Monies and Inflation in a Finite Horizon Model

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Abstract

Economists and Central Bankers were surprised by the extraordinary inflation of 2021-22, just like they were by the worse inflation of the 1970s, for which there is still no definitive explanation. Many models have banished the quantity of money and even money itself (in favor of interest rates), blinding us to the inexorably emerging new monies and the steps central bankers must take to accomodate them without unleashing a new inflation wave.

We argue that a robust predictive model of price levels must monitor disaggregated transactions between heterogeneous agents, including the methods of payment, just like it would for relative prices. Incorporating so much detail in an infinite horizon model is computationally intractable. We propose a tractable finite horizon model with inside and outside money and rational agents in which money prices are endogenous, despite money giving no direct utility, that we use to study price levels, interest rates, and inflation. This model could be calibrated to real world transactions data at any level of disaggregation.

We show in our model that, depending on the architecture of payment systems, new monies like credit cards can cause a huge increase in prices, on the order of the 1970s inflation when credit cards emerged in full use. We also prove in the model that hyperinflation is inevitable when fiscal spending out of printed money is high enough, unless the central bank pursues the counterintuitive policy of looser lending. We show that fiscal spending with printed money is highly inflationary. By contrast, massive injections of money through bond purchases are not inflationary when it is believed that the central bank will eventually run down its bond holdings as they mature (a liquidity trap). This contrast aligns with the inflation after the 2020-21 Covid spending, and the non-inflation after the 2010- quantitative easing.

Our main contribution is to provide a framework and parsimonious notation for a model with many monies, like fiat money, credit cards, debit cards, money market cards, stable coins, and a host of potential future monies. Surprisingly, we prove that the viability of old fashioned cash and bank money is not destroyed by any of these new monies except stable coins, but that inflation will result without Fed action.

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Introduction

Inflation is the change in the commodity price of money. Yet the supply of money has virtually disappeared from modern economic theory. Curiously this has happened just as we are getting more monies. Instead of modeling the new monies, we exiled all money.

The orthodox models of inflation use too little data, have almost no general theorems, and are bad at making predictions except when inflation doesn't change. Most economists and the Fed failed to predict the 2021-22 inflation. For the first ten months the Fed stood idle while inflation surged from under 2% to close to 8%. This was followed by initial disagreement (failure) in predicting how fast inflation would come down. One explanation is that Covid Supply side disturbances are so unprecedented that failure to predict their consequences is unsurprising. However, there was also disagreement about whether 2008-2014 monetary injections (QE) would create inflation.[1]

Economists also did not foresee the magnitude of the 1970s inflation. The failure to predict inflation is less bad than the inability to understand it after the fact. Fifty years later, there is still no satisfactory explanation for the 1970s. It can't be just the Vietnam war, or the 1974 and 1979 oil shocks, because the war ended and oil is not a big enough part of the economy. It can't be the Anchovies debacle of the late 70s, though that was much ballyhooed at the time. What then was it?

1 What's Missing from Inflation Models

Prices refer to transactions between two instruments. Yet there are no transactions in most inflation models. In real life, nothing is purchased without either money or a promise in exchange. Promises end in payments, usually of money, except when payments are netted.

The ratio of transactions to money and other means of payment, that is to liquidity, would seem to be related to price levels and interest rates. The Fed injects money into the system every Christmas because of this. If it matters at Christmas, why not every other day?

Not only does the total amount of liquidity matter to prices, but so does how it is divided among agents and what their desired purchases are. A key factor in Covid inflation was the transition of demand from services to the goods sector. If a model does not explicitly include different sectors, it cannot possibly predict the effects of

[1] A crucial factor might have been the interest the Fed began tp pay on reserves. But do models really incorporate this factor; how much would inflation have changed if there were limits on the amount of reserves that could earn interest?

such demand shocks. Supply chain disruptions are also said to have been a crucial driver of covid inflation. Needless to say, without an explicit model of supply chains, and substitute inputs, and the elasticity of final demands, it is impossible to predict the effects of a supply bottleneck on prices.

A truly satisfactory model of price levels must keep track of who is buying what and with what (money or credit), that is, it must keep track of transactions at a finely disaggregated level. Such a model would not only follow the value and quantity of purchases, but also the means of payment, including the type of credit and the eventual delivery. One example of such a model is the Andersen et al (2023) study of purchases in Denmark, though it is a little short on credit data. Such agent based models are becoming entirely feasible. Even little hedge funds can follow mortgage decisions of almost every household. Why can't the Fed follow who has money and how many transactions various agents make?[2]

General purpose credit cards are perhaps the most ubiquitous financial innovation in the last fifty years. In the last five or ten years we are beginning to see a plethora of new monies like money market charging, and crypto currencies. Many more on the way. Aba Lerner called money a creature of the state, since nobody else can create it.[3] We had better recognize that this is no longer true. Do the new monies matter? A model that does not recognize them cannot answer.

1.1 Inflation is not entirely a Disequilibrium Phenomenon

Many mainstream models of inflation concentrate on the labor market, implicitly presuming that inflation is driven by disequilibrium in the labor market. The models often combine rational optimizing agents and market clearing in most goods, with fixed prices and ad hoc rationing mechanisms in the labor market, or the other way around. They often rely on special production functions and special kind of monopolistic competition.When prices do move they are sometimes governed by an exogenously given Phillips Curve or by exogenous menu costs or exogenous attention probabilities.

There is no single model of inflation. One stand alone model uses the Beveridge Curve to predict wage increases and thus inflation from the ratio of vacancies to unemployment, even without a precise mechanism that explains the relationship, and even though the relationship is often shifting. The Fed uses different models for different purposes. While that seems laudable, from another perspective it reveals there is no unified framework. There are no general theorems. Everything is special production functions, and special kinds of monopolistic competition.

Perhaps there is room for another complementary model that does not rely on special functional forms, that allows for general theorems, and that has a conventional market clearing mechanism for explaining equilibrium.

[2] The models must keep track of agent transactions, and they must also attach a utility or behavioral rule to each agent, in order to make (counterfactual) predictions. This is precisely what Wall Street prepayent and default models do.

[3] See Lerner [27].

1.2 Radical Proposal

It is impossible to keep track of many agent-based transactions in an infinite horizon model. We propose a **finite** horizon equilibrium model instead that allows for arbitrary production and utility functions, unlimited agent heterogeneity and unlimited commodity variety. If there is only one period, or a small finite number of periods, computational advances in general equilibrium make the model tractable even with huge heterogeneity. Fine detail agent based modeling is possible in a finite horizon model.

Unlike fantastical Walrasian budget sets, in which a purchase of an egg is paid for by the simultaneous sale of a banana to a completely different agent, all real world purchases give something tangible to the seller at the moment of purchase that the seller can use *later* for her subsequent purchase. This might be cash, or a check. Or a debit card, that transfers money electronically, without the hassle of cash bills or coins. A credit card purchase is payment via a promise to deliver money later. The credit card promise is a kind of money that enables purchases. We propose to study the effect of this continuing broadening of the notion of money.

Credit cards are by now a well established kind of money surrogate. But new monies are on the way. There are money market cards which are like debit cards except that it is possible to charge on assets held in a money market instead of money held in a bank account. Currently this ends up in a transfer of money to the recipients bank account. But we could imagine in the future that the correct amount of bonds are transferred from the buyer's money market - mutual fund to the seller's fund, without any money used at all. Another step is that the buyer might be able to choose the kind of bonds, for example their maturity bucket, that he charges on. And one could further imagine that there could be stock market charging cards where the buyer could charge on his index stock account without any conventional money payments. Stable coins that are backed by stocks guaranteeing a stable money value are exactly like that. At the moment, there are very few issuers of stable coins. But in the future we could imagine that any holder of a stock index could issue stable coins.

There are two critical distinctions in these new monies. The first is whether the ultimate delivery is in old fashioned money, or in securities. The second is whether, in the latter case, the securities themselves are bonds that promise conventional money, or whether they are real assets like stocks.

We build a model that includes all these innovations and we ask a fundamental question. Does the march to more monies create inflation? Does it foreshadow the inevitable nonviability of money? If credit became so easily available, how could money play any role?

Standard macroeconomic models nowadays pretty much ignore conventional money, and definitely new monies. The models presume that everything is determined by interest rates. One of our goals is to show that this is wrong. Changes in the stocks of money and access to credit will have real effects, and price level effects, even if the Fed holds the interest rates constant. If the Fed remains passive, inflation is inevitable.

We begin by explaining how money can have value in a finite horizon model, so long as the gains to trade from autarky are big enough. We give a precise definition of gains to trade, and a general existence *theorem* for the positive value of money. To introduce the notation and the main ideas, we start with a one period model, for which the positive value of money is the most surprising.

Next we give a *theorem* showing that there will be an accelerating difference, as m grows larger, in price levels caused by two policies: one, the fiscal expenditure of m dollars out of printed money (i.e. a combination of fiscal and monetary policy called helicopter money) and two, the open market purchase of m dollars worth of bonds (i.e. pure monetary policy). This might shed some light on the inflation that followed the unprecedented Trump-Biden helicopter drop during the Covid crisis.

We prove a *theorem* demonstrating that as the helicopter drop m gets large enough, hyperinflation must arise, in which the price level goes to infinity at a finite level of m. As m rises, the central bank will be tempted to raise interest rates even more to slow the price level increase. At first this will be helpful, but we prove that eventually the only way the fed can slow hyperinflation into inflation is by counterintuitively loosening lending.

Next in the one period model we show again in a *theorem* why credit cards do not destroy the viabiility of money, even if everyone thinks they are more convenient than cash. However, we prove that the widespread introduction of credit cards leads to a big jump in the price level, of the same size of the cumulative 1970s inflation, especially if the Fed passively maintains the same interest rate. We suggest that economists should investigate whether the missing explanation for the 1970s inflation is the contemporaneous rapid spread of credit cards.

With these preliminaries out of the way, we move to the multiperiod finite horizon model, in which new monies may be gradually coming on board. The notation is flexible enough to allow the period length to be minutes, corresponding to transactions processing durations, or years in order to reduce the dimensionality if the model encompasses many agents.

We first ask under what conditions conventional money (cash or bank account money) can have value? One might think with all the new monies, like money market credit cards, one would not need money. Surprisingly, we *prove* that even if purchases can be made by transferring bond assets, without any money changing hands, money still must have value if there are enough gains to trade and if the central bank does not set interest rates too high. The reason is that no matter how many bonds are accumulated, the bonds will not cover all the payments people need to make; if rising prices reduce the viability of money, they will also curtail the usefulness of the bonds as a substitute means of transaction.

Though they will not destroy the viability of old money, the technological move to bond purchases will be inflationary, just like the invention of credit cards. So are bit coins.

The situation is quite different with money backed by real assets. If it becomes possible to purchase commodities via a transfer of shares of stock, which stable coins effectively achieve, the viability of conventional money will be undermined.

As broader access is gained to such stock market credit cards, prices will rapidly rise without Fed intervention.

We close our paper by describing the liquidity trap. If the Fed injects large quantities of money through say mortgage backed securities purchases, and if it is generally believed that the Fed will eventually let its bonds pay down, then we prove there will be no inflationary effect beyond a certain point. This squares nicely with the Japanese liquidity trap during the end of the 20th century, and the American Quantitative Easing programs after the Great Recession of 2010.

1.3 The Hahn Problem

All this raises an old question, often called the Hahn problem. How can money in positive supply have value in a finite horizon equilibrium model? In the last period, nobody will want to hold it, so its value would apparently be zero. But then nobody would want to end up with it at the end of the second to last period. Working backwards, it would seem that money could never have value even if cash was necessary for trade. The economy would break down to autarky.

For this reason economists have embraced infinite horizon economies, so there is no last period. Unfortunately, then detailed records of heterogeneous agent based transactions become unmanageable.

Several devices for giving money value in finite horizon economies have been tried before. One is to assume money gives direct utility. The trouble is, that makes the price level essentially exogenous, rendering the model useless for studying inflation. Another device, suggested by Aba Lerner, is to assume that the supply of money is exactly equal to the taxes owed to the government, and payable only in money. Aside from the implausible equality between the past accumulation of money and the current tax debt, this "fiat money as a creature of the state" device robs fiscal policy of its force. A transfer of cash printed by the central bank and handed to the citizens by the government, such as we saw during Covid, loses much of its inflationary effect if it must be offset by taxes, analogous to the difference between the Keynesian multiplier and the Keynesian budget balanced multiplier. Finally, in some models all agents are obliged to sell everything for money at every moment, which they can simultaneously repurchase. Aside from the absurdity of this device, it

We give another explanation for the value of money by introducing a central bank. Far from propping up the value of money, the central bank will inject more money into the economy. The bank does not have anything to offer in exchange for money. It does, however, have the power to lend money to voluntary borrowers, and to enforce the collection of the ensuing debts. That collection power, together with the necessity of trading via money, guarantees money has value, provided the potential gains to trade from autarky are big enough.

The value of money puzzle is greatest when the time horizon is shortest. So we begin with a one period Walrasian economy, with the cash-in-advance stipulation, suggested by Clower in 1965, that purchases can only be made via old fashioned cash or fiat money. Agents $h \in H$ have endowments of money m^h that they own free and clear. But agents can also voluntarily borrow money from the central bank. By proving the existence of a one period equilibrium in which money has value, we show that the value comes from its transactions role alone, quite apart from any store-of-value role it might also have in multiperiod models. Needless to say, the model becomes more realistic, and more interesting, when there are more periods. But the logic is starkest and simplest in the one period model.

Agents who wish to spend more than their endowments of cash can borrow money from the bank, at some endogenously determined interest rate. They repay the bank out of the cash receipts from the sale of goods. The banking sector is endowed with an exogenously fixed stock of money, M. In equilibrium, which exists under quite general conditions, the bank interest rate r adjusts so that demand for money is equal to M, and the value of money is positive. As exogenous M rises, equilibrium r falls. Thus we could equivalently describe bank policy as fixing r, and accomodating whatever the demand M is at r. The bank could allow deposits at r as well; in the one period model no agent would want to deposit, but in the multiperiod model this becomes important. By adding a banking sector and a cash-in-advance constraint to the traditional general-equilibrium model, we are able to show what neither modification alone can generate: that money has value because of its transactions role.

The crucial idea behind our analysis is that agents, who do not initially owe the bank anything, are voluntarily driven by their own optimizing behavior to borrow at a positive interest rate r and incur debts $(1+r)M$ to the bank, that exceed what they borrowed M. They can only repay this by using (sending the bank) their own endowments $\sum_h m^h$ of money. Although the result is superficially similar to the Lerner model, in that all endowment money is finally owed to an external agency, each agent in our model begins with money yet no offsetting tax debt.[4]

When the state injects M (via a central bank) into the private sector in exchange for a promise to repay (i.e. a bond), its arrival foreshadows its departure, and we call it inside money. Money that is owned free and clear, with no countervailing obligation, like private endowments $\sum_h m^h$ of money or money injected by the state into the private sector as a transfer, or in exchange for a commodity (which gives no claim on future repayment), is called outside money. Our definitions of inside and outside money are taken from Gurley and Shaw [22]. Fiat money in our model corresponds to the green paper used as cash in the real economy, or to bank deposits. (We assume there is no other money.) All cash of course looks the same, regardless of whether it was originally injected as inside money or as outside money. But, in our model, the origin of the money plays a critical role in determining the endogenous real and monetary variables in equilibrium.[5] Keeping the total money constant, but shifting part of it from outside to inside will necessarily alter relative prices and the allocation of goods, as well as the interest rate and the price level.

When $\sum_h m^h > 0$, the set of monetary equilibria is determinate; i.e., there

[4] Shubik and Wilson (1977) first introduced a banking sector into a general equilibrium model. They computed the equilibrium in an example, but did not give a general statement or proof of the existence of equilibrium. Nor did they allow for private endowments of money.

[5] Many authors emphasize one of outside or inside money at the expense of the other. But it is the *interplay* between them that gives rise to many of the phenomena that form the focus of this lecture.

are (generically) only a finite number of equilibria. Supply and demand determine not only the relative prices but also the level of prices and the interest rate. As $M/\sum_h m^h \to \infty$, the interest rate goes to 0, and the monetary equilibrium commodity allocations converge to the Walrasian equilibrium allocations of the underlying nonmonetary economy.

Our existence theorem for monetary equilibrium provides a completely new proof of the existence of Walrasian equilibrium, by taking the limit as $M \to \infty$ (see Dubey and Geanakoplos 1989b). When $\sum_h m^h = 0$, and agents honor all their debts, the price level is indeterminate, but all monetary equilibrium allocations are Walrasian allocations.[6]

In our model the banking sector "earns" a positive rate of interest on worthless paper by exploiting the agents' need to transact through money. The bank profits end up equal to the original private endowments of money $\sum_h m^h$. One might think of the endowments $\sum_h m^h$ as having come from distributions from the Treasury, either as transfers or purchases of labor in the past. These Treasury expenses are thus effectively financed by central bank seigniorage profits.

The positive rate of money interest r puts a wedge between buying and selling prices, inhibiting trade. We are thus led to introduce a measure $\gamma^*(x)$ for the available gains to trade from any allocation $x = (x^h)_{h \in H}$, which may be of some value in its own right. Gerard Debreu (1951) defined the inefficiency of an allocation x by the biggest fraction $\delta(x)$ that could be thrown away so that the remaining $(1 - \delta(x))x$ could be properly reallocated to leave everyone at least as well off as they were at x. Our definition $\gamma^*(x)$ is the biggest fraction of *trades* that could be thrown away while still permitting trades to make everyone strictly better off than they were at x. Debreu's coefficient of resource allocation $\delta(x)$ is a global measure of the inefficiency of x, while our gains to trade $\gamma^*(x)$ is a local measure of the inefficiency of x.

Neither $\delta(x)$ nor $\gamma^*(x)$ has anything to do with money. But $\gamma^*(x)$ involves overcoming impediments to *trade*, and money facilitates trade while introducing an interest rate impediment. It turns out that the existence theorem for monetary equilibrium (which implies a positive price for money) follows from the hypothesis that the gains to trade $\gamma^*(e)$ at the initial endowment e exceeds $\sum_h m^h/M$. Under circumstances that we describe, $\gamma^*(e) > \sum_h m^h/M$ is necessary and sufficient for the existence of monetary equilibrium. This makes precise the link between monetary equilibrium and gains to trade.

Our model displays some of the rudimentary properties of a full-fledged monetary economy when the private stocks of money are positive. Injections of bank money are not neutral. They tend to cause inflation, but to lower the interest rate. On the other hand, gifts of fiat money to agents also cause inflation, but raise the interest rate. The former injections are analogous to open-market operations and the latter to fiscal policy.

In this simple model, injections of bank money tend to push the economy toward the Pareto frontier by lowering the rate of interest which is an inefficient wedge

[6] By contrast, in the Lerner model, the monetary equilibrium commodity allocations are always indeterminate.(See also Balasko and Shell 1983.)

between buying and selling. One wonders what prevents the central bank from indefinitely increasing M? One answer is that the central bank does not want to create too much inflation. Moreover, the resulting price increases may make agents with large endowments m^h of fiat money worse off; they will oppose such money increases.

It turns out that injections of inside money and outside money have a profoundly different quantitative effect on inflation, though they both raise prices. Injecting more and more inside money M into the economy eventually raises prices at a steady rate; prices rise linearly in M. Increases in $\sum_h m^h$ raise prices at a faster and faster rate. *Transfers of money by the Treasury are more inflationary than the same amounts of money injected through central bank loans (purchases of bonds).* Treasury money transfers eventually drive prices up so fast as to cause a *hyperinflation*. Prices go to infinity at a finite threshold of $\sum_h m^h/M$.

1.4 Credit Cards

We add credit cards to our one period model of money. If the credit card payments come due before the bank loans, then the same dollar can do double duty, enabling Jack to pay Jill for her apple and then enabling Jill to pay off her credit card purchase of Jack's banana. In a multiperiod setting the same logic holds. Money that might otherwise be idle can be used more frequently when credit cards are settled.

1.5 More Monies and More Time periods

A more realistic model of a monetary economy involves many possible periods of trade, and uncertainty. Money as a durable good would then have a store-of-value role as well as a transactions role to play. With other assets in the economy, money must compete with them in each agent's portfolio, and we can speak of the speculative demand for money, and the velocity of money as well.

One interesting phenomenon that emerges in the multiperiod model is the *liquidity trap*. As the central bank purchases more and more bonds at some time t, it will raise prices. However, if the economy understands that these injections of bank money are temporary, and that by some future time the central bank will return to its usual bond purchases, then the inflationary effect of time t purchases will eventually stop. Interest rates at time t will fall all the way to 0, and further injections of bank money will have no effect on prices or real activity. Such a phenomenon appears to have occurred in Japan in the 2000s and after the enormous quantitative easing in the US following the financial crisis of 2008-10. Fiscal transfers of money from the Treasury will nonetheless still have a powerful inflationary effect. We seem to have seen this after the great Covid stimulas packages of 2020-2021.

2 A One Period Model with Fiat Money

Consider an economy in which fiat money is the *sole* medium of exchange. Furthermore, suppose that there is just one round of trade between money and commodities. Since the money receipts from commodity sales come after the round is over, let us add the possibility of borrowing money prior to the trading round and repaying it after. Thus the period is divided into three time intervals: borrowing, trading, and repaying.[7]

As will be the case throughout, we allow for an arbitrary number of commodities and agents. All cash flows between agents explicitly enter the model. Since there is only one time period, it is easy to compute (approximate) equilibria with enormous agent heterogeneity over a large range of commodities.

2.1 The Underlying Economy

We first analyze a pure exchange economy which has only private goods (commodities) $L = \{1, ..., L\}$. The agents in the economy are households $H = \{1, ..., H\}$. Each $h \in H$ has an endowment of commodities $e^h \in \mathbb{R}^L_+$ and a utility of consumption $u^h : \mathbb{R}^L_+ \to \mathbb{R}$. We assume: (1) $e^h \neq 0$ for all $h \in H$, i.e., every household has at least some endowment (e.g., its own labor) and $\sum_h e^h >> 0$, so all goods are present; u^h is (2*) strictly increasing in each variable, (3) continuous, and (4) concave, for all $h \in H$. The underlying economy, which constitutes the real sector of our model, is denoted $\mathcal{E} \equiv (u^h, e^h)_{h \in H}$.

2.2 Inside and Outside Fiat Money

Money is fiat and gives no direct utility of consumption to the households; they value money only insofar as it enables them to acquire commodities for consumption. Money enters the economy in two ways: as private endowment $m^h \geq 0$ of household $h \in H$ and as injections M from a (central) bank. Apart from households, the bank is the only other agent in our model, but it has a passive role. The money endowments $m \equiv \{m^h\}_{h \in H}$ are exogenously fixed as part of the data of the model. The sum $\bar{m} \equiv \sum_{h \in H} m^h$ constitutes the stock of *outside money*, which households own free and clear of debt, at the start of the economy. The bank injection M is *inside money* and is always accompanied by debt when it comes into households' hands.

In one version of bank policy, the bank stands ready to lend $M > 0$ to households at an interest rate r that is determined endogenously in equilibrium. In that case we denote the monetary economy by

$$(\mathcal{E}, m, M) \equiv ((u^h, e^h, m^h)_{h \in H}, M);$$

and its private sector by $(\mathcal{E}, m) \equiv (u^h, e^h, m^h)_{h \in H}$.

Alternatively, the central bank can set the interest rate r at which agents are allowed to borrow whatever money M they want, or even to deposit whatever private money they want. In that case, r is exogenous and M becomes endogenous. The monetary economy is then denoted by

$$(\mathcal{E}, m, M) \equiv ((u^h, e^h, m^h)_{h \in H}, r);$$

The period, as was said, is divided into three time intervals. In the first interval, households borrow money from the bank. In effect, households sell IOU notes or

[7] Later we take up multiple trading rounds.

bonds to the bank in exchange for cash. In the second interval, they sell commodities for money and simultaneously buy goods with cash. In the third interval, they repay bank loans with money and consume.

All commodity markets meet simultaneously in the second interval. Households are required to pay money to purchase commodities at the different markets. It is only in the third interval, after these markets close, that revenue from the sales of commodities comes into households' hands, by which time it is too late to use this revenue for purchases. Those households who find their endowment m^h of money insufficient will need to borrow money from the bank to finance purchases, and will defray the loan out of their sales revenue.

2.3 Macrovariables: Prices and Quantities

Let $p_\ell > 0$ denote the price of commodity $\ell \in L$ in terms of money, and let $r \geq 0$ denote the money rate of interest on the bank loan. Money is borrowed by selling bonds to the bank. Each bond constitutes a promise to pay 1 dollar after commodity trade. Thus the price before commodity trade of a bond is $1/(1+r)$.

The vector $(p, r) \in \mathbb{R}^L_{++} \times \mathbb{R}_+$ will be referred to as "market prices." The price of money is $1/p_\ell$ in terms of commodity ℓ, and $(1+r)$ in terms of the bond. The value of money is reflected by these prices. As $p \to \infty$, money loses all value (in terms of commodities). As $r \to -1$, money-now loses all value (in terms of money-later) and as $r \to \infty$ money-later loses all value (in terms of money-now). Our focus is on p, since it is determined by the interaction of the real sector $\mathcal{E}$ and monetary sector (m, M) of the economy, and not so much on r, which is determined entirely by the monetary sector.[8]

2.4 Microvariables: Expenditures, Sales, and Deliveries

We denote money by m (without confusing it with the vector $m \equiv (m^1, ..., m^H)$ of household endowments) and bonds by b. A bond promises \$1 of delivery. Since money is the sole medium of exchange, the vector q^h of market actions of household h has $2L+1$ components (where $\ell \in L$):

$q^h_{bm} \equiv$ quantity of bonds sold by h to the bank for money
$q^h_{m\ell} \equiv$ money spent by h to purchase ℓ
$q^h_{\ell m} \equiv$ quantity of ℓ sold by h for money

It is evident, on account of their being just one period, that no household would improve its consumption by depositing money at the bank to earn interest. So, we suppress deposits, i.e., the purchase of bonds q^h_{mb}.

By real income q we mean the vector of aggregate commodity sales, with components $q_\ell = \sum_{h \in H} q^h_{\ell m}$. By nominal income we mean the value of real income

$$Y = p \cdot q \equiv \sum_{\ell \in L} \sum_{h \in H} p_\ell q^h_{\ell m}.$$

[8]In the multiperiod setting, which we study in a later lecture, there is a term structure of interest rates determined by the interaction of the real and monetary sectors, and our focus shifts to both p and r. When default penalties λ^h are low enough to allow default, r becomes endogenous.

Notice that income corresponds to sales and not to endowments. Since households are not obliged to sell their endowments, real income is genuinely endogenous. Nominal income appears doubly endogenous, since both p and q are endogenous, but often it can be deduced from monetary considerations alone.

Irving Fisher introduced a famous formula for the velocity of money, v, which in our context becomes

$$(M + \bar{m})v = p \cdot q \equiv Y.$$

In a one-period model the velocity of money is not very interesting. If all the money is spent, then $v = 1$ and nominal income is determined. If some of the money is unspent, v may be less than 1 and Y becomes endogenous.

2.5 The Budget Set of a Household

We consider the case of a perfectly competitive household sector. Each $h \in H$ regards market prices $(p, r) \in \mathbb{R}^L_{++} \times \mathbb{R}_+$ as fixed (uninfluenced by its own actions). The *monetary budget set* $B(p, r, e^h, m^h) \subset \mathbb{R}^{2L+1}_+ \times \mathbb{R}^L_+$ consists of all market actions and consumptions $(q^h, x^h) \in \mathbb{R}^{2L+1}_+ \times \mathbb{R}^L_+$ that satisfy the budget constraints (1), (2), (5), and (3ℓ), (4ℓ), (6ℓ) for all $\ell \in L$. The residual variables $\tilde{x}^h = \tilde{x}^h(q^h, p)$ and $\tilde{m}^h = \tilde{m}^h(q^h, r)$ are determined automatically by q^h, p, r.

$$\tilde{m}^h \equiv \frac{q^h_{bm}}{1+r} \tag{1}$$

$$\sum_{\ell \in L} q^h_{m\ell} \leq m^h + \tilde{m}^h \tag{2}$$

$$q^h_{\ell m} \leq e^h_\ell \tag{3ℓ}$$

$$\tilde{x}^h_\ell \equiv \frac{q^h_{m\ell}}{p_\ell} \tag{4ℓ}$$

$$q^h_{bm} \leq \Delta(2) + \sum_{\ell \in L} p_\ell q^h_{\ell m} \tag{5ℓ}$$

$$x^h_\ell \leq (\Delta 3\ell) + \tilde{x}^h_\ell. \tag{6ℓ}$$

Here $\Delta(\alpha)$ is the difference between the right and left sides of inequality (α). The interpretation is clear: (1) says that household h borrows $\tilde{m}^h$ dollars by promising to pay $q^h_{bm} = (1+r)\tilde{m}^h$ dollars after commodity trade, i.e., by selling q^h_{bm} bonds; (2) says that total money spent on purchases cannot exceed the money on hand, i.e., money endowed plus money borrowed; (3ℓ) says that no household can sell more of any commodity than it is endowed with; (4ℓ) says that households purchase commodities $\tilde{x}^h$ with money at market prices p; (5) says that there can be no default and deliveries of money to the bank must come out of the cash on hand; (6ℓ) says that consumption cannot exceed what a household winds up with after trade.

The budget set describes constraints on the flows of money and commodities that a household may send to market. Implicitly, these flows define changes in the household stocks of money and commodities after trade. The budget set ensures that the stocks are always nonnegative.

2.6 Monetary Equilibrium

A vector of prices and household actions

$$\langle p, r, (q^h, x^h)_{h\in H}\rangle \in \mathbb{R}^L_{++} \times \mathbb{R}_+ \times (\mathbb{R}^{2L+1}_+ \times \mathbb{R}^L_+)^H$$

is a *pre-monetary equilibrium* (preME) of $(\mathcal{E}, m, M)$ if all household actions are in their budget sets, i.e.,

$$(q^h, x^h) \in B(p, r, e^h, m^h) \tag{7}$$

and demand equals supply for the loan market and for all commodity markets, i.e.,

$$\begin{aligned} &\text{(a)} \sum_{h\in H} \tilde{m}^h(q^h, r) = M. \\ &\text{(b)} \sum_{h\in H} \tilde{x}^h_\ell(q^h, p) = \sum_{h\in H} q^h_{\ell m},\ \ell \in L \end{aligned} \tag{8}$$

It is worth noting that in a pre-monetary equilibrium, the total stock of money and commodities held collectively in the hands of the bank and the households is conserved in all three time intervals into which the period is divided. At the start, the bank holds M and households hold $\bar{m}$ of money. Money market clearing (8a) guarantees that the bank stock M flows to households at the end of the first interval. Commodity market clearing (8b) guarantees that the total stock of commodities is conserved and redistributed among the households during the second time interval. And (8b), multiplied by p_ℓ, shows that the total stock of money is conserved and redistributed among the households during the second time interval. Thus at the end of the first and second intervals, all of $M + \bar{m}$ is with households. The no-default condition (5) implies that the total bonds sold by households do not exceed $M + \bar{m}$. At the end of the third interval in a preME, the bank holds $(1 + r)M \le M + \bar{m}$, and households hold the balance $\bar{m} - rM$.

A preME $\langle p, r, (q^h, x^h)_{h\in H}\rangle$ is a *monetary equilibrium* (ME) iff

$$u^h(x^h) \ge u^h(\underline{x}^h) \text{ for all } (\underline{q}^h, \underline{x}^h) \in B(p, r, e^h, m^h).$$

This says that, taking into account the penalty for default, agents optimize in their budget sets.

2.6.1 Equilibrium Interest r

In any ME, at the end of the third interval, after repaying the bank, no household will be left with unowed cash, otherwise it should have borrowed more money and then spent it to purchase commodities, improving its utility (but also raising demand for M and thus raising r). Hence at least $(1 + r)M \ge M + \bar{m}$ is owed to the bank. But no more could be owed, if default is not permitted. (Anticipating its default, some agent would reduce its borrowing, presumably lowering r). Thus $(1 + r)M \le M + \bar{m}$ at any ME, if there is no default. Hence with no default permitted, $r = \bar{m}/M$.

This shows that when there is no default, the rate of interest r in (our one-period) monetary equilibrium is determined solely by the stocks of inside and outside money, and is unaffected by the real sector $\mathcal{E}$. Equivalently, if the central bank sets r, then M is determined in equilibrium by the same equation. In a multiperiod setting there would be a genuine interaction between the real and monetary sectors that determines the interest rates, as we shall see.

2.6.2 Endogenous Prices p

In contrast, even with one period, p is determined by a genuine interaction between the real and monetary sectors. Notice that since the components of p at any ME must be finite by definition, money will have positive value at an ME. Thus the existence of an ME is tantamount to a resolution of the Hahn paradox.

2.7 Another View of the No-Default Budget Set

We denote the set of *no-default budget-feasible consumptions* for household h by

$$B_C(p, r, e^h, m^h) = \{x^h \in \mathbb{R}_+^L : \exists q^h \in \mathbb{R}_+^{2L+1} \text{ with } (q^h, x^h) \in B(p, r, e^h, m^h)\}.$$

Note that B_C is homogeneous in p, m^h; i.e., for any $\lambda > 0$,

$$B_C(\lambda p, r, e^h, \lambda m^h) = B_C(p, r, e^h, m^h).^9$$

We can picture the budget-feasible consumptions for a household h, endowed with money and both goods 1 and 2, in the diagram below:

[9] Indeed, if $(q^h, x^h) \in B^{ND}(p, r, e^h, m^h)$, then $(\tilde{q}^h, x^h) \in B^{ND}(\lambda p, r, e^h, \lambda m^h)$ where $\tilde{q}^h_{\ell m} = q^h_{\ell m}$, $\tilde{q}^h_{bm} = \lambda q^h_{bm}$ and $\tilde{q}^h_{m\ell} = \lambda q^h_{m\ell}$.

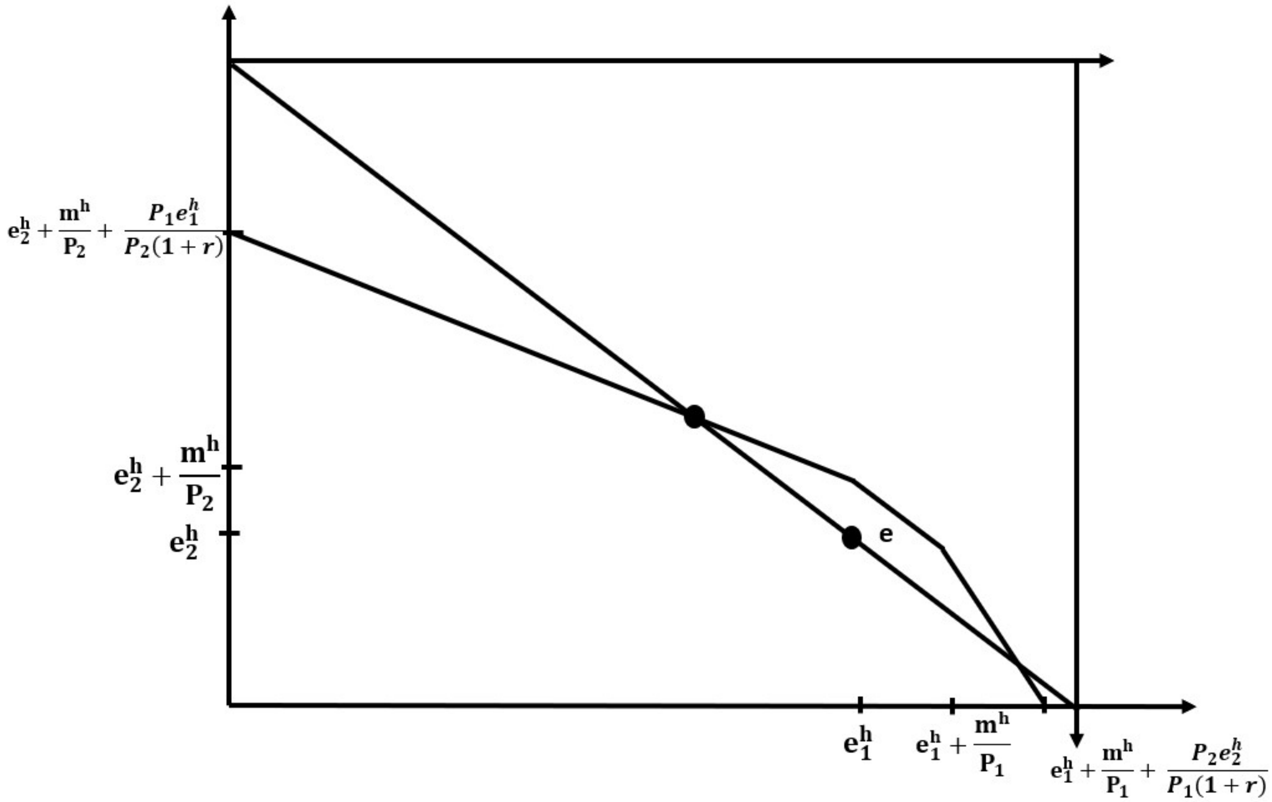


Diagram 1: Budget-Feasible Consumptions

2.7.1 Consumption Budget Set: Think Walrasian

In monetary equilibrium we defined the budget set by sequential restrictions on the q's, and then finally by inequalities connecting the q's to the x's. Here we rewrite the budget set with simultaneous constraints on the x's alone. In short, we show that restricted to the part of the budget set where we know which goods the agent is buying and which she is selling, the complicated monetary budget set can be recast as a Walrasian budget set. For example, in the picture above, if we restrict attention to the right side of the budget set in which the agent is buying good 1 and selling good 2, the budget set looks exactly like a Walrasian budget set. The great advantage of this transformation is that we know how to compute Walrasian demand, for example for Cobb-Douglas utilities. This will allow us to compute equilibrium for non-symmetric monetary economies.[10]

[10] The idea of rewriting a sequential budget set as a Walrasian budget set, with altred prices, was pioneered by Ken Arrow in a more complicated context that we describe in a later lecture.

Given prices $p >> 0$ and interest rate $r \geq 0$.

$$B_C(p, r, e^h, m^h) = \{x^h \in \mathbb{R}^L_+ : \sum_{\{\ell \in L: x_\ell \geq e^h_\ell\}} p_\ell(x_\ell - e^h_\ell) + \frac{1}{1+r} \sum_{\{\ell \in L: x_\ell < e^h_\ell\}} p_\ell(x_\ell - e^h_\ell) \leq m^h\}$$

$$\begin{aligned} B_C(p, r, e^h, m^h) = \{x^h \in \mathbb{R}^L_+ : & \sum_{\{\ell \in L: x_\ell \geq e^h_\ell\}} p_\ell x_\ell + \frac{1}{1+r} \sum_{\{\ell \in L: x_\ell < e^h_\ell\}} p_\ell x_\ell \\ & \leq \sum_{\{\ell \in L: x_\ell \geq e^h_\ell\}} p_\ell e^h_\ell + \frac{1}{1+r} \sum_{\{\ell \in L: x_\ell < e^h_\ell\}} p_\ell e^h_\ell + m^h\} \end{aligned}$$

The upshot is that we get the usual sort of Walrasian budget set, but with selling prices adjusted by a discount. One way to remember this formula is that buying takes place first and repaying the bank (which is the purpose of selling) takes place later and so is discounted.

2.7.2 Maximum Marginal Utility Conditions

Once we have the budget set written this way, we can easily derive the monetary first order conditions.

Whenever h buys i and sells j,

$$\frac{\frac{\partial u^h(x)}{\partial x_i}}{p_i} = (1+r) \frac{\frac{\partial u^h(x)}{\partial x_j}}{p_j}.$$

2.8 A Doubly Symmetric Monetary Equilibrium

Consider an exchange economy with two agents and two goods. Let the agents have identical utilities

$$u^h(x_1, x_2) = \log x_1 + \log x_2, \qquad h = 1, 2$$

Let

$$e^1 = (3, 1)$$
$$e^2 = (1, 3)$$

The symmetry in this example between agents and commodities makes it very easy to compute Walrasian and monetary equilibrium.

In Walrasian equilibrium, we would have $p = (\lambda, \lambda)$ for any $\lambda > 0$, and $x^1 = x^2 = (2, 2)$. As is well known, Walrasian economics ignores fiat money, and therefore cannot pin down the price *level*; nor does it make room for the nominal rate of interest r.

We first add inside money, then outside money, to the model.

Suppose default penalties are infinite. Suppose $M = 2$ and $m^1 = m^2 = 0$. Then it is easy to see that one monetary equilibrium goes like this: each agent borrows \$1 from the bank at interest rate zero; commodity prices are $p_{1m} = p_{2m} = 1$; agent h spends \$1 buying one unit of commodity $-h \neq h$, while simultaneously selling one unit of commodity h, for $h = 1, 2$; each h takes the \$1 he got from selling and uses it to repay his bank loan.

There is no equilibrium in which the rate of interest $r > 0$. For then the bank would be owed $M(1+r)$, but with only M dollars in existence, there would necessarily be default on bank loans, which is not permitted. There are however many other equilibria with $r = 0$ in which $(p_1, p_2) = \alpha(1, 1)$ with $0 < \alpha \leq 1$. Each agent h borrows \$1 as before, sells one unit of good h, and spends \$$\alpha$ on the other good (good $-h$). He simply hoards the other \$$(1 - \lambda)$ returning it unspent, together with the \$$\alpha$ obtained from selling one unit of h. Thus adding inside money M to the economy only reproduces the Walrasian indeterminancy of equilibrium price levels, with an upper bound on λ.

Monetary equilibrium becomes more interesting once we take $\bar{m} \equiv m_1 + m_2 > 0$. Then we get a unique equilibrium and positive interest rate, provided the gains to trade at the initial endowment e exceed $\bar{m}/M$. Let us suppose that $m^1 = m^2 > 0$, and that $M > \bar{m}/2$ is arbitrary.[11]

A picture of the Edgeworth Box for both agents is given below.

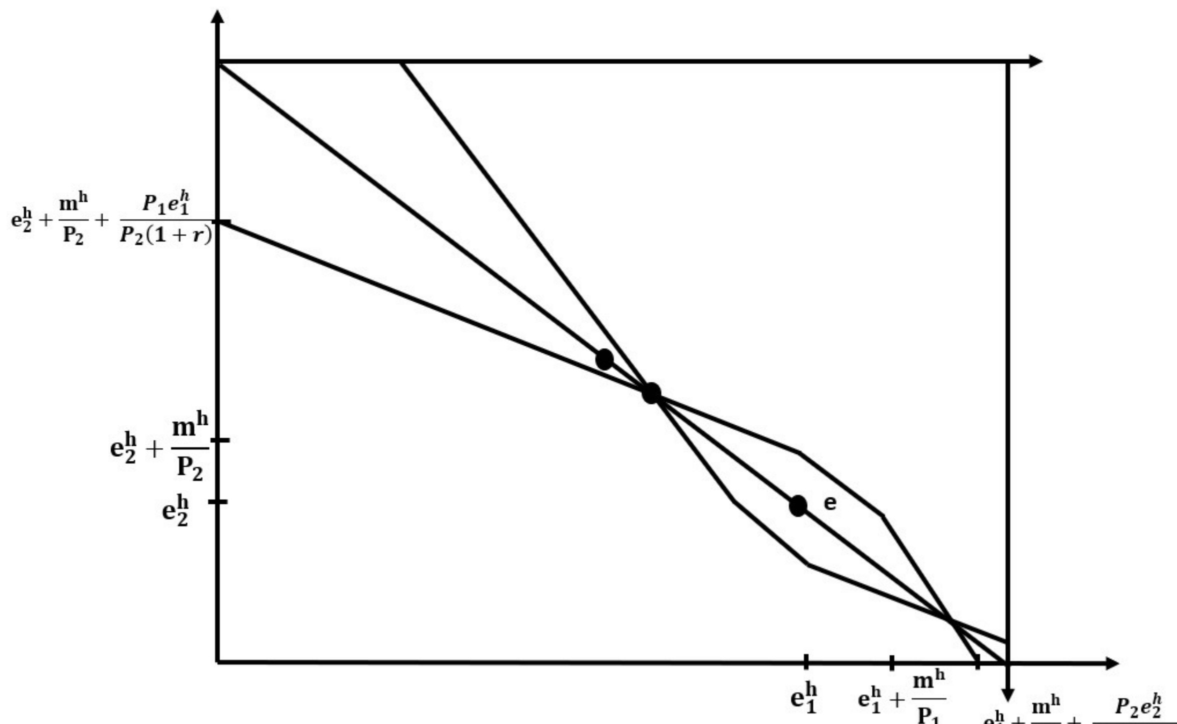


Diagram 2

[11] The gains to trade at e can be immediately computed from the formula in Dubey-Geanakoplos (2003)as $\sqrt{3/1 \cdot 3/1} = 1+\gamma(e)$, so $\gamma(e) = 2$. Thus monetary equilibrium exists for all $(m_1+m_2)/M = \bar{m}/M < 2$, i.e., for $M > \bar{m}//2$.

The stock of inside bank money M, and the outside money (m^1, m^2), in conjunction with the real economy, will determine price levels and the rate of interest. In fact the equilibrium rate of interest must be

$$r = \frac{\bar{m}}{M}$$

We can confidently guess that in equilibrium each agent will sell the good of which she owns 3 and buy the good of which she owns 1. In equilibrium, agent 1 is selling a units of good 1 and buying b units of good 2. By agent symmetry, agent 2 is selling a units of good 2 and buying b units of good 1. From market clearing, $a = b$. Hence we have $x^1 = (3 - \tau, 1 + \tau)$, $x^2 = (1 + \tau, 3 - \tau)$ for some $0 < \tau = a = b \leq 1$.

By symmetry of the goods, $p_1 = p_2$. Since there is only one period, agents will spend all their money. So we must have

$$p = p_1 = p_2 = \frac{M + \bar{m}}{2\tau}$$

Finally, in equilibrium we must have that the ratio of the marginal utilities of consumption are equal to the ratio of prices, distorted by the interest rate wedge. From the last section, we must have:

$$\begin{aligned}
\frac{\frac{\partial u^h(x)}{\partial x_i}}{p_i} &= (1+r)\frac{\frac{\partial u^h(x)}{\partial x_h}}{p_j} \\
\frac{\partial u^h(x)}{\partial x_i} &= (1+r)\frac{\partial u^h(x)}{\partial x_h}
\end{aligned}$$

$$\begin{aligned}
\frac{1}{1+\tau}/p_1 &= (1+r)\frac{1}{3-\tau}/p_2 \\
\frac{3-\tau}{1+\tau} &= \frac{p_1}{p_2}(1+r) = (1+r)
\end{aligned}$$

Solving for τ in terms of r, we see that

$$\begin{aligned}
3 - \tau &= (1+r)(1+\tau) \\
2 - r &= \tau + (1+r)\tau = (2+r)\tau
\end{aligned}$$

$$\begin{aligned}
r(\bar{m}, M) &= \frac{\bar{m}}{M} \\
\tau(\bar{m}, M) &= \frac{2-r}{2+r} = \frac{2 - \frac{\bar{m}}{M}}{2 + \frac{\bar{m}}{M}} = \frac{2M - \bar{m}}{2M + \bar{m}} \\
p(\bar{m}, M) &= \frac{(M+\bar{m})(2M+\bar{m})}{4M - 2\bar{m}}
\end{aligned}$$

As can be seen, the need for money to make transactions introduces an inefficiency into the sytem. The efficient trade level is $\tau = 1$, but in the monetary equilibrium $\tau(M)$ is always less than 1.

As M increases (holding $\bar{m}$ fixed) the nominal rate of interest charged by the bank, $r = \bar{m}/M$, decreases toward 0, and the level of trade $\tau(M) \to 1$ as $M \to \infty$.

Unfortunately, as M increases, the price level $p(M)$ also increases. If the central bank does not want to allow too high a price level, it will need to keep M low.

For example, if $\bar{m} = 2$ and $M = 20$, then $r = (1+1)/20 = 10\%$, $\tau = 19/21 \approx 0.90$ and $p = 21(22)/38 \approx 12.16$.

If we take $\bar{m} = 2$ and $M = 20 + \sqrt{20^2 + 2(20)} = 40.98$, then $r \approx 4.9\%$, $\tau \approx .95$, and $p_m \approx 22.56$. The increase in efficiency from $\tau = .90$ to $\tau = .95$, engineered by an increase in the stock of bank money M from 20 to 40.98, has been at the cost of almost a 90% inflation of prices.

The model thus incorporates a simple trade-off between efficiency and the price level, which we shall use as a proxy for the very important trade-off the Federal Reserve faces between employment (or output) and inflation. In a non-symmetric economy, agents with relatively high cash endowments m^h could be opposed to policy that reduces interest rates and increases trading efficiency, because the higher prices would diminish the value of their cash endowments.

Prices also increase if $\bar{m}$ is increased while holding M fixed. Indeed the rise in prices is much faster, and accelerating. A \$1 increase in $\bar{m}$ increases the numerator about as much as a \$1 increase in M. However, the increase in $\bar{m}$ also decreases trade and the denominator, leading to a faster increase in prices. Indeed as $\bar{m}$ approaches $2M$, trades go to zero and prices go to infinity. We get a hyperinflation. We shall see shortly that the ratio $\frac{\bar{m}}{M}$ near which hyperinflation occurs is determined by the gains to trade at the endowment e.

2.9 Gains to Trade and Monetary Equilibrium

At first glance the cash-in-advance constraint (embodied in budget constraint (2)) and the presence of the bank seem to provide a way out of the Hahn paradox: the bank, as was said, is an agent that demands money for its own sake, and households will need to hold money at the end in order to repay their loans to the bank. This argument would be fine if we could guarantee that households took out bank loans in the first place. But, unless money already has value to begin with, why should anyone want to take out loans? In a representative agent economy, for instance, nobody would take out loans and money would have no value. Thus the bank, while necessary, does not in and of itself ensure that money will have value. Something more is needed because otherwise equilibrium may not exist.

Let us draw the Edgeworth Box for a two agent economy with money. One can see that there may be a difficulty for monetary equilibrium. Clearly there can be no equilibrium in which every agent spends his cash without borrowing. As the diagram makes clear, any agent who has $m^h > 0$ will necessarily end up with utility higher than her initial endowment, if money has positive value. It follows that if the initial endowment is Pareto efficient, money cannot have positive value.

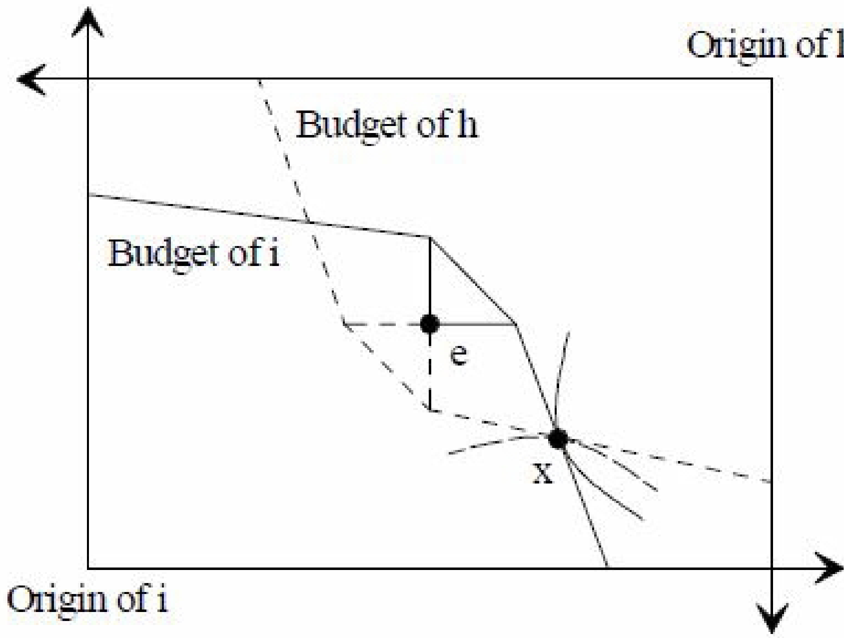


Figure 4: ME Allocation n in the Edgeworth Box

So what can we do? One device that gives money value is to oblige households to put up some positive fraction of their endowment for sale against money (i.e., require in condition (3) of the budget set that $\alpha e_\ell^h \leq q_{\ell m}^h \leq e_\ell^h$ for some $0 < \alpha \leq 1$). Indeed the case when the entire endowment must be put up for sale (i.e., $\alpha = 1$) is considered by Lucas [29], [30], and Magill–Quinzii [31]. Such forced sales, of course, ensure that money will buy something of value in equilibrium (i.e., an ME exists, see Remark 2). But the trouble is that some of these sales *must* be forced. With even the tiniest transactions cost, households would strictly prefer not to sell and buy back the same commodities. For any $\alpha > 0$, if any $e^h \gg 0$, household h would not voluntarily undertake to sell αe^h, for then there would be a commodity ℓ which h would be buying as well as selling. In our model there is no transaction cost; but there is a positive rate of interest at any ME if $\bar{m} > 0$. Households are loath to indulge in wash sales, because they would lose the interest float.

Another device to guarantee that money has value is to introduce a government, ready to defend the sanctity of its fiat money by putting up some exogenous stock of commodities (e.g., gold) for sale against money. By this device we could again get ME without much ado: government sales of gold back the fiat money and guarantee its purchasing power.

We do not have to take recourse to such extraneous and drastic measures as forced sales of commodities, or gold-backed money, in order to guarantee that money has value. What is required is an intrinsic "gains to trade hypothesis."

Fiat money is wanted only for trading commodities. It follows that the value of money should depend on households' motivation to trade commodities with each other. In this next lecture we develop a measure of this motivation called gains to trade and show that, whenever they are strong enough, monetary equilibrium exists. Money is valued and used to move commodities through markets.

2.9.1 Gains to Trade

Consider an economy of agents $h \in H$ with utilities over L goods

$$u^h : \mathbb{R}_+^L \to \mathbb{R}$$

We wish to provide a scalar measure $\gamma(e)$ of the available gains to trade starting from any initial allocation of goods $e = (e^h)_{h \in H} \in \mathbb{R}_+^{LH}$. Such a number could be useful in many contexts. For example, fiat money is wanted only for trading commodities. It follows that the value of money should depend on households' motivation to trade commodities with each other. We develop a measure of this motivation called local gains to trade and show that, whenever they are strong enough, monetary equilibrium exists.

Gerard Debreu (1951) defined the inefficiency of an allocation e by the biggest fraction $\delta(e)$ of e that could be thrown away so that the remainder could be properly reallocated to leave everyone at least as well off as they were at e. Our definition $\gamma(x)$ indicates the biggest fraction $\frac{\gamma(x)}{1+\gamma(x)} \approx \gamma(x)$ of *trades* that could be thrown away while still permitting trades to make everyone strictly better off than they were at x.[12] Debreu's coefficient of resource allocation $\delta(x)$ is a global measure of the inefficiency of x, while our gains to trade $\gamma(x)$ is a local measure of the inefficiency of x.

Neither $\delta(x)$ nor $\gamma(x)$ has anything to do with money. But $\gamma(x)$ involves overcoming impediments to *trade*, and money facilitates trade while introducing an interest rate impediment. It turns out that the existence theorem for monetary equilibrium (which implies a positive price for money) follows from the hypothesis that the gains to trade $\gamma(e)$ at the initial endowment e exceeds $\sum_h m^h/M$. Under circumstances that we describe, $\gamma(e) > \sum_h m^h/M$ is necessary and sufficient for the existence of monetary equilibrium. This makes precise the link between monetary equilibrium and gains to trade.

2.9.2 Local Gains to Trade Defined

One can think of the agents h living on separate islands. There are transportation losses from moving commodities between islands. When an agent h ships goods to another island he gives up all the goods he ships, but only the fraction $\frac{1}{1+\gamma}$ of those goods reach the target island; in other words, the fraction $\frac{\gamma}{1+\gamma}$ is lost. The bigger the transportation losses can be without eliminating the desire to trade, the bigger the gains to trade must have been.

Let $\tau^h \in \mathbb{R}^L$ be a trade vector of h (with positive components representing receivables and negative components representing sendables). For any scalar $\gamma \geq 0$,

[12] The reason for denoting the fraction of lost trades by $\frac{\gamma^*(x)}{1+\gamma^*(x)}$ instead of $\gamma^*(x)$ will become clear shortly.

define

$$\tau_\ell^h(\gamma) = \min\{\tau_\ell^h, \tau_\ell^h/(1+\gamma)\}$$

Note $\tau_\ell^h(\gamma) = \tau_\ell^h$ if $\tau_\ell^h < 0$, $\tau_\ell^h(\gamma) = \tau_\ell^h/(1+\gamma)$ if $\tau_\ell^h > 0$. Thus $\tau^h(\gamma)$ entails a diminution of receivables in τ^h by the fraction $\gamma/(1+\gamma)$.

We say that there are *gains to γ-diminished trade* at $x \equiv (x^h)_{h\in H} \in (\mathbb{R}_+^L)^H$ if there exist trades $(\tau^h)_{h\in H}$ such that:

(a) $\sum_{h\in H} \tau^h = 0$

(b) $x^h + \tau^h \in \mathbb{R}_+^L$ for all $h \in H$

(c) $u^h(x^h + \tau^h(\gamma)) > u^h(x^h)$ for all[13] $h \in H$.

In other words, it should be possible — in spite of the "γ-handicap" on trade — for households to Pareto-improve on x. We define $\gamma(x)$ as the supremum of all handicaps that permit Pareto improvement.

Definition The **gains to trade at x** are given by

$$\begin{aligned}\gamma(x) &\equiv \sup\{\gamma : \text{there are gains to } \gamma\text{-diminished trade at } x\} \\ &= \min\{\gamma : \text{there are } \textit{not} \text{ gains to } \gamma\text{-diminished trade at } x\}.\end{aligned}$$

2.10 Existence of Monetary Equilibrium

Theorem *Consider a monetary economy $((u^h, e^h, m^h)_{h\in H}, M)$ satisfying A1,A2*,A3,A4 in which $\bar{m} \equiv \sum_{h\in H} m^h > 0$ and the Gains to Trade Hypothesis holds, i.e., $\gamma(e) > \bar{m}/M$. Then a monetary equilibrium without default exists and, at any monetary equilibrium, the interest rate $r = \bar{m}/M$.*

Conversely, if each u^h is additively separable,

$$u^h(x_1, ..., x_L) = \sum_{\ell=1}^{L} u_\ell^h(x_\ell)$$

then under the same hypotheses except that $\gamma(e) \le \bar{m}/M$, monetary equilibrium does not exist.

According to the theorem, increasing the stock of inside money M must eventually guarantee the orderly functioning of markets, if the initial endowment is not Pareto-optimal. Equilibrium exists once M exceeds the finite threshold $\bar{m}/\gamma(e)$. In our model, inside money is indeed "the grease that turns the wheels of commerce."

As bank money M approaches infinity, the final allocation of goods becomes essentially no different from the Walrasian allocation obtained in an idealized world without any money at all, and in which prices really only have meaning as exchange rates between pairs of commodities. Levels of bank money beyond $\bar{m}/\gamma(e)$, but short of infinity, give a large domain in which the real sector $\mathcal{E} = (u^h, e^h)_{h\in H}$ and the financial sector (m, M) influence each other, as we show in the next lecture.

[13] Since utilities are strictly monotonic, this is equivalent to requiring that some household is strictly better off and none are worse off.

2.10.1 Proof of Existence of Monetary Equilibrium

The intuition behind the proof is as follows. Imagine that every agent is forced to sell the fraction ε of each real good she is endowed with against money. That will guarantee money has value by a fixed point argument. Letting $\varepsilon \to 0$ gives a sequence of monetary equilibria. If they stay bounded, then there must be a convergent subsequence whose limit would be a genuine monetary equilibrium with no sales requirement and a positive value of money.

The only thing that could go wrong is that the monetary equilibrium prices $p(\varepsilon) \to \infty$. That would imply that money loses all its value and all trades go to zero as $\varepsilon \to 0$. But then we can show there would be a convergent subsequence of *relative* prices at which the endowment is an autarkic $r-$equilibrium, with $r = \bar{m}/M$. But the gains to trade hypothesis and the gains to trade theorem at autarkic r-equilibrium rules that out, showing that prices stay bounded and monetary equilibrium exists.

Now we give the details. Without default, we know that in any monetary equilibrium, interest $r = \sum_{h \in H} m^h/M$. So fix $r > 0$ there.

Fix a very small $\varepsilon > 0$. We shall describe a continuous function F_ε

$$F_\varepsilon : P_\varepsilon \times S_\varepsilon^1 \times ... \times S_\varepsilon^H \to P_\varepsilon \times S_\varepsilon^1 \times ... \times S_\varepsilon^H$$

$$F_\varepsilon(p, (q^1, x^1), ..., (q^H, x^H)) = (\hat{p}, (\hat{q}^1, \hat{x}^1), ..., (\hat{q}^H, \hat{x}^H))$$

$$P_\varepsilon = \{p \in \mathbb{R}^L_{++} : \frac{\varepsilon \bar{m}}{\sum_{i \in H} e^i_\ell} \leq p_\ell \leq \frac{H(M + \bar{m})}{\varepsilon \sum_{i \in H} e^i_\ell}, \forall \ell\}$$

$$S^h_\varepsilon = \{(q^h, x^h) \in \mathbb{R}^{2L+1}_+ \times \mathbb{R}^L_+ : q^h_{bm} \leq M + \bar{m}; \sum_\ell q^h_{m\ell} \leq \frac{q^h_{bm}}{1+r} + m^h;$$

$$q^h_{m\ell} \geq \varepsilon m^h; \varepsilon e^h_\ell \leq q^h_{\ell m} \leq e^h_\ell, x^h_\ell \leq \sum_{i \in H} e^i_\ell + 1, \forall \ell\}$$

Clearly P_ε and each S^h_ε is compact and convex. Notice that the ε forces a positive amount of sales of each good and a positive amount of money expenditures on each good.

Define F_ε by

$$\hat{p}_\ell = \frac{\sum_h q^h_{m\ell}}{\sum_h q^h_{\ell m}}$$

$$(\hat{q}^h, \hat{x}^h) = Arg \max_{(\hat{q}^h, \hat{x}^h) \in B(p, r, e^h, m^h) \cap S^h_\varepsilon} [u^h(\hat{x}^h) - ||\hat{q}^h - q^h||^2 - ||\hat{x}^h - x^h||^2]$$

From the maximum principle and the strct concavity of the maximand and the compactness and convexity of the domain, and because the constraint qualification is satisfied, the demands are continuous functions over strictly positive prices.

The novel part is the price formation mechanism, obtained not by the usual Debreu price player but more simply by taking the ratios of expenditures to sales.[14]

[14] The Debreu price player approach does not work in monetary economies because Walras Law fails. Every agent can spend more than the value of his endowment of goods by using his endowment of money.

This function is continuous because the denominator can never be 0 on account of the forced ε sales. Notice also that the bounds established by the domains S_ε^h on the choices $\hat{q}^h$ guarantee that $\hat{p} \in P_\varepsilon$.

Hence by Brouwer, the function F_ε has a fixed point $(p(\varepsilon), (q^1(\varepsilon), x^1(\varepsilon)), ..., (q^H(\varepsilon), x^H(\varepsilon)))$. At the fixed point every agent is fully optimizing her utility u^h because the quadratic perturbations are irrelevant, as per the Geanakoplos lemma. Moreover, rewriting the price part of the fixed point

$$p_\ell(\varepsilon) = \frac{\sum_h q^h_{m\ell}(\varepsilon)}{\sum_h q^h_{\ell m}(\varepsilon)} \Longleftrightarrow \frac{\sum_h q^h_{m\ell}(\varepsilon)}{p_\ell(\varepsilon)} = \sum_h q^h_{\ell m}(\varepsilon)$$

we get demand equals supply. As we let $\varepsilon \to 0$, the ε bounds on agent behavior will become irrelevant. If we could show that the variables $(p(\varepsilon), (q^1(\varepsilon), x^1(\varepsilon)), ..., (q^H(\varepsilon), x^H(\varepsilon)))$ stay bounded as $\varepsilon \to 0$, we could pass to a convergent subsequence whose limit would be a genuine monetary equilibrium. All the $(q^h(\varepsilon), x^h(\varepsilon))$ are indeed bounded independent of ε. Unfortunately, the prices might tend to infinity as $\varepsilon \to 0$.

Money naturally has value with the forced ε sales. As $\varepsilon \to 0$ the artificial boost to money value goes away and there is a genuine danger that $p(\varepsilon) \to \infty$, meaning that a dollar buys less and less goods.

Note that no $p_i(\varepsilon) \to 0$, for otherwise by strict monotonicity any household h with $m^h > 0$ would be buying more of some good ℓ than there is, contradicting the feasibility of ε-ME.

By the same strict monotonicity argument, we cannot have $p_\ell(\varepsilon)/p_i(\varepsilon) \to \infty$, because then for some $\varepsilon > 0$ some agent could have done better by borrowing money and purchasing more i than exists, paying back the bank with a sliver of ℓ sales.

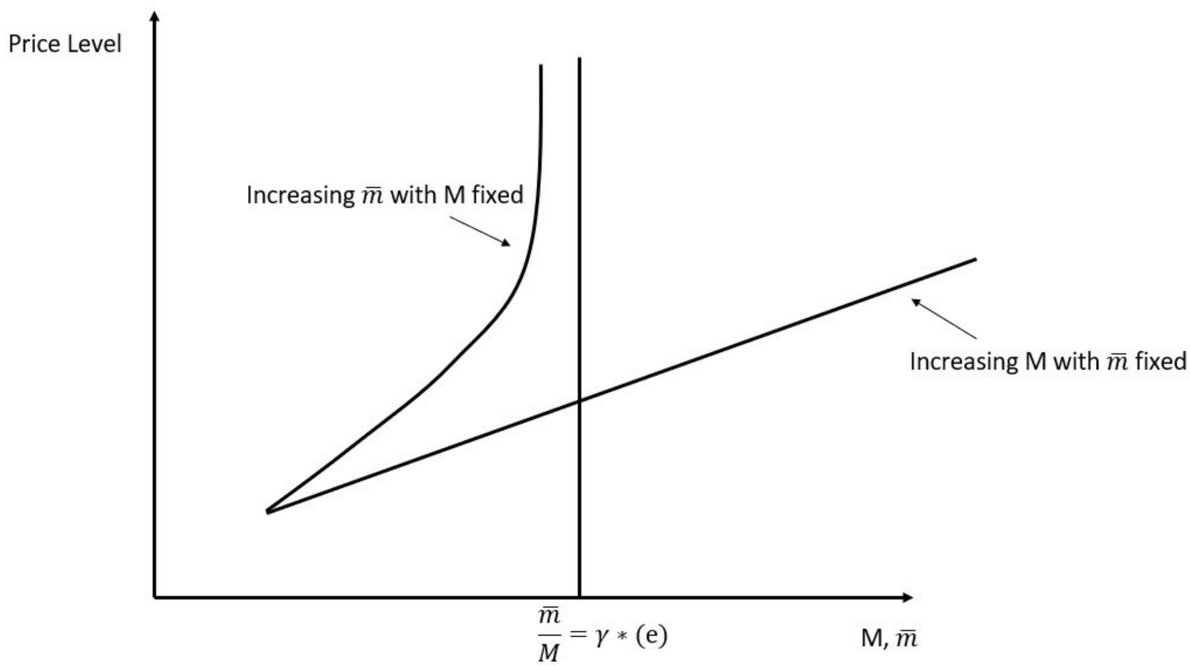
So suppose $p_\ell(\varepsilon) \to \infty$ for all ℓ. Then, since the total money in the system is bounded and since money is the sole medium of exchange, trade in all goods $\to 0$ as $p(\varepsilon) \to \infty$. Hence households end up consuming their initial endowment e^h in the limit. At the same time, notice that with $p(\varepsilon) \to \infty$, the purchasing power of the endowed money m goes to zero and may be ignored. Consider now the limiting price *ratios* (on some subsequence of ε) given by p, where $p_\ell = \lim_{\varepsilon \to 0} p_\ell(\varepsilon)/\sum_{k\in L} p_k(\varepsilon)$. The trading opportunity for any household (at the limit) is effectively to purchase goods solely out of borrowed money and to pay the loan back, at the interest rate $r \equiv \bar{m}/M$ out of his sales revenue (conducting all trade via money, of course, at the prices p). A little reflection reveals that this is tantamount to saying that $(p, (e^h)_{h\in H})$ is an autarkic r-Walrasian equilibrium. This implies that $\gamma(e) \leq r = \bar{m}/M$, contradicting the gains-to-trade hypothesis. So $p(\varepsilon) \nrightarrow \infty$, finishing the proof.■

For the proof that monetary equilibrium does not exist when $\gamma(e) \leq \bar{m}/M$, see Dubey-Geanakoplos 2003.

2.11 Money and Prices

Doubling the m^h and also M leaves real trade τ and the interest rate r unchanged, while doubling the price level. So balanced increases in all money are neutral on real variables.

Increases in the m^h can be thought of as expansionary *fiscal policy*. As can be seen in the formulas, fiscal policy always has non-neutral effects. Fiscal policy always increases the nominal interest rate r, increases nominal trade or income Y, increases the price level p, and reduces efficiency (i.e. increases the unexploited gains to trade at equilibrium, and in this example reduces real trade τ).



The model thus incorporates a simple trade-off between efficiency and the price level, which we shall use as a proxy for the very important trade-off the Federal Reserve faces between employment (or output) and inflation. In a non-symmetric economy, agents with relatively high cash endowments m^h could be opposed to policy that reduces interest rates and increases trading efficiency, because the higher prices would diminish the value of their cash endowments.

The increase in price levels from expansionary monetary policy is somewhat muted because while the numerator (money spent) increases, so does the denominator (goods sold) because of the efficiency gains from lower r. Eventually the denominator converges to the Walrasian sales, and thus prices rise *linearly* in M.

Prices also increase if $\bar{m}$ is increased while holding M fixed. Indeed the rise in prices is much faster, and accelerating. A \$1 increase in $\bar{m}$ increases the numerator as much as a \$1 increase in M. However, the increase in $\bar{m}$ also decreases trade and the denominator, leading to a faster increase in prices. Indeed as $\bar{m}$ approaches $2M$, trades go to zero and prices go to infinity. We get a hyperinflation.

2.12 Fiscal and Monetary Policy

A change in M alone, or in m alone, or in both but in different proportions, will invariably affect real trades. The injection of bank money (with private endowments of money held fixed) corresponds to a form of elementary monetary policy in our model. It is evident that this policy will lower the interest rate (since $r = \bar{m}/M$) and alter commodity allocations, moving them "closer" to Pareto-efficiency since the unexploited gains to trade "left on the table" become smaller. However, increases in M will also eventually raise equilibrium price levels p. Households that began with relatively large endowments m^h of money will be hurt, since their cash endowments lose purchasing power. These households could be expected to use their influence on the central bank to resist such expansionary monetary policy.

Gifts of fiat money to households constitute fiscal policy. They will cause interest rates to rise, and the ensuing ME allocations are bound to be affected, becoming less (locally) efficient in the process. Of course households that were the primary recipients of the fiscal gifts may be better off than before.

The welfare-reducing impact of fiscal injections is most pronounced in the setting of exchange economies with private goods and complete markets. When there is production and incomplete markets, fiscal injections may be Pareto improving. But we deal with this important issue elsewhere [13]. Fiscal injections can also be Pareto improving when there are public goods.[15]

Central bank purchases of bonds (i.e. lending to individuals via higher M) constitutes expansionary monetary policy. Over the usual domains, this increases prices. In other words, over the usual domain contractionary monetary policy, shrinking M, will lower prices.

2.13 Inflation and Hyperinflation

Let us fix $\bar{m}$ and start with M so large that $M/\bar{m}$ is well to the right of $1/\gamma(e)$. Equilibrium exists, and with very low r, it is nearly Walrasian. Increasing M still further has little real effect; nearly exactly the same real trades are conducted. Since all of $M + \bar{m}$ is spent on practically the same purchases, the price level rises linearly with M, and we have the linear inflation depicted in the diagram.

2.13.1 Fiscal Policy and the 2020-2021 Fiscal Transfers

In 2020 and then in 2021 Presidents Trump and Biden pushed through massive fiscal transfers to households, small businesses, and local governments that they felt needed the money in the midst of Covid. The total transfers of the Covid relief packages of \$1 Trilliom and then \$1.9 Trillion or \$3 Trillion amounted to about 14% of GDP, perhaps 5 times bigger than any other transfer (as a fraction of GDP) in American history. A crude calculation might be that if the money was spent once over the next year, then we should expect about a 14% increase in prices. Many recipients

[15] In this richer setting Dubey and Geanakoplos show how to derive Hicks' famous IS-LM curves in a completely general monetary equilibrium model with arbitrary heterogeneous agents.

saved the money without spending it on any commodities. Other parts of the money got spent multiple times in the same year. Without knowing the average velocity of money, it is difficult to predict the size of the price change.

Nonetheless, our one period model can offer some qualitative conclusions. We know that the prices will rise faster from a fiscal injection than from the same size injection of bank money through bond purchases.

2.13.2 Fiscal Policy and Hyperinlation

Suppose the economy has strictly concave and separable utilities. As $\bar{m}$ rises toward $M\gamma^*(e)$, what happens to equilibrium price levels? We know that equilibrium fails to exist when $\bar{m}/M = \gamma^*(e)$. If price levels stay bounded as $\bar{m}$ rises toward $M\gamma^*(e)$, then there will be a convergent subsequence of prices that do converge. The limit would be an equilibrium for $\bar{m}/M = \gamma^*(e)$. Hence the prices *must* go to infinity, as in the Diagram. We call the phenomenon when prices go to infinity at finite levels of inside and outside money a *hyperinflation*.

There have been many hyperinflations in history. The most famous example is in 1923 in Weimar Germany, but there were many others in Africa and South America. They used to be quite common. The monetary equilibrium model gives an explanation. Hyperinflation is caused when the government tries to spend too much money, say for soldiers or other boondoggles, by printing it instead of raising it through taxes.

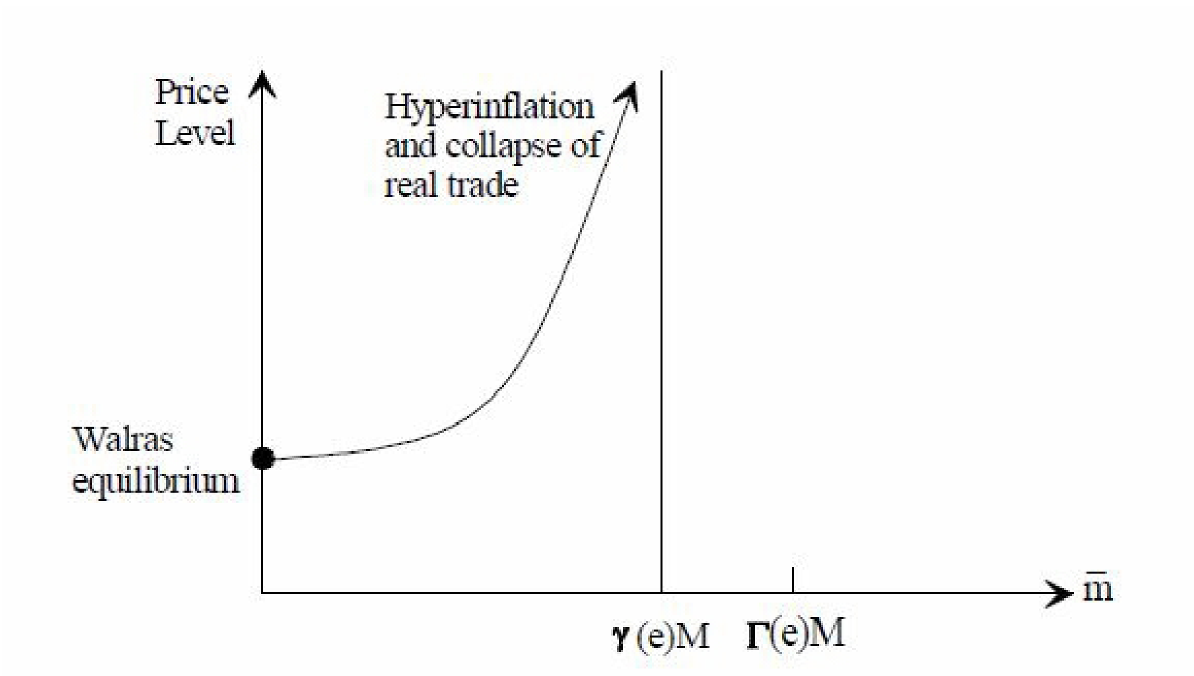


Figure 7: M Fixed

2.13.3 Hyperinflation via Reductions in Bank Money

A decline in money M suggests that price levels would fall. But as M falls, $r = \bar{m}/M$ rises, discouraging trade and moving us to less efficient allocations. Smaller volumes of trade Q make for higher price levels. Which effect dominates?

Suppose the economy has strictly concave and separable utilities. Consider a sequence of ME with bank money $M(n)$ and equilibrium prices $p(n)$. Suppose $M(n)$ converges to $\bar{m}/\gamma(e)$ from above. If $p(n)$ remains bounded, then by passing to a convergent subsequence, we could show the existence of ME at $M = \bar{m}/\gamma(e)$, contradicting our theorem form the last lecture. Hence $p(n) \to \infty$ and the price level *must* look something like the following:

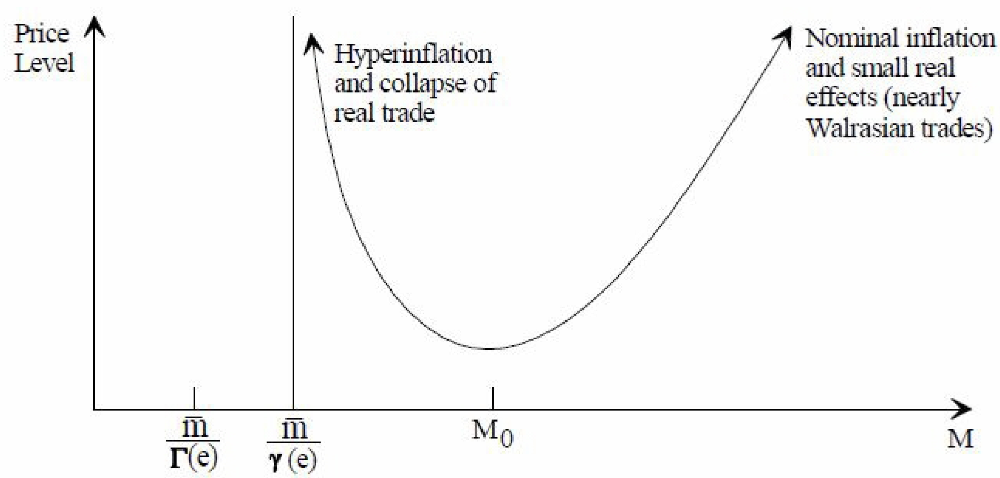


Figure 6: m Fixed

Our analysis has the paradoxical feature that there is some stock M_0 of bank money which minimizes the price level. If the bank eases, and lends more money, inflation will creep in, though the equilibrium allocation will improve somewhat. If the bank tightens its policy, lending less than M_0, inflation will again occur, and eventually price levels will *rise* much more rapidly (i.e., much faster than linearly, since they reach infinity over a finite move $M_0 - \bar{m}/\gamma(e)$). We call this explosion of prices, a hyperinflation.

The diagnosis of hyperinflation is now the following. Too much spending by the government starts raising prices. The central bank could normally compensate by lowering M and restoring the old price levels without too much loss in efficiency. However, once m/M gets too high, the usual central bank contractionary monetary policy has the paradoxical effect of speeding up the hyperinflation. One might check historically whether hyperinflations involved massive government deficits (spending beyond taxation) together with passive or contractionary monetary policy apart from the printed money for the spending.

3 Credit Cards

3.1 Introduction

We argue that the introduction and widespread use of credit cards increases trading efficiency but must cause an increase in price levels provided that r (or equivalently M) is held fixed. This contradicts the commonly held belief that money or monies only matter insofar as they affect interest rates. Government monetary intervention sufficient to stop these price increases might undo much of the efficiency gains that credit cards bring. Things are worse if there is default on credit cards: the price increases are greater, and the monetary authority might have to engineer even more reductions in trading efficiency to bring back the old price levels. The surge in price levels in the United States in the 1970s and early 1980s coincided with the introduction of credit cards. Our model provides a theoretical possibility of a causal connection.

In modern economies, more and more transactions take place via credit cards. They are perhaps the single most visible and talked about economic innovation in the last 50 years. Yet credit cards have not been extensively studied by general equilibrium theorists or monetary theorists, presumably because it has been thought that the effects of credit cards are negligible, or easily managed by monetary interventions. After all, credit cards only postpone the need for money, so one might wonder whether they have any effect. An older macroeconomic literature in the 1950s and 60s did raise these issues about "near monies", but this was before the advent of credit cards, in an intellectual era of reduced form models in which it would have been impossible to directly analyze credit cards anyway.[16]

We introduce a one-period general equilibrium model in which all agents have easy access to bank loans and to credit cards. They choose whether to buy goods with cash or credit cards, and prices adjust in order to clear all markets. No assumptions are needed on the number of commodities or the form of the utilities (beyond the usual general equilibrium hypotheses of continuity and concavity). We show in a series of theorems that credit cards must have a profound inflationary effect on price levels, and that monetary interventions to prevent price increases can be problematic. We do not deal with the transition from the regime without credit cards to the new regime with credit cards, preferring to keep the analysis as simple as possible by restricting ourselves to the comparative statics of a one-period model. In a multi-period equilibrium we would expect to see several periods of rapid inflation after credit cards are introduced, tapering off only when prices settle down at much higher levels; after the inflationary transition, credit cards would continue to enable efficient trade, but would no longer contribute to inflation.

[16] See for example Gurley and Shaw [1960], Brainard and Tobin [1963], Tobin [1963], and Brainard [1964].

In order to bring out the inflationary effect of credit cards in the starkest manner, our model makes the extreme assumption that all agents have the same easy access to bank loans and credit cards. Under this assumption credit cards double the velocity of money, because the cash proceeds from the sale of goods can be used again to defray the debt on credit card purchases. The same dollar in effect can be used by one agent for purchases in the cash-commodity market and simultaneously by another for purchases in the credit card-commodity market. This creates a massive inflation, on the order of 100%.

Credit cards are introduced in Subsection 2, and for simplicity, we examine the idealized situation where default does not occur. Consumers choose whether to buy goods with cash or credit cards, raising the question whether money can survive. Indeed many commentators refer to the coming "cashless" economy in which the supply of inside and outside money (i.e., cash) will be irrelevant. It is tempting to think that if credit cards became available to all households for the purchase of all commodities, and if there were no credit limits, then virtually all transactions would be conducted via credit cards, eventually eliminating the use of money altogether. Who would borrow money at positive interest to buy with cash when he could pay by credit card without interest? The puzzle is resolved in our one-period model because credit card prices are higher than cash prices, or in other words, cash purchases are made at a discount. (In a multi-period model, equality of cash and credit card prices could be maintained, provided there are consumers who pay interest on their credit card debt). We are able to show quite generally in Theorem 2 that money remains viable with credit cards (though less valuable because of the inflation), i.e. that an equilibrium exists in which money has positive (albeit diminished) value.

Theorem 2 also shows that credit cards improve trading efficiency. This is not because we suppose that going to the bank to get cash wears down shoe leather. In fact, bank transactions nowadays are done sitting at the computer, or with debit cards, and this is reflected in our model by postulating costless transactions. The real source of the efficiency gains is not saving shoe leather, but that credit cards increase the velocity of money. Credit cards are much more than pieces of plastic: they are backed by a sophisticated and expensive computer network that settles accounts by transferring money obtained from cash sales to defray credit card purchases. This enables money to do double work.[17]

Credit card purchases do crowd out some cash transactions, but they do not threaten the viability of money; indeed they enhance it. By improving the efficiency of transactions, credit cards paradoxically create circumstances or parameter values for which money could not have any positive value on its own, but does after credit cards are added.

However, the threat to money from credit cards is not without basis. What is crucial is how credit cards are settled. With unrestricted netting and no credit limits, money would cease to have value. But in the natural case of Theorem 2,

[17] Though we do not pursue this in our model, one could further imagine two credit cards, which were settled sequentially, which would enable money to do triple work, creating yet more inflation and trading efficiency.

corresponding roughly to the situation faced by the typical consumer today, credit card debts and receipts are not "netted." A consumer who gets his credit card bill must find the cash to pay, perhaps by writing a check on his bank account or paying with a debit card.[18] He cannot point out that as a merchant he has sold goods to customers charged on their credit cards, who owe him as much money as he himself owes. Without netting, credit cards do not alleviate the need for money, they only postpone it.

In Subsection 3 we examine the inflation caused by credit cards, when the monetary authority remains passive, and the potential stagflation when it tries to intervene. Within the confines of our simple model these effects are dramatic. In Theorem 3 we prove that when there is no default or credit limits in a one-period economy, aggregate cash expenditures necessarily equal aggregate credit card expenditures. Theorem 4 shows that the introduction of credit cards creates a new equilibrium in which the price level is higher, on the order of about 100%. Furthermore, their introduction has the identical effect on cash prices and commodity allocations as the infusion of vastly more inside money (almost double when the outside money is small).

A monetary authority, alarmed by the inflation, might try to undo it by tightening the money supply. We show in Theorem 5 that the authority can indeed cut the money supply to reproduce the pre-credit card equilibrium cash prices. But at the same time it will have to reduce trade to the pre-credit card equilibrium levels. This means giving up *all* the efficiency gains created by the credit cards.

Furthermore, this tightening does not completely undo the inflation because credit cards and money are not perfect substitutes: if the cash prices are brought back to their pre-credit card levels, trade will be back to where it was before, but the credit card prices will have to be slightly higher. Thus the average (credit card and cash) price would not be restored. This is a touch of stagflation.

In Subection 4 we work out an example illustrating Theorems 1-5.

3.2 The Credit Card Economy

Let us now imagine that credit cards are introduced into our monetary economy without default, discussed in the last lecture. Households can buy commodities directly with the credit cards, without having to borrow any money in advance. Of course the seller then gets a promise, and not cash, for his good. The simplest timing is to suppose that credit card purchases are made simultaneously with cash purchases, but that credit card debts must be repaid just before bank loans come due.

Denote by c the promise of one unit of money via the credit card. For $\ell \in L$, let

$p_{\ell c} \equiv$ credit card price of $\ell \equiv$ price of ℓ in terms of c

$p_{\ell m} \equiv$ cash price of $\ell \equiv$ price of ℓ in terms of money

[18] In the model we shall shortly describe, we allow for cash and for credit cards, but not for example for debit cards or checking accounts. These extra instruments are similar to cash; indeed their differences from cash can only be rigorously modeled in a multiperiod setting. In any case, with or without them, the issues connected with credit card purchases are quite similar, so we treat those issues in the simplest setting, without debit cards or checking accounts.

Note that the cash price $p_{\ell m}$ of a commodity need not be the same as its credit card price $p_{\ell c}$. By selling for cash, one gets the money sooner. So it might well be that $p_{\ell m} < p_{\ell c}$. (In practice goods can often be purchased at a discount with cash.) Thus market prices are now given by a longer vector $(p, r) \equiv ((p_{\ell m})_{\ell \in L}, (p_{\ell c})_{\ell \in L}, r)$ with $2L + 1$ components.

We denote the credit card economy by $(\mathcal{E}^c, m, M)$. Once again, this is equivalent (in our one period economy) to the economy $(\mathcal{E}^c, m, r)$ in which the central bank fixes the interest rate $r = \sum_h m^h / M$ instead of the money supply M.

The vector q^h of market actions of household h now has[19] $2L+2L+1$ components, with

$q^h_{\alpha\beta} \equiv$ quantity of α sent by h to the market $\alpha\beta$

where the markets are nm (the bank loan market), $(\ell c)_{\ell \in L}$ (the credit card-commodity markets), $(\ell m)_{\ell \in L}$ (the cash-commodity markets).

3.3 Credit Card Budget Set

Given (p, r), the vector (q^h, x^h) must satisfy:

$$\tilde{m}^h \equiv \frac{q^h_{bm}}{1+r} \tag{1*}$$

$$\sum_{\ell \in L} q^h_{m\ell} \leq m^h + \tilde{m}^h \tag{2*}$$

$$q^h_{\ell c} + q^h_{\ell m} \leq e^h_\ell \tag{3*ℓ}$$

$$\tilde{x}^h_\ell(m) \equiv \frac{q^h_{m\ell}}{p_{\ell m}},\ \tilde{x}^h_\ell(c) \equiv \frac{q^h_{c\ell}}{p_{\ell c}} \tag{4*ℓ}$$

$$\sum_{\ell \in L} q^h_{c\ell} \leq \Delta(2^*) + \sum_{\ell \in L} p_{\ell m} q^h_{\ell m} \tag{5*}$$

$$q^h_{bm} \leq \Delta(5^*) + \sum_{\ell \in L} p_{\ell c} q^h_{\ell c} \tag{6*}$$

$$x^h_\ell \leq \Delta(3\ell^*) + \tilde{x}^h_\ell(c) + \tilde{x}^h_\ell(m) \tag{7*ℓ}$$

The budget set $B^h_c(p, r)$ of household h consists of all (q^h, x^h) that satisfy constraints (1^*) to (7^*). (1^*) and (2^*) are as in the monetary economy. $(3^*\ell)$ says that household h sells good ℓ separately against cash and credit cards, but cannot sell in total more than it has. $(4^*\ell)$ says that household h buys good ℓ separately with cash and credit cards. (5^*) requires that credit card debts be paid in full with money, before cash receipts from credit card sales become available, and before bank loans come due. (6^*) says that after receiving all the money from sales against credit cards, there is enough money to pay off all bank loans. $(7^*\ell)$ constrains household h to consume no more of commodity ℓ than it has after trade.

[19] Since bank deposits are returned after clearing debts on the credit card, once again no household will want to deposit money at the bank, and we suppress deposits.

The critical thing to notice is that the same dollar can be used to repay two different debts: one household uses the dollar to pay his credit card debt, and the recipient uses the same dollar to pay his bank loan. Thus credit cards enable money to do extra work. This inevitably causes inflation, as we shall see shortly.

3.3.1 Credit Card Equilibrium

We say that $\langle p, r, (q^h, x^h)_{h\in H}\rangle$ is a credit card equilibrium of the economy $(\mathcal{E}^c, m, M)$ if:

$$\text{(a) } (q^h, x^h) \in B_c^h(p, r) \tag{8*}$$

$$\text{(b) } u^h(x^h) \geq u^h(\underline{x}^h) \text{ for all } (\underline{q}^h, \underline{x}^h) \in B_c^h(p, r); \tag{7}$$

for all $h \in H$, i.e., all agents optimize on their budget sets and

$$\text{(a) } \sum_{h\in H} \tilde{m}^h = M \tag{9*}$$

$$\text{(b) } \sum_{h\in H} \tilde{x}_\ell^h(m) = \sum_{h\in H} q_{\ell m}^h \quad \forall \ell \in L \tag{7}$$

$$\text{(c) } \sum_{h\in H} \tilde{x}_\ell^h(c) = \sum_{h\in H} q_{\ell c}^h \quad \forall \ell \in L \tag{1}$$

i.e., all markets clear.

3.3.2 Existence and Efficiency of Credit Card Equilibrium

Theorem 2 *Let $(\mathcal{E}^c, m, M)$ be a credit card economy and assume that $\gamma(e) > \sqrt{1+\frac{\bar{m}}{M}} - 1$. Then a credit card equilibrium exists. Moreover, if $\langle p, r, (q^h, x^h)_{h\in H}\rangle$ is a credit card equilibrium, then we must have $r = \bar{m}/M$, and $p_{\ell c} = \left(\sqrt{1+\frac{\bar{m}}{M}}\right) p_{\ell m}$ for all $\ell \in L$. Finally, if the utilities $(u^h)_{h\in H}$ are smooth, then the unexploited gains to trade $\gamma((x^h)_{h\in H}) = \sqrt{1+\frac{\bar{m}}{M}} - 1$.*

The proof of Theorem 2 is contained in the proof of Theorem 4 (given in Section 7).

Since $\frac{\bar{m}}{M} > 0$, it follows that $\sqrt{1+\frac{\bar{m}}{M}} < 1 + \frac{\bar{m}}{M}$, so that credit card equilibrium exists whenever monetary equilibrium exists, and continues to exist when $\sqrt{1+\frac{\bar{m}}{M}} < 1 + \gamma(e) < 1 + \frac{\bar{m}}{M}$, even where monetary equilibrium might not. The introduction of credit cards thus enhances the viability of money.

It is often said that credit cards will drive out money, and that we are headed toward a cashless economy. Observe, however, that the reasons which make credit card purchases more attractive than cash purchases to the buyer often make them less attractive to the seller. Buyers prefer to pay later rather than earlier, and a credit card purchase enables them to defer the transfer of cash. But for precisely this reason, sellers prefer cash buyers. The presumption that credit cards must eventually drive out cash neglects half the market, since it ignores the sellers. The key is to recognize that cash and credit cards will coexist if cash prices are lower than credit card prices.

Notice that in our model the introduction of credit cards does not affect the bank rate of interest $r = \bar{m}/M$. Nevertheless, since the gains to trade remaining at credit card equilibrium are lower than the gains to trade remaining at monetary equilibrium, credit cards lead to more efficient trade. This is because the "effective" rate of interest paid in credit card equilibrium is $\sqrt{1 + \bar{m}/M} - 1$, which is lower than the rate $\bar{m}/M$ prevailing in pure monetary equilibrium.

In the pure monetary economy, households borrowed money from the bank to purchase commodities and sold commodities for cash in order to defray the bank loan. As a result, buyers who borrowed a dollar to spend on goods had to sell goods worth $(1 + r)$ dollars in order to repay the bank. The introduction of credit cards enables households to reduce this wedge to $\sqrt{(1+r)}$ by engaging in either of two equivalent trading strategies. From Theorem 2, credit card prices are precisely $\sqrt{(1+r)}$ higher than cash prices. Any household can purchase commodities, whose cash prices are \$1, by charging $\$\sqrt{(1+r)}$ on a credit card and then selling goods for cash worth $\$\sqrt{(1+r)}$ to defray the credit card debt, indeed reducing the wedge to $\sqrt{(1+r)}$. Or else, he can borrow \$1 from the bank, spend the cash on goods, while simultaneously raising enough money to repay the bank by selling other commodities against credit cards for $\$(1 + r)$. Since the credit card sales have cash value equal to $\$(1 + r)/\sqrt{(1+r)} = \$\sqrt{(1+r)}$, the wedge is again $\sqrt{(1+r)}$.

We shall call these two trading strategies *buy-credit/sell-cash and buy-cash/sell-credit*, respectively. They are reflected in (i) and (ii) of Theorem 3 below.

3.3.3 Flow of Funds in Credit Card Equilibrium

Theorem 3 *Let $(\mathcal{E}^c, m, M)$ be a credit card economy and assume that $\langle p, r, (q^h, x^h)_{h\in H}\rangle$ is a credit card equilibrium. Then (i) individual credit card debt equals individual cash receipts, i.e., $\sum_{\ell=1}^{L} q^h_{c\ell} = \sum_{\ell=1}^{L} p_{\ell m} q^h_{\ell m}$, and so aggregate credit card debt is equal to aggregate cash receipts, i.e, $\sum_{h=1}^{H}\sum_{\ell=1}^{L} q^h_{c\ell} = \sum_{h=1}^{H}\sum_{\ell=1}^{L} p_{\ell m} q^h_{\ell m}$. Similarly, (ii) individual bank debt equals individual credit card receipts, i.e. $q^h_{bm} = \sum_{\ell=1}^{L} p_{c\ell} q^h_{\ell c}$, and so aggregate bank debt equals aggregate credit card receipts $\sum_{h=1}^{H} q^h_{bm} = \sum_{h=1}^{H}\sum_{\ell=1}^{L} p_{c\ell} q^h_{\ell c}$.*

The proof of Theorem 3 also follows from the proof of Theorem 4.

Households sell commodities for cash (at prices lower than they could get by selling against credit cards) only in order to defray their own credit card debt, incurrred in the course of following the first trading strategy described above. This is the content of (i) of Theorem 3.

Since all the sales revenue from the commodity-cash markets are used to redeem credit card debts, households must repay their bank loans out of the sales revenue from commodity-credit card markets. This explains (ii) of Theorem 3.

Since total cash receipts must equal total cash expenditures, and since in a one-period model all cash is spent, both must equal $(M + \bar{m})$. From (i) we conclude that aggregate credit card debt is equal to $(M+\bar{m})$. But credit card debt is another word for credit card expenditures, hence we have

Corollary to Theorem 3: *In any credit card equilibrium, total expenditures on cash markets = total expenditures on credit card markets (albeit calculated at different prices). In short,* $\sum_{h=1}^{H}\sum_{\ell=1}^{L} q_{c\ell}^{h} = \sum_{h=1}^{H}\sum_{\ell=1}^{L} q_{m\ell}^{h} = (M+\bar{m})$. *Thus total expenditures are* $2(M+\bar{m})$.

The introduction of credit cards in our one-period model doubles expenditures, independent of the interest rate or preferences of the agents. Since aggregate cash expenditures are the same as aggregate credit card expenditures, but credit card prices are uniformly higher than cash prices, it follows that more than half of all sales are against cash. Credit card purchases indeed crowd out cash purchases, but never more than half (in our one-period model).

3.4 Credit Cards and Inflation

The main effect of credit cards is to increase prices, i.e, they lower the value of money even as they increase its viability. The following theorem shows that the introduction of credit cards is tantamount to an infusion of a huge amount of bank money. As was said, its proof, in conjunction with Theorem 1, also yields a constructive proof of Theorem 2, as well as a proof of Theorem 3.

Theorem 4 *Consider a credit card economy* $(\mathcal{E}^c, m, M)$. *Let* $(\mathcal{E}, m, M^*)$ *be a pure monetary economy with more bank money*

$$M^* \equiv M + \sqrt{M^2 + \bar{m}M}$$

Then the equilibria of the two economies coincide in the following sense. For every credit card equilibrium $(p, r, (q^h, x^h)_{h\in H})$ *of* $(\mathcal{E}^c, m, M)$, *there exists a pure monetary equilibrium* $(p^*, r^*, ({}^*q^h, {}^*x^h)_{h\in H})$ *of* $(\mathcal{E}, m, M^*)$ *with the same consumption* $({}^*x^h)_{h\in H} = (x^h)_{h\in H}$ *and the same cash prices* $(p^*_{\ell m})_{\ell\in L} = (p_{\ell m})_{\ell\in L}$ *but a lower interest rate* $(1+r^*) = \sqrt{1+r}$. *And vice versa.*

Theorem 4 (in conjunction with Theorem 1) yields

Corollary 1 to Theorem 4 *For generic smooth utilities, endowments, and money stocks, the credit card economy* $(\mathcal{E}^c, m, M) = ((u^h, e^h, m^h)_{h\in H}, M)$ *has finitely many credit card equilibrium allocations and prices.*

Note that determinacy is claimed here for equilibrium outcomes, not actions. Typically it will be possible to shift some households' trade from credit card purchases/cash sales into cash purchases/credit card sales, while moving other households in the reverse direction, without disturbing the equilibrium.

Corollary 2 to Theorem 4 *Let $\mathcal{E}$ be a smooth underlying economy with a unique Walrasian equilibrium. Let m be fixed. Then for all sufficiently large M, any pure monetary equilibrium of $(\mathcal{E}, m, M)$ has prices nearly proportional to Walrasian, and trades nearly equal to Walrasian. When credit cards are added to $(\mathcal{E}, m, M)$, prices will nearly double, without much change in the trade.*

According to the Corollary of Theorem 3, the introduction of credit cards literally doubles the spending (via cash and credit cards) on traded goods. Part of the increase in the money value of trade after credit cards are introduced is due to the increased real trade permitted by more efficient exchange. But the great bulk of the increase comes from higher prices. When $\bar{m}/M$ is very low, as in the scenario of Corollary 2 to Theorem 4, monetary equilibrium trades are necessarily close to Walrasian (and, since the wedge is lower, even closer after credit cards are introduced). Thus almost the entire increase in spending is on exactly the same trades, causing prices to double. Credit cards cause inflation.

3.4.1 A Touch of Stagflation

We have seen that credit cards increase the efficiency of trade but cause massive inflation. It is natural to imagine a monetary authority that would try to stem this inflation by tightening the money supply and raising interest rates.

What is surprising is that in order to restore the old cash prices, it is necessary to abandon all the gains to trade engendered by the credit cards. In fact, strictly speaking, since the credit card prices are higher than the cash prices, it is actually necessary to reduce trade *below* the original pre-credit card levels in order that the average cash/credit card price be no higher than before. Curiously, a financial innovation (like credit cards) that creates the potential for more efficient trade might end up reducing trade if the monetary authority is committed to preventing *all* inflation.

A conservative monetary authority might well compromise by tolerating a small increase in average prices. But if the increase were small enough, there would necessarily be a drop in efficiency. This is stagflation, though perhaps just a semblance.

Theorem 5 follows immediately from Theorem 4 (taking $M^* = M$ and $M = \hat{M}$) and Theorem 2.

Theorem 5 *Consider a monetary economy $(\mathcal{E}, m, M)$ with an equilibrium $(p, r, (q^h, x^h)_{h \in H})$. If credit cards are added to the economy, there is always a reduction in the bank money supply to $\hat{M} < M$ solving $M = \hat{M} + \sqrt{\hat{M}^2 + \bar{m}\hat{M}}$ such that the credit card economy $(\mathcal{E}^c, m, \hat{M})$ has an equilibrium $(\hat{p}, \hat{r}, (\hat{q}^h, \hat{x}^h)_{h \in H})$, where consumption is what it was before credit cards, $(\hat{x}^h)_{h \in H} = (x^h)_{h \in H}$, and cash prices are restored to their pre-credit card levels, $\hat{p}_{\ell m} = p_{\ell m}$, but credit card prices are higher than cash prices, $\hat{p}_{\ell c} = (\sqrt{1 + (\bar{m}/\hat{M})})\hat{p}_{\ell m} > p_{\ell m}$ for all $\ell \in L$. A further reduction in money supply to $\tilde{M}$ will lower average prices in the credit card equilibrium $(\tilde{p}, \tilde{r}, (\tilde{q}^h, \tilde{x}^h)_{h \in H})$ to their pre-credit card levels, but (assuming utilities are smooth), at the cost of leaving* *more unexploited gains to trade than there were before credit cards were introduced* $\gamma((\tilde{x}^h)_{h\in H}) > \gamma((x^h)_{h\in H})$.

Even in our one-period setting, we have argued in Dubey-Geanakoplos () that we can get genuine stagflation, not just a semblance of it, if there is default on credit cards. The reason is that the default can eliminate most of the efficiency gains of credit cards, while at the same increasing the inflation caused by credit cards. A monetary authority that tries to cut inflation will have to *reduce* real trade.

3.5 Example

Suppose credit cards are suddenly introduced into our monetary equilibrium example. They improve the transactions technology in our model by enabling an extra period of cash flow between commodity trade and bank repayment. This represents the overnight electronic transfer of credit card accounts in the real world. Now the same paper money can do more work, and inevitably price levels rise.

The transactions technology improvement will reduce the wedge between buying and selling, and thus tend to improve welfare. On the other hand, the price level increase will reduce the purchasing power of the each household's stock of outside money m^h, which tends to reduce welfare. When there is no default on credit cards, the benefits from superior technology outweigh the costs of higher prices. But as we shall see, when credit cards default, some of the benefits are frittered away, and welfare tends to go down on account of the inflation.

To find money prices and credit card prices we must consider the sales each agent makes of his abundant good against credit cards τ_c and of the same good against money τ_m. His total sales are then $\tau^* = \tau_c + \tau_m$. Then, by symmetry, we must have

$$\begin{aligned}
\frac{M+\bar{m}}{2\tau_m} &= p_m \\
p_m\tau_m &= p_c\tau_c \\
\frac{(1+r)p_m}{p_c} &= \frac{p_c}{p_m} \\
\frac{(3-\tau_m-\tau_c)}{(1+\tau_m+\tau_c)} &= \sqrt{(1+r)} \\
r &= \frac{\bar{m}}{M}
\end{aligned}$$

The first equation comes from the fact that all the money $M + \bar{m}$ is spent equally in the two cash markets. The second equation says that cash receipts are used entirely to pay off credit card debt. The third equation, which may also be written $p_c/p_m = \sqrt{(1+r)}$, equates the two strategies of trading in a CCE. On the left hand side is the quantity of good that must be sold on the credit card market in order to repay the bank loan required to buy an incremental unit on the cash market. On the right hand side is the quantity of good that must be sold on the cash market in order to repay the debt incurred buying an incremental unit on the credit card market. Since each agent is buying on both markets, these must be equal.

The fourth equation says that the wedge between buying with borrowed cash and selling on credit cards (or vice versa), which is $\sqrt{(1+r)}$, as we just saw, must equal the ratio of the marginal utility of buying and selling. The last equation holds as before because there is no bank default.

Solving these equations for $M = 20, m^1 = m^2 = 1$, we see that trade is more efficient, but the cash prices nearly double, and the credit card prices are even higher:

$$
\begin{aligned}
\tau_m &= 0.488 \\
\tau_c &= 0.465 \\
\tau^* &= 0.953 > .90 \\
p_m &= 22.56 > 12.16 \\
p_c &= 23.66 \\
r &= .1
\end{aligned}
$$

The introduction of credit cards does not change the rate of interest at the bank. But let us denote the "wedge" between buying and selling (or the "effective interest rate") by $1 + r^* \equiv \sqrt{1 + \frac{m^1+m^2}{M}}$, so, $1 + r^* < (1 + r^*)^2 = 1 + r$, where r is the bank rate of interest (both before, and after, the introduction of credit cards). For $M = 20$, $r = 10\%$ and $r^* \approx 4.9\%$.

The fourth equation may now be recast

$$
\begin{aligned}
\frac{3-\tau^*}{1+\tau^*} &= 1 + r^* \\
\tau^* &= \frac{2-r^*}{2+r^*} \approx 0.95 > 0.90 \approx \frac{2-r}{2+r} = \tau
\end{aligned}
$$

revealing that trade is more efficient because $r^* < r$.

These trades and cash prices are exactly the same as were obtained without credit cards in the pure monetary economy with inside money $M = 40.98$, as stated in Theorem 4.

Now we verify that there are indeed corresponding credit card equilibrium actions.

Let each agent h borrow $M/2 = \$10$ from the bank and spend $\$11 = \$1 + \$10$ on the cash market for good $-h$, and thus buying $11/p_m \approx 0.49$ of good $-h$ via cash. Let h also spend \$11 on the credit card market for good $-h$ thus buying $11/p_c \approx 0.46$ of good $-h$ via the credit card. Finally, let each agent h sell 0.49 units of good h against cash, and 0.46 units of good h against the credit card promise.[20]

Notice that all markets clear. Also, each agent h is able to repay his credit card debt of \$11 from the revenue he obtains from his cash sales. Furthermore, since each agent h borrows \$10 from the bank at interest rate 10%, he must repay \$11 to the bank. But this is precisely what he is paid for his sales against the credit card just before he must go to the bank. The cash flows mirror what is stated in Theorem 3.

[20] Thinking of each type as many identical agents, we are here describing a type-symmetric equilibrium in actions. But, as was said after Corollary 1 to Theorem 4, this can be recast in a more realistic manner: some of each type could buy exclusively via credit cards and the rest exclusively via cash without distrurbing the equilibrium allocation.

These choices are not only consistent, but also optimal for each household h. By selling a unit of good h for cash at price p_m and buying $1/(1+r^*)$ units of good $-h$ via a credit card at price $p_c = (1+r^*)p_m$ (repaying the credit card debt later with the cash receipt) h faces a wedge of $(1+r^*)$ between buying and selling. This is also the case if he borrows $p_m/(1+r^*)$ from the bank at interest rate r, then uses the money to purchase $1/(1+r^*)$ units of good $-h$ for cash at price p_m, while simultaneously selling one unit of good h against the credit card at price $p_c = (1+r^*)p_m$. The money obtained from the credit card sale will be just enough to repay the bank debt $(1+r)p_m/(1+r^*) = (1+r^*)p_m$.

Optimality requires that the ratio of marginal utilities is equal to the wedge $(1+r^*)$, i.e.

$$\frac{x_h^h}{x_{-h}^h} = (1+r^*)$$

This is exactly how we derived the formula for trades τ^*.

As stated in Corollary 1 to Theorem 4, the introduction of credit cards has created efficiency gains in trade, but at the cost of much higher price levels. Unfortunately, the only way to reduce prices to their old levels is to give back all of the gains in trade, and even a little more!

To get back to the old efficiency levels of trade means an effective interest rate of $r = 10\%$ again. But as we just saw, in the presence of credit cards, that requires a higher bank interest rate of

$$1 + \hat{r} = (1+r)^2 = 1.1^2 = 1.21$$

which in turn implies

$$\hat{M} = \frac{\bar{m}}{\hat{r}} = \frac{2}{0.21} = 9.5,$$

a drastic reduction in the money supply.

The price levels will then be

$$\begin{aligned} \hat{p}_m &= 12.16 = p_m \\ \hat{p}_c &= (1+r)\hat{p}_m = 13.38 \end{aligned}$$

While the cash prices are restored to their old levels, the credit card prices are 10% higher. Since half the expenditures are by credit card, there is an overall inflation of about 5%. The only way to get rid of this inflation is to reduce the bank supply even more. But this will cut trading efficiency below the levels that prevailed prior to the introduction of credit cards. This illustrates Theorem 5.

It is easy to imagine that the monetary authority might be reluctant to cut money supply so far. It might well stop at a point where average trades are just a tad below 0.90 and prices are a tad above 12.16. This is a touch of stagflation.

Since the interest rate in our model corresponds to a transactions interest rate in the real world, it is likely to be on the order of one week or one month's interest. That would be a small number, perhaps on the order of .5% instead of 10%. One can easily see that in this case the efficiency gain from adding credit cards is quite small, reducing the wedge by .25% instead of by 5% as in our example. But the inflation of nearly 100% with credit cards holds no matter what the original interest. In fact it is closer to exactly 100% the lower is r.

4 The Multiperiod Model

We consider a finite horizon, pure exchange economy in which bonds can be used partially in lieu of money, they do not bypass money since they promise money in the future which must be fully delivered when the bonds come due. Moreover we postulate that there are enough crucial trades left that can only be conducted with money.

4.1 The Real Sector

Let $T = \{1, \ldots, T\}, L = \{1, \ldots, L\}$ and $H = \{1, ..., H\}$ denote the set of time periods, commodities and agent-types (or, household-types) respectively[21]. Each $h \in H$ has an initial endowment of commodities $e^h \in \mathbb{R}_+^{T \times L}$ and a utility of consumption $u^h : \mathbb{R}_+^{T \times L} \rightarrow \mathbb{R}$. We assume throughout that u^h is continuous, concave, and weakly monotonic, for all $h \in H$. (More detailed conditions will be spelt out later). To incorporate durable goods, we also postulate that when h consumes $x_t^h = \left(x_{t1}^h, \ldots, x_{tL}^h\right) \in \mathbb{R}_+^L$ in period t, then x_t^h not only yields utility to h but gets transformed into the bundle[22] $\left(f_{(t+1)l}^h\left(x_t^h\right)\right)_{l \in L} \in \mathbb{R}_+^L$ that is made available to h at the start of period $t+1$, over and above his endowment $e_{t+1}^h = \left(e_{(t+1)l}^h\right)_{l \in L} \in \mathbb{R}_+^L$ there. The collection of functions $\left\{f_{(t+1)l}^h : t = 1, \ldots, T-1; l \in L\right\}$ is denoted f^h, and each $f_{(t+1)l}^h : \mathbb{R}_+^L \longrightarrow \mathbb{R}$ is assumed to be 0 at 0, continuous, concave and weakly monotonic.

The real sector of the economy is denoted $(u^h, e^h, f^h)_{h \in H}$.

4.1.1 Trade and Consumption

Consider an agent h who conducts the trade $\tau \in \mathbb{R}^{T \times L}$ (with positive components representing commodities received and negative components representing commodities sent). If commodities were perishable (i.e., all $f_{tl}^h = 0$), then τ would induce the familiar consumption $e^h + \tau$ (provided, of course, that τ is "feasible" in the sense that h never has to send out more than he has on hand, i.e., $e^h + \tau \in \mathbb{R}_+^{T \times L}$). In the

[21] We assume that there is a unit mass of agents of each type h with identical characteristics, and throughout confine attention to type-symmetric behavior. For brevity, we shall refer to "agent h" from now on, but it should be understood that we actually mean every agent of type h.The agents constitute a perfectly competitive sector of the economy, and in particular they are "price-takers", since prices depend on aggregate behavior which is unaffected by any single agent. See section ??? for details.

[22] In our pure exchange model, the future bundle $\left(f_{(t+1)l}^h\left(x_t^h\right)\right)_{l \in L}$ is *not* enhanced by reducing the current consumption x_t^h . That would be tantamount to production (which we include later, see Remark ???)

presence of durable commodities, we need to replace $e^h + \tau$ with $e^h \oplus \tau \equiv z \in \mathbb{R}_+^{T \times L}$, defined recursively for $t = 1, \ldots, T$ (starting with $f_{1l}(\centerdot) = 0$) as follows[23]:

$$z_{tl} = f_{tl}^h(z_{t-1}) + e_{tl}^h + \tau_{tl}$$

In particular, the initial endowment e^h induces the *initial consumption*

$$\widetilde{e}^h = e^h \oplus 0 = 0 \oplus e^h$$

of agent h (the second equality states that it is merely a matter of nomenclature whether h "has" an extra e_t^h in period t – over and above his inventory of goods – and trades nothing, or has nothing extra and "receives" e_t^h in trade).

4.2 The Monetary Sector

4.2.1 Inside and Outside Money

Money is fiat and gives no direct utility of consumption to the agents: they value money only insofar as it enables them to acquire commodities for consumption.

Money enters the economy in two ways: "outside" or "inside" money.

Outside money is money that is owned by agents, free and clear of debt (e.g., inheritance from an unmodeled past, or given by the government). We model outside money as private endowment $m^h = (m_t^h)_{t \in T} \in \mathbb{R}_+^T$ of agent $h \in H$, where m_t^h denotes the money that accrues to h in period t. The vector $m = (m^h)_{h \in H}$ denotes *outside money*. We assume that $(m_1^h)_{h \in H} \neq 0$, i.e., outside money is present in the aggregate at the start of the economy. Outside money may further enter the economy when the government creates it to pay off buyers of its *bonds* (see the next section for details).

In contrast, *inside money* is money that is owed by agents from the moment it comes into their hands. It is injected by the government into the economy via the purchase of agents' bonds.

4.2.2 Bonds

A unit of bond n is a promise to pay one dollar in (a future) period n. Bonds come into being when they are created by the government, or by agents as promises, and are sold on markets in exchange for money or commodities. The quantity of bonds sold by government at a market can either be exogenously fixed by it, or else determined endogenously in order to maintain the fixed interest rate that it has announced at that market (see (1),(2),(3) below for the precise details). In contrast, agents may create as many bonds as they like, though they cannot create the money which is needed to deliver on those bonds. Conventional money is a "creature of the state" and its creation remains the sole perogative of the government at all times.

Sales of newly created bonds are called *primary* sales. Once bonds come into being, they can — just like money — be inventoried costlessly by agents across time

[23] Notice that $e^h \oplus \tau = 0 \oplus (e^h + \tau)$ since e^h is tantamount to a trade where h is receiving $e_t^h \in \mathbb{R}_+^L$ in every period t.

and retraded on markets in the future. Sales of bonds bought in the past are called *secondary* sales.

When period n arrives, money due on bond n must be delivered *in full* by all its primary sellers, i.e., there is no default. There is, moreover, *no netting* permitted on these deliveries: an agent cannot claim that if he owes \$100 on his primary sales of bond n, but is simultaneously owed \$300 on bonds n (by virtue of bonds n inventoried by him from the past), then he need deliver nothing and just collect the net \$200. The "no netting condition" is significant because money due on bond n must be delivered prior to "market exchange" in period n, while money receivable on bond n comes.to hand after "market exchange" (see the next section for details).

For simplicity we assume that the government intervenes, via its central bank, only on some pre-selected *government bond (vs. money) markets* (i.e., it does not buy or sell commodities) and that all its sales of bonds are primary sales. Government intervention occurs in *one* of the following three disjoint ways:

(1) It directly fixes an exogenous price of the bond (in terms of money), i.e., the interest rate, and — depending on the ensuing aggregate supply of, and demand for, the bond by the agents — it either puts up money or bonds (but not both) to clear the market. The amounts so put up by it are thus endogenously determined.

(2) It puts up a fixed exogenous quantity of money for the purchase of the bond.

(3) It puts up a fixed exogenous quantity of the bond for primary sale against money (creating the money later to deliver on the bond in full when the bond comes due).

In cases (2) and (3), the market-clearing price of the bond (its interest rate) is determined endogenously by agents' aggregate supply and demand of the bond . In all three cases, the money put up by the government enters the economy immediately as inside money; and the money it pays out to deliver on the bonds it sold — when they come due in the future — enters the economy as outside money.

Denote by $\mathcal{F}$ (or$\mathcal{M}$,or $\mathcal{B}$) the set of money-bond markets $\{tnm\}$ at which the interest rates r_{tnm} (or money M_{tmn} loaned, or bonds B_{tnm} sold) are exogenously specified by the central bank. Thus $\mathcal{F} \cup \mathcal{M} \cup \mathcal{B}$ constitutes a partition of government bond markets.

Denote commodity, money, bonds by l, m, n respectively.[24]; and for any pair $\alpha, \beta \in \{l, m, n\}$, denote the market (if it exists) for the bilateral trade of α and β by the unordered triple $\{t\alpha\beta\}$, and denote by $p_{t\alpha\beta}$ denote the price of α in terms of β at this market (so that $p_{t\beta\alpha} = (p_{t\alpha\beta})^{-1}$, where $t\alpha\beta$ is an ordered triple).

An agent can enter any market $\{t\alpha\beta\}$ by selling the quantity $q_{t\alpha\beta}$ of α in exchange for β.

For the government's actions, we use two different symbols interchangeably, in keeping with common parlance and mathematical consistency: $p_{tnm} = (1 + r_{tnm})^{-1}$ (where r_{tnm} is the *interest rate* set on bond n fixed at the market $\{tnm\}$, as in (1)); or $Q_{tmn} = M_{tmn}$ for the money government puts up, as in (2); or $Q_{tnm} = B_{tnm}$ for

[24] We earlier used the symbol m for the vector of agents' endowments of outside money, but there should be no confusion, as the meaning of m will always be clear from the context.

the bonds it puts up, as in (3). *Monetary policy* is thus specified by a triple of vectors

$$(r, M, B)$$

whose components correspond to $\mathcal{F}$, $\mathcal{M}$, $\mathcal{B}$ respectively.

Let us next spell out the constraints on an agent's market actions. The sale $q_{tm\beta}$ (for $\beta = n, l$) of money, or $q_{tl\beta}$ (for $\beta = m, n$) of commodities, that an agent can make must be out of the stock of money or commodity he has on hand (i.e., the "Clower constraint" holds). However, as was said, the sale of bonds does not face this constraint. An agent can not only sell out of his inventoried stock of *secondary* bonds n, but in addition he can also create arbitrary amounts of primary bonds n for sale at the markets $\{tnm\}$ or $\{tnl\}$ in order to acquire money or commodities. Since deliveries on bonds are made by its primary sellers, it will be useful to split the notation for an agent's sale of bonds: q_{tnm} (resp., b_{tnm}) for the sale of secondary (resp., primary) at the market $\{tnm\}$, so that his total sale is $q_{tnm} + b_{tnm}$; and, similarly, q_{tnl}, b_{tnl}, $q_{tnl} + b_{tnl}$ at the market $\{tnl\}$. (It might be helpful to keep the following picture in mind: every primary issue of a bond is a "promissory note" signed by its issuer (agent or government), which may be retraded later but without obliterating the signature. Thus, at any time, all bonds in the economy bear the signature of their primary issuers who are obliged to deliver in full the money they promised when the bonds come due.) In common parlance $q_{tnm} + b_{tnm}$ corresponds to "*money charging privileges*" and $q_{tnl} + b_{tnl}$ to "*credit card purchases*".

Note that at markets $\{tnl\}$ the bonds sold could be a mix of secondary and primary bonds of the agents; and at markets $\{tnm\}$) there could in addition be primary bonds of the government. The buyer of these bonds is indifferent about the kind of bond he buys, since no default is permitted and all the bonds are delivered upon in full (by their primary issuers).[25]

4.2.3 Market Exchange: Stocks and Flows

We now describe precisely the "market exchange" between the agents and/or government. There are two aspects to it: spot market trades and exchange of money via bond deliveries.

(a) There is bilateral trade of of α and β at each spot market $\{t\alpha\beta\}$ that is available in period t. The price $p_{t\alpha\beta}$ mediates trade between the agents at $\{t\alpha\beta\}$. An agent who *supplies* $q_{t\alpha\beta}$ units of α *to* the market $\{t\alpha\beta\}$ has an effective *demand* of $q_{t\alpha\beta}p_{t\alpha\beta}$ units of β *from* that market.[26]

(b) In every period t, all primary issuers of bond t must *supply* (deliver) the money they promised *to* the exchange; and all holders of bond t *demand* for money

[25] There is room within our model to include default on bond deliveries by the agents, in conjunction with penalties for those who default (in the spirit of ??? —our default papers???). In this setting it is crucial for the buyer to know which pool of bonds he is buying, for different pools will have different delivery rates and prices. We plan to examine default in future work.

[26] For general prices, aggregate supply and demand need not be equal (taking government's actions into account whenever relevant), and there could be excess demand for α or β. Equality obtains at markets only in equilibrium (see section ??/), and in this event prices are said to be "market clearing".

from the exchange equal to the bonds they hold. Here again, there might be excess supply or demand of deliveries, except at equilibrium.

In (a), commodities, money and bonds are exchanged between the agents (and, possibly, the government). In (b), only money is exchanged. Putting (a) and (b) together, we may think of the market exchange as a *flow,* where supply is an inflow and demand is met by an outflow. (Of course, except at equilibrium, there is no conservation of l, m, n; the exchange may need to create or destroy l, m, n in order to honor supply and demand)[27].

Each agent has a *stock* of l, m, n at the start of period t (prior to market exchange) which is altered by the flow into a new stock at the end of period t (after market exchange) to be carried over to the start of period $t+1$.

Our stock-flow model implies that an agent who sells bond n for money in period t realizes the proceeds from his sale in his stock at the start of period $t+1$. Similarly an agent who buys bond n with money realizes the proceeds from his purchase in his stock at the start of period $n+1$. It follows that no purpose is served for any agent by selling bond n in period $n-1$, because whatever he gets by way of proceeds in period n will all be owed simultaneously; indeed often more will be owed[28]. Thus bond n is not traded after period $n-2$.

(In common parlance, the sale (resp., purchase) of bond n corresponds to borrowing (resp., depositing) money on loan n; and the duration of all loans is of *at least* two periods. We too shall often use this terminology.)

4.3 Missing Actions

Our model allows for considerable flexibility in the sequencing of spot market trades and bond deliveries. When markets are missing, or the government is abstaining from them, or else when deliveries are not called for, we simply take all the corresponding actions to be 0.

To keep matters simple, we assume throughout that if any commodity-money market $\{tml\}$ or commodity-bond market $\{tnl\}$ is present, then all commodity-money

[27] This "Walrasian" view is in sharp contrast to the dual "(Nash) strategic market game" view (which we shall turn to, in our proofs in the Appendix). In the game-theoretic approach, prices are formed by agents' actions to always clear markets (i.e., they equate supply and demand), though agents actions are not be optimal (or even feasible) in the budget sets determined by the prices, except at Nash equilibrium. In the Walrasian view, optimal actions are formed by prices but the prices may not be market-clearing except at Walras equilibrium. To the extent that trades are mediated by prices at markets every day, and the world merrily goes on, *regardless of what — if anything — has motivated agents' market actions,* the game-theoretic view has something to recommend it.

[28] Clearly the price π of any bond n (in terms of money in period $t < n$) cannot exceed 1, otherwise an agent can do arbitrage as follows. He can sell x units of the bond, get πx dollars immediately in period $t+1$, inventory x dollars into the future period n to payoff the bond, and be left with a profit of $(\pi - 1)x$ dollars for arbirarily large x. Since $\pi = (1+r)^{-1}$, where r denotes the interest rate on the bond, this is equivalent to saying that interest rates must be non-negative. Now, if the interest rate is positive, an agent who sells bond n in period $n-1$ will owe more money on the bond than he receives from it, rendering the sale totally meaningless. And, even if the interest rate is zero, the slightest transactions cost makes the sale a losing proposition.

markets $\{tkm\}$ are present; moreover, these commodities are universally liked and present in the aggregate. Precisely

(AI) **ASSUMPTION ON COMMODITIES.** Let $L_t = \{l \in L : \text{market } \{tml\} \text{ exists}\}$ denote the set of commodities marketed for money in period t. Then, if either market $\{tmk\}$ or $\{tnk\}$ exists for some commodity k, some bond n, and some $t = 1, \ldots, T-1$, this implies $L_t = L$. Furthermore $\mathcal{L}_T = L$ unconditionally. Finally, if $\mathcal{L}_t = L$, then $\sum_{h \in H} \widetilde{e}^h >> 0$ and (for all $l \in L$ and all $h \in H$) $\widetilde{e}^h_t \neq 0$ and $u^h(x)$ is strictly monotonic in the variables x_{tl} for all $h \in H$ and all $h \in H$ have access to all commodity-money markets whenever they exist.

4.4 Full Span of Government Bond Markets $\mathcal{F} \cup \mathcal{M}$

We add a period 0 at the start of the economy and a period $T+1$ at the end. The only activity in period 0 is that agents can borrow money by selling a fixed-rate bond. And the only activity in period $T+1$ is that agents can deliver money on their past sale of government bonds.

Consider an ordered set $\mathcal{C} = \{t_i n_i m\}_{i=1,\ldots,k}$ in $\mathcal{F} \cup \mathcal{M}$ (where, of course, $n_i \geq t_i + 2$). We say that $\mathcal{C}$ *is overlapping and spans the time interval* $[t, t^*] = [t, t+1, \ldots, t^*-1, t^*] \subset \{1, \ldots, T\}$ if $t_1 < t$, $n_k > t^*$ and $t_{i+1} < n_i$ for all i. More generally, for any $\mathcal{A} \subset \mathcal{F} \cup \mathcal{M}$,.we say that A *spans* $[t, t^*]$ if $\mathcal{A}$ contains such an ordered set $\mathcal{C}$

(AII) **FULL SPAN:** $\mathcal{F} \cup \mathcal{M}$ spans $[1, \ldots, T]$

In the case that $\mathcal{C} \subset \mathcal{F}$, define the compound interest rate $r(\mathcal{C}) = (1+r_1)(1+r_2)\ldots(1+r_k)$, where $r_i = r_{t_i n_i m}$,and define *the effective interest rate* $r(t, t^*)$ *of the time interval* $[t, t^*]$ by

$$r([t, t^*]) = \min\{r(\mathcal{C}) : \mathcal{C} \subset \mathcal{F} \text{ and } \mathcal{C} \text{ is overlapping and spans } [t, t^*]\}$$

It is evident an agent can buy \$1worth of any commodity in any period $t' \in [t, t^*]$ and then (by rolling over the loans if necessary) defray that loan by raising $\$\,(1 + r([t, t^*]))$ through the sale of any other commodity in period t'' where $t' \leq t'' \leq t^*$ (i.e., the sale can happen at the same time t' or anytime later in the time interval).

4.5 Strict No-Arbitrage on Fixed-Rate Bond Markets $\mathcal{F}$

Consider an imaginary "player" (corresponding to the collective of all the agents in H) who has no money to begin with, but has full and free access to the fixed-rate bond markets in $\mathcal{F}$, where he can borrow or deposit money as he likes, rolling over the loans and inventorying money with no constraints (except, of course, that he cannot create money, but must earn it through interest on deposits of the money he borrowed). The standard "no-arbitrage hypothesis" would say that the player cannot end up with positive money at the end after repaying all his loans (both principal and interest). We strengthen this hypothesis *slightly* by replacing "positive" with "non-negative". In other words we postulate

(AIII) **STRICT NO ARBITRAGE: If the player becomes active on the fixed-rate bond markets, by borrowing money at any one of them, then**

it is not possible for him to repay all the loans he takes out, i.e., he must default on at least one of them.

Let us note a consequence of this hypothesis. Denote by S_M the compact set of all strategies of the player that entail a total borrowing of $\$M$ across all the markets in $\mathcal{F}$.(viewing any strategy as a vector whose components correspond to how much money he is borrowing, depositing, inventorying and repaying on each loan in $\mathcal{F}$). For any $\sigma \in S_M$ let $\Delta(\sigma)$ denote the vector of defaults, incurred via σ, on the repayment of loans[29] in $\mathcal{F}$, and note

(a) $S_M = MS_1 = \{M\sigma : \sigma \in S_1\}$

(b) $\Delta(M\sigma) = M\Delta(\sigma)$

Define

$$f : S_1 \longrightarrow \mathbb{R}_+$$

by

$$f(\sigma) = \text{ maximum component of } \Delta(\sigma)$$

Now Δ is continuous (clearly) and strictly positive (by AIII) on its compact domain S_1, therefore

$$\gamma = \min_{\sigma \in S_1} f(\sigma) > 0$$

and (using (b))

$$\min_{\sigma \in S_M} f(\sigma) = \gamma M \tag{2}$$

4.6 MONETARY EQUILIBRIUM

4.6.1 The Budget Set $B^h(p)$

We already introduced the notation $q_{t\alpha\beta}$, b_{tnm}, b_{tnl} for agents' market actions (see section ???). We add one more action variable x_{tl} for the consumption of tl out of stock on hand. For their stocks, *after* market exchange and consumption in period t, denote

$\mu_t =$ stock of money in period t

$y_{t\ell} =$ stock of commodity ℓ in period t

$\beta_{tn} =$ stock of bonds n in period t

$\bar{b}_{tn} =$ stock of accumulated primary bond sales n made before, or in, period t

Let $p = \{p_{t\alpha\beta} : \text{market } \{t\alpha\beta\} \text{ exists}\}$ denote the vector of market prices. Then the *budget set* $B^h(p)$ of agent h consists of all his market actions $q_{t\alpha\beta}$, b_{tnm}, b_{tnl} and consumptions x_{tl} which satisy the following constraints (where $\Delta(\theta)$ denotes the difference between the right and left hand sides of inequality (θ), and $\mu_0 = e^h_{0\ell} =$

[29] w.l.o.g. we assume that no more is repaid on a loan than is due (in any strategy in S_M)

$\beta_{0,n} = 0$

$$\sum_{\ell} q_{tm\ell} + \sum_{n>t+1} q_{tmn} + \bar{b}_{t-1,t} \leq \mu_{t-1} + m_t^h....(1t)$$

$$q_{t\ell m} + \sum_{n>t+1} q_{t\ell n} \leq f_{t\ell}^h(y_{t-1}) + e_{t\ell}^h....(2tl)$$

$$q_{tnm} + \sum_{\ell} q_{tn\ell} \leq \beta_{t-1,n} \text{ for all } n > t+1....(3t)$$

$$x_{t\ell} \leq y_{tl}$$

$$\mu_t = \Delta(1) + \beta_{t-1,t} + \sum_{\ell} p_{t\ell m} q_{t\ell m} + \sum_{n>t+1} p_{tnm} q_{tnm} + \sum_{n>t} p_{tnm} b_{tnm}...(4t)$$

$$y_{tl} = \Delta(2l) + \frac{q_{tm\ell}}{p_{t\ell m}} + \sum_{n>t+1} \frac{q_{tn\ell}}{p_{tn\ell}} + \sum_{n>t+1} \frac{b_{tn\ell}}{p_{tn\ell}} \text{ for all } \ell$$

$$\beta_{t,n} = \Delta(3) + \frac{q_{tmn}}{p_{tnm}} + \sum_{\ell} p_{t\ell n} q_{t\ell n} \text{ for all } n > t+1$$

$$\bar{b}_{t,n} = \bar{b}_{t-1,n} + b_{tnm} + \sum_{\ell} b_{tn\ell} \text{ for all } n > t+1; \ 0 \text{ for all } n \leq t+1$$

The first inequality says that an agent can spend money on commodities or bonds, or delivering on his primary bond sales, out of the money he carried over from last period plus his new endowment of money.

The second inequality says that the agent cannot sell more of a commodity than he has carried over from last period plus his new endowment.

The third inequality says that no agent's sale of secondary bonds can exceed the stock he has on hand. (He can, in addition, make primary sales without limit, on which he will have to deliver in full when they come due.)

The fourth inequality says that the money inventoried (after market exchange) from period t to $t+1$ is comprised of the money he did not spend in (1) plus the payments he receives from his portfolio of bonds, plus the money received from his sales of commodities and his sales of old and new bonds.

The fifth inequality says that commodity l inventoried (after market exchange) from period t to $t+1$ consists of what he had and did not sell in $(2l)$ plus the commodities he bought via money, or secondary bonds, or primary bonds.

The sixth inequality says that the portfolio of bonds inventoried (after market exchange) from period t to $t+1$ consists the bonds he had and did not sell in (3) plus the bonds (secondary and primary) he just bought via money or commodity sales.

The seventh inequality says that the bond debts (promises) he ends up with in period t are the sum of the promises he had made before plus the new promises made in period t in exchange for money or commodities.

It is evident (recalling the concavity of the functions $f_{t\ell}^h$) that the budget set $B^h(p)$ is convex.

4.6.2 Monetary Equilibrium (ME)

The definition of monetary equilibrium is (note that, $\mu^h, \beta^h, \bar{b}^h$ are residual variables defined by equalities)

$$(((q^h, b^h, x^h)_{h\in H}, p), (r, M, B))$$

such that

$$\begin{aligned}
\frac{M_{tmn}}{p_{tnm}} + \sum_h \frac{q^h_{tmn}}{p_{tnm}} &= B_{tnm} + \sum_h q^h_{tnm} \\
\sum_h \frac{q^h_{tm\ell}}{p_{t\ell m}} &= \sum_h q^h_{t\ell m} \\
\sum_h \frac{q^h_{tn\ell}}{p_{t\ell n}} + \sum_h \frac{b^h_{tn\ell}}{p_{t\ell n}} &= \sum_h q^h_{t\ell n} \\
\frac{M_{tmn}}{p_{tnm}} + \sum_h \frac{q^h_{tmn}}{p_{tnm}} &= B_{tnm} + \sum_h q^h_{tnm} \\
(q^h, b^h, x^h) &\in B^h(p, r) \\
(q, b, x) \in B^h(p, r) &\Longrightarrow u^h(x^h) \geq u^h(x)
\end{aligned}$$

and the following two additional qualifications hold: $\{tnm\} \in \mathcal{F}$, with $p_{tnm} = (1 + r_{tnm})^{-1}$ and $M_{tmn}B_{tnm} = 0$ (in the first equality); $\{tnm\} \in \mathcal{M} \cup \mathcal{B}$, with $M_{tmn} = 0$ if $\{tnm\} \in \mathcal{B}$ and $B_{tnm} = 0$ if $\{tnm\} \in \mathcal{M}$ (in the second equality)

To establish the existence of ME we shall need an assumption (see below) to the effect that there are sufficient "gains to trade" at the initial endowment (for which agents will desire to borrow on some government bond market money, despite its interest rate and despite astronomical prices of the commodities).

4.6.3 Feasible Outcomes

$(((q^h, b^h, x^h)_{h\in H}, p), (r, M, B))$ is called a *feasible outcome* of the economy if it satisfies all the conditions of a monetary equilibrium except possibly the last condition of utility maximization.

4.6.4 Gains to Trade

Fiat money is ultimately wanted only for trading commodities. It follows that the value of money should depend on agents' motivation to trade commodities. We develop a measure of this motivation called "gains to trade" and show that, whenever it is strong enough, monetary equilibrium exists (i.e., money is valued and used to trade commodities at markets).

For any trade vector $\tau \in \mathbb{R}^{T\times L}$ and any scalar $\gamma > -1$, define

$$\tau_{t\ell}(\gamma) = \min\{\tau_{t\ell}, \tau_{t\ell}/(1 + \gamma)\}$$

Note $\tau_{t\ell}(\gamma) = \tau_{t\ell}$ if $\tau_{t\ell} < 0$, $\tau_{t\ell}(\gamma) = \tau_{t\ell}/(1+\gamma)$ if $\tau_{t\ell} > 0$. Thus $\tau(\gamma)$ entails a diminution of commodities received in τ by the fraction $\gamma/(1+\gamma)$, but no diminution in the commodities sent out.

Let $x = \left(e^h \oplus \widetilde{\tau}^h\right)_{h\in H} \in \left(\mathbb{R}_+^{T\times L}\right)^H$ be a consumption that is induced by the trades $\left(\widetilde{\tau}^h\right)_{h\in H} \in \left(\mathbb{R}^{T\times L}\right)^H$ in the economy $\mathcal{E} = \left((u^h, e^h, f^h, m^h)_{h\in H}, M, \mathcal{M}\right)$. (It is understood, of course, that $\widetilde{\tau}^h_{tl} = 0$ if $tl \notin \mathcal{M}$.) We say that there are *gains to γ-diminished trades* at x *in the time interval* $[t, t^*]$ if there exist trades $(\tau^h)_{h\in H} \in \left(\mathbb{R}^{T\times L}\right)^H$ such that[30]

(a) $\sum_{h\in H} \tau^h = 0$ and $\tau^h_{t\ell} = 0$ if $t \notin [t, t^*]$

(b) $e^h \oplus \left(\widetilde{\tau}^h + \tau^h(\gamma)\right) \in \mathbb{R}_+^{T\times L}$ for all $h \in H$

(c) $u^h(e^h \oplus \left(\widetilde{\tau}^h + \tau^h(\gamma)\right)) > u^h(e^h \oplus \widetilde{\tau}^h)$ for all[31] $h \in H$.

In short, it should be possible — despite the "γ-handicap" on trades — for agents to Pareto-improve on x with trades confined to the interval $[t, t^*]$..

4.6.5 Gains to Trade Assumption for $\mathcal{F}$

(AIV)($\mathcal{F}$) **GAINS TO TRADE UNDER** $\mathcal{F}$: There exists a time interval $[t, t^*] \subset \{1, \ldots, T\}$ that is spanned by $\mathcal{F}$ and there exist gains to $r([t, t^*])$-diminished trades at the initial consumption $\widetilde{e} = \left(\widetilde{e}^h\right)_{h\in H}$ in $[t, t^*]$.

Remark If there are gains to r_t-diminished trade at $\widetilde{e}$ in any period $t \in \{1, \ldots, T\}$ (where r_t is the fixed interest on any govt loan that spans t), then AIV(a) is satisfied.

4.6.6 Gains to Trade Assumption for $\mathcal{M}$

(AIV)($\mathcal{M}$) **GAINS TO TRADE UNDER** $\mathcal{M}$: For every bond market $tnm \in \mathcal{M}$, there exists gains to γ_{tnm}-diminished trades at the initial consumption $\widetilde{e} = \left(\widetilde{e}^h\right)_{h\in H}$ in the time interval spanned by $\{tnm\}$ (with all $\gamma_{tnm} \geq 0$). Moreover at least *one* of the following inequalities 5_t (defined recursively for $n = 1, \ldots, T$ starting with $\Delta(5_0) = 0$) is violated

$$\sum_{\nu<t} (1+\gamma_{\nu tm}) M_{\nu mt} \leq \Delta(5_{t-1}) + \sum_{n>t} M_{tnm} + \sum_h m^h_t + \textstyle\sum_{\nu<t} B_{\nu(t-1)m} \tag{5_t}$$

, i.e., $>$ holds in place of $\leq$ for some $t \in \{1, \ldots, T\}$.

(It is understood here that $M_{tmn} = 0$ if $\{tnm\} \notin \mathcal{M}$ and $B_{tnm} = 0$ if $\{tnm\} \notin \mathcal{B}$.) [32] Also recall that $\Delta(5_{t-1})$ denotes the difference between the right and left sides of inequality (5_{t-1}).)

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[30] A much more restrictive, albeit also much simpler-looking, definition would be to simply replace "$\oplus$" by the standard addition "+" throughout, thereby postulating that agents improve their utilities even after throwing into the sea all the additonal durable commodities that they get from the incremental trades.

[31] Since utilities are strictly monotonic, this is equivalent to requiring that some household is strictly better off and none are worse off.

[32] Thus, for example, $\sum_{v<t} B_{\nu(t-1)m}$ is the same as $\sum_{v<t-2} B_{\nu(t-1)m}$ because $B_{\nu(t-1)m} = B_{\nu(t-1)m} = 0$ on account of the fact the markets $\{t-1, t-1, m\}$ and $\{t-2, t-1, m\}$ do not exist.

4.6.7 Gains to Trade assumption for $\mathcal{F} \cup \mathcal{M}$

(AIV) **GAINS TO TRADE UNDER** $\mathcal{F} \cup \mathcal{M}$: Either $\mathcal{F} \neq \varnothing$ and (AIV)($\mathcal{F}$) holds; or $\mathcal{F} = \varnothing$ and (AIV)($\mathcal{M}$) holds.

4.7 Existence of Monetary Equilibrium

THEOREM I: *Suppose that Assumptions AI, AII, AIII, AIV hold. Then a monetary equilibrium exists.*

Proof: See the Appendix

4.7.1 Interpretation of AIV ($\mathcal{M}$)

Let $(r_{tnm})_{\{tnm\}\in\mathcal{M}}$ be the interest rates prevailing at any ME of the economy and assume $\mathcal{F} = \varnothing$ and (AIV)($\mathcal{M}$) holds. Consider the set of inequalities $\{5_t : t = 1, \ldots, T\}$ with each $\gamma_{\nu tm}$ replaced by $r_{\nu tm}$. Then the LHS of 5_t is the money collectively owed to the government by all the agents in H in period t. The RHS of 5_t is the money collectively in the hands of H at the time of repayment to the government in period t (i.e., the money they have collectively inventoried from $t-1$ + money collectively borrowed from government in t + money collectively endowed in t + money delivered by the government in period t on its past sales of bonds). At any ME of $\mathcal{E}$, there is no default on loans, hence the inequality 5_t must hold; and, since agents pay back no more than they owe, $\Delta(5_t) =$ RHS-LHS must be collectively carried over into period $t+1$ by H. Iterating this argument, we see that entire set of inequalities must hold.

Now consider the inequalities in their original form, reinstating $\gamma_{\nu tm}$ in place of the interest rates r_{vtm}. If $\gamma_{vtm} \leq r_{vtm}$ for all $vtm \in \mathcal{M}$, then it is clear that the validity of the inequalities $\{5_t : t = 1, \ldots, T\}$ for r_{vtm} implies their validity for γ_{vtm} as well. Assumption (AIV)($\mathcal{M}$) rules this out, and thus *guarantees that* $\gamma_{vt} > r_{vt}$ *for at least one* $\{\nu tm\} \in \mathcal{M}$ at any ME of the economy.

Note that (AIV)($\mathcal{M}$) relates the time-distribution $\{\gamma_{vt}\}_{vt\in\mathcal{B}}$ of the gains-to-trade at $\widetilde{e}$ (arising from the real sector of $\mathcal{E}$) to the time-distribution $\{M_{vmt}\}_{\{\nu tm\}\in\mathcal{M}}$ and $\{m_t^h\}_{t\in T, h\in H}$, $\{B_{\nu tm}\}_{\{\nu tm\}\in\mathcal{B}}$ of the inside and outside money (arising from the monetary sector of the economy).

Indeed, it is worth noting that the above discussion holds even if we replace ME by any *feasible outcome* of the economy (since agents do not repay more than they owe on their loans, nor do they default on them, in our definition of "feasible outcome").

4.7.2 A Stronger Version of AIV ($\mathcal{M}$)

Let $\overline{m} + \overline{B} = \sum_{t\in T}\sum_{h\in H} m_t^h + \sum_{\{tnm\}\in\mathcal{B}} B_{tnm}$denote the total outside money in the economy and let γ_{vtm} be as in (AIV)($\mathcal{M}$) for all $\nu tm \in \mathcal{M}$. Then

$$\sum_{\{tnm\}\in\mathcal{M}} \gamma_{tnm} M_{tmn} > \overline{m} + \overline{B}$$

Remark It is readily checked that this is a stronger version because summing the inequalities 5_t over t yields $\sum_{\{tnm\}\in\mathcal{M}} \gamma_{tnm} M_{tmn} \leq \overline{m} + \overline{B}$. Thus, while this assumption is simpler than (AIV)($\mathcal{M}$), it is also a stronger. Note further that it clearly implies that, for any solution $r = (r_{tnm})_{\{tnm\}\in\mathcal{M}}$ of the inequalities

$$\sum_{tn\in\mathcal{M}} r_{tnm}M_{tmn} \leq \overline{m} + \overline{B}$$

there exists $\{t^*n^*m\} \in \mathcal{M}$ such that

$$\gamma_{t^*n^*m} > r_{t^*n^*m}$$

REMARK: Theorem 1 holds even if we allow netting on the deliveries of primary bonds that do not trade against commodities (but only against money). Call these red bonds. Steps 1 and 2 remain true as before, using the even stronger no-arbitrage condition which allows the player against the bank to do such netting. Next note that interest rates on red bonds (that are issued outside of the fixed-rate bond markets) must also be bounded, for if they went too high, no one would borrow on them, as they could borrow more cheaply on the fixed-rate markets (using the full span Assumption I). But then primary sales of such bonds are also bounded, since money is bounded. The rest of the argument is as before .... Netting on bonds that trade against commodities will — as you pointed out — destroy money.

4.8 The Liquidity Trap

Suppose the central bank wishes to provide stimulus to the economy by making money plentifully available to the agents. To this end it can:

Scenario (i): decrease interest rates r_{tnm} to 0 on markets $\{tnm\} \in \mathcal{F}^* \subset \mathcal{F}$ (and, for simplicity, keep all other policy variables fixed)

Scenario (ii): increase bond purchases (i.e., the money loaned) M_{tnm} to ∞ on markets $\{tnm\} \in \mathcal{M}^* \subset \mathcal{M}$ (and, again for simplicity, keep all other policy variables fixed)

We shall focus on these scenarios when the expansionary monetary policy is carried out by the central bank in the short run, but not maintained by it in the long run, i.e.

all the bonds traded at markets $\mathcal{F}^* \cup \mathcal{M}^*$ are due by some early period $t << T$

It turns out then that the monetary policy is ineffectual. No matter how drastic decreases in the interest rates (or increases in the bond purchases) are, there comes a threshold beyond which the real sector is hardly budged. The amounts borrowed and spent (in ME) in scenario (i) converge to a constant (as do all the real outcomes of the ME) even as the interest rates in $\mathcal{F}^*$ are lowered to 0: agents can simply *not* be lured to enhance their expenditures beyond this threshold. A similar phenomenon occurs in case (ii): the interest rate collapses to 0, after the money injected by the central bank crosses a certain threshold; and, though all the money put up by the central bank continues to be borrowed, the portion of it deployed by agents on markets becomes constant (along with the ME outcomes), while the rest of the borrowed money is simply hoarded and returned. In short, a myopic expansionary policy leads to the onset of a *liquidity trap* in both cases.

We shall state our Liquidity Trap theorems in two special scenarios, where they are simple to state (see, however, Remark ?? for a more general version). For the analysis, we need to first strengthen assumptions III and IV.

(AIII)* ($\mathcal{F}^*$) The **Strict No Arbitrage Assumption (AIII) holds for all the varying interest rates** $0 < r^*_{tnm} \leq r_{tnm}$ **on markets** $\{tnm\} \in \mathcal{F}^*$ **(in conjunction with the unaltered rates** r_{tnm} **on markets** $\{tnm\} \in \mathcal{F} \diagdown \mathcal{F}^*$**)**

Let

$$\mathcal{A}(t) = \text{ the set of all commodity allocations achievable without any commodity trade in periods } t' \geq t$$

(AIV)*($\mathcal{F}^*$) **GAINS TO TRADE UNDER** $\mathcal{F}$ **: at every allocation in** $\mathcal{A}(t)$ **, there exist gains to** $r[t_*, t^*]$**-diminished trades for some time interval** $[t_*, t^*] \subset [t, T]$ **(where, recall,** $r[t_*, t^*]$ **is based on the rates** r_{tnm} **on markets** $\{tnm\} \in \mathcal{F}$**)**

Note that $\mathcal{A}(t)$ is a "small" subset of the set of all allocations when $t << T$; and therefore (AIV)*($\mathcal{F}^*$) is not a very demanding assumption.

THEOREM II ($\mathcal{F}$) *Suppose, in Scenario (i), that Span* $F = \{1, \ldots, T\}$ *, and that assumptions AI, (AIII)*,(AIV)* hold. Then there exists a Liquidity Trap, i.e., any convergent sequence of ME has has a finite limit.*

PROOF: See the Appendix.

For Scenario (ii), we need a variant of (AIV)*($\mathcal{F}^*$)

(AIV)*($\mathcal{M}^*$) **GAINS TO TRADE UNDER** $\mathcal{M}$: For every bond market $tnm \in \mathcal{M} \diagdown \mathcal{M}^*$, such that $n > t$, and every $x \in \mathcal{A}(t)$, there exists gains to γ_{tnm}-diminished trades at x in the time interval spanned by $\{tnm\}$ (with all $\gamma_{tnm} \geq 0$). Moreover at least *one* of the following inequalities 6_ς (defined recursively below for $\varsigma = t+1, \ldots, T$ starting with $\Delta(6_t) = \sum_{\varsigma \leq t} \sum_h m^h_\varsigma + \sum_{\varsigma < t} \sum_{v < \varsigma} B_{\nu m \varsigma})$ is violated

$$\sum_{\nu < \varsigma} (1 + \gamma_{\nu \varsigma m}) M_{\nu m \varsigma} \leq \Delta(5_{\varsigma - 1}) + \sum_{n > \varsigma} M_{\varsigma m n} + \sum_h m^h_\varsigma + \textstyle\sum_{v < \varsigma} B_{\nu m (\varsigma - 1)} \qquad (5_t)$$

, i.e., $>$ holds in place of $\leq$ for some $t \in \{1, \ldots, T\}$. **at every allocation in** $\mathcal{A}(t)$

THEOREM II ($\mathcal{M}$) *Suppose, in Scenario (ii), that* $F = \varnothing$ *and Span* $M = \{1, \ldots, T\}$ *, and that assumptions AI, (AIV)* hold. Then there exists a Liquidity Trap, i.e., any convergent sequence of ME has has a finite limit.*

PROOF: See the Appendix (to be written)

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5 Appendix

5.1 Proof of Theorem I

For any $\varepsilon > 0$, we establish the existence of an "ε-monetary equilibrium" (ε-ME) whose limit (as $\varepsilon \to 0$) will yield an ME.

An ε-ME is a type-symmetric strategic (Nash) equilibrium of the following generalized game $\mathcal{G}_\varepsilon$ with a continuum of players. Replace each $h \in H$ by a continuum $(h-1, h]$ of identical agents. Each α in the interval $(h-1, h]$ is of "type h" and has the characteristics

$$(e^\alpha, m^\alpha, u^\alpha, f^\alpha) \equiv (e^h, m^h, u^h, f^h)$$

Throughout we shall focus on *type-symmetric* strategies. The strategy of an agent of type h consists of vectors $\left(b^h, q^h, x^h\right)$. The *ambient* strategy-set is the same for all agents, regardless of their type and given by $B(\varepsilon) = \{\left(b^h, q^h, x^h\right)$: every component is between 0 and $1/\varepsilon\}$. (On account of the type-symmetry assumption, the notation $\left(b^h, q^h, x^h\right)$ can — and will — be used in three different senses: as the *vector* which is the common individual strategy chosen by each agent $\alpha \in (h-1, h]$ of type h; as the constant *function* which maps each $\alpha \in (h-1, h]$ to the vector $\left(b^h, q^h, x^h\right)$ and describes the symmetric strategy-selection by agents of type h; and as the *integral* of this constant function on the unit interval $(h-1, h]$, which gives the aggregate strategy of agents of type h. The sense of the usage will initially be indicated, and later on be always clear from the context.)

Denoting a choice of strategies by the functions $\sigma = \left(\sigma^h\right)_{h\in H} = \left(b^h, q^h, x^h\right)_{h\in H}$, market prices $p_{t\alpha\beta}(\sigma)$ or government actions $Q_{t\beta\alpha}$, $Q_{t\alpha\beta}$ form in the game $\mathcal{G}_\varepsilon$ i n accordance with government policy and the rule[33]:

$$p_{t\alpha\beta}(\sigma) = \frac{\varepsilon + Q_{t\beta\alpha} + \sum_h q^h_{t\beta\alpha}}{\varepsilon + Q_{t\alpha\beta} + \sum_h q^h_{t\alpha\beta}} \text{ for market } \{t\alpha\beta\}$$

The symbols in the fractions on the right, with the exception of ε and Q's , represent integrals of agents' choices. Thus in the game $\mathcal{G}_\varepsilon$, we imagine an *external agent* who is a "strategic dummy" and puts up ε units on both sides of every market , as indicated in the formulae displayed above; and who does not default on his delivery of money or commodities (creating them, if necessary, to fulfil his obligations). Note that prices form to clear all markets, taking the external agent into account, and trade occurs in conformity with these prices. Thus we have a closed market system in the following sense: for an arbitrary choice of strategies by the agents, the total supply of α at the market $\{t\alpha\beta\}$ is disbursed to the agents on the other side of the market in proportion to their β-supplies. Note also that the money and commodities injected into the system by the external agent is of the order of ε

Of course, given an arbitrary σ, it may well happen that at the emergent prices $p(\sigma)$ agents do not balance their budgets. This leads us to consider a generalized

[33] Recall that $Q_{t\beta\alpha}$ is understood to be 0 when the government abstains from the market $\{t\beta\alpha\}$, or when it has chosen $Q_{t\alpha\beta} > 0$.

game with strategy sets that depend on agents' choices. For each each $\alpha \in (h-1, h]$, define his *feasible* strategy-set (or, the *truncated budget set*) by:

$$B^h_\varepsilon(p(\sigma)) \equiv B(\varepsilon) \cap B^h(p(\sigma));$$

and his payoff by

$$u^h(\sigma^h) = u^h(x^h).$$

where x^h is the consumption component of σ^h. This completes the definition of the the generalized game $\mathcal{G}_\varepsilon$ and of ε-ME (which, recall, denotes a type-symmetric strategic equilibrium of $\mathcal{G}_\varepsilon$.)

From now on, assume that ε is sufficiently small so that, at any feasible outcome of $\mathcal{G}_\varepsilon$, the aggregate of any commodity in the system has an upper bound $\mu < 1/\varepsilon$.

Also define

$$u^h_* = u^h(\mu, ..., \mu)$$

and let μ^* be chosen to guarantee that

$$u^h(0, ..., 0, \mu^*, 0, ..., 0) > u^h_*$$

for μ^* in any component. (W.l.o.g.[34] we may suppose that such a μ^* exists.)

Let $B(\varepsilon) = ((B(\varepsilon))^H$ and define the "best reply" correspondence $\psi_\varepsilon : B(\varepsilon) \rightrightarrows B(\varepsilon)$ by

$$\psi_\varepsilon = \psi^1_\varepsilon \times \cdots \times \psi^H_\varepsilon.$$

where, for $\sigma = \left(\sigma^h\right)_{h\in H}$ and $h \in H$,

$$\psi^h_\varepsilon(\sigma) = \arg\max\{u^h(\sigma^h) : \sigma^h \in B^h_\varepsilon(p(\sigma))\}$$

Clearly $B^h_\varepsilon(p(\sigma))$ is non-empty, compact, and convex. On account of the external agent's ε, all prices $p_{t\alpha\beta}(\sigma)$ are positive, and it follows that B^h_ε is a continuous correspondence in σ. Hence each ψ^h is non-empty, convex, and upper semi-continuous.

By Kakutani's Theorem ψ_ε has a fixed point

$$(\sigma(\varepsilon)) = \left(\sigma^h(\varepsilon)\right)_{h\in H} \equiv \left(b^h(\varepsilon), q^h(\varepsilon), s^h(\varepsilon), x^h(\varepsilon)\right)_{h\in H}$$

with induced prices $p(\varepsilon)$, $r(\varepsilon)$. The vector $\sigma(\varepsilon)$ is clearly an ε-ME.

Notice that, at any[35] ε-ME, (1) we have a physically closed system, in which all the money or commodities or bonds sent to market are conserved and redistributed

[34] Let $\square$ be the cube in $\mathbb{R}^{T\times L}_+$ with sides of length μ. Recall $u^h : \square \to \mathbb{R}$ is strictly increasing in the variables $x_{t\ell}$ for any $t\ell \in \mathcal{M}$. Define $\tilde{u}^h : \mathbb{R}^{T\times L}_+ \to R$ by $\tilde{u}^h(y) = \inf\{L_x(y) : x \in \square,\ L_x$ is an affine function representing a supporting hyperplane to the graph of u^h at the point $(x, u^h(x))\}$. Then it is clear that (a) $\tilde{u}^h$ is concave, strictly monotonic in the variables $\{x_{tl} : tl \in \mathcal{M}\}$, and coincides with u^h on $\square$, hence ME of our economy are unaltered if we replace u^h by $\tilde{u}^h$; and (b) there exists a μ^* such that $\tilde{u}^h(0, ..., 0, \mu^*, 0, ..., 0) > u^h_*$ for μ^* in any component.

[35] And, indeed, at any *feasible outcome* of $\mathcal{G}_\varepsilon$ i.e., at any $\sigma = \left(\sigma^h\right)_{h\in H}$ such that each $\sigma^h \in B^h_\varepsilon(\sigma)$ for all h.

amongst the agents (including the external agent) and the government; (2) all agents view $p(\sigma)$ as fixed, since their individual actions do not affect the integrals involved in forming prices; (3) each agent chooses optimal strategies in his feasible strategy set (or, truncated budget-set) $B_{\varepsilon}^{h}(p(\sigma))$.

Select a subsequence of $\sigma(\varepsilon)$ as $\varepsilon \to 0$ to ensure that all its components and all ratios of all components converge (possibly to zero or infinity).

We now examine the subsequence below, it being understood throughout that *all assertions pertain to small enough* ε, *and all limits and bounds are taken with* $\varepsilon \longrightarrow 0$;.and show, through a series of claims, that its limit is an ME (monetary equilibrium).

CLAIM 1. The money in the system is bounded in all time periods for all ε.

PROOF.Suppose the money in the system goes to ∞. Consider an imaginary player (as in the strict no arbitrage hypothesis) whose actions are the integral of agents' actions at the fixed-rate (money-bond) markets in $\mathcal{F}$. The money injected by the central bank via all the government bond markets other than $\mathcal{F}$ (i.e., markets that are *not* fixed-rate) is given by M and B and does not vary with ε, nor does the aggregate initial endowment $\sum_h m^h$; and, of course, all the money injected by the ε-player is going to zero. Thus the sum of all the money (other than the money coming from $\mathcal{F}$) is bounded, even after this sum is deemed to earn the maximum conceivable interest via $\mathcal{F}$ from period 0 till $T+1$). So the money in the system that is going to ∞ is being borrowed on loans in$\mathcal{F}$ by the imaginary player.[36] By the strict no arbitrage hypothesis (see the discussion just after its definition, in particular (2)), the imaginary player's default must go to ∞ on some loan in $\mathcal{F}$. Since his default is the aggregate of the defaults of all the agents, the default of some agent-type goes to ∞ in the sequence of ε-ME, which is a contradiction since, at any ε-ME, all agents must be in their feasible strategy sets and hence not defaulting..

CLAIM 2 All market actions are bounded in the sequence of ε-ME.

PROOF.We have already argued that the total money and commodities are bounded above at any feasible outcome of $\mathcal{G}_{\varepsilon}$. Thus all components of $\sigma(\varepsilon)$, except possibly the bonds sold by agents, are bounded. To show that these are also bounded, first consider primary bonds. They have to be delivered upon in full and, since money is bounded, their sales must also be bounded. It follows that sales of secondary bonds are also bounded.

CLAIM 3. $\sigma^h(\varepsilon)$ is optimal in the entire budget set $B^h(p(\varepsilon))$, not just on the truncated budget set $B_{\varepsilon}^{h}(p(\varepsilon))$.

PROOF.By claim 2, the bound of $1/\varepsilon$ on agents' optimal actions is not binding for small enough ε. The claim now follows from the fact that their utilities are concave, and the budget sets are convex.

CLAIM 4. All interest rates $r_{tnm}(\varepsilon)$ are non-negative and they are also bounded above at all markets in $\mathcal{M}$.

[36] As was said, on government bond markets other than $\mathcal{F}$, the money injected is given by M or B and therefore bounded. As for private money-bond markets, the money borrowed on them is loaned by agents who cannot create money (and only ε by the external agent), so if these agents are lending huge amounts they must have acquired it at some market in $\mathcal{M}$.

PROOF. Suppose some $r_{tnm}(\varepsilon) < 0$. Then let h increase b^h_{tn} by a positive δ, obtaining $\delta(1 + r_{tn}(\varepsilon))^{-1} > \delta$ units of bank money[37]. Let him inventory δ to repay this additional loan in period n and spend the surplus money to buy commodities in period T. (Recall that commodity markets exist in period T by Assumption I.) This improves his utility, contradicting Claim 3. We conclude that $r_{tnm}(\varepsilon) \geq 0$

Next $r_{tnm}(\varepsilon) \longrightarrow \infty$ for some $\{tnm\} \in \mathcal{M}$, then (since the external player borrows $\varepsilon / (1 + r_{tnm}(\varepsilon)) \leq \varepsilon$, the money owed in period n (by primary issuers of bond n at the market $\{tnm\}$) is at least $r_{tnm}(\varepsilon)(M_{tmn} - \varepsilon) \longrightarrow \infty$, hence (by Claim 1) one of them must be defaulting, a contradiction.

CLAIM 5.Prices $p_{tkm}(\varepsilon)$ and $p_{tkn}(\varepsilon)$ are bounded away from 0.

PROOF. Any h with $m^h_0 > 0$ can spend m^h_0 in period t (inventorying the money if $t > 1$) to buy $m^h_0 / p_{tkm}(\varepsilon)$ units of tk. (Such an h exists since $\sum_h m^h_1 > 0$ by assumption). However $\left(m^h_0 / p_{tkm}(\varepsilon)\right) < \mu^*$ otherwise h would be obtaining more utility than is feasible by consuming all the commodities in the economy, again contradicting claim 3. This shows that $p_{tkm}(\varepsilon)$ is bounded away from 0. Next, note that if the market $\{tkn\}$ exists, then so does the market $\{tkm\}$ by Assumption AI. But $p_{tkm}(\varepsilon) \leq p_{tkn}(\varepsilon)$ since bond n delivers a dollar later which is clearly not more valuable than a dollar now.

CLAIM 6 Price ratios $p_{tkm}(\varepsilon)/p_{v\ell m}(\varepsilon)$ are bounded for any t, k, v, l.

PROOF. All interest rates in $\mathcal{F} \cup \mathcal{M}$ are bounded above (see Claim 4), therefore the full span of $\mathcal{F} \cup \mathcal{M}$ (see Assumption II) implies the exisence of effective interest rate $r^{\#} < \infty$ at which agents can borrow money by rolling over the loans in $\mathcal{F} \cup \mathcal{M}$. Now, by Assumption AI, that there exists an agent h who can sell $\widetilde{e}^h_{tk} > 0$ to obtain $\left(1 + r^{\#}\right)^{-1} \left(p_{tkm}(\varepsilon)\widetilde{e}^h_{tk} / p_{v\ell m}(\varepsilon)\right)$ units of commodity νl, but by Claim 3 this term must be less than μ^*.

CLAIM 7 All $p_{tkm}(\varepsilon)$ and $p_{tkn}(\varepsilon)$ converge to finite limits.

PROOF. Suppose some $p_{tkn}(\varepsilon) \longrightarrow \infty$. The agent h with $\widetilde{e}^h_{tk} > 0$ can sell tk at the market $\{tkn\}$ to obtain $p_{tkn}(\varepsilon)\widetilde{e}^h_{tk}$ units of money in period $n + 1 \leq T + 1$. Anticipating this income, he can buy $(1 + r^*)^{-1} \left(p_{tkn}(\varepsilon)\widetilde{e}^h_{tk} / p_{tlm}(\varepsilon)\right)$ units of any $l \in L$ at the market $\{tlm\}$ via money borrowed and rolled over on But the amount he buys cannot exceed μ^*, which implies $p_{tlm}(\varepsilon) \longrightarrow \infty$. Therefore, to establish claim 7, it suffices to show no commodity-money price goes to infinity.

Suppose some commodity-money price goes to infinity. By Claim 6, *all* commodity-money prices go to infinity; and therefore so do all commodity-bond prices (since they are at least as high as the corresponding commodity-money prices). Then, by Claim 2, commodity trades go to 0 and consumptions converge to $\widetilde{e} = \left(\widetilde{e}^h\right)_{h \in H}$.

We will show that this contradicts the Gains to Trade assumption IV.

First consider the case where $\mathcal{F} \neq \varnothing$. By Assumption IV there exist gains to $r([t, t^*])$-diminished trade at $\widetilde{e}$ in some time interval $[t, t^*] = T^*$. Denote the trades that have (actually) occurred at the ε-ME by $\widetilde{\tau}^h(\varepsilon)$, and abbreviate $r([t, t^*])$ by r^* and $[t, t^*]$ by T^*. By the continuity of the utilities $u^h$and the fact that all $\widetilde{\tau}^h(\varepsilon) \longrightarrow 0$, Assumption IV implies that there exist trades $(\tau^h)_{h \in H} \in \left(\mathbb{R}^{T \times L}\right)^H$ such that, for

[37] This action is feasible by virtue of Claim 3 (as are the other actions of the agents in the proofs that follow, though we shall not keep saying so).

sufficiently small ε,

(a) $\sum_{h \in H} \tau^h = 0$ and $\tau^h_{t\ell} = 0$ if $t \notin T^*$

(b) $e^h \oplus \left(\widetilde{\tau}^h(\varepsilon) + \tau^h(r^*)\right) \in \mathbb{R}_+^{T \times L}$ for all $h \in H$

(c) $u^h(e^h \oplus \left(\widetilde{\tau}^h(\varepsilon) + \tau^h(r^*)\right)) > u^h(e^h \oplus \widetilde{\tau}^h(\varepsilon))$ for all $h \in H$.

Now (c) implies that $p(\varepsilon) .\tau^h > 0$ for otherwise (as is easily verified) $e^h \oplus \left(\widetilde{\tau}^h(\varepsilon) + \tau^h(\gamma_{t\#n})\right)$ would be in the budget set of h contradicting that $e^h \oplus \widetilde{\tau}^h(\varepsilon)$ maximizes his utility there. Summing over h we obtain $p(\varepsilon) . \sum_{h \in H} \tau^h > 0$ which contradicts (a).

Next consider the case where $\mathcal{F} = \varnothing$ and assumption AIV($\mathcal{M}$) holds. Now the only money injected by the government is via the markets in $\mathcal{M}$ and $\mathcal{B}$. On the markets in $\mathcal{M}$ the government enters as a pure lender and agents cannot *collectively* earn any money from the government as depositers. On any market $\{tnm\} \in \mathcal{B}$ agents collectively earn the amount B_{tnm} in period n from their deposits of money at $\{tnm\}$ regardless of the interest rate r_{tnm}. Thus the outside money in the system at the time of repayment of loans in time t is as given on the RHS of inequality (5_t). It follows that there exists a market $\{tnm\} \in \mathcal{M}$ such that $\gamma_{tnm} > r_{tnm}(\varepsilon)$ for sufficiently small ε. This contradicts the fact that, at there are gains to γ_{tnm}-diminished trade in the time interval spanned by $\{tnm\}$. (The argument is exactly as before).

CLAIM 8 The limit of ε-ME is an ME.

PROOF This is immediate from the previous claims and the continuity of the u^h.

5.2 Proof of Theorem II ($\mathcal{F}$)

During the descent of the interest rates on $\mathcal{F}^*$ to 0 (keeping the rates on $\mathcal{F} \diagdown \mathcal{F}^*$ fixed), the time span of the fixed-rate bond markets clearly does not change; the Gains to Trade Assumption AIV($\mathcal{F}$) holds even more easily than before, since the effective interest rates (induced via $\mathcal{F}$ on any time interval) either stay the same or else go down, and finally — by Assumption (AIII)^* ($\mathcal{F}^*$) — the Strict No Arbitrage assumption holds for every level of the interest rates. Therefore, by Theorem I, an ME exists throughout the descent of the interest rates.

Consider a convergent sequence of ME as the interest rates on $\mathcal{F}^*$ go to 0.Proof of Theorem II ($\mathcal{F}$)

CLAIM A. The money borrowed on $\mathcal{F}^*$ stays bounded.

PROOF. Suppose the money borrowed on $\mathcal{F}^*$ goes to ∞. Then, since this money is borrowed at positive interest rates, none of it can be kept idle, hoarded and returned. Now if any amount X of this money was used to purely deposit and earn interest, this would not even cover the borrowing cost of X on $\mathcal{F}^*$ for otherwise the borrower would have done arbitrage, contradicting Assumption $\text{(AIII)}^*(\mathcal{F}^*)$. Thus the expenditure on buying commodities (out of the money borrowed on $\mathcal{F}^*$) goes to ∞ in some period $t' < t$. Since the sales of commodities are bounded, the price of some commodity in period t' must then also go to ∞. But the effective interest rates, induced via $\mathcal{F}$, are bounded on all time intervals as we lower interest rates on on $\mathcal{F}^*$ (indeed, as was said, they stay the same or fall), therefore all price ratios (across commodities and time periods) are bounded as in Claim 6 (in the proof of theorem 1). We conclude that *all* commodity-money prices go to ∞.

However, the money in the system is bounded after period t. To see this, note that all the money borrowed on $\mathcal{F}^*$ leaves the system by time t. If the money goes to ∞ after t, it must have been borrowed on $\mathcal{F}\diagdown\mathcal{F}^*$. By (2) the default must also be going to infinity on $\mathcal{F}\diagdown\mathcal{F}^*$, which leads to a contradiction as in Claim 1 (in the proof of theorem 1)[38].

It follows that trade must be going to zero period t onward, which contradicts $(\text{AIV})^*(\mathcal{F}^*)$ exactly as in the proof of Claim 7 (in the proof of theorem 1).

CLAIM B The ME converge to a finite limit

PROOF This is now obvious.

5.3 Proof of Theorem II $(\mathcal{M})$

This is analogous to the proof of Proof of Theorem II $(\mathcal{F})$, .so we shall just sketch it.

First we show that an ME exists (with Assumption AIV $(\mathcal{M}^*)$ in place of Assumption AIV $(\mathcal{M})$). An ε-ME exists as before. Let $\varepsilon \longrightarrow 0$ (holding *fixed* for now all the M_{tnm} for $\{tnm\} \in \mathcal{M}$.) Note that the total money in the system is bounded in any period by $\overline{m} + \overline{B} + \sum_{\{tnm\}\in\mathcal{M}} M_{tnm}$ plus the vanishingly small amounts injected by the external ε-agent, where (recall) $\overline{m} + \overline{B} = \sum_{t\in T}\sum_{h\in H} m_t^h + \sum_{\{tnm\}\in\mathcal{B}} B_{tnm}$ denotes the aggregate outside money. Since the total interest payment on all the loans in $\mathcal{M}$ cannot exceed $\overline{m} + \overline{B}$ plus the ε-agent's contribution, all interest rates are also bounded as $\varepsilon \longrightarrow 0$. It follows (by the standard arguments done earlier) that all actions are bounded and price ratios are bounded. Now if the spending out of money borrowed on $\mathcal{M}^*$ goes to ∞, then some commodity price before t must go to ∞ and therefore all commodity prices go to ∞. Then the ε-ME allocations converge to $\widetilde{e}$. But since $\widetilde{e} \in \mathcal{A}(t)$, Assumption AIV$(\mathcal{M}^*)$ implies (along the lines of the proof of Theorem I $(\mathcal{M})$) that Assumption AIV $(\mathcal{M}^*)$ is violated in some interval of $[t, T]$. This proves that prices converge to finite limits, and the limit of the ε-ME is an ME.

Now let $M_{tnm} \longrightarrow \infty$ for $\{tnm\} \in \mathcal{M}^*$ (holding fixed the other M_{tnm}) and consider a convergent sequence of ME. As already argued in the previous paragraph, interest rates (therefore price ratios) remain bounded throughout the sequence, as also the money in the economy after period t. Therefore if the money spent, out of money borrowing on $\mathcal{M}^*$ goes to ∞, so will all the commodity prices, with the upshot that the ME allocations will converge to a point in $\mathcal{A}(t)$ (with possible trade before period t but not otherwise). Arguing as in Claim 6 of the proof of Theorem I, this contradicts Assumption AIV $(\mathcal{M}^*)$.

Thus the ME converge to a finite limit.

[38] To see this, note that the strategy σ in (2) allows the imaginary player to abstain from any subset (e.g., $\mathcal{F}^*$) of markets in $\mathcal{F}$ and act only on its complement (e.g.,.$\mathcal{F}\diagdown\mathcal{F}^*$). Thus (2) implies that, if M dollars are borrowed on *any* subset $\mathcal{G} \subset \mathcal{F}$ by the imaginary player, then the default of the player is at least λM on one of the loans in $\mathcal{G}$, *where λ is invariant of the choice of the subset $\mathcal{G}$ of $\mathcal{F}$.*