Econ 897 (math camp). Part I

Exam

August 21

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You should work on the exam independently. The exam is closed book, so you are not allowed to use any material. Questions about this part are not allowed. If you believe something is ambiguous, clearly identify it in your answer and explain the interpretation you assigned (it should be reasonable) and proceed accordingly. This part is supposed to take one hour.

Each question in each problem has the same weight (3 points). However **the maximal score you can get on this part is** 40 (so your final score is minimum of 40 and what you've actually scored). **That means you don't have to answer all questions to get full score**. You can also use results from previous questions even if you have not solved them. Good luck!

Problem 1 (15 points)**.** Recall that the cocountable topology on a set X consists of the empty set and all sets whose complement in X is countable:

$$\tau_X^c = \{A \subset X \mid A = \varnothing \text{ or } X \setminus A \text{ is countable}\}.$$

(a) Explain shortly why this is indeed well defined topology.

(b) Which sets belong to $\tau_{\mathbb{N}}^c$ (cocountabale topology on the set of all natural numbers)? Does there exist a metric function on $\mathbb{N}$ such that topology generated by this metric function coincides with $\tau_{\mathbb{N}}^c$?

(c) Does there exist a metric function on $\mathbb{R}$ such that topology generated by this metric function coincides with $\tau_{\mathbb{R}}^c$?

(d) Consider sequence $\{x_i = 1/i\}_{i\in\mathbb{N}}$ in $\mathbb{R}$. Does it converge in $\mathbb{R}$ with the topology $\tau_{\mathbb{R}}^c$?

(e) Show that every non-empty open set in $\tau_{\mathbb{R}}^c$ is connected. Is it true that every closed set except X and $\varnothing$ is not connected?

Problem 2 (21 points)**.** Let's define vector space X over $\mathbb{R}$ where $X \subset \mathbb{R}^\infty$ is a set of all sequences $x = (x_0, x_1, \ldots)$ such that sequence $\{x_i\}_{i\in\mathbb{N}}$ is bounded:

$$X = \{(x_0, x_1, \ldots) \in \mathbb{R}^\infty \mid \exists\, C \in \mathbb{R}_+ \,\forall\, i \in \mathbb{N}\, x_i \in [-C, C]\}\,.$$

(a) Explain shortly why it is indeed a vector space. Consider the following norm on this vector space:

$$\|x\|_\bullet = \sum_{i\in\mathbb{N}} \frac{|x_i|}{2^i}.$$

Show that it is a well defined norm on X.

(b) Is X bounded with a metric $d_\bullet$ generated from this norm ($d_\bullet(x,y) = \|x-y\|_\bullet$)?

(c) Consider the subspace $Y \subset X$ of all sequences in X such that they converge (in standard Euclidean metric sense) to a number that belongs to $[-1, 1]$:

$$Y = \left\{ (x_0, x_1, \ldots) \in X \mid \exists\, a \in [-1, 1]\; a = \lim_{i \to +\infty} x_i \right\}.$$

Is Y bounded with the metric $d_\bullet$?

(d) Is Y open, closed or neither with the topology generated by $d_\bullet$?

(e) Is Y open, closed or neither if we change metric to the sup-metric ($d_{\sup}(x, y) = \sup_{i\in\mathbb{N}} |x_i - y_i|$)?

(f) Now consider the subspace $Z \subset X$ of all sequences such that every element of any sequence in Z belongs to $[-1, 1]$:

$$Z = \{(x_0, x_1, \ldots) \in X \mid \forall\, i \in \mathbb{N}\; x_i \in [-1, 1]\}.$$

Is Z compact in the topology generated by the first metric ($d_\bullet$)?

(g) Is Z compact in the topology generated by the sup-metric?

Problem 3 (15 points)**.** Consider the following correspondence $\phi : [0, 10] \rightrightarrows [0, 10]$:

$$\phi(x) = \begin{cases} [0, 10], & \text{if } x \leqslant 1; \\ \{2x - 2, 12 - 2x\} \cup (4, 6), & \text{if } 1 < x \leqslant 3; \\ (4, 6), & \text{if } 3 < x < 4; \\ (8 - x, x - 2), & \text{if } 4 \leqslant x < 6; \\ [2, 4] \cup [6, 8], & \text{if } 6 \leqslant x < 8; \\ [2, 8], & \text{if } 8 \leqslant x < 9; \\ \mathbb{Q} \cap [2, 8], & \text{if } 9 \leqslant x < 10; \\ [2, 8], & \text{if } x = 10. \end{cases}$$

(a) For each of the points in $[0, 10]$ describe whether ϕ is lower hemicontinuous at it.

(b) For each of the points in $[0, 10]$ describe whether ϕ is upper hemicontinuous at it.

(c) Show that ϕ has a continuous selector (precise functional description is not nessesary).

(d) Does ϕ has a closed graph? Name all $x \in [0, 10]$ such that $\phi(x)$ is closed.

(e) Is $\phi([0, 10])$ compact? Does there exist a pair $a < b$ such that $\phi([a, b])$ is not compact?

Upenn Math Camp Part II

Final

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If you are not able to answer an item of a question you can continue to the next assuming that the previous item holds.

1. **(20 points)** In this exercise, you will decompose a multivariate normal likelihood into the product of conditional normals using the Cholesky Decomposition. Let $x \in \mathbb{R}^K$ be a vector of variables, $\mu \in \mathbb{R}^K$ a vector of means and let Σ, a $(K \times K)$ positive definite matrix, represent the covariance matrix.

$$f(x, \mu, \Sigma) = \frac{1}{(2\pi)^{K/2} \det(\Sigma)^{1/2}} e^{-\frac{1}{2}(x-\mu)^t \Sigma^{-1}(x-\mu)}$$

   (a) (4 points) Show that since Σ is positive definite, the matrix Σ^{-1} is also positive definite.

   (b) (1 point) Define $z := (x-\mu)^t \Sigma^{-1}(x-\mu)$. Show that

$$z = (x-\mu)^t L L^t (x-\mu)$$

   where L is a lower triangular, full rank matrix with entries l_{sk}.

   (c) (5 points) Define $y := L^t(x-\mu) \in \mathbb{R}^K$. Show that $y_k = \sum_{s=k}^K l_{sk}(x_s - \mu_s)$.

   (d) (2 points) Show that $z = \sum_{k=1}^K y_k^2$.

   (e) (3 points) Show that $l_{kk} \neq 0$.

   [Hint: Determinant of lower triangular matrix equal to product of diagonals]

   (f) (2 points) Define $\sigma_k := \frac{1}{l_{kk}}$. Show that $y_K = \frac{1}{\sigma_K}(x_K - \mu_K)$ and that for $k = \{1, \dots, K-1\}$,

$$y_k = \frac{1}{\sigma_k}\left((x_k - \mu_k) + \sum_{s=k+1}^{K} \frac{l_{sk}}{l_{kk}}(x_s - \mu_s)\right)$$

   (g) (3 points) Take it as given that $\det(\Sigma) = \prod_{k=1}^K \sigma_k^2$ (Can be shown). Combine this with your previous results to show that:

$$f(x, \mu, \Sigma) = \prod_{k=1}^{K} \frac{1}{(2\pi)^{1/2}\sigma_k} e^{-\frac{1}{2}y_k^2}$$

2. **(10 points)** This part focuses on the implicit function Theorem. Consider an individual optimization problem:

$$\max_{x_1,x_2} u(x_1, x_2)$$

$$s.t. p_1 x_1 + p_2 x_2 = I$$

Suppose u is concave, twice differentiable.

(a) (3 points) Write down individual's first order condition.

(b) (7 points) Compute the price effect $\frac{\partial x_i}{\partial p_j}$ and income effect $\frac{\partial x_i}{I}$.

3. **(10 points)** One version of the separating hyperplane theorems states: Let $D \subseteq \mathbb{R}^n$ be compact and convex, and $E \subseteq \mathbb{R}^n$ be closed and convex. Assume $D \cap E = \emptyset$. Then, there exists a hyperplane $H(p, a)$ such that $p \cdot e < a$ for all $e \in E$ and $p \cdot d > a$ for all $d \in D$.

(a) (2 points) Give an example where D is compact but not convex, and there does not exist such a strictly separating hyperplane.

(b) (3 points) Give an example where the restriction of D being compact is replaced by being bounded (while still being convex), and there does not exist such a strictly separating hyperplane.

(c) (5 points) Give an example where the restriction of D being compact is replaced by being closed (while still being convex), and there does not exist such a strictly separating hyperplane.

For all three subquestions, you could provide either numeric examples or graphic illustrations (make sure you label key properties clearly).

Math Camp Part III   University of Pennsylvania

Final Exam

Instructions

1. This exam is closed book.
2. Please restraint yourself from commenting the solutions with other students.
3. In all True or False questions, if the statement is True, provide a sketch proof. Else, provide a counterexample.
4. Unless stated otherwise, every function involved in each problem is twice differentiable.
5. This exam consists of three questions, each one being of 50 points. Please read the instructions at the beginning of each question.

Method of Moments Estimation (50 Points)

A random variable X follows a **Gamma Distribution**, written $X \sim G(\alpha, \lambda)$, if its PDF is given by:

$$f_X(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} \quad \text{with} \quad x > 0,$$

considering parameters $\lambda > 0, \alpha > 0$. The constant $\Gamma(\alpha)$ is not relevant for this problem. The moment generating function of a Gamma random variable is given by:

$$M_X(t) = \left[\frac{\lambda}{\lambda - t}\right]^\alpha .$$

1. (10 Points) Consider that $X_1 \sim G(\alpha_1, \lambda), X_2 \sim G(\alpha_2, \lambda)$ are independent from each other. Compute the distribution of $Y = X_1 + X_2$.

2. (10 Points) Use the Moment Generating function to compute the first moment of a Gamma distributed random variable.

3. (15 Points) Now, imagine that you have observed n realizations of a random variable $X \sim G(\alpha, \lambda)$, i.e., we know the value of $\{X_1, ..., X_n\}$. The problem is now that we want to estimate the values of (α, λ) that generated these observations. In order to do this, we will follow the Method of Moments, which consists of finding the values $(\hat{\alpha}, \hat{\lambda})$ that solve the following system of equations:

$$\mathbb{E}[X] = \frac{1}{n} \sum_{i=1}^{n} X_i,$$

$$\mathbb{E}[X^2] = \frac{1}{n} \sum_{i=1}^{n} X_i^2.$$

   Use the Method of Moments to estimate $(\hat{\alpha}, \hat{\lambda})$ considering that $\mathbb{E}[X^2] = \alpha/\lambda^2$.

4. (15 Points) We say that the estimator of a parameter θ, $\hat{\theta}$, is consistent if $\hat{\theta} \to_p \theta$. Show that both $(\hat{\alpha}, \hat{\lambda})$ are consistent estimators.

Duality (50 Points)

Consider the following maximization problem:

$$\max \quad x + \alpha y \quad \text{subject to:}$$

$$x^2 + \beta y^2 \leq 1,$$
$$0 \leq x,$$
$$0 \leq y,$$

with $\alpha > 0, \beta > 0$.

1. (5 Points) Briefly argue that this problem is concave and it is such that the Duality gap will be zero.
2. (10 Points) Compute $\mathcal{L}^\star(\mu_1, \mu_2, \mu_3)$.
3. (10 Points) Using a graph, argue that $\mu_2^\star = \mu_3^\star = 0$.
4. (15 Points) Solve this optimization problem by solving the Dual Lagrange function $\mathcal{L}^\star(\mu_1) = \mathcal{L}^\star(\mu_1, 0, 0)$.
5. (10 Points) **Using the Envelope Theorem**, compute $\nabla f \star (\alpha, \beta)$.

Miscellaneous Problems (20 Points)

Answer TWO of the following quick problems.

1. (10 Points) Vivian wants to approximate a function $f : [a, b] \to \mathbb{R}$. She wishes to make the "best" linear approximation according to the $p \in [1, \infty)$ norm, i.e., she wishes to find $\alpha, \beta \in \mathbb{R}$ such that they solve:
$$\min \int_{[a,b]} |f(x) - \alpha - \beta x|^p \, d\lambda.$$
Vivian knows that this problem is convex, and hence the first order conditions are sufficient. Which assumptions does Vivian need to impose on her problem in order to be able to use this first-order conditions approach?

2. (10 Points) Briefly explain why defining the CDF of a random variable X is enough for us to be able to compute the probability that $X \in B$ as long as B is Borelean.

3. (10 Points) True or False: Consider the following maximization problem:
$$\max \quad f(x) \quad \text{subject to:}$$
$$g(x) \geq 0.$$
If we assume that f is concave then the set of maxima $\mathcal{D}^\star$ is convex.