Redistribution and Investment

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Abstract

This paper studies the trade-offs associated with income redistribution in an overlapping generations model in which marginal savings rates increase with permanent income. By transferring permanent income from high savers to low savers, redistribution lowers aggregate savings, and depresses investment in capital. If more capital is welfare improving, the government faces a trade-off between redistribution and investment which depends on the distribution of marginal propensities to save. I estimate this distribution using U.S. household panel data, and use my estimates to calibrate a quantitative overlapping generations model with un-insurable idiosyncratic earnings risk. I study the effects of a simple labor income redistribution policy in this calibrated model and a standard model with homogeneous marginal savings rates. The direct effect of the policy on the permanent income distribution has no effect on capital in a standard model, but a large effect in the calibrated model. This channel accounts for 23% of the drop in aggregate consumption following the policy.

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1 Introduction

What is the optimal amount of income redistribution? The existing literature has answered this question primarily by focusing on trade-offs between greater equity and inefficiencies introduced by distortionary taxation (Mirrlees 1971, Piketty and Saez 2013a, Werning, 2007). At the same time, empirical evidence suggests that marginal savings rates increase with permanent income (Dynan et al., 2004, De Nardi and Fella, 2017, Straub, 2019). This implies that redistribution may have additional effects on welfare by changing the permanent income distribution and lowering aggregate savings. In this paper, I explore the consequences of this savings behavior for the trade-offs associated with income redistribution in overlapping generations (OLG) models. In particular, I show that when marginal propensities to save (MPS) out of permanent income increase over the income distribution, an additional trade-off arises between redistribution and investment.

Intuitively, if lifetime savings increases with permanent income, all permanent redistributive policies — including non-distortionary lump-sum redistribution — will result in a transfer from households with a high MPS to households with a lower MPS, lowering aggregate savings and putting upward pressure on interest rates in a closed or large open economy (Straub, 2019, Mian et al., 2020).[1] This increase in borrowing costs will curb firms' capital investment, reducing the productive capacity of the economy. It is this potential trade-off between permanent income redistribution and optimal capital accumulation that will be the focus of this paper.[2]

I make several contributions towards better understanding this trade-off. The first set of contributions are theoretical. In a simple OLG model, I first present sufficient conditions for a welfare trade-off between non-distortionary (lump-sum) permanent income redistribution and capital accumulation in both the short and long run. I show that whether such a trade-off exists depends both on whether high-income households have higher MPS out of permanent income, and on the *desirability* of additional investment.[3] Intuitively, for there to be a trade-off, it must both be the case that redistribution dampens investment *and that boosting investment is welfare improving.* I explore how these conditions change when the government is given the ability to tax/subsidize capital, make inter-generational transfers, and issue debt, policies which are well known to alter the savings level.

I show how the size of this channel depends not only on the distribution of MPS out of

[1] As long as the economy is not a small open economy and the domestic savings supply has some impact on interest rates.

[2] As noted in Piketty and Saez (2013b) and Atkinson and Sandmo (1980), this trade-off is conceptually orthogonal to inefficiency concerns.

[3] Importantly, the presence of a trade-off does not depend on *why* high income households have greater marginal propensities to save.

permanent income – *how much more* would a rich household save out of a marginal dollar – but also on the interest rate elasticities of firms and savers. If the interest rate elasticity of firm investment is low, permanent income redistribution will simply drive up interest rates with little effect on aggregate capital.[4] Additionally, if the domestic economy is open and small relative to the rest of the world, a decline in aggregate domestic savings has little impact on the total supply of savings available to domestic firms, muting the size of the redistribution investment trade-off.

The final theoretical point I make, is that when the above sufficient conditions are satisfied, this trade-off applies to *all* permanent redistribution policies. Using a standard redistributive labor income tax as an illustrative case, I decompose the steady state welfare impact of a small increase in the tax into the benefits of greater equality, the well-understood efficiency costs of distorted labor supply, and the costs associated with the permanent income redistribution investment trade-off.

The second contribution of the paper is empirical. I use data from the Panel Study of Income Dynamics to estimate the distribution of marginal propensities to save out of permanent income by permanent income quintile. I borrow the techniques use in Dynan et al. (2004) to estimate household permanent income, but make use of modern PSID consumption data to directly compute (rather than indirectly impute) household savings. I find that MPS out of permanent income rise sharply over the distribution, ranging from near 0 for the bottom income quintile to over .5 for the highest quintile.

With these estimates in hand, I am able to quantify the size of the redistribution investment channel. I solve a quantitative OLG model with un-insurable idiosyncratic labor income risk, heterogeneous time preferences, and non-homothetic preferences over savings and bequests, similar in form to the model in De Nardi and Yang (2014). I calibrate the baseline model to the United States in 2019 and choose the degree of non-homotheticity and time preference heterogeneity to target my estimated MPS. I compare the effects of redistribution in this environment to a standard model with homogeneous MPS out of permanent income. In particular, I study the effect of a simple labor income redistribution scheme, in which a linear labor income tax is levied to fund a uniform lump-sum transfer.

I first isolate the direct effect of this policy on the permanent income distribution – before any general equilibrium responses – and consider its effect on capital. In a standard model, capital is unchanged, while in my baseline calibrated model, the direct effect on a 1 percentage point increase in the linear tax on the permanent income distribution decreases capital by .8 percentage points. Relative to the total effect of the tax, this channel can

[4] In Straub (2019) and Mian et al. (2021), an increase in permanent income inequality lowers interest rates in an economy without capital when savings behavior is non-homothetic.

account for around 20 percent of the decline in aggregate consumption.

Framework and Methodology. I begin by studying a simple closed-economy OLG model with a possible motive for bequests and two labor productivity types. The model nests several prominent micro-foundations for marginal propensities to save that increase with permanent income. In particular both bequests and consumption later in life can be considered luxury goods (De Nardi, 2004, Mian et al., 2021, Straub, 2019), and high-productivity households may discount the future less than low-productivity households (De Nardi and Fella, 2017).[5]

I first consider the impact of lump-sum permanent income redistribution from the high-productivity households to the low-productivity households on steady state welfare, defined as the Pareto weighted sum of each type's lifetime utility. An *unconstrained* planner who could choose any feasible allocation would redistribute resources until the Pareto-weighted marginal utility of consumption was equal across households — the *first best* level of inequality. A welfare trade-off exists whenever the optimal redistribution policy for a fiscal policy maker *constrained* to using the lump-sum tax results in more inequality than first best.

I find that a long-run trade-off exists whenever MPS out of permanent income are higher for the high productivity types *and* when the steady state with the first-best level of inequality is dynamically efficient. Intuitively, suppose the fiscal authority were to set redistribution policy to implement the first best level of inequality. If MPS increase with permanent income and this steady state is dynamically efficient, reducing the amount of redistribution slightly will improve welfare by boosting savings, investment, and ultimately aggregate consumption.

I then consider the entire transition path, such that welfare is defined as the Pareto weighted *discounted* sum of the lifetime utility of all present and future generations. Now, the planner cares about both the short and long-run. In this case, the presence of a trade-off depends not only on the sufficient conditions for a long-run (steady state) trade-off, but also on the rate at which the planner discounts future generations. For a welfare trade-off to exist, the planner must put sufficiently high weight on future generations for the benefit of greater future capital to outweigh the costs of more inequality today. Finally, I show that these sufficient conditions extend to a setting in which the the government has access to a broader set of fiscal policy tools, including debt, inter-generational transfers, and capital subsidies.[6]

Assuming the sufficient conditions derived above are satisfied, a natural question is how the size of this trade-off depends on other features of the economic environment. I perform

[5] The simple model does not take into account earnings risk or heterogeneous rates of return. I consider earnings risk in the quantitative model.

[6] These tools can be used to alter the savings supply in life cycle models, and could possibly be employed to offset the effect of changing the permanent income distribution (Diamond, 1965).

several comparative static exercises in the simple model, varying both the substitutability of capital and labor, as well as the size of the domestic economy relative to the world economy. As the substitutability of capital changes, both the elasticity of firm investment to the interest rate and the distributional effects of capital change, both of which determine the size of my channel. Meanwhile, as the size of the domestic economy shrinks relative to the rest of the world – and as the interest rate elasticity of foreign savings increases – the size of the trade-off decreases and the optimal amount of redistribution approaches first-best.

I then estimate the distribution of marginal propensities to save out of permanent income using PSID data starting 1999. Doing so allows me to measure savings directly using the consumption data introduced that year. I follow Dynan et al. (2004) and use lagged and future income as proxies for permanent income, and use within-quintile cross sectional variation to estimate the effect of higher permanent income on savings. I find that MPS increase from near zero for the first income quintile, to over .5 for the highest quintile.

Finally, in order to quantify the size of my trade-off, I solve a quantitative version of the simple life-cycle model with idiosyncratic income risk calibrated to the United States economy in 2019. I calibrate the parameters governing savings behavior to target my estimated MPS distribution, first using only non-homothetic preferences over lifetime consumption and bequests, and then in my baseline model, using both non-homothetic preferences and heterogeneous discount rates. I consider the same simple policy experiment as in the simple model: a linear labor income tax that funds a uniform transfer.

I first consider the *direct* effects of this tax on the permanent income distribution – before any equilibrium adjustments – on steady state aggregates in 3 environments: A standard model with homothetic preferences and homogeneous time preferences (and therefore homogeneous MPS), a model with only non-homothetic preferences, and the baseline model with both non-homothetic preferences and heterogeneous discount rates. To do so, I construct a set of lump-sum taxes/transfers set equal to the direct effect of the labor income tax on a household's permanent income. Doing so allows me to isolate the effect of changing the permanent income distribution from the effects of the tax on idiosyncratic risk, intergenerational transfers, and labor supply distortions.

As expected, the change in the permanent income distribution has no effect on capital in the standard model, as households marginal propensities to save out of permanent income are homogeneous. However, there is a substantial effect on capital in models with realistic distributions of MPS. The direct effect of a .5 percentage point increase in the labor income tax causes capital to fall by .4 percent in both versions of the targeted model. For larger changes, the effect on capital in the model with only non-homothetic preferences is smaller than the model with type-dependent time preferences, presumably because as the degree of redistribution increases, the distribution of MPS compresses more in a model with only non-homothetic preferences, muting the effect on capital.

Finally, I find that in the baseline model, the direct effect of the change in the permanent income distribution on capital can account for around 20 percent of the decline in total aggregate consumption resulting from the tax, suggesting that optimal policy calculations that ignore this channel may underestimate the costs of redistribution.

Related Literature This paper is related to the substantial literature on redistributive taxation. In their review of the optimal labor income tax literature, Piketty and Saez (2013a) note that researchers typically focus on ‘ the classical trade-off between equity and efficiency which is at the core of the optimal labor income tax problem.’ Similarly, Piketty and Saez (2013b) analyze the optimal inheritance tax through the lens of an equity-efficiency trade-off, noting that their results are orthogonal to concerns over optimal capital accumulation.[7] Werning (2007) considers the equity-efficiency trade-off in a dynamic economy subject to aggregate shocks.

Golosov et al. (2016) focuses on the trade-offs between efficiency and both equity and insurance in a model with idiosyncratic household labor income shocks. Heathcote et al. (2017) focus on the trade-off of between the benefits of equity and insurance and the costs of labor supply distortions and disincentives to invest in skills. Imrohoroglu et al. (2018) study the trade-off between greater equity through taxing top earners and entrepreneurial activity. I depart from much of the literature in considering the *non-distortionary* effects of redistributing permanent income on optimal capital accumulation.

This paper is certainly not the first to consider a trade-off between capital accumulation and taxation (Atkinson and Sandmo 1980; Hamada 1972, Pizzo 2023). However, this paper is one of only a few that analyze a trade-off between redistribution and capital accumulation while abstracting away from inefficiency concerns (notably Pestieau and Possen (1978) and Okuno and Yakita (1981)).

A growing literature studies the effect heterogeneous savings behavior on optimal tax policy in various settings. Golosov et al. (2013) solve a static model with preference heterogeneity. Pestieau and Possen (1978) and Judd (1985) consider a ‘two-class’ model with capitalists and workers. Sheshinski (1976) considers a model with infinitely lived agents. Several papers take a Mirrleesian approach (Saez and Stantcheva (2018); Gerritsen et al. (2020); Schulz (2021)). This literature primarily studies the effect of *distortionary* taxa-

[7] Note that if the economy is dynamically efficient, less capital may be sub-optimal for a *given* set of Pareto weights, but is not *inefficient*. That is, one can find Pareto weights such that a lower level of capital is optimal, namely by weighting current generations more.

tion in infinitely lived models. In this paper, I highlight a new channel through which even *non-distortionary* permanent income redistribution impacts welfare in life-cycle models.

This paper also contributes the the empirical literature studying the relationship between permanent household income and savings. I use the Panel Study of Income Dynamics (PSID) to estimate marginal propensities to save (MPS) by permanent income type, and find significantly higher MPS for high income households. These findings echo those of Dynan et al. (2004), who use the PSID in combination with several other data sets to estimate both average and marginal savings rates by permanent income group. Relative to this study, I take advantage of the fact that the PSID added consumption data in 1999 in order to generate a more straightforward measure of 'active' savings. Straub (2019) uses the same data set to estimate a related statistic: the *elasticity of consumption* with respect to permanent income. Using the elasticity of savings implied by his findings in conjunction with estimates of savings *rates* by income group, I can generate additional estimates of the MPS and find that they are very similar to my direct estimates.

Finally, this paper contributes to a small recent literature on the macroeconomic effects of heterogeneous household savings behavior. Straub (2019) shows that non-homothetic savings behavior and increased inequality can explain falling interest rates. Blanco and Diz (2021) study the effects of non-homothetic preferences on optimal monetary policy. Mian et al. (2021) show how non-homothetic preferences have contributed to increased indebtedness and dampened aggregate demand in the long run. Doerr et al. (2023) show that because high-income households save relatively more in stocks and bonds, which increases relative borrowing costs for bank-dependent firms, lowering their employment share.

Layout. The rest of the paper proceeds as follows. In Section 2, I lay out the baseline overlapping generations model and establish its key properties. I derive sufficient conditions for a welfare trade-off between permanent income redistribution and capital accumulation. In Section 3, I estimate the distribution of marginal propensities to save out of permanent income. In Section 4, I present the quantitative model and results. Section 5 concludes.

2 The Redistribution-Investment Trade-off

In this section I derive sufficient conditions for the existence of a welfare trade-off between non-distortionary redistribution and capital accumulation in a simple overlapping generations model. The model nests several leading micro-foundations for MPS that increase over the income distribution. To derive the conditions, I consider the problem of a planner who aims to maximize social welfare, defined as the Pareto weighted sum of household utility. A planner free to choose any feasible allocation would allocate resources between households to generate an *ideal* (first best) level of equality. On the other hand, a constrained fiscal planner faces a *trade-off* between lump-sum redistribution and capital accumulation whenever it is optimal for fiscal policy to tolerate more inequality than the ideal level. I characterize this trade-off in both the short and long run, show how key features of the economic environment determine its *size*, and show how it impacts the welfare costs of a more realistic distortionary labor income tax.

2.1 Environment

I begin with a variant of the canonical 2-generation overlapping generations closed-economy model with fixed exogenous labor supply. Time is discrete. Agents have perfect foresight over future variables and there is no uncertainty.

Households. There is a unit mass of households who each live for 2 periods, $j \in \{y, o\}$ and have heterogeneous labor productivity types, θ_i for $i \in \{L, H\}$ where $\theta_L < \theta_H$. There is a constant fraction, π_i of each productivity type with an equal share, $\pi_j = 1/2$ of each generation (no population growth). While young, households supply a single unit of labor to firms and receive $w_t\theta_i$ in labor income. The weighted sum of labor productivity is normalized to 1. Households can borrow and save at gross rate of return R_t and may leave bequests $a^o_{i,t}$ to households with the same productivity type in the next period. Households born at time t receive $R_t a^o_{i,t-1}$ in inheritance when they are young. Capital depreciates at rate δ. Households also receive a type-specific lump-sum tax(transfer) T_{it} when young. A type-i household born in year t has lifetime utility given by equation (1).

$$U(c^y_{it}, c^o_{it+1}, a^o_{it+1}) = \frac{(c^y_{it})^{1-\sigma_y}}{1-\sigma_y} + \beta_i \left(\frac{(c^o_{it+1})^{1-\sigma_o}}{1-\sigma_o} + \psi_a \frac{(a^o_{it+1})^{1+\eta}}{1+\eta} \right) \tag{1}$$

Note that the discount factor, β_i may be type-specific and that the parameters σ_y, σ_o, and η, which govern the elasticity of inter-temporal substitution and bequests, may differ from one another. Households choose consumption when young and old and bequests to maximize (1) subject to their lifetime budget constraint:

$$c^y_{it} + \frac{c^o_{i,t+1} + a^o_{it+1}}{R_{t+1}} = R_t a^o_{it-1} + w_t\theta_i + T_{it} = PI_{it} \tag{2}$$

I define the right hand side of equation (2) as the household's permanent income, PI_{it}. Let a household's change in assets, $a^j_i - a^{j-1}_{it-1} = s^j_t$, their savings at age j. Note that when $\psi_a = 0$

households do not leave bequests and $s_t^y = a_t^y$.

Firms. There is a continuum of perfectly competitive firms who rent capital and labor from households and produce output in order to maximize profit subject to a constant elasticity of substitution production function (3).

$$F(K_t, L) = \left(\alpha_K K^\gamma + \alpha_L L^\gamma\right)^{\frac{1}{\gamma}} \tag{3}$$

Government. The government runs a balanced budget each period. The transfer T_{it} is defined in terms of each generation's lifetime income. The government cannot make net transfers between living generations, and can only transfer resources between household types in the same generation. I consider the case of a government with access to inter-generational transfers and debt policy in the next section. The government budget constraint is given by the following expression.

$$\sum_{i \in I} \pi_i T_{it} = 0$$

Equilibrium. I define an allocation $\mathcal{A} \equiv \{\{c_{it}^y, c_{it}^o\}_{i \in I}, K_t\}_{t \geq 0}$. An equilibrium, $\mathcal{X}$ is an allocation, a sequences of financial positions, $\{a_{it}^o, a_{it}^y\}_{i \in I, t \geq 0}$ a sequence of prices, $\{R_t, w_t\}_{t \geq 0}$, and policies $T \equiv \{T_{Lt}, T_{Ht}\}_{t \geq 0}$ such that the household first order conditions and budget constraint, the firms' first order conditions, and the government's budget constraint are satisfied, the labor market clears ($L_t = 1$), and the resource constraint (4) and asset market clearing condition (5) are satisfied.

$$\sum_I \pi_{ij}(c_{it}^y + c_{it}^o) + K_{t+1} = F(K_t, 1) + (1 - \delta)K_t \tag{4}$$

$$K_{t+1} = \sum_I \pi_{ij}(a_{it}^y + a_{it}^o) \tag{5}$$

I define the set of all *feasible* allocations, $\mathcal{X}^f$ as the set of allocations that satisfy the resource constraint (4). I define the set of all *implementable* allocations, $\mathcal{X}^I$ as the set of allocations for which prices and policies exist that implement all $\mathcal{A} \in \mathcal{X}^I$ as an equilibrium. When policy is held constant, the economy converges monotonically to the unique steady state. Let $\mathcal{X}_s^f$ denote the set of all feasible steady state allocations and $\mathcal{X}_s^I$ be the set of all implementable steady state allocations.

Discussion of preferences. The parameters σ_y, σ_o, and η govern the elasticity of substitution between consumption over the life-cycle and bequests. I make the following assumption about these parameters and the discount factor, β_i.

Assumption. I assume that $\sigma_y \geq \sigma_o \geq \eta$ and that $\beta_H \geq \beta_L$.

The above assumption allows for the possibility that households with higher incomes have a higher propensity to save out of their lifetime income. When any of the above inequalities are strict, as long as the high productivity types have higher permanent income, the marginal propensity to save out of permanent income for high-productivity type households, $\frac{\partial s^j_{Ht}}{\partial PI_H}$ is greater than for low-productivity households at both ages.

When all elasticity parameters are equal and discount factors are uniform across types, the marginal propensity to save out of permanent income is constant over types. In this case, any lump-sum transfer from the high-types to the low-types has no affect on aggregate savings or the interest rate. Therefore, the steady state capital stock is unaffected by fiscal policy. When the marginal propensity to save is higher for high-permanent-income households, a greater lump-sum transfer to the low types, T_L reduces aggregate savings. This puts upward pressure on the steady state interest rate R and reduces steady state capital, K. These results are summarized in the following Lemma.

Lemma 1 *If either $\sigma_y > \sigma_o$ or $\beta_H > \beta_L$, and $PI_{Ht} > PI_{Lt}$, the marginal propensity to save out of permanent income is higher for high-productivity types, $\frac{\partial s^y_{Ht}}{\partial PI_{Ht}} > \frac{\partial s^y_{Lt}}{\partial PI_{Lt}}$.*

If $\psi_a > 0$, and either $\sigma_y > \sigma_o$ or $\beta_H > \beta_L$ or $\sigma_o > \eta$, and $PI_{Ht} > PI_{Lt}$, the marginal propensity to save out of permanent income higher for high-productivity types $\frac{\partial s^j_{Ht}}{\partial PI_{Ht}} > \frac{\partial s^j_{Lt}}{\partial PI_{Lt}}$ for $j \in (y, o)$.

For a proof, see Appendix A.1.

Lemma 1 shows that this simple life-cycle model nests several major explanations for non-homothetic savings behavior. When $\sigma_y > \sigma_o$, consumption later in life is considered a luxury, and households consume a greater share in the second period as their lifetime income increases (Straub, 2019).[8] Whenever $\sigma_o > \eta$, leaving bequests is a luxury good, and households leave larger bequests as their lifetime income increases (De Nardi, 2004, Straub, 2019, Mian et al., 2021). Finally, I allow for the possibility that high-productivity

[8]For example, more lavish retirements, private school for children, and out-of-pocket medical expenses are all luxury goods purchased later in life.

households are simply more patient, which may explain some of the observed differences in savings rates over the income distribution (De Nardi and Fella, 2017). Such *type dependent* preferences can capture other type-dependent channels – for example, entrepreneurial talent – in a simple way.

2.2 Redistribution and Steady State Welfare

I begin by considering the effect of an incremental change in steady state transfers to the low-productivity households, T_L on steady state social welfare. Consider a social planner with Pareto weights λ_i for each household type. Define steady state social welfare as in equation (6).

$$SW_s = \sum_I \lambda_i \pi_i \left(\frac{(c_i^y)^{1-\sigma_y}}{1-\sigma_y} + \beta_i \frac{(c_i^o)^{1-\sigma_o}}{1-\sigma_o} + \beta_i \psi_a \frac{(a_i^o)^{1-\eta}}{1-\eta} \right) \tag{6}$$

The product $\lambda_i (c_i^y)^{-\sigma_y}$ reflects the marginal value from the planner's perspective of giving additional resources to a type-i household.[9] For a given set of Pareto weights, as the consumption of type-i households falls, their marginal utility of consumption, and therefore their welfare weight increases. For a given allocation therefore, the ratio between the welfare weights of the two household types characterizes the degree of inequality. I make the following assumption about the Pareto weights.

Assumption. I assume that $\lambda_H \geq \lambda_L$.

By assuming that the Pareto weights assigned to the high-productivity households are always higher, I ensure that the planner will never prefer allocations in which the low-productivity types consume *more* than the high-productivity types, and therefore that the permanent income and the marginal propensity to save for the high types will always be higher than that of the low types. I define the unconstrained first-best steady state allocation, A_s^u and the optimal constrained steady state allocation, A_s^*.

Definition. Define the optimal *unconstrained* steady state allocation, $\mathcal{A}_s^u \equiv \{\{c_i^{yu}, c_i^{ou}\}_{i\in I}, K^u\} \equiv \underset{\mathcal{A}\in\mathcal{X}_s^f}{\text{argmax}}\ SW_s$ and the optimal *constrained* allocation, $\mathcal{A}_s^* \equiv \{\{c_i^{yu}, c_i^{ou}\}_{i\in I}, K^u\} \equiv \underset{\mathcal{A}\in\chi_s^I}{\text{argmax}}\ SW_s$.

[9] Note that in equilibrium, the households' inter-temporal conditions ensure that this term is proportional to the change in welfare of type-i households receiving additional consumption when old or being able to leave greater bequests.

The optimal unconstrained allocation is the allocation that maximizes steady state social welfare subject to the allocation satisfying the resource constraint (feasibility). The optimal constrained allocation maximizes steady state social welfare subject to implementability. That is, there must exist prices, financial positions, and a policy that implement this allocation as a competitive equilibrium. The ratio of the marginal utility of the two types characterizes the degree of inequality for a particular allocation.

Lemma 2 *(First best inequality) In the unconstrained first best steady state allocation, $\mathcal{A}_s^u$, inequality is given by (7).*

$$\frac{(c_H^{yu})^{-\sigma)y}}{(c_L^{yu})^{-\sigma_y}} = \frac{\lambda_H}{\lambda_L} \tag{7}$$

For a proof, see Appendix A.2.

Lemma 2 says that the unconstrained planner will choose an allocation that sets the marginal social welfare benefit of redistributing to the low-type equal to the cost of redistributing away from the high type. If the ratio of the marginal utilities for a given allocation is *higher* than this ratio – that is the marginal utility of the low types is relatively higher – this implies a greater level of inequality.

How does inequality in the constrained optimum differ from the first best allocation? To build intuition, it is helpful to first examine how a small increase in the lump-sum transfer from high-types to low types affect steady state welfare. First, the transfer affects welfare directly by shifting resources between households with different welfare weights. If $\lambda_L(c_L^y)^{\sigma_y} > \lambda_H(c_H^y)^{\sigma_y}$, this direct effect of the policy will be positive. Second, if $\psi_a > 0$ and households have a bequest motive, the redistribution will equalize the bequest distribution as well, further increasing the lifetime resources of the low-productivity households. Finally, the policy may change the steady state level of aggregate capital, which in turn would affect welfare by increasing the total level of bequests, and through general equilibrium effects on household income. These results are summarized in Lemma 3.

Lemma 3 *(Welfare impact of redistribution)*
Denote K_{PI} as the semi-elasticity of the steady state capital stock to the amount of lump-sum redistribution, $\frac{dK}{dT_L}\frac{1}{K}$. Define $\omega_i = \lambda_i (c_i^y)^{-\sigma_y}$ for $i \in I$.

The steady state change in social welfare, dSW_s from a small increase in T_L is:

$$dSW_s = \underbrace{\sum_I \pi_i \omega_i dT_i}_{\text{Direct Effect}} + \underbrace{RK\frac{1}{2}\sum_I \pi_i \omega_i d\Gamma_i^b}_{\text{Bequest Distribution}} + \underbrace{\left(\frac{1}{2}\sum_I \pi_i \omega_i a_i^0 R + wL\Theta w_K\right)K_{PI}}_{\text{Change in Capital}}$$

Here, $\Theta \equiv \sum_I \omega_i \pi_i \left(\frac{\pi_y \theta_i}{L} - \left(\frac{a_i^y}{KR} + \frac{a_i^o}{K}\right)\right)$, and Γ_i^b denotes type-i household bequest's share of total capital.

For a proof, see Appendix A.3.3.

Lemma 3 says that the total effect of the transfer can be decomposed into the direct effect, the effect on the distribution of bequests, and the effect of changing the aggregate level of capital.[10] The change in steady state capital affects welfare in two ways. First, through the effect of a change in capital on factor prices, and second, whenever $\psi_a > 0$, through changes in aggregate bequests left. The welfare impact of the change in factor prices is summarized by the term Θ. What the total effect of these changes are on aggregate welfare depends on whether Θ is positive, which in turn depends both on the the rate of return, R at the current steady state and on the current steady state distribution of capital and labor income.

When the steady state gross rate of return, $R > 1$, the steady state is dynamically efficient, and more capital increases average consumption. Furthermore, when savings rates are increasing in permanent income, high-productivity households have a greater share of aggregate capital income than aggregate labor income. Therefore, if the welfare weight of the low-productivity households is higher than that of the high productivity households, when $R > 1$ and savings rates increase with permanent income, an increase in capital improves welfare ($\Theta > 0$) by both increasing average consumption and by increasing wage income relative to capital income, disproportionately benefiting low-income households.

Whether the planner faces a redistribution-investment trade-off depends on the welfare impact of additional capital at the steady state associated with the first best level of inequality. At this steady state, the direct benefit of redistributing resources from the high

[10]Note that this result relies on a standard application of the envelope theorem. Households are already optimizing with respect to bequests and therefore the policy has no first order effect on utility associated with bequests.

to the low types has been exhausted. That is, the economy is at the ideal level of equality. A small reduction in the amount of redistribution would therefore have no *direct* effect on steady state social welfare. If MPS out of permanent income are higher for high-types *and* additional capital at this steady state would increase welfare ($\Theta > 0$), the planner could improve welfare by reducing the degree of redistribution and tolerating a slightly higher level of inequality. Proposition 1 summarizes this result.

Proposition 1 *(Redistribution-Investment Trade-off)*
Let $\bar{K}$ be the steady state level of capital at the equilibrium, $\mathcal{X}^e$ with first-best equality such that $\frac{(c_H^y)^\sigma}{(c_L^y)^\sigma} = \frac{\lambda_H}{\lambda_L}$. Let $\mathcal{A}_s^* \equiv \{\{c_i^{y*}, c_i^{o*}\}_{i \in I}, K^*\}$ be the constrained optimal steady state allocation.

(1) If $\beta_L = \beta_H$ and $\sigma_y = \sigma_o = \eta$, then $\frac{(c_H^{y*})^\sigma}{(c_L^{y*})^\sigma} = \frac{\lambda_H}{\lambda_L}$.

(2) If $F_K(\bar{K}) > \delta$ and either (a) $\beta_H > \beta_L$ or (b) $\sigma_y > \sigma_o$ and $\lambda_H > \lambda_L$ or (c) $\sigma_o > \eta$, $\psi_a > 0$, and $\lambda_H > \lambda_L$, then $\frac{(c_H^{y*})^\sigma}{(c_L^{y*})^\sigma} > \frac{\lambda_H}{\lambda_L}$.

For a proof, see Appendix A.3.

Proposition 1 states that when MPS are uniform, the degree of inequality in the constrained optimal allocation is identical to first-best. Intuitively, redistribution has no effect on aggregate capital in this case, and the planner faces no trade-off between additional redistribution and capital accumulation. However, when the high types have higher MPS at the allocation with first-best equality, and the steady state corresponding to the first best level of inequality is dynamically efficient, the constrained optimal level of inequality is greater than first best.

To see why, suppose that the government were to use T_L to implement the first-best level of inequality. Because the choice of redistribution policy pins down the level of steady state capital, it may be the case that the marginal product of this level of capital, $F_K(\bar{K})$ is higher than the depreciation rate.[11] In this case, reducing T_L would boost the capital stock and increase average consumption. At the same time, the resulting small increase in inequality would have no *direct* impact on welfare, as the economy is currently optimizing with respect to the inequality level. Therefore, implementing the first-best level of inequality is not optimal, and the planner should reduce T_L until the costs of greater inequality equal the benefit of additional capital.

[11]Note that unconstrained the first-best level is the Golden Rule capital stock, in which $F_K(K) = \delta$.

2.3 Comparative Statics

While the conditions outlined above are sufficient for the *existence* of a trade-off between non-distortionary permanent income and investment, the *size* of the trade-off – *how much* equality should the constrained planner sacrifice – depends on features of economic environment. In particular, it depends on both the interest rate elasticity of firms and of savers. Intuitively, suppose the savings supply contracts in response to redistribution and borrowing costs rise, but firm investment is inelastic to the interest rate. In this case, the welfare cost of redistribution will be muted, implying greater equality at the optimal allocation. Similarly, if savings are highly elastic to the interest rate, then as interest rates rise in response to redistribution, redistribution may simply crowd in household savings and crowd out household debt rather than curbing capital investment. In this case, the size of the redistribution-investment trade-off also shrinks. The following exercises illustrate these ideas.

2.3.1 Capital-Labor Substitutability

Recall that firms produce using a CES production function with elasticity parameter $\gamma \in (-\infty, 1)$. As γ approaches $-\infty$, capital and labor becomes perfect complements, and as γ approaches 1, they become perfect substitutes. Varying γ affects the size of the trade-off by varying firm's interest rate elasticity of capital as well as the distributional effects of a higher capital stock. Proposition 2 summarizes this result.

Proposition 2 *Let $\bar{K}$ be the steady state level of capital at the equilibrium, $\mathcal{X}^e$ with first-best equality such that $\frac{(c_H^y)^\sigma}{(c_L^y)^\sigma} = \frac{\lambda_H}{\lambda_L}$. Let $\mathcal{A}_s^* \equiv \{\{c_i^{y*}, c_i^{o*}\}_{i \in I}, K^*\}$ be the constrained optimal steady state allocation.*

If $F_K(\bar{K}) > \delta$ and either (a) $\beta_H > \beta_L$ or (b) $\sigma_y > \sigma_o$ and $\lambda_H > \lambda_L$ or (c) $\sigma_o > \eta$ and $\lambda_H > \lambda_L$, then $\frac{(c_H^{y*})^\sigma}{(c_L^{y*})^\sigma} - \frac{\lambda_H}{\lambda_L} > 0$ and is maximized at some $\gamma \in (-\infty, 1)$.

For a proof, see Appendix A.5.

Intuitively, as the production function approaches perfect complements, firms' interest rate elasticity approaches 0. Instead of crowding out capital, redistribution simply pushes up the equilibrium interest rate, crowding in household savings to maintain firms' original capital stock. The cost of redistribution declines as a result, implying greater equality in the constrained optimum. As $\gamma \to 1$, firms becomes more responsive to higher borrowing costs, and redistribution crowds out more capital. However, the elasticity of wages to capital declines, lessening the distributional benefits of additional capital. These two countervailing forces imply that the welfare costs of redistribution are maximized at some $\gamma \in (\infty, 1)$.

2.3.2 Open Economy

How does the size of the trade-off change when the domestic economy is open to foreign savings? Suppose the size of the domestic population were scaled by $\rho \geq 1$, so that the measure of type-i age-j households $= \pi_{ij}/\rho$. Let the measure of the foreign population, $\pi^f = (\rho - 1)/\rho$. I define the domestic asset demand from foreigners, $a^f(R)$ and assume a constant interest rate elasticity, a^f_R. The size of the redistribution-investment trade-off depends on both the scale of the domestic population, as well as the interest rate elasticity of foreign households.

Proposition 3 *(Open Economy) Let $\bar{K}$ be the steady state level of capital at the equilibrium, $\mathcal{X}^e$ with first-best equality such that $\frac{(c^y_H)^\sigma}{(c^y_L)^\sigma} = \frac{\lambda_H}{\lambda_L}$. Let $\mathcal{A}^*_s \equiv \{\{c^{y*}_i, c^{o*}_i\}_{i \in I}, K^*\}$ be the constrained optimal steady state allocation.*

If $F_K(\bar{K}) > \delta$ and either (a) $\beta_H > \beta_L$ or (b) $\sigma_y > \sigma_o$ and $\lambda_H > \lambda_L$ or (c) $\sigma_o > \eta$ and $\lambda_H > \lambda_L$, then $\frac{(c^{y*}_H)^\sigma}{(c^{y*}_L)^\sigma} - \frac{\lambda_H}{\lambda_L} > 0$ and decreasing in ρ and a^f_R.

For a proof, see Appendix A.4.

Intuitively, as ρ approaches ∞, we approach the small economy limit. In this case, the supply of domestic savings makes up such a small share of the savings available to firms, that a decrease in aggregate savings following additional redistribution will have no affect on the domestic capital stock. Similarly, as the interest rate elasticity of foreign households increases, an increase the interest rate following additional redistribution will crowd in foreign savings and mute the effect on investment.

2.4 Redistribution and Welfare Along the Transition

To see how taking into account short run welfare affects the trade-off, I consider the problem of a social planner who weights each household type with Pareto weights, λ_i and discounts generations at constant rate, ν. Here, social welfare is defined as the infinite Pareto-weighted discounted sum of the lifetime utility of all households as in equation (8).

$$SW(x) = \sum_I \pi_i \lambda_i \sum_{t=0}^{\infty} \nu^t \left(\frac{(c^y_{it})^{1-\sigma_y}}{1-\sigma_y} + \nu^{-1} \beta_i \frac{(c^o_{it})^{1-\sigma_o}}{1-\sigma_o} + \beta_i \nu^{-1} \psi_a \frac{(a^o_{it})^{1-\eta}}{1-\eta} \right) \tag{8}$$

I characterize a second set of sufficient conditions for the existence of a redistribution-investment trade-off analogous to the one presented the previous section. That is, I outline conditions under which the optimal redistribution policy results in a level of intra-generational inequality that is greater than first best. As in the previous section, whether it is optimal for fiscal policy to implement the first best level of intra-generational inequality depends on the distribution of MPS out of permanent income, and on whether the steady state associated with the first-best level of inequality is dynamically efficient. However, now the existence of a trade-off also depends on the rate at which the planner discounts the future. These results are summarized in Proposition 4.

Proposition 4 *(Redistribution investment trade-off in the short run)*
Let $\bar{K}$ be the steady state level of capital at the equilibrium, $\mathcal{X}^e$ with first-best equality such that $\frac{(c^y_H)^\sigma}{(c^y_L)^\sigma} = \frac{\lambda_H}{\lambda_L}$. Let $\mathcal{A}^* \equiv \{\{c^{y*}_{it}, c^{o*}_{it}\}_{i \in I}, K^*_{t+1}\}_{t \geq 0}$ be the constrained optimal allocation given the initial capital stock, K_0.

(1) If $\beta_L = \beta_H$ and $\sigma_y = \sigma_o = \eta$, then $\frac{(c^{y*}_{Ht})^\sigma}{(c^{y*}_{Lt})^\sigma} = \frac{\lambda_H}{\lambda_L}$ for all $t \geq 0$.

(2) If $F_K(\bar{K}) > \delta$ and either (a) $\beta_H > \beta_L$ or (b) $\sigma_y > \sigma_o$ and $\lambda_H > \lambda_L$ or (c) $\sigma_o > \eta$ and $\lambda_H > \lambda_L$, then there exists a $\hat{\nu} \in (0, 1)$ and $\tau > 0$ such that if $\nu > \hat{\nu}$, $\frac{(c^{y*}_{Ht})^\sigma}{(c^{y*}_{Lt})^\sigma} > \frac{\lambda_H}{\lambda_L}$ for some $t \geq 0$.

For a proof, see Appendix A.6

Proposition 4 states that when MPS are uniform, the optimal allocation features the first-best level of inequality at every time horizon. Intuitively, because redistribution has no effect on capital accumulation, the planner faces no trade-off between redistribution and investment, and therefore it is optimal to simply use redistribution to achieve the first-best level of equality in every period. However, when the high types have higher MPS out of permanent income at the steady state corresponding to the first best level of inequality, and this steady state is dynamically efficient, then as long as the planner puts *sufficiently high weight on future generations*, a trade-off exists between permanent income redistribution and investment, and the first best level of inequality is not optimal.

To see why, again consider a government who sets T_{Lt} every period in order to achieve the first-best path of inequality for all $t \geq 0$. Such a policy pins down a particular path for capital, $\{K_t\}_{t \geq 1}$. No matter the level of initial capital, K_0, by setting such a policy, the planner ensures that K_t will eventually converge to $\bar{K}$, the steady state level of capital associated with the first best level of equality. If this level of capital is dynamically efficient, then as K_t approaches $\bar{K}$, K_t will eventually become dynamically efficient for all $t \geq \tau$. T_{Lt} has been set to achieve optimal equality at time t, but because the high types have higher MPS, reducing T_{Lt} today will increase the entire future path of capital, and increase average consumption, C_t for all $t \geq \tau$. So long as the planner *does not discount the future too heavily*, the marginal benefit of additional future capital will outweigh the costs of higher inequality and lower aggregate consumption today, and the first-best level of inequality is not optimal at all horizons.

Intuitively, the planner can use redistribution policy in order to target a future path for capital. When deciding whether to increase savings to boost the future capital stock, they must weigh the benefits against the costs of greater inequality today. As long as capital is guaranteed to produce greater aggregate consumption in the future and the weight put on future generations is sufficiently large, a trade-off emerges and optimal policy will implement higher-than-first best levels of inequality in order to boost the future capital stock at some point.

2.5 Redistribution with More Fiscal Policy Tools

In the previous 2 subsections, when high-income households had higher MPS out of permanent income, a trade-off emerged between redistribution and capital accumulation because the income distribution was the single tool available to alter the savings level. In reality, governments have many fiscal tools available that allow them to alter the level of savings, including debt management, inter-generational transfers, and capital taxes/subsidies. In this section, I show how the sufficient conditions needed for an redistribution-investment trade-off along the transition change when the government has access to a larger set of policy tools.

Government. Suppose the government can now issue age-specific lump-sum taxes(transfers), T_{jt} in addition to type-specific lump-sum taxes(transfers), T_{it}. The government can also tax or subsidize capital directly, τ_{Kt} and borrow at the prevailing interest rate. The government's per-period budget constraint is given by Equation (9).

$$\sum_{I} \pi_i T_{it} + \sum_{A} \pi_j T_{jt} + \tau_{Kt-1} R_t K_t = R_t B_{t-1} - B_t \tag{9}$$

Crucially, the government also faces a set of *political constraints*. Equation (10) implies that the government cannot redistribute lump-sum from the current old to the current young. Equation (11) says that the government can issue debt but cannot invest directly.

$$T_{yt} \geq 0 \geq T_{ot} \tag{10}$$

$$0 \geq B_t \tag{11}$$

I adopt these assumptions for their realism. To my knowledge no scheme to redistribute directly from the current old to the current young – essentially reverse pay-as-you-go social security – exists. The few permanent government surpluses we observe in the data tend to be the result of state-owned natural resources rather than fiscal policy. If these restrictions are relaxed, and the government had a complete set of policy tools, I show below that they can implement the first best allocation and there is no trade-off.

Households. Households are identical to those in the previous section. For simplicity, we begin by considering the case without bequests in which $\psi_a = 0$. Household utility is still given by equation (1), however given the new fiscal policy, the type-i households' lifetime budget constraint is now given by the following.

$$c^y_{it} + \frac{c^o_{i,t+1}}{R_{t+1}(1-\tau_{Kt})} = w_t\theta_i + \frac{T_{ot+1}}{R_{t+1}(1-\tau_{Kt})} + T_{it} + T_{yt}$$

Equilibrium. An equilibrium, $\mathcal{X}$ is again defined as an allocation, $\mathcal{A}$, a sequence of prices, $\{R_t, w_t\}_{t\geq 0}$, and policies $\kappa \equiv \{T_{Lt}, T_{Ht}, T_{yt}, T_{ot}, B_t\}_{t\geq 0}$ such that the household first order conditions, the firms' first order conditions, and the government's budget constraint (9) are satisfied, the labor market clears ($L_t = 1$), and the resource constraint (4) and asset market clearing condition (5) are satisfied. I again define the set of all *feasible* allocations, $\mathcal{X}$ and the set of all *implementable* allocations, $\mathcal{X}^I$ as before.

As in the previous section, when MPS out of permanent income grow over the income distribution, redistribution decreases steady state capital, as resources are transferred from those with a high propensity to save to those with a lower propensity. However, now the fiscal planner has additional tools to influence the savings rate. A well known feature of overlapping generations models is the ability of fiscal policy that transfers resources from the current young to the current old to change the savings supply (Diamond 1965; Samuelson 1975). Both pay-as-you-go social security schemes and debt transfer resources from the saving young to the non-saving old, resulting in a lower capital stock. These results are presented in Lemma 4.

Lemma 4 *(Debt and Social Security Lower Savings)*
When $\psi_a = 0$, for a given set of policies, $T_{Ht}, T_{Lt}, \tau_{Kt}$, steady state capital, K is decreasing in steady state inter-generational transfers, $T_{yt} - T_{ot}$ and steady state debt, B_t.

A proof of 4 can be found in Appendix **??**.

Lemma 4 implies that once the fiscal planner's political constraints bind, they can no longer rely on debt management or inter-generational transfers to increase the capital stock, and must trade-off the benefits of equality against the cost of lower future capital accumulation and distortions associated with a capital tax/subsidy. However, when the political constraints do not bind the planner has all the tools needed to achieve the optimal amount of capital accumulation while also achieving the first best level of inequality. Therefore, the presence of a redistribution-capital accumulation trade-off now depends on whether savings behavior is non-homothetic and on whether the steady state with the first best level of inequality, no capital tax, and *binding political constraints* is dynamically efficient. If so, then for sufficiently high γ, the optimal redistribution policy will result in a higher than first-best level of inequality. These results are summarized in Proposition 5.

Proposition 5 *Assume $\psi_a = 0$. Let $\bar{K}$ be the steady state level of capital at the equilibrium, $\mathcal{X}^e$ with first-best equality such that $\frac{(c_H^y)^\sigma}{(c_L^y)^\sigma} = \frac{\lambda_H}{\lambda_L}$, $\tau_K = 0$ and binding political constraints. Let $\mathcal{A}^* \equiv \{\{c_{it}^{y*}, c_{it}^{o*}\}_{i \in I}, K_{t+1}^*\}_{t \geq 0}$ be the constrained optimal allocation given the initial capital stock, K_0.*

(1) If $\beta_L = \beta_H$ and $\sigma_y = \sigma_o = \eta$, then $\frac{(c_{Ht}^{y*})^\sigma}{(c_{Lt}^{y*})^\sigma} = \frac{\lambda_H}{\lambda_L}$ for all $t \geq 0$.

(2) If $F_K(\bar{K}) > \delta$ and either (a) $\beta_H > \beta_L$ or (b) $\sigma_y > \sigma_o$ and $\lambda_H > \lambda_L$ or (c) $\sigma_o > \eta$ and $\lambda_H > \lambda_L$, then there exists a $\hat{\nu} \in (0,1)$ and $\tau > 0$ such that if $\nu > \hat{\nu}$, $\frac{(c_{Ht}^{y*})^\sigma}{(c_{Lt}^{y*})^\sigma} > \frac{\lambda_H}{\lambda_L}$ for some $t \geq 0$.

For a proof, see Appendix A.7.

To see why, consider a hypothetical steady state in which $\tau_{Kt} = 0$, T_{Lt} and T_{Ht} are set to implement the first-best level of intra-generational equality, and both political constraints bind. That is, $T_{yt} = T_{ot} = 0$ and $B_t = \bar{B}$ for all $t \geq 0$. This set of policies is associated with a unique steady state level of capital, $\bar{K}$. If $\bar{K}$ is greater than or equal to the modified golden rule (first-best) level of capital, then the planner can use debt or transfers from the young to the old to lower the capital stock, while using T_{Lt} and T_{Ht} to achieve the first best level of inequality.

If instead $F_K(\bar{K}) > \delta$ and the steady state is dynamically efficient, then if debt, inter-generational transfers, and τ_{Kt} remain unchanged, K_t will eventually converge to $\bar{K}$ and the economy will eventually become dynamically efficient. That is, there exists some future period τ such that for all $t \geq \tau$, increasing capital increases aggregate consumption. If future generations are given sufficient weight – as $\nu \to 1$, the welfare impact of this additional capital outweighs the costs of reduced consumption and greater inequality today. In this case, the planner will choose to keep inter-generational transfers and debt at their constrained level, so as to not further reduce the investment rate. They will set intra-generational redistribution policy and capital subsidies so that the benefits of capital to future generations equals the cost of deviating from the first best level of equality and the distortions created by the capital tax.

2.6 Labor Income Redistribution

As alluded to in the introduction, the trade-off between permanent income redistribution and investment will apply to any redistributive policy that affects permanent income. I consider a simple labor income redistribution scheme as an illustrative case, and make the following small modifications to the simple model.

Households. Households' labor supply is now endogenous and supplied only in the first period of life. Type-i households choose ℓ_{it} and receive $(1-\tau_\ell)w_t\ell_{it}\theta_i$ in after-tax labor income while young and are retired when old. As in the previous section, households can borrow or save each period with gross rate of return R_{t+1}. For simplicity, I consider the version of the model without bequests ($\psi_a = 0$), however none of the results presented below depend on this assumption. Household lifetime utility is now given by equation (12).

$$u(c^y_{it}, c^o_{it}, \ell_{it}) = \frac{(c^y_{it})^{1-\sigma_y}}{1-\sigma_y} + \beta_i \frac{(c^o_{it})^{1-\sigma_o}}{1-\sigma_o} - \frac{(\ell^h_{it})^{1+\gamma}}{1+\gamma} \tag{12}$$

In addition to paying labor income taxes, households receive a uniform lump-sum transfer, T. The type-i households' lifetime budget constraint is given equation (13).

$$c^y_{it} + \frac{c^o_{i,t+1}}{R_{t+1}} = (1-\tau_{\ell t})w_t\theta_i\ell_{it} + T = PI_{it} \tag{13}$$

Government. The government runs a balanced budget each period and can fund lump-sum transfers using linear taxes on labor income, $\tau_{\ell i}$.

$$\sum_{i\in I} \pi_i T_i = \sum_{i\in I} \pi_i \bigg(w_t \ell_{it}\theta_i\tau_\ell \bigg)$$

Equilibrium. An equilibrium is a sequence of quantities, $\{\{c^h_{it}, a^h_{it}\}_{i\in I, h\in H}, K_t, L_t\}_{t\geq 0}$, prices, $\{R_t, w_t\}$, and policies $\{T, \tau_\ell\}$ such that the household first order conditions, the firms' first order conditions and the government's budget constraint are satisfied, the labor market clears, and the goods and asset markets clear.

Again defining steady state social welfare as the Pareto weighted sum of household utility, I consider the incremental impact on steady state welfare of a small budget-balancing increasing in the uniform lump-sum transfer, dT, funded by a small increase in the labor income tax, $d\tau_\ell$. I show that the welfare effect this change in fiscal policy can be decomposed into the direct effects of redistributing labor income from those with higher-than-average labor income to those with lower-than average, the effects of distorted labor supply, and the non-homothetic savings channel. This decomposition is presented in theoremion 6.

Proposition 6 *(Effect of labor income redistribution on steady stare welfare) Define Θ, w_K as in Lemma 2, and let w_L be the labor elasticity of the wage. Define K_{PI} as the semi-elasticity of capital to the direct effects of the tax and L_{τ_ℓ} is the labor supply semi-elasticity with respect to τ_ℓ. Let $\omega_i = \lambda_i (c_i^y)^{-\sigma_y}$. Then the impact of an incremental increase in τ_ℓ on social welfare is:*

$$dSW = \underbrace{\sum_I \omega_i \pi_i (wL - w\theta_i \ell_i) d\tau_\ell}_{\textit{Direct Effect of Redistribution}} + \underbrace{wL(\Theta + \tau_\ell) w_K K_{PI}}_{\textit{Direct Effect of NH Savings}}$$
$$\underbrace{wL(\Theta + \tau_\ell)\left(w_L + \frac{\tau_\ell}{\Theta + \tau_\ell}\right) L_{\tau_\ell}}_{\textit{Direct Effect of Labor Distortion}} + \underbrace{wL(\Theta + \tau_\ell)(\mathcal{L} + \mathcal{K})}_{\textit{Feedback Effects}}$$

Where $\mathcal{L}$ and $\mathcal{K}$ are defined as in equation (A.19) and (A.22).

For a proof, see Appendix A.8.

Like lump-sum redistribution, the labor income tax affects social welfare directly by transferring lifetime income between households with potentially different social welfare weights, ω_i. Now however, because labor is endogenous, the redistribution policy also distorts the aggregate labor supply, which lowers taxable labor income directly and indirectly through changes in the equilibrium wage. When marginal propensities to save, $\frac{\partial s_{it}^j}{\partial PI_i}$ co-varies with labor productivity, an additional welfare channel emerges that is proportional to K_{PI}, the semi-elasticity of steady state capital to the direct effect of the redistribution on the permanent income distribution.

Finally, the policy impacts welfare through interaction between the latter two channels. The distortion of labor supply caused by τ_ℓ impacts firms' incentives to invest in capital and households' incentive and ability to save. These effects are captured in the term $\mathcal{K}$. At the same time, the decline in capital affects firms' labor demand and households' incentives to work. These effects are summarized in the term $\mathcal{L}$.

3 Estimating Marginal Propensities to Save

In this subsection, I estimate the distribution of marginal propensities to save out of permanent labor income flows and each households tax burden using data from the Panel Survey of Income Dynamics (PSID). This term requires estimates of the lifetime average marginal propensity to save (MPS) out of annual permanent income flow for each labor-productivity type, as well as the difference between average annual labor income and annual labor income for each age-productivity group. Estimating the latter is straightforward using labor income data.

Both Straub (2019) and Dynan et al. (2004) (henceforth DSZ) use the PSID to explore the relationship between savings behavior and permanent income. Straub (2019) uses consumption data beginning in 1999 to estimate the elasticity of consumption to permanent income. These estimates can be used to generate an implied savings *elasticity*.[12] While this is not the statistic in my formula, in principle, these implied savings elasticities could be combined with savings *rates* out of permanent income by permanent income type to generate marginal propensities to save. I report the results of combining this implied savings elasticity with my own estimates of savings out of permanent income.

DSZ use the PSID to estimate the marginal propensity to save out of permanent income by permanent income type in 2 ways. First, they use variation in the cross section and simply divide the change in median savings rates between income quintiles by the change in median income to trace out a marginal savings schedule. Second, they use time-series variation and regress the change in average individual household savings between an earlier and later sample on the change in household income. Their cross sectional MPS estimates use a change-of-wealth savings measure, which includes capital gains and therefore may not accurately reflect the supply of loanable funds available to firms (Gale and Potter, 2002). They provide time-series estimates for both the change-of-wealth measure and an 'active' savings measure corresponding to the change in wealth minus capital gains, corrected for inflation and reporting error.

I follow DSZ and exploit cross-sectional differences in permanent income. However, my approach differs from theirs in that I use variation in permanent income *within* a permanent income quintile rather than *across* permanent income quintiles, as my formula calls for the within-group MPS. Furthermore, I use a more direct measure of active savings: income less consumption ($Y_{it} - C_{it}$). The reason they were unable to consider a more straightforward income less consumption measure of active savings is that consumption data did not appear in the PSID until 1999. Thankfully, the introduction of a set of consumption questions into

[12] For example, a consumption elasticity of .7 implies an approximate savings elasticity of 1.3.

the survey in 1999, and an additional set in 2005, means that it is possible to observe both many years of a household's active savings – measured simply as income less consumption – as well as many years of past and future income.

Empirical Strategy. To estimate $\frac{\partial s_i}{\partial PI_i}$ and $\frac{wL - w\theta_i^h \ell_i^h}{wL}$, I first define productivity types as permanent labor income quintiles. I do not observe permanent labor income in the data, and therefore must estimate it. I provide detail on this estimation procedure below. I measure the savings of household j at time t (who is type-i, age-h), S_{hijt} as current total income less consumption and taxes. This measures so called 'active' savings – as opposed to changes in total wealth – which is the closest analog to s_i in the model and captures the change in loanable funds available to firms.[13] With an estimate for permanent income, $\hat{PI}_{ijt}$ in hand, I estimate a quintile's average MPS using the following equation for each quintile.

$$S_{hijt} = \beta_{0i} + \beta_{1i}\hat{PI}_{ijt} + X_{hijt} + \epsilon_{hijt} \tag{14}$$

Here, X_{hijt} is a vector of controls, ϵ_{hijt} is an error term, and β_{0i} is a constant. The estimated coefficient $\hat{\beta}_{1i}$ can be used as the estimate of $\frac{\partial s_i}{\partial PI_i}$.

This strategy identifies MPS out of permanent income using within quintile cross-sectional variation. For this specification to be valid, it must be the case that there is no third factor driving both household permanent income and savings behavior. An obvious candidate of such a factor would be age, and I include the age of the household's reference person in the set of controls X_{hijt}. Similarly, general macroeconomic conditions over a household's sample would affect both my estimate of their permanent income as well as their savings behavior. To address this I also include average annual labor income at year t.

Other factors such as lifestyle or innate preferences could also impact both savings and permanent income. In a second set of regressions I add additional household level controls for the reference person's education, martial status, and family size to attempt to capture these factors.

In the data, I only observe a household's total annual labor income flow, and cannot directly observe what fraction of their current income reflects permanent rather than transitory income. Furthermore, because current income is used to construct my measure of current savings, any measurement error in current income will bias my estimate of the marginal propensity to save. To deal with these issues, I follow DSZ and use both lagged labor income and future labor income as a proxy for a household's permanent labor income flow. Let Y_{pij}

[13] Measuring savings as changes in total wealth include capital gains. If the value of a household's assets, in particular their house, appreciates, this would increase total wealth but would not reflect new resources available for investment.

be the average labor income of household j for a sample of four years prior to(after) the start of their sample.[14] For a sample sufficiently far in the past(future) and for a sufficiently low persistence transitory income process, Y_{pij} should be correlated only with the permanent component of Y_{hijt}. I regress current after-tax labor income on lagged(future) income and an age group dummy variable as in equation (15), and use the fitted values $\hat{\beta}_1$ and $\hat{\beta}_2$ to predict $\hat{PI}_{hijt}$.

$$\hat{Y}_{hijt} = \beta_0 + \beta_1 Y_{pij} + \beta_2 \mathbb{1}_{\text{Age Group}} + \epsilon_{hijt} \tag{15}$$

Because the dataset is not balanced across age, for both measures of permanent income I calculate the permanent income quintile for each age separately. That is, household j is put in quintile i if their estimated permanent income is in the ith quintile *for their age group.* All regressions control for age group and average annual labor income in year t. I run an additional set with controls for household characteristics including marital status, family size, and education of the response person. PSID sample weights are used in all regressions and summary statistics, and robust standard errors are used to correct for heteroskedasticity.

The Data. The PSID is a longitudinal household survey which began in 1968. The survey was conducted annually until 1997, at which point it became bi-annual. In 1999, the survey added a large group of questions about household consumption, covering about 70% of the categories in the Consumption Expenditure Survey (CEX). In 2005, an additional set of categories was included. Because the 'active' saving concept I use is income less consumption, I will only be able to measure savings starting in 1999. However, I use income data starting in 1995 to construct my lagged income proxy for permanent income.

To construct my consumption measures, I simply add up all consumption categories together using the 1999 and 2005 set of categories respectively. The PSID total family income measure includes all taxable income of both the respondent and their spouse, as well as all transfer income and social security income. Savings is then simply calculated as total family income less taxes and consumption. I exclude households with missing data or with unrealistically high levels of any individual consumption category.[15] All variables are then put in terms of 2019 dollars. I drop households with missing income data, households younger than 25, households with fewer than 5 years of responses, and households with less than $1,000 of total family income. I split the sample into 4 equally sized age groups, g.

The PSID reports only pre-tax income, however I need post-tax income in order to

[14] For example, if I observe a household starting in 2001, I begin that household's sample in 2005 and set the proxy Y_{pij} to be their average labor income in the years prior to 2005.

[15] Specifically, any household that spends more than a million dollars on any category.

Table 1: PSID Summary Statistics

	Quintile 1	Quintile 2	Quintile 3	Quintile 4	Quintile 5
			All Ages		
Median Income	34,147	50,644	70,193	103,175	174,726
Saving Rate ('99)	0.12	0.16	0.19	0.25	0.36
Saving Rate ('05)	0.00	0.02	0.06	0.12	0.23
			Ages 20-35		
Median Income	22238	29,178	45,502	75,098	121,959
Saving Rate ('99)	0.05	0.07	0.12	0.18	0.31
Saving Rate ('05)	-0.14	-0.09	-0.04	0.06	0.18
			Ages 35-50		
Median Income	33923	55,640	80,221	111,767	181,068
Saving Rate ('99)	0.08	0.15	0.19	0.26	0.35
Saving Rate ('05)	-0.07	0.02	0.05	0.12	0.22
			Ages 50-65		
Median Income	36,973	62,369	87,852	121,391	200,100
Saving Rate ('99)	0.16	0.24	0.28	0.33	0.41
Saving Rate ('05)	0.05	0.12	0.18	0.18	0.28
Observations ('99)	7,966	8,943	9,249	8,021	6,789
Observations ('05)	5989	7,164	7,500	6,367	5,321

This table reports summary statistics for the PSID data by age group and permanent income quintile. Saving is calculated as annual total post-tax income less consumption. The Savings rate is equal to savings over current total income. Saving Rate ('99) is the average savings rate using only the 1999 consumption measures. Saving ('05) uses the 2005 consumption measures.

properly measure household active saving. To estimate each household's annual total tax payment, I use the NBER TAXSIM program.

Results. Table 2 reports the results from the estimation procedure outlined above. Columns labeled '1999' and '2005' report estimates using the 1999 and 2005 measures of consumption respectively. Columns with household controls control for education, marital status, and family size in addition to age group and average annual labor income in year t. The left four columns report the estimated MPS for each income quintile when the symmetric income average is used to construct the proxy for permanent income. A clear pattern emerges. Household in higher income quintiles tend to have higher marginal propensities to save out of permanent income. The differences are especially pronounced between the top quintile and the 3rd and 4th quintile, and between the 3rd and 4th quintile, and the bottom 2 quintiles. The panels on the right report estimates using lagged labor income to construct the proxy for permanent income. The results are largely similar.

The final column takes average savings rates out of current income for each permanent income quintile and multiplies them by 1.3 – the permanent income elasticity of savings

Table 2: Marginal Propensity to Save Out of Permanent Income

	Lagged Income				Future Income				Implied by
	1999	2005	1999	2005	1999	2005	1999	2005	Straub (2019)
Quintile 1	0.20	0.16	0.10	0.09	0.04	0.08	0.00	0.07	0.16
	(0.04)	(0.04)	(0.04)	(0.05)	(0.09)	(0.10)	(0.09)	(0.10)	
Quintile 2	0.40	0.26	0.23	0.12	0.30	0.23	0.03	0.04	0.21
	(0.06)	(0.07)	(0.06)	(0.07)	(0.20)	(0.22)	(0.06)	(0.07)	
Quintile 3	0.33	0.12	0.26	0.06	0.47	0.31	0.40	0.28	0.25
	(0.07)	(0.07)	(0.07)	(0.07)	(0.07)	(0.08)	(0.07)	(0.08)	
Quintile 4	0.49	0.39	0.38	0.31	0.32	0.24	0.31	0.26	0.33
	(0.06)	(0.06)	(0.06)	(0.06)	(0.08)	(0.09)	(0.09)	(0.09)	
Quintile 5	0.59	0.56	0.59	0.55	0.60	0.43	0.57	0.55	0.47
	(0.01)	(0.01)	(0.01)	(0.02)	(0.01)	(0.01)	(0.01)	(0.01)	
Household Controls	No	No	Yes	Yes	No	No	Yes	Yes	

Note. This table reports estimated marginal propensities to consume out of permanent income by permanent income quintile using PSID data. All regressions control for average age group and average labor income for a given year. All regressions use PSID sample weights and heteroskedasticity robust standard errors. Columns marked 1999 or 2005 use the 1999 or 2005 consumption data respectively. Household controls include marital status, family size, and education. The final columns multiplies the savings elasticity implied by Straub (2019) by average savings rates.

implied by Straub (2019). Multiplying a rate by an elasticity generates the derivative required by the formula. The estimated MPS implied by this procedure are very similar to the direct estimates.

4 Quantification.

In this section, I use the distribution of MPS out of permanent income estimated above to calibrate a richer quantitative OLG model with a realistic earnings life-cycle and idiosyncratic labor productivity shocks. As in the simple model, I use a combination of non-homothetic preferences over lifetime consumption and bequests, as well as type-dependent time preferences to match the estimated MPS out of permanent income. A potentially important model ingredient missing from the baseline quantitative model is heterogeneous rates of return. Empirical evidence shows that wealthier households receive higher returns to financial wealth (Fagereng et al. (2016), Bach et al. (2020)), which may be compensation for greater risk taking or evidence of greater investment skill. Idiosyncratic returns may help explain higher savings rates at the top of the income distribution (Benhabib et al. (2019), De Nardi and Fella (2017)).

I use this model to quantify the size of the trade-off between permanent income redistribution and investment in the steady state. In particular, I study the same simple labor income redistribution scheme as in Section 2.6, a linear labor income tax that funds a lump-sum transfer. In addition to redistributing from high to low productivity types and distorting labor supply, this tax impacts welfare by (i) providing insurance through redistributing resources from households with higher realized productivity to households with lower realizations, (ii) lowering precautionary savings and the capital stock as a result, and (iii) redistributing resources from older higher earning households to younger households, potentially impacting the capital stock.

I first show that in a standard model with homogeneous MPS out of permanent income, the direct effect of this tax on the permanent income distribution has no effect on the capital stock, confirming the results from Proposition 1. In my baseline model in which I directly target the distribution of MPS, this channel is present and quantitatively larger. I show that for small changes in the degree of redistribution, the size of the channel does not depend on *which* model primitives are used to target the distribution of MPS. The estimated MPS are sufficient statistics. However, as the size of the tax increases, the model primitives begin to matter.

Finally, I use the baseline model to compare the welfare costs associated with the redistribution-investment trade-off to the other channels present in the model. The change in social welfare as a result of the tax can be decomposed into the benefits of a more *equal distribution* of consumption over permanent productivity types and idiosyncratic states, and the costs of a decrease in aggregate consumption. My channel alone can account for about 20 percent of the decline in aggregate consumption. The welfare costs of this decline are equal to X percent of the welfare benefits of greater equality.

4.1 Environment.

Households. There is a mass of households indexed by $k \in [0, 1]$ who each live for J periods. Households supply labor to firms in all but the final retirement period, in which they receive type-specific social security income SS_i. There are I permanent labor productivity types. A household j born in year t who is age h and is type-i has labor productivity, $\theta_i^j e_{ik,t+j}$ where θ_i^j is the permanent component of labor income for type-i age-j households and $e_{ik,t+h}$ is that household's idiosyncratic labor shock that evolves according to an AR(1) process with persistence ρ_e and standard deviation σ_e.

Households receive $(1 - \tau_{\ell ijkt}(w_t \ell_{ik,t+j}^j \theta_i^j e_{ik,t+j}))w_t \ell_{ikt+j}^j \theta_i^j e_{ik,t+j}$ in after-tax labor income each period. Here, $\tau_{\ell ijkt}(w_{t+j} \ell_{ik,t+j}^h \theta_i^j e_{ik,t+j})$ is a progressive labor income tax following

Bénabou (2002) and Heathcote et al. (2017), given by equation (16). Here, $\bar{\tau}_\ell$ parameterizes the average level of the labor income tax, while γ_ℓ determines the degree of progressivity.

$$\tau_{\ell ijkt}(w_{t+j}\ell^j_{ik,t+j}\theta^j_i e_{ijt+h}) = 1 - (w_{t+j}\ell^j_{ij,t+j}\theta^j_i e_{ik,t+j})^{-\gamma_\ell}\bar{\tau}_\ell \tag{16}$$

Households can save or borrow in a one-period bond, $a^j_{ik,t+j}$, buy government bonds, B_t or capital K_{t+1} at gross after-tax interest rate $R_{t+1} = 1 + (1-\tau_K)r_{t+1}$. Households face a borrowing constraint $\underline{a}$ such that $\underline{a} < a^j_{ik,t+j}$. Type-i household receive share σ^i_π of profit flows each period, as well as $R_t a^0_{it}(1-\tau_b)$ in after-tax bequest income when the are born. Note that all type-i households recieve the same bequest transfer equal to the average type-i bequest the year before they are born, $a^0_{it} = a^J_{it-1}$.[16] Finally, households may receive a lump-sum transfer, T_t. Lifetime utility for a household born at time t is given by (17).

$$u(c^j_{ik,t+j}, \ell^j_{ik,t+j}, a^J_{ik,t+J}) = \sum_{j=1}^{J}\beta_i^{j-1}\left(\frac{(c^j_{ik,t+j})^{1-\sigma_j}}{1-\sigma_j} - \psi_\ell\frac{(\ell^j_{ik,t+j})^{1+\sigma_\ell}}{1+\sigma_\ell}\right) + \beta^J\psi_a\frac{(a^J_{ik,t+J}+\bar{a})^{1-\eta}}{1-\eta} \tag{17}$$

Note that this model nests the same three sources of non-homothetic savings behavior as the simple model presented in Section 2. I follow Straub (2019) and include the term $\bar{a}$ in households' bequest motive in order to generate a mass of households who give no bequests. Let $R^j_{t+j} = \prod_{\tau=0}^{j} R_{t+\tau}$. A type-i household born at time t has the following lifetime budget constraint, given by equation (18).

$$a^J_{ik,t+J} + \sum_J \frac{c^j_{ik,t+j}}{R^j_{t+j}} = R_t a^0_{it}(1-\tau_b) + \sum_J \frac{(1-\tau_{\ell ijkt})w_{t+j}\ell^j_{ijkt}\theta^j_i e_{ik,t+j} + T_{t+j}}{R^j_{t+j}} + \frac{SS_i}{R^J_{t+J}} \tag{18}$$

Firms. There is unit mass of monopolistically competitive intermediate goods firms indexed by $m \in [0,1]$ who rent labor and capital from households and produce a differentiated intermediate good, y^m_t according to a CES production function (19). A competitive final

$$y^m_t = Z_t\left(\alpha k^{m\gamma}_t + (1-\alpha)\ell^{m\gamma}_t\right)^{1/\gamma} \tag{19}$$

goods firm aggregates the intermediate goods using a Dixit-Stiglitz CES aggregator (20). This specification generates standard expression for demand for each intermediate good

[16] I make this simplifying assumption to avoid having to keep track of the history of idiosyncratic shocks across generations.

(21). Because there are no nominal rigidities, intermediate goods firms all produce the same

$$Y_t = \left(\int_0^1 {y_t^m}^{\frac{\epsilon-1}{\epsilon}} dm \right)^{\frac{\epsilon}{\epsilon-1}} \tag{20}$$

$$y_t^m = Y_t \left(\frac{p_t^m}{P_t} \right)^{-\epsilon} \tag{21}$$

level of output, employ the same labor and capital, and charge the same markup $\mu = \frac{\epsilon}{\epsilon-1}$ over marginal cost.

Government. The government taxes returns on capital, bequests, and labor income, and issues debt, B_t, pays social security, issues uniform lump-sum transfers, T_t. The government's per-period budget constraint is given by (22).

$$B_t + w_t \int_K \tau_{\ell ijkt} \theta_i^j e_{ijkt} \ell_{ijkt}^j dk + A_t r_t \tau_K + \sum_I \pi_{Ji} a_{it-1}^J \tau_b = T_t + R_t B_{t-1} + \sum_I \pi_{Ji} SS_i \tag{22}$$

Equilibrium. An equilibrium defined as a sequence of prices, $\{R_t, w_t\}_{t\geq 0}$, individual and aggregate financial positions, $\{\{a_{ijkt}^j\}_{k\in K, i\in I, j\in J}, A_t\}_{t\geq 0}$, policies, $\{\tau_{\ell t}, \tau_{bt}, \tau_{Kt}, B_t, G_t, T_t\}$, individual household and firm allocations, $\{\{c_{ikt}^j, \ell_{ikt}^j\}_{i\in I, j\in J, k\in K}, \{y_t^m, n_t^m, k_t^m\}_{m\in[0,1]}\}_{t\geq 0}$, and aggregate allocation, $\{K_t, L_t, C_t\}_{t\geq 0}$ such that the following conditions hold. Households' first order conditions and budget constraints hold for each productivity type, generation, and history of shocks. The intermediate goods firms' first order conditions (23) and (24), production function and demand hold. The competitive goods firm's technology constraint

$$w_t \mu = \left(\frac{y_t^m}{\ell_t^m} \right)^{\frac{1}{\rho}} (1-\alpha) Z_t \tag{23}$$

$$(r_t + \delta)\mu = \left(\frac{y_t^m}{k_t^m} \right)^{\frac{1}{\rho}} \alpha Z_t \tag{24}$$

holds. Finally, the government budget constraint, labor market clearing, asset market clearing, and resource constraint hold.

4.2 Calibration

Macroeconomic targets. A period in the model is 15 years. I set the depreciation rate, δ to target an investment share of 18%. I set the markup, μ to target a 7.5% profit share, and α to target a labor share of .67. In the baseline model I assume $\gamma = 0$ and therefore the production function in Cobb-Douglas. I normalize the supply of labor to 1, set K so that the capital to *annual* output share is 2.5, and set Z to normalize aggregate output in each period to 1. Net foreign assets are set to clear asset markets, which in the calibration results in a value of $NFA/K = .5$. The *annual* net rate of return is $r = .03$.

Government Policy. I set average labor income taxes, $\bar{\tau}_\ell$ to .35 and γ_ℓ to .15 following Heathcote et al. (2017). I follow De Nardi (2004) and Straub (2019) and set the bequest tax equal to .1. I set the capital tax to .4. I follow Huggett and Ventura (2000), De Nardi and Yang (2014), Straub (2019) in setting social security payments by income quintile (see Appendix X for details). Lump-sum transfers are initially set to 0. Government debt relative to GDP, B/Y is set to .65. G is set to satisfy the government's budget constraint, which in the calibration results in a value of government spending to output, $G/Y = .27$.

Household Income. Labor productivity by permanent income type and age, θ_i^j are set so that $\sum \pi_{ij}\theta_i^j = 1$ and so that relative labor productivity matches relative income by type and age in the data. The parameters ρ_e and σ_e are set as in De Nardi and Yang (2014).[17] The AR(1) process is discretized into a Markov process with 2 states. I set the equity shares to match the 2019 wealth Lorenz curve (Aladangady and Forde, 2021). Households live for 60 years (4 periods), ages 25 to 85. In order to match my own estimates from the PSID in the previous section, I assume $I = 5$ and that permanent labor productivity stays constant for 3 distinct 15-year periods, at which point households retire and have no labor earnings in the final 15-year period.

Household Preferences. I set the inverse Frisch elasticity, γ to 1/.82 following Chetty et al. (2011), and the inverse elasticity of inter-temporal substitution for the median age, $\bar{\sigma}$ to 2.5 following the literature. The weight on the disutility of labor, ψ_ℓ and the weight on the bequest motive, ψ_a are set to clear labor markets and target bequests as 10% of GDP respectively. The remaining parameters, $\{\beta_i\}_{i\in I}$, η, $\bar{a}$, and σ_{nh}, where $\sigma_j = \sigma_{nh}\sigma_{j+1}$, are all jointly calibrated so that the marginal savings rates by age and labor productivity type out of the *average annual permanent income flow* in the model target their counterparts in the data.

Figure 1 plots the baseline estimated MPS to save out of the average permanent income flow from Section 3, alongside their model counterparts for 2 versions of the model. In the baseline model I use a combination of non-homothetic preferences over lifetime consumption and bequests, as well as heterogeneous time preferences, β_i to try to match the estimated

[17]These parameters are correspond to a 5-year labor earnings shock. Households in my model have the same labor earnings for 15 years.

MPS. In the second version, I impose uniform β_i and only use non-homothetic preferences. Specifically, I iterate over a grid of possible values for $\sigma_{nh}, \beta_i, \bar{a}$, and η, select the combination of the parameters than clears asset markets and minimizes the sum of squared differences between the MPS out the average annual permanent income flow in the model, and the empirical counterpart.

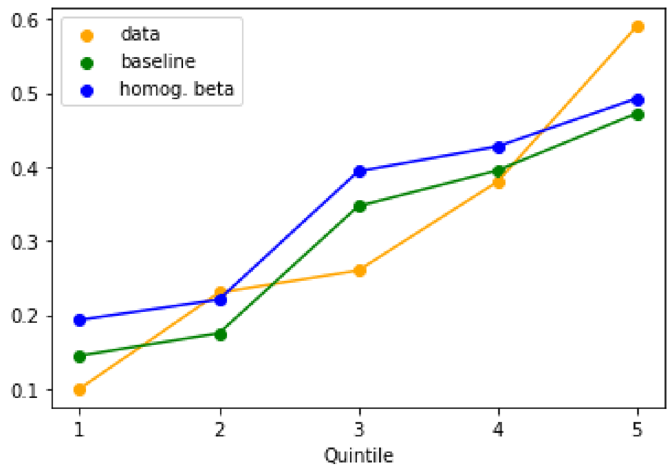

Figure 1: Marginal Propensities to Save out of Permanent Income

This Figure plots the estimated marginal propensities to save (MPS) out of permanent income by permanent labor income quintile from Table 2 (yellow). See Section 3 for details on this estimation procedure. The Figure also plots the MPS by permanent labor income type in the baseline quantitative model (green) and the model with homogeneous time preferences (blue).

Both strategies do well in matching the basic shape of the empirical distribution and have approximately the same slope.[18]

4.3 Policy Experiment

Suppose the fiscal authority increased the average labor income tax rate, $\bar{\tau}_\ell$, keeping the degree of progressivity γ_ℓ constant, in order to fund a budget-balancing uniform lump-sum transfer, T. I first consider the *direct* effect of this policy on the distribution of permanent income. Before any general equilibrium adjustment in prices or labor supply from their steady state levels, the effect of the policy on a household's expected permanent income is given by the following expression.

$$\Delta E[PI_i] = \tau \sum_J \frac{(w^s L^s - w^s \theta_i^j \ell_i^{j\,s})}{(1+r^s)^j}$$

[18]The distribution with homogeneous discount rates is slightly higher than that of the baseline model, as this version of the model relies more heavily on non-homothetic preferences.

Table 3: Calibration for Baseline Model

Parameter	Description	Value	Source or Target
Distribution of Income			
J	Age Groups	4	
I	Income Groups	5	
$\{\theta_i^y\}_{i\in I}$	Labor productivity (young)	$\{.35,.46,.72,.1.18,1.92\}$	PSID data
$\{\theta_i^m\}_{i\in I}$	Labor productivity (middle)	$\{.53,.88,1.26,1.76,2.85\}$	PSID data
$\{\theta_i^o\}_{i\in I}$	Labor productivity (old)	$\{.58,.98,1.38,1.91,3.15\}$	PSID data
ρ_e	Persistence e	.8	De Nardi and Yang (2014). See text.
σ_e	Standard deviation e	.3	De Nardi and Yang (2014). See text.
$\{\pi_i\}_{i\in I}$	Distribution of profit income	$\{0,.05,.1,.15,.7\}$	US Wealth Lorenz Curve 2019
σ_ℓ	Inverse Frisch	$1/.82$	Chetty et al. (2011)
Macro Parameters			
μ	Markup	1.08	Profit share of 7.5% (BEA)
α	Capital share after profits	.24	Labor share of .67
δ	Capital depreciation	.07	Investment share of .1
Z	Aggregate Productivity	.76	Set Y=1
NFA	Net Foreign Assets	.54	U.S. NFA/GDP in 2019
Endogenously Calibrated Micro parameters			
$\bar{a}$	Bequest utility when $a_{it}^J=0$	.1	See text
ψ_a	Bequest utility parameter	.12	Bequests/GDP of .05
ψ_ℓ	Labor disutility weight	.99	L=1
$\{\sigma_j\}_{j\in J}$	1/EIS	$\{2.42,2.14,1.84,1.52\}$	See text
$\{\beta\}_{i\in I}$	Discount factor	$\{.83,.91,.99,1.08,1.18\}$	See text
$\underline{a}$	Borrowing limit	-.15	10% constrained
η	1/elasticity of bequests	1.5	See text
Fiscal Policy			
τ_K	Capital tax	.1	
$\bar{\tau}_\ell$	Average Labor Tax	.35	PSZ. See text.
τ_b	Bequest tax	.1	De Nardi (2004)
B	Government debt	.65	Debt held by public/GDP in 2019
S	Social security transfers	{.06,.13,.21,.33,.41}	See text
G	Government Spending	.27	See text
γ_ℓ	Labor tax progressivity	.4	CEX (see text)

Note. This table contains the model parameters, their values, and their source or target in the data. Details on how the endogenously calibrated were calculated can be found in the text.

In order to isolate the impact of the change in the permanent income distribution, I construct a set of lump sum taxes and transfers, $\{T_{ij}\}$ that equal the change in a household's expected permanent income. Furthermore, the taxes are constructed to satisfy the following 2 conditions.

$$\sum_J \frac{T_{ij}}{(1+r)^j} = \Delta E[PI_i]$$

$$T_i = \frac{T_{ij}}{(1+r)^j} \text{ for all } j \in J$$

By constructing the lump-sum taxes in this way, I ensure that they reflect the change in a type-i household's permanent income without redistributing resources over the life-cycle. I study the direct effect of my channel in 3 versions of the model. The first is a standard model, with homothetic preferences ($\sigma_{nh} = 1$), $\eta = \sigma$, $\bar{a} = 0$) homogeneous time preferences ($\beta_i = \beta$), and therefore homogeneous marginal propensities to save. The second is my baseline model, and the third is the model with non-homothetic preferences but homogeneous discount rates. Table 5 reports the direct effects of my channel on aggregate capital, labor, output and consumption in the 2 models. The effects are reported in terms of the semi-elasticity of each variable to τ. That is, dlnX= $\frac{dlnX}{d\tau}$.

Table 4. Effect of Permanent Income Redistribution

	$\tau = .005$				$\tau = .03$			
Model	dlnK	dlnL	dlnY	dlnC	dlnK	dlnL	dlnY	dlnC
Standard	0.00	-0.41	-0.29	-0.15	0.00	-0.38	-0.26	-0.14
Baseline	-0.80	-0.48	-0.58	-0.30	-0.77	-0.45	-0.54	-0.29
Uniform β_i	-0.76	-0.50	-0.57	-0.29	-0.67	-0.47	-0.52	-0.27

As predicted, the direct effect of the change in permanent income has no effect on aggregate capital in the standard model. The direct effect of a .5 percent increase in τ results in a .4 percent decline in capital in the baseline model, and a slightly smaller .76 percent decline in the model with uniform β_i. For this small policy change, which model primitives were used to generate the distribution of MPS appears to not matter very much for the size of the redistribution-investment trade-off. The distribution itself is a sufficient statistic.

For larger changes however, the model with uniform discount rates generates a smaller decline in capital than the model with heterogeneous discount rates: the gap in the semi-elasticities of capital more than double. One likely explanation, is that when only non-homothetic preferences are used to generate the distribution of MPS out of permanent income, the distribution of MPS is more dependent on the degree of redistribution. That is, as redistribution increases, the permanent income distribution compresses, compressing the distribution of MPS. On the other hand, if savings behavior is *type dependent* and not only *scale dependent* then the distribution of MPS compresses less in response to redistribution. This exercise highlights the importance for researchers to determine which micro-foundations generate the savings behavior observed in the data.

Finally, I consider the size of the direct effect of the change in the permanent income distribution relative to the total effect of the tax. When $\tau = .005$, the decline in consumption stemming from the direct effect of my channel. The total decline in consumption can be decomposed into the decline resulting from the general equilibrium drop in labor, the drop in capital generated by permanent income redistribution, $d\hat{K}$, and the remaining drop in capital.

$$dC = \frac{dC}{dL}dL + \frac{dC}{dK}dK = \frac{dC}{dL}dL + \frac{dC}{dK}(dK - d\hat{K}) + \frac{dC}{dK}d\hat{K}$$

The decline attributed to my channel accounts for 23 percent of the total decline in consumption. This result suggests that any optimal policy analysis performed without taking into account the impact of heterogeneous marginal propensities to save out of permanent income may miss a quantitatively important share of the costs of redistribution.

4.4 The short run trade-off.

To be added.

5 Conclusion

The aim of this paper is to study the effect of heterogenous savings behavior on the trade-offs associated with income redistribution in OLG models. When high permanent income households have relatively higher marginal propensities to save out of permanent income, all permanent redistribution policies transfer resources from high savers to low savers, lowering both the aggregate savings level and investment. This savings behavior generates a novel welfare trade-off between redistribution and capital accumulation that is present for all forms of redistribution policy. This trade-off is distinct from the well-studied equity-efficiency trade-off, in which distortionary redistribution shifts an economy inside the Pareto frontier. I argue that *all* redistribution results in a movement along the Pareto frontier away from allocations that put weight on future generations.

I show that the existence of such a trade-off depends both on whether the rich do indeed have higher marginal propensities to save out of permanent income, and on whether achieving the first-best level of inequality would result in a savings level and aggregate capital stock below the golden-rule level. Using U.S. household panel data, I provide empirical evidence that this is the case.

Using a quantitative life-cycle model, I confirm that this trade-off is absent in standard models in which MPS out of permanent income are uniform, but substantial when the distribution of MPS is explicitly targeted. For small policy changes, the distribution itself is a sufficient statistic, and the size of the channel does not depend on the model primitives used to achieve it. When the policy change is larger however, a model with only *scale* dependent MPS generates a smaller trade-off than a model with *type* dependent MPS.[19] In the baseline model, my channel accounts for 20 percent of the decline in aggregate consumption resulting from a small increase in labor income redistribution.

The next step in this project is to use the quantitative model to study the importance of this channel in the short run. Redistribution affects long run capital by lowering the savings supply and pushing up interest rates. If firms are slow to respond to these increases – for example due to capital adjustment costs – the long run costs associated with my channel may take many years to materialize. In this case, the weight placed on future generations in the welfare calculus would become a more important ingredient in determining optimal redistribution.

[19] Gaillard et al. (2023) make a similar point.

References

Aditya Aladangady and Akila Forde. Wealth Inequality and the Racial Wealth Gap, 2021.

Anthony Atkinson and Agnar Sandmo. Welfare implications of the taxation of savings. *Economic Journal*, 90(359):529–49, 1980.

Laurent Bach, Laurent E. Calvet, and Paolo Sodini. Rich pickings? risk, return, and skill in household wealth. *American Economic Review*, 110(9):2703–47, September 2020.

Roland Bénabou. Tax and education policy in a heterogeneous-agent economy: What levels of redistribution maximize growth and efficiency? *Econometrica*, 70(2):481–517, 2002. ISSN 0012-9682. doi: 10.1111/1468-0262.00293.

Jess Benhabib, Alberto Bisin, and Mi Luo. Wealth distribution and social mobility in the us: A quantitative approach. *American Economic Review*, 109(5):1623–47, May 2019.

Cesar Blanco and Sebastian Diz. Optimal monetary policy with non-homothetic preferences. MPRA Paper 107427, University Library of Munich, Germany, 2021.

Raj Chetty, Adam Guren, Day Manoli, and Andrea Weber. Are micro and macro labor supply elasticities consistent? a review of evidence on the intensive and extensive margins. *American Economic Review*, 101(3):471–75, May 2011. doi: 10.1257/aer.101.3.471.

Mariacristina De Nardi. Wealth inequality and intergenerational links. *Review of Economic Studies*, 71(3), 2004.

Mariacristina De Nardi and Giulio Fella. Saving and Wealth Inequality. *Review of Economic Dynamics*, 26:280–300, October 2017.

Mariacristina De Nardi and Fang Yang. Bequests and heterogeneity in retirement wealth. *European Economic Review*, 72:182–196, 2014. ISSN 0014-2921.

Peter A. Diamond. National debt in a neoclassical growth model. *The American Economic Review*, 55(5):1126–1150, 1965.

Sebastian Doerr, Thomas Drechsel, and Donggyu Lee. Income Inequality and Job Creation, 2023.

Karen E. Dynan, Jonathan Skinner, and Stephen P. Zeldes. Do the rich save more? *Journal of Political Economy*, 112(2):397–444, 2004.

Andreas Fagereng, Luigi Guiso, Davide Malacrino, and Luigi Pistaferri. Heterogeneity in returns to wealth and the measurement of wealth inequality. *American Economic Review*, 106(5):651–55, May 2016.

Alexandre Gaillard, Christian Hellwig, Philipp Wangner, and Nicolas Werquin. Inequality Accounting: A Tale of Tails, 2023.

William G. Gale and Samara R. Potter. An Economic Evaluation of the Economic Growth and Tax Relief Reconciliation Act of 2001. *National Tax Journal*, 55(1):133–186, March 2002. doi: 10.17310/ntj.2002.1.08.

Aart Gerritsen, Bas Jacobs, Alexandra Rusu, and Kevin Spiritus. Optimal Taxation of Capital Income with Heterogeneous Rates of Return. June 2020.

Mikhail Golosov, Maxim Troshkin, Aleh Tsyvinski, and Matthew Weinzierl. Preference heterogeneity and optimal capital income taxation. *Journal of Public Economics*, 97: 160–175, 2013.

Mikhail Golosov, Maxim Troshkin, and Aleh Tsyvinski. Redistribution and social insurance. *American Economic Review*, 106(2):359–86, February 2016.

Koichi Hamada. Lifetime equity and dynamic efficiency on the balanced growth path. *Journal of Public Economics*, 1(3):379–396, 1972. ISSN 0047-2727.

Jonathan Heathcote, Kjetil Storesletten, and Giovanni L. Violante. Optimal Tax Progressivity: An Analytical Framework*. *The Quarterly Journal of Economics*, 132(4):1693–1754, 06 2017.

Mark Huggett and Gustavo Ventura. Understanding why high income households save more than low income households. *Journal of Monetary Economics*, 45(2):361–397, 2000. ISSN 0304-3932. doi: https://doi.org/10.1016/S0304-3932(99)00058-6.

Ayse Imrohoroglu, Cagri S. Kumru, and Arm Nakornthab. Revisiting Tax on Top Income. Anu working papers in economics and econometrics, Australian National University, College of Business and Economics, School of Economics, February 2018.

Kenneth L. Judd. Redistributive taxation in a simple perfect foresight model. *Journal of Public Economics*, 28(1):59–83, 1985. ISSN 0047-2727.

Atif Mian, Ludwig Straub, and Amir Sufi. Indebted Demand*. *The Quarterly Journal of Economics*, 136(4):2243–2307, 03 2021. ISSN 0033-5533. doi: 10.1093/qje/qjab007.

Atif R Mian, Ludwig Straub, and Amir Sufi. The saving glut of the rich. Working Paper 26941, National Bureau of Economic Research, April 2020.

J. A. Mirrlees. An exploration in the theory of optimum income taxation. *The Review of Economic Studies*, 38(2):175–208, 1971.

Nobuhiro Okuno and Akira Yakita. Public investment and income distribution: A note. *The Quarterly Journal of Economics*, 96(1):171–176, 1981.

Pierre Pestieau and Uri M. Possen. Optimal growth and distribution policies. *Journal of Public Economics*, 9(3):357–372, 1978. ISSN 0047-2727.

Thomas Piketty and Emmanuel Saez. Chapter 7 - optimal labor income taxation. In Alan J. Auerbach, Raj Chetty, Martin Feldstein, and Emmanuel Saez, editors, *handbook of public economics, vol. 5*, volume 5 of *Handbook of Public Economics*, pages 391–474. Elsevier, 2013a.

Thomas Piketty and Emmanuel Saez. A theory of optimal inheritance taxation. *Econometrica*, 81(5):1851–1886, 2013b.

Alessandra Pizzo. The welfare effects of tax progressivity with frictional labor markets. *Review of Economic Dynamics*, 49:123–146, July 2023. doi: 10.1016/j.red.2022.07.004.

Emmanuel Saez and Stefanie Stantcheva. A simpler theory of optimal capital taxation. *Journal of Public Economics [also: NBER Working Paper 22664]*, 162:120–142, 2018.

Paul A. Samuelson. Optimum social security in a life-cycle growth model. *International Economic Review*, 16(3):539–544, 1975.

Karl Schulz. Redistribution and Return Inequality. June 2021.

Eytan Sheshinski. Income taxation and capital accumulation. *The Quarterly Journal of Economics*, 90(1):138–149, 1976.

Ludwig Straub. Consumption, savings, and the distribution of permanent income, 2019. Revise and resubmit at Econometrica.

Iván Werning. Optimal Fiscal Policy with Redistribution. *The Quarterly Journal of Economics*, 122(3):925–967, 08 2007.

A Appendix

A.1 Proof of Lemma 1.

Show that when $\sigma_y > \sigma_o$ and / or when $\psi_a > 0$ and $\sigma_o > \eta$ that savings at each age, s_t^h is increasing in permanent income, PI_i.

The household's problem is the following:

$$\max_{\{c_{it}^y, c_{i,t+1}^o, a_{i,t+1}^o\}_{h\in\{0,1\}}} \frac{(c_{it}^y)^{1-\sigma_y}}{1-\sigma_y} + \beta_i \left(\frac{(c_{it+1}^o)^{1-\sigma_o}}{1-\sigma_o} + \psi_a \frac{(a_{it+1}^o)^{1+\eta}}{1+\eta} \right)$$
$$\text{s.t. } c_{it}^y + \frac{c_{i,t+1}^o + a_{it+1}^o}{R_{i,t+1}} = R_{it} a_{it-1}^o + w_t \theta_i + T_{it}$$

If $\psi_a = 0$, the households' first order condition is:

$$(c_{it}^0)^{-\sigma_y} = \beta_i R_{it} (c_{i,t+1}^1)^{-\sigma_o}$$

If $\psi_a = 1$, the household's first order condition is:

$$(c_{it}^0)^{-\sigma_y} = \beta_i R_{it} (c_{i,t+1}^1)^{-\sigma_o}$$
$$(c_{i,t+1}^1)^{-\sigma_o} = \psi_a (a_{i,t+1}^1)^{-\eta}$$

Permanent income, PI_i is defined as:

$$PI_i = R_{it} a_{i,t-1}^o + w_t \theta_i + T_{it}$$

For each household type, the derivative of steady state savings to permanent income is given by the following expressions:

Case 1 ($\psi_a = 0$):

$$c_{it}^y + R_{i,t+1}^{\frac{1}{\sigma_o}-1} \beta_i^{\frac{1}{\sigma_o}} (c_{it}^y)^{\frac{\sigma_y}{\sigma_o}} = PI_{it}$$
$$\frac{\partial c_{it}^y}{\partial PI_{it}} = \left((1 + \frac{\sigma_y}{\sigma_o} (c_{it}^y)^{\frac{\sigma_y}{\sigma_o}-1} R_{i,t+1}^{\frac{1}{\sigma_o}-1} \beta_i^{\frac{1}{\sigma_o}} \right)^{-1} \tag{A.1}$$

This term is positive, meaning consumption increases with permanent income. When permanent income increases, c_{it}^y increases. From A.1 we can see then that $\frac{\partial c_{it}^y}{\partial PI_i}$ is straightforwardly decreasing in PI_i whenever $\sigma_y > \sigma_o$. This derivative is also decreasing in β_i. This implies that $\frac{\partial a_{it}^y}{\partial PI_i} = \frac{\partial s_t^h}{\partial PI_i}$ is *increasing* in β_i and increasing in PI_{it} whenever $\sigma_y > \sigma_o$.

Case 2 ($\psi_a > 0$):

$$c_{it}^y + R_{it}^{\frac{1}{\sigma_o}-1} \beta_i^{\frac{1}{\sigma_o}} (c_{it}^y)^{\frac{\sigma_y}{\sigma_o}} + R_{it}^{\frac{1}{\eta}-1} (\psi_a \beta_i)^{\frac{1}{\eta}} (c_{it}^y)^{\frac{\sigma_y}{\eta}} = PI_{it}$$

$$\frac{\partial c_{it}^y}{\partial PI_{it}} = \left((1 + \frac{\sigma_y}{\sigma_o} (c_{it}^y)^{\frac{\sigma_y}{\sigma_o}-1} R_{it+1}^{\frac{1}{\sigma_o}-1} \beta_i^{\frac{1}{\sigma_o}} + \frac{\sigma_y}{\eta} (c_{it}^y)^{\frac{\sigma_y}{\eta}-1} R_{it+1}^{\frac{1}{\eta}-1} (\psi_a \beta_i)^{\frac{1}{\eta}} \right)^{-1} \tag{A.2}$$

Again, this term is positive, meaning the derivative is decreasing in permanent income whenever $\sigma_y > \sigma_o$ or $\sigma_o > \eta$. It is also straightforwardly decreasing in β_i. The derivative of bequests, a_{it+t}^o with respect to permanent income is:

$$\frac{\partial a_{it+1}^o}{\partial PI_{it}} = \left(\frac{\eta}{\sigma_y} (\psi_a \beta_i R_{it+1})^{-1/\sigma_y} (a_{it+1}^o)^{\frac{\eta}{\sigma_y}-1} + \frac{\eta}{\sigma_o} (\psi_a^{-1/\sigma_o} R_{it+1}^{-1}) (a_{it+1}^o)^{\frac{\eta}{\sigma_o}-1} + \frac{1}{R} \right)^{-1} \tag{A.3}$$

Because this derivative is positive, bequests increase with permanent income. Therefore, whenever permanent income increases and $\eta < \sigma_y$ or $\eta < \sigma_o$, the denominator in A.3 decreases and $\frac{\partial a_{it+1}^0}{\partial PI_{it}}$ increases. Therefore, the derivative of bequests

Because the derivative of consumption with respect to PI is decreasing in PI, $\frac{\partial a_{it}^y}{\partial PI_{it}} = \frac{\partial s_{it}^y}{\partial PI_{it}}$ is increasing in PI_{it}.

A.2 Proof that $\lambda_H (c_H^{yu})^{-\sigma_y} = \lambda_L (c_L^{yu})^{-\sigma_y}$

The unconstrained planner's problem is to maximize social welfare, given by:

$$SW_s = \sum_I \pi_{ih} \lambda_i \left(\frac{(c_i^y)^{1-\sigma_y}}{1-\sigma_y} + \beta_i \gamma \frac{(c_i^o)^{1-\sigma_o}}{1-\sigma_o} + \beta_i \gamma \psi_a \frac{(a_i^o)^{1-\eta}}{1-\eta} \right)$$

subject only to the resource constraint (4) in steady state.

Let μ be the lagrange multiplier on the The first order conditions with respect to c_i^y are:

$$\pi_{iy} \lambda_i (c_i^y)^{-\sigma_y} = \pi_{iy} \mu$$

This implies that $\lambda_L (c_L^y)^{-\sigma_y} = \lambda_H (c_H^y)^{-\sigma_y}$.

A.3 Proof of Proposition 1.

A.3.1 Properties of CES Production Function.

I assume that the representative firm produces output with capital and fixed labor, $L = 1$ according to a CES production function (A.4) where α_L and α_K are positive constants and $\gamma \in (-\infty, 1)$.

$$Y(K, L) = \Big(\alpha_K K^\gamma + \alpha_L L^\gamma\Big)^{\frac{1}{\gamma}} \tag{A.4}$$

As in the text, I assume that $\delta \approx 0$, so $R \approx \frac{\partial Y}{\partial K}$. In this case, the elasticity of firm capital to R, which I denote as K_R is given by the following.

$$K_R = \frac{\partial \text{log} K}{\partial \text{log} R} = \left(\frac{1}{1-\gamma}\right)\left(\left(\frac{K}{Y}\right)^\gamma \alpha_K - 1\right)^{-1}$$

The elasticity of the wage to capital, w_K is:

$$w_K = \frac{\partial \text{log} w}{\partial \text{log} K} = (1-\gamma)\alpha_K \left(\frac{K}{Y}\right)^\gamma$$

A.3.2 Solve for steady state capital as a function of the transfer, $\frac{dK}{dT_L}$ and useful properties of $K(T_L)$ function.

Case 1 ($\psi_a = 0$)**:**
First, use the firms' steady state optimality conditions to write $R = (1 + F_K(K) - \delta)$ and $w(K) = F_L(K)$. Then use the households' Euler equations and budget constraints to write a_i^y as a function of $w(K)$, T_L, and $R_t(K_t)$. Then use the government budget constraint to write $T_i(T_L)$ for $i \in \{L, H\}$, and the steady state asset market clearing condition to write the entire system in terms of K and T_L.

$$K(R) = \sum_I \pi_i a_i^y \Big(T_i(T_L), w(K), R(K)\Big)$$

Take the total derivative with respect to T_j :

$$\frac{dK}{dT_L} = \frac{dK}{dR}\left(\sum_I \pi_i \frac{\partial a_i^y}{\partial PI_i}\frac{\partial T_i}{\partial T_L}\right)\left(\frac{dK}{dR}(1 - \sum_I \pi_i \frac{\partial a_i^y}{\partial w}\frac{\partial w}{\partial K}) - \sum_I \pi_i \frac{\partial a_i^y}{\partial R}\frac{\partial R}{\partial K}\right)^{-1}$$

Define $X_Y = \frac{\partial \text{log} X}{\partial \text{log} Y}$ as the elasticity of X with respect to Y, for any two variables X and Y.

Then the above can be rewritten in the following way.

$$\frac{dK}{dT_L} = K_R \Bigg(\sum_I \pi_i \frac{\partial a_i^y}{\partial PI_i} \frac{\partial T_i}{\partial T_L} \Bigg) \Bigg(K_R(1 - A_w w_K) - A_R \Bigg)^{-1}$$

Because $\sum_I \pi_i a_i^y = A = K$, $\sum_I \pi_i \frac{\partial a_i^y}{\partial w} \frac{\partial w}{\partial K} = \frac{\partial \text{log} A}{\partial \text{log} w} \frac{\partial \text{log} w}{\partial \text{log} K} = A_w w_K$ where the elasticity of aggregate assets to the wage, A_w is positive and bounded.

Using the results from section A.3.1, we know that,

$$K_R w_K = \Bigg(\frac{\alpha_K (K/Y)^\gamma}{\alpha_K (K/Y)^\gamma - 1} \Bigg)$$

Because in this case, $K_R w_K$ is bounded, the limit of $\frac{dK}{dT_L}$ as $\gamma \to -\infty$ is 0.

The limit as $\gamma \to 1$ of $w_K = 0$, while the limit of $K_R = -\infty$. In this case, the limit of $\frac{dK}{dT_L}$ as $\gamma \to 1$ is:

$$\frac{dK}{dT_L} = \Bigg(\sum_I \pi_i \frac{\partial a_i^y}{\partial PI_i} \frac{\partial T_i}{\partial T_L} \Bigg)$$

Finally, to see that the term $K_R/(K_R(1 - A_w w_K) - A_R)$ is always positive, note that this can be written as:

$$K_R \Bigg(\frac{\frac{1}{\gamma - 1} - A_w \alpha_K (K/Y)^\gamma}{1 - (K/Y)^\gamma \alpha_K} \Bigg)$$

Note that by definition, $\alpha_K (K/Y)^\gamma = 1 - (L/Y)^\gamma \alpha_L \in (0, 1)$ by assumption (as L=1), and therefore the numerator of this term is positive. Because $\gamma < 1$, $A_w > 0$, and $K_R < 0$, the numerator is also positive.

Therefore, whether K is increasing or decreasing in T_L depends on whether $\frac{\partial a_H^y}{\partial PI_H} > \frac{\partial a_L^y}{\partial PI_L}$. That is,

$$\frac{\partial K}{\partial T_L} < 0 \text{ iff } \frac{\partial a_H^y}{\partial PI_H} > \frac{\partial a_L^y}{\partial PI_L}$$

Furthermore, when $\beta_L = \beta_H$, capital reaches its lowest point when permanent incomes are equal. To see this, define $\bar{T}_L$ as the policy that generates equal permanent income share. That is, $PI_L(\bar{T}_L) = PI_H(\bar{T}_H)$. Using Lemma 1, this implies that $\frac{\partial a_L^y}{\partial PI_L}(\bar{T}_L) = \frac{\partial a_H^y}{\partial PI_H}(\bar{T}_H)$, and therefore that $\frac{\partial K}{\partial T_L} = 0$. To see that this is a minimum, note that $\frac{\partial K}{\partial T_L} > 0$ for all $T_L > \bar{T}_L$ (as the type L households now have greater permanent income and a greater MPS) and $\frac{\partial K}{\partial T_L} < 0$ for all $T_L < \bar{T}_L$.

Finally, we can use equation (A.1) to show that as long as $\beta_L = \beta_H$, then $c^y_L(\bar{PI}) = c^y_H(\bar{PI})$ for any $\bar{PI}$, and therefore $a^y_L(\bar{PI}) = a^y_H(\bar{PI})$. Let PI'_L and PI'_H be the permanent incomes resulting from policy T'. Consider values for permanent income $PI''_L = P'_H$ and $PI''_H = PI'_L$ (swapping the respective permanent incomes). Then it follows that $a^y_L(PI'_L) = a^y_H(PI''_H)$ and $a^y_H(PI'_H) = a^y_L(PI''_L)$, implying the same level of total capital and total income. Therefore, PI''_H and PI''_L can be implemented by some policy T''_L, and $K(T''_L) = K(T'_L)$.

Case 2 ($\psi_a > 0$): Again, use the firms' FOC to write $R = (1 + F_K(K) - \delta)$ and $w(K) = F_L(K)$. Then use the households' Euler equations and budget constraints to write a^y_i as a function of $w(K)$, T_i, a^o_i, $R(K)$, and a^o_i, and $R(K)$ for $i \in \{L, H\}$. Now, use household's optimal bequest condition and second period budget constraint to write a^y_i as a function of $w(K), a^o_i, R(K)$, and T_i. These two equations jointly determine a^y_i and a^o_i as a function of $w(K)$, $w_t(K_t)$, $R(K)$, $R(K)$, a^o_i, and T_i. The asset market clearing condition is:

$$2K = \sum_I \pi_i \bigg(a^y_i(w(K), R(K), a^o_i, T_i) + a^o_i((w(K), R(K)), a^o_i, T_i) \bigg)$$

Taking the total derivative with respect to T_j :

$$\frac{dK}{dT_L} = K_R \sum_I \pi_i \bigg(\frac{\partial(a^y_i + a^o_i)}{\partial PI_i} \frac{\partial T_i}{\partial T_L} \bigg) \bigg(K_R(1 - A_w w_K) - A_R \bigg)^{-1}$$

Using the same logic as in Case 1, the denominator of $\frac{dK}{dT_L}$ is always positive, and therefore the sign of this derivative depends only on whether $\sum_I \pi_i \bigg(\frac{\partial(a^y_i + a^o_i)}{\partial PI_{it}} \bigg) > 0$.

Using the results in Lemma 1, it is clear to see that again, when $\beta_L = \beta_H$, $\bar{T}_L$ is the argmin of the $K(T_L)$ function and that $K(T'_L) = K(T''_L)$ if $PI'_H = PI''_L$ and $PI'_L = PI''_H$.

A.3.3 Proof of Lemma 3

Social welfare in the steady state is defined as:

$$SW_s = \sum_I \pi_{ij} \lambda_i \bigg(\frac{(c^y_i)^{1-\sigma_y}}{1 - \sigma_y} + \beta_i \frac{(c^o_i)^{1-\sigma_o}}{1 - \sigma_o} + \beta_i \psi_a \frac{(a^o_i)^{1-\eta}}{1 - \eta} \bigg)$$

The lifetime budget constraint is given by:

$$c_i^y + \frac{c_i^o + a_i^o}{R} = Ra_i^o + w\theta_i + T_i$$

I can express a change in SW_s following a change in T_i as:

$$\frac{dSW_s}{dT_L} = \sum_I \pi_{ij}\lambda_i \Bigg((c_i^y)^{-\sigma_y} \frac{dc_i^y}{dT_i}\frac{dT_i}{dT_L} + \beta_i (c_i^o)^{-\sigma_o} \frac{dc_i^o}{dT_i}\frac{dT_i}{dT_L} + \beta_i \psi_a (a_i^o)^{-\eta} \frac{da_i^o}{dT_i}\frac{dT_i}{dT_L} \Bigg)$$

Using the household's budget constraint:

$$\begin{aligned} dc_i^y &= Rda_i^o + a_i^o dR + \theta_i dw + dT_i - da_i^y \\ dc_i^o &= Rda_i^y + a_i^y dR - da_i^o \end{aligned}$$

The household's optimality condition for bequests is $(c_i^o)^{-\sigma_o} = \psi(a_i^o)^{-\eta}$ and Euler equation is: $(c_i^y)^{-\sigma_y} = \beta_i R (c_i^o)^{-\sigma_o}$. Defining $\omega_i = \lambda_i (c_i^y)^{-\sigma_y}$ as in the text, and subbing in the bequest condition and Euler equation, this change can be written as:

$$dSW_s = \sum_I \pi_{ij}\omega_i \Bigg(Rda_i^o + a_i^o dR + \theta_i dw + dT_i \Bigg) + \pi_{ih}\omega_{io} \Bigg(a_i^y dR \Bigg)$$

Note that $R\omega_{io} = \lambda_i R \beta (c_i^o)^{-\sigma_o} = \omega_i$. Therefore, $\omega_{io} = \omega_i / R$. This is equivalent to:

$$dSW_s = \sum_I \pi_{ij}\omega_i dT_i + \sum_I \pi_{ij}\omega_i \Bigg(Rda_i^o + a_i^o dR + \theta_i dw + (a_i^y)\frac{dR}{R} \Bigg)$$

Let Γ_i^b be type-i households' bequests as a share of total capital:

$$\begin{aligned} dSW_s = \sum_I \pi_{ij}\omega_i dT_i + \sum_I \pi_{ij}\omega_i RK d\Gamma_i^b + R\sum_I \pi_{ij}(\omega_i \Gamma_i^b) dK + \\ \sum_I \omega_i \pi_{ij} \Bigg((a_i^o + \frac{a_i^y}{R}) dR + \theta_i dw \Bigg) \end{aligned}$$

Note that for CES production functions, $dR/dw = -L/K$. Therefore, the above can be written as:

$$\begin{aligned} dSW_s = \sum_I \pi_{ij}\omega_i dT_i + \sum_I \pi_{ij}\omega_i RK d\Gamma_i^b + R\sum_I \pi_{ij}(\omega_i \Gamma_i^b) dK + \\ \sum_I \omega_i \pi_{ij} \Bigg((-\frac{a_i^o}{K} + \frac{a_i^y}{KR}) + \frac{\theta_i}{L} \Bigg) dwL \end{aligned}$$

Defining Θ and K_{PI} as in the text,

$$\Theta = \sum_I \omega_i \pi_{ij} \left(\frac{\theta_i}{L} - \left(\frac{a_i^o}{K} + \frac{a_i^y}{KR} \right) \right) \tag{A.5}$$

Finally, multiply $R\sum_I \pi_{ij}(\omega_i \Gamma_i^b)dK$ by K/K, and note that $\Gamma_i^b K = a_i^o$ to get:

$$dSW_s = \sum_I \pi_{ij}\omega_i dT_i^y + \sum_I \pi_{ij}\omega_i RK d\Gamma_i^b + \left(R\sum_I \pi_{ij}(\omega_i a_i^o) + wL\Theta w_K \right) dK \tag{A.6}$$

A.3.4 Consider each combination of parameters.

Case 1: $\sigma_y = \sigma_o = \eta$ **and** $\beta_L = \beta_H$

From Lemma 1, we know that in this case, $\frac{dK}{dT_j} = 0$ for all allocations. Therefore, the necessary condition for the constrained optimum is:

$$\frac{dSW_s}{dT_L} = \left(1 + \pi_j RK \frac{d\Gamma_j^b}{dT_L} \right) (\pi_L \omega_L - \pi_H \omega_H) = 0$$

This condition straightforwardly holds only when $\omega_L = \lambda_L (c_L^y)^{-\sigma_y} = \lambda_H (c_H^y)^{-\sigma_y} = \omega_H$.

Case 2: $\sigma_y > \sigma_o$ **and / or** $\psi_a > 0$ **and** $\sigma_o > \eta$ **and** $\beta_L = \beta_H$

(a) Suppose $\lambda_H > \lambda_L$. In this case, an allocation in which $\omega_H = \omega_L$ implies $\frac{\lambda_H}{\lambda_L} = \left(\frac{c_H^y}{c_L^y} \right)^{\sigma_y} > 1$ and therefore more permanent income for type-H households. Let T_L^E be the transfer that implements this allocation. Then using the results from Lemma 1, we have that at this allocation,

$$\frac{\partial a_H^h}{\partial T_L}(T_L^E) - \frac{\partial a_L^h}{\partial T_L}(T_L^E) < 0 \text{ for } h \in \{y, o\}$$

and therefore, $\frac{\partial K}{\partial T_L} < 0$.

Recall that Θ is defined in the following way.

$$\Theta = \sum_I \omega_i \pi_{ij} \left(\frac{\theta_i}{L} - \left(\frac{a_i^o}{K} + \frac{a_i^y}{KR} \right) \right)$$

By evaluating $\Theta(T_L^E)$, where $\omega_H = \omega_L$, we see that as long as $R > 1$, it follows that $\Theta > 0$, and therefore that $\omega_H = \omega_L$ cannot be an optimum.

(b) Next, evaluate equation (A.6) at some policy $T' = T_L^E + \epsilon$ at some $\epsilon \neq 0$:

$$\frac{dSW_s}{dT_L} = \left(1 + \pi_j R' \frac{d\Gamma_j^b}{dT_L}(T')\right)(\omega_L' - \omega_H') + \left(R'\pi_j \sum_I (\omega_i' a_i^{o\prime}) + w' L \Theta' w_K\right)\frac{dK}{dT_L}(T') \quad \text{(A.7)}$$

To see that at the optimal allocation, $\omega_H' < \omega_L'$ (and therefore less redistribution toward type-L households), assume the contradiction.

Suppose instead at the optimal allocation $\lambda_L(c_L^{y\prime})^{-\sigma_y} < \lambda_H(c_H^{y\prime})^{\sigma_y}$. Let T' be the policy that implements this allocation, and let $K'(T')$ be the level of capital and PI_L' and PI_H' be the permanent incomes associated with this alternative policy. Using the results from subsection A.3.2, a second allocation is feasible, implemented by T'' with $K'' = K'$, $PI_H'' = PI_L'$ and $PI_L'' = PI_H'$, and therefore $c_L^{y\prime\prime} = c_H^{y\prime}$ and $c_L^{y\prime\prime} = c_H^{y\prime}$. Because $\lambda_H > \lambda_L$, this feasible allocation strictly dominates the first allocation in terms of welfare, a contradiction.

(c) By a symmetric argument, if $\lambda_L > \lambda_H$, then the optimal allocation is characterized by $\omega_L < \omega_H$, that is, more redistribution towards the type-L households.

Case 3: $\lambda_H \geq \lambda_L$ **and** $\beta_H > \beta_L$**.**

(a) Using identical logic as in case 2(a), if $R(K(T_L^E)) > 1$, ω_H cannot equal ω_L at the optimal allocation.

To see why, note that if $\lambda_H > \lambda_L$, the first term in equation A.7 is 0 by construction, $\Theta > 0$, and $\frac{\partial K}{\partial T_L}$ is positive, as both $\beta_H > \beta_L$ and the permanent income of type-H households is greater than type-L households when $\omega_H = \omega_L$, violating the necessary condition for an optimum.

(b) To see that $\omega_H^* < \omega_L^*$ at the optimal allocation, assume the contradiction.

A.4 Proof of Proposition 3

As in the text, assume that we vary the size of the domestic population relative to a foreign population who can invest in domestic capital. The foreign population does not work in the domestic economy or receive transfers. Therefore, the asset demand of foreigners, $a^F(R)$ is simply a function of the interest rate offered to them. Denote $\rho \geq 1$ as the scale of the domestic economy's population. That is, $\pi_i(\rho) = \pi_i/\rho$. The foreign population is a mass $\frac{\rho-1}{\rho}$. The asset market clearing condition is now given by equation (A.8).

$$K(R) = \frac{1}{\rho}\sum_I \pi_{iy} a_i^y(w(K), R(K), T_L) + \frac{\rho-1}{\rho} a^F(R(K)) \tag{A.8}$$

Using the same procedure as before, we can solve for the general equilibrium change in capital from a change in policy, T_L.

$$\frac{dK}{dR}\frac{dR}{dT_L}dT_L = \frac{1}{\rho}\sum_I \pi_{iy}\left(\frac{\partial a_i^y}{\partial PI_i}\left(\frac{\partial T_i}{\partial T_L}dT_L + \theta_i \frac{dw}{dK}\frac{dK}{dR}\frac{dR}{dT_L}dT_L\right) + \frac{\partial a_i^y}{\partial R}\frac{dR}{dT_L}dT_L\right) + \frac{\rho-1}{\rho}\frac{\partial a^F}{\partial R}\frac{dR}{dT_L}dT_L$$

Collecting all the $\frac{dR}{dT_L}$ terms on one side gives the following.

$$\frac{dR}{dT_L}\left(\frac{dK}{dR} - \frac{\partial A}{\partial w}\frac{dw}{dK}\frac{dK}{dR} - \frac{1}{\rho}\sum_I \pi_{iy}\frac{\partial a_i^y}{\partial R} - \frac{\rho-1}{\rho}\frac{\partial a^F}{\partial R}\right) = \frac{1}{\rho}\sum_I \pi_{iy}\left(\frac{\partial a_i^y}{\partial PI_i}\frac{\partial T_i}{\partial T_L}\right)$$

Rearranging to solve for $\frac{dK}{dT_j}$:

$$\frac{dK}{dT_L} = \frac{\frac{\partial K}{\partial R}\frac{R}{K}\frac{1}{\rho}\sum_I \pi_{iy}\left(\frac{\partial a_i^y}{\partial PI_i}\frac{\partial T_i}{\partial T_L}\right)}{\frac{\partial K}{\partial R}\frac{R}{K}\left(1 - \frac{\partial A}{\partial w}\frac{dw}{dK}\frac{w}{w}\frac{A}{A}\frac{K}{K}\right) - \sum_I \frac{1}{\rho}\pi_{iy}\frac{\partial a_i^y}{\partial R}\frac{a_i^y}{a_i^y}\frac{R}{K} - \frac{\rho-1}{\rho}\frac{\partial a^F}{\partial R}\frac{a^F}{a^F}\frac{R}{K}}$$

Let $X_Y = \frac{\partial X}{\partial Y}\frac{Y}{X}$. Then the above can be expressed as: Rearranging to solve for $\frac{dK}{dT_j}$:

$$\frac{dK}{dT_L} = \frac{K_R\frac{1}{\rho}\sum_I \pi_{iy}\frac{\partial s_i^y}{\partial PI_i}dT_i}{K_R\left(1 - A_w w_K \frac{A}{K}\right) - \frac{A}{K}A_R - \frac{NFA}{K}a_R^F}$$

Here, $\frac{A}{K} = \frac{\sum_I \pi_{iy}(\rho)a_i^y}{K}$, the initial domestic asset share and $\frac{NFA}{K} = \frac{\pi^F(\rho)a^F}{K}$ is the initial foreign asset share.

Note that again, the denominator is unambiguously negative, while the numerator is positive whenever $\frac{\partial s_H^y}{\partial PI_H} > \frac{\partial s_L^y}{\partial PI_L}$. However, the numerator is also unambiguously decreasing as $\rho \to \infty$. Note also that for a given ρ, the absolute values of the denominator is unambiguously increasing in a_R^F.

Therefore, as $\rho \to \infty$ or as $a_R^f \to \infty$, $\frac{dK}{dT_L} \to 0$. Plugging this result into the constrained planner's first order condition, (A.6), at the optimal allocation $\omega_L = \omega_H$.

A.5 Proof of Proposition 2

From the constrained planner's first order condition, the cost of redistribution is proportional to:

$$\frac{w_K K_R}{K_R(1 - A_w w_K) - A_R}\left(\sum_I \pi_{ih}\frac{\partial s_i^y}{\partial PI_i} dT_i\right)$$

From the results in Section A.3.1, we have that $w_K \frac{dK}{dT_L}$ can be expressed as:

A.6 Proof of Proposition 4.

A.6.1 Derivative of K_{t+1} with respect to T_{jt}

Use the firms' FOC to write $R_{t+1} = (1 + F_K(K_{t+1}) - \delta)$ and $w_t(K_t) = F_L(K_t)$. Then use the households' Euler equations and budget constraints to write a_{it}^y as a function of $w_t(K_t)$, T_{it}, and $R_{t+1}(K_{t+1})$.

$$a_{it}^y = c_{it}^y + R_{i,t+1}^{\frac{1}{\sigma_o}-1}\beta_i^{\frac{1}{\sigma_o}}(c_{it}^y)^{\frac{\sigma_y}{\sigma_o}} = PI_{it}$$

Use the government budget constraint to write $T_{Ht}(T_{Lt})$ and the asset market clearing condition to write the entire system in terms of K_{t+1}, K_t, and T_{it}.

$$K_{t+1} = \sum_I \pi_i a_{it}^y\Big(T_{it}(T_{jt}), w_t(K_t), R_{t+1}(K_{t+1})\Big) = K_{t+1}(K_t, T_{jt}) \tag{A.9}$$

Take the total derivative with respect to T_{jt} for a given K_t.

$$\frac{dK_{t+1}}{dT_{jt}} = \left(\sum_I \pi_i \frac{\partial a_{it}^y}{\partial PI_{it}}\frac{dT_i}{dT_j}\right)\left(1 - \sum_I \pi_i \frac{\partial a_{it}^y}{\partial R_{t+1}}\frac{\partial R_{t+1}}{\partial K_{t+1}}\right)^{-1}$$

Note that because savings is increasing the interest rate, and the interest rate is decreasing in K_{t+1} by the firms' FOC, the entire denominator is positive. The derivative of K_{t+1} with respect to $K_t = \sum_I \pi_i \frac{\partial a_{it}^y}{\partial w}\frac{\partial w_t}{\partial K_t} > 0$.

Therefore, Equation (A.9) defines K_{t+1} as an implicit function of T_{jt} and K_t such that $\frac{dK_{t+1}}{dK_t} > 0$ and $\frac{dK_{t+1}}{dT_{jt}}$ is positive whenever $\frac{\partial a^{jt}}{\partial PI_{jt}} > \frac{\partial a^{it}}{\partial PI_{it}}$.

Next, we can write a type-i generation-t household's share of consumption when young, Γ^y_{it} and share of total consumption when old, Γ^o_{it+1} as implicit functions of K_t, K_{t+1}, C_t, and T_{jt}. Using the household's Euler equation and lifetime budget constraint, and substituting in the firm's first order conditions for w_t and R_t, both $c^y_{it}(K_t, K_{t+1}, T_{it})$ and $c^o_{i,t+1}(K_t, K_{t+1}, T_{it})$ can be expressed as an implicit function of K_t, K_{t+1}, T_{it}.

The derivative of c^y_{it} with respect to K_t, $\frac{\partial c^y_{it}}{\partial K_t} = \frac{\partial c^y_{it}}{\partial PI_{it}}\theta_i\frac{\partial w_t(K_t)}{\partial K_t}$ while $\frac{\partial c^o_{it+1}}{\partial K_t} = \frac{\partial c^o_{it+1}}{\partial PI_{it}}\theta_i\frac{\partial w_t(K_t)}{\partial K_t}$. Whether $\frac{\partial c^y_{it}}{\partial K_t} > \frac{\partial c^y_{jt}}{\partial K_t}$ – and therefore whether $\frac{\Gamma^y_{it}}{\partial K_t} > \frac{\Gamma^y_{jt}}{\partial K_t}$ – depends on whether the product of a type-i household's marginal propensity to consume out of their permanent income and their labor productivity is higher than that of a type-j household.

We also have that, $\frac{\partial c^y_{it}}{\partial K_{t+1}} = \frac{\partial c^y_{it}}{\partial R_{t+1}}\frac{\partial R_{t+1}}{\partial K_{t+1}}$ and $\frac{\partial c^o_{it+1}}{\partial K_{t+1}} = \frac{\partial c^o_{it+1}}{\partial R_{t+1}}\frac{\partial R_{t+1}}{\partial K_{t+1}}$. Whether $\frac{\partial \Gamma^o_{it+1}}{\partial K_{t+1}} > \frac{\partial \Gamma^o_{jt+1}}{\partial K_{t+1}}$ depends on whether the interest rate elasticity of savings of type-i households is higher than that of type-j households. The derivative of consumption and the consumption share at either age for a type-i household is unambiguously increasing in T_{it}.

Because by construction $\pi_i c^y_{it}(K_t, K_{t+1}, T_{it}) = \Gamma^y_{it} C_t$ and $\pi_i c^o_{it+1}(K_t, K_{t+1}, T_{it}) = \Gamma^o_{it+1} C_{t+1}$, so we can rearrange and write Γ^y_{it} and Γ^o_{it+1} as implicit functions of K_t, K_{t+1}, T_{it} and C_t and C_{t+1}. respectively.

$$\Gamma^y_{it}(K_t, K_{t+1}, T_{it}, C_t) = \frac{\pi_i}{C_t} c^y_{it}(K_t, K_{t+1}, T_{it}) \tag{A.10}$$

$$\Gamma^o_{it+1}(K_t, K_{t+1}, T_{it}, C_{t+1}) = \frac{\pi_i}{C_{t+1}} c^o_{it+1}(K_t, K_{t+1}, T_{it}) \tag{A.11}$$

Equilibrium Definition. Recall from the text that an equilibrium in this simple model is a sequence of financial positions $\{a^y_{it}, a^y_{jt}\}_{t\geq 0}$, prices $\{w_t, R_t\}_{t\geq 0}$, taxes $\{T_{it}, T_{jt}\}_{t\geq 0}$, and allocations $\{K_t, c^y_{it}, c^o_{it}\}_{t\geq 0}$ such that (i) households' Euler equations, intra-temporal conditions, and budget constraints are satisfied, (ii) the firm's optimality conditions for labor and capital are satisfied, (iii) the government's budget constraint is satisfied, (iv) and the resource constraint is satisfied for all $t \geq 0$.

A.6.2 Necessary and Sufficient Conditions for an Equilibrium.

In this Lemma, I show that equations (A.9), (A.10), and (A.11), along with the resource constraint (A.12) are necessary and sufficient for an equilibrium.

$$C_t = F(K_t, L_t) + (1-\delta)K_t - K_{t+1} \tag{A.12}$$

Necessary. Follows directly from the construction of the functions.

Sufficient. To show sufficiency, I find a set of prices and financial positions such that any allocation that satisfies equations (A.9)-(A.12) for policies, $\{T_{jt}\}_{t\geq 0}$ can be implemented as a competitive equilibrium, meaning that at these prices and policies, the allocation satisfies the firm's optimality conditions, the households' optimality conditions and budget constraints, and the government budget balances, the resource constraint, and asset market conditions clear. To satisfy the government's budget constraint, set $T_{it} = \frac{\pi_j}{\pi_i} T_{jt}$. Set $w_t = F_L(K_t)$ and $R_t = 1 - \delta + F_K(K_t)$. In doing so, the firm's first order conditions are satisfied. Set $a^y_{it} = \theta_i w_t + T_{it} - \Gamma^y_{it} C_t$ and $a^y_{jt} = \theta_j w_t + T_{jt} - \Gamma^y_{jt} C_t$. By construction, at these prices the household's Euler equation and budget constraint are satisfied as long as equations A.10 and A.11 are satisfied. By construction, equation A.9 guarantees the asset market clears. Finally, by assumption, the resource constraint A.12 is satisfied.

A.6.3 Constrained Planner's Problem.

The constrained planner's problem is to choose T_{jt} to maximize social welfare (A.13) subject to equations (A.9)-(A.12).

$$SW = \sum_I \lambda_i \pi_i \sum_{t=0}^{\infty} \nu^t \left(\frac{\Gamma^y_{it} C_t^{1-\sigma_y}}{1-\sigma_y} + \beta_i \frac{\Gamma^o_{it+1} C_{t+1}^{1-\sigma_o}}{1-\sigma_o} \right) + \frac{1}{\nu} \beta_i \frac{\Gamma^o_{i0} C_0^{1-\sigma_o}}{1-\sigma_o} \tag{A.13}$$

Define λ_t as the planner's lagrange multiplier with respect to the resource constraint at time t. Define $\omega_{it} = \lambda_i \nu^t (c^y_{it})^{-\sigma_y}$ and $\omega^o_{it} = \beta_i \lambda_i \nu^{t-1} (c^o_{it})^{-\sigma_o} = \omega_{it-1}/R_t$ as in the text. Let The planner's first order condition with respect to T_{Lt} is:

$$\sum_{j\geq 0}^{\infty} C_{t+j} \left(\pi_L \omega^y_{Lt+j} \frac{\partial \Gamma^y_{Lt+j}}{\partial T_{Lt}} + \pi_H \omega^y_{Ht+j} \frac{\partial \Gamma^y_{Ht+j}}{\partial T_{Lt}} + \pi_L \omega^o_{Lt+j} \frac{\partial \Gamma^o_{Lt+j}}{\partial T_{Lt}} + \pi_H \omega^o_{Ht+j} \frac{\partial \Gamma^o_{Ht+j}}{\partial T_{Lt}} \right) + \sum_{t\geq 0}^{\infty} \frac{\partial K_{t+1}}{\partial T_{Lt}} \left(\lambda_{t+1} (F_K(K_{t+1}) + 1 - \delta) - \lambda_t \right) = 0$$

The first order condition with respect to C_t is:

$$\sum_I \lambda_i \pi_i \left(\nu^{t-1} (\Gamma^o_{it})^{1-\sigma_o} C_t^{-\sigma_o} + \nu^t (\Gamma^y_{it})^{1-\sigma_y} C_t^{-\sigma_y} \right) = \lambda_t$$

A.6.4 Uniform β_i and $\sigma_y = \sigma_o$.

A.6.5 Rule out any allocation in which $c_L^j > c_H^j$.

A.6.6 Show that for $\nu > \hat{\nu}$, first-best equality allocation is not optimal.

Here I show that for sufficiently high ν, the full equality allocation in which $(c_{it}/c_{jt})^{\sigma_y} = \frac{\lambda_i}{\lambda_j}$ for all $t \geq 0$ can not be the optimal allocation whenever the following conditions hold.

Consider the steady state corresponding to fiscal policy setting the population-weighted welfare weights equal across types. Let $\bar{T}_L$ be the steady state fiscal policy that implements this allocation.

(a) Suppose $K_0 = \bar{K}$. Consider $\{\bar{T}_L\}_{t\geq 0}$ (and therefore $\{\bar{K}\}_{t\geq 0}$ as a potential solution to our planner's problem. In this case, the allocation would remain unchanged, and therefore, using the FOC with respect to C_t, $\lambda_t \nu = \lambda_{t+1}$. Recall that we have assumed that $F_K(\bar{K} + 1 - \delta) > \frac{1}{\nu}$. Plugging this into the FOC with respect to T_{Lt} :

$$\sum_{j\geq 0}^{\infty} C\left(\pi_L \omega_{Lt+j}\frac{\partial \Gamma^y_{Lt+j}}{\partial T_{Lt}} + \pi_H \omega_{Ht+j}\frac{\partial \Gamma^y_{Ht+j}}{\partial T_{Lt}} + \pi_L \omega^o_{Lt+j}\frac{\partial \Gamma^o_{Lt+j}}{\partial T_{Lt}} + \pi_H \omega^o_{Ht+j}\frac{\partial \Gamma^o_{Ht+j}}{\partial T_{Lt}}\right) = -\sum_{t\geq 0}^{\infty}\frac{\partial K_{t+1}}{\partial T_{Lt}}\left((F_K(\bar{K}) + 1 - \delta) - \frac{1}{\nu}\right)\lambda_{t+1}$$

Using the fact that $\pi_L \omega_{Lt+j} = \pi_H \omega_{Ht+j}$ and $\omega^o_{it+j} = \omega_{it+j-1}/R(\bar{K}) = \omega_{it+j}\frac{1}{\nu}/R(\bar{K})$, we can re-write the above as:

$$\pi_L \omega_{Lt}\left[\sum_{j\geq 1}^{\infty} C\nu^j\left(\frac{\partial \Gamma^y_{Lt+j}}{\partial T_{Lt}} + \frac{\partial \Gamma^y_{Ht+j}}{\partial T_{Lt}} + \frac{1/\nu}{R(\bar{K})}\frac{\partial \Gamma^o_{Lt+j}}{\partial T_{Lt}} + \frac{1/\nu}{R(\bar{K})}\frac{\partial \Gamma^o_{Ht+j}}{\partial T_{Lt}}\right) + \left(\frac{\partial \Gamma^y_{Lt}}{\partial T_{Lt}} + \frac{\partial \Gamma^y_{Ht}}{\partial T_{Lt}}\right)\right] = -\lambda_{t+1}\sum_{j\geq 0}^{\infty}\frac{\partial K_{t+1+j}}{\partial T_{Lt}}\left((F_K(\bar{K}) + 1 - \delta) - \frac{1}{\nu}\right)\nu^j$$

Note that because, by assumption, $\frac{1/\nu}{R(\bar{K})} < 1$ and therefore the left hand side of the equation is non-zero. T_{Lt} only affects the consumption share of generation t directly, but changes the consumption share of every subsequent generation by decreasing the capital stock and increasing the interest rate. Therefore, the first term above is negative, as the increasing interest rate decreases the consumption share of the young (which is weighted more heavily, as $\frac{1/\nu}{R(\bar{K})} < 1$) than the increase in the consumption share when old. Note that the right hand side is positive, as $\lambda_t > 0$, $\frac{-\partial K_{t+1+j}}{\partial T_{Lt}} > 0$, and $F_K(\bar{K}) + 1 - \delta - \frac{1}{\nu}$ is positive by assumption.

As $\nu \to 1$, the first term gets larger in terms of absolute value and approaches $-\infty$. Therefore, for sufficiently large $\hat{\nu}$, the first order condition is not satisfied, as the right hand side becomes an increasingly large positive number, while the left hand side becomes an increasingly large negative number.

(b) Now suppose $K_0 \neq \bar{K}$. Consider a path for fiscal policy, $\{\tilde{T}_{Lt}\}$, now with a time subscript, that implements the first best level of inequality at each period.

Prove that $\tilde{T}_{Lt}$ converges to $\bar{T}_L$ and therefore K_t converges to $\bar{K}$. To show that the path of fiscal policy converges to $\bar{T}_L$, the policy associated with the first-best steady state I show that the following are true:

1. The ratio $\frac{K_{t+1}}{K_t}$ is monotonically decreasing in K_t for a given fixed T_L. Therefore, if $K_t > \bar{K}$, $\frac{I(K_t)}{K_t} - \delta = \frac{K_{t+1}}{K_t} - 1 < 0$ and capital decreases, while the opposite is true if $K_t < \bar{K}$. Therefore, steady states are unique and stable.
2. If $K_t = \tilde{K}_t > \bar{K}$, then then T^*_{Lt} associated with first best equality in this case is higher that the $\tilde{T}_L$ associated with the steady state level of capital of $\tilde{K}$. Similarly, if $K_t = \tilde{K} < \bar{K}$, then $T^*_{Lt} < \tilde{T}_L$.
3. Using (1) and (2) Setting T^*_{Lt} at time t will bring K_{t+1} closer to $\bar{K}$. This implies that the T^*_{Lt+1} is closer to $\bar{T}_L$.

(1) Proof that $\frac{K_{t+1}}{K_t}$ is monotonically decreasing in K_t :

The derivative, $\frac{\partial K_{t+1}/K_t}{\partial K_t}$ is equal to:

$$\frac{\partial K_{t+1}/\partial K_t - K_{t+1}/K_t}{K_t} < 0$$

If we multiply by, $\frac{K_t}{K_{t+1}}$, we can see that this derivative is negative whenever:

$$\frac{\partial K_{t+1} K_t}{\partial K_t K_{t+1}} < 1$$

Using the implicit function theorem and the asset market clearing condition, this can be written in the following way, where A_r, A_w, and w_K are defined as in Section **??**:

$$\frac{\partial K_{t+1} K_t}{\partial K_t K_{t+1}} = \frac{\sum_I \pi_i \frac{a_{it}}{A_t} A_w w_K}{\frac{K_{t+1}}{A_t} - \sum_I \pi_i \frac{a_{it}}{A_t} A_r R_K} = \frac{A_w w_K}{1 - A_r R_K}$$

Where the final expression comes from the fact that $K_{t+1} = A_t$. Becuase $A_r > 0$ and $R_K < 0$, the denominator is above 1. Using the result from the previous section, that $A_w w_K < 1$, this fraction is less than one, meaning the derivative is negative, and the steady state is unique and stable.

(2) Take the first case where $K_t = \tilde{K} > \bar{K}$. Suppose that $T^*_{Lt} = \tilde{T}_L$. Then $K_{t+1} = K_t = \tilde{K}$. In this case $\tilde{T}_L < \bar{T}_L$, which implies that $\tilde{c}^y_{Lt} < \bar{c}^y_L$ and $\tilde{c}^y_{Ht} > \bar{c}^y_H$ and therefore $\tilde{\omega}_L > \tilde{\omega}_H$. A contradiction. Suppose $T^*_{Lt} < \tilde{T}_L$. Using the convergence result from Part 1, this implies that $K_{t+1} > K_t > \bar{K}$. This implies that $c^y_{Lt} < \bar{c}^y_L$ and $c^y_{Ht} > \bar{c}^y_H$ and therefore $\tilde{\omega}_L > \tilde{\omega}_H$. A contradiction. Therefore, it must be that $T^*_{Lt} > \tilde{T}_L$. By a symmetric argument, if $K_t = \tilde{K} < \bar{K}$, then $T^*_{Lt} < \tilde{T}_L$.

(3) Combining parts (1) and (2) implies that if $K_t > \bar{K}$, then by (2) $T^*_{Lt} > \tilde{T}_L$. Therefore, using (1), $K_{t+1} < K_t$ and $|K_{t+1} - \bar{K}| < |K_t - \bar{K}|$. This implies that K_t converges to $\bar{K}$ and therefore that T^*_{Lt} converges to $\bar{T}_L$.

Again consider the first order condition with respect to consumption:

$$\sum_I \lambda_i \pi_i \left(\nu^{t-1} (\Gamma^o_{it})^{1-\sigma_o} C_t^{-\sigma_o} + \nu^t (\Gamma^y_{it})^{1-\sigma_y} C_t^{-\sigma_y} \right) = \lambda_t$$

Note that as $K_{t+1} \to \bar{K}$, $C_{t+1} \to C$, and therefore for any arbitrary small ϵ, there exists a $\tau \geq 0$, such that for $t \geq 0$, $|\lambda_{t+1} - \lambda_t| < \epsilon$ for all $t \geq \tau$.

Again using the fact that $\pi_L \omega_{Lt+j} = \pi_H \omega_{Ht+j}$ by assumption, we can re-write the first order condition with respect to T_{Lt} as:

$$\pi_L \sum_{j \geq 1}^{\infty} C_{t+j} \nu^j \left(\omega_{Lt} \frac{\partial \Gamma^y_{Lt+j}}{\partial T_{Lt}} + \omega_{Lt} \frac{\partial \Gamma^y_{Ht+j}}{\partial T_{Lt}} + \omega^o_{Lt} \frac{\partial \Gamma^o_{Lt+j}}{\partial T_{Lt}} + \omega^o_{Lt} \frac{\partial \Gamma^o_{Ht+j}}{\partial T_{Lt}} \right) +$$
$$\left(\frac{\partial \Gamma^y_{Lt}}{\partial T_{Lt}} + \frac{\partial \Gamma^y_{Ht}}{\partial T_{Lt}} \right) = -\lambda_{t+1} \sum_{j \geq 0}^{\infty} \frac{\partial K_{t+1+j}}{\partial T_{Lt}} \left(\lambda_{t+1} (F_K(\bar{K}) + 1 - \delta) - \lambda_t \right)$$

This term can be broken further into the (finite) portion before $t = \tau$ and the (infinite) portion for $t \geq \tau$.

$$\pi_L \sum_{j\geq\tau}^{\infty} C_{t+j}\nu^j \left(\omega_{Lt}\frac{\partial\Gamma^y_{Lt+j}}{\partial T_{Lt}} + \omega_{Lt}\frac{\partial\Gamma^y_{Ht+j}}{\partial T_{Lt}} + \omega^o_{Lt}\frac{\partial\Gamma^o_{Lt+j}}{\partial T_{Lt}} + \omega^o_{Lt}\frac{\partial\Gamma^o_{Ht+j}}{\partial T_{Lt}}\right)+$$
$$\pi_L \sum_{j\geq1}^{\tau-1} C_{t+j}\nu^j \left(\omega_{Lt}\frac{\partial\Gamma^y_{Lt+j}}{\partial T_{Lt}} + \omega_{Lt}\frac{\partial\Gamma^y_{Ht+j}}{\partial T_{Lt}} + \omega^o_{Lt}\frac{\partial\Gamma^o_{Lt+j}}{\partial T_{Lt}} + \omega^o_{Lt}\frac{\partial\Gamma^o_{Ht+j}}{\partial T_{Lt}}\right)+$$
$$\left(\frac{\partial\Gamma^y_{Lt}}{\partial T_{Lt}} + \frac{\partial\Gamma^y_{Ht}}{\partial T_{Lt}}\right) = -\sum_{j\geq0}^{\tau-1}\frac{\partial K_{t+1+j}}{\partial T_{Lt}}\left(\lambda_{t+1}(F_K(\bar{K})+1-\delta)-\lambda_t\right)$$
$$-\sum_{j\geq\tau}^{\infty}\frac{\partial K_{t+1+j}}{\partial T_{Lt}}\left(\lambda_{t+1}(F_K(\bar{K})+1-\delta)-\lambda_t\right)$$

Because after $t \geq \tau$, C_t, K_{t+1}, and λ_{t+t}/λ_t are all arbitrarily close to their steady state counterparts, the above can be written as:

$$\pi_L\omega_{Lt}\left[\sum_{j\geq1}^{\infty} C\nu^j \left(\frac{\partial\Gamma^y_{Lt+j}}{\partial T_{Lt}} + \frac{\partial\Gamma^y_{Ht+j}}{\partial T_{Lt}} + \frac{1/\nu}{R(\bar{K})}\frac{\partial\Gamma^o_{Lt+j}}{\partial T_{Lt}} + \frac{1/\nu}{R(\bar{K})}\frac{\partial\Gamma^o_{Ht+j}}{\partial T_{Lt}}\right)+\right.$$
$$\pi_L \sum_{j\geq1}^{\tau-1} C_{t+j}\nu^j \left(\omega_{Lt}\frac{\partial\Gamma^y_{Lt+j}}{\partial T_{Lt}} + \omega_{Lt}\frac{\partial\Gamma^y_{Ht+j}}{\partial T_{Lt}} + \omega^o_{Lt}\frac{\partial\Gamma^o_{Lt+j}}{\partial T_{Lt}} + \omega^o_{Lt}\frac{\partial\Gamma^o_{Ht+j}}{\partial T_{Lt}}\right)+$$
$$\left(\frac{\partial\Gamma^y_{Lt}}{\partial T_{Lt}} + \frac{\partial\Gamma^y_{Ht}}{\partial T_{Lt}}\right) = -\sum_{j\geq0}^{\tau-1}\frac{\partial K_{t+1+j}}{\partial T_{Lt}}\left(\lambda_{t+1}(F_K(\bar{K})+1-\delta)-\lambda_t\right)$$
$$-\lambda_{t+1}\sum_{j\geq0}^{\infty}\frac{\partial K_{t+1+j}}{\partial T_{Lt}}\left((F_K(\bar{K})+1-\delta)-\frac{1}{\nu}\right)\nu^j$$

Again, as $\nu \to 1$, the right hand side becomes an (infinite large) positive number, while the left hand side is a negative number. Therefore, there exists a $\hat{\nu} \in (0,1)$ such that if $\nu > \hat{\nu}$ implementing the first best level of inequality violates the planner's first order conditions.

Case 2 ($\psi_a > 0$): The optimal implementable allocation x^*_I maximizes the following expression:

$$\max_{\{C_t\}_{t\geq0},T_L} \sum_I \lambda_i \sum_{t=0}^{\infty}\nu^t\left(\frac{(\Gamma^y_{it}C_t)^{1-\sigma_y}}{1-\sigma_y} + \nu^{-1}\beta_i\frac{(\Gamma^o_{it}C_t)^{1-\sigma_o}}{1-\sigma_o} + \beta_i\nu^{-1}\psi_a\frac{(a^o_i)^{1-\eta}}{1-\eta}\right)$$

subject to:

$$\begin{aligned}
\Gamma^y_{it} &= \Gamma^y_{it}(K_t, K_{t+1}, T_{it}, C_t, a^o_{t-1}) \;\; \text{for } i \in \{L, H\} \\
\Gamma^o_{it+1} &= \Gamma^y_{it}(K_t, K_{t+1}, T_{it}, C_{t+1}, a^o_{t-1}) \;\; \text{for } i \in \{L, H\} \\
K_{t+1} &= K_{t+1}(K_t, T_{Lt}, \{a^o_{it-1}, a^o_{it-2\}_{i\in I}}) \\
C_t + K_{t+1} &= F(K_t, 1) + (1-\delta)K_t \\
a^o_{it+1} &= a^o_{it+1}(K_{t+1}, K_t, a^o_{it-1}, T_{it})
\end{aligned}$$

Let μ_t be the lagrange multiplier for the resource constraint. The first order condition with respect to C_t is:

$$\sum_I \left(\omega_{it}(\Gamma^y_{it}) + \frac{\omega_{it-1}}{R_t}(\Gamma^o_{it}) \right) = \mu_t \tag{A.14}$$

The first order condition with respect to T_{Lt} is:

$$\sum_{j\geq 0}^{\infty} C_t \Bigg(\omega_{Lt+j}\frac{\partial \Gamma^y_{Lt+j}}{\partial T_{Lt}} + \omega_{Ht+j}\frac{\partial \Gamma^y_{Ht+j}}{\partial T_{Lt}} + \omega^o_{Lt+j}\frac{\partial \Gamma^o_{Lt+j}}{\partial T_{Lt}} + \omega^o_{Ht+j}\frac{\partial \Gamma^o_{Ht+j}}{\partial T_{Lt}} =$$

$$+\omega^o_{Lt+j}\frac{\partial a^o_{Lt+j}}{\partial T_{Lt}} + \omega^o_{Ht+j}\frac{\partial a^o_{Ht+j}}{\partial T_{Lt}} \Bigg) - \sum_{t\geq 0}^{\infty} \frac{\partial K_{t+1}}{\partial T_{Lt}} \left((F_K(K_{t+j}) + 1 - \delta) - \frac{1}{\nu} \right) \mu_{t+1} \tag{A.15}$$

Combining equations A.14 and A.15:

$$\sum_{j\geq 0}^{\infty} C_{t+j} \left(\omega_{Lt+j}\frac{\partial \Gamma^y_{Lt+j}}{\partial T_{Lt}} + \omega_{Ht+j}\frac{\partial \Gamma^y_{Ht+j}}{\partial T_{Lt}} + \omega^o_{Lt+j}\frac{\partial \Gamma^o_{Lt+j}}{\partial T_{Lt}} + \omega^o_{Ht+j}\frac{\partial \Gamma^o_{Ht+j}}{\partial T_{Lt}} + \omega^o_{Lt+j}\frac{\partial a^o_{Lt+j}}{\partial T_{Lt}} + \omega^o_{Ht+j}\frac{\partial a^o_{Ht+j}}{\partial T_{Lt}} \right)$$

$$= -\sum_{t\geq 0}^{\infty} \frac{\partial K_{t+j+1}}{\partial T_{Lt}} \left((F_K(K_{t+j}) + 1 - \delta) - \frac{1}{\nu} \right) \left(\sum_I \omega_{it+j}(\Gamma^y_{it+j}) + \frac{\omega_{it+j-1}}{R_{t+j}}(\Gamma^o_{it+j}) \right)$$

Again, suppose $K_0 = \bar{K}$ and consider a solution where $\omega_{Lt+j} = \omega_{Ht+j} = \omega_L$ for all $j \geq 0$. Recall that $\omega^o_{it} = \frac{\omega^y_{it-1}}{R_t}$. Plugging this into the above, pulling out and canceling the ω_L gives you:

$$\sum_{j\geq 0}^{\infty} \bar{C}\nu^j \left(\frac{\partial \Gamma^y_{Lt+j}}{\partial T_{Lt}} + \frac{\partial \Gamma^y_{Ht+j}}{\partial T_{Lt}} + \frac{1/\nu}{R(\bar{K})} \left(\frac{\partial \Gamma^o_{Lt+j}}{\partial T_{Lt}} + \frac{\partial \Gamma^o_{Ht+j}}{\partial T_{Lt}} + \frac{\partial a^o_{Lt+j}}{\partial T_{Lt}} + \frac{\partial a^o_{Ht+j}}{\partial T_{Lt}} \right) \right)$$

$$= -\sum_{t\geq 0}^{\infty} \frac{\partial K_{t+j+1}}{\partial T_{Lt}} \left((F_K(K_{t+j}) + 1 - \delta) - \frac{1}{\nu} \right) \nu^j \left(\Gamma^y_{Lt+j} + \Gamma^y_{Ht+j} + \frac{1/\nu}{R(\bar{K})}(\Gamma^o_{Lt+j} + \Gamma^o_{Ht+j}) \right)$$

This can be rewritten as:

$$\sum_{j\geq 1}^{\infty}\bar{C}\nu^j\left(\frac{\partial\Gamma^y_{Lt+j}}{\partial T_{Lt}}+\frac{\partial\Gamma^y_{Ht+j}}{\partial T_{Lt}}+\frac{1/\nu}{R(\bar{K})}\left(\frac{\partial\Gamma^o_{Lt+j}}{\partial T_{Lt}}+\frac{\partial\Gamma^o_{Ht+j}}{\partial T_{Lt}}+\frac{\partial a^o_{Lt+j}}{\partial T_{Lt}}+\frac{\partial a^o_{Ht+j}}{\partial T_{Lt}}\right)\right)$$

$$=-\sum_{t\geq 0}^{\infty}\frac{\partial K_{t+j+1}}{\partial T_{Lt}}\left((F_K(K_{t+j})+1-\delta)-\frac{1}{\nu}\right)\nu^j\left(\Gamma^y_{Lt+j}+\Gamma^y_{Ht+j}+\frac{1/\nu}{R(\bar{K})}(\Gamma^o_{Lt+j}+\Gamma^o_{Ht+j})\right)$$

$$+\bar{C}\left(\frac{\partial\Gamma^y_{Lt}}{\partial T_{Lt}}+\frac{\partial\Gamma^y_{Ht}}{\partial T_{Lt}}+\frac{1/\nu}{R(\bar{K})}\left(\frac{\partial\Gamma^o_{Lt}}{\partial T_{Lt}}+\frac{\partial\Gamma^o_{Ht}}{\partial T_{Lt}}+\frac{\partial a^o_{Lt}}{\partial T_{Lt}}+\frac{\partial a^o_{Ht}}{\partial T_{Lt}}\right)\right)$$

Note that by assumption $\frac{1/\nu}{R(\bar{K})}<1$. Note also that T_{Lt} only affects the consumption share and bequests of generation t directly, but changes the consumption share and bequests of every subsequent generation by decreasing the capital stock and increasing the interest rate. Therefore, the first term above is negative. To see why, note that a lower capital stock at each horizon decreases income and therefore decreases bequests left for both low and high type (that is $\frac{\partial a^o_{it+j}}{\partial T_{Lt}}<0$). To see that the remaining part of this term is also negative, note that a decline in K_{t+1} increases R_{t+1}, increasing the consumption share when old, but because $\frac{1}{\nu}/R(\bar{K})<1$ by assumption, the decline in the consumption share of the young is weighted more heavily. As before, the right hand side is positive, as $\mu_t>0$, $\frac{-\partial K_{t+1+j}}{\partial T_{Lt}}>0$, and $F_K(\bar{K})+1-\delta-\frac{1}{\nu}$ is positive by assumption.

Therefore, as $\nu\to 1$, the first term gets larger in terms of absolute value and approaches $-\infty$. Therefore, for sufficiently large $\hat{\nu}$, the first order condition is not satisfied, as the right hand side becomes an increasingly large positive number, while the left hand side becomes an increasingly large negative number.

Suppose $K_0\neq\bar{K}$. By an identical argument as in the $\psi_a=0$ case, because the policy that would implement ideal equality, $\{T^*_{Lt}\}_{t\geq 0}$ eventually converges to $\bar{T}^*_L$, as $\nu\to 1$, the planner's first order condition would have an ever large negative number on the left hand side and an ever larger positive number on the right hand side. Therefore, $\omega^*_{Lt}=\omega^*_{Ht}$ for all t is ruled out as a solution in this case as well.

A.7 Proof of Proposition 5

To be added.

A.8 Endogenous Labor + Labor Income Redistribution.

In the case with no bequest motive, the derivative of total steady-state social welfare with respect to τ_ℓ is defined as the following.

$$\frac{dSW}{d\tau_\ell}d\tau_\ell = \frac{1}{2}\sum_I \Bigg(\pi_i\lambda_i(c_i^y)^{-\sigma_y}\Bigg((1-\tau_\ell)\theta_i(\ell_i\frac{dw}{d\tau_\ell} + w\frac{d\ell_i}{d\tau_\ell}) - \theta_i w\ell_i + \frac{dT}{d\tau_\ell} - \frac{da_i^y}{d\tau_\ell}\Bigg)$$

$$-\pi_i\lambda_i\ell_i^\gamma\frac{d\ell_i}{d\tau_\ell} + \pi_i\lambda_i\beta_i(c_i^o)^{-\sigma_o}\Bigg(\frac{da_i^y}{d\tau_\ell}R + a_i^y\frac{dR}{d\tau_\ell}\Bigg)\Bigg)d\tau_\ell$$

The households' labor supply condition is given in equation (A.16) and Euler equation is given by equation (A.17).

$$(1-\tau_\ell)w(c_i^y)^{-\sigma_y} = \ell_i^\gamma \tag{A.16}$$

$$(c_i^y)^{-\sigma_y} = \beta_i R(c_i^o)^{-\sigma_o} \tag{A.17}$$

Plugging these conditions into the change in social welfare gives the following.

$$dSW = \frac{1}{2}\sum_I \Bigg(\pi_i\lambda_i(c_i^y)^{-\sigma_y}\Bigg((1-\tau_\ell)\theta_i\ell_i\frac{dw}{d\tau_\ell} - \theta_i w\ell_i + \frac{dT}{d\tau_\ell}\Bigg) + \pi_i\lambda_i\beta_i(c_i^o)^{-\sigma_o}a_i^y\frac{dR}{d\tau_\ell})\Bigg)d\tau_\ell$$

Finally, the fiscal authority's balanced budget implies,

$$dT = d\tau_\ell wL + \tau_\ell(Ldw + wdL)$$

Plugging this and the households' Euler equation into the change in social welfare gives the following.

$$= \frac{1}{2}\sum_I \Bigg(\pi_i\lambda_i(c_i^y)^{-\sigma_y}\Bigg((1-\tau_\ell)\theta_i\ell_i\frac{dw}{d\tau_\ell} + (wL - \theta_i w\ell_i) + \tau_\ell(L\frac{dw}{d\tau_\ell} + w\frac{dL}{d\tau_\ell}) + a_i^y\frac{dR}{Rd\tau_\ell}\Bigg)d\tau_\ell$$

Define Θ exactly as in the previous section.

$$\Theta = \sum_I \pi_i\lambda_i(c_i^y)^{-\sigma_y}\Bigg(\theta_i\frac{\ell_i}{L} - \frac{a_i^y}{RK}\Bigg)$$

Then, using the fact that for CES production, $\frac{dR}{dw} = -\frac{L}{K}$, the change in social welfare is:

$$dSW = \underbrace{\sum_I \omega_i \Big(L - \ell_i \theta_i \Big) w d\tau_\ell}_{\text{Direct Effects}} + \underbrace{L\Big(\Theta - \tau_\ell \sum_I \omega_i \Big(\frac{\theta_i \ell_i}{L} \Big) \Big) dw + \tau_\ell d(wL)}_{\text{General Equilibrium Costs}}$$

This can then be written as:

$$= \underbrace{\sum_I \omega_i \Big(wL - w\ell_i \theta_i \Big) d\tau_\ell}_{\text{Direct Effects}} + \underbrace{wL\Big((\Theta - \sum_I \omega_i \Big(\frac{\theta_i \ell_i}{L} \Big) \tau_\ell + \tau_\ell) \Big(w_K \frac{dK}{K} + w_L \frac{dL}{L} \Big) + \tau_\ell \frac{dL}{L} \Big)}_{\text{General Equilibrium Costs}}$$

Define $K_\tau = \frac{dK}{K} \frac{1}{d\tau_\ell}$ as the general equilibrium semi-elasticity of capital with respect to the labor income tax. Define L_τ analogously. Then the change in social welfare can be written in the following way.

$$= \sum_I \omega_i \Big(\frac{wL - w\ell_i \theta_i}{wL} \Big) d\tau_\ell + \Theta \Big(w_K K_\tau + w_L L_\tau \Big) + \tau_\ell \Big(L_\tau + w_K K_\tau + w_L L_\tau - \sum_I \omega_i \Big(\frac{\theta_i \ell_i}{L} \Big) \Big)$$

A.8.1 Solving for general equilibrium elasticities as functions of measurable sufficient statistics.

Saving, a_i^y is a direct function of permanent income and the gross interest rate, R.

$$a_i^y = PI_i - c_i^y(R(K), PI_i)$$

Where permanent income is defined as in the text.

$$PI_i = (1 - \tau_\ell)\theta_i w(K, L)\ell_i(c_i^y, (1 - \tau_\ell)w(K, L)) + w(K, L)L\tau_\ell$$

The general equilibrium derivative of permanent income with respect to the tax, scaled to average labor income, is given by equation (A.18).

$$\begin{aligned} \frac{dPI_i}{wL} &= \Big(\frac{wL - \theta_i w \ell_i}{wL} \Big) + (1 - \tau_\ell) \frac{\theta_i \ell_i w}{wL} \Big(\frac{d\ell_i}{\ell_i d\tau_\ell} + \frac{dw}{wd\tau_\ell} \Big) + \tau_\ell \Big(\frac{dL}{Ld\tau_\ell} + \frac{dw}{wd\tau_\ell} \Big) \\ &= \Big(\frac{wL - \theta_i w \ell_i}{wL} \Big) + \Big((1 - \tau_\ell) \frac{\theta_i \ell_i w}{wL} + \tau_\ell \Big) \Big(w_K \frac{dK}{Kd\tau_\ell} + w_L \frac{dL}{Ld\tau_\ell} \Big) \\ &\quad + (1 - \tau_\ell) \frac{\theta_i \ell_i w}{wL} \frac{d\ell_i}{\ell_i d\tau_\ell} + \tau_\ell \frac{dL}{Ld\tau_\ell} \end{aligned} \tag{A.18}$$

Above, I am using the government's budget constraint and the household's labor supply condition at each age to write household labor as an implicit function of permanent income, assets, and tax policy, $\ell(.) = \ell_i(PI_i, \tau_\ell, a_i^y)$:

$$\ell_i^\alpha = w(K,L)(1-\tau_\ell)\theta_i(c_i^y(\tau_\ell, R(K), w(K,L)))^{-\sigma_y} = \ell^\alpha(\tau_\ell, K, L)$$

This implies that aggregate labor, L can also be expressed as an implicit function of the linear tax and K.

$$L = \sum_I \pi_{iy}\theta_i\ell_i(\tau_\ell, K, L)$$

Note that $\frac{dL}{L}$ can be decomposed into the direct effects of the linear tax, $\frac{\partial L}{L}\frac{1}{d\tau_\ell}$ and $\mathcal{K}$.

$$\mathcal{K} = \frac{\partial L}{L}\frac{1}{\partial K} \tag{A.19}$$

The asset market clearing condition is given by the following.

$$K = \sum_I \pi_{iy} a_i^y(PI_i, R)$$

The total derivative of capital with respect to the tax is given in equation (A.20), which is divided by K to give the elasticity.

$$\frac{dK}{K}\frac{1}{d\tau_\ell} = \frac{1}{2}\sum_I \pi_i\left(\frac{\partial a_i^y}{\partial PI_i}\frac{dPI_i}{wL}\frac{wL}{Y}\frac{Y}{K} + \frac{\partial a_i^y}{\partial R}\frac{\partial R}{\partial K}\frac{dK}{K}\right) \tag{A.20}$$

Plugging equation A.18 into equation A.20 and isolating $\frac{dK}{K}$ gives the following expression. Here w_K and w_L are the elasticities of the wage to changes in capital and aggregate labor respectively. Similarly, A_{PI} is the elasticity of savings to changes in permanent income, A_r is the elasticity of household savings to the interest rate, and K_r is the elasticity of capital to R-1.

$$K_\tau = \frac{\sum_I \pi_{iy}\frac{\partial a_i^y}{\partial PI_i}\left(\frac{wL-\theta_i w\ell_i}{wL}\right) + \mathcal{L}}{1 - \sum_I \pi_{iy}\frac{\partial a_i^y}{\partial PI_i}\left(\frac{(1-\tau_\ell)\theta_i\ell_i w}{wL} + \tau_\ell\right)w_K\frac{wL}{Y}\frac{Y}{K} - A_r K_r^{-1}} \tag{A.21}$$

Here, the term $\mathcal{L}$ summarizes the effect of the general equilibrium change in labor supply on assets:

$$\mathcal{L} = \sum_I \pi_{iy} \frac{\partial a_i^y}{\partial PI_i} \left(w_L L_{\tau_\ell} \left(\frac{(1-\tau_\ell)\theta_i \ell_i w}{wL} + \tau_\ell \right) + \frac{(1-\tau_\ell)\theta_i \ell_i}{L} \frac{d\ell_i}{\ell_i d\tau_i} \right) \tag{A.22}$$