm{D}t}\mathbb{E}_{z'}\left[u_{\mathrm{D}t}(z') - u_{\mathrm{D}t}(z)|z\right] \geq 0\right\}.$$

Observe that if $\mathbb{I}^j_{\mathrm{D}t}(z;w) = 1$ for a given w and all z, it is easy to check that $h^{\mathbb{I}^j=1}_{\mathrm{D}t}(z;w) > u_{\mathrm{D}t}(z)\ \forall z \in \mathbb{WD}^h_{\mathrm{D}t}(w)$. Thus, the set $\mathbb{WD}^h_{\mathrm{D}t}(w)$ characterizes the productivity levels for which continuation is a weakly dominating strategy for the worker. Taking the limit, we have that

$$\mathcal{Z}^h(w) = \{z : h(z;w) > u(z) \text{ or } e^w - ru(z) + g\frac{¶u(z)}{¶z} + \frac{s^2}{2}\frac{¶^2u(z)}{¶z^2} \geq 0\}.$$

Applying the same argument to the firm's problem, we have that

$$\mathcal{Z}^h(w) = \{z : j(z;w) > 0 \text{ or } e^z - e^w \geq 0\}.$$

II Additional Results for Section 3: Aggregate Shocks in Labor Markets with Inefficient Turnover

II.1 A Monetary Economy with Exogenous Money Supply

We modify four aspects of the baseline model. First, we introduce preferences over real money holdings:

$$\mathbb{E}_0\left[\int_{t=0}^{¥} e^{-rt}\left(C_{it} + m\log\left(\frac{\hat{M}_{it}}{P_t}\right)\right)\mathrm{d}t\right], \tag{II.1}$$

where $\hat{M}_{it}$ denotes the money holdings of worker i, P_t is the relative price of the consumption good in terms of money, and m is a preference weight on real money holdings.

Second, workers face a budget constraint that reflects ownership of firms and access to complete financial markets. Given a history of labor market decisions regarding job search, job acceptance, and job dissolution, $Im_i^t := \{Im_{it'}\}_{t'=0}^{t}$, a worker's private income is $Y_t(Im_i^t)$, which equals the nominal value of the wage while employed and the nominal value of home production while unemployed. In addition, each worker receives transfers of T_{it} from the government and fully diversified claims on firms' profits. On the spending side, a worker pays for consumption expenditures P_tC_{it} and the opportunity cost of holding money $i_t\hat{M}_{it}$ at a given interest rate $i_t \geq 0$. Letting Q_t denote the time-0 Arrow-Debreu price under complete markets, the worker's budget constraint is

$$\mathbb{E}_0\left[\int_{t=0}^{¥} Q_t\left(P_tC_{it} + i_t\hat{M}_{it} - Y_t(Im_i^t) - T_{it}\right)\mathrm{d}t\right] \leq M_{i0}. \tag{II.2}$$

The worker's problem is to choose a consumption stream $\{C_{it}\}_{t=0}^{¥}$, labor market decisions $\{Im_{it}\}_{t=0}^{¥}$, and money holdings $\{\hat{M}_{it}\}_{t=0}^{¥}$ to maximize utility (II.1) subject to the budget constraint (II.2) at time 0.

Third, the economy is subject to shocks to the aggregate money supply M_t. We assume that the log of the aggregate money supply m_t follows a Brownian motion with drift p and volatility z:

$$dm_t = p\,dt + z\,d\mathcal{W}_t^m,$$

where $\mathcal{W}_t^m$ is a Wiener process. Fourth and finally, we assume that the vacancy posting cost $\tilde{K}Z_t$ and the value of home production $\tilde{B}Z_t$ are both denominated in real terms.

Given these modifications, the market-clearing conditions for goods and money, respectively, are

$$\int_0^1 \left(C_{it} + q_{it}\mathbb{1}[E_{it} = u]\tilde{K}Z_{it}\right) di = \int_0^1 \left(Z_{it}\mathbb{1}[E_{it} = h] + \tilde{B}Z_{it}\mathbb{1}[E_{it} = u]\right) di, \tag{II.3}$$

$$\int_0^1 \hat{M}_{it}\, di = M_t, \tag{II.4}$$

where $\mathbb{1}[\cdot]$ is an indicator function that takes a logical expression as its argument. Equation (II.3) states that the sum of real consumption and recruiting expenses must equal the total market and home production of the good. Equation (II.4) states that the total demand of nominal money holdings across workers equals the aggregate money supply.

The following proposition characterizes the worker's problem in this monetary economy.

Proposition II.1. *Let $Q_0 = 1$ be the numéraire and assume $m = r + p - z^2/2$. Then, $P_t = M_t$ and the value of a worker at time 0 is*

$$V_0 = \max_{\{Im_{it}\}_{t=0}^{¥}} \mathbb{E}_0 \left[\int_0^{¥} e^{-rt} \frac{Y(Im_i^t)}{P_t}\, dt\right] + k,$$

where k is a constant independent of the worker's choices, capturing the present discounted value of financial wealth.

Proposition II.1 shows that the price level equals the aggregate money supply and that maximizing (II.1) subject to (II.2) is equivalent to maximizing expected discounted real income. The result relies on three assumptions: (i) complete markets, (ii) worker preferences that are quasi-linear in consumption, and (iii) the log of aggregate money supply following a random walk with drift. The first two assumptions imply a constant marginal value of nominal wealth, which, combined with the last assumption, leads to a constant real interest rate and a one-for-one pass-through of money shocks to inflation.

Proof. Let V_0 be the present discounted value of the optimal plan. The worker's value is given by

$$V_0 = \max_{\{C_{it}, \hat{M}_{it}, Im_{it}\}_{t=0}^{\infty}} \mathbb{E}_0 \left[\int_{t=0}^{\infty} e^{-rt} \left(C_{it} + m \log \left(\frac{\hat{M}_{it}}{P_t} \right) \right) \mathrm{d}t \right],$$

subject to

$$\mathbb{E}_0 \left[\int_{t=0}^{\infty} Q_t \left(P_t C_{it} + i_t \hat{M}_{it} - Y(Im_i^t) - T_{it} \right) \mathrm{d}t \right] \leq M_{i0}. \tag{II.5}$$

The first-order conditions for consumption and money holdings, combined with the definition of the nominal interest rate, are given by

$$e^{-rt} = \mathsf{L}_i Q_t P_t, \tag{II.6}$$

$$m \frac{e^{-rt}}{\hat{M}_{it}} = \mathsf{L}_i Q_t i_t, \tag{II.7}$$

$$\mathbb{E}[\mathrm{d}Q_t] = -i_t Q_t \, \mathrm{d}t. \tag{II.8}$$

Here, L_i is the Lagrange multiplier of (II.5) for each worker. Equation (II.6) shows that $\mathsf{L}_i = \mathsf{L}$ for all i. Taking integrals over (II.7), we can replace $\hat{M}_{ti} = M_t$. With these results, we guess and verify the following equilibrium outcomes

$$\begin{aligned} P_t &= A^p M_t, \\ i_t &= A^i, \\ Q_t &= \frac{A^Q e^{-rt}}{M_t}. \end{aligned} \tag{II.9}$$

given a set of constants A^p, A^i, and A^Q. Using the guess in (II.6) and (II.7)

$$1 = \mathsf{L} A^Q A^p, \tag{II.10}$$

$$m = \mathsf{L} A^Q A^i. \tag{II.11}$$

Equations (II.10) and (II.11) provide the equilibrium values for A^Q and A^p given A^i. Applying Ito's lemma and using the guess over (II.8)

$$\begin{aligned} \mathrm{d}Q_t &= A^Q \, \mathrm{d} \left(\frac{e^{-rt}}{e^{\log(M_t)}} \right), \\ &= -r A^Q \left(\frac{e^{-rt}}{e^{\log(M_t)}} \right) \mathrm{d}t - A^Q \frac{e^{-rt}}{e^{\log(M_t)}} \, \mathrm{d}\log(M_t) + A^Q \frac{e^{-rt}}{2e^{\log(M_t)}} (\mathrm{d}\log(M_t))^2, \end{aligned}$$

$$= -rQ_t\,\mathrm{d}t - pQ_t\,\mathrm{d}t - zQ_t\,\mathrm{d}\mathcal{W}_t^m + \frac{z^2}{2}Q_t\,\mathrm{d}t.$$

Thus, using the guess (II.9) and $\mathbb{E}[\mathrm{d}\mathcal{W}_t^m] = 0$

$$\mathbb{E}[\mathrm{d}Q_t] = -\underbrace{\left(r + p - \frac{z^2}{2}\right)}_{=A^i} Q_t\,\mathrm{d}t.$$

If we take as numéraire $Q_0 = 1$, then we verify the guess with $m = r + p - \frac{z^2}{2}$:

$$\begin{aligned}
A^Q &= M_0,\\
A^i &= r + p - \frac{z^2}{2} = m,\\
\mathsf{L} &= \frac{m}{M_0(r + p - z^2/2)} = \frac{1}{M_0},\\
A^p &= \frac{r + p - z^2/2}{m} = 1.
\end{aligned}$$

Using the budget constraint (II.5)

$$\begin{aligned}
&\mathbb{E}_0\left[\int_0^{¥} Q_t\left(P_tC_{it} + i_t\hat{M}_{it} - Y(Im_i^t) - T_{it}\right)\mathrm{d}t\right] = M_{i0} \iff\\
&\mathbb{E}_0\left[\int_0^{¥} \frac{M_0 e^{-rt}}{M_t}\left(M_tC_{it} + mM_t - Y(Im_i^t) - T_{it}\right)\mathrm{d}t\right] = M_{i0} \iff\\
&M_0\mathbb{E}_0\left[\int_0^{¥} e^{-rt}C_{it}\,\mathrm{d}t\right] = M_{i0} + M_0\mathbb{E}_0\left[\int_0^{¥} e^{-rt}\frac{Y(Im_i^t)}{M_t}\right] + M_0\mathbb{E}_0\left[\int_0^{¥} e^{-rt}\frac{T_{it}}{M_t}\,\mathrm{d}t\right] - \frac{M_0}{r}m \iff\\
\\
&\mathbb{E}_0\left[\int_0^{¥} e^{-rt}C_{it}\,\mathrm{d}t\right] = \mathbb{E}_0\left[\int_0^{¥} e^{-rt}\frac{Y(Im_i^t)}{M_t}\right] + k_i,
\end{aligned}$$

where k_i is a constant independent of the worker's policies. Thus,

$$\begin{aligned}
V_0 &= \max_{\{C_{it},\hat{M}_{it},Im_{it}\}_{t=0}^{¥}} \mathbb{E}_0\left[\int_0^{¥} e^{-rt}\left(C_{it} + m\log\left(\frac{\hat{M}_{it}}{P_t}\right)\right)\mathrm{d}t\right],\\
&= \max_{\{C_{it},Im_{it}\}_{t=0}^{¥}} \mathbb{E}_0\left[\int_0^{¥} e^{-rt}\left(C_{it} + m\log\left(\frac{m}{r + p - z^2/2}\right)\right)\mathrm{d}t\right],\\
&= \max_{\{C_{it},Im_{it}\}_{t=0}^{¥}} \mathbb{E}_0\left[\int_0^{¥} e^{-rt}C_{it}\,\mathrm{d}t\right],\\
&= \max_{\{Im_{it}\}_{t=0}^{¥}} \mathbb{E}_0\left[\int_0^{¥} e^{-rt}\frac{Y(Im_i^t)}{M_t}\right] + k_i.
\end{aligned}$$

□

II.2 A Monetary Economy with a Taylor Rule and Interest Rate Shocks

We now show that our previous environment is isomorphic to an economy in which the monetary authority sets the interest rate by following a Taylor rule. As in Galí (2015), we study a cashless economy in discrete time where $t \in \mathbb{T} = \{0, \Delta, 2\Delta, 3\Delta, \ldots\}$.

Preferences are the same as in Section 2:

$$\mathbb{E}_0 \left[\sum_{t \in \mathbb{T}} e^{-\rho t} C_{it} \Delta \right], \tag{II.12}$$

where $e^{-\rho t}$ denotes the discount factor. Workers face a budget constraint that reflects ownership of firms and access to complete financial markets. Given a history of labor market decisions regarding job search, job acceptance, and job dissolution, $Im_i^t := \{Im_{it'}\}_{t'=0}^{t}$, a worker's private income is $Y_t(Im_i^t)$, which equals the nominal value of the wage while employed and the nominal value of home production while unemployed. In addition, each worker receives transfers of T_{it} from the government and fully diversified claims on firms' profits. On the spending side, a worker pays for consumption expenditures $P_t C_{it}$. Letting Q_t denote the time-0 Arrow-Debreu price under complete markets, the worker's budget constraint is

$$\mathbb{E}_0 \left[\sum_{t \in \mathbb{T}} Q_t \left(P_t C_{it} - Y_t(Im_i^t) - T_{it} \right) \Delta \right] \leq 0. \tag{II.13}$$

The worker's problem is to choose a consumption stream $\{C_{it}\}_{t=0}^{\infty}$ and labor market decisions $\{Im_{it}\}_{t=0}^{\infty}$ to maximize utility (II.12) subject to the budget constraint (II.13) at time 0.

In this microfoundation, the central bank sets the nominal interest rate following a Taylor rule given by

$$i_t = \rho + \bar{\pi} + \phi_\pi (\pi_t - \bar{\pi}) + \iota_t$$

Here, i_t is the nominal interest rate, $\bar{\pi}$ is the inflation target, and ι_t is a compound Poisson process such that with probability $e^{-\lambda\Delta}$ it is equal to zero and with probability $1 - e^{-\lambda\Delta}$ it is equal to $\epsilon_t \sigma_i / \Delta$, where ϵ_t is an i.i.d. random variable with mean zero and standard deviation of 1.

Finally, we assume that the vacancy posting cost $\tilde{K} Z_t$ and the value of home production $\tilde{B} Z_t$ are both denominated in real terms. The market-clearing condition for the goods market is still given by (II.3).

The following proposition characterizes the worker's problem in this monetary economy.

Proposition II.2. *Take the limit* $\mathsf{D} \downarrow 0$*. Then,*

$$\mathrm{d}\log(P_t) = \bar{p}\,\mathrm{d}t - \frac{s_i}{f_p} e_t\,\mathrm{d}\mathcal{N}_t,$$

where $\mathcal{N}_t$ *is a Poisson process with intensity* l*. The value of a worker at time* 0 *is*

$$V_0 = \max_{\{Im_{it}\}_{t=0}^{¥}} \mathbb{E}_0 \left[\int_0^{¥} e^{-rt} \frac{Y(Im_i^t)}{P_t}\,\mathrm{d}t \right] + k_i,$$

where k_i *is a constant independent of the worker's choices, which captures the present discounted value of financial wealth.*

Proof. Let V_0 be the present discounted value of the optimal plan. The worker's value is given by

$$V_0 = \max_{\{C_{it}, Im_{it}\}_{t=0}^{¥}} \mathbb{E}_0 \left[\text{å}_{t \in \mathbb{T}} e^{-rt} C_{it} \mathsf{D} \right],$$

subject to

$$\mathbb{E}_0 \left[\text{å}_{t \in \mathbb{T}} Q_t \left(P_t C_{it} - Y_t(Im_i^t) - T_{it} \right) \mathsf{D} \right] \leq 0. \tag{II.14}$$

The first-order condition for consumption is given by

$$e^{-rt} = \mathsf{L}_i Q_t P_t, \tag{II.15}$$

Here, L_i is the Lagrange multiplier of (II.14) for worker i. Equation (II.15) shows that $\mathsf{L}_i = \mathsf{L}$ for all i. Evaluating (II.15) at periods t and $t + \mathsf{D}$ and taking their ratio, we have

$$e^{-r\mathsf{D}} = \frac{Q_{t+\mathsf{D}} P_{t+\mathsf{D}}}{Q_t P_t} \iff \mathbb{E}_t \left[\frac{P_t}{P_{t+\mathsf{D}}} e^{-r\mathsf{D}} \right] = \mathbb{E}_t \left[\frac{Q_{t+\mathsf{D}}}{Q_t} \right]$$

By definition of the interest rate, the lack of arbitrage opportunities with the nominal bond offered by the monetary authority $\mathbb{E}_t \left[\frac{Q_{t+\mathsf{D}}}{Q_t} \right] = e^{-i_t \mathsf{D}}$, and the Taylor rule, we have the following system of equations:

$$1 = e^{i_t \mathsf{D}} e^{-r\mathsf{D}} \mathbb{E}_t \left[\frac{P_t}{P_{t+\mathsf{D}}} \right] \quad \text{and} \quad i_t = r + \bar{p} + f_p (p_t - \bar{p}) + i_t.$$

Since $P_{t+\mathsf{D}} = P_t e^{p_{t+\mathsf{D}} \mathsf{D}}$, from the first equation we have that $e^{r\mathsf{D}} = e^{i_t \mathsf{D}} \mathbb{E}_t \left[e^{-p_{t+\mathsf{D}} \mathsf{D}} \right]$. Making a first order Taylor approximation when $\mathsf{D} \downarrow 0$, we obtain $r = i_t - \mathbb{E}_t\left[p_{t+\mathsf{D}}\right]$. Replacing in the expression for i_t from the Taylor rule,

$$p_t = \bar{p} - \frac{i_t}{f_p} + \frac{1}{f_p}\mathbb{E}_t\left[p_{t+\mathsf{D}} - \bar{p}\right].$$

Iterating this equation forward, inflation can be expressed as a function of the current and future shocks:

$$p_t - \bar{p} = -\mathbb{E}_t\left[\mathop{\text{å}}_{j=0}^{¥}\left(\frac{1}{f_p}\right)^{j\mathsf{D}}\frac{i_{t+j}}{f_p}\right].$$

Since $\mathbb{E}_t\left[i_{t+j}\right] = 0$, we have that $p_t - \bar{p} = -\frac{i_t}{f_p}$, and therefore

$$\log(P_{t+\mathsf{D}}) = \log(P_t) + p_{t+\mathsf{D}}\mathsf{D} = \log(P_t) + \bar{p}\mathsf{D} - \frac{i_t\mathsf{D}}{f_p} = \log(P_t) + \bar{p}\mathsf{D} - \mathbb{B}_t\frac{s_i}{f_p}e_t,$$

where $\mathbb{B}_t$ is a random variable equal to one with probability $1 - e^{-l\mathsf{D}}$ and zero otherwise. Taking the limit $\mathsf{D} \downarrow 0$, we have a continuous-time compound Poisson process for the aggregate consumer price index

$$\mathrm{dlog}(P_t) = \bar{p}\,\mathrm{d}t - \frac{s_i}{f_p}e_t\,\mathrm{d}\mathcal{N}_t,$$

where N_t is a Poisson process with intensity l. Combining the fact that $e^{-rt} = \mathsf{L}Q_tP_t$ with the worker's budget constraint:

$$\begin{aligned}
\mathbb{E}_0\left[\mathop{\text{å}}_{t\in\mathbb{T}} e^{-rt}C_{it}\mathsf{D}\right] &= \mathsf{L}\mathbb{E}_0\left[\mathop{\text{å}}_{t\in\mathbb{T}} Q_tP_tC_{it}\mathsf{D}\right] \\
&= \mathsf{L}\mathbb{E}_0\left[\mathop{\text{å}}_{t\in\mathbb{T}} Q_t(Y_t(lm_i^t) + T_{it})\mathsf{D}\right] \\
&= \mathsf{L}\mathbb{E}_0\left[\mathop{\text{å}}_{t\in\mathbb{T}} Q_tY_t(lm_i^t)\mathsf{D}\right] + \underbrace{\mathsf{L}\mathbb{E}_0\left[\mathop{\text{å}}_{t\in\mathbb{T}} Q_tT_{it}\mathsf{D}\right]}_{=k_i} \\
&= \mathbb{E}_0\left[\mathop{\text{å}}_{t\in\mathbb{T}} e^{-rt}\frac{Y_t(lm_i^t)}{P_t}\mathsf{D}\right] + k_i.
\end{aligned}$$

Taking the limit when $\mathsf{D} \downarrow 0$, we have the desired result.

□

II.3 Model Extension: Staggered Wage Renegotiations

In this section, we generalize our model to the case of staggered wage renegotiations, which we assume follow a Nash bargaining protocol with worker weight a and to occur at rate $d^r \geq 0$ à la Calvo (1983). The generalized model nests as a special case the economy with fully rigid wages ($d^r \to 0$) presented in the main text and also the polar opposite case with fully flexible wages ($d^r \to$ ¥). By convexifying between these two cases, the generalized model allows for arbitrary frequencies of wage changes in employment that can be matched to the data. The generalized model with staggered wage renegotiations yields several results but our main conclusion is that all of our key insights extend to an environment with $0 < d^r <$ ¥ subject to minor modifications.

In the baseline model, wages are completely rigid within a worker-firm match. We have chosen to present and analyze this simplified model to enhance clarity and provide a first theoretical foundation for analyzing several core aspects of the model. Nevertheless, this assumption is unrealistic and affects our results. To address this limitation, we follow the approach of Gertler and Trigari (2009) and extend the baseline model by incorporating staggered wage renegotiations à la Calvo (1983) in a way consistent with the Hosios (1990) condition. The resulting insight is that all our key findings can be extended to this more general environment. Importantly, almost all the proofs can be extended to the model involving wage adjustments within a match with only minor modifications. Next, we present a summary of how wage renegotiations within a match affect our results.

Environment.

Here, we modify the baseline setup exclusively to permit wage renegotiations within a match. The preferences, technology, and search frictions remain the same as those outlined in the main text. Wage bargaining within a match is modeled by a Poisson process with a rate denoted by $d^r \geq 0$. We assume that these renegotiations entail setting the wage within a worker-firm match according to Nash bargaining, with the worker's weight over the prevailing surplus at the time of renegotiation given by a (i.e., equal to the elasticity of the matching function to satisfy Hosios condition). As $d^r \to 0$, we recover the baseline model with fully rigid wages within a match. As $d^r \to$ ¥, the model transitions to the opposite extreme where wages are flexible and continuously reset.

Recursive Formulation.

Value Functions. The Hamilton-Jacobi-Bellman (HJB) equation of an unemployed worker is still given by

$$ru(z) = \tilde{B}e^{z} + g\frac{\P u(z)}{\P z} + \frac{s^2}{2}\frac{\P^2 u(z)}{\P z^2} + \max_{w} f(q(z;w))[h(z;w) - u(z)].$$

The HJB equation of a worker employed at log wage w with log productivity $z \in \mathcal{Z}^{j}(w)$, for which the firm prefers to continue, is now

$$rh(z;w) = \max\left\{e^{w} + g\frac{\P h(z;w)}{\P z} + \frac{s^2}{2}\frac{\P^2 h(z;w)}{\P z^2} + d^{r}\left[h(z;w^{*}(z)) - h(z;w)\right] - d[h(z;w) - u(z)]\,,\, ru(z)\right\},$$

and for productivities $z \notin \mathcal{Z}^{j}(w)$ (i.e., at productivity for which the firm prefers to dissolve the match), the HJB is given by

$$h(z;w) = u(z) \qquad \forall z \in (\mathcal{Z}^{j*}(w))^{c},$$

with $h(\cdot;w) \in \mathbb{C}^1(\mathcal{Z}^{j*}(w)) \cap \mathbb{C}(\mathbb{R})$. The only difference relative to the baseline model is the term $d^{r}\left[h(z;w^{*}(z)) - h(z;w)\right]$, which captures the expected capital gain from wage renegotiation.

Similarly, the HJB equation of a firm employing a worker at log wage w with log productivity $z \in \mathcal{Z}^{h}(w)$, for which the worker prefers to continue the match, is now given by

$$rj(z;w) = \max\left\{e^{z} - e^{w} + g\frac{\P j(z;w)}{\P z} + \frac{s^2}{2}\frac{\P^2 j(z;w)}{\P z^2} + d^{r}[j(z;w^{*}(z)) - j(z;w)] - dj(z;w)\,,\, 0\right\}.$$

and the HJB equation evaluated at log productivity $z \notin \mathcal{Z}^{h}(w)$, when the worker prefers to dissolve the match, is given by

$$j(z;w) = 0 \qquad \forall z \in (\mathcal{Z}^{h*}(w))^{c},$$

with $j(\cdot;w) \in \mathbb{C}^1(\mathcal{Z}^{h*}(w)) \cap \mathbb{C}(\mathbb{R})$. Again, the only difference is the term $d^{r}[j(z;w^{*}(z)) - j(z;w)]$, which captures the expected capital gain that a firm experiences when renegotiating the wage with the worker.

Continuation Sets. The firm's and worker's optimal continuation sets are

$$\mathcal{Z}^{j*}(w) = int\left\{z \in \mathbb{R} : j(z;w) > 0 \text{ or } e^{z} - e^{w} > 0\right\},$$
$$\mathcal{Z}^{h*}(w) = int\left\{z \in \mathbb{R} : h(z;w) > u(z) \text{ or } 0 < e^{w} - ru(z) + g\frac{\P u(z)}{\P z} + \frac{s^2}{2}\frac{\P^2 u(z)}{\P z^2}\right\}.$$

Observe the following property: Allowing for wage renegotiation within a match does not affect the equilibrium conditions characterizing the continuation sets. We first explain the intuition behind this result and then we formalize it.

To understand the intuition, we follow the same steps as in Online Appendix I.3. We do so by focusing on the continuation set of the worker—a similar logic applies to the continuation set of the firm. Take a discrete time approximation $\mathsf{D}t$ of our model. Let $\mathbb{I}^j_{\mathsf{D}t}(z;w)$ denote the firm's continuation policy, which is equal to 1 if the firm continues the match and zero if the firm lays the worker off. Continuing in the match dominates separating from it whenever

$$h_{\mathsf{D}t}(z;w) = e^w \mathsf{D}t + e^{-r\mathsf{D}t}\mathbb{E}_{(z;w)}[h_{\mathsf{D}t}(z';w')\mathbb{I}^j(z';w') + u_{\mathsf{D}t}(z')(1-\mathbb{I}^j(z';w'))] > u_{\mathsf{D}t}(z), \quad \text{(II.16)}$$

for the firm's stopping policy $\mathbb{I}^j(z';w')$. Here, $\mathbb{E}_{(z;w)}[\cdot]$ denotes the conditional expectation given wage renegotiations, the law of motion of productivity, and exogenous separations. Since the worker is optimally choosing to stay or quit, we have $h^{\mathsf{D}}(z;w) \geq u^{\mathsf{D}}(z)$ for all z. Since condition (II.16) holds for any policy $\mathbb{I}^j(z';w')$, it must also hold for $\mathbb{I}^j(z';w') = 0$ for all $(z';w')$. Therefore,

$$e^w + \frac{e^{-r\mathsf{D}t}\mathbb{E}_{(z;w)}[u_{\mathsf{D}t}(z;w)(z';w')] - u(z)}{\mathsf{D}t} > 0.$$

Taking the limit as $\mathsf{D}t \downarrow 0$, we have

$$e^w - ru(z) + g\frac{¶u(z)}{¶z} + \frac{s^2}{2}\frac{¶^2u(z)}{¶z^2} > 0,$$

which is the same condition derived from our baseline model.

The reason why wage renegotiations do not affect the condition for continuation to be a weakly dominating strategy is that they do not directly affect the worker's or the firm's value conditional on a separation (i.e., $u(z)$ and 0). Formally, define the law motion of the worker's state variables as

$$\mathrm{d}w_t = (\hat{w}^*(z) - \hat{w}_{t^-})\,\mathrm{d}N_t,$$
$$\mathrm{d}z_t = g\,\mathrm{d}t + s\,\mathrm{d}\mathcal{W}^z_t,$$

where $\mathcal{N}_t$ is a Poisson counter with arrival rate d^r and $\hat{w}^*(z)$ is the bargained wage (see below). Given the law of motion of the state and a stopping time t^m, the value functions of the worker and the firm are

$$h(w,z) = \mathbb{E}_0\left[\int_0^{t^m} e^{-rt}e^{w_t}\,\mathrm{d}t + e^{-rt^m}u(z_{t^m})\right],$$
$$j(w,z) = \mathbb{E}_0\left[\int_0^{t^m} e^{-rt}\left[e^{z_t} - e^{w_t}\right] + e^{-rt^m}\times 0\,\mathrm{d}t\right].$$

Let $\mathcal{A}$ be the characteristic operator of (w_t, z_t) adjusted by discounting—i.e., $\mathcal{A}(f) = -r\,f(w,z) + d^r(f(w^*(z),z) - f(w,z)) + g\frac{\P f(\cdot)}{\P z} + \frac{s^2}{2}\frac{\P^2 f(\cdot)}{\P z^2}$. Then, the pairs (w,z) for which the worker and the firm prefer to continue for every stopping time are given by

$$e^w + \mathcal{A}(u) > 0,$$
$$1 - e^w + \mathcal{A}(0) > 0.$$

Wage Renegotiations. Let $w^*(z)$ be the solution to a Nash bargaining problem with worker's bargaining weight given by a, which satisfies the Hosios (1990) condition:

$$w^*(z) = \arg\max_w \left\{(h(z;w) - u(z))^a\, j(z;w)^{1-a}\right\}.$$

We conclude this section with a discussion on the wage-renegotiation protocol. Our aim is to expand our model by incorporating the on-the-job bargaining framework as presented in Gertler and Trigari (2009), while keeping the economic environment unchanged. However, it is important to make some comments. First, the opportunity cost for each agent during bargaining is the corresponding value of separation ($u(z)$ for the worker and 0 for the firm), not the corresponding value at the current wage. This implies that agents commit to separate from the match in the off-equilibrium event that bargaining fails—the conventional assumption adopted in the literature (e.g., Shimer, 2005; Gertler and Trigari, 2009). For a deviation from this assumption, refer to Blanco and Drenik (2022). Second, we adhere to the Hosios (1990) condition. The consequence of this assumption is that the entry wage coincides with the bargained wage. While a deviation from the Hosios (1990) condition would break this equality, the economic mechanisms affecting the entry and bargained wages remain the same.

Equilibrium Characterization.

Using the change of notation adopted in Lemma 1—$\hat{w} := w - z$, $\hat{r} := r - g - s^2/2$ and

$\hat{g} := g + s^2$—we define

$$(\hat{U}, \hat{J}(w-z), \hat{W}(w-z), \hat{q}(w-z)) = \left(\frac{u(z)}{e^z}, \frac{j(z;w)}{e^z}, \frac{h(z;w)-u(z)}{e^z}, q(z;w)\right).$$

Rewriting the HJB equations using this change of notation, we get

$$\hat{r}\hat{U} = \tilde{B} + \max_{\hat{w}} f(\hat{q}(\hat{w}))\hat{W}(\hat{w})$$
$$(\hat{r}+d)\hat{W}(\hat{w}) = \max\left\{e^{\hat{w}} - \hat{r}\hat{U} + d^r\left(\hat{W}(\hat{w}^*) - \hat{W}(\hat{w})\right) - \hat{g}\hat{W}'(\hat{w}) + \frac{s^2}{2}\hat{W}''(\hat{w}), 0\right\}$$
$$(\hat{r}+d)\hat{J}(\hat{w}) = \max\left\{1 - e^{\hat{w}} + d^r\left(\hat{J}(\hat{w}^*) - \hat{J}(\hat{w})\right) - \hat{g}\hat{J}'(\hat{w}) + \frac{s^2}{2}\hat{J}''(\hat{w}), 0\right\}.$$

Here, the terms $d^r\hat{W}(\hat{w}^*)$ and $d^r\hat{J}(\hat{w}^*)$ are constant, since the reset wage $\hat{w}^*$ does not depend on the current value of $\hat{w}$. Therefore, the problem is identical to that in the baseline model with completely rigid wages (i.e., with $d^r = 0$), with the exception of three aspects. First, the effective discount rate for both workers and firms becomes $\hat{r} + d + d^r$ instead of the previous expression $\hat{r} + d$. Second, the worker's flow value is now given by $e^{\hat{w}} + d^r\hat{W}(\hat{w}^*) - \hat{r}\hat{U}$ instead of the previous expression $e^{\hat{w}} - \hat{r}\hat{U}$. Third, the firm's flow value is now given by $1 - e^{\hat{w}} + d^r\hat{J}(\hat{w}^*)$ instead of the previous expression $1 - e^{\hat{w}}$. It is important to note that these expressions simplify to those from the baseline model as $d^r \to 0$.

We now extend our key results to the case of on-the-job wage renegotiations.

Equilibrium Policies. We now analyze how on-the-job wage renegotiation affects equilibrium policies associated with job creation and job destruction. In particular, we extend Proposition 2-Parts 1 to 3, which focus on job creation. We skip Part 4 of that proposition since its extension to a setting with wage renegotiation is trivial.

Proposition II.3. *With wage renegotiations à la Calvo (1983), the BRE has the following properties:*

1. *The joint match surplus satisfies*

$$\hat{S}(\hat{w}) = \frac{1 - \hat{r}\hat{U}}{1 - d^r\mathcal{T}(\hat{w}^*, \hat{r} + d^r)}\mathcal{T}(\hat{w}, \hat{r} + d^r),$$

where

$$\mathcal{T}(\hat{w}, \hat{r} + d^r) := \mathbb{E}_{\hat{w}}\left[\int_0^{t^{m*}} e^{-(\hat{r}+d^r)t}\,\mathrm{d}t\right]$$

is the expected discounted duration of current wages and $1 > \hat{r}\hat{U} > \tilde{B}$*. The following properties hold:*

$$\lim_{d^r \downarrow 0} \hat{S}(\hat{w}) = (1 - \hat{r}\hat{U})\mathcal{T}(\hat{w}, \hat{r}) \text{ and } \lim_{d^r \to ¥} \hat{S}(\hat{w}) = \frac{(1-\hat{r}\hat{U})}{\hat{r}+d} \tag{II.17}$$

i.e., endogenous separations do not affect the surplus (and thus, the entry wage) when $d^r \to ¥$*. s*

2. *The competitive entry wage—i.e.,* $\hat{w}^* = \arg\max_{\hat{w}} f(\hat{q}(\hat{w}))\hat{W}(\hat{w})$*—exist and is unique. Moreover, it solves:*

$$\hat{w}^* = \arg\max_{\hat{w}} \left\{ \hat{W}(\hat{w})^a \hat{J}(\hat{w})^{1-a} \right\} = \arg\max_{\hat{w}} \left\{ h(\hat{w})^a (1-h(\hat{w}))^{1-a} \mathcal{T}(\hat{w}, \hat{r} + d^r) \right\},$$

with optimality condition

$$\underbrace{h'(\hat{w}^*)\left(\frac{a}{h(\hat{w}^*)} - \frac{1-a}{1-h(\hat{w}^*)}\right)}_{\text{Share channel}} = -\underbrace{\frac{\mathcal{T}_{\hat{w}}(\hat{w}^*, \hat{r}+d^r)}{\mathcal{T}(\hat{w}^*, \hat{r}+d^r)}}_{\text{Surplus channel}}.$$

with $h(\hat{w}^*) = a$ *as* $d^r \to ¥$*.*

3. *Given* $h(\hat{w}^*)$ *and* $\mathcal{T}(\hat{w}^*, \hat{r} + d^r)$*, the equilibrium job finding rate* $f(\hat{q}(\hat{w}^*))$ *and the flow opportunity cost of employment* $\hat{r}\hat{U}$ *are given by*

$$f(\hat{q}(\hat{w}^*)) = \left[(1-h(\hat{w}^*)) \frac{1-\hat{r}\hat{U}}{1-d^r\mathcal{T}(\hat{w}^*, \hat{r}+d^r)} \mathcal{T}(\hat{w}, \hat{r}+d^r)/\tilde{K} \right]^{\frac{1-a}{a}},$$
$$\hat{r}\hat{U} = \tilde{B} + \left(\tilde{K}^{a-1} (1-h(\hat{w}))^{1-a} h(\hat{w})^a \frac{1-\hat{r}\hat{U}}{1-d^r\mathcal{T}(\hat{w}^*, \hat{r}+d^r)} \mathcal{T}(\hat{w}, \hat{r}+d^r) \right)^{\frac{1}{a}}.$$

Proof. Now, we prove each equilibrium property.

1. The fact that $\hat{r}U \geq \tilde{B}$ follows from the same argument as before. Combining the sequence and recursive formulations of the value functions, we have

$$\hat{W}(\hat{w}) = \mathbb{E}_{\hat{w}} \left[\int_0^{t^{m*}} e^{-(\hat{r}+d^r)t} (e^{\hat{w}_t} + d^r \hat{W}(\hat{w}^*) - \hat{r}\hat{U}) \, dt \right]$$
$$\hat{J}(\hat{w}) = \mathbb{E}_{\hat{w}} \left[\int_0^{t^{m*}} e^{-(\hat{r}+d^r)t} (1 - e^{\hat{w}_t} + d^r \hat{J}(\hat{w}^*)) \, dt \right]$$

where t^{m*} is the optimal stopping time that determines match duration. Summing up, we have

$$\hat{S}(\hat{w}) = \hat{W}(\hat{w}) + \hat{J}(\hat{w}) = (1 + d^r \hat{S}(\hat{w}^*) - \hat{r}\hat{U})\mathcal{T}(\hat{w}, \hat{r} + d^r).$$

Evaluating the expression for match surplus $\hat{S}(\hat{w})$ at $\hat{w}^*$, we get

$$\hat{S}(\hat{w}^*) = (1 + d^r \hat{S}(\hat{w}^*) - \hat{r}\hat{U})\mathcal{T}(\hat{w}^*, \hat{r} + d^r)$$

and thus

$$\hat{S}(\hat{w}^*) = \frac{1 - \hat{r}\hat{U}}{1 - d^r \mathcal{T}(\hat{w}^*, \hat{r} + d^r)} \mathcal{T}(\hat{w}^*, \hat{r} + d^r).$$

Plugging this back into the above expression, we obtain

$$\hat{S}(\hat{w}) = \frac{1 - \hat{r}\hat{U}}{1 - d^r \mathcal{T}(\hat{w}^*, \hat{r} + d^r)} \mathcal{T}(\hat{w}, \hat{r} + d^r),$$

which is an expression for $\hat{S}(\hat{w})$ that depends only on $\hat{U}$, $\mathcal{T}(\hat{w}^*, \hat{r} + d^r)$ and $\mathcal{T}(\hat{w}, \hat{r} + d^r)$, but not on $\hat{S}(\hat{w}^*)$. Since $\hat{W}(\hat{w}), \hat{J}(\hat{w}) \geq 0$, then $\hat{S}(\hat{w}) \geq 0$ and thus

$$0 \leq \hat{S}(\hat{w}^*) = (1 - \hat{r}\hat{U}) \underbrace{\frac{\mathcal{T}(\hat{w}^*, \hat{r} + d^r)}{1 - d^r \mathcal{T}(\hat{w}^*, \hat{r} + d^r)}}_{>0} \iff 0 \leq 1 - \hat{r}\hat{U} \iff 1 \geq \hat{r}\hat{U}.$$

Therefore, $1 \geq \hat{r}\hat{U} \geq \tilde{B}$. To go from weak to strict inequalities, we follow the same steps as in the baseline model.

Observe that, if $\hat{t}^{m*}$ denotes only the stopping times arising from endogenous separations, then

$$\mathcal{T}(\hat{w}, \hat{r} + d^r) = \frac{1}{\hat{r} + d + d^r} \mathbb{E}_{\hat{w}} \left[1 - e^{-(\hat{r} + d + d^r)\hat{t}^{m*}} \right]$$

Using these results and the algebra of limits, we obtain

$$\lim_{d^r \to ¥} \hat{S}(\hat{w}) = \frac{1 - \hat{r}\hat{U}}{\hat{r} + d}.$$

2. The proof is analogous. For log-concavity of the value functions, the Ricatti equation continues

to hold following the redefinition of variables, but now

$$F(t) \equiv \frac{2}{s^2}\left[(\hat{r}+d+d^r) - \frac{(e^{t+\underline{\hat{w}}} - \hat{r}\hat{U}) + d^r \hat{W}(\underline{\hat{w}^*})}{\hat{W}(t+\underline{\hat{w}})}\right].$$

3. The same equilibrium conditions apply.

□

Discussion of the Effect of Wage Renegotiations on Job Creation and the Entry Wage. Before discussing Proposition II.3, it is important to highlight a distinction between the baseline model and the model with on-the-job bargaining. In the baseline model, $\mathcal{T}(\hat{w}, \hat{r})$ represented the expected discounted duration of a match, but now, $\mathcal{T}(\hat{w}, \hat{r}+d^r)$ denotes the expected discounted duration of the current wage. While these objects are trivially identical when $d^r = 0$, as shown in equation (II.17), they differ when $d^r > 0$.

Proposition II.3 formalizes a simple intuition: As the frequency of bargaining increases, the economic mechanisms influencing job creation resemble those in existing models of directed search. To illustrate this, Figure II1 shows the value functions of the worker and the firm, the surplus of the match, and the objective function $\hat{J}(\hat{w})^{1-a}\hat{W}(\hat{w})^a$ that the bargained wage $\hat{w}^*$ maximizes. As we can see in Panel A, when $d^r = 0$, the surplus function exhibits curvature at $\hat{w} = \hat{w}^*$ since the entry wage affects the likelihood of future separations. Instead, when $d^r = 0.2$, the surplus function becomes independent of the wage for $\hat{w}$ close to $\hat{w}^*$ since the probability of an inefficient separation is small (i.e., it is quite likely that the wage will be renegotiated before the match gets endogenously dissolved). In the limit, as $d^r \to ¥$, the surplus function becomes a constant independent of $\hat{w}$.

Figure II2 plots the worker's share evaluated at the entry wage and the flow opportunity cost as a function of the renegotiation rate d^r. As proposition II.3 shows, when $d^r \to ¥$, the worker's share of surplus $h(\hat{w}^*) \to a$ and the flow opportunity cost $\hat{r}\hat{U}^*$ converges to the solution of the following equation

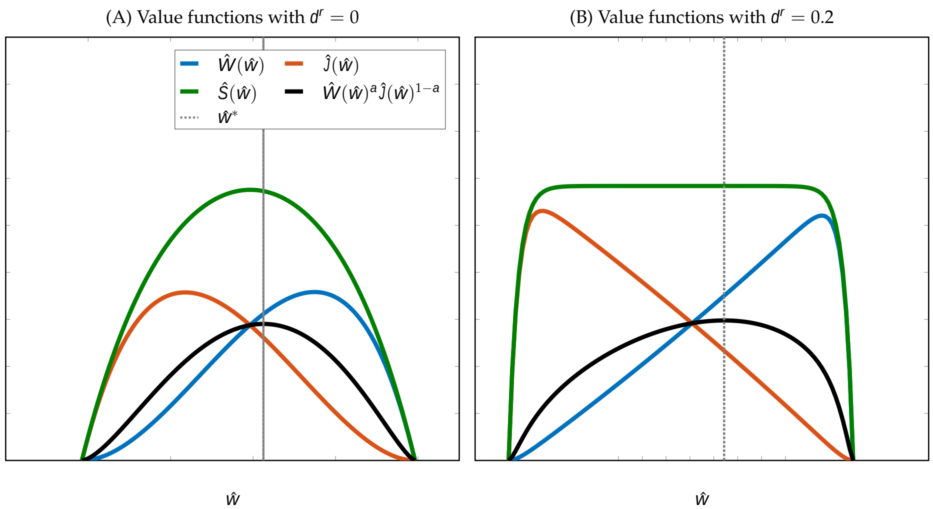
$$\hat{r}\hat{U}^{fb} = \tilde{B} + \left(\tilde{K}^{a-1}(1-a)^{1-a}a^a \frac{1-\hat{r}\hat{U}^{fb}}{\hat{r}+d}\right)^{\frac{1}{a}},$$

i.e., the flow opportunity cost when there are no inefficient separations.

Static and Dynamic Considerations Behind Equilibrium Policies. We now extend Propositions 3, 4, and 5 within the model with wage renegotiations.

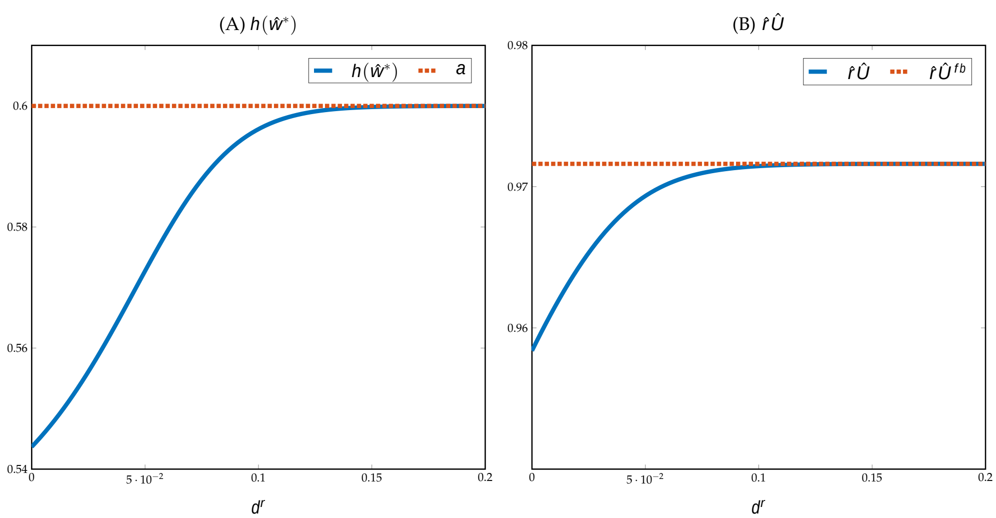
Proposition II.4. *The following properties hold:*

FIGURE II1. EQUILIBRIUM VALUES AND CONTINUATION SETS IN $\hat{w}$-SPACE



Notes: The figure plots the equilibrium value functions of the firm $\hat{J}(\hat{w})$ (red line) and the employed worker $\hat{W}(\hat{w})$ (blue line), the surplus of the match $\hat{S}(\hat{w})$ (green line), and the "Nash bargaining" objective function $\hat{J}(\hat{w})^{1-a}\hat{W}(\hat{w})^a$ (black line) as a function of $\hat{w} = w - z$ for $d^r = 0$ and $d^r = 0.2$. The vertical lines mark the boundaries of the firms' continuation set $(-¥, \hat{w}^+)$ and the worker's continuation set $(\hat{w}^-, ¥)$.

FIGURE II2. EQUILIBRIUM POLICIES FOR DIFFERENT VALUES OF d^r



Notes: Panels A and B plot the worker's share evaluated at the entry wage $h(\hat{w}^*)$ and the flow opportunity cost $\hat{r}\hat{U}$ as a function of the renegotiation rate d^r, respectively. The solid blue lines show the equilibrium values of $h(\hat{w}^*)$ and $\hat{r}\hat{U}$, while the dotted red lines show the corresponding values when $d^r \to ¥$.

1. *If $g = s = 0$, then the optimal policies are given by*

$$(\hat{w}^-, \hat{w}^*, \hat{w}^+) = \log\left(\hat{r}\hat{U} - \frac{d^r}{\hat{r}+d} a(1-\hat{r}\hat{U}), a + (1-a)\hat{r}\hat{U}, 1 + \frac{d^r}{\hat{r}+d}(1-a)(1-\hat{r}\hat{U}).\right)$$

with $h(\hat{w}^*) = a$ *and* $\mathcal{T}(\hat{w}^*, \hat{r} + d^r) = 1/(\hat{r} + d)$.

2. *Assume* $\hat{g} = 0$ *and* $a = 1/2$, *and a first-order approximation of the flow payoffs around* $\hat{w}^*$. *Then* $\hat{w}^{\pm} = \hat{w}^* \pm h(j, \mathsf{F}, \frac{d^r}{d+\hat{r}})$ *with* $j = \sqrt{2\frac{\hat{r}+d+d^r}{s^2}}$ *and* $\mathsf{F} = \frac{1-\hat{r}\hat{U}}{1+\hat{r}\hat{U}}$. *If* $d^r \to$ ¥, *then* $h\left(\sqrt{2\frac{\hat{r}+d+d^r}{s^2}}, \mathsf{F}, \frac{d^r}{d+\hat{r}}\right) \to$ ¥. *Fix* j *such that* $j\,h\left(j, \mathsf{F}, \frac{d^r}{d+\hat{r}}\right) < 1.606107734475270$, *then for a given* j, $h(j, \mathsf{F}, \cdot)$ *is decreasing in* $\frac{d^r}{d+\hat{r}}$. *Furthermore,* $h(\hat{w}^*) = a$ *and*

$$\mathcal{T}(\hat{w}^*, \hat{r} + d^r) = \frac{1 - \mathit{sech}\left(j\,h\left(j, \mathsf{F}, \frac{d^r}{d+\hat{r}}\right)\right)}{d + \hat{r} + d^r}.$$

3. *Assume* $s^2 = 0$ *and* $\hat{g} \geq 0$. *Then,*

$$\hat{w}^- = \log(\hat{r}\hat{U} - d^r \hat{W}(\hat{w}^*)).$$

There exists a $\bar{d}^r$ *satisfying* $\frac{\hat{r}\hat{U}}{e^{\hat{w}^*} - \hat{r}\hat{U}}(\hat{r} + d) < \bar{d}^r < \frac{\hat{r}\hat{U}e^{(\hat{r}+d)T(\hat{w}^*)}}{\int_0^{T(\hat{w}^*)}\left(e^{\hat{w}^* - \hat{g}s} - \hat{r}\hat{U}\right)\mathrm{d}s}$ *such that if* $d^r \uparrow \bar{d}^r$, *then* $\hat{w}^- \to -$¥.

Proof. We depart from the equilibrium conditions:

$$(\hat{r} + d + d^r)\hat{W}(\hat{w}) = e^{\hat{w}} - \hat{r}\hat{U} + d^r \hat{W}(\hat{w}^*) - \hat{g}\hat{W}'(\hat{w}) + \frac{s^2}{2}\hat{W}''(\hat{w}), \ \ \forall \hat{w} \in (\hat{w}^-, \hat{w}^+)$$
$$(\hat{r} + d + d^r)\hat{J}(\hat{w}) = 1 - e^{\hat{w}} + d^r \hat{J}(\hat{w}^*) - \hat{g}\hat{J}'(\hat{w}) + \frac{s^2}{2}\hat{J}''(\hat{w}), \ \ \forall \hat{w} \in (\hat{w}^-, \hat{w}^+)$$
$$(1 - a)\frac{\mathrm{d}\log \hat{J}(\hat{w}^*)}{\mathrm{d}\hat{w}} = -a\frac{\mathrm{d}\log \hat{W}(\hat{w}^*)}{\mathrm{d}\hat{w}},$$
$$\hat{W}(\hat{w}^-) = \hat{J}(\hat{w}^-) = \hat{W}(\hat{w}^+) = \hat{J}(\hat{w}^+) = 0$$
$$\hat{W}'(\hat{w}^-) = \hat{J}'(\hat{w}^+) = 0$$

whenever $\hat{g} \neq 0$ or $s > 0$. When $\hat{g} = s = 0$, we have the variation inequality holding without a smooth pasting condition. We now show the properties of equilibrium policies.

Case $g = s = 0$: When $g = s = 0$, we have that

$$(\hat{r} + d + d^r)\hat{W}(\hat{w}) = \max\{e^{\hat{w}} - \hat{r}\hat{U} + d^r \hat{W}(\hat{w}^*), 0\} \tag{II.18}$$
$$(\hat{r} + d + d^r)\hat{J}(\hat{w}) = \max\{1 - e^{\hat{w}} + d^r \hat{J}(\hat{w}^*), 0\} \tag{II.19}$$
$$(1 - a)\frac{\mathrm{d}\log \hat{J}(\hat{w}^*)}{\mathrm{d}\hat{w}} = -a\frac{\mathrm{d}\log \hat{W}(\hat{w}^*)}{\mathrm{d}\hat{w}}. \tag{II.20}$$

Evaluating the equilibrium conditions (II.18) and (II.19) at $\hat{w}^*$

$$\hat{W}(\hat{w}^*) = \frac{e^{\hat{w}^*} - \hat{r}\hat{U}}{\hat{r} + d}, \;\; \hat{J}(\hat{w}^*) = \frac{1 - e^{\hat{w}^*}}{\hat{r} + d}.$$

and using the equilibrium conditions again, we obtain

$$\hat{W}(\hat{w}) = \frac{e^{\hat{w}} - \hat{r}\hat{U}}{\hat{r} + d + d^r} + \frac{d^r}{\hat{r} + d + d^r}\frac{e^{\hat{w}^*} - \hat{r}\hat{U}}{\hat{r} + d}, \;\; \hat{J}(\hat{w}) = \frac{1 - e^{\hat{w}}}{\hat{r} + d + d^r} + \frac{d^r}{\hat{r} + d + d^r}\frac{1 - e^{\hat{w}^*}}{\hat{r} + d}.$$

Next, we compute the reset wage. Given value functions, the equilibrium condition (II.20) yields

$$\begin{aligned}
(1-a)\frac{\frac{-e^{\hat{w}^*}}{\hat{r}+d+d^r}}{\frac{1-e^{\hat{w}^*}}{\hat{r}+d+d^r} + \frac{d^r}{\hat{r}+d+d^r}\frac{1-e^{\hat{w}^*}}{\hat{r}+d}} &= -a\frac{\frac{e^{\hat{w}^*}}{\hat{r}+d+d^r}}{\frac{e^{\hat{w}^*}-\hat{r}\hat{U}}{\hat{r}+d+d^r} + \frac{d^r}{\hat{r}+d+d^r}\frac{e^{\hat{w}^*}-\hat{r}\hat{U}}{\hat{r}+d}} \iff \\
\left(1 + \frac{d^r}{\hat{r}+d}\right)\left(a + (1-a)\hat{r}\hat{U}\right) &= e^{\hat{w}^*}(a + (1-a))\left(1 + \frac{d^r}{\hat{r}+d}\right) \\
e^{\hat{w}^*} &= a + (1-a)\hat{r}\hat{U}.
\end{aligned}$$

The boundaries of the continuation region $\hat{w}^-$ and $\hat{w}^+$ are given by:

$$\begin{aligned}
\hat{W}(\hat{w}^-) &= 0 \\
\frac{e^{\hat{w}^-} - \hat{r}\hat{U}}{\hat{r} + d + d^r} + \frac{d^r}{\hat{r} + d + d^r}\frac{e^{\hat{w}^*} - \hat{r}\hat{U}}{\hat{r} + d} &= 0 \\
\frac{e^{\hat{w}^-} - \hat{r}\hat{U}}{\hat{r} + d + d^r} + \frac{d^r}{\hat{r} + d + d^r}\frac{a + (1-a)\hat{r}\hat{U} - \hat{r}\hat{U}}{\hat{r} + d} &= 0 \\
e^{\hat{w}^-} - \hat{r}\hat{U} + \frac{d^r}{\hat{r}+d}a(1 - \hat{r}\hat{U}) &= 0 \\
e^{\hat{w}^-} &= \hat{r}\hat{U} - \frac{d^r}{\hat{r}+d}a(1 - \hat{r}\hat{U}).
\end{aligned}$$

Similarly, $\hat{w}^+$ is given by

$$e^{\hat{w}^+} = 1 + \frac{d^r}{\hat{r}+d}(1-a)(1 - \hat{r}\hat{U}).$$

The rest of the proof is similar to the proof in the baseline model without renegotiation.

Case $\hat{g} = 0$ and $a = 1/2$: We follow the same strategy as in Proposition 4. Let us guess and verify that $e^{\hat{w}^*} = \frac{1+\hat{r}\hat{U}}{2}$, $\hat{w}^- = \hat{w}^* - h$ and $\hat{w}^+ = \hat{w}^* + h$ for a given h. Using a Taylor approximation of the flow profits around $\hat{w}^*$, we have that

$$(\hat{r} + d + d^r)\hat{W}(\hat{w}) = \frac{1 - \hat{r}\hat{U}}{2} + e^{\hat{w}^*}(\hat{w} - \hat{w}^*) + d^r\hat{W}(\hat{w}^*) + \frac{s^2}{2}\hat{W}''(\hat{w}), \;\; \forall \hat{w} \in (\hat{w}^* - h, \hat{w}^* + h)$$

$$(\hat{r}+d+d^r)\hat{J}(\hat{w}) = \frac{1-\hat{r}\hat{U}}{2} - e^{\hat{w}^*}(\hat{w}-\hat{w}^*) + d^r\hat{J}(\hat{w}^*) + \frac{s^2}{2}\hat{J}''(\hat{w}), \ \ \forall \hat{w} \in (\hat{w}^*-h, \hat{w}^*+h),$$

with the border conditions given by the value matching and smooth pasting conditions. It is easy to check that when $d^r \to$ ¥, $\hat{W}(\hat{w}) \to \hat{W}(\hat{w}^*)$, so h converges to ¥. Define $J(x) = \frac{\hat{J}(x+\hat{w}^*)-\frac{1-\hat{r}\hat{U}}{2(\hat{r}+d+d^r)}}{e^{\hat{w}^*}}$ and $W(x) = \frac{\hat{W}(x+\hat{w}^*)-\frac{1-\hat{r}\hat{U}}{2(\hat{r}+d+d^r)}}{e^{\hat{w}^*}}$. Following the same steps as in Proposition 4, we have that $J(x) = W(-x)$ with

$$(\hat{r}+d+d^r)W(x) = x + d^r W(0) + \frac{s^2}{2}W''(x), \ \ \forall x \in (-h,h)$$
$$W(-h) = W(h) = -\frac{F}{\hat{r}+d+d^r}; \ \ W'(-h) = 0.$$

Notice that an increase in the renegotiation arrival rate d^r, increases the effective discount factor and, at the same time, it increases the worker's flow value. It is easy to show that the solution of the previous differential equation is given by

$$W(x) = Ae^{jx} + Be^{-jx} + \frac{x}{\hat{r}+d+d^r} + \frac{d^r}{d+\hat{r}}(A+B),$$

where $j = \sqrt{2\frac{\hat{r}+d+d^r}{s^2}}$. Writing the value matching conditions and operating, we obtain

$$A\left(e^{jh}+\frac{d^r}{d+\hat{r}}\right) + B\left(e^{-jh}+\frac{d^r}{d+\hat{r}}\right) = \frac{-F-h}{\hat{r}+d+d^r}$$
$$A\left(e^{-jh}+\frac{d^r}{d+\hat{r}}\right) + B\left(e^{jh}+\frac{d^r}{d+\hat{r}}\right) = \frac{-F+h}{\hat{r}+d+d^r}.$$

Solving for A and B,

$$A = -\frac{1}{\hat{r}+d+d^r}\frac{\left(e^{jh}+\frac{d^r}{d+\hat{r}}\right)(F+h)+\left(e^{-jh}+\frac{d^r}{d+\hat{r}}\right)(h-F)}{\left(e^{jh}+\frac{d^r}{d+\hat{r}}\right)^2-\left(e^{-jh}+\frac{d^r}{d+\hat{r}}\right)^2}$$
$$B = \frac{1}{\hat{r}+d+d^r}\frac{\left(e^{jh}+\frac{d^r}{d+\hat{r}}\right)(h-F)+\left(e^{-jh}+\frac{d^r}{d+\hat{r}}\right)(h+F)}{\left(e^{jh}+\frac{d^r}{d+\hat{r}}\right)^2-\left(e^{-jh}+\frac{d^r}{d+\hat{r}}\right)^2}.$$

Thus,

$$-W(x) = \frac{\frac{\left(e^{jh}+\frac{d^r}{d+\hat{r}}\right)(F+h)+\left(e^{-jh}+\frac{d^r}{d+\hat{r}}\right)(h-F)}{\left(e^{jh}+\frac{d^r}{d+\hat{r}}\right)^2-\left(e^{-jh}+\frac{d^r}{d+\hat{r}}\right)^2}\left(e^{jx}+\frac{d^r}{d+\hat{r}}\right)}{\hat{r}+d+d^r}$$

$$+\frac{\frac{\left(e^{jh}+\frac{d^r}{d+\hat{r}}\right)(h-\mathsf{F})+\left(e^{-jh}+\frac{d^r}{d+\hat{r}}\right)(h+\mathsf{F})}{\left(e^{jh}+\frac{d^r}{d+\hat{r}}\right)^2-\left(e^{-jh}+\frac{d^r}{d+\hat{r}}\right)^2}\left(e^{-jx}+\frac{d^r}{d+\hat{r}}\right)+x}{\hat{r}+d+d^r}.$$

Evaluating the smooth pasting condition, we obtain

$$-\frac{\left(e^{jh}+\frac{d^r}{d+\hat{r}}\right)(\mathsf{F}+h)+\left(e^{-jh}+\frac{d^r}{d+\hat{r}}\right)(h-\mathsf{F})}{\left(e^{jh}+\frac{d^r}{d+\hat{r}}\right)^2-\left(e^{-jh}+\frac{d^r}{d+\hat{r}}\right)^2}\, j e^{-jh}$$
$$-\frac{\left(e^{jh}+\frac{d^r}{d+\hat{r}}\right)(h-\mathsf{F})+\left(e^{-jh}+\frac{d^r}{d+\hat{r}}\right)(h+\mathsf{F})}{\left(e^{jh}+\frac{d^r}{d+\hat{r}}\right)^2-\left(e^{-jh}+\frac{d^r}{d+\hat{r}}\right)^2}\, j e^{jh}+1=0.$$

Operating on this expression and defining $q = jh$, we get

$$\mathsf{F}\, j2+\frac{4d^r}{d+\hat{r}}\frac{\sinh(q)-q\cosh(q)}{\cosh(2q)-1}=-\frac{2\sinh(2q)-2q(\cosh(2q)+1)}{\cosh(2q)-1}.$$

Thus,

$$\mathsf{F}\, j2=-\frac{4d^r}{d+\hat{r}}\frac{\sinh(q)-q\cosh(q)}{\cosh(2q)-1}-\frac{2\sinh(2q)-2q(\cosh(2q)+1)}{\cosh(2q)-1}$$

The following properties hold: $-\frac{\sinh(q)-q\cosh(q)}{\cosh(2q)-1}$ converges to 0 when $q \downarrow 0$, it increases until $q \approx 1.606$ and then decreases to 0 for $q > 1.606$. Since $-\frac{\sinh(q)-q\cosh(q)}{\cosh(2q)-1}$ is increasing in q if the solution q is lower than 1.606, we have that $q(\mathsf{F}\, j2, \frac{4d^r}{d+\hat{r}})$ is decreasing in the second argument.

As in the baseline model, due to symmetry $\hat{w}^{\pm} = \hat{w}^* \pm h(j, \mathsf{F}, \frac{d^r}{d+\hat{r}})$, we have that $T'_{\hat{w}}(\hat{w}^*, \hat{r}) = 0$ and $\mathcal{T}(\hat{w}^*, \hat{r}+d^r) = \frac{1-sech(jh(j,\mathsf{F},\frac{d^r}{d+\hat{r}}))}{d+\hat{r}+d^r}$.

Case $\hat{g} > 0$ and $s = 0$: In this case, the stopping time is a deterministic function; hence, it is easier to work with the sequential formulation:

$$\hat{W}(\hat{w}) = \max_T \int_0^T e^{-(\hat{r}+d+d^r)s}\left(e^{\hat{w}-\hat{g}s}+d^r\hat{W}(\hat{w}^*)-\hat{r}\hat{U}\right)\mathrm{d}s,$$
$$\hat{J}(\hat{w}) = \int_0^{T(\hat{w})} e^{-(\hat{r}+d+d^r)s}\left(1+d^r\hat{J}(\hat{w}^*)-e^{\hat{w}-\hat{g}s}\right)\mathrm{d}s.$$

In equation (A.28), $T(\hat{w})$ is the worker's optimal policy. Taking the the FOC with respect to $T(\hat{w})$:

$$e^{\hat{w}-\hat{g}T(\hat{w})} = \hat{r}\hat{U} - d^r\hat{W}(\hat{w}^*).$$

Solving the previous equation,

$$T(\hat{w}) = \frac{\hat{w} - \log\left(\hat{r}\hat{U} - d^r W(\hat{w}^*)\right)}{\hat{g}}.$$

Thus, if $\hat{w} = \hat{w}^*$, we have that $\hat{w}^- = \hat{w}^* - \hat{g}T(\hat{w}^*)$ satisfies

$$\hat{w}^- = \log(\hat{r}\hat{U} - d^r \hat{W}(\hat{w}^*)).$$

Following similar steps as in the baseline model, it is easy to show how $\hat{W}(\hat{w}^*)$ depends on $\hat{g}$ and $\hat{r} + d$. Now, we prove the last property—i.e., there exists a $\bar{d}^r <$ such that $\lim_{d^r \to \bar{d}^r} \hat{w}^- = -¥$. That is, if the bargaining probability is high enough but finite, then the continuation region becomes unbounded. Observe that if $d^r \uparrow \bar{d}^r$, where $\bar{d}^r := \frac{\hat{r}\hat{U}}{\hat{W}(\hat{w}^*)}$, then $\hat{w}^- \to -¥$. Since $\hat{W}(\hat{w}^*) < \frac{e^{\hat{w}^*} - \hat{r}\hat{U}}{\hat{r}+d}$, we have that

$$0 = \hat{r}\hat{U} - \bar{d}^r \hat{W}(\hat{w}^*) > \hat{r}\hat{U} - \bar{d}^r \frac{e^{\hat{w}^*} - \hat{r}\hat{U}}{\hat{r}+d}, \iff \bar{d}^r > \frac{\hat{r}\hat{U}}{e^{\hat{w}^*} - \hat{r}\hat{U}}(\hat{r}+d).$$

Thus, we have a lower bound for $\bar{d}^r$. To find an upper bound, we compute the value function without on-the-job bargaining. In this case, we have that

$$\hat{W}(\hat{w}^*) > \int_0^{T(\hat{w}^*)} e^{-(\hat{r}+d)s}\left(e^{\hat{w}-\hat{g}s} - \hat{r}\hat{U}\right) ds > e^{-(\hat{r}+d)T(\hat{w}^*)} \int_0^{T(\hat{w}^*)} \left(e^{\hat{w}-\hat{g}s} - \hat{r}\hat{U}\right) ds$$

Thus,

$$\bar{d}^r < \frac{\hat{r}\hat{U}e^{(\hat{r}+d)T(\hat{w}^*)}}{\int_0^{T(\hat{w}^*)}\left(e^{\hat{w}^*-\hat{g}s} - \hat{r}\hat{U}\right) ds}.$$

□

Discussion of Static and Dynamic Consideration for Equilibrium Policies. Intuition suggests that an increase in the frequency of bargaining will lead to a change in the quit and layoff triggers, ultimately resulting in an extended match duration. Proposition II.4-Part 1 demonstrates that this intuition holds true whenever there is no drift or shocks in idiosyncratic productivity. Furthermore, it illustrates how the continuation set of the match changes as a function of the primitives $(d^r, \hat{r} + d, a, 1 - \hat{r}\hat{U})$. Importantly, the size of the surplus does not affect the separation thresholds when $d^r = 0$, but also affects the marginal effect of the frequency of bargaining since both terms appear multiplicatively. Also, observe that when $d^r \to ¥$, then $\hat{w}^- \to -¥$ and $\hat{w}^+ \to ¥$. This property also holds for $\hat{g} > 0$ and $s > 0$.

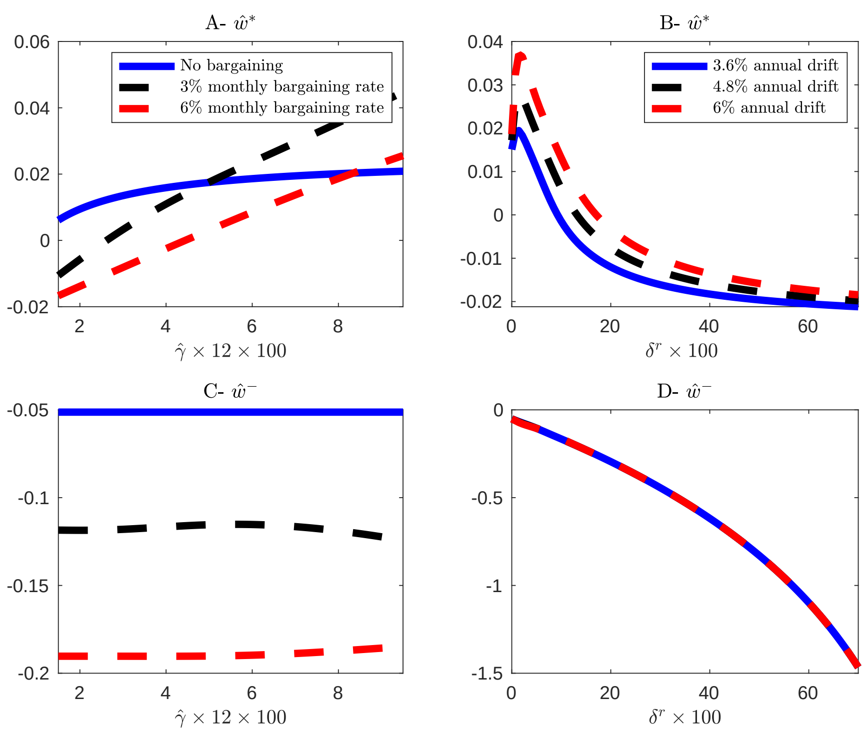
Proposition II.4-Part 2 characterizes the interaction between the option value effect and the frequency of on-the-job bargaining. The width of the continuation region depends on $\frac{s^2}{\hat{r}+d+d^r}$, the surplus, and the frequency of on-the-job bargaining. The first result, where we fix the value of s^2, shows that if the frequency of bargaining increases, then the width of the continuation set converges to infinity.

In the second result, we construct a specific case that highlights a counter-intuitive finding: If there is an increase in d^r and s^2 while keeping $j = \sqrt{2\frac{\hat{r}+d+d^r}{s^2}}$ fixed, then the width of the continuation region decreases. The intuition behind this result is not straightforward. To understand it, first recall that in the baseline model, the width of the inaction region does not grow unboundedly with the volatility of shocks due to lack of commitment. A firm paying a high $\hat{w}$ is not willing to wait before dissolving the match when the volatility of productivity shocks is high because of the associated high probability of a sufficiently large and positive shock that would make the worker quit. In this version of the model, this willingness to wait before dissolving the match decreases when the frequency of bargaining increases. This is because a higher bargaining rate is associated with a $\hat{w}$ that tends to fluctuate well within the boundaries of the continuation region of the match. Therefore, for the same increase in the volatility of productivity shocks, the probability of experiencing a shock positive enough that makes the worker quit is much higher; thus, the firm decides to dissolve the match sooner.

Proposition II.4-Part 3 characterizes the anticipatory effect of the drift on the quit threshold. When $d^r = 0$, we recover the result presented for the baseline model: The quit threshold is fully static; i.e., the worker quits when $\hat{w} < \log(\hat{r}\hat{U})$. When $d^r > 0$, the quit threshold is dynamic and depends on the value of renegotiating the wage. A novel result arises: If the renegotiation frequency is large enough, but not necessarily infinitely large, then the worker will never quit her job. The intuition is that if the incentives to wait offered by wage renegotiation are significantly larger than the opportunity cost, then the worker will never find it optimal to quit.

We finish this discussion with a joint analysis of equilibrium policies when $s = 0$ and the worker's opportunity cost is kept constant. Figure II3-Panels A and C show the effect of the drift for different values of the monthly frequency of wage renegotiation. As we can see, the entry wage is increasing in the drift—as in the main text. In addition, the quit threshold is almost independent of the value of the drift—as in the main text—and it depends mainly on the probability of resetting the wage within the match. This is the result of two opposing forces almost balancing each other perfectly: i) for a fixed bargained wage, a larger drift and the associated higher quit probability reduce the value of the worker,and ii) a larger drift results in a higher bargained wage and, thus, a higher value when evaluated at that wage. Figure II3-Panels B and D show the effect of the frequency of bargaining for different values of the annual drift. As we can see, for a frequency large enough, the reset wage decreases—due a weaker anticipatory effect—and it converges to the static Nash bargaining solution. Once the entry wage converges, the elasticity of the quit threshold with respect to the renegotiation rate will become unboundedly large (recall that $\hat{w}^{-} = \log(\hat{r}\hat{U} - d^r\hat{W}(\hat{w}^*)))$.

FIGURE II3. COMPARATIVE STATICS WITH RESPECT TO THE DRIFT



Notes: Panels A and C plot the entry wage and the quit threshold as a function of the drift for three values of the monthly frequency of wage renegotiation, respectively. Panels B and D plot the entry wage and the quit threshold as a function of the frequency of bargaining for three values of the annual drift, respectively.

The CIR of Employment with Flexible and Sticky Entry Wages.

Before presenting the new results, we define new notation. Let t^m denote the duration of the current wage spell. Let t^r denote the time elapsed until the arrival of an opportunity to renegotiate the wage. Observe that, if $t^m < t^r$, then the match finishes in a separation. Otherwise, if $t^m = t^r$, then the current wage is renegotiated. Furthermore, as in the baseline model, let $g^h(\mathrm{D}z)$ and $g^u(\mathrm{D}z)$ be the distributions of $\mathrm{D}z$ across employed and unemployed workers, respectively. Observe that

$\mathsf{D}z$ now represents the cumulative shocks to revenue productivity $z + p$ that the match experienced since either its inception or its last wage renegotiation. The support of $g^h(\mathsf{D}z)$ is given by $[-\mathsf{D}^-, \mathsf{D}^+]$, where $\mathsf{D}^- := \hat{w}^* - \hat{w}^-$ and $\mathsf{D}^+ := \hat{w}^+ - \hat{w}^*$. We denote by $\mathbb{E}_h[\cdot]$ and $\mathbb{E}_u[\cdot]$ the expectation operators under the distributions $g^h(\mathsf{D}z)$ and $g^u(\mathsf{D}z)$, respectively.

Below, we describe the KFEs characterizing $g^h(\mathsf{D}z)$ and $g^u(\mathsf{D}z)$:

$$(d + d^r)\, g^h(\mathsf{D}z) = (g + c)(g^h)'(\mathsf{D}z) + \frac{s^2}{2}(g^h)''(\mathsf{D}z) \quad \text{for all } \mathsf{D}z \in (-\mathsf{D}^-, \mathsf{D}^+)/\{0\}, \tag{II.21}$$

$$f(\hat{q}(\hat{w}^*))g^u(\mathsf{D}z) = (g + c)(g^u)'(\mathsf{D}z) + \frac{s^2}{2}(g^u)''(\mathsf{D}z) \quad \text{for all } \mathsf{D}z \in (-¥, ¥)/\{0\}. \tag{II.22}$$

$$g^h(\mathsf{D}z) = 0, \text{ for all } \mathsf{D}z \notin (-\mathsf{D}^-, \mathsf{D}^+) \tag{II.23}$$

$$\lim_{\mathsf{D}z \to -¥} g^u(\mathsf{D}z) = \lim_{\mathsf{D}z \to ¥} g^u(\mathsf{D}z) = 0. \tag{II.24}$$

$$1 = \int_{-¥}^{¥} g^u(\mathsf{D}z)\, \mathrm{d}\mathsf{D}z + \int_{-\mathsf{D}^-}^{\mathsf{D}^+} g^h(\mathsf{D}z)\, \mathrm{d}\mathsf{D}z, \tag{II.25}$$

$$f(\hat{q}(\hat{w}^*))(1 - \mathcal{E}) = d\mathcal{E} + \frac{s^2}{2}\left[\lim_{\mathsf{D}z \downarrow -\mathsf{D}^-}(g^h)'(\mathsf{D}z) - \lim_{\mathsf{D}z \uparrow \mathsf{D}^+}(g^h)'(\mathsf{D}z)\right], \tag{II.26}$$

$$g^u(\mathsf{D}z) \in \mathbb{C}, \mathbb{C}^2((-¥, ¥)/\{0\}), \;\; g^h(\mathsf{D}z), \mathbb{C}, \mathbb{C}^2((-\mathsf{D}^-, \mathsf{D}^+)/\{0\})$$

The main difference with our baseline analysis is the additional d^r term in the KFE for $g^h(\mathsf{D}z)$. For this to hold, the renegotiated wage must be the same as the entry wage from unemployment, which results from the assumption that the worker's bargaining weight equals the elasticity of the matching function.

We divide the proof of the extension of Proposition 6 to the case of wage renegotiations into three propositions. Proposition II.5 relates the CIR to a perturbation of two Bellman equations describing future employment fluctuations for initially employed and unemployed workers. This proposition covers both the case with flexible and sticky entry wages. Proposition II.6 relates steady-state moments of the perturbed Bellman equations to steady-state moments of the distribution of $\mathsf{D}z$. Finally, Proposition II.7 related the steady-state moments of $\mathsf{D}z$ to observable moments in the steady-state.

Taken together, Propositions II.6 and II.7 extend Proposition 6 for the case with wage changes within a job. Finally, Proposition II.8 extends Lemma 2 for the case with wage changes within a job. Whenever the steps of the proof are the same as those in the baseline model, we omit them.

Proposition II.5. *Given steady-state policies* $(\hat{w}^-, \hat{w}^*, \hat{w}^+)$ *and distributions* $(g^h(\mathsf{D}z), g^u(\mathsf{D}z))$*, the CIR* *is*

$$CIR_{\mathcal{E}}(z) = \int_{-¥}^{¥} m_{\mathcal{E},h}(\mathsf{D}z) g^h(\mathsf{D}z + z)\, \mathrm{d}\mathsf{D}z + \int_{-¥}^{¥} m_{\mathcal{E},u}(\mathsf{D}z, z) g^u(\mathsf{D}z + z)\, \mathrm{d}\mathsf{D}z,$$

where the value functions $m_{\mathcal{E},h}(\mathsf{D}z)$ *and* $m_{\mathcal{E},u}(\mathsf{D}z, z)$ *are characterized by:*

$$\begin{aligned} 0 &= 1 - \mathcal{E}_{ss} - (g + c)\frac{\mathrm{d}m_{\mathcal{E},h}(\mathsf{D}z)}{\mathrm{d}\mathsf{D}z} + \frac{s^2}{2}\frac{d^2 m_{\mathcal{E},h}(\mathsf{D}z)}{\mathrm{d}\mathsf{D}z^2} + d(m_{\mathcal{E},u}(0,0) - m_{\mathcal{E},h}(\mathsf{D}z)) \\ &\quad + d^r(m_{\mathcal{E},h}(0) - m_{\mathcal{E},h}(\mathsf{D}z)), \end{aligned} \tag{II.27}$$

$$\begin{aligned} 0 &= -\mathcal{E}_{ss} - (g + c)\frac{\mathrm{d}m_{\mathcal{E},u}(\mathsf{D}z, z)}{\mathrm{d}\mathsf{D}z} + \frac{s^2}{2}\frac{d^2 m_{\mathcal{E},u}(\mathsf{D}z, z)}{\mathrm{d}\mathsf{D}z^2} + f(\hat{q}(\hat{w}^* - z))(m_{\mathcal{E},h}(-z) \\ &\quad - m_{\mathcal{E},u}(\mathsf{D}z, z)) \end{aligned} \tag{II.28}$$

$$m_{\mathcal{E},u}(0,0) = m_{\mathcal{E},h}(\mathsf{D}z),\ \forall \mathsf{D}z \notin (-\mathsf{D}^-, \mathsf{D}^+)$$

$$0 = \lim_{\mathsf{D}z \to -¥} \frac{\mathrm{d}m_{\mathcal{E},u}(\mathsf{D}z, z)}{\mathrm{d}\mathsf{D}z} = \lim_{\mathsf{D}z \to ¥} \frac{\mathrm{d}m_{\mathcal{E},u}(\mathsf{D}z, z)}{\mathrm{d}\mathsf{D}z} \tag{II.29}$$

$$0 = \int_{-¥}^{¥} m_{\mathcal{E},h}(\mathsf{D}z) g^h(\mathsf{D}z)\, \mathrm{d}\mathsf{D}z + \int_{-¥}^{¥} m_{\mathcal{E},u}(\mathsf{D}z, 0) g^u(\mathsf{D}z)\, \mathrm{d}\mathsf{D}z. \tag{II.30}$$

Proof. We define the CIR of aggregate employment to an aggregate TFPR shock as

$$\mathrm{CIR}_{\mathcal{E}}(z) = \int_0^{¥} \int_{-¥}^{¥} \left(g^h(\mathsf{D}z, z, t) - g^h(\mathsf{D}z) \right) \mathrm{d}\mathsf{D}z\, \mathrm{d}t.$$

Here, $\mathcal{E}_t = \int_{-¥}^{¥} g^h(\mathsf{D}z, z, t)\, \mathrm{d}\mathsf{D}z$ is a function of z since aggregate shocks affect net flows into employment. As in the main proof of Proposition 6, starting from the definition of the CIR, we can derive

$$\mathrm{CIR}_{\mathcal{E}}(z) = \int_{-¥}^{¥} \lim_{\mathcal{T} \to ¥} m_{\mathcal{E},h}(\mathsf{D}z, \mathcal{T}) g^h(\mathsf{D}z + z)\, \mathrm{d}\mathsf{D}z + \int_{-¥}^{¥} \lim_{\mathcal{T} \to ¥} m_{\mathcal{E},u}(\mathsf{D}z, z, \mathcal{T}) g^u(\mathsf{D}z + z)\, \mathrm{d}\mathsf{D}z,$$

where we define

$$\begin{aligned} m_{\mathcal{E},h}(\mathsf{D}z_0, \mathcal{T}) &\equiv \int_0^{\mathcal{T}} \left[\int_{-¥}^{¥} \left[(1 - \mathcal{E}_{ss})\, g^h(\mathsf{D}z, t | \mathsf{D}z_0, h) + (-\mathcal{E}_{ss}) g^u(\mathsf{D}z, t | \mathsf{D}z_0, h) \right] \mathrm{d}\mathsf{D}z\, \mathrm{d}t \right] \\ m_{\mathcal{E},u}(\mathsf{D}z_0, z, \mathcal{T}) &\equiv \int_0^{\mathcal{T}} \left[\int_{-¥}^{¥} \left[(1 - \mathcal{E}_{ss})\, g^h(\mathsf{D}z, z, t | \mathsf{D}z_0, u) + (-\mathcal{E}_{ss}) g^u(\mathsf{D}z, z, t | \mathsf{D}z_0, u) \right] \mathrm{d}\mathsf{D}z\, \mathrm{d}t \right]. \end{aligned}$$

Taking the limit as $\mathcal{T} \to ¥$, we have that the value functions $m_{\mathcal{E},h}(\mathsf{D}z_0)$ and $m_{\mathcal{E},u}(\mathsf{D}z_0, z)$ satisfy the the condition in equation (II.27) to (II.30). □

Notice that the main difference between equations (II.27)–(II.30) and equations (B.4)–(B.8) is the extra term in the HJB characterizing $m_{\mathcal{E},h}(\mathrm{D}z_0)$, which takes into account staggered wage renegotiations.

The CIR of Employment with Flexible Entry Wages and Wage Renegotiations.

Proposition II.6. *Assume flexible entry wages. Up to first order, the CIR of employment is given by:*

$$\frac{CIR_{\mathcal{E}}(z)}{z} = -(1-\mathcal{E}_{ss})\frac{[(g+c)\mathbb{E}_h[a]+\mathbb{E}_h[\mathrm{D}z]]}{s^2}$$
$$-\frac{\mathcal{E}_{ss}}{s^2 f(\hat{q}(\hat{w}^*))}\left((g+c)\mathcal{E}_{ss}\left(1-\frac{\mathbb{E}_{\mathcal{D}}\left[t^m \mathbb{1}\{t^m < t^{d^r}\}\right]}{\mathbb{E}_{\mathcal{D}}\left[t^m\right]}\right)+\mathbb{E}_h[\mathrm{D}z]\mathcal{F}^{\mathrm{D}w}\right)+o(z),$$

where $\mathcal{F}^{\mathrm{D}w}$ denotes the observed frequency of wage renegotiation within a match in the data.

Proof. The proof proceeds in three steps. Step 1 computes the value function for an unemployed worker $m_{\mathcal{E},u}(\mathrm{D}z)$ (when entry wages are flexible, the job-finding rate and this value function are independent of the shock z, so we omit this argument). Step 2 computes the value for the employed worker at $\mathrm{D}z = 0$—i.e., $m_{\mathcal{E},h}(0)$. Step 3 characterizes the CIR as a function of steady-state aggregate variables and moments.

Step 1. The CIR is given by

$$\mathrm{CIR}_{\mathcal{E}}(z) = \int_{-¥}^{¥} m_{\mathcal{E},h}(\mathrm{D}z) g^h(\mathrm{D}z + z)\,\mathrm{d}\mathrm{D}z + \left(-\frac{\mathcal{E}_{ss}}{f(\hat{q}(\hat{w}^*))} + m_{\mathcal{E},h}(0)\right)(1-\mathcal{E}_{ss}),$$

with

$$0 = 1-\mathcal{E}_{ss} - (g+c)\frac{\mathrm{d}m_{\mathcal{E},h}(\mathrm{D}z)}{\mathrm{d}\mathrm{D}z} + \frac{s^2}{2}\frac{d^2 m_{\mathcal{E},h}(\mathrm{D}z)}{\mathrm{d}\mathrm{D}z^2} + d\left(-\frac{\mathcal{E}_{ss}}{f(\hat{q}(\hat{w}^*))} + m_{\mathcal{E},h}(0) - m_{\mathcal{E},h}(\mathrm{D}z)\right)$$
$$+ d^r(m_{\mathcal{E},h}(0) - m_{\mathcal{E},h}(\mathrm{D}z)),$$
$$m_{\mathcal{E},h}(\mathrm{D}z) = -\frac{\mathcal{E}_{ss}}{f(\hat{q}(\hat{w}^*))} + m_{\mathcal{E},h}(0),\ \forall \mathrm{D}z \notin (-\mathrm{D}^-, \mathrm{D}^+)$$
$$0 = \int_{-¥}^{¥} m_{\mathcal{E},h}(\mathrm{D}z) g^h(\mathrm{D}z)\,\mathrm{d}\mathrm{D}z + \left(-\frac{\mathcal{E}_{ss}}{f(\hat{q}(\hat{w}^*))} + m_{\mathcal{E},h}(0)\right)(1-\mathcal{E}_{ss}). \tag{II.31}$$

Proof of Step 1. To show this result, observe that the solution to (II.28) and (II.29) is

$$m_{\mathcal{E},u}(\mathrm{D}z) = m_{\mathcal{E},u}(0),\ \forall \mathrm{D}z.$$

Thus,

$$0 = -\mathcal{E}_{ss} + f(\hat{q}(\hat{w}^*))(m_{\mathcal{E},h}(0) - m_{\mathcal{E},u}(0)) \iff m_{\mathcal{E},u}(0) = -\frac{\mathcal{E}_{ss}}{f(\hat{q}(\hat{w}^*))} + m_{\mathcal{E},h}(0). \tag{II.32}$$

Replacing (II.32) into the CIR, we have the result.

Step 2. We show that $m_{\mathcal{E},h}(0) = \frac{\mathcal{E}_{ss}}{f(\hat{q}(\hat{w}^*))}\left((1-\mathcal{E}_{ss}) + \mathcal{E}_{ss}\frac{\mathbb{E}_{\mathcal{D}}[t^m \mathbb{1}\{t^m < t^{d^r}\}]}{\mathbb{E}_{\mathcal{D}}[t^m]}\right) - (1-\mathcal{E}_{ss})\mathbb{E}_h[a]$, where $\mathbb{E}_h[a]$ is the cross-sectional expected age of the match or the worker's tenure at the current match.

Proof of Step 2. Observe that $m_{\mathcal{E},h}(\mathsf{D}z)$ satisfies the following recursive representation

$$m_{\mathcal{E},h}(\mathsf{D}z) = \mathbb{E}\left[\int_0^{t^m} (1-\mathcal{E}_{ss})\, dt + \left(-\mathbb{1}\{t^m < t^{d^r}\}\frac{\mathcal{E}_{ss}}{f(\hat{q}(\hat{w}^*))} + m_{\mathcal{E},h}(0)\right)\middle| \mathsf{D}z_0 = \mathsf{D}z\right]. \tag{II.33}$$

Define the following auxiliary function

$$\mathsf{Y}(\mathsf{D}z|\,j) \;=\; \mathbb{E}\left[\int_0^{t^m} e^{jt}(1-\mathcal{E}_{ss})\, dt + e^{j t^m}\left(-\mathbb{1}\{t^m < t^{d^r}\}\frac{\mathcal{E}_{ss}}{f(\hat{q}(\hat{w}^*))} + m_{\mathcal{E},h}(0)\right)\middle| \mathsf{D}z_0 = \mathsf{D}z\right]. \tag{II.34}$$

and note that $\mathsf{Y}(\mathsf{D}z|0) = m_{\mathcal{E},h}(\mathsf{D}z)$. Then $\mathsf{Y}(\mathsf{D}z|\,j)$ satisfies the following HJB and border conditions:

$$\begin{aligned} &-j\,\mathsf{Y}(\mathsf{D}z|\,j) + (d+d^r)\left(\mathsf{Y}(\mathsf{D}z|\,j) - m_{\mathcal{E},h}(0)\right) + d\frac{\mathcal{E}_{ss}}{f(\hat{q}(\hat{w}^*))} \\ &= (1-\mathcal{E}_{ss}) - (g+c)\frac{\P \mathsf{Y}(\mathsf{D}z|\,j)}{\P \mathsf{D}z} + \frac{s^2}{2}\frac{\P^2 \mathsf{Y}(\mathsf{D}z|\,j)}{\P \mathsf{D}z^2}, \\ \mathsf{Y}(\mathsf{D}z, j) &= \left(-\frac{\mathcal{E}_{ss}}{f(\hat{q}(\hat{w}^*))} + m_{\mathcal{E},h}(0)\right) \;\forall \mathsf{D}z \notin (-\mathsf{D}^-, \mathsf{D}^+). \end{aligned} \tag{II.35}$$

Taking the derivative with respect to j in (II.35), we have that

$$\begin{aligned} (d+d^r-j)\frac{\P \mathsf{Y}(\mathsf{D}z|\,j)}{\P j} - \mathsf{Y}(\mathsf{D}z|\,j) &= -(g+c)\frac{\P^2 \mathsf{Y}(\mathsf{D}z, j)}{\P \mathsf{D}z \P j} + \frac{s^2}{2}\frac{\P^3 \mathsf{Y}(\mathsf{D}z|\,j)}{\P \mathsf{D}z^2 \P j}, \\ \frac{\P \mathsf{Y}(\mathsf{D}z|\,j)}{\P j} &= 0 \;\forall \mathsf{D}z \notin (-\mathsf{D}^-, \mathsf{D}^+). \end{aligned}$$

Using Schwarz's Theorem to exchange partial derivatives, evaluating at $j = 0$, and using $\mathsf{Y}(\mathsf{D}z|0) = m_{\mathcal{E},h}(\mathsf{D}z)$, we obtain

$$(d+d^r)\frac{\P \mathsf{Y}(\mathsf{D}z|0)}{\P j} - m_{\mathcal{E},h}(\mathsf{D}z) = -(g+c)\frac{\P}{\P \mathsf{D}z}\left(\frac{\P \mathsf{Y}(\mathsf{D}z|0)}{\P j}\right) + \frac{s^2}{2}\frac{\P^2}{\P \mathsf{D}z^2}\left(\frac{\P \mathsf{Y}(\mathsf{D}z|0)}{\P j}\right), \tag{II.36}$$

$$\frac{\P\mathsf{Y}(-\mathsf{D}^-|0)}{\P j} = \frac{\P\mathsf{Y}(\mathsf{D}^+|0)}{\P j} = 0. \tag{II.37}$$

Equations (II.36) and (II.37) correspond to the HJB and border conditions of the function $\frac{\P\mathsf{Y}(\mathsf{D}z|0)}{\P j} = \mathbb{E}\left[\int_0^{t^m} m_{\mathcal{E},h}(\mathsf{D}z_t)\,\mathrm{d}t \middle| \mathsf{D}z_0 = \mathsf{D}z\right]$. Evaluating $\frac{\P\mathsf{Y}(\mathsf{D}z|0)}{\P j}$ at $\mathsf{D}z = 0$, using the occupancy measure and result (II.31), we write the previous equation as:

$$\begin{aligned}
\frac{\P\mathsf{Y}(0|0)}{\P j} &= \mathbb{E}\left[\int_0^{t^m} m_{\mathcal{E},h}(\mathsf{D}z_t)\,\mathrm{d}t \middle| \mathsf{D}z_0 = 0\right] \\
&= \mathbb{E}_{\mathcal{D}}[t^m]\frac{\int_{-¥}^{¥} m_{\mathcal{E},h}(\mathsf{D}z)g^h(\mathsf{D}z)\,\mathrm{d}\mathsf{D}z}{\mathcal{E}_{ss}} \\
&= \mathbb{E}_{\mathcal{D}}[t^m]\left(\frac{\mathcal{E}_{ss}}{f(\hat{q}(\hat{w}^*))} - m_{\mathcal{E},h}(0)\right)\frac{(1-\mathcal{E}_{ss})}{\mathcal{E}_{ss}},
\end{aligned} \tag{II.38}$$

where $\mathbb{E}_{\mathcal{D}}[t^m]$ is the mean duration of completed *wage* spells (the subscript highlights that the moment can be computed from the data).[21] From (II.34), we also have that

$$\frac{\P\mathsf{Y}(0|0)}{\P j} = \mathbb{E}_{\mathcal{D}}\left[t^m\right]\left[(1-\mathcal{E}_{ss})\frac{\mathbb{E}_h[a]}{\mathcal{E}_{ss}} + m_{\mathcal{E},h}(0)\right] - \mathbb{E}\left[t^m \mathbb{1}\{t^m < t^{d^r}\}|\mathsf{D}z_0 = 0\right]\frac{\mathcal{E}_{ss}}{f(\hat{q}(\hat{w}^*))} \tag{II.39}$$

Combining (II.38) and (II.39), and solving for $m_{\mathcal{E},h}(0)$ we obtain:

$$m_{\mathcal{E},h}(0) = \frac{\mathcal{E}_{ss}}{f(\hat{q}(\hat{w}^*))}\left((1-\mathcal{E}_{ss}) + \mathcal{E}_{ss}\frac{\mathbb{E}\left[t^m\mathbb{1}\{t^m < t^{d^r}\}|\mathsf{D}z_0 = 0\right]}{\mathbb{E}_{\mathcal{D}}\left[t^m\right]}\right) - (1-\mathcal{E}_{ss})\mathbb{E}_h[a].$$

Observe that $\mathbb{E}\left[t^m\mathbb{1}\{t^m < t^{d^r}\}|\mathsf{D}z_0 = 0\right]$ is equal to $\mathbb{E}_{\mathcal{D}}\left[t^m\mathbb{1}\{t^m < t^{d^r}\}\right]$—i.e., the average duration of wage spells that ended in a job separation.

Step 3. Up to a first-order approximation, the CIR is given by:

$$\begin{aligned}
\mathrm{CIR}_{\mathcal{E}}(z) = &-(1-\mathcal{E}_{ss})\frac{[(g+c)\mathbb{E}_h[a] + \mathbb{E}_h[\mathsf{D}z]]}{s^2}z \\
&- \frac{\mathcal{E}_{ss}}{s^2 f(\hat{q}(\hat{w}^*))}\left((g+c)\mathcal{E}_{ss}\left(1 - \frac{\mathbb{E}_{\mathcal{D}}\left[t^m\mathbb{1}\{t^m < t^{d^r}\}\right]}{\mathbb{E}_{\mathcal{D}}\left[t^m\right]}\right) + \mathbb{E}_h[\mathsf{D}z]\mathcal{F}^{\mathsf{D}w}\right)z + O(z^2).
\end{aligned}$$

Proof of Step 3. To help the reader, we summarize below the conditions used in this step of the proof.

$$\mathrm{CIR}_{\mathcal{E}}(z) = \int_{-¥}^{¥} m_{\mathcal{E},h}(\mathsf{D}z)g^h(\mathsf{D}z + z)\,\mathrm{d}\mathsf{D}z + \left(-\frac{\mathcal{E}_{ss}}{f(\hat{q}(\hat{w}^*))} + m_{\mathcal{E},h}(0)\right)(1-\mathcal{E}_{ss}) \tag{II.40}$$

[21] A completed wage spell starts when the worker earns a new wage (i.e., finding a new job or renegotiating the wage with the current employer) and ends either when the match dissolves or the wage is renegotiated.

with

$$(d + d^r)\, m_{\mathcal{E},h}(\mathsf{D}z) = 1 - \mathcal{E}_{ss} - (g + c)\frac{\mathrm{d}m_{\mathcal{E},h}(\mathsf{D}z)}{\mathrm{d}\mathsf{D}z} + \frac{s^2}{2}\frac{d^2 m_{\mathcal{E},h}(\mathsf{D}z)}{\mathrm{d}\mathsf{D}z^2} + d m_{\mathcal{E},u}(0) + d^r m_{\mathcal{E},h}(0), \tag{II.41}$$

$$m_{\mathcal{E},u}(0) = m_{\mathcal{E},h}(\mathsf{D}z)\ \forall \mathsf{D}z \notin (-\mathsf{D}^-, \mathsf{D}^+)$$

$$0 = \int_{-\yen}^{\yen} m_{\mathcal{E},h}(\mathsf{D}z) g^h(\mathsf{D}z)\, \mathrm{d}\mathsf{D}z + m_{\mathcal{E},u}(0)(1 - \mathcal{E}_{ss}). \tag{II.42}$$

1. **Zeroth Order:** If $z = 0$, condition (II.42) implies

$$\mathrm{CIR}_{\mathcal{E}}(0) = \int_{-\yen}^{\yen} m_{\mathcal{E},h}(\mathsf{D}z) g^h(\mathsf{D}z)\, \mathrm{d}\mathsf{D}z + \left(-\frac{\mathcal{E}_{ss}}{f(\hat{q}(\hat{w}^*))} + m_{\mathcal{E},h}(0)\right)(1 - \mathcal{E}_{ss}) = 0.$$

2. **First Order:** Taking the derivative of (II.40) we obtain

$$\mathrm{CIR}'_{\mathcal{E}}(z) = \int_{-\yen}^{\yen} m_{\mathcal{E},h}(\mathsf{D}z)(g^h)'(\mathsf{D}z + z)\, \mathrm{d}\mathsf{D}z,$$

which evaluated at $z = 0$ becomes

$$\mathrm{CIR}'_{\mathcal{E}}(0) = \int_{-\mathsf{D}^-}^{\mathsf{D}^+} m_{\mathcal{E},h}(\mathsf{D}z)(g^h)'(\mathsf{D}z)\, \mathrm{d}\mathsf{D}z.$$

Using condition (II.21) to replace $(d + d^r) = \frac{(g+c)(g^h)'(\mathsf{D}z) + \frac{s^2}{2}(g^h)''(\mathsf{D}z)}{g^h(\mathsf{D}z)}$ into equation (II.41), we obtain

$$\begin{aligned}
&\frac{(g + c)(g^h)'(\mathsf{D}z) + \frac{s^2}{2}(g^h)''(\mathsf{D}z)}{g^h(\mathsf{D}z)} m_{\mathcal{E},h}(\mathsf{D}z) \\
=&1 - \mathcal{E}_{ss} - (g + c)m'_{\mathcal{E},h}(\mathsf{D}z) + \frac{s^2}{2}m''_{\mathcal{E},h}(\mathsf{D}z) + \left(\frac{(g + c)g'(\mathsf{D}z) + \frac{s^2}{2}g''(\mathsf{D}z)}{g(\mathsf{D}z)} - d^r\right) m_{\mathcal{E},u}(0) \\
&+ d^r m_{\mathcal{E},h}(0).
\end{aligned}$$

Multiplying both sides by $g^h(\mathsf{D}z)\mathsf{D}z$ and integrating between $-\mathsf{D}^-$ and D^+,

$$0 = (1 - \mathcal{E}_{ss})\mathbb{E}_h[\mathsf{D}z] - (g + c)T_1 + \frac{s^2}{2}T_2 + m_{\mathcal{E},u}(0)T_3 + d^r\,(m_{\mathcal{E},h}(0) - m_{\mathcal{E},u}(0))\,\mathbb{E}_h[\mathsf{D}z] \tag{II.43}$$

$$T_1 = \int_{-\mathsf{D}^-}^{\mathsf{D}^+} \mathsf{D}z \left[m'_{\mathcal{E},h}(\mathsf{D}z)g^h(\mathsf{D}z) + m_{\mathcal{E},h}(\mathsf{D}z)(g^h)'(\mathsf{D}z)\right] \mathrm{d}\mathsf{D}z$$

$$T_2 = \int_{-\mathsf{D}^-}^{\mathsf{D}^+} \mathsf{D}z \left[m''_{\mathcal{E},h}(\mathsf{D}z)g^h(\mathsf{D}z) - m_{\mathcal{E},h}(\mathsf{D}z)(g^h)''(\mathsf{D}z)\right] \mathrm{d}\mathsf{D}z$$

$$T_3 = \int_{-\mathsf{D}^-}^{\mathsf{D}^+} \mathsf{D}z \left((g+c)(g^h)'(\mathsf{D}z) + \frac{s^2}{2}(g^h)''(\mathsf{D}z) \right) \mathrm{d}\mathsf{D}z.$$

Operating on the terms T_1, T_2, and T_3, we get

$$T_1 = m_{\mathcal{E},u}(0)(1-\mathcal{E}_{ss}), \tag{II.44}$$

$$T_2 = -m_{\mathcal{E},u}(0)\ \mathsf{D}z(g^h)'(\mathsf{D}z)\Big|_{\mathsf{D}^-}^{\mathsf{D}^+} + 2\int_{\mathsf{D}^-}^{\mathsf{D}^+} m_{\mathcal{E},h}(\mathsf{D}z)g'(\mathsf{D}z)\,\mathrm{d}\mathsf{D}z, \tag{II.45}$$

$$T_3 = -(g+c)\mathcal{E}_{ss} + \frac{s^2}{2}\left[\mathsf{D}z(g^h)'(\mathsf{D}z)\Big|_{\mathsf{D}^-}^{\mathsf{D}^+}\right]. \tag{II.46}$$

Combining (II.43), (II.44), (II.45), (II.46), and the results from Step 2, we obtain

$$\begin{aligned}
0 &= (1-\mathcal{E}_{ss})\mathbb{E}_h[\mathsf{D}z] - (g+c)T_1 + \frac{s^2}{2}T_2 + m_{\mathcal{E},u}(0)T_3 + d^r\left(m_{\mathcal{E},h}(0) - m_{\mathcal{E},u}(0)\right)\mathbb{E}_h[\mathsf{D}z] \\
&= (1-\mathcal{E}_{ss})\mathbb{E}_h[\mathsf{D}z] - (g+c)m_{\mathcal{E},u}(0)(1-\mathcal{E}_{ss}) \\
&\quad + \frac{s^2}{2}\left[-m_{\mathcal{E},u}(0)\ \mathsf{D}z(g^h)'(\mathsf{D}z)\Big|_{-\mathsf{D}^-}^{\mathsf{D}^+} + 2\int_{-\mathsf{D}^-}^{\mathsf{D}^+} m_{\mathcal{E},h}(\mathsf{D}z)(g^h)'(\mathsf{D}z)\,\mathrm{d}\mathsf{D}z\right] + \dots \\
&\quad + m_{\mathcal{E},u}(0)\left[-(g+c)\mathcal{E}_{ss} + \frac{s^2}{2}\left[\mathsf{D}z(g^h)'(\mathsf{D}z)\Big|_{-\mathsf{D}^-}^{\mathsf{D}^+}\right]\right] + d^r\frac{\mathcal{E}_{ss}}{f(\hat{q}(\hat{w}^*))}\mathbb{E}_h[\mathsf{D}z] \\
&= (1-\mathcal{E}_{ss})\mathbb{E}_h[\mathsf{D}z] - (g+c)m_{\mathcal{E},u}(0) + s^2\int_{-\mathsf{D}^-}^{\mathsf{D}^+} m_{\mathcal{E},h}(\mathsf{D}z)(g^h)'(\mathsf{D}z)\,\mathrm{d}\mathsf{D}z + d^r\frac{\mathcal{E}_{ss}}{f(\hat{q}(\hat{w}^*))}\mathbb{E}_h[\mathsf{D}z],
\end{aligned}$$

which implies

$$\begin{aligned}
& s^2\int_{-\mathsf{D}^-}^{\mathsf{D}^+} m_{\mathcal{E},h}(\mathsf{D}z)(g^h)'(\mathsf{D}z)\,\mathrm{d}\mathsf{D}z \\
=&(g+c)\left(-\frac{\mathcal{E}_{ss}}{f(\hat{q}(\hat{w}^*))} + \frac{\mathcal{E}_{ss}}{f(\hat{q}(\hat{w}^*))}\left((1-\mathcal{E}_{ss}) + \mathcal{E}_{ss}\frac{\mathbb{E}_{\mathcal{D}}\left[t^m\mathbb{1}\{t^m < t^{d^r}\}\right]}{\mathbb{E}_{\mathcal{D}}\left[t^m\right]}\right) - (1-\mathcal{E}_{ss})\mathbb{E}_h[a]\right) \\
& - (1-\mathcal{E}_{ss})\mathbb{E}_h[\mathsf{D}z] - d^r\frac{\mathcal{E}_{ss}}{f(\hat{q}(\hat{w}^*))}\mathbb{E}_h[\mathsf{D}z],
\end{aligned}$$

and

$$\begin{aligned}
&\int_{-\mathsf{D}^-}^{\mathsf{D}^+} m_{\mathcal{E},h}(\mathsf{D}z)(g^h)'(\mathsf{D}z)\,\mathrm{d}\mathsf{D}z \\
&= -(1-\mathcal{E}_{ss})\frac{[(g+c)\mathbb{E}_h[a] + \mathbb{E}_h[\mathsf{D}z]]}{s^2}
\end{aligned}$$

$$-\frac{\mathcal{E}_{ss}}{s^2 f(\hat{q}(\hat{w}^*))}\left((g+c)\mathcal{E}_{ss}\left(1-\frac{\mathbb{E}_{\mathcal{D}}\left[t^m \mathbb{1}\{t^m < t^{d^r}\}\right]}{\mathbb{E}_{\mathcal{D}}\left[t^m\right]}\right)+\mathbb{E}_h[\mathrm{D}z]d^r\right).$$

Finally, since the probability of wage renegotiation is independent of the state of the match, we have

$$\mathcal{F}^{\mathrm{D}w}\,\mathrm{d}t = \int_{-\mathrm{D}^-}^{\mathrm{D}^+} Pr\,(\text{bargaining in } [t, t+\mathrm{d}t)|\mathrm{D}z)\,\frac{g(\mathrm{D}z)}{\mathcal{E}_{ss}}\,\mathrm{d}\mathrm{D}z = d^r\,\mathrm{d}t$$

and, therefore,

$$\mathcal{F}^{\mathrm{D}w} = d^r.$$

□

Next, we write the CIR as a function of observable moments. Let $\mathrm{D}w_B$ denote the log wage change following a wage renegotiation and let $I^B(\mathrm{D}w)$ denote its distribution. In addition, let $I^{EUE}(\mathrm{D}w)$ be the distribution of wage changes following a separation (i.e., wage changes between two consecutive jobs). Our objective is to recover $\mathbb{E}_h[a]$ and $\mathbb{E}_h[\mathrm{D}z]$ from observable micro-data on wage changes. To simplify the discussion, from now on we focus on the case with $g+c=0$. Under this parametric restriction, we only need to recover one moment: $\mathbb{E}_h[\mathrm{D}z]$. The CIR of employment is given by:

$$\begin{aligned}\frac{\mathrm{CIR}_{\mathcal{E}}(z)}{z} &= -(1-\mathcal{E}_{ss})\frac{[\mathbb{E}_h[\mathrm{D}z]]}{s^2} - \frac{\mathcal{E}_{ss}}{s^2 f(\hat{q}(\hat{w}^*))}\left(\mathbb{E}_h[\mathrm{D}z]\mathcal{F}^{\mathrm{D}w}\right) + o(z), \\ &= -\frac{\mathbb{E}_h[\mathrm{D}z]}{s^2}\left(1-\mathcal{E}_{ss}+\mathcal{E}_{ss}\frac{\mathcal{F}^{\mathrm{D}w}}{f(\hat{q}(\hat{w}^*))}\right).\end{aligned}$$

Two moments are informative of $\mathbb{E}_h[\mathrm{D}z]$. The first one is obtained from the distribution of wage changes within a match $\mathrm{D}\hat{w}_B$. Under our assumptions that the bargaining process satisfies the Hosios (1990) condition and the renegotiation hazard is independent of the idiosyncratic state, we have that

$$\frac{\mathbb{E}_h[\mathrm{D}z]}{\mathcal{E}_{ss}} = -\mathbb{E}_{\mathcal{D}}[\mathrm{D}w_B].$$

In addition, the next proposition provides an alternative expression that remains valid when these assumptions do not hold. Importantly, it is easy to show that the following proposition still holds when the hazard rate for renegotiation is state-dependent.

Proposition II.7. *Assume that $g+c=0$. Up to first order, the $CIR_{\mathcal{E}}(z)$ can be expressed in terms of data* *moments as follows:*

$$\frac{CIR_{\mathcal{E}}(z)}{z} = \frac{1}{3f(\hat{q}(\hat{w}^*))} \frac{\left(1-\frac{\mathcal{F}^{\mathsf{D}w}}{s+\mathcal{F}^{\mathsf{D}w}}\right)\mathbb{E}_{\mathcal{D}}\left[\mathsf{D}w_{EUE}^3\right] + \frac{\mathcal{F}^{\mathsf{D}w}}{s+\mathcal{F}^{\mathsf{D}w}}\mathbb{E}_{\mathcal{D}}[\mathsf{D}w_B^3]}{\left[\left(1-\frac{\mathcal{F}^{\mathsf{D}w}}{s+\mathcal{F}^{\mathsf{D}w}}\right)\mathbb{E}_{\mathcal{D}}\left[\mathsf{D}w_{EUE}^2\right] + \frac{\mathcal{F}^{\mathsf{D}w}}{s+\mathcal{F}^{\mathsf{D}w}}\mathbb{E}_{\mathcal{D}}[\mathsf{D}w_B^2]\right]^2} + o(z).$$

Proof. The goal is to express the sufficient statistics of the CIR, $\mathbb{E}_h[\mathsf{D}z]$, in terms of moments of the distribution of $\mathsf{D}w^{EUE}$, $\mathsf{D}w^B$, and (t^u, t^m) when $(g+c)=0$. Let $\tilde{x} \equiv x/\mathbb{E}_{\mathcal{D}}[x]$ denote random variable x relative to its mean in the data.

Our starting point is the KFE for the distribution of employed workers:

$$\left(d+\mathcal{F}^{\mathsf{D}w}\right)g_h(\mathsf{D}z) = \frac{s^2}{2}g_h''(\mathsf{D}z),$$

where we used the result that $\mathcal{F}^{\mathsf{D}w} = d^r$. Since the arrival rate of bargaining opportunities is independent of the state of the match, we have that $\mathcal{F}^{\mathsf{D}w}g_h(\mathsf{D}z) = \mathcal{F}^{\mathsf{D}w}\mathcal{E}_{ss}l^B(-\mathsf{D}z)$. Thus,

$$dg_h(\mathsf{D}z) + \mathcal{F}^{\mathsf{D}w}\mathcal{E}_{ss}l^B(-\mathsf{D}z) = \frac{s^2}{2}g_h''(\mathsf{D}z).$$

Multiplying both sides of the equation by $\mathsf{D}z^2$ and integrating, we

$$d\int_{-\mathsf{D}^-}^{\mathsf{D}^+}\mathsf{D}z^2 g_h(\mathsf{D}z)\,\mathrm{d}\mathsf{D}z + \mathcal{F}^{\mathsf{D}w}\mathcal{E}_{ss}\int_{-\mathsf{D}^-}^{\mathsf{D}^+}\mathsf{D}z^2 l^B(-\mathsf{D}z)\,\mathrm{d}\mathsf{D}z = \frac{s^2}{2}\int_{-\mathsf{D}^-}^{\mathsf{D}^+}\mathsf{D}z^2 g_h''(\mathsf{D}z)\,\mathrm{d}\mathsf{D}z.$$

Notice that $\mathcal{E}_{ss}\int_{-\mathsf{D}^-}^{\mathsf{D}^+}\mathsf{D}z^2 l^B(-\mathsf{D}z)\,\mathrm{d}\mathsf{D}z = \mathcal{E}_{ss}\mathbb{E}_{\mathcal{D}}[\mathsf{D}w_B^2]$. Integrating $\int_{-\mathsf{D}^-}^{\mathsf{D}^+}\mathsf{D}z^2 g_h''(\mathsf{D}z)\,\mathrm{d}\mathsf{D}z$ by parts, we have

$$\begin{aligned}\int_{-\mathsf{D}^-}^{\mathsf{D}^+}\mathsf{D}z^2 g_h''(\mathsf{D}z)\,\mathrm{d}\mathsf{D}z &= \mathsf{D}z^2 g_h'(\mathsf{D}z)\big|_{-\mathsf{D}^-}^{\mathsf{D}^+} - 2\int_{-\mathsf{D}^-}^{\mathsf{D}^+}\mathsf{D}z g_h'(\mathsf{D}z)\,\mathrm{d}\mathsf{D}z \\ &= \mathsf{D}z^2 g_h'(\mathsf{D}z)\big|_{-\mathsf{D}^-}^{\mathsf{D}^+} - 2\underbrace{\mathsf{D}z g_h(\mathsf{D}z)\big|_{-\mathsf{D}^-}^{\mathsf{D}^+}}_{=0} + 2\underbrace{\int_{-\mathsf{D}^-}^{\mathsf{D}^+} g_h(\mathsf{D}z)\,\mathrm{d}\mathsf{D}z}_{=\mathcal{E}_{ss}}\end{aligned}$$

Operating, we have that

$$\begin{aligned}d\int_{-\mathsf{D}^-}^{\mathsf{D}^+}\mathsf{D}z^2 g_h(\mathsf{D}z)\,\mathrm{d}\mathsf{D}z - \frac{s^2}{2}\,\mathsf{D}z^2 g_h'(\mathsf{D}z)\big|_{-\mathsf{D}^-}^{\mathsf{D}^+} + \mathcal{E}_{ss}\mathcal{F}^{\mathsf{D}w}\mathbb{E}_{\mathcal{D}}[\mathsf{D}w_B^2] = s^2\mathcal{E}_{ss} \iff \\ s\mathcal{E}_{ss}\bar{\mathbb{E}}_h[\mathsf{D}z^2] + \mathcal{E}_{ss}\mathcal{F}^{\mathsf{D}w}\mathbb{E}_{\mathcal{D}}[\mathsf{D}w_B^2] = s^2\mathcal{E}_{ss}, \iff\end{aligned}$$

$$\bar{\mathbb{E}}_h[\mathsf{D}z^2] + \frac{\mathcal{F}^{\mathsf{D}w}}{s}\mathbb{E}_{\mathcal{D}}[\mathsf{D}w_B^2] = \frac{s^2}{s}$$

Following Proposition B.3 when $g + c = 0$, $\mathbb{E}_{\mathcal{D}}\left[\mathsf{D}w_{EUE}^2\right] = \left[\bar{\mathbb{E}}_u\left[\mathsf{D}z^2\right] + 2\bar{\mathbb{E}}_h\left[\mathsf{D}z\right]\bar{\mathbb{E}}_u\left[\mathsf{D}z\right] + \bar{\mathbb{E}}_h\left[\mathsf{D}z^2\right]\right] = \bar{\mathbb{E}}_u\left[\mathsf{D}z^2\right] + \bar{\mathbb{E}}_h\left[\mathsf{D}z^2\right]$ with $\bar{\mathbb{E}}_u\left[\mathsf{D}z^2\right] = \frac{s^2}{f(\hat{q}(\hat{w}^*))}$. Thus,

$$\mathbb{E}_{\mathcal{D}}\left[\mathsf{D}w_{EUE}^2\right] = \bar{\mathbb{E}}_h\left[\mathsf{D}z^2\right] + \frac{s^2}{f(\hat{q}(\hat{w}^*))}.$$

Combining the previous two results, we have that

$$\mathbb{E}_{\mathcal{D}}\left[\mathsf{D}w_{EUE}^2\right] + \frac{\mathcal{F}^{\mathsf{D}w}}{s}\mathbb{E}_{\mathcal{D}}[\mathsf{D}w_B^2] = \frac{s^2}{s} + \frac{s^2}{f(\hat{q}(\hat{w}^*))},$$

or equivalently

$$s^2 = \frac{s\mathbb{E}_{\mathcal{D}}\left[\mathsf{D}w_{EUE}^2\right] + \mathcal{F}^{\mathsf{D}w}\mathbb{E}_{\mathcal{D}}[\mathsf{D}w_B^2]}{1 + \frac{s}{f(\hat{q}(\hat{w}^*))}}.$$

With this expression, we are ready to compute $\mathbb{E}_h[\mathsf{D}z]$. Repeating the same steps as before but with $\mathsf{D}z^3$, we have that

$$s\mathcal{E}_{ss}\bar{\mathbb{E}}_h[\mathsf{D}z^3] - \mathcal{F}^{\mathsf{D}w}\mathcal{E}_{ss}\mathbb{E}_{\mathcal{D}}[\mathsf{D}w_B^3] = \frac{s^2}{2}6\mathbb{E}_h[\mathsf{D}z]$$

Following similar steps as in Proposition B.3, we have that

$$\bar{\mathbb{E}}_h\left[\mathsf{D}z^3\right] = -\mathbb{E}_{\mathcal{D}}\left[\mathsf{D}w_{EUE}^3\right]$$

Thus,

$$\mathbb{E}_h[\mathsf{D}z] = -\mathcal{E}_{ss}\frac{s\mathbb{E}_{\mathcal{D}}\left[\mathsf{D}w_{EUE}^3\right] + \mathcal{F}^{\mathsf{D}w}\mathbb{E}_{\mathcal{D}}[\mathsf{D}w_B^3]}{3s^2}.$$

Combining this expression with the one for s^2, we have that

$$\begin{aligned}\frac{\mathbb{E}_h[\mathsf{D}z]}{s^2} &= -\mathcal{E}_{ss}\frac{s\mathbb{E}_{\mathcal{D}}\left[\mathsf{D}w_{EUE}^3\right] + \mathcal{F}^{\mathsf{D}w}\mathbb{E}_{\mathcal{D}}[\mathsf{D}w_B^3]}{3(s^2)^2} \\ &= -\frac{\mathcal{E}_{ss}}{3}\left(1 + \frac{s}{f(\hat{q}(\hat{w}^*))}\right)^2\frac{s\mathbb{E}_{\mathcal{D}}\left[\mathsf{D}w_{EUE}^3\right] + \mathcal{F}^{\mathsf{D}w}\mathbb{E}_{\mathcal{D}}[\mathsf{D}w_B^3]}{\left[s\mathbb{E}_{\mathcal{D}}\left[\mathsf{D}w_{EUE}^2\right] + \mathcal{F}^{\mathsf{D}w}\mathbb{E}_{\mathcal{D}}[\mathsf{D}w_B^2]\right]^2}.\end{aligned}$$

Therefore, the CIR for employment is given by

$$\frac{\mathrm{CIR}_{\mathcal{E}}(z)}{z} = -\frac{\mathbb{E}_h[\mathsf{D}z]}{s^2}\left(1 - \mathcal{E}_{ss} + \mathcal{E}_{ss}\frac{\mathcal{F}^{\mathsf{D}w}}{f(\hat{q}(\hat{w}^*))}\right) + o(z)$$

$$= \frac{1}{3f(\hat{q}(\hat{w}^*))} \frac{\left(1 - \frac{\mathcal{F}^{\mathrm{D}w}}{s+\mathcal{F}^{\mathrm{D}w}}\right) \mathbb{E}_{\mathcal{D}}\left[\mathrm{D}w^3_{EUE}\right] + \frac{\mathcal{F}^{\mathrm{D}w}}{s+\mathcal{F}^{\mathrm{D}w}} \mathbb{E}_{\mathcal{D}}[\mathrm{D}w^3_B]}{\left[\left(1 - \frac{\mathcal{F}^{\mathrm{D}w}}{s+\mathcal{F}^{\mathrm{D}w}}\right) \mathbb{E}_{\mathcal{D}}\left[\mathrm{D}w^2_{EUE}\right] + \frac{\mathcal{F}^{\mathrm{D}w}}{s+\mathcal{F}^{\mathrm{D}w}} \mathbb{E}_{\mathcal{D}}[\mathrm{D}w^2_B]\right]^2} + o(z).$$

□

Discussion of Employment Dynamics With Flexible Entry Wages. How do infrequent wage adjustments within the match affect the business cycle dynamics of employment when entry wages are flexible? The CIR of employment provides valuable theoretical insight to answer this question. Intuitively, with all other parameters in the model held constant, a higher frequency of wage renegotiation decreases the CIR of employment: Some workers will be able to reset their wages before transitioning into unemployment. The theory presented shows how the CIR of employment is affected by on-the-job renegotiation, which is now given by

$$\frac{\mathrm{CIR}_{\mathcal{E}}(z)}{z} = \frac{1}{3f(\hat{q}(\hat{w}^*))} \frac{\left(1 - \frac{\mathcal{F}^{\mathrm{D}w}}{s+\mathcal{F}^{\mathrm{D}w}}\right) \mathbb{E}_{\mathcal{D}}\left[\mathrm{D}w^3_{EUE}\right] + \frac{\mathcal{F}^{\mathrm{D}w}}{s+\mathcal{F}^{\mathrm{D}w}} \bar{\mathbb{E}}[\mathrm{D}w^3_B]}{\left[\left(1 - \frac{\mathcal{F}^{\mathrm{D}w}}{s+\mathcal{F}^{\mathrm{D}w}}\right) \mathbb{E}_{\mathcal{D}}\left[\mathrm{D}w^2_{EUE}\right] + \frac{\mathcal{F}^{\mathrm{D}w}}{s+\mathcal{F}^{\mathrm{D}w}} \mathbb{E}[\mathrm{D}w^2_B]\right]^2} + o(z).$$

Notice that, without wage bargaining (i.e., $d^r = \mathcal{F}^{\mathrm{D}w} = 0$), we recover the expression in Proposition 6 for the baseline case:

$$\frac{\mathrm{CIR}_{\mathcal{E}}(z)}{z} = \frac{1}{3f(\hat{q}(\hat{w}^*))} \frac{\mathbb{E}_{\mathcal{D}}\left[\mathrm{D}w^3_{EUE}\right]}{\mathbb{E}_{\mathcal{D}}\left[\mathrm{D}w^2_{EUE}\right]^2} + o(z).$$

A key result from this analysis is that the relevant micro-moments in the labor market are the second and third moments of the distributions of wage changes within and across jobs, alongside the probabilities of wage renegotiation and job-finding. The intuition for the relevance of the job-finding rate is the same as in the baseline model. The reason the weighted sum of moments of wage changes within and across jobs appear in the CIR is the simple application of Bayes' law:

$$\frac{\mathbb{E}[\mathrm{D}w^3]}{\mathbb{E}[\mathrm{D}w^2]^2} = \frac{\mathbb{E}[\mathrm{D}w^3|EUE]Pr(EUE) + \mathbb{E}[\mathrm{D}w^3|bargaining]Pr(bargaining)}{(\mathbb{E}[\mathrm{D}w^2|EUE]Pr(EUE) + \mathbb{E}[\mathrm{D}w^2|bargaining]Pr(bargaining))^2},$$

where $Pr(EUE) + Pr(bargaining) = 1$. Thus, the CIR of employment following a TFPR shock is fully captured by the moments of the distribution of $\mathrm{D}z$, which can be recovered with micro-data on wage changes workers experience within and across jobs.

The CIR of Employment with Sticky Entry Wages and Wage Renegotiations.

As in the previous subsection, we focus on the case with no drift; i.e., $g + c = 0$. Furthermore, for pedagogical purposes, we consider the case with symmetric separation thresholds $\mathsf{D}^- = \mathsf{D}^+$.

Proposition II.8. *Assume sticky entry wages. Up to first order, the CIR of employment is given by*

$$\frac{CIR_{\mathcal{E}}(z)}{z} = \frac{1}{f(\hat{q}(\hat{w}^*)) + s} \frac{h'(\hat{w}^*)}{h(\hat{w}^*)} + o(z), \tag{II.47}$$

where

1. *If* $\mathsf{D}^+ \to$ ¥*, then*

$$\left.\frac{\mathrm{dlog}(h(\hat{w}))}{\mathrm{d}\hat{w}}\right|_{\hat{w}=\hat{w}^*} = \frac{\hat{r} + d}{\hat{r} + d + \mathcal{N}^{\mathsf{D}w}} \frac{\left[a + (1-a)\hat{r}\hat{U}\right]}{a(1-\hat{r}\hat{U})}.$$

2. *If* $d^r \to$ ¥*, then*

$$\left.\frac{\mathrm{dlog}(h(\hat{w}))}{\mathrm{d}\hat{w}}\right|_{\hat{w}=\hat{w}^*} = 0.$$

3. *If* D^+ *small enough, then*

$$\left.\frac{\mathrm{dlog}(h(\hat{w}))}{\mathrm{d}\hat{w}}\right|_{\hat{w}=\hat{w}^*} = \frac{\sqrt{s^{end}}}{2as}.$$

Proof. We divide the proof into three steps. Step 1 characterizes $m_{\mathcal{E},u}(\mathsf{D}z, z)$. Steps 2 uses the equilibrium conditions to show (II.47). Step 3 extends proposition B.5 for the case with on-the-job bargaining.

The starting point is the CIR for employment, which is given by

$$\mathrm{CIR}_{\mathcal{E}}(z) = \int_{-¥}^{¥} m_{\mathcal{E},h}(\mathsf{D}z) g^h(\mathsf{D}z + z)\,\mathrm{d}\mathsf{D}z + \int_{-¥}^{¥} m_{\mathcal{E},u}(\mathsf{D}z, z) g^u(\mathsf{D}z + z)\,\mathrm{d}\mathsf{D}z, \tag{II.48}$$

with

$$0 = 1 - \mathcal{E}_{ss} + \frac{s^2}{2}\frac{d^2 m_{\mathcal{E},h}(\mathsf{D}z)}{\mathsf{d}\mathsf{D}z^2} + d(m_{\mathcal{E},u}(0,0) - m_{\mathcal{E},h}(\mathsf{D}z))$$

$$+ d^r(m_{\mathcal{E},h}(0) - m_{\mathcal{E},h}(\mathsf{D}z)),\ \forall \mathsf{D}z \in (-\mathsf{D}^+, \mathsf{D}^+)$$

$$0 = -\mathcal{E}_{ss} + \frac{s^2}{2}\frac{d^2 m_{\mathcal{E},u}(\mathsf{D}z, z)}{\mathsf{d}\mathsf{D}z^2} + f(\hat{q}(\hat{w}^* - z))(m_{\mathcal{E},h}(-z) - m_{\mathcal{E},u}(\mathsf{D}z, z)) \tag{II.49}$$

$$m_{\mathcal{E},u}(0,0) = m_{\mathcal{E},h}(\mathsf{D}z),\ \forall \mathsf{D}z \notin (-\mathsf{D}^+, \mathsf{D}^+) \tag{II.50}$$

$$0 = \lim_{\mathsf{D}z \to -¥} \frac{\mathsf{d}m_{\mathcal{E},u}(\mathsf{D}z, z)}{\mathsf{d}\mathsf{D}z} = \lim_{\mathsf{D}z \to ¥} \frac{\mathsf{d}m_{\mathcal{E},u}(\mathsf{D}z, z)}{\mathsf{d}\mathsf{D}z} \tag{II.51}$$

$$0 = \int_{-¥}^{¥} m_{\mathcal{E},h}(\mathsf{D}z) g^h(\mathsf{D}z)\,\mathsf{d}\mathsf{D}z + \int_{-¥}^{¥} m_{\mathcal{E},u}(\mathsf{D}z) g^u(\mathsf{D}z)\,\mathsf{d}\mathsf{D}z \tag{II.52}$$

Step 1. The value function $m_{\mathcal{E},u}(\mathsf{D}z, z)$ is independent of $\mathsf{D}z$ and satisfies

$$m_{\mathcal{E},u}(\mathsf{D}z, z) = -\frac{\mathcal{E}_{ss}}{f(\hat{q}(\hat{w}^* - z))} + m_{\mathcal{E},h}(-z).$$

Proof of Step 1. We guess and verify that $m_{\mathcal{E},u}(\mathsf{D}z, z) = m_{\mathcal{E},u}(0, z)$ for all $\mathsf{D}z$. From the equilibrium conditions (II.49) and (II.51),

$$0 = -\mathcal{E}_{ss} + f(\hat{q}(\hat{w}^* - z))(m_{\mathcal{E},h}(-z) - m_{\mathcal{E},u}(0, z)).$$

Thus,

$$m_{\mathcal{E},u}(0, z) = m_{\mathcal{E},u}(\mathsf{D}z, z) = -\frac{\mathcal{E}_{ss}}{f(\hat{q}(\hat{w}^* - z))} + m_{\mathcal{E},h}(-z).$$

Step 2. Up to a first-order approximation, the CIR is given by:

$$\mathrm{CIR}_{\mathcal{E}}(z) = \frac{1}{f(\hat{q}(\hat{w}^*)) + s} \frac{h'(\hat{w}^*)}{h(\hat{w}^*)} z + O(z^2).$$

Proof of Step 2. From Step 1, we have that

$$\mathrm{CIR}'_{\mathcal{E}}(0) = \int_{-¥}^{¥} m_{\mathcal{E},h}(\mathsf{D}z)(g^h)'(\mathsf{D}z)\, \mathrm{d}\mathsf{D}z + \left(-\frac{\mathcal{E}_{ss} f_{\hat{w}}(\hat{q}(\hat{w}^*))}{f(\hat{q}(\hat{w}^*))^2} - m'_{\mathcal{E},h}(0) \right) (1 - \mathcal{E}_{ss}).$$

Since $\int_{-¥}^{¥} m_{\mathcal{E},h}(\mathsf{D}z)(g^h)'(\mathsf{D}z)\, \mathrm{d}\mathsf{D}z$ satisfies the same system of functional equations as the CIR of employment with flexible entry wages and wage renegotiations characterized in Online Appendix II.3,

$$\int_{-¥}^{¥} m_{\mathcal{E},h}(\mathsf{D}z)(g^h)'(\mathsf{D}z)\, \mathrm{d}\mathsf{D}z = -\frac{\mathbb{E}_h[\mathsf{D}z]}{s^2} \left(1 - \mathcal{E}_{ss} + \mathcal{E}_{ss} \frac{\mathcal{F}^{\mathsf{D}w}}{f(\hat{q}(\hat{w}^*))} \right).$$

By the symmetry of separation thresholds, we have that

$$\int_{-¥}^{¥} m_{\mathcal{E},h}(\mathsf{D}z)(g^h)'(\mathsf{D}z)\, \mathrm{d}\mathsf{D}z = 0.$$

and $m'_{\mathcal{E},h}(0) = 0$. Thus,

$$\mathrm{CIR}'_{\mathcal{E}}(0) = -\frac{\mathcal{E}_{ss} f_{\hat{w}}(\hat{q}(\hat{w}^*))}{f(\hat{q}(\hat{w}^*))^2}(1 - \mathcal{E}_{ss})$$

Since under symmetry $h(\hat{w}^*) = a$ and $\mathcal{T}_{\hat{w}}(\hat{w}^*, \hat{r} + d^r) = 0$, from free entry and optimality of $\hat{w}^*$,

$$\frac{f_{\hat{w}}(\hat{q}(\hat{w}^*))}{f(\hat{q}(\hat{w}^*))} = -\frac{h'(\hat{w}^*)}{h(\hat{w}^*)}.$$

Thus,

$$\mathrm{CIR}'_{\mathcal{E}}(0) = \frac{1}{f(\hat{q}(\hat{w}^*)) + s} \frac{h'(\hat{w}^*)}{h(\hat{w}^*)}.$$

Step 3. Define

$$t^{end} = \inf\{t \geq 0 : \mathsf{G}_t \notin (\hat{w}^-, \hat{w}^+)\}$$

where $(\hat{w}^-, \hat{w}^+)$ is a Nash equilibrium. Then, the worker's share $h(\hat{w})$ satisfies the Bellman equation

$$h(\hat{w}) = \mathbb{E}\Bigg[\int_0^{t^{end}} e^{-(\hat{r}+d+d^r)t} (\hat{r} + d + d^r) \left(\frac{e^{\mathsf{G}_t} - 1}{1 - \hat{r}\hat{U}} (1 - d^r \mathcal{T}(\hat{w}^*, \hat{r} + d^r)) + 1 \right) \mathrm{d}t \\ + e^{-(\hat{r}+d+d^r)t^{end}} \mathbb{1}[\mathsf{D}z_{t^{end}} = \mathsf{D}^+] | \mathsf{G}_0 = \hat{w} \Bigg]$$

with

$$\mathrm{d}\mathsf{G}_t = (\hat{r} + d + d^r) s^2 \mathcal{T}'_{\hat{w}}(\mathsf{G}_t, \hat{r} + d^r)\, \mathrm{d}t + s\sqrt{\mathcal{T}(\mathsf{G}_t, \hat{r} + d^r)(\hat{r} + d + d^r)}\, \mathrm{d}\mathcal{W}^z_t.$$

Proof of step 3. The HJB equations for the worker's value and the surplus of the match are

$$(\hat{r} + d + d^r)\hat{W}(\hat{w}) = e^{\hat{w}} + d^r \hat{W}(\hat{w}^*) - \hat{r}\hat{U} + \frac{s^2}{2}\hat{W}''(\hat{w}) \quad \forall \hat{w} \in (\hat{w}^-, \hat{w}^+)$$
$$(\hat{r} + d + d^r)\hat{S}(\hat{w}) = 1 + d^r \hat{S}(\hat{w}^*) - \hat{r}\hat{U} + \frac{s^2}{2}\hat{S}''(\hat{w}) \quad \forall \hat{w} \in (\hat{w}^-, \hat{w}^+),$$

respectively. Replacing $h(\hat{w}) = \hat{W}(\hat{w})/\hat{S}(\hat{w})$ in the worker's value function, we have $\forall \hat{w} \in (\hat{w}^-, \hat{w}^+)$:

$$(\hat{r}+d+d^r)(h(\hat{w})\hat{S}(\hat{w})) = e^{\hat{w}} + d^r h(\hat{w}^*)\hat{S}(\hat{w}^*) - \hat{r}\hat{U} + \frac{s^2}{2}\left(h(\hat{w})\hat{S}''(\hat{w}) + 2h'(\hat{w})\hat{S}'(\hat{w}) + h''(\hat{w})\hat{S}(\hat{w})\right).$$

Using the HJB equation of the surplus to replace $(\hat{r} + d)\hat{S}(\hat{w})$ on the left hand side,

$$(1 - \hat{r}\hat{U} + d^r\hat{S}(\hat{w}^*))h(\hat{w}) = e^{\hat{w}} - \hat{r}\hat{U} + d^r\hat{S}(\hat{w}^*) + h'(\hat{w})s^2\hat{S}'(\hat{w}) + h''(\hat{w})\frac{s^2}{2}\hat{S}(\hat{w}) \quad \forall \hat{w} \in (\hat{w}^-, \hat{w}^+).$$

Since $\hat{S}(\hat{w}) = \frac{1-\hat{r}\hat{U}}{1-d^r\mathcal{T}(\hat{w}^*,\hat{r}+d^r)}\mathcal{T}(\hat{w},\hat{r}+d^r)$, operating from the left hand side $\forall \hat{w} \in (\hat{w}^-, \hat{w}^+)$, we have

$$
\begin{aligned}
&(1-\hat{r}\hat{U}+d^r\hat{S}(\hat{w}^*))h(\hat{w})\\
=&e^{\hat{w}}-1+\frac{1-\hat{r}\hat{U}}{1-d^r\mathcal{T}(\hat{w}^*,\hat{r}+d^r)}+h'(\hat{w})s^2\frac{(1-\hat{r}\hat{U})\mathcal{T}'(\hat{w},\hat{r}+d^r)}{1-d^r\mathcal{T}(\hat{w}^*,\hat{r}+d^r)}+h''(\hat{w})\frac{s^2}{2}\frac{(1-\hat{r}\hat{U})\mathcal{T}(\hat{w},\hat{r}+d^r)}{1-d^r\mathcal{T}(\hat{w}^*,\hat{r}+d^r)}
\end{aligned}
$$

In conclusion, we arrive at

$$
h(\hat{w}) = (1-d^r\mathcal{T}(\hat{w}^*,\hat{r}+d^r))\frac{e^{\hat{w}}-1}{1-\hat{r}\hat{U}}+1+h'(\hat{w})s^2\mathcal{T}'(\hat{w},\hat{r}+d^r)+h''(\hat{w})\frac{s^2}{2}\mathcal{T}(\hat{w},\hat{r}+d^r).
$$

Multiplying by $(\hat{r}+d+d^r)$, we have, $\forall \hat{w} \in (\hat{w}^-, \hat{w}^+)$:

$$
\begin{aligned}
(\hat{r}+d+d^r)h(\hat{w}) = (\hat{r}+d+d^r)&\left((1-d^r\mathcal{T}(\hat{w}^*,\hat{r}+d^r))\frac{e^{\hat{w}}-1}{1-\hat{r}\hat{U}}+1\right)\\
&+h'(\hat{w})s^2(\hat{r}+d+d^r)\mathcal{T}'(\hat{w},\hat{r}+d^r)+h''(\hat{w})\frac{s^2}{2}(\hat{r}+d+d^r)\mathcal{T}(\hat{w},\hat{r}+d^r).
\end{aligned}
$$

Finally, recall the value-matching and smooth-pasting conditions

$$
\hat{W}(\hat{w}^-) = \hat{J}(\hat{w}^-) = \hat{W}(\hat{w}^+) = \hat{J}(\hat{w}^+) = 0, \quad \hat{W}'(-\mathsf{D}^-) = \hat{J}'(\mathsf{D}^+) = 0.
$$

By L'Hôpital's rule,

$$
\begin{aligned}
\lim_{\hat{w}\downarrow\hat{w}^-} h(\hat{w}) &= \lim_{\hat{w}\downarrow\hat{w}^-}\frac{\hat{W}(\hat{w})}{\hat{S}(\hat{w})} = \lim_{\hat{w}\downarrow\hat{w}^-}\frac{\hat{W}'(\hat{w})}{\hat{J}'(\hat{w})} = 0\\
\lim_{\hat{w}\uparrow\hat{w}^+} h(\hat{w}) &= \lim_{\hat{w}\uparrow\hat{w}^+}\frac{\hat{W}(\hat{w})}{\hat{S}(\hat{w})} = \lim_{\hat{w}\uparrow\hat{w}^+}\frac{\hat{W}'(\hat{w})}{\hat{W}'(\hat{w})} = 1,
\end{aligned}
$$

which are the boundary values for the worker's share at the separation triggers.

Finally, the equivalence of the combined Dirichlet-Poisson problem (i.e., the mapping from the corresponding HJB equations and boundary conditions of $h(\hat{w})$ to the sequential formulation) gives us the following Bellman equation

$$
\begin{aligned}
h(\hat{w}) = \mathbb{E}\Bigg[\int_0^{t^{end}} & e^{-(\hat{r}+d+d^r)t}(\hat{r}+d+d^r)\left(\frac{e^{\mathsf{G}_t}-1}{1-\hat{r}\hat{U}}(1-d^r\mathcal{T}(\hat{w}^*,\hat{r}+d^r))+1\right)\mathrm{d}t\\
&+e^{-(\hat{r}+d+d^r)t^{end}}\mathbb{1}[\mathsf{D}z_{t^{end}} = \mathsf{D}^+]|\mathsf{G}_0 = \hat{w}\Bigg]
\end{aligned}
$$

where

$$t^{end} = \inf\{t \geq 0 : \mathsf{G}_t \notin (\hat{w}^-, \hat{w}^+)\}$$

and

$$\mathrm{d}\mathsf{G}_t = (\hat{r} + d + d^r)s^2 \mathcal{T}'_{\hat{w}}(\mathsf{G}_t, \hat{r} + d^r)\,\mathrm{d}t + s\sqrt{\mathcal{T}(\mathsf{G}_t, \hat{r} + d^r)(\hat{r} + d + d^r)}\,\mathrm{d}\mathcal{W}^z_t.$$

Step 4. The following results hold:

1. If $\mathsf{D}^+ \to ¥$, then

$$\left.\frac{\mathrm{d}\log(h(\hat{w}))}{\mathrm{d}\hat{w}}\right|_{\hat{w}=\hat{w}^*} = \frac{\hat{r} + d}{\hat{r} + d + \mathcal{N}^{\mathsf{D}w}} \frac{\left[a + (1-a)\hat{r}\hat{U}\right]}{a(1 - \hat{r}\hat{U})}.$$

2. If $d^r \to ¥$, then

$$\left.\frac{\mathrm{d}\log(h(\hat{w}))}{\mathrm{d}\hat{w}}\right|_{\hat{w}=\hat{w}^*} = 0.$$

3. If D^+ small enough, then

$$\left.\frac{\mathrm{d}\log(h(\hat{w}))}{\mathrm{d}\hat{w}}\right|_{\hat{w}=\hat{w}^*} = \frac{\sqrt{s^{end}}}{2as}.$$

Proof of step 4.

Next, we prove the first two results. If $\mathsf{D}^+ \to ¥$, then $\mathcal{T}(\hat{w}, \hat{r}) = \int_0^{¥} e^{-(\hat{r}+d+d^r)t}\,\mathrm{d}t = \frac{1}{\hat{r}+d+d^r}$ and $\mathcal{T}'_{\hat{w}^*}(\hat{w}^*, \hat{r} + d^r) = 0$. Similarly, if $d^r \to ¥$, then $\lim_{d^r \to ¥} \frac{\mathcal{T}(\hat{w}, \hat{r}+d^r)}{1 - d^r\mathcal{T}(\hat{w}^*, \hat{r}+d^r)} = \frac{1}{\hat{r}+d}$; thus, $\mathcal{T}'_{\hat{w}^*}(\hat{w}^*, \hat{r} + d^r) = 0$. Therefore, by the definition of $h(\hat{w})$,

$$a = h(\hat{w}^*) = \frac{\hat{W}(\hat{w}^*)}{S(\hat{w}^*)} = \frac{\mathbb{E}\left[\int_0^{t^m} e^{-(\hat{r}+d^r)t + w_t} | \hat{w}_0 = \hat{w}^*\right] + (d^r \overbrace{\hat{W}(\hat{w}^*)}^{a\hat{S}(\hat{w}^*)} - \hat{r}\hat{U})\mathcal{T}(\hat{w}^*, \hat{r} + d^r)}{\frac{1-\hat{r}\hat{U}}{1-d^r\mathcal{T}(\hat{w}^*, \hat{r}+d^r)}\mathcal{T}(\hat{w}^*, \hat{r} + d^r)}$$

which gives, for both limits $\mathsf{D}^+ \to ¥$ and $d^r \to ¥$:

$$\mathbb{E}\left[\int_0^{t^m} e^{-(\hat{r}+d^r)t + w_t} | \hat{w}_0 = \hat{w}^*\right] = \left(a + (1-a)\hat{r}\hat{U}\right)\mathcal{T}(\hat{w}^*, \hat{r} + d^r)$$

Take the limit $\mathsf{D}^+ \to ¥$ and following similar steps as in the proof in the baseline model, we obtain

$$h'(\hat{w}^*) = (\hat{r} + d + d^r)(1 - d^r\mathcal{T}(\hat{w}^*, \hat{r} + d^r))\frac{\mathbb{E}\left[\int_0^{t^m} e^{-(\hat{r}+d^r)t + \hat{w}_t}\,\mathrm{d}t | \hat{w}_0 = \hat{w}^*\right]}{1 - \hat{r}\hat{U}}.$$

Combining all these results, we finally obtain

$$\frac{h'(\hat{w}^*)}{h(\hat{w}^*)} = \frac{\hat{r}+d}{\hat{r}+d+d^r}\frac{\left(a+(1-a)\hat{r}\hat{U}\right)}{a(1-\hat{r}\hat{U})}.$$

Regarding the second limit $d^r \to ¥$, it is easy to show that $h'(\hat{w}^*) = 0$.

Following similar steps as in the proof for the baseline model, we have that if D^+ is small, then

$$\frac{h'(\hat{w}^*)}{h(\hat{w}^*)} = \frac{1}{2aD^+}.$$

Since $\frac{1}{s+\mathcal{F}^{Dw}} = \mathcal{T}(0,d^r) \approx \frac{1}{d+d^r+(s/D^+)^2}$ and letting $s^{end} = \frac{s}{D^+)^2}$ denote the rate of endogenous separations,

$$\frac{h'(\hat{w}^*)}{h(\hat{w}^*)} = \frac{\sqrt{s^{end}}}{2as}.$$

□

Discussion of Employment Dynamics With Sticky Entry Wages. Several insights emerge from this analysis. From the first part of Proposition II.8, we learn that the presence of on-the-job wage renegotiations affects the response of the job-finding rate to the TFPR shock. The intuition is that the possibility of wage renegotiations reduces the effect of the TFPR shock on job creation: Shocks stop affecting the real normalized wage following the first wage renegotiation. In the extreme case, when the frequency of bargaining tends to ¥, job creation does not respond to the shock since wages become renegotiated and reflect the occurrence of the shock immediately after the match is created. The last part of Proposition II.8 shows that the possibility of wage renegotiations does not affect the shape of the sufficient statistic for the elasticity of the worker's share to the entry wage, which continues to be determined by the frequency of endogenous separations. However, wage renegotiations does affect the value of the elasticity of the worker's share to the entry wage because a higher frequency of wage renegotiations reduces the frequency of endogenous separations.

III Additional Results for Section 4: Mapping the Model to Labor Market Microdata

III.1 Characterizing $g^h(\mathsf{D}z)$ and $g^u(\mathsf{D}z)$

Proposition III.1. *Assume $d > 0$. Then, $g^h(\mathsf{D}z)$ and $g^u(\mathsf{D}z)$ are given by*

$$g^h(\mathsf{D}z) = \mathcal{E}\mathcal{G}_h \begin{cases} \frac{e^{b_1(d)(\mathsf{D}z+\mathsf{D}^-)} - e^{b_2(d)(\mathsf{D}z+\mathsf{D}^-)}}{e^{b_1(d)\mathsf{D}^-} - e^{b_2(d)\mathsf{D}^-}} & \text{if } \mathsf{D}z \in (-\mathsf{D}^-, 0] \\ \frac{e^{b_1(d)(\mathsf{D}z-\mathsf{D}^+)} - e^{b_2(d)(\mathsf{D}z-\mathsf{D}^+)}}{e^{-b_1(d)\mathsf{D}^+} - e^{-b_2(d)\mathsf{D}^+}} & \text{if } \mathsf{D}z \in [0, \mathsf{D}^+) \end{cases}$$

$$g^u(\mathsf{D}z) = (1-\mathcal{E})\mathcal{G}_u \begin{cases} e^{b_2(f(\hat{q}(\hat{w}^*)))\mathsf{D}z} & \text{if } \mathsf{D}z \in (-\text{¥}, 0] \\ e^{b_1(f(\hat{q}(\hat{w}^*)))\mathsf{D}z} & \text{if } \mathsf{D}z \in [0, \text{¥}) \end{cases}$$

where

$$b_1(x) = \frac{-g - \sqrt{g^2 + 2s^2x}}{s^2}, b_2(x) = \frac{-g + \sqrt{g^2 + 2s^2x}}{s^2},$$

$$\mathcal{E} = \frac{f(\hat{q}(\hat{w}^*))}{f(\hat{q}(\hat{w}^*)) + d + \frac{s^2}{2}\mathcal{G}_h\left[\frac{b_1(d)-b_2(d)}{e^{b_1(d)\mathsf{D}^-} - e^{b_2(d)\mathsf{D}^-}} - \frac{b_1(d)-b_2(d)}{e^{-b_1(d)\mathsf{D}^+} - e^{-b_2(d)\mathsf{D}^+}}\right]},$$

$$\mathcal{G}_h = \left[\frac{\frac{e^{b_1(d)\mathsf{D}^-}-1}{b_1(d)} - \frac{e^{b_2(d)\mathsf{D}^-}-1}{b_2(d)}}{e^{b_1(d)\mathsf{D}^-} - e^{b_2(d)\mathsf{D}^-}} + \frac{\frac{1-e^{-b_1\mathsf{D}^+}}{b_1(d)} - \frac{1-e^{-b_2\mathsf{D}^+}}{b_2(d)}}{e^{-b_1(d)\mathsf{D}^+} - e^{-b_2(d)\mathsf{D}^+}}\right]^{-1},$$

$$\mathcal{G}_u = \left[-b_1(f(\hat{q}(\hat{w}^*)))^{-1} + b_2(f(\hat{q}(\hat{w}^*)))^{-1}\right]^{-1}.$$

Proof. Let us write the KFE and border conditions:

$$dg^h(\mathsf{D}z) = g(g^h)'(\mathsf{D}z) + \frac{s^2}{2}(g^h)''(\mathsf{D}z) \quad \forall \mathsf{D}z \in (-\mathsf{D}^-, \mathsf{D}^+)/\{0\} \tag{III.1}$$

$$g^h(-\mathsf{D}^-) = g^h(\mathsf{D}^+) = 0, \tag{III.2}$$

$$f(\hat{q}(\hat{w}^*))g^u(\mathsf{D}z) = g(g^u)'(\mathsf{D}z) + \frac{s^2}{2}(g^u)''(\mathsf{D}z) \quad \forall \mathsf{D}z \in (-\text{¥}, \text{¥})/\{0\}, \tag{III.3}$$

$$\lim_{\mathsf{D}z \to -\text{¥}} g^u(\mathsf{D}z) = \lim_{\mathsf{D}z \to \text{¥}} g^u(\mathsf{D}z) = 0, \tag{III.4}$$

$$1 = \int_{-\text{¥}}^{\text{¥}} g^u(\mathsf{D}z)\,\mathrm{d}\mathsf{D}z + \int_{-\mathsf{D}^-}^{\mathsf{D}^+} g^h(\mathsf{D}z)\,\mathrm{d}\mathsf{D}z, \tag{III.5}$$

$$f(\hat{q}(\hat{w}^*))(1-\mathcal{E}) = d\mathcal{E} + \frac{s^2}{2}\left[\lim_{\mathsf{D}z \downarrow -\mathsf{D}^-}(g^h)'(\mathsf{D}z) - \lim_{\mathsf{D}z \uparrow \mathsf{D}^+}(g^h)'(\mathsf{D}z)\right], \tag{III.6}$$

$$g^h(\mathsf{D}z), g^u(\mathsf{D}z) \in \mathbb{C}.$$

We guess and verify the proposed solution. Substituting the guess for $g^h(\mathsf{D}z)$ in (III.1) for $\mathsf{D}z < 0$, we have

$$0 = -d\mathcal{E}\mathcal{G}_h \frac{e^{b_1(d)(\mathsf{D}z+\mathsf{D}^-)}}{e^{b_1(d)\mathsf{D}^-} - e^{b_2(d)\mathsf{D}^-}} + g b_1(d)\mathcal{E}\mathcal{G}_h \frac{e^{b_1(d)(\mathsf{D}z+\mathsf{D}^-)}}{e^{b_1(d)\mathsf{D}^-} - e^{b_2(d)\mathsf{D}^-}} + \mathcal{E}\mathcal{G}_h \frac{s^2}{2} b_1(d)^2 \frac{e^{b_1(d)(\mathsf{D}z+\mathsf{D}^-)}}{e^{b_1(d)\mathsf{D}^-} - e^{b_2(d)\mathsf{D}^-}} \iff$$

$$0 = -d + g b_1(d) + \frac{s^2}{2} b_1(d)^2,$$

mutatis mutandis, for the terms that include $b_2(d)$. Given the definition of $b_1(d)$, the guess satisfies (III.1). A similar argument applies when (III.1) is evaluated at $\mathsf{D}z > 0$. It is easy to verify that the boundary conditions (III.2) are satisfied and that $g^h(\mathsf{D}z)$ is continuous at $\mathsf{D}z = 0$. Following the same steps for $g^u(\mathsf{D}z)$, we verify conditions (III.3) and (III.4). Next, we verify condition (III.5):

$$\begin{aligned}
&\int_{-\text{¥}}^{\text{¥}} g^u(\mathsf{D}z)\,\mathrm{d}\mathsf{D}z + \int_{-\mathsf{D}^-}^{\mathsf{D}^+} g^h(\mathsf{D}z)\,\mathrm{d}\mathsf{D}z \\
&= (1-\mathcal{E})\mathcal{G}_u \left[\int_{-\text{¥}}^{0} e^{b_2(f(\hat{q}(\hat{w}^*)))\mathsf{D}z}\,\mathrm{d}\mathsf{D}z + \int_{0}^{\text{¥}} e^{b_1(f(\hat{q}(\hat{w}^*)))\mathsf{D}z}\,\mathrm{d}\mathsf{D}z \right] + \dots \\
&\quad + \mathcal{E}\mathcal{G}_h \left[\int_{-\mathsf{D}^-}^{0} \frac{e^{b_1(d)(\mathsf{D}z+\mathsf{D}^-)} - e^{b_2(d)(\mathsf{D}z+\mathsf{D}^-)}}{e^{b_1(d)\mathsf{D}^-} - e^{b_2(d)\mathsf{D}^-}}\,\mathrm{d}\mathsf{D}z + \int_{0}^{\mathsf{D}^+} \frac{e^{b_1(d)(\mathsf{D}z-\mathsf{D}^+)} - e^{b_2(d)(\mathsf{D}z-\mathsf{D}^+)}}{e^{-b_1(d)\mathsf{D}^+} - e^{-b_2(d)\mathsf{D}^+}}\,\mathrm{d}\mathsf{D}z \right] \\
&= (1-\mathcal{E})\mathcal{G}_u \left[\frac{1 - \lim_{\mathsf{D}z\to-\text{¥}} e^{b_2(f(\hat{q}(\hat{w}^*)))\mathsf{D}z}}{b_2(f(\hat{q}(\hat{w}^*)))} + \frac{\lim_{\mathsf{D}z\to\text{¥}} e^{b_1(f(\hat{q}(\hat{w}^*)))\mathsf{D}z} - 1}{b_1(f(\hat{q}(\hat{w}^*)))} \right] + \dots \\
&\quad + \mathcal{E}\mathcal{G}_h \left[\frac{\frac{e^{b_1(d)\mathsf{D}^-}-1}{b_1(d)} - \frac{e^{b_2(d)\mathsf{D}^-}-1}{b_2(d)}}{e^{b_1(d)\mathsf{D}^-} - e^{b_2(d)\mathsf{D}^-}} + \frac{\frac{1-e^{-b_1\mathsf{D}^+}}{b_1(d)} - \frac{1-e^{-b_2\mathsf{D}^+}}{b_2(d)}}{e^{-b_1(d)\mathsf{D}^+} - e^{-b_2(d)\mathsf{D}^+}} \right] = (1-\mathcal{E}) + \mathcal{E} = 1.
\end{aligned}$$

Finally, combining condition (III.6) with the definition of $g^h(\mathsf{D}z)$, the employment rate is

$$\mathcal{E} = \frac{f(\hat{q}(\hat{w}^*))}{f(\hat{q}(\hat{w}^*)) + d + \frac{s^2}{2}\mathcal{G}_h \left[\frac{b_1(d)-b_2(d)}{e^{b_1(d)\mathsf{D}^-} - e^{b_2(d)\mathsf{D}^-}} - \frac{b_1(d)-b_2(d)}{e^{-b_1(d)\mathsf{D}^+} - e^{-b_2(d)\mathsf{D}^+}} \right]}.$$

□

III.2 Characterizing $l^w(\mathsf{D}w)$

Proposition III.2. *The distribution of log nominal wage changes satisfies*

$$l^w(\mathsf{D}w) = \mathcal{G}_u \left[b_2(f(\hat{q}(\hat{w}^*)))e^{-b_2(f(\hat{q}(\hat{w}^*)))\mathsf{D}w} \mathsf{G}_2(\mathsf{D}w) + b_1(f(\hat{q}(\hat{w}^*)))e^{-b_1(f(\hat{q}(\hat{w}^*)))\mathsf{D}w} \mathsf{G}_1(\mathsf{D}w) \right]$$

with

$$(\mathsf{G}_1(c), \mathsf{G}_2(c)) = \left(\int_{-¥}^{-c} e^{-b_1(f(\hat{q}(\hat{w}^*))))x} \bar{G}^h(x)\, \mathrm{d}x, \int_{-c}^{¥} e^{-b_2(f(\hat{q}(\hat{w}^*)))x} \bar{G}^h(x)\, \mathrm{d}x \right).$$

Proof. Fix a date t_0 and focus on a newly hired worker. Then, the distribution of wage changes between two new jobs is given by

$$\begin{aligned}
Pr(\mathsf{D}w \leq c) &= Pr(w_{t_0+t^m+t^u} - w_{t_0} \leq c) \\
&=^{(1)} Pr(w_{t_0+t^m+t^u} - z_{t_0+t^m+t^u} - (w_{t_0} - z_{t_0}) + (z_{t_0+t^m+t^u} - z_{t_0}) \leq c) \\
&=^{(2)} Pr(\hat{w}^* - \hat{w}^* + (z_{t_0+t^m+t^u} - z_{t_0}) \leq c) \\
&=^{(3)} Pr(-(\mathsf{D}z^h + \mathsf{D}z^u) \leq c),
\end{aligned}$$

where $\mathsf{D}z^h$ and $\mathsf{D}z^u$ denote cumulative productivity shocks during completed employment and unemployment spells, respectively. Here, step (1) adds and subtracts productivity at the beginning of both job spells. In step (2), we use the result that $\hat{w}^*$ is constant across jobs. Step 3 uses the facts that t^u and the Brownian motion increments are independent of the filtration $\mathcal{F}_{t_u}$. Therefore, the distributions of cumulative productivity shocks for completed employment and unemployment spells are given by

$$\bar{G}^h(\mathsf{D}z) = \begin{cases} 1 & \text{if } \mathsf{D}z \in [\mathsf{D}^+, ¥) \\ \frac{1}{s\mathcal{E}} \left[\frac{s^2}{2} \lim_{\mathsf{D}z \downarrow -\mathsf{D}^-} (g^h)'(\mathsf{D}z) + d \int_{-\mathsf{D}^-}^{\mathsf{D}z} g^h(x)\, \mathrm{d}x \right] & \text{if } \mathsf{D}z \in [-\mathsf{D}^-, \mathsf{D}^+) \\ 0 & \text{if } \mathsf{D}z \in (-¥, -\mathsf{D}^-) \end{cases}$$

$$\bar{g}^u(\mathsf{D}z) = \mathcal{G}_u \begin{cases} e^{b_2(f(\hat{q}(\hat{w}^*)))\mathsf{D}z} & \text{if } \mathsf{D}z \in (-¥, 0] \\ e^{b_1(f(\hat{q}(\hat{w}^*)))\mathsf{D}z} & \text{if } \mathsf{D}z \in [0, ¥) \end{cases}$$

Thus,

$$\begin{aligned}
&\Pr(\mathsf{D}w \leq c) \\
&= \Pr(-(\mathsf{D}z^u + \mathsf{D}z^h) \leq c) \\
&= 1 - \Pr(\mathsf{D}z^u + \mathsf{D}z^h \leq -c) \\
&=^{(1)} 1 - \int_{-¥}^{¥} \bar{G}^h(-(c + \mathsf{D}z)) \bar{g}^u(\mathsf{D}z)\, \mathrm{d}\mathsf{D}z \\
&=^{(2)} 1 - \mathcal{G}_u \left[\int_{-¥}^{0} e^{b_2(f(\hat{q}(\hat{w}^*)))\mathsf{D}z} \bar{G}^h(-(c + \mathsf{D}z))\, \mathrm{d}\mathsf{D}z + \int_{0}^{¥} e^{b_1(f(\hat{q}(\hat{w}^*)))\mathsf{D}z} \bar{G}^h(-(c + \mathsf{D}z))\, \mathrm{d}\mathsf{D}z \right]
\end{aligned}$$

$$
\begin{aligned}
&=^{(3)} 1+\mathcal{G}_u\left[\int_{¥}^{-c} e^{-b_2(f(\hat{q}(\hat{w}^*)))(c+x)}\bar{G}^h(x)\,\mathrm{d}x+\int_{-c}^{-¥} e^{-b_1(f(\hat{q}(\hat{w}^*)))(c+x)}\bar{G}^h(x)\,\mathrm{d}x\right]\\
&=^{(4)} 1-\mathcal{G}_u\left[e^{-b_2(f(\hat{q}(\hat{w}^*)))c}\int_{-c}^{¥} e^{-b_2(f(\hat{q}(\hat{w}^*)))x}\bar{G}^h(x)\,\mathrm{d}x+e^{-b_1(f(\hat{q}(\hat{w}^*)))c}\int_{-¥}^{-c} e^{-b_1(f(\hat{q}(\hat{w}^*)))x}\bar{G}^h(x)\,\mathrm{d}x\right].
\end{aligned}
$$

In step (1), we use the independence of $\mathsf{D}z^u$ and $\mathsf{D}z^h$. In step (2), we use the definition of $\bar{g}^u(\mathsf{D}z)$. In step (3), we integrate by substituting $x=-c-\mathsf{D}z$, and in step (4), we use the properties of an integral. The last step involves defining

$$
(\mathsf{G}_1(c),\mathsf{G}_2(c))=\left(\int_{-¥}^{-c} e^{-b_1(f(\hat{q}(\hat{w}^*)))x}\bar{G}^h(x)\,\mathrm{d}x,\int_{-c}^{¥} e^{-b_2(f(\hat{q}(\hat{w}^*)))x}\bar{G}^h(x)\,\mathrm{d}x\right).
$$

□

III.3 Characterizing $\mathbb{E}_h[\mathsf{D}z^n]$

Let $\bar{\mathbb{E}}_h[\cdot]$ and $\bar{\mathbb{E}}_u[\cdot]$ be the expectation operators under the distributions $\bar{g}^h(\mathsf{D}z)$ and $\bar{g}^u(\mathsf{D}z)$, respectively.

Proposition III.3. *Define the weights* $w^{hn}(\mathsf{D}z)=\frac{\mathsf{D}z^n}{\bar{\mathbb{E}}_h[\mathsf{D}z^n]}$ *with the property that*

$$
\bar{\mathbb{E}}_h\left[w^{hn}(\mathsf{D}z)\right]=1.
$$

If $g+c=0$*, then* $\mathbb{E}_h[(\mathsf{D}z)^n]$ *can be recovered from*

$$
\mathbb{E}_h[(\mathsf{D}z)^n]=\frac{2\mathcal{E}}{(n+1)(n+2)}\bar{\mathbb{E}}_h\left[(\mathsf{D}z)^n w^{h2}(\mathsf{D}z)\right]. \tag{III.7}
$$

If $g+c\neq 0$*, then* $\mathbb{E}_h[(\mathsf{D}z)^n]$ *can be recovered recursively from*

$$
\mathbb{E}_h[(\mathsf{D}z)^n]=\frac{\mathcal{E}}{n+1}\bar{\mathbb{E}}_h[(\mathsf{D}z)^n w^{h1}(\mathsf{D}z)]+\frac{s^2 n}{2g}\mathbb{E}_h[(\mathsf{D}z)^{n-1}].
$$

The moments $\bar{\mathbb{E}}_h\left[(\mathsf{D}z)^n w^{hk}(\mathsf{D}z)\right]=\frac{\bar{\mathbb{E}}_h\left[(\mathsf{D}z)^{n+k}\right]}{\bar{\mathbb{E}}_h\left[(\mathsf{D}z)^k\right]}$ *can be recovered from the following linear system of equations:*

$$
\mathbb{E}_{\mathcal{D}}[\mathsf{D}w^n]=(-1)^n \overset{n}{\underset{i=0}{å}}\begin{pmatrix} n\\ i\end{pmatrix}\bar{\mathbb{E}}_h[\mathsf{D}z^i]\bar{\mathbb{E}}_u[\mathsf{D}z^{n-i}],
$$

$$\bar{\mathbb{E}}_u[(\mathrm{D}z)^{n-i}] = \frac{(n-i)!}{\mathcal{L}_1^{n-i}\left(\mathcal{L}_2 + \mathcal{L}_2^{-1}\right)}\left(\mathcal{L}_2^{-(n-i+1)} - (-\mathcal{L}_2)^{(n-i+1)}\right),$$

where

$$\mathcal{L}_1 = \sqrt{\frac{2f(\hat{q}(\hat{w}^*))}{s^2}} \text{ and } \mathcal{L}_2 = \sqrt{\frac{(g+c) + \sqrt{(g+c)^2 + 2s^2 f(\hat{q}(\hat{w}^*))}}{-(g+c) + \sqrt{(g+c)^2 + 2s^2 f(\hat{q}(\hat{w}^*))}}}.$$

Proof. We divide the proof into 3 steps.

Step 1. We first show that

$$\mathbb{E}_h[(\mathrm{D}z)^n] = \frac{\mathcal{E}}{n+1}\bar{\mathbb{E}}_h[(\mathrm{D}z)^n w^{h1}(\mathrm{D}z)] - \frac{s^2 n}{2(g+c)}\mathbb{E}_h[(\mathrm{D}z)^{n-1}].$$

when $(g+c) \neq 0$. For the case with $(g+c) = 0$, see Baley and Blanco (2021).

Let us define $Y_t = (\mathrm{D}z_t)^n$. The law of motion for $\mathrm{D}z_t$ is given by $\mathrm{d}\mathrm{D}z_t = -(g+c)\,\mathrm{d}t + s\,\mathrm{d}\mathcal{W}_t^z$. Applying Itô's Lemma, we obtain

$$\begin{aligned}\mathrm{d}Y_t &= n(\mathrm{D}z_t)^{n-1}\,\mathrm{d}\mathrm{D}z_t + \frac{1}{2}n(n-1)(\mathrm{D}z_t)^{n-2}(\mathrm{d}\mathrm{D}z_t)^2 \\ &= \left[-(g+c)n(\mathrm{D}z_t)^{n-1} + \frac{s^2}{2}n(n-1)(\mathrm{D}z_t)^{n-2}\right]\mathrm{d}t + ns(\mathrm{D}z_t)^{n-1}\,\mathrm{d}\mathcal{W}_t^z\end{aligned}$$

Thus,

$$(\mathrm{D}z_{t^m})^n = -(g+c)n\int_0^{t^m}(\mathrm{D}z_t)^{n-1}\,\mathrm{d}t + \frac{s^2}{2}n(n-1)\int_0^{t^m}(\mathrm{D}z_t)^{n-2}\,\mathrm{d}t + n\int_0^{t^m}(\mathrm{D}z_t)^{n-1}s\,\mathrm{d}\mathcal{W}_t^z.$$

Following the same arguments as in the proof of Proposition C.1 and using the Renewal Principle to have $\mathbb{E}_{\mathcal{D}}[t^m] = 1/s$, we obtain

$$\bar{\mathbb{E}}_h[(\mathrm{D}z)^n] = -(g+c)n\mathbb{E}_{\mathcal{D}}[t^m]\frac{\mathbb{E}_h[(\mathrm{D}z)^{n-1}]}{\mathcal{E}} + \frac{s^2 n(n-1)}{2s}\frac{\mathbb{E}_h[(\mathrm{D}z)^{n-2}]}{\mathcal{E}}$$

or equivalently

$$\mathbb{E}_h[(\mathrm{D}z)^n] = -\frac{\mathcal{E}}{(g+c)\mathbb{E}_{\mathcal{D}}[t^m]}\frac{\bar{\mathbb{E}}_h[(\mathrm{D}z)^{n+1}]}{n+1} + \frac{s^2 n}{2(g+c)}\mathbb{E}_h[(\mathrm{D}z)^{n-1}].$$

From Propostion C.1, we have $(g+c)\mathbb{E}_{\mathcal{D}}[t^m] = -\bar{\mathbb{E}}_h[(\mathrm{D}z)]$ and $\frac{\bar{\mathbb{E}}_h[(\mathrm{D}z)^{n+1}]}{\bar{\mathbb{E}}_h[(\mathrm{D}z)]} = \bar{\mathbb{E}}_h[(\mathrm{D}z)^n w^{h1}(\mathrm{D}z)]$.

Thus,

$$\mathbb{E}_h[(\mathsf{D}z)^n] = \frac{\mathcal{E}}{n+1}\bar{\mathbb{E}}_h[(\mathsf{D}z)^n w^{h1}(\mathsf{D}z)] + \frac{s^2 n}{2(g+c)}\mathbb{E}_h[(\mathsf{D}z)^{n-1}].$$

Step 2. Here we show that

$$\mathbb{E}_{\mathcal{D}}[\mathsf{D}w^n] = (-1)^n \overset{n}{\underset{i=0}{\mathring{a}}} \begin{pmatrix} n \\ i \end{pmatrix} \bar{\mathbb{E}}_h[\mathsf{D}z^n]\bar{\mathbb{E}}_u[\mathsf{D}z^{n-i}].$$

Using the independence of cumulative productivity shocks during employment and unemployment,

$$\begin{aligned}
\mathbb{E}_{\mathcal{D}}[\mathsf{D}w^n] &= \bar{\mathbb{E}}[(-\mathsf{D}z^h - \mathsf{D}z^u)^n], \\
&= \overset{n}{\underset{i=0}{\mathring{a}}} \begin{pmatrix} n \\ i \end{pmatrix} \bar{\mathbb{E}}[(-\mathsf{D}z^h)^i(-\mathsf{D}u^u)^{n-i}], \\
&= \overset{n}{\underset{i=0}{\mathring{a}}} \begin{pmatrix} n \\ i \end{pmatrix} \bar{\mathbb{E}}_h[(-\mathsf{D}z)^i]\bar{\mathbb{E}}_u[(-\mathsf{D}z)^{n-i}], \\
&= (-1)^n \overset{n}{\underset{i=0}{\mathring{a}}} \begin{pmatrix} n \\ i \end{pmatrix} \bar{\mathbb{E}}_h[\mathsf{D}z^i]\bar{\mathbb{E}}_u[\mathsf{D}z^{n-i}],
\end{aligned}$$

Step 3. Here we show that

$$\bar{\mathbb{E}}_u[(\mathsf{D}z)^{n-i}] = \frac{(n-i)!}{\mathcal{L}_1^{n-i}\left(\mathcal{L}_2 + \mathcal{L}_2^{-1}\right)}\left(\mathcal{L}_2^{-(n-i+1)} - (-\mathcal{L}_2)^{(n-i+1)}\right).$$

Let us depart from the definition of $\bar{g}^u(\mathsf{D}z)$, which is given by

$$\bar{g}^u(\mathsf{D}z) = \left[-b_1(f(\hat{q}(\hat{w}^*)))^{-1} + b_2(f(\hat{q}(\hat{w}^*)))^{-1}\right]^{-1} \begin{cases} e^{b_2(f(\hat{q}(\hat{w}^*)))\mathsf{D}z} & \text{if } \mathsf{D}z \in (-¥, 0] \\ e^{b_1(f(\hat{q}(\hat{w}^*)))\mathsf{D}z} & \text{if } \mathsf{D}z \in [0, ¥) \end{cases}$$

where $b_1(x) = \frac{-(g+c)-\sqrt{(g+c)^2+2s^2x}}{s^2}$ and $b_2(x) = \frac{-(g+c)+\sqrt{(g+c)^2+2s^2x}}{s^2}$. This step consist of showing that $\bar{g}^u(\mathsf{D}z)$ is an asymmetric Laplace distribution with parameters

$$\mathcal{L}_1 = \sqrt{\frac{2f(\hat{q}(\hat{w}^*))}{s^2}} \quad \text{and} \quad \mathcal{L}_2 = \sqrt{\frac{(g+c)+\sqrt{(g+c)^2+2s^2f(\hat{q}(\hat{w}^*))}}{-(g+c)+\sqrt{(g+c)^2+2s^2f(\hat{q}(\hat{w}^*))}}}$$

The ratio between $\mathcal{L}_1$ and $\mathcal{L}_2$ is

$$\frac{\mathcal{L}_1}{\mathcal{L}_2} = b_2(f(\hat{q}(\hat{w}^*))).$$

The negative of the product between $\mathcal{L}_1$ and $\mathcal{L}_2$ is

$$-\mathcal{L}_1\mathcal{L}_2 = b_1(f(\hat{q}(\hat{w}^*))).$$

Therefore, we can write $\bar{g}^u(\mathsf{D}z)$

$$\bar{g}^u(\mathsf{D}z) = \frac{\mathcal{L}_1}{\mathcal{L}_2 + \mathcal{L}_2^{-1}} \begin{cases} e^{\frac{\mathcal{L}_1}{\mathcal{L}_2}\mathsf{D}z} & \text{if } \mathsf{D}z \in (-¥, 0] \\ e^{-\mathcal{L}_1\mathcal{L}_2\mathsf{D}z} & \text{if } \mathsf{D}z \in [0, ¥), \end{cases}$$

which is the probability distribution function of an asymmetric Laplace distribution. It is a standard result that the n-th moment for an asymmetric Laplace distribution is given by

$$\bar{\mathbb{E}}_u[(\mathsf{D}z)^n] = \frac{n!}{\mathcal{L}_1^n\left(\mathcal{L}_2 + \mathcal{L}_2^{-1}\right)}\left(\mathcal{L}_2^{-(n+1)} - (-\mathcal{L}_2)^{(n+1)}\right).$$

□

References for the Online Appendix

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