arXiv:2308.10138v5 [econ.EM] 23 Mar 2024

On the Inconsistency of Cluster-Robust Inference and How Subsampling Can Fix It*

Harold D. Chiang† Yuya Sasaki‡ Yulong Wang§

Abstract

Conventional methods of cluster-robust inference are inconsistent in the presence of unignorably large clusters. We formalize this claim by establishing a necessary and sufficient condition for the consistency of the conventional methods. We find that this condition for the consistency is rejected for a majority of empirical research papers. In this light, we propose a novel score subsampling method that achieves uniform size control over a broad class of data generating processes, covering that fails the conventional method. Simulation studies support these claims. With real data used by an empirical paper, we showcase that the conventional methods conclude significance while our proposed method concludes insignificance.

Keywords: cluster-robust inference, score subsampling, unignorably large cluster

JEL Code: C12, C18, C46

*First arXiv date: August 23, 2023. We benefited from comments and discussions with A. Colin Cameron, Ivan Canay, Bruce Hansen, Seojeong (Jay) Lee, James MacKinnon, Francesca Molinari, Ulrich Müller, Jack Porter, Xiaoxia Shi, Kevin Song, Max Tabord-Meehan, seminar participants at numerous institutions and conferences. We would like to thank Hong Xu and Siyuan Xu for excellent research assistance. All remaining errors are ours.

†Assistant Professor of Economics, University of Wisconsin-Madison. hdchiang@wisc.edu

‡Brian and Charlotte Grove Chair and Professor of Economics, Vanderbilt University. Email: yuya.sasaki@vanderbilt.edu

§Associate Professor of Economics, Syracuse University. Email: ywang402@syr.edu

1 Introduction

Cluster-robust (CR) standard errors account for within-cluster correlations. Such correlations often arise by construction within an industry (Hersch, 1998) or within a state (Bertrand, Duflo, and Mullainathan, 2004), to list a couple of the earliest examples. Today, even if a model may not induce cluster dependence by construction, applying CR methods by observable group identifiers is quite common in practice.

The initial theory (White, 1984; Liang and Zeger, 1986; Arellano, 1987) for CR inference methods assumes small cluster sizes N_g (uniformly bounded above by $\overline{N} < \infty$) under a large number $G \to \infty$ of clusters, where N_g denotes the number of entities in the g-th cluster for $g \in \{1, 2, \ldots, G\}$. The procedure based on this theory is implemented by the '`cluster()`' and '`vce(cluster)`' options by Stata and is used by almost all, if not all, empirical papers that report CR standard errors.

It has been known that a large cluster size N_g could lead to a large CR standard error (e.g., Cameron and Miller, 2015, p. 324). More recent theory (Djogbenou, MacKinnon, and Nielsen, 2019; Hansen and Lee, 2019; Hansen, 2022b) accommodates larger cluster sizes N_g. They no longer require $N_g \leqslant \overline{N}$ and thus widen the scope of applications in which the '`cluster()`' and '`vce(cluster)`' options, among others, work. With this said, they still impose the restriction $\max_g N_g^2/N \to 0$ of vanishing maximum cluster size relative to the whole sample size $N = \sum_{g=1}^{G} N_g$ as $G \to \infty$.

A natural question is whether this relaxed condition $\max_g N_g^2/N \to 0$ accommodates a wide range of data sets. To answer it, we analyze 31 published articles.[1] They all use the aforementioned Stata options for CR standard errors, and hence they implicitly assume $\max_g N_g^2/N \to 0$. Table 1 summarizes the number of articles with $\max_g N_g^2/N$ falling in each of the bins in the logarithmic scale. Observe that 55 percent (respectively, 39, 29 and 16 percents) of them use data sets entailing $\max_g N_g^2/N \geqslant 1$ (respectively, $\geqslant 10$, $\geqslant 100$ and $\geqslant 1000$). In other words, the condition $\max_g N_g^2/N \to 0$ for the validity of the conventional

[1] We studied all the articles published in *American Economic Review* and *Econometrica* between 2020 and 2021. Among them, we extracted a list of those papers that report estimation and inference results based on regressions, IV regressions, and their variants. Furthermore, we focus on those articles that use publicly available data sets for replication. See Section 3 for further details of this study.

The Distribution of $\max_g N_g^2/N$ in Empirical Economic Research: 2020–2021

$\max_g N_g^2/N$	<0.1	0.1–1	1–10	10–100	100–1000	⩾1000
American Economic Review	4	8	4	1	3	1
Econometrica	2	0	1	2	1	4
Total	6	8	5	3	4	5
	(19%)	(26%)	(15%)	(10%)	(13%)	(16%)

Table 1: Number of articles with $\max_g N_g^2/N$ falling in each of the bins $[0, 0.1)$, $[0.1, 1)$, $[1, 10)$, $[10, 100)$, $[100, 1000)$ and $[1000, \infty)$ in the logarithmic scale. The articles are drawn from those papers published in *American Economic Review* and *Econometrica* during the period of 2020-2021. We focus on those papers that report CR standard errors for regression and IV regression estimates with publicly available data sets for replication. For each paper running more than one regression, we take the largest $\max_g N_g^2/N$ among the multiple regressions.

CR inference may well fail for a nontrivial portion of the list of these published articles.

It is important to emphasize that the condition $\max_g N_g^2/N \to 0$ is merely sufficient but not necessary for asymptotic normality to hold, and thus one cannot analyze the adequacy of normality-based confidence intervals and testing results in those papers only based on discussion on the plausibility of this condition. To overcome this difficulty, in this paper, we first formally establish a necessary and sufficient condition under which the conventional CR methods of inference are valid. Specifically, an implication of this necessary and sufficient condition is that the limiting distribution is normal if and only if the score of the largest cluster is ignorable. In the presence of *unignorably large* clusters, regression estimates have non-normal limiting distributions such as those illustrated in Figure 1.[2] Based on this precise characterization of asymptotic normality, we then conduct formal statistical tests based on Sasaki and Wang (2023) to show that the null hypothesis that the limiting distribution being normal is rejected for 24 out of those 31 papers (i.e., 77 percent) reported in Table 1.

Non-normal limiting distributions imply that the conventional critical values, such as 1.96, and bootstrap critical values are invalid. For instance, erroneously using the critical value of 1.96 would result in the sizes of 0.053, 0.087, and 0.250 (as opposed to the desired

[2]We provide details of the non-normal distributions illustrated in this figure in Section 4.2 after presenting our formal theory.

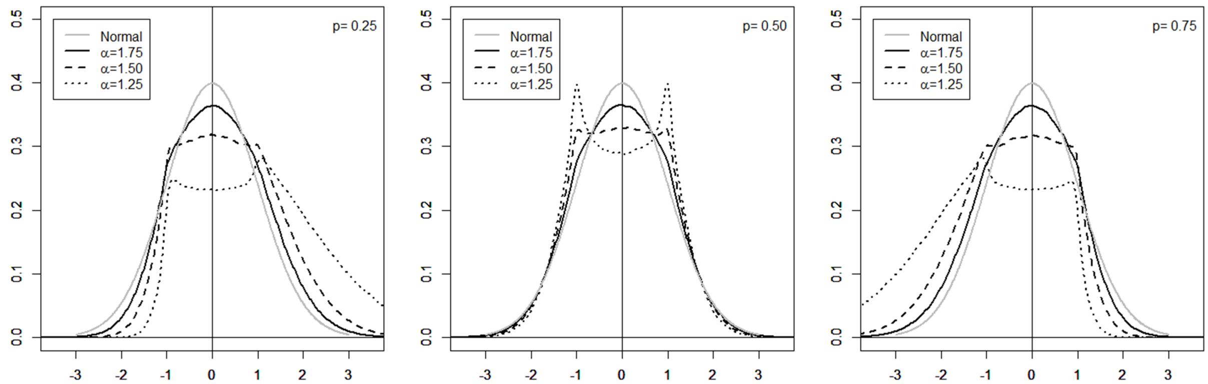

Figure 1: Illustration of non-normal limiting distributions in the presence of unignorably large clusters. Details about the different shapes indexed by α and p are provided in Section 4.2.

size of 0.050) when the limiting distribution has nuisance parameter α (to be defined in Section 2) takes value of $\alpha = 1.75$, 1.50, and 1.25, respectively, in the left panel of Figure 1. The empirical bootstrap also fails in these scenarios where the score has an infinite variance. Furthermore, we show that the popular wild cluster bootstrap is inconsistent in these scenarios as well. In light of this negative discovery, one may ask how to fix the problem. We propose a novel score subsampling for clustered data, and formally show that it yields correct critical values regardless of whether the limiting distribution is normal or not. In particular, this new method robustly delivers valid inference under any of the limiting distributions illustrated in Figure 1 in an adaptive manner.

Moreover, we demonstrate the adaptability of the proposed score subsampling inference procedure to asymptotic scenarios featuring both normal and non-normal limiting distributions. This adaptability is established reliably through the affirmation of a uniform size control property across an extensive array of models. Notably, this encompasses data-generating processes involving both ignorable and notably large clusters. To underpin this uniform validity, we introduce a novel convergence in distribution result for row-wise independently and identically distributed (i.i.d.) triangular arrays. These arrays exhibit heavy tails characterized by varying yet converging tail exponents, adding a layer of theoretical interest. To our knowledge, this is the first theoretical result on the uniformity property of subsampling for statistical models with potentially infinite variance. Finally, our simulation studies support our theoretical findings.

Related Literature: The literature of cluster robust inference has a long history dating back to White (1984), Liang and Zeger (1986), and Arellano (1987). For a thorough review of the literature, we refer the readers to Cameron and Miller (2015) and MacKinnon, Nielsen, and Webb (2023). The sampling frameworks in which cluster sizes are treated as a random variable have been recently investigated by Bugni, Canay, Shaikh, and Tabord-Meehan (2022), Cavaliere, Mikosch, Rahbek, and Vilandt (2024), and Bai, Liu, Shaikh, and Tabord-Meehan (2022). We consider a model-based perspective with an increasing number of clusters and unrestricted intra-cluster dependence, as the vast majority of the papers did in this literature. An alternative framework is a fixed number of clusters with growing cluster sizes and manage to derive asymptotic normality under some extra assumptions on weak intra-cluster cluster dependence following Canay, Santos, and Shaikh (2021), as well as design-based asymptotics [3] under some stronger treatment assignment rules, such as randomized experiments, considered by Abadie, Athey, Imbens, and Wooldridge (2023).

In an insightful recent work, Kojevnikov and Song (2023) obtain an impossibility result on consistent estimation for asymptotic variance when there is only a single large cluster in the sample under a triangular array setup. They also provide a necessary and sufficient condition for the cluster structure that the asymptotic variance is consistently estimable. Our findings complement their impossibility result of normal approximation for t-statistics in the presence of unignorably large clusters. Our proposed procedure overcomes such an impossibility as it does not require consistent estimation of the variance. Indeed, we show that the variance estimator, after normalization by an unknown rate, is convergent in distribution and formally derive its limiting stable distribution in such scenarios. In addition, such unknown rate is not necessary for the implementation of the proposed score subsampling inference procedure, due to the self-normalizing nature of the test statistics.

Our key distributional approximation results for the self-normalized sums are due to Logan, Mallows, Rice, and Shepp (1973), LePage, Woodroofe, and Zinn (1981), and Giné,

[3] See Reichardt and Gollob (1999) for an in-depth philosophical discussion on the model-based versus design-based perspectives.

Götze, and Mason (1997). For theoretical details of the underlying foundations of probability and statistics for heavy-tailed distributions, we refer the reader to Resnick (1987) and Resnick (2007). Our uniformity result relies on the general uniformity theory for subsampling studied in Romano and Shaikh (2012). For the failure of empirical bootstrap for means of random variables with infinite variances, see, e.g., Athreya (1987), Arcones and Giné (1989), and Knight (1989). Our inference procedure relies on the theory of resampling method developed in Politis and Romano (1994) and Romano and Wolf (1999). Also, see Politis, Romano, and Wolf (1999) for a comprehensive treatment.

2 The Model

While the idea extends to a general class of econometric models, we consider the linear model

$$Y_{gi} = X_{gi}'\theta + U_{gi} \qquad \mathbb{E}[U_g|X_g] = 0$$

for ease of exposition, where $X_g = (X_{g1}, \ldots, X_{gN_g})'$, $U_g = (U_{g1}, \ldots, U_{gN_g})'$, $g \in \{1, \ldots, G\}$ indexes clusters, and N_g denotes the size of the g-th cluster. Define the OLS estimator and its cluster-robust (CR) variance estimator by

$$\widehat{\theta} = \left(\sum_{g=1}^{G} \sum_{i=1}^{N_g} X_{gi} X_{gi}' \right)^{-1} \sum_{g=1}^{G} \sum_{i=1}^{N_g} X_{gi} Y_{gi} = \left(\sum_{g=1}^{G} X_g' X_g \right)^{-1} \sum_{g=1}^{G} (X_g' X_g \theta + S_g) \quad \text{and}$$
$$\widehat{V}^{\text{CR}} = a_G \left(\sum_{g=1}^{G} X_g' X_g \right)^{-1} \left(\sum_{g=1}^{G} \widehat{S}_g \widehat{S}_g' \right) \left(\sum_{g=1}^{G} X_g' X_g \right)^{-1},$$

respectively, for some finite sample adjustment factor a_G such that $a_G \to 1$ as $G \to \infty$, where $S_g = \sum_{i=1}^{N_g} X_{gi} U_{gi}$, $\widehat{S}_g = \sum_{i=1}^{N_g} X_{gi} \widehat{U}_{gi}$, and $\widehat{U}_{gi} = Y_{gi} - X_{gi}'\widehat{\theta}$. For simplicity of writing, we set $a_G = 1$ throughout.

Consider a linear transformation $\delta = r'\theta$, such that $r \in \mathbb{R}^{\dim(\theta)}$ and $\|r\| = 1$, as the parameter of interest. Let the corresponding estimator and its CR standard error be denoted by

$$\begin{aligned}\widehat{\delta} =& r'\widehat{\theta} \quad \text{and} \\ \widehat{\sigma}^2 =& r'\left(\sum_{g=1}^{G} X_g' X_g\right)^{-1}\left(\sum_{g=1}^{G} \widehat{S}_g \widehat{S}_g'\right)\left(\sum_{g=1}^{G} X_g' X_g\right)^{-1} r,\end{aligned}$$

respectively. We are interested in conducting inference for δ using the t-statistic

$$\frac{(\widehat{\delta}-\delta)}{\widehat{\sigma}} = \frac{r'(\widehat{\theta}-\theta)}{\sqrt{r'\left(\sum_{g=1}^{G} X_g' X_g\right)^{-1}\left(\sum_{g=1}^{G} \widehat{S}_g \widehat{S}_g'\right)\left(\sum_{g=1}^{G} X_g' X_g\right)^{-1} r}} \tag{2.1}$$

based on the CR standard error.

To state our model assumption, we introduce a few definitions. A random variable η is said to be *stable* if it has a domain of attraction in that there exists a sequence of i.i.d. random variables $\xi_1, \xi_2, \ldots$ and sequences of positive numbers A_G and real numbers D_G such that

$$\frac{\sum_{g=1}^{G} \xi_g - D_G}{A_G} \xrightarrow{d} \eta \qquad \text{as } G \to \infty.$$

A function $L(\cdot)$ is said to be *slowly varying* at ∞ if $\lim_{t\to\infty} L(yt)/L(t) = 1$ for all $y > 0$. If η is stable, then A_G takes the form of $G^{1/\alpha}L(G)$ for some $\alpha \in (0,2]$ and some slowly varying function $L(\cdot)$ at ∞ (cf. Proposition 2.2.13 in Embrechts, Klüppelberg, and Mikosch 1997). If $\alpha \in (1,2]$, then D_G can be chosen to be $G \cdot \mathbb{E}[\xi_g]$. The number α is called the *index of stability*, and η is said to be *α-stable*. In such a case, ξ_g is said to belong to the *domain of attraction* of an *α-stable distribution*. Although this concept may look esoteric to some readers, it essentially states that a sum of i.i.d. random variables, after being suitably centered and normalized, converges in distribution to a limiting random variable, and it, in particular, encompasses the standard cases where central limit theorems (CLTs) hold. We shall focus on $\alpha \in (1,2]$, since even the first moment of ξ_g would not be well-defined otherwise.

Assumption 1. $(X_g' X_g, S_g)_{g=1}^G$ are i.i.d., $\mathbb{E}[N_g] = c \in (0, \infty)$, and the design matrix satisfies

$$\frac{1}{G} \sum_{g=1}^{G} X_g' X_g = Q + o_p(1)$$

for a finite positive definite matrix Q. For $v = r'Q^{-1}$ and for all $u_1, u_2 \in \mathbb{R}^{\dim(\theta)}$ with unit length, $v'S_g$ and $u_1' X_g' X_g u_2$ belong to the domain of attraction of stable laws with an index of stability $\alpha \in (1, 2]$.

Assumption 1 accommodates a broad class of both standard and non-standard cases considered in econometrics. First, the case of $\alpha = 2$ encompasses the conventional assumption under which $r'(\widehat{\theta} - \theta)$ enjoys the standard convergence rate of $\sqrt{G}$ through CLTs. In this case, the limiting α-stable distribution must be normal (cf. Geluk and de Haan, 2000, Theorem 2). Furthermore, it also covers some non-standard cases with a normal limiting distribution but without a finite variance, e.g., a Pareto random variable with the shape parameter (Pareto exponent) of 2.

Second, on the other hand, the case of $\alpha < 2$ entails the power law (de la Peña, Lai, and Shao, 2009, Theorem 2.24), i.e.,

$$P(|v'S_g| > t) = t^{-\alpha} L_1(t) \qquad \text{and} \qquad P(|u_1 X_g' X_g u_2| > t) = t^{-\alpha} L_2(t) \tag{2.2}$$

for some slowly varying functions, $L_1(\cdot)$ and $L_2(\cdot)$, where $L_2(\cdot)$ may depend on u_1 and u_2. In this case of $\alpha < 2$, the index α of stability coincides with the tail exponent α_T in the sense that $\alpha = \min\{\alpha_T, 2\}$. Thus, the case of $\alpha < 2$ implies infinite variance of the score. See Theorem 4 in Appendix A.1 for more precise details. In this case, *unignorably large* clusters are literally unignorable because the sample sum of the (scaled) scores becomes asymptotically proportional to the (scaled) score of the largest cluster - see Remark 4 in Appendix A.2 for more discussions. Hence, the asymptotic distribution cannot be normal.

To simplify the writings, we focus on the case where $v'S_g$ and $u_1' X_g' X_g u_2$ share the common index α of stability. This simple setting is rationalized if the tail shape of their distributions are driven by the tail shape of the distribution of cluster sizes N_g – see Sasaki and Wang (2022, Theorem 1). With this said, we emphasize that this setting can be relaxed only at the cost of more cumbersome writing.

The i.i.d. requirement in Assumption 1 is standard in this literature, (cf. Bugni et al. 2022; Cavaliere et al. 2024; Bai et al. 2022). It is mild because 1) the conditional distributions of S_g and $X_g' X_g$ given $N_g = n_g$ can be heterogeneous across n_g; and 2) the distributions of individuals within each cluster can be non-identical. In addition, S_g and X_g can be arbitrarily correlated with the cluster size N_g so long as the exogeneity condition for the regression is respected.

3 Fragility of the Conventional CR Methods

In this section, we argue that the conventional methods of CR inference work if and only if $\alpha = 2$. In other words, they are doomed to fail if $\alpha < 2$. We then discuss how often researchers encounter cases with $\alpha < 2$ in empirical economic studies.

Theorem 1 (Necessary and sufficient condition)**.** *Suppose that Assumption 1 is satisfied for an* $\alpha \in (1, 2]$*, then the t-statistic* (2.1) *is asymptotically Gaussian if and only if* $\alpha = 2$*.*

A proof is found in Appendix B.3. This theorem implies that the conventional inference based on the common variance estimators, such as CR1, CR2, CR3, and jackknife, together with the Gaussian critical values (e.g., ≈1.96 for the 97.5-th percentile) fails if $\alpha < 2$.

There is certainly another class of conventional approaches, namely the bootstraps. However, it is well established that the empirical bootstrap is inconsistent when the variance of the score is infinite (cf. Athreya, 1987; Knight, 1989). In light of the power law characterization (2.2), therefore, the empirical cluster bootstrap, also known as the pairs cluster bootstrap, is inconsistent under Assumption 1 with $\alpha < 2$. Furthermore, we formally show in Appendix A.3 that the wild cluster bootstrap is also inconsistent under Assumption 1 with $\alpha < 2$.

Provided that the case of $\alpha < 2$ fails all these conventional methods of CR inference, our natural question now is how common it is to encounter $\alpha < 2$ in empirical studies in economics. We analyzed all the articles published in a couple of journals (*American Economic Review* and *Econometrica*) between 2020 and 2021. Among them, we extracted a list of those papers that report estimation and inference results based on regressions, IV regressions, and their variants. Furthermore, we focus on those articles that use publicly available data sets for replication.

For these articles, we test the null hypothesis $H_0 : \alpha = 2$ against the alternative hypothesis $H_1 : \alpha < 2$ for the score. Such a test can be conducted via the likelihood ratio test (Sasaki and Wang, 2023) of the surrogate null hypothesis $H_0 : \alpha_T \geqslant 2$ against the alternative $H_1 : \alpha_T < 2$ in light of (2.2), where α_T denotes the tail exponent of the score.[4]

Table 2 summarizes the list of all the papers we studied. The first two columns list the journals and years of publication. The following column "All #" indicates the total number of eligible articles according to the above selection criteria. The column group under "Cluster" collects articles in which CR inference is used for at least one regression result. Under this column group, the column "#" shows the numbers of articles, and the column "Test $\alpha < 2$" shows the fractions of those articles for which the test rejects the null hypothesis for at least one regression specification. The final row displays the summary of each column.

During 2020–2021, *American Economic Review* published 30 articles meeting our selection criteria. Out of them, 21 articles report CR standard errors. We reject the null hypothesis for 16 of these 21 articles. In other words, the inference results may be misleading for 76% of those articles that employ the conventional CR method of inference.

During 2020–2021, *Econometrica* published 14 articles meeting our selection criteria. Out of them, 10 articles report CR standard errors. We reject the null hypothesis for 8 of these 9 articles. In other words, the inference results may be misleading for 80% of those articles that employ the conventional CR method of inference.

Combining the two journals together, we suspect potentially misleading inference results for as many as 77% of those articles that employ the conventional CR method. Hence, problematic practice is prevalent even in these highly influential journals.[5]

[4] The test of $H_0 : \alpha_T \geqslant 2$ against $H_1 : \alpha_T < 2$ is implemented with the Stata command "`testout y x1 x2 ..., cluster(cid)`" for regressions and "`testout y x1 x2 ..., iv(z) cluster(cid)`" for IV regressions based on Sasaki and Wang (2023).

[5] Spreadsheets of all the test results with specific papers and specific equations are available upon request from the authors under certain conditions.

Journal	Year of Publication	All #	Cluster #	Test $\alpha < 2$	
American Economic Review	2020	15	10	7/10	(70%)
American Economic Review	2021	15	11	9/11	(82%)
	Subtotal	30	21	16/21	(76%)
Econometrica	2020	12	7	7/8	(88%)
Econometrica	2021	3	2	1/2	(50%)
	Subtotal	15	10	8/10	(80%)
	Total	45	31	24/31	(77%)

Table 2: The column "All – #" indicates the total number of eligible articles that use regressions or IV regressions with publicly available data for replication. The column "Cluster – #" indicates the number of the eligible articles that use CR inference. The column "Cluster – Test $\alpha < 2$" indicates the rate of rejecting the null hypothesis $\alpha = 2$ among those articles that use CR inference. The tests of the null hypothesis $\alpha = 2$ against the alternative hypothesis $\alpha < 2$ is implemented with the Stata command "`testout y x1 x2 ..., cluster(cid)`" for regressions and "`testout y x1 x2 ..., iv(z) cluster(cid)`" for IV regressions based on Sasaki and Wang (2023).

4 Score Subsampling as a Reliable Solution

In light of the issue with the conventional methods of CR inference reported in the previous section, we now propose a novel method of score subsampling as a reliable solution. We first present the proposed method without theoretical details in Section 4.1. Its theoretical support follows in Section 4.2. Finally, Section A.4 presents practical details.

4.1 The Method

Our objective is to conduct statistical inference for δ using the self-normalized t-statistic (2.1). Let the CDF J_G^* of the sampling distribution of the t-statistic be given by

$$J_G^*(t) = P\left((\widehat{\delta} - \delta)/\widehat{\sigma} \leqslant t\right).$$

We will show that it converges to the CDF J^* of a limiting distribution under suitable conditions. Consider a sequence of subsample sizes $b = b_G$ that grows with $b/G = o(1)$ as

$G \to \infty$. Let $B_G = \binom{G}{b}$ denote the total possible number of subsamples of b clusters. For a given b and $j \in \{1, ..., B_G\}$, let $S_j \subset \{1, .., G\}$ be one of the B_G subsamples of the cluster indices with $|S_j| = b$, and define the score-subsampled estimators

$$\widehat{\delta}_{b,j} = r'\widehat{\theta}_{b,j} = \left(\frac{G}{b}\right) r' \left(\sum_{g=1}^{G} X_g' X_g\right)^{-1} \sum_{g \in S_j} X_g' Y_g \qquad \text{and}$$

$$\widehat{\sigma}^2_{b,j} = \left(\frac{G}{b}\right)^2 r' \left(\sum_{g=1}^{G} X_g' X_g\right)^{-1} \left(\sum_{g \in S_j} \widehat{S}_{g,j} \widehat{S}'_{g,j}\right) \left(\sum_{g=1}^{G} X_g' X_g\right)^{-1} r,$$

where $\widehat{S}_{g,j} = X_g'(Y_g - X_g \widehat{\theta}_{b,j})$. Observe that the inverse factor $(\sum_{g=1}^{G} X_g' X_g)^{-1}$ is calculated based on the full sample while the linear component and its variance are computed based on the subsample S_j. We discuss practical motivations for this feature in Remark 2 below.

Define the empirical CDF $L^*_{G,b}$ of $(\widehat{\delta}_{b,j} - \widehat{\delta})/\widehat{\sigma}_{b,j}$ based on all possible B_G-subsamples by

$$L^*_{G,b}(t) = \frac{1}{B_G} \sum_{j=1}^{B_G} \mathbb{1}\left((\widehat{\delta}_{b,j} - \widehat{\delta})/\widehat{\sigma}_{b,j} \leqslant t\right).$$

It will be shown that J^* can be approximated by $L^*_{G,b}$ uniformly as the number G of clusters grows under suitable conditions. In practice, however, $L^*_{G,b}$ is computationally infeasible when G and b are both large. Thus, we randomly draw M such subsamples of b clusters with replacement, and define

$$\widehat{L}_{G,b}(t) = \frac{1}{M} \sum_{j=1}^{M} \mathbb{1}\left((\widehat{\delta}_{b,j} - \widehat{\delta})/\widehat{\sigma}_{b,j} \leqslant t\right).$$

As M grows with the number G of clusters, this $\widehat{L}_{G,b}$ can be used in place of $L^*_{G,b}$.

For any $a \in (0, 1)$, define the critical value

$$\widehat{c}_{G,b}(1-a) = \inf\left\{t \in \mathbb{R} : \widehat{L}_{G,b}(t) \geqslant 1 - a\right\}.$$

Since $J^*(\cdot)$ has no point mass as we shall show, it follows that

$$P\left((\widehat{\delta} - \delta)/\widehat{\sigma} \leqslant \widehat{c}_{G,b}(1-a)\right) \to 1 - a$$

as $G \to \infty$. Therefore, this critical value leads to theoretically valid tests. In addition, a confidence region can be obtained by test-inversion.

Practical Implication: For the estimate $\widehat{\delta}$, one can continue to use the conventional CR "standard error" $\widehat{\sigma}$.[6] However, instead of using the conventional critical values, $\Phi^{-1}(0.025) \approx -1.96$ and $\Phi^{-1}(0.975) \approx 1.96$, one should use $\widehat{c}_{G,b}(0.025)$ and $\widehat{c}_{G,b}(0.975)$ obtained by the score subsampling to construct a 95% confidence interval for example.

Remark 1 (Practicality of the method)**.** The convergence rate of $\widehat{\delta} - \delta$ is unknown, but the inference is robust to the unknown rate due to the use of the self-normalized statistic. In particular, this implies that our inference procedure does *not* require an estimation of the unknown tail exponent. Furthermore, it is not necessary to estimate the unknown slowly varying function either. These features are practical advantages of our proposed method. ▲

Remark 2 (Finite sample non-invertibility of other cluster-based resampling methods)**.** In comparison with the (conventional) subsampling, the score subsampling has two major advantages. First, as it does not require to recompute the inverse factor for each subsample, the score subsampling is computationally more efficient than the subsampling. Second, in finite samples, when regressors contain a cluster-specific binary treatment variable or other dummies variables that are highly correlated within a cluster, $\sum_{g \in S_j} X_g' X_g$ can be often singular especially for small $b = |S_j|$, and thus the subsampled OLS may not behave well for a non-negligible proportion of subsamples. This issue is also faced by other cluster-based resampling methods, such as the jackknife and bootstrap. In practice, several *ad hoc* 'fixes,' such as the use of generalized inverse or dropping such realizations, are employed. However, their theoretical implications remain unclear. Our cluster-robust score subsampling procedure avoids such an issue in a theoretically supported manner. ▲

4.2 Theoretical Properties

The following theorem formally justifies that the subsampling method of inference presented in Section 4.1 is asymptotically valid even if $\alpha < 2$, as well as when $\alpha = 2$.

[6] Note that the "standard error" $\widehat{\sigma}$ does not converge in probability when $\alpha < 2$.

Theorem 2 (Cluster robust inference by score subsampling)**.** *Suppose that Assumption 1 is satisfied for $\alpha \in (1, 2]$. If $b \to \infty$ and $b/G = o(1)$ as $G \to \infty$, then*

$$\sup_{t \in \mathbb{R}} |L^*_{G,b}(t) - J^*(t)| \xrightarrow{p} 0$$

and the limiting distribution $J^*(\cdot)$ is continuous. In addition, if $M \to \infty$, then

$$\sup_{t \in \mathbb{R}} |\widehat{L}_{G,b}(t) - J^*(t)| \xrightarrow{p} 0,$$

and thus

$$P\left((\widehat{\delta} - \delta)/\widehat{\sigma} \leqslant \widehat{c}_{G,b}(1-a)\right) \to 1 - a.$$

The proof branches into two cases. First, we focus on the pathological case with $\alpha < 2$. The statement for this case is presented as Lemma 1 in Appendix A.2, which is proved in Appendix B.1. Appendix B.2 proves the statement for the case with $\alpha = 2$, and combines the two cases to establish Theorem 2.

The limiting distribution, which is approximated by our proposed method of score subsampling, is not unique. It varies with the values of two parameters. One is the index α of stability, and the other is

$$p = \lim_{t \to \infty} \frac{P\left(v' S_g > t\right)}{P\left(|v' S_g| > t\right)},$$

where v is defined in Assumption 1. See the proof of Lemma 1 in Appendix B.1 for details. This second parameter p measures the tail asymmetry of the distribution of $v' S_g$. One need not know the true limiting distribution in practice as the score subsampling can be used to approximate it. With that said, we can now discuss Figure 1 shown in the introductory section. Namely, the left, middle, and right panels of Figure 1 illustrate the limiting distributions under p = 0.25, 0.50, and 0.75, respectively. In each of these three panels, three non-normal limiting distributions under α = 1.25, 1.50, and 1.75 are drawn with distinct line styles, along with the normal reference (α = 2.00). The main takeaway is that the conventional CR inference based on the normal approximation becomes less precise as α becomes smaller and p deviates from 0.5.

Choice of b: We close this section with discussions on the choice of b in practice. While the theory requires $b \to \infty$ and $b/G = o(1)$ as $G \to \infty$, a researcher needs to choose some value of b in a finite sample. We suggest to adapt the minimum volatility method (Algorithm 9.3.3 in Politis et al., 1999, Section 9.3.2) to our framework. Appendix A.4 provides a detailed algorithmic procedure that a practitioner can readily implement. We also employ this method to choose b in the numerical studies presented below.

4.3 Uniformity

So far all the asymptotic properties considered are derived for a fixed data-generating process (DGP). We now discuss the uniformity properties of the proposed score subsampling for cluster robust inference. To simplify the notations and assumptions, we focus on inference for the mean of a scalar random variable in the current subsection. Consider a triangular array setup: for each $G \in \mathbb{N}$, suppose that we have an i.i.d. sequence $(S_g)_{g=1}^G = (S_{g,G})_{g=1}^G$, whose distribution is now $P = P_G$. Recall that

$$\widehat{\delta} - \delta = \frac{1}{G}\sum_{g=1}^{G} S_g \quad \text{and} \quad \widehat{\sigma}^2 = \frac{1}{G}\sum_{g=1}^{G} \widehat{S}_g^2,$$

where $\widehat{S}_g = S_g - G^{-1}\sum_{g=1}^G S_g$. The test statistic of interest is again the t-ratio $(\widehat{\delta} - \delta)/\widehat{\sigma}$.

Henceforth, we will let $\mathbb{E}_P[\cdot]$ denote the expectation with respect to the DGP, P, if we are to emphasize such a dependence. For any $\varepsilon \in [0, 1)$, define $\mathbf{P}_1(\varepsilon)$ as the set of all the DGPs, P, such that $\mathbb{E}_P[S_g] = 0$, and there exist some $p \in [0, 1]$ and $\alpha \in [1 + \varepsilon, 2)$ such that

$$\lim_{t\to\infty} \frac{P(S_g > t)}{P(|S_g| > t)} = p, \quad \text{and} \tag{4.1}$$

$$P(|S_g| > t) = t^{-\alpha} L_P(t) \quad \text{as } t \to \infty \tag{4.2}$$

for an $L_P(\cdot)$ slowly varying at ∞ that can depend on $P = P_G$. In addition, define $\mathbf{P}_2$ as the set of all DGPs satisfying $\mathbb{E}_P[S_g] = 0$ and the following uniform integrability condition

$$\lim_{\lambda\to\infty} \sup_{P\in\mathbf{P}_2} \mathbb{E}_P\left[\frac{|S_g - \mathbb{E}_P[S_g]|^2}{\sigma^2(P)}\mathbb{1}\left\{\frac{|S_g - \mathbb{E}_P[S_g]|}{\sigma(P)} > \lambda\right\}\right] = 0,$$

where $\sigma^2(P) = \mathbb{E}_P[S_g^2]$ is finite. Finally, define $\mathbf{P}(\varepsilon) = \mathbf{P}_1(\varepsilon) \cup \mathbf{P}_2$. The first set $\mathbf{P}_1(\varepsilon)$ covers the DGPs with heavy tail distributions and with regularly varying tail probabilities so that the variances of S_g are infinite. The second set $\mathbf{P}_2$ covers a rich subset of DGPs in which the variances of S_g are always finite and contains, in particular, the set of DGPs with $2 + \epsilon$ moments for any $\epsilon > 0$. It rules out certain examples such as those in the classical Bahadur-Savage example under which the t-test fails its size control for every sample size; see Romano (2004) for more details.

First, we note that when $\alpha = 1$, the t-ratio does not converge in distribution in general, except in very special situations. The following is a direct implication of Logan et al. (1973, p. 790).

Proposition 1. *When $\alpha = 1$ in Equation (4.2), the t-ratio $(\widehat{\delta} - \delta)/\widehat{\sigma}$ converges weakly to a nondegenerate limiting distribution only if S_g follows a (translation of) Cauchy distribution. Hence, no confidence set constructed using quantiles of the asymptotic distribution of the t-ratio can achieve uniform size control over $\mathbf{P}(0)$.*

Nonetheless, we show a next best result holds true: our proposed cluster score subsampling inference controls size uniformly over the set $\mathbf{P}(\varepsilon)$ if $\varepsilon > 0$. Note that the score subsampling coincides with (conventional) subsampling for sample means. Denote

$$L_G(x, P) = \frac{1}{B_G}\sum_{j=1}^{B_G} \mathbb{1}\left\{\frac{\widehat{\delta}_{b,j} - \delta}{\widehat{\sigma}_{b,j}} \leqslant x\right\}, \quad \widehat{L}_G(x) = \frac{1}{B_G}\sum_{j=1}^{B_G} \mathbb{1}\left\{\frac{\widehat{\delta}_{b,j} - \widehat{\delta}}{\widehat{\sigma}_{b,j}} \leqslant x\right\}.$$

Further, let the a-th quantile of $\widehat{L}_G(\cdot)$ be denoted by $\widehat{L}_G^{-1}(a)$.

Theorem 3 (Uniformity of the cluster score subsampling)**.** *For any $\varepsilon \in (0, 1]$, the confidence sets constructed based on cluster score subsampling achieves asymptotically uniform size control over $\mathbf{P}(\varepsilon)$. Explicitly, for any nonnegative a_1 and a_2 such that $0 \leqslant a_1 + a_2 < 1$, we* *have*

$$\lim_{G\to\infty} \inf_{P\in\mathbf{P}} P\left(\widehat{L}_G^{-1}(a_1) \leqslant \frac{\widehat{\delta}-\delta}{\widehat{\sigma}} \leqslant \widehat{L}_G^{-1}(1-a_2)\right) = 1 - a_1 - a_2.$$

A proof can be found in Appendix B.5. The proof utilizes the general results in Romano and Shaikh (2012) under high-level conditions together with our Lemma 2 in Appendix B.5. This new lemma establishes a novel convergence in distribution result for row-wise i.i.d. triangular arrays. Specifically, we consider the sequence of indices $\alpha_G \to \alpha_0 \in [1+\varepsilon, 2]$ as $G \to \infty$, covering the cases with both normal ($\alpha_0 = 2$) and non-normal ($\alpha_0 < 2$) limiting distributions. Recall that the t-test is not uniformly valid over the set of all DGPs with finite second moments, while it controls size uniformly over the set of all DGPs with finite $2 + \epsilon$ moments for any $\epsilon > 0$ (see e.g. Romano 2004). Our result with $\mathbf{P}(\varepsilon)$ for all $\varepsilon > 0$ is analogous to this classic result, although it extends the scope of uniformity to a much larger class of DGPs with potentially infinite second moments and non-Gaussian limiting distributions.

Finally, it is noteworthy that our uniform size control property exhibits resemblances to certain instances in the existing literature. An example is the AR(1) model presented in Example 1 of Andrews, Cheng, and Guggenberger (2020), where uniform size control persists across DGPs leading to either normal or non-normal limiting distributions. In that example, Andrews et al. (2020) demonstrate the continuity of their limiting distribution in a local parameter h throughout its support, akin to the role served by our nuisance parameters (α, p) in our asymptotic theory. Notably, while infinite variance poses no hindrance in Andrews et al. (2020), its presence significantly complicates the analytical framework within our study. To the best of our knowledge, Theorem 3 stands as the first theoretical result addressing the uniformity property of subsampling for statistical models that may exhibit potentially infinite variance.

5 Simulations

In this section, we present simulation studies to evaluate the finite sample performance of our proposed score subsampling method of CR inference in comparison with the conventional CR methods.

The data-generating design is defined as follows. We consider the cluster treatment model with individual covariates

$$Y_{gi} = \theta_0 + \theta_1 T_g + \sum_{j=1}^{K} \theta_j X_{g,i,j+1} + U_{gi}$$

following MacKinnon, Nielsen, and Webb (2022, Equation (40)) among others. The binary treatment variable T_g takes the value of one for $\lceil 0.2G \rceil$ clusters and zero for the remaining clusters $G - \lceil 0.2G \rceil$, where $\lceil a \rceil$ denotes the smallest integer greater than or equal to a. We draw cluster sizes $N_g \sim \lceil \text{Pareto}(1, \alpha) \rceil$ independently for $g \in \{1, \cdots, G\}$. For each $g \in \{1, \cdots, G\}$, we independently draw N_g-variate random vectors, $(\widetilde{X}_{g1j}, \cdots, \widetilde{X}_{gN_g j})' \sim N(0, \Omega)$ for $j \in \{1, \cdots, K\}$ and $(\widetilde{U}_{g1}, \cdots, \widetilde{U}_{gN_g})' \sim N(0, \Omega)$ in the baseline design, where Ω is an $N_g \times N_g$ variance-covariance matrix such that $\Omega_{ii} = 1$ for all $i \in \{1, \cdots, N_g\}$ and $\Omega_{ii'} = 1/2$ whenever $i \neq i'$. The controls are constructed by $X_{gij} = 0.2 F^{-1}_{\text{Beta}(2,2)} \circ \Phi(\widetilde{X}_{gij})$, where $F_{\text{Beta}(2,2)}$ and Φ denote the CDFs of the Beta$(2, 2)$ and standard normal distributions, respectively. The errors are heteroskedastically constructed by $U_{gi} = 0.2\widetilde{U}_{gi}$ if $T_g = 0$ and $U_{gi} = \widetilde{U}_{gi}$ if $T_g = 1$.

We vary values of the exponent parameter $\alpha \in \{1.1, 1.2, \cdots, 1.9, 2.0\}$ across sets of simulations. The regression coefficients are fixed to $(\theta_0, \theta_1, \theta_2, \cdots, \theta_{K+1})' = (1, 1, 1, \cdots, 1)'$ throughout, whereas the dimension K of covariates vary as $K \in \{0, 5, 10\}$. We set the sample size (i.e., the number of clusters) to $G = 50$ across sets of simulations, which is close to the number of states in the U.S. we discussed as an example earlier. Each set of simulations consists of 5,000 Monte Carlo iterations.

Figure 2 illustrates the Monte Carlo coverage frequencies. The horizontal axis measures the value of α, and the vertical axis measures the coverage frequency. In the legend, 'SUB' (respectively, 'WCB', 'JACK' and 'CR1') indicates the score subsampling (respectively, wild cluster bootstrap, jackknife standard error with normal critical value, and CR1 standard

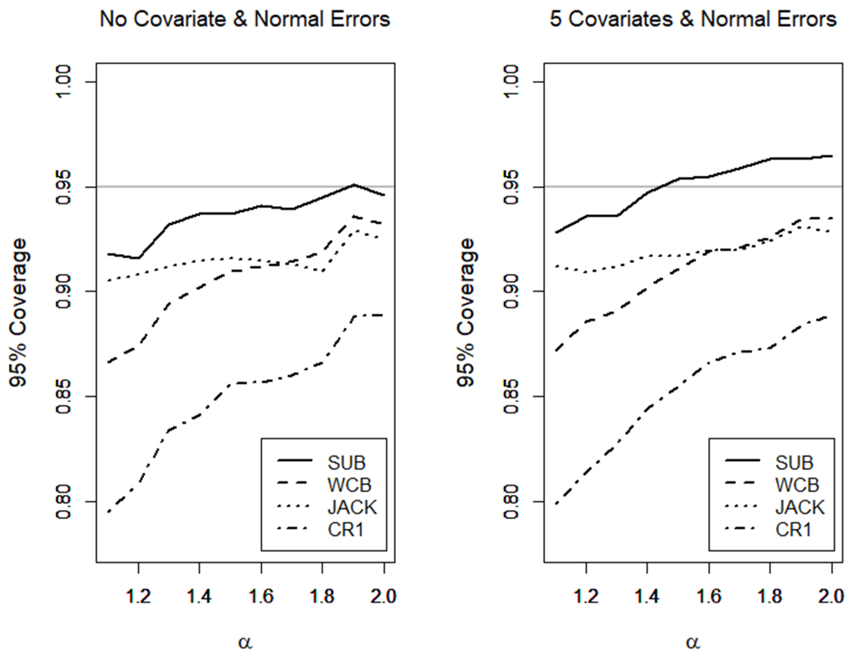

Figure 2: Monte Carlo coverage frequencies for the baseline design with normal errors. 'SUB' (respectively, 'WCB', 'JACK' and 'CR1') indicates the score subsampling (respectively, wild cluster bootstrap, jackknife standard error with normal critical value, and CR1 standard error with normal critical value). The nominal coverage probability of 95% is indicated by the horizontal gray line.

error with normal critical value). The nominal coverage probability of 95% is indicated by the horizontal gray line at 0.95.

When α is small, say $\alpha < 1.6$, the score subsampling performs the best, followed by the jackknife, the WCB, and the CR1. When α is larger, say $\alpha > 1.6$, the score subsampling still performs the best, followed by the WCB, the jackknife, and the CR1. Overall, the score subsampling robustly yields the coverage frequencies closest to the nominal probability of 95% across various values of α. All the conventional methods suffer from sever under-coverage especially for small values of α.

6 An Empirical Illustration

Akhtari, Moreira, and Trucco (2022) study the effects of political turnover on a range of outcomes measuring the quality of public services in Brazil. In the original paper (Table 3, Column 5), they estimate the following linear model by the OLS:

$$\text{score}_{gi+1} = \theta_0 + \theta_1 \mathbb{1}\{\text{IVM}_g < 0\} + \theta_2 \text{IVM}_g + \theta_3 \mathbb{1}\{\text{IVM}_g < 0\}\text{IVM}_g + \theta_4 \text{score}_{gi} + U_{gi}.$$

The dependent variable score_{gi+1} is the test score of fourth-grade students in the next year after an election. The main explanatory variable IVM_g is the incumbent vote margin. Thus, $\mathbb{1}\{\text{IVM}_g < 0\}$ equals 1 when the incumbent party loses the election. The parameter of interest is $\delta = \theta_1$, which measures the effects of political turnover on the test score.[7] While the original paper considers alternative bandwidths, we focus on the bandwidth 0.110 for the maximum sample size following the prior work (MacKinnon et al., 2022, Section 7.2) which replicates the above regression.

The original paper clusters the standard errors at the municipality level, and we follow this definition of a cluster unit. There are $G = 2101$ municipalities in the data, and $\max_{1 \leqslant g \leqslant G} N_g^2/N \approx 26$. Thus, the assumption $\max_{1 \leqslant g \leqslant G} N_g^2/N \to 0$ under which the conventional methods of CR inference are guaranteed work is hard to justify for this application.

Table 3 reports the p-values for $\delta = \theta_1$ based on the alternative inference method. Column HC1 reports the p-value with the conventional method inference without clustering, i.e., the HC1 standard error. Columns CR1, JACK, and WCB report the p-values with the conventional methods of CR inference, namely the CR1 standard error, the jackknife standard error, and the wild cluster bootstrap,[8] respectively, along with the normal approximation. Finally, column SUB reports the p-value based on our proposed inference method with the score subsampling. We use the same code used for simulation studies in Section 5, including the choice of b based on the minimum volatility method.

Observe that the p-value is zero up to the third digit when standard errors are not clustered (HC1). Conventional methods of CR inference (CR1, JACK, and WCB) with the normal approximation certainly yield larger p-values, but do not change the statistical

[7] Effectively, it implements a sharp regression discontinuity design, but the original paper estimates the effect by the OLS for this linear estimating equation.

[8] While there are four alternative methods of WCB (cf. MacKinnon et al., 2022, Section 7.2), they all produce the same p-values up to the displayed digits, and hence we summarize them in a single column.

	HC1	CR1	JACK	WCB	SUB
p-value	0.000	0.006	0.007	0.006	0.129

Table 3: p-values for the effect $\delta = \theta_1$ of political turnover on the test score of fourth-grade students based on the conventional methods of inference (HC1, CR1, JACK, WCR) and our proposed method of score subsampling (SUB).

significance. Now, observe that our proposed method of inference with the score subsampling yields a much larger p-value, and entails statistical insignificance of the effect $\delta = \theta_1$ of interest, unlike any of the conventional methods. These results showcase that not correctly accounting for the potentially non-normality in the limiting distributions in the presence of unignorably large clusters could lead to erroneous statistical conclusions.

Appendix

A Omitted Details

This appendix section collectes technical details that are omitted from the main text.

A.1 Alternative Characterization of ξ_g Belonging to the Domain of Attraction of an α-Stable Distribution for $\alpha < 2$

Citing a result from the existing literature, this section presents complete details about the power law characterization (2.2) discussed in Section 2 in the main text.

Theorem 4 (de la Peña et al., 2009, Theorem 2.24)**.** *Suppose $\alpha < 2$. Then, ξ_g belongs to the domain of attraction of an α-stable distribution if and only if*

$$P(|\xi_g| > t) = t^{-\alpha} L(t) \qquad \textit{and}$$
$$\lim_{t\to\infty} \frac{P(\xi_g > t)}{P(|\xi_g| > t)} = p, \quad p \in [0, 1],$$

for some slowly varying function $L(\cdot)$.

The first condition means that the tail limit of the absolute value of the random variable of interest has an approximately Pareto tail, or so-called power law. Known as the balancing condition, the second condition in this alternative characterization imposes a mild restriction on the existence of limiting ratios of one-sided tail probabilities over the two-sided tail probability; it rules out some irregular, infinitely oscillating type situations such that these limiting ratios do not exist. This condition only imposes restrictions in the limit and accommodates a wide range of tail behaviors as p are permitted to be either 0 or 1.

A.2 Auxiliary Theory Focusing on Cases with $\alpha < 2$

This section presents a lemma that we state and prove on the way to proving Theorem 2 in Section 4.2 in the main text. Namely, for ease of writing, we state our main result focusing on cases with $\alpha \in (1, 2)$. An extension of this result to the general cases with $\alpha \in (1, 2]$ follows as Theorem 2 with additionally accounting for the case with $\alpha = 2$.

Lemma 1 (Cluster robust inference by score subsampling)**.** *Suppose that Assumption 1 is satisfied for $\alpha \in (1, 2)$. If $b \to \infty$ and $b/G = o(1)$ as $G \to \infty$, then*

$$\sup_{t \in \mathbb{R}} |L^*_{G,b}(t) - J^*(t)| \xrightarrow{p} 0$$

and the limiting distribution $J^*(\cdot)$ is continuous. In addition, if $M \to \infty$, then

$$\sup_{t \in \mathbb{R}} |\widehat{L}_{G,b}(t) - J^*(t)| \xrightarrow{p} 0,$$

and thus

$$P\left((\widehat{\delta} - \delta)/\widehat{\sigma} \leqslant \widehat{c}_{G,b}(1 - a)\right) \to 1 - a.$$

A proof is provided in Appendix B.1.

Remark 3 (Heavy-tailed cluster sums)**.** In this lemma, we essentially assume that the tails of the distributions of $\|S_g\|$ and $\|X_g' X_g\|$ both follow the power law with the shape parameter (Pareto exponent) in $(1, 2)$, which implies that the variances of S_g and $(X_g' X_g)$ do not exist. See Appendix A.1. This is a rather general condition in the sense that the heavy tail can come from the distribution of cluster sizes N_g, the distribution of individuals' (X'_{gi}, U_{gi}), or both. ▲

Remark 4 (Unignorability and impossibility of normal approximation)**.** An inspection of the proof of Lemma 1, combined with Remark 2 in LePage et al. (1981), unveils that, when $\alpha < 2$, the tails of the first component of representation (B.3) satisfies

$$P\left(|\epsilon_1 Z_1 - (2p-1)\mathbb{E}[Z_1 \mathbb{1}(Z_1 < 1)]| > t\right) \sim P\left(\left|\sum_{k=1}^{\infty}\{\epsilon_k Z_k - (2p-1)\mathbb{E}[Z_k \mathbb{1}(Z_k < 1)]\}\right| > t\right)$$

as $t \to \infty$. Since the term $|\epsilon_1 Z_1 - (2p-1)\mathbb{E}[Z_1 \mathbb{1}(Z_1 < 1)]|$ corresponds to the limiting distribution of the absolute value of the scaled score of the largest cluster, it has an asymptotically unignorable influence on the limiting α-stable distribution – see also Section 1.4 in Samorodnitsky and Taqqu (1994). For ease of illustration, suppose that the regressor and error distributions are uniformly bounded and $\mathrm{Cov}(X_{gi}U_{gi}, X_{gi}U_{gi'}|N_g) \geqslant \underline{c} > 0$ for all $i = 1, ..., N_g$ with probability one. This then implies

$$\frac{\max_{g=1,\dots,G}\|S_g\|}{G} \sim_p \frac{\max_{g=1,\dots,G} N_g}{G} \gg 0,$$

which directly violates the necessary and sufficient condition for the asymptotic variance to be estimable derived in Corollary 4.1 in Kojevnikov and Song (2023), as well as the conventional assumption

$$\frac{\max_{g=1,\dots,G} N_g^2}{G} = o_p(1),$$

required in the literature (e.g. Assumption 2 in Hansen and Lee 2019) for normal approximation.[9]

In addition, a necessary and sufficient condition for the limiting distribution of sums of independent random variables to be normal is the uniform asymptotic negligibility condition, i.e., the largest summand in absolute value has an asymptotically negligible contribution to

[9]It is assumed in the literature of CR inference based on the normal approximation that $\frac{\max_{g=1,\dots,G} N_g^2}{N} = o_p(1)$. When $\mathbb{E}[N_g] = c > 0$ exists, this assumption is equivalent to $\frac{\max_{g=1,\dots,G} N_g^2}{G} = o_p(1)$.

the sum (cf. Davidson, 1994, Theorem 23.13). Thus, it is impossible to derive a theoretically valid normal-approximation-based procedure of inference in the presence of unignorably large clusters without imposing restrictions on within-cluster dependence. ▲

Remark 5 (On CR standard error estimation)**.** The test statistic we consider is the standard t-statistic used in the literature. Its denominator consists of a CR standard error without imposing a null hypothesis. When $\alpha < 2$, the asymptotic variance does not exist, and nor is this "standard error" consistent but remains random asymptotically. This is similar in spirit to the fixed-b asymptotics (e.g., Kiefer and Vogelsang, 2002) in the literature of long-run variance estimation, although the underlying theory is completely different as the fixed-b asymptotics crucially relies on normal approximation and the functional central limit theorem. Showing that this "standard error" with estimated residuals has negligible impact on the asymptotic distribution requires a completely different proof strategy from the conventional approach of those taken in the proof of Theorem 7.6 in Hansen (2022a) for example. ▲

A.3 Inconsistency of the Wild Cluster Bootstrap under $\alpha < 2$.

The wild cluster bootstrap (Cameron, Gelbach, and Miller, 2008) is a popular alternative resampling method of CR inference. It has been shown in various simulation studies to behave well under $\alpha = 2$. Validity of the wild cluster bootstrap in cases of $\alpha = 2$ has been shown in Djogbenou et al. (2019) under fairly general conditions. As their proof relies crucially on Lyapunov's CLT, however, their arguments do not hold under $\alpha < 2$ – see Remark 4. A remaining and potentially more interesting question is whether one can prove its validity using an alternative argument. The following result suggests that such efforts are ill-fated when $\alpha < 2$.

For simplicity of illustration, consider the case of a univariate regression with only the intercept, i.e. a cluster sampled mean $\widehat{\theta} = N^{-1}\sum_{g=1}^{G}\sum_{i=1}^{N_g} Y_{gi}$ with the cluster specific population mean normalized to $\theta = \mathbb{E}\left[\sum_{i=1}^{N_g} Y_{gi}\right] = 0$ without loss of generality. Suppose that the parameter of inference is θ. Under the null hypothesis $H_0 : \theta = 0$, the standard CR t-statistic can be formed as

$$T_G = \frac{\sum_{g=1}^{G} \sum_{i=1}^{N_g} Y_{gi}}{\sqrt{\sum_{g=1}^{G} \left(\sum_{i=1}^{N_g} (Y_{gi} - \widehat{\theta}) \right)^2}}.$$

The wild-cluster-bootstrap version of the estimator is defined by $\widehat{\theta}^* = N^{-1} \sum_{g=1}^{G} v_g^* \sum_{i=1}^{N_g} Y_{gi}$, where $(v_g^*)_{g=1}^G$ are i.i.d. Rademacher auxiliary random variables generated by a researcher independently from the observed data $\{\{Y_{gi}\}_{i=1}^{N_g}\}_{g=1}^G$. The null-imposed wild cluster bootstrap test statistic is defined by

$$T_G^* = \frac{\sum_{g=1}^{G} v_g^* \sum_{i=1}^{N_g} Y_{gi}}{\sqrt{\sum_{g=1}^{G} \left(v_g^* \sum_{i=1}^{N_g} (Y_{gi} - \widehat{\theta}^*) \right)^2}}.$$

We introduce the notation $Y_{1:G} = (Y_{gi} : g = 1, ..., G, i = 1, ..., N_g)$ for convenience. As Lemma 1 implies continuity of the limiting distribution of T_G, the wild cluster bootstrap is consistent if

$$\sup_{t \in \mathbb{R}} |P(T_G^* \leqslant t | Y_{1:G}) - P(T_G \leqslant t)| = o_p(1) \qquad \text{as } G \to \infty.$$

Theorem 5 (Inconsistency of the wild cluster bootstrap)**.** *Under the above setup and Assumption 1, if $\alpha \in (1, 2)$, then the wild cluster bootstrap with Rademacher auxiliary random variables is inconsistent.*

A proof can be found in Appendix B.4

A.4 Choosing the Number b of Cluster Subsamples

For the choice of b in practice, we adapt the minimum volatility method (Algorithm 9.3.3 in Politis et al., 1999, Section 9.3.2) to our framework of cluster-robust inference.

For subsampling to be valid, b needs to grow with the number G of clusters but at a slower rate. If b is too close to G, then all the subsampled t-statistics will be almost identical to the full-sample t-statistic, resulting in a subsampling distribution being too tight and thus in under-coverage by confidence intervals. On the other hand, if b is too small, then the subsampled t-statistics will be noisy and can result in either under-coverage or over-coverage. Thus, intuitively, we wish to select a b that is in a stable range for the test statistic. The following algorithm formalizes such an idea.

Algorithm 1 (Minimum volatility method for cluster-robust inference)**.**

1. *For* $b \in \{b_{small}, b_{small} + 1, ..., b_{big}\}$*, compute the critical value* $\widehat{c}_{G,b}(1-a)$ *at a desired significance level* a*.*

2. *For* $b \in \{b_{small}+k, b_{small}+k+1, ..., b_{big}-k\}$*, compute a volatility index* VI_b *of the critical value, i.e., the standard deviation of the values* $\widehat{c}_{G,b-k}(1-a), ..., \widehat{c}_{G,b}(1-a), ..., \widehat{c}_{G,b+k}(1-a)$ *for a small positive integer* k*.*

3. *Pick* b^* *that has the smallest* VI_{b*} *and the corresponding confidence interval.*

Remark 6. As pointed out by Romano and Wolf (1999, Section 11.5), empirical bootstrap is not valid in the presence of heavy-tailed distributions. Thus, the common calibration method for the choice of subsampling block size cannot be used in our setup. ▲

B Mathematical Proofs

This section collects all the mathematical proofs. The order in which the proofs appear below differs from the order in which the corresponding statements appear. Namely, the proof of Theorem 2 uses Lemma 1, and hence we present the proof of Lemma 1 before the proof of Theorem 2. Furthermore, the proof of Theorem 1 uses Lemma 1 and Theorem 2, and hence we present the proofs of Lemma 1 and Theorem 2 before the proof of Theorem 1.

B.1 Proof of Lemma 1

Proof of Lemma 1. Without loss of generality, suppose that X_{gi} is a scalar and $r = 1$, and hence $\delta = \theta$. The proof is divided into two steps. In the first step, we derive the asymptotic distribution of the self-normalized sums that consist of the linear component of the influence function of the estimator. In the second step, we derive the validity of the proposed subsampling inference procedure.

Step 1. Recall that

$$\widehat{\theta} - \theta = \left(\sum_{g=1}^{G} X_g' X_g \right)^{-1} \sum_{g=1}^{G} S_g.$$

We shall derive the asymptotic distribution for the following self-normalized sums of the linear component $\sum_{g=1}^{G} S_g$:

$$SN_{1G}(\theta) := \frac{\sum_{g=1}^{G} S_g}{\sqrt{\sum_{g=1}^{G} S_g^2}}, \qquad SN_{2G}(\theta) := \frac{\sum_{g=1}^{G} S_g}{\sqrt{\sum_{g=1}^{G} \widehat{S}_g^2}}, \tag{B.1}$$

where $\widehat{S}_g = X_g' \widehat{U}_g$. The asymptotic distribution of a properly re-scaled $(\widehat{\theta} - \theta)$ will then follow straightforwardly from the multiplication of Q^{-1} on both the numerator and the denominator. Since $\alpha \in (1, 2)$, Corollary 1 in LePage et al. (1981) yields

$$SN_{1G}(\theta) \xrightarrow{d} \frac{\sum_{k=1}^{\infty} \{\epsilon_k Z_k - (2p-1)\mathbb{E}[Z_k \mathbb{1}(Z_k < 1)]\}}{\sqrt{\sum_{k=1}^{\infty} Z_k^2}} \tag{B.2}$$

as $G \to \infty$, where

$$p = \lim_{t \to \infty} \frac{P\left(S_g > t\right)}{P\left(|S_g| > t\right)},$$

$Z_k = (E_1 + ... + E_k)^{-1/\alpha}$ for each k, $\{E_k\}_k$ are i.i.d. standard exponential random variables, and $\{\epsilon_k\}_k$ are i.i.d. random variables that take the value of 1 with probability p and -1 with probability $(1-p)$ and are independent of $\{Z_k\}_k$.

We now claim that $SN_{2G}(\theta)$ converges in distribution to the same limiting distribution as (B.2). By Theorems 1 and $1'$ in LePage et al. (1981),

$$\left(\frac{1}{A_G} \sum_{g=1}^{G} S_g, \frac{1}{A_G^2} \sum_{g=1}^{G} S_g^2 \right) \xrightarrow{d} (S, V) := \left(\sum_{k=1}^{\infty} \{\epsilon_k Z_k - (2p-1)\mathbb{E}[Z_k \mathbb{1}(Z_k < 1)]\}, \sum_{k=1}^{\infty} Z_k^2 \right) = O_p(1) \tag{B.3}$$

holds for $A_G = G^{1/\alpha} L_1(G)$, where Z_k, ϵ_k, and p are defined below Equation (B.2), and $L_1(\cdot)$

is slowly varying at ∞; and

$$\frac{1}{(A'_G)^2}\sum_{g=1}^{G}(X'_gX_g)^2 \xrightarrow{d} \sum_{k=1}^{\infty}\widetilde{Z}_k^2 = O_p(1) \tag{B.4}$$

holds where $A'_G = G^{1/\alpha}L_2(G)$, $\widetilde{Z}_k = (\widetilde{E}_1 + ... + \widetilde{E}_k)^{-1/\alpha}$ for each k, $\{\widetilde{E}_k\}_k$ are i.i.d. standard exponential random variables, and $L_2(\cdot)$ is slowly varying at ∞. Because $\alpha \in (1,2)$ and L_1 is slowly varying at ∞, Equation (B.3) implies the consistency

$$\|\widehat{\theta} - \theta\| = \left\|\left(\sum_{g=1}^{G} X'_gX_g\right)^{-1}\sum_{g=1}^{G} S_g\right\| = O_p(L_1(G)G^{-(1-1/\alpha)}) = o_p(1) \tag{B.5}$$

under Assumption 1. Using $\widehat{U}_g = U_g + X_g(\theta - \widehat{\theta})$ and $\widehat{S}_g = S_g + X'_gX_g(\theta - \widehat{\theta})$, where $\widehat{U}_g = (\widehat{U}_{g1}, ..., \widehat{U}_{gN_g})'$, we can write

$$\begin{aligned}\frac{1}{A_G^2}\sum_{g=1}^{G}\widehat{S}_g^2 =& \frac{1}{A_G^2}\sum_{g=1}^{G}S_g^2 + \frac{1}{A_G^2}\sum_{g=1}^{G}\left(\widehat{S}_g - S_g\right)\widehat{S}_g + \frac{1}{A_G^2}\sum_{g=1}^{G}S_g\left(\widehat{S}_g - S_g\right)\\ =& \frac{1}{A_G^2}\sum_{g=1}^{G}S_g^2 + (1) + (2).\end{aligned}$$

We are going to show that the terms (1) and (2) are $o_p(1)$. First,

$$\begin{aligned}\|(1)\| =& \left\|\frac{1}{A_G^2}\sum_{g=1}^{G}(S_g + X'_gX_g(\theta - \widehat{\theta}))(X'_gX_g(\theta - \widehat{\theta}))'\right\|\\ \leqslant& \left\|\frac{1}{A_G^2}\sum_{g=1}^{G}S_gX'_gX_g\right\|\|\widehat{\theta} - \theta\| + \left\|\frac{1}{A_G^2}\sum_{g=1}^{G}(X'_gX_g)^2\right\|\|\widehat{\theta} - \theta\|^2\\ \leqslant& \underbrace{\sqrt{\frac{1}{A_G^2}\sum_{g=1}^{G}S_g^2}}_{=O_p(1)}\underbrace{\sqrt{\frac{1}{A_G^2}\sum_{g=1}^{G}(X'_gX_g)^2}}_{=O_p(1)}\underbrace{\|\widehat{\theta} - \theta\|}_{=o_p(1)} + \underbrace{\frac{1}{A_G^2}\sum_{g=1}^{G}(X'_gX_g)^2}_{=O_p(1)}\underbrace{\|\widehat{\theta} - \theta\|^2}_{=o_p(1)}\\ =& o_p(1)\end{aligned}$$

holds, where the second inequality follows from the Cauchy-Schwarz inequality and the stochastic orders are due to Equations (B.3), (B.4), and (B.5). Second, similar lines of calculations yield

$$\|(2)\| = \left\| \frac{1}{A_G^2} \sum_{g=1}^{G} S_g (X_g' X_g (\theta - \widehat{\theta}))' \right\| = o_p(1).$$

We have now established that

$$\frac{1}{A_G^2} \sum_{g=1}^{G} \widehat{S}_g^2 = \frac{1}{A_G^2} \sum_{g=1}^{G} S_g^2 + o_p(1),$$

and consequently, $SN_{1G}(\theta)$ is asymptotically equivalent to $SN_{2G}(\theta)$.

Step 2. We next show the validity of cluster robust score subsampling procedure. Define the conventional subsampling estimator

$$\breve{\theta}_{b,j} = \left(\sum_{g \in S_j} X_g' X_g \right)^{-1} \sum_{g \in S_j} X_g' Y_g.$$

Since $B^{-1} - A^{-1} = A^{-1}(A - B)B^{-1}$, we have

$$\begin{aligned}
\breve{\theta}_{b,j} - \widehat{\theta}_{b,j} &= \left(\sum_{g \in S_j} X_g' X_g \right)^{-1} \sum_{g \in S_j} X_g' Y_g - \left(\frac{G}{b} \right) \left(\sum_{g=1}^{G} X_g' X_g \right)^{-1} \sum_{g \in S_j} X_g' Y_g \\
&= \left(\frac{1}{G} \sum_{g=1}^{G} X_g' X_g \right)^{-1} \left(\frac{1}{G} \sum_{g=1}^{G} X_g X_g - \frac{1}{b} \sum_{g \in S_j} X_g' X_g \right) \left(\frac{1}{b} \sum_{g \in S_j} X_g' X_g \right)^{-1} \frac{1}{b} \sum_{g \in S_j} X_g' Y_g \\
&= o_p(1) \cdot \breve{\theta}_{b,j}
\end{aligned}$$

This implies $\widehat{\theta}_{b,j} = \breve{\theta}_{b,j}(1 + o_p(1))$. Therefore, in the subsampling process, $\breve{\theta}_{b,j}$ can be replaced by $\widehat{\theta}_{b,j}$ without changing the asymptotic behavior. Thus, it suffices to establish validity of subsampling procedure based on the conventional subsampling estimator $\breve{\theta}_{b,j}$.

Now, since the stable distributions S and V defined in the previous step are both continuous and $V > 0$ with probability 1, $S/V^{1/2}$ is continuously distributed and $J^*(\cdot)$ is continuous.

Hence, by invoking Theorem 11.3.1 in Politis et al. (1999), we have

$$\sup_{t\in\mathbb{R}} |L^*_{G,b}(t) - J^*(t)| = o_p(1)$$

as $G \to \infty$, $b \to \infty$, and $b/G = o(1)$. Next, note that $\widehat{L}_{G,b}$ is an empirical CDF consisting of M i.i.d. summands as we randomly sample the subsample clusters with replacement. By Dvoretzky-Kiefer- Wolfowitz inequality, therefore, we have the uniform convergence of the empirical CDF:

$$\sup_{t\in\mathbb{R}} |\widehat{L}_{G,b}(t) - J^*(t)| = o_p(1)$$

as $M \to \infty$ and $G \to \infty$ This concludes the proof. □

B.2 Proof of Theorem 2

Proof of Theorem 2. The case of $\alpha < 2$ follows directly from Lemma 1. For $\alpha = 2$, the proof is similar to the proof of Lemma 1 with the following minor modifications. First, when $\alpha = 2$, S_g is in the domain of attraction of the normal distribution and hence Theorem 3.4 in Giné et al. (1997) yields

$$SN_{1G}(\theta) \xrightarrow{d} N(0,1).$$

Second, to show the asymptotic equivalence of $SN_1(\theta)$ and $SN_2(\theta)$, note that both S_g and $(X_g'X_g)$ belong to the domain of attraction of the normal law when $\alpha = 2$. We branch into two cases. In case that both S_g and $(X_g'X_g)$ have finite variances, we have

$$\frac{1}{G}\sum_{g=1}^{G} \widehat{S}_g^2 = \frac{1}{G}\sum_{g=1}^{G} S_g^2 + o_p(1) \xrightarrow{p} \mathrm{Var}(S_g)$$

by following the standard argument for consistency of the CR variance estimator. In case their variances do not exist, Lemma 3.1 in Giné et al. (1997) yields

$$\frac{1}{A_G^2}\sum_{g=1}^{G} S_g^2 \xrightarrow{p} 1$$

for A_G such that

$$\frac{1}{A_G}\sum_{g=1}^{G}(S_g - \mathbb{E}[S_g]) \xrightarrow{d} N(0,1).$$

A similar argument holds when S_g is replaced by $(X_g' X_g)$. Then, the arguments for bounding $\|(1)\|$ and $\|(2)\|$ in the proof of Lemma 1 still go through, and thus for the self-normalized sums defined in Equation (B.6), it holds that $SN_2(\theta) = SN_1(\theta) + o_p(1)$. Finally, for the validity of the subsampling procedure, we now invoke Theorem 2.2.1 in Politis et al. (1999) and note that the limiting distribution is normal and hence continuous. □

B.3 Proof of Theorem 1

Proof of Theorem 1. The if part of the statement follows from the proof of Theorem 2. The only if part is a direct implication of Theorem 3.4 in Giné et al. (1997) and the fact that for any $\alpha \in (1, 2]$, the self-normalized sums defined in Equation (B.6) satisfy $SN_2(\theta) = SN_1(\theta) + o_p(1)$, as shown in the proofs for Lemma 1 and Theorem 2. □

B.4 Proof of Theorem 5

Proof of Theorem 5. Write

$$T_G = \frac{S_G}{\sqrt{V_G}} := \frac{A_G^{-1}\sum_{g=1}^{G}\left(\sum_{i=1}^{N_g} Y_{gi}\right)}{\sqrt{A_G^{-2}\sum_{g=1}^{G}\left(\sum_{i=1}^{N_g}(Y_{gi}-\widehat{\theta})\right)^2}} \quad \text{and}$$

$$T_G^* = \frac{S_G^*}{\sqrt{V_G^*}} := \frac{A_G^{-1}\sum_{g=1}^{G} v_g^*\left(\sum_{i=1}^{N_g} Y_{gi}\right)}{\sqrt{A_G^{-2}\sum_{g=1}^{G}\left(v_g^*\sum_{i=1}^{N_g}(Y_{gi}-\widehat{\theta}^*)\right)^2}}.$$

Let P denote the probability measure for the data and P^* denote the probability measure of Rademacher auxiliary random variables. Define

$$p = \lim_{t \to \infty} \frac{P\left(\sum_{i=1}^{N_g} Y_{gi} > t\right)}{P\left(\left|\sum_{i=1}^{N_g} Y_{gi}\right| > t\right)}.$$

Write $W_g = \left|\sum_{i=1}^{N_g} Y_{gi}\right|$ and the order statistics of $W_1, ..., W_G$ as follows:

$$W_{G1} \geqslant W_{G2} \geqslant ... \geqslant W_{GG}.$$

The rescaled counterpart is denoted by $Z_{Gg} = A_G^{-1} W_{Gg}$, for $g = 1, ..., G$ – recall that $A_G = G^{1/\alpha} L(G)$ for a slow varying $L(\cdot)$ is defined right before Assumption 1. For each G, we can collect them into a countably long vector

$$Z^G = (Z_{G1}, ..., Z_{GG}, 0, 0, ...) \in \mathbb{R}^\infty.$$

Similarly defined is the countably long sign vector

$$\epsilon^G = (\epsilon_{G1}, ..., \epsilon_{GG}, 1, 1, ...) \in \mathbb{R}^\infty,$$

where ϵ_{Gg} indicates the sign such that $\sum_{i=1}^{N_h} Y_{hi} = \epsilon_{Gg} W_{Gg}$ for the cluster h that corresponds to the g-th order statistic W_{Gg} for each $g = 1, ..., G$, $G \in \mathbb{N}$. By Lemmas 1 and 2 in LePage et al. (1981), we have

$$Z^G \xrightarrow{d} Z = (Z_1, Z_2, ...) \quad \text{and} \quad \epsilon^G \xrightarrow{d} \epsilon = (\epsilon_1, \epsilon_2, ...),$$

where $\{Z_k\}_k$ and $\{\epsilon_k\}$ are defined in the proof for Lemma 1. In addition, since $\mathbb{R}^\infty$ is a complete separable metric space under the metric

$$d((x_1, x_2, ...), (y_1, y_2, ...)) = \sum_{k=1}^{\infty} \frac{1}{2^k} \cdot \frac{|x_k - y_k|}{1 + |x_k - y_k|},$$

following Skorohod's representation theorem, on an adequately chosen probability space,

$$d(Z^G, Z) \to 0 \quad \text{and} \quad d(\epsilon^G, \epsilon) \to 0$$

P-almost surely. Denote the countable vector of i.i.d. Rademacher random variables by $v^* = (v_1^*, v_2^*, ...) \in \mathbb{R}^\infty$, which is invariant of G. We now claim that the weak convergence

$$S_G^* = \sum_{g=1}^{G} \epsilon_{Gg} Z_{Gg} v_g^* \xrightarrow{d^*} S^* := \sum_{k=1}^{\infty} \epsilon_k Z_k v_k^*$$

for (Z, ϵ) with P-probability one, where the convergence in distribution $\xrightarrow{d^*}$ is with respect to P^*. Note that the limiting random variable on the right-hand side is well-defined since

$$\begin{aligned} &\mathbb{E}^* \left[\epsilon_k Z_k v_k^*\right] = 0 \text{ for all } k \text{ and} \\ &\sum_{k=1}^{\infty} \mathbb{E}^* \left[(\epsilon_k Z_k v_k^*)^2\right] = \sum_{k=1}^{\infty} Z_k^2 < \infty \end{aligned}$$

P-almost surely. The convergence in distribution is shown following the same arguments as in the proof of Theorem 2 in Knight (1989) with i.i.d. Rademacher random variables v_k^* in place of their centered i.i.d. Poisson random variables $(M_k^* - 1)$. Specifically, observe that $Z_k \to 0$ as $k \to \infty$ P-almost surely. Following Equation (12) in the proof of Theorem 1 in LePage et al. (1981), define $\mathcal{Z} \subset \mathbb{R}^\infty$ be the subspace consists of countable sequences $z = (z_1, z_2, ...)$ such that $z_1 \geqslant z_2 \geqslant ... \geqslant 0$ (note that $\mathcal{Z}$ is also a complete separable space with the inherited topology). Subsequently, for a fixed $\varepsilon > 0$, define $\phi : \mathcal{Z} \times \{-1, 1\}^\infty \times \{-1, 1\}^\infty$ by

$$\phi(z, \epsilon, v^*) = \begin{cases} \sum_{k=1}^{\infty} \epsilon_k z_k \mathbb{1}(z_k > \epsilon) v_k^* & \text{if } z_k \to 0 \text{ as } k \to \infty, \\ 0 & \text{otherwise.} \end{cases}$$

Then ϕ is a continuous mapping with respect to the product topology. Thus by the continuous mapping theorem as well as the convergences of $d(Z^G, Z) \to 0$ and $d(\epsilon^G, \epsilon) \to 0$ with

P-probability one established earlier, for any $\varepsilon > 0$,

$$\sum_{g=1}^{G} \epsilon_{Gg} Z_{Gg} \mathbb{1}(Z_{Gg} > \varepsilon) v_g^* \xrightarrow{d^*} \sum_{k=1}^{\infty} \epsilon_k Z_k \mathbb{1}(Z_k > \varepsilon) v_k^*$$

for (Z, ϵ) with P-probability one. In addition, note that

$$\mathbb{E}^* \left[\left(\sum_{g=1}^{G} \epsilon_{Gg} Z_{Gg} \mathbb{1}(Z_{Gg} \leqslant \varepsilon) v_g^* \right)^2 \right] = \sum_{g=1}^{G} Z_{Gg}^2 \mathbb{1}(Z_{Gg} \leqslant \varepsilon) \mathrm{Var}^*(v_k^*) \leqslant \sum_{k=1}^{\infty} Z_k^2 \mathbb{1}(Z_k \leqslant \varepsilon)$$

holds almost surely in P and the right-hand side converges to zero as $\varepsilon \to 0$, which implies via Markov's inequality that, for any $\delta > 0$,

$$\lim_{\varepsilon \to 0} \limsup_{G \to \infty} P^* \left(\left| \sum_{k=1}^{\infty} \epsilon_{Gk} Z_{Gk} \mathbb{1}(Z_{Gk} \leqslant \varepsilon) v_k^* \right| > \delta \right) = 0$$

P-almost surely. Finally, for any $\delta > 0$,

$$\lim_{\varepsilon \to 0} P^* \left(\left| \sum_{k=1}^{\infty} \epsilon_k Z_k \mathbb{1}(Z_k \leqslant \varepsilon) v_k^* \right| > \delta \right) = 0$$

P-almost surely, which follows immediately from the fact that

$$\mathbb{E}^* \left[\left(\sum_{k=1}^{\infty} \epsilon_k Z_k \mathbb{1}(Z_k \leqslant \varepsilon) v_k^* \right)^2 \right] = \sum_{k=1}^{\infty} Z_k^2 \mathbb{1}(Z_k \leqslant \varepsilon) \to 0$$

P-almost surely as $\varepsilon \to 0$. Combining these results yields that

$$S_G^* \xrightarrow{d^*} S^* = \sum_{k=1}^{\infty} \epsilon_k Z_k v_k^*$$

for (Z, ϵ) with P-probability one. On the other hand, recall from Step 1 in the proof of Lemma 1 that

$$S_G = \sum_{g=1}^{G} \epsilon_{Gg} Z_{Gg} \xrightarrow{d} S := \sum_{k=1}^{\infty} \{\epsilon_k Z_k - (2p-1)\mathbb{E}[Z_k \mathbb{1}(Z_k \leqslant 1)]\},$$

by Theorem 1 in LePage et al. (1981). Note that Z_k, ϵ_k, and v_k^* are all mutually independent from each other. Therefore, the limiting distribution of S_G^* given $Y_{1:G}$, i.e. S^* conditionally on (Z, ϵ), differs from, S, the limiting α-stable distribution of S_G with positive P-probability.

Next, to cope with the denominator term of S_G^*, note that, combined with the law of large numbers, the above weak convergence of S_G^* also implies

$$\begin{aligned}\widehat{\theta}^* =& \frac{1}{N}\sum_{g=1}^{G}\epsilon_{Gg}W_{Gg}v_g^* \\ =& \frac{1}{c+o_p(1)}\cdot\frac{1}{G}\sum_{g=1}^{G}\epsilon_{Gg}W_{Gg}v_g^* \\ =& \underbrace{\frac{1}{c+o_p(1)}}_{=O_p(1)}\cdot\underbrace{\frac{A_G}{G}}_{=\frac{L(G)}{G^{1-1/\alpha}}}\cdot\underbrace{\sum_{g=1}^{G}\epsilon_{Gg}Z_{Gg}v_g^*}_{=O_p(1)} = o_p(1).\end{aligned}$$

Thus, the denominator term, $(V_G^*)^{1/2}$, of S_G^* turns out to be asymptotically independent of the auxiliary Rademacher random variables v_g^*:

$$V_G^* = \frac{1}{A_G^2}\sum_{g=1}^{G}\left(v_g^*\sum_{i=1}^{N_g}(Y_{gi}+o_p(1))\right)^2 = \sum_{g=1}^{G}Z_{Gg}^2 + o_p(1).$$

Given $Y_{1:G}$, the denominator is asymptotically constant. Following Step 1 in the proof of Lemma 1, we have

$$V_G = \sum_{g=1}^{G}Z_{Gg}^2 + o_p(1) \xrightarrow{d} \sum_{k=1}^{\infty}Z_k^2 = O_p(1).$$

Thus, given $Y_{1:G}$, the denominator term $(V_G^*)^{1/2}$ is a fixed value, while the original limit of the denominator is an $(\alpha/2)$-stable, non-degenerate continuous distribution. Hence, the limiting distribution of V_G^* given $Y_{1:G}$ and the unconditional limiting distribution of V_G differs with non-zero P-probability.

Finally, note that $V_G^* > 0$ P-almost surely. Thus, the fact that

$$(S_G^*, V_G^*) \xrightarrow{d^*} \left(\sum_{k=1}^{\infty} \epsilon_k Z_k v_k^*, \sum_{k=1}^{\infty} Z_k^2 \right)$$

for almost every (Z, ϵ) and the continuous mapping theorem yield that

$$T_G^* \xrightarrow{d^*} \frac{\sum_{k=1}^{\infty} \epsilon_k Z_k v_k^*}{\sqrt{\sum_{k=1}^{\infty} Z_k^2}}$$

for (Z, ϵ) with P-probability one. This, together with the unconditional limiting distribution of T_G implies the conclusion that the unconditional limiting distribution of T_G and the conditional limiting distribution of T_G^* differs with positive P-probability. The inconsistency then follows. □

B.5 Proof of Theorem 3

We shall derive the asymptotic distribution for the following self-normalized sums of S_g:

$$R_{1G} := \frac{\sum_{g=1}^{G} S_g}{\sqrt{\sum_{g=1}^{G} S_g^2}} \quad \text{and} \quad R_{2G} := \frac{\widehat{\delta} - \delta}{\widehat{\sigma}} = \frac{\sum_{g=1}^{G} S_g}{\sqrt{\sum_{g=1}^{G} \widehat{S}_g^2}}. \tag{B.6}$$

Following Eq (1.3) in Logan et al. (1973), we obtain

$$R_{2G} = R_{1G} \left(\frac{G}{G - R_{1G}^2} \right)^{1/2}.$$

Thus, if we can show that R_{1G} converges in distribution, then the limiting distribution of R_{2G} coincides with the one of R_{1G}.

Let us first introduce an important theoretical result that will be crucial for establishing the uniformity. The following auxiliary lemma shows a weak convergence result for triangular arrays consisting of $P_G \in \mathbf{P}$.

Lemma 2 (Weak convergence of triangular arrays)**.** *For any sequence of $P_G \in \mathbf{P}$ such that*

$\alpha_G \to \alpha_0 \in [1+\varepsilon, 2]$ and $p_G \to p_0 \in [0,1]$ as $G \to \infty$, we have

$$R_{1G} \xrightarrow{d} \mathbb{S}_{\alpha_0, p_0}.$$

Proof of Lemma 2. First, consider the case of $\alpha_0 < 2$. Denote $S_g = S_g(\alpha, p)$ to emphasize the dependence of the DGP on the index α of stability and the tail balancing parameter p (it does not suggest that the DGP is uniquely defined by these two parameters). For each DGP, $P_G \in \{P_G : G \geqslant 1\} \subset \mathbf{P}_1(\varepsilon)$, with indices (α_M, p_M) for an auxiliary index $M = G$, define

$$X_{Mn} = \frac{\sum_{g=1}^{n} S_g(\alpha_M, p_M)}{\sqrt{\sum_{g=1}^{n} S_g^2(\alpha_M, p_M)}}$$

for each $n \geqslant 1$. Since (α_M, p_M) is fixed over n for each M, we can apply Theorem 1$'$ in LePage et al. (1981) to obtain that, for each M as $n \to \infty$, there exists some positive sequence $A_{Mn} \to \infty$ such that

$$\begin{aligned} &\left(\frac{1}{A_{Mn}} \sum_{g=1}^{n} S_g(\alpha_M, p_M), \frac{1}{A_{Mn}^2} \sum_{g=1}^{n} S_g^2(\alpha_M, p_M) \right) \\ &\xrightarrow{d} \left(\sum_{k=1}^{\infty} \{\epsilon_k(p_M) Z_k(\alpha_M) - (2p_M - 1)\mathbb{E}[Z_k(\alpha_M)\mathbb{1}(Z_k(\alpha_M) < 1)]\}, \sum_{k=1}^{\infty} Z_k^2(\alpha_M) \right) = (S_M, V_M) \end{aligned}$$

as $n \to \infty$, where $Z_k(\alpha_M) = (E_1 + ... + E_k)^{-1/\alpha_M}$ for each k, $\{E_k\}_k$ are i.i.d. standard exponential random variables, and $\{\epsilon_k(p_M)\}_k$ are i.i.d. random variables that take the value of 1 with probability p_M and -1 with probability $(1-p_M)$ and are independent of $\{Z_k(\alpha_M)\}_k$. Note that the distributions of both S_M and V_M are stable with indices of stability of α_M and $\alpha_M/2$, respectively. Furthermore, it follows from Corollary 1 in LePage et al. (1981) that

$$X_{Mn} \xrightarrow{d} X_M \overset{d}{=} \frac{\sum_{k=1}^{\infty} \{\epsilon_k(p_M) Z_k(\alpha_M) - (2p_M - 1)\mathbb{E}[Z_k(\alpha_M)\mathbb{1}(Z_k(\alpha_M) < 1)]\}}{\sqrt{\sum_{k=1}^{\infty} Z_k^2(\alpha_M)}}.$$

Let the limiting distribution on the right-hand side be denoted by $\mathbb{S}_{\alpha_M, p_M}$. Also, note that

$(\alpha_M, p_M) \to (\alpha_0, p_0)$ by our construction, and thus,

$$X_M \xrightarrow{d} X \sim \mathbb{S}_{\alpha_0, p_0}$$

follows from the convergence of the sequence of the characteristic functions of the stable distributions S_M and V_M, as these characteristic functions are continuous in (α, p) over $(1,2) \times [0,1]$ (cf. Remark 4 on pp.7 in Samorodnitsky and Taqqu 1994) and V_M is positive with probability one for all $\alpha \in (1,2)$.

Next, by invoking the Skorohod's representation theorem (as $\mathbb{R}$ is a separable metric space), there exist versions of X_{Mn} and X_M such that $X_{Mn} \xrightarrow{a.s.} X_M$ for each M and as $n \to \infty$, and $X_M \xrightarrow{a.s.} X$ as $M \to \infty$. Now, for such X_{Mn}, define $Y_n = X_{nn}$. By construction, we have $Y_n \overset{d}{=} R_{1n}$ for all $n \geqslant 1$. Also, it follows from the almost sure converges that

$$\lim_{M \to \infty} \limsup_{n \to \infty} P(|X_{Mn} - Y_n| \geqslant \varepsilon) = 0$$

for all $\varepsilon > 0$. Applying Lemma 4 in Appendix C, we have $Y_n \xrightarrow{d} X$ as $n \to \infty$. Thus, we conclude $R_{1n} \xrightarrow{d} X$.

Now, consider the case of $\alpha_0 = 2$. We only need to consider the case where we have $\alpha_G < 2$ for at least one G, as, otherwise, $\alpha_G = 2$ for all G and

$$R_{1G} \xrightarrow{d} N(0,1)$$

follows immediately from the Lindeberg-Feller CLT. Now, for those $\alpha_M < 2$, construct X_{Mn} as in the previous case. By Corollary in LePage et al. (1981), we have

$$X_{Mn} = \frac{\sum_{g=1}^{n} S_g(\alpha_M, p_M)}{\sqrt{\sum_{g=1}^{n} S_g^2(\alpha_M, p_M)}} \xrightarrow{d} X_M \sim \mathbb{S}_{\alpha_M, p_M}.$$

By Assertion (vi) in Section 5 and Equation (5.13) in Logan et al. (1973), the density $f_{\alpha_M, p_M}(\cdot)$ of $\mathbb{S}_{\alpha_M, p_M}$ exists and is bounded everywhere except on a set with measure zero, and, as $\alpha_M \to 2$, $f_{\alpha_M, p_M} \to \varphi$, the standard normal density on the real line. Thus, by the bounded convergence theorem, the CDF $F_{\alpha_M, p_M}(x)$ of $\mathbb{S}_{\alpha_M, p_M}$ converges to the standard normal distribution function $\Phi(x)$ for all $x \in \mathbb{R}$, i.e. $X_M \xrightarrow{d} X \sim N(0,1)$. Using the same construction of Y_n as above, we conclude $R_{1n} \xrightarrow{d} N(0,1)$ by Lemma 4 in Appendix C. $\square$

Proof of the Theorem. The proof follows a similar structure to the one for Theorem 3.1 in Romano and Shaikh (2012). We will apply our Lemma 3 in Appendix C with

$$R_G = \frac{\widehat{\delta} - \delta}{\widehat{\sigma}} \quad \text{and} \quad \widehat{R}_b = \frac{\widehat{\delta}_{b,j} - \widehat{\delta}}{\widehat{\sigma}_{b,j}}.$$

First, we verify

$$\sup_{P \in \mathbf{P}} \sup_{x \in \mathbb{R}} |J_b(x,P) - J_G(x,P)| \to 0 \tag{B.7}$$

as $b, G \to \infty$ with $b/G = o(1)$. By way of contradiction, assume that it fails. Then, there exists a subsequence G_l and some $(\alpha, p) \in [1+\varepsilon, 2] \times [0,1]$ such that either

$$\sup_{x \in \mathbb{R}} |J_{b_{G_l}}(x, P_{G_l}) - F_{\alpha,p}(x)| \nrightarrow 0 \quad \text{or} \quad \sup_{x \in \mathbb{R}} |J_{G_l}(x, P_{G_l}) - F_{\alpha,p}(x)| \nrightarrow 0.$$

Recall that $\mathbb{S}_{\alpha,p} \sim F_{\alpha,p}$ has a continuous distribution (almost everywhere). Yet, either of these would violate Lemma 2. Thus Condition (B.7) must hold.

We will next verify the condition that

$$\sup_{P \in \mathbf{P}} P\left(\sup_{x \in \mathbb{R}} \left|\widehat{L}_G(x) - L_G(x,P)\right| > \varepsilon'\right) = o(1)$$

for all $\varepsilon' > 0$. Consider any sequence $\{P_G \in \mathbf{P} : G \geqslant 1\}$. For any $\eta > 0$, we have

$$\begin{aligned}
&\sup_{x \in \mathbb{R}}\{\widehat{L}_G(x) - L_G(x,P_G)\} \\
\leqslant &\sup_{x \in \mathbb{R}}\{\widehat{L}_G(x) - L_G(x+\eta,P_G)\} + \sup_{x \in \mathbb{R}}\{L_G(x+\eta,P_G) - L_G(x,P_G)\} \\
\leqslant &\sup_{x \in \mathbb{R}}\{\widehat{L}_G(x) - L_G(x+\eta,P_G)\} + \sup_{x \in \mathbb{R}}\{L_G(x+\eta,P_G) - L_b(x+\eta,P_G)\} \\
+ &\sup_{x \in \mathbb{R}}\{L_b(x,P_G) - L_G(x,P_G)\} + \sup_{x \in \mathbb{R}}\{L_b(x+\eta,P_G) - L_b(x,P_G)\} \\
= &(i) + (ii) + (iii) + (iv).
\end{aligned}$$

Note that (ii) and (iii) are both $o_{P_G}(1)$ by Lemma 4.5 in Romano and Shaikh (2012). Furthermore, (iv) converges to zero as $\eta \to 0$.

Finally, we will verify $(i) = o_{P_G}(1)$ as $\eta \to 0$. By considering a subsequence, if necessary, one may assume without loss of generality that P_G is such that $\alpha_G \to \alpha$ and $p_G \to p$. The proof for this statement utilizes an argument similar to those taken in Theorem 11.3.1 in Politis et al. (1999). By its definition,

$$\begin{aligned}\widehat{L}_G(x) =& \frac{1}{B_G}\sum_{j=1}^{B_G} \mathbb{1}\left\{\frac{\widehat{\delta}_{b,j} - \widehat{\delta}}{\widehat{\sigma}_{b,j}} \leqslant x\right\} \\ \leqslant& \frac{1}{B_G}\sum_{j=1}^{B_G} \mathbb{1}\left\{\frac{\widehat{\delta}_{b,j} - \delta}{\widehat{\sigma}_{b,j}} \leqslant x + \frac{\widehat{\delta} - \delta}{\widehat{\sigma}_{b,j}}\right\} \\ \leqslant& \frac{1}{B_G}\sum_{j=1}^{B_G} \mathbb{1}\left\{\frac{\widehat{\delta}_{b,j} - \delta}{\widehat{\sigma}_{b,j}} \leqslant x + \eta\right\} + (1 - R_G(\eta)),\end{aligned}$$

where $R_G(\eta)$ is defined for $\eta > 0$ as

$$\begin{aligned}R_G(\eta)) =& \frac{1}{B_G}\sum_{j=1}^{B_G} \mathbb{1}\left\{\frac{\widehat{\delta} - \delta}{\widehat{\sigma}_{b,j}} \leqslant \eta\right\} \\ =& \frac{1}{B_G}\sum_{j=1}^{B_G} \mathbb{1}\left\{(b/A_b)\widehat{\sigma}_{b,j} \geqslant (b/A_b)(\widehat{\delta} - \delta)/\eta\right\},\end{aligned}$$

$A_b = b^{1/\alpha}L(b)$ for some slow varying L at infinity. As $A_G/A_b \to 0$, for any $\varepsilon'' > 0$, it holds that $(b/A_b)(\widehat{\delta} - \delta) \leqslant \varepsilon''$ with probability approaching one along P_G. This is because $\widehat{\delta}$ is the full sample estimator and thus $(G/A_G)(\widehat{\delta} - \delta) = O_{P_G}(1)$ follows from the proof of Lemma 2. As such, following the proof of Lemma 2, we have

$$R_G(\eta) \geqslant \frac{1}{B_G}\sum_{j=1}^{B_G} \mathbb{1}\left\{(b/A_b)\widehat{\sigma}_{b,j} \geqslant \varepsilon''/\eta\right\} \overset{P_G}{\to} P_G(V \geqslant \varepsilon''/\eta)$$

as $G \to \infty$, where V is the stable distribution with index of stability of $\alpha/2$. By Theorem $1'$ in LePage et al. (1981) for example, V has the representation $V = \sum_{k=1}^{\infty} Z_k^2$, where $Z_k = (E_1 + ... + E_k)^{-1/\alpha}$ for each k, $\{E_k\}_k$ are i.i.d. standard exponential random variables, and $\{\epsilon_k\}_k$ are i.i.d. random variables that take the value of 1 with probability p and -1 with probability $(1-p)$ and are independent of $\{Z_k\}_k$. As ε'' can be arbitrarily small, we have $R_G(\eta) = 1 + o_{P_G}(1)$. Thus, we have

$$\begin{aligned}\widehat{L}_G(x) \leqslant & \frac{1}{B_G}\sum_{j=1}^{B_G} \mathbb{1}\left\{\frac{\widehat{\delta}_{b,j}-\delta}{\widehat{\sigma}_{b,j}} \leqslant x+\eta\right\} + (1-R_G(\eta)) \\ \leqslant & L_G(x+\eta, P_G) + o_{P_G}(1).\end{aligned}$$

A similar argument derives $\widehat{L}_G(x) \geqslant L_G(x+\eta, P_G) + o_{P_G}(1)$. This shows $(i) = o_{P_G}(1)$ as $\eta \to 0$, and hence concludes the proof. □

C Auxiliary Lemmas

Let $X^{(G)} = (X_1, ..., X_G)$ be a sequence of i.i.d. random variables with distribution $P \in \mathbf{P}$ and let the distribution of a real-valued root $R_G = R_G(X^{(G)}, P)$ under P be denoted by $J_G(x, P)$. In addition, for a subsample size $b = b_G < G$ such that $b = o(G)$, define $B_G = \binom{G}{b}$. For $j = 1, ..., B_G$, let $X^{G,(b),j}$ denote the j-th subsample of size b. Define

$$\begin{aligned}L_G(x, P) = & \frac{1}{B_G}\sum_{j=1}^{B_G} \mathbb{1}\{R_b(X^{G,(b),j}, P) \leqslant x\} \quad \text{and} \\ \widehat{L}_G(x) = & \frac{1}{B_G}\sum_{j=1}^{B_G} \mathbb{1}\{\widehat{R}_b(X^{G,(b),j}) \leqslant x\},\end{aligned}$$

where $\widehat{R}_b$ is a feasible estimator of R_b, which depends on the unknown P.

The following lemma restates Theorems 2.1 and 2.2 as well as Remark 2.1 in Romano and Shaikh (2012) for convenience of reference.

Lemma 3 (High-level uniformity). *Under the current setup,*

$$\lim_{G\to 0}\sup_{P\in\mathbf{P}}\sup_{x\in\mathbb{R}} |J_b(x,P) - J_G(x,P)| = 0,$$

implies

$$\liminf_{G\to\infty}\inf_{P\in\mathbf{P}} P\left(L_G^{-1}(a_1, P) \leqslant R_G \leqslant L_G^{-1}(1-a_2, P)\right) \geqslant 1 - a_1 - a_2$$

for any nonnegative a_1 and a_2 such that $0 \leqslant a_1 + a_2 < 1$. In addition, if $J_G(x, P)$ tends in distribution to a limiting distribution $J(x, P)$ that is continuous, then

$$\lim_{G\to\infty} \inf_{P\in\mathbf{P}} P\left(L_G^{-1}(a_1, P) \leqslant R_G \leqslant L_G^{-1}(1 - a_2, P)\right) = 1 - a_1 - a_2.$$

Finally, if

$$\sup_{P\in\mathbf{P}} P\left(\sup_{x\in\mathbb{R}} \left|\widehat{L}_G(x) - L_G(x, P)\right| > \varepsilon\right) = o(1)$$

for all $\varepsilon > 0$, then

$$\lim_{G\to\infty} \inf_{P\in\mathbf{P}} P\left(\widehat{L}_G^{-1}(a_1) \leqslant R_G \leqslant \widehat{L}_G^{-1}(1 - a_2)\right) = 1 - a_1 - a_2.$$

The next result is taken from Theorem 3.5 in Resnick (2007).

Lemma 4 (Second converging together theorem)**.** *Suppose that $\{X_{Mn}, X_M, X, Y_n : n \geqslant 1, M \geqslant 1\}$ are random elements of the metric space $(\mathbb{S}, \mathcal{S})$ with a metric $d(\cdot, \cdot)$ that are defined on a common domain. Assume that for each M, as $n \to \infty$, $X_{Mn} \rightsquigarrow X_M$, and as $M \to \infty$, $X_M \rightsquigarrow X$, Further suppose that for all $\varepsilon > 0$,*

$$\lim_{M\to\infty} \limsup_{n\to\infty} P(d(X_{Mn}, Y_n) \geqslant \varepsilon) = 0.$$

Then, as $n \to \infty$, we have $Y_n \rightsquigarrow X$, where $\rightsquigarrow$ denotes weak convergence.

References

Abadie, A., S. Athey, G. W. Imbens, and J. M. Wooldridge (2023): "When should you adjust standard errors for clustering?" *Quarterly Journal of Economics*, 138, 1–35.

Akhtari, M., D. Moreira, and L. Trucco (2022): "Political turnover, bureaucratic turnover, and the quality of public services," *American Economic Review*, 112, 442–493.

Andrews, D. W., X. Cheng, and P. Guggenberger (2020): "Generic results for establishing the asymptotic size of confidence sets and tests," *Journal of Econometrics*, 218, 496–531.

Arcones, M. A. and E. Giné (1989): "The bootstrap of the mean with arbitrary bootstrap sample size," in *Annales de l'IHP Probabilités et Statistiques*, vol. 25, 457–481.

Arellano, M. (1987): "Computing robust standard errors for within-groups estimators," *Oxford Bulletin of Economics and Statistics*, 49, 431–434.

Athreya, K. (1987): "Bootstrap of the mean in the infinite variance case," *Annals of Statistics*, 724–731.

Bai, Y., J. Liu, A. M. Shaikh, and M. Tabord-Meehan (2022): "Inference in Cluster Randomized Trials with Matched Pairs," *arXiv preprint arXiv:2211.14903*.

Bertrand, M., E. Duflo, and S. Mullainathan (2004): "How much should we trust differences-in-differences estimates?" *Quarterly Journal of Economics*, 119, 249–275.

Bugni, F., I. Canay, A. Shaikh, and M. Tabord-Meehan (2022): "Inference for cluster randomized experiments with non-ignorable cluster sizes," *arXiv preprint arXiv:2204.08356*.

Cameron, A. C., J. B. Gelbach, and D. L. Miller (2008): "Bootstrap-based improvements for inference with clustered errors," *Review of Economics and Statistics*, 90, 414–427.

Cameron, A. C. and D. L. Miller (2015): "A practitioner's guide to cluster-robust inference," *Journal of Human Resources*, 50, 317–372.

Canay, I. A., A. Santos, and A. M. Shaikh (2021): "The wild bootstrap with a "small" number of "large" clusters," *Review of Economics and Statistics*, 103, 346–363.

Cavaliere, G., T. Mikosch, A. Rahbek, and F. Vilandt (2024): "Tail behavior of ACD models and consequences for likelihood-based estimation," *Journal of Econometrics*, 238, 105613.

Davidson, J. (1994): *Stochastic Limit Theory: An Introduction for Econometricians*, OUP Oxford.

de la Peña, V. H., T. L. Lai, and Q.-M. Shao (2009): *Self-Normalized Processes: Limit Theory and Statistical Applications*, Springer.

Djogbenou, A. A., J. G. MacKinnon, and M. Ø. Nielsen (2019): "Asymptotic theory and wild bootstrap inference with clustered errors," *Journal of Econometrics*, 212, 393–412.

Embrechts, P., C. Klüppelberg, and T. Mikosch (1997): *Modelling Extremal Events: for Insurance and Finance*, Springer Science & Business Media.

Geluk, J. and L. de Haan (2000): "Stable probability distributions and their domains of attraction: a direct approach," *Probability and Mathematical Statistics*, 20, 169–188.

Giné, E., F. Götze, and D. M. Mason (1997): "When is the Student t-statistic asymptotically standard normal?" *Annals of Probability*, 25, 1514–1531.

Hansen, B. (2022a): *Econometrics*, Princeton University Press.

Hansen, B. E. (2022b): "Jackknife standard errors for clustered regression," *Working Paper*.

Hansen, B. E. and S. Lee (2019): "Asymptotic theory for clustered samples," *Journal of Econometrics*, 210, 268–290.

Hersch, J. (1998): "Compensating differentials for gender-specific job injury risks," *American Economic Review*, 88, 598–607.

Kiefer, N. M. and T. J. Vogelsang (2002): "Heteroskedasticity-autocorrelation robust standard errors using the Bartlett kernel without truncation," *Econometrica*, 70, 2093–2095.

Knight, K. (1989): "On the bootstrap of the sample mean in the infinite variance case," *Annals of Statistics*, 1168–1175.

Kojevnikov, D. and K. Song (2023): "Some Impossibility Results for Inference With Cluster Dependence with Large Clusters," *Journal of Econometrics*, Forthcoming.

LePage, R., M. Woodroofe, and J. Zinn (1981): "Convergence to a stable distribution via order statistics," *Annals of Probability*, 9, 624–632.

Liang, K.-Y. and S. L. Zeger (1986): "Longitudinal data analysis using generalized linear models," *Biometrika*, 73, 13–22.

Logan, B. F., C. Mallows, S. Rice, and L. A. Shepp (1973): "Limit distributions of self-normalized sums," *Annals of Probability*, 1, 788–809.

MacKinnon, J. G., M. Ø. Nielsen, and M. D. Webb (2022): "Fast and reliable jackknife and bootstrap methods for cluster-robust inference," *Working Paper*.

——— (2023): "Cluster-robust inference: A guide to empirical practice," *Journal of Econometrics*, 232, 272–299.

Politis, D. N. and J. P. Romano (1994): "Large sample confidence regions based on subsamples under minimal assumptions," *Annals of Statistics*, 22, 2031–2050.

Politis, D. N., J. P. Romano, and M. Wolf (1999): *Subsampling*, Springer Science & Business Media.

Reichardt, C. S. and H. F. Gollob (1999): "Justifying the use and increasing the power of at test for a randomized experiment with a convenience sample." *Psychological Methods*, 4, 117.

Resnick, S. (1987): *Extreme Values, Regular Variation and Point Processes*, Springer-Verlag.

Resnick, S. I. (2007): *Heavy-Tail Phenomena: Probabilistic and Statistical Modeling*, Springer Science & Business Media.

Romano, J. P. (2004): "On Non-parametric Testing, the Uniform Behaviour of the t-test, and Related Problems," *Scandinavian Journal of Statistics*, 31, 567–584.

Romano, J. P. and A. M. Shaikh (2012): "On the uniform asymptotic validity of subsampling and the bootstrap," *The Annals of Statistics*, 40, 2798–2822.

Romano, J. P. and M. Wolf (1999): "Subsampling inference for the mean in the heavy-tailed case," *Metrika*, 50, 55–69.

Samorodnitsky, G. and M. Taqqu (1994): *Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance*, vol. 1, CRC Press.

Sasaki, Y. and Y. Wang (2022): "Non-robustness of the cluster-robust inference: with a proposal of a new robust method," *arXiv preprint arXiv:2210.16991*.

——— (2023): "Diagnostic Testing of Finite Moment Conditions for the Consistency and Root-N Asymptotic Normality of the GMM and M Estimators," *Journal of Business & Economic Statistics*, 41, 339–348.

White, H. (1984): *Asymptotic Theory for Econometricians*, Academic Press.