Income Anonymity*

Maya Eden[†] Luis Mota Freitas[‡]

August 7, 2023

Abstract

This paper characterizes a normative criterion for ranking income distributions based on two axioms: Pareto and Income Anonymity. Pareto requires that, if everyone supports a simultaneous change in the distribution of income and in prices, then that change is socially desirable. Income Anonymity requires that, whenever everyone faces the same prices, social welfare can be evaluated based on the anonymized distribution of income. When individuals have heterogeneous preferences, there exists at most one social preference relation that satisfies both axioms. Given current expenditure patterns in the United States, this social welfare function ranks income distributions approximately according to the sum of log incomes.

JEL Classification: D6
Keywords: resource equality, additive separability, non-utilitarian welfare criteria

*We are grateful for helpful conversations with Matthias Doepke, Simon Fuchs, Antoine Germain, Erwin Ooghe, Francois Maniquet, Nicola Persico, Paolo Piacquadio, Uzi Segal, Itai Sher, Peter Wakker and John Weymark, and for comments from seminar participants at the University of Oklahoma, the University of Oslo, the Atlanta Fed, the University of Cologne, the Welfare Economics and Economic Policy virtual seminar, Columbia University, Birkbeck University of London, the University of Zurich, the PSE workshop on "Around and Beyond GDP" and Virginia Commonwealth University.

[†]Brandeis University; Centre for Economic Policy Research (CEPR); and the University of Oxford's Global Priorities Institute. Contact address: Economics Department, Brandeis University, 415 South Street, Waltham, MA 02453; Email: meden@brandeis.edu.

[‡]Northwestern University and Global Priorities Institute, University of Oxford. Email: luismotafreitas@u.northwestern.edu.

1 Introduction

Economic analysis often aims to identify the optimal policy in various contexts, such as determining the best trade agreements or designing a fair taxation system. The success of this analysis depends on a clear understanding of the social objective. The overarching goal of welfare economics in general, and this paper in particular, is to narrow down the set of plausible social welfare criteria based on ethical principles.

One such principle is that everyone should be treated equally. In practice, applied welfare analysis usually reflects this principle by assigning equal value to the income of each individual. We study this approach through a new axiom, Income Anonymity, which states that each person's income matters equally for social welfare. Although we are the first to formally study this axiom (to the best of our knowledge), it is implicit in any welfare criterion that is based on the statistical properties of the income distribution, such as the Gini index, GDP per-capita or the Atkinson Index (see, for example, Dollar et al. [2015] or Kraay et al. [2023]).

Our main result is that, if individual preferences are heterogeneous, then there exists at most one social preference relation that satisfies both Pareto and Income Anonymity. This means that when we agree on the applicability of these principles, then we must also agree on everything else, from optimal trade agreements to optimal redistributive policy.

The level of inequality aversion implied by this welfare criterion depends on the joint distribution of individual preferences. For example, when preferences are homothetic, then the social preference ranks income distributions according to the sum of log incomes. When individual preferences are quasilinear, then the social ranking exhibits no inequality aversion, and ranks income distributions according to per-capita income.

For certain individual preference profiles, the two axioms are inconsistent. Consistency requires that individual preferences satisfy a certain separability condition. Loosely speaking, the condition requires that we can *imagine* that any two individuals who face the same budget have the same marginal utility of income. That is, there is nothing about the profile of ordinal preferences that can reject this hypothesis.

We calibrate our social welfare criterion based on the cross sectional distribution of consumption expenditures, using data from the United States Consumer Expenditure survey. We find that our social welfare criterion ranks income distributions approximately according to the sum of log incomes. This means that, if we accept Pareto and Income Anonymity, then, at current prices, policymakers should strive to maximize the geometric mean of the income distribution.

This paper contributes to a rich literature on the axiomatic characterization of social preferences. The approach here is closely related to the money-metric utility approach (Deaton and Muellbauer [1980], Fleurbaey and Maniquet [2011], Fleurbaey and Maniquet [2018]). The money-metric utility approach proposes a social preference relation that is an aggregation of individuals' equivalent incomes at common reference prices ("money-metric utilities"). There are many social welfare functions that satisfy anonymity with respect to the distribution of money-metric utilities. The anonymity condition that we consider is more restrictive, as it requires anonymity with respect to the distribution of income at any prices – not just at the reference price. The social preferences must therefore be symmetric in money-metric utilities for any reference price.

This paper is also related to the literature on price-independent welfare prescriptions (Roberts [1980], Slesnick [1991], Blackorby et al. [1993] and Fleurbaey and Blanchet [2013]). Roberts [1980] studies the conditions under which income distributions can be ranked irrespective of prices, and finds that they are "highly restrictive".[1] This sparked a debate about whether it is appropriate to require welfare prescriptions to be independent of prevailing prices (Fleurbaey and Blanchet [2013]). The social preference relation that we derive generates price-independent welfare prescriptions only in special cases; in general, the amount of inequality aversion depends on the prices that consumers face.

In addition, this paper is related to the literature on the inconsistency of Pareto and other normative principles. Sen [1970a] and Kaplow and Shavell [2001] show that the Pareto principle leaves limited room for expressing a concern for non-welfarist normative principles such as freedom, justice or procedural fairness (see also Sher [2021]). In the context of resource allocation problems, Sen [1970b], Suzumura [1981a], Suzumura [1981b], Suzumura [1983], Tadenuma [2002] and Fleurbaey and Trannoy [2003] uncover tensions between Pareto and various egalitarian principles pertaining to the fair allocation of resources. Fleurbaey and Trannoy [2003] show that, whenever preferences are heterogeneous, Pareto is inconsistent with a social preference for redistributing resources from rich to poor (the Pigou-Dalton principle).[2] We add to these impossibility results by showing that, for some preference profiles, Pareto is also inconsistent with Income Anonymity, which can be interpreted as a procedural fairness requirement. However, in light of the negative results in this literature, our positive results are perhaps more surprising: we show that, under standard assumptions, there is no conflict between Income Anonymity and Pareto, even when preferences are heterogeneous.

Finally, this paper is related to the literature on the axiomatic foundations of additively-

[1] He finds that, unless the welfare criterion is dictatorial, both individual and social preferences must be homothetic.

[2] More precisely, the Pigou-Dalton principle states that if one person has more of every good than another person, then any transfer of goods that maintains this ordering but reduces inequality is a socially-desirable transfer.

separable utility functions (Gorman [1968], Wakker [1989], Blackorby et al. [1998] and Qin and Rommeswinkel [2022]). Here, we establish that if the social preference relation satisfies Income Anonymity and Pareto (and individual preferences are heterogeneous), then it must be additively separable in individual budgets. Unlike previous results in this literature, our result does not rely on the assumption of weaker separability conditions (i.e., that it is possible to rank subgroups of variables independently from one another). Here, additive separability is obtained from a combination of a monotonicity condition (Pareto) and a symmetry requirement (Income Anonymity).

2 Preliminaries

There are $2 \leq I < \infty$ individuals indexed $i = 1, ..., I$, and $2 \leq J < \infty$ goods indexed $j = 1, ..., J$. Throughout, we use subscripts for indicating individuals and superscripts for indicating goods; for example, c_i^j is individual i's consumption of good j.

Individuals' preferences over goods are represented by the utility functions, $\{u_i\}_{i=1}^I$. Define the indirect utility function, v_i, as

$$v_i(m, p) = \max_{c^1, ..., c^J} u(c^1, ..., c^J) \text{ s.t. } \sum_{j=1}^{J} p^j c^j \leq m$$

Here, $m \in \mathbb{R}_{++}$ is income and $p = (p^1, ..., p^J) \in \mathbb{R}_{++}^J$ is a vector of prices. We assume that the indirect utility functions, $\{v_i\}$, are continuously differentiable, strictly increasing in m, and weakly decreasing in all prices.[3] The ranking $\preceq_i$ on $\mathbb{R}_{++} \times \mathbb{R}_{++}^J$ denotes the individual's indirect ranking of combinations of income and prices. Note that each $(m, p) \in \mathbb{R}_{++} \times \mathbb{R}_{++}^J$ represents a budget constraint; thus, $\preceq_i$ represents the individual's preferences over budget sets.

Throughout, we use bold letters to denote vectors of length I (the number of individuals). The social preference ranking, $\preceq$ (with no subscript), is defined over elements of the form $(\mathbf{m}, \mathbf{p})$, where $\mathbf{m} = (m_1, ..., m_I) \in \mathbb{R}_{++}^I$ is the distribution of income and $\mathbf{p} = (p_1, ..., p_I) \in \left(\mathbb{R}_{++}^J\right)^I$ are individual price vectors (individual i faces the prices $p_i = (p_i^1, ..., p_i^J)$). For a price vector p, the notation $(\mathbf{m}, p)$ is used as a shorthand for the allocation $(\mathbf{m}, (p, ..., p))$, in which everyone faces the same prices.

[3]Note that, as indirect preferences are always homogeneous of degree 0 and strictly increasing in m, it follows that they must be locally strictly decreasing in at least one price.

2.1 Axioms

We consider two axioms on the social preference relation. The first is the standard Pareto condition, which is stated as follows.

Axiom (Pareto)**.** *For each* $\mathbf{m}, \mathbf{m}', \mathbf{p}, \mathbf{p}'$*, (a) if* $(m_i, p_i) \preceq_i (m_i', p_i')$ *for all* i*, then* $(\mathbf{m}, \mathbf{p}) \preceq (\mathbf{m}', \mathbf{p}')$*; and (b) if, in addition,* $(m_i, p_i) \prec_i (m_i', p_i')$ *for some* i*, then* $(\mathbf{m}, \mathbf{p}) \prec (\mathbf{m}', \mathbf{p}')$*.*

This axiom is sometimes referred to as "unanimity". It states that, if all individuals support a certain change in prices and incomes, then it should be considered socially desirable.

The second axiom is Income Anonymity:

Axiom (Income Anonymity)**.** *For every common price vector,* $p \in \mathbb{R}^J_{++}$*, and income distribution,* $\mathbf{m} \in \mathbb{R}^I_{++}$*, it holds that*

$$(\mathbf{m}, p) \sim ((m_{\sigma(1)}, ..., m_{\sigma(I)}), p)$$

for any permutation $\sigma : \{1, ..., I\} \mapsto \{1, ..., I\}$*.*

Income Anonymity states that the normative ranking of income distributions should not depend on which person receives which income. Instead, income distributions can be ranked anonymously. This anonymity condition is ubiquitous in applied welfare analysis. Below, we discuss two alternative justifications for it.

Informational constraints. According to some prominent welfare criteria, what matters is the distribution of utilities, not incomes. The problem is that we currently lack the tools to ascertain the individual mappings between consumption and cardinal utilities. Therefore, for practical reasons, we assume that any two people who face the same budget constraint are equally "productive" in converting consumption into utility; or, at least, that there is no good reason to think that one person is better at producing utils than any other. This type of assumption is captured by Income Anonymity.

The normative argument. Alternatively, this anonymity condition reflects the normative argument in Dworkin [1981].[4] Dworkin [1981] argues that fairness should be judged based on the distribution of income, regardless of individuals' preferences, and regardless of prevailing prices.[5] According to Dworkin, what matters is the value of the resources devoted

[4] Varian [1976] also considers the question of how to define a fair allocation when people have heterogeneous preferences. He discusses the sense in which an equal distribution of income results in a better market allocation than an unequal distribution of income.

[5] See Keller [2002] for a discussion. A similar notion of equality is also reflected in the Laisser-Faire axiom in Fleurbaey and Maniquet [2006], which postulates that there is no scope for redistribution between two people who face the same earning opportunities, even when they choose to work different amounts.

to each person's life – and not how that individual chooses to use those resources, or the utility that he derives from their use.

Dworkin writes, "the true measure of the social resources devoted to the life of one person is fixed by asking how important, in fact, that resource is for others." This implies a measure of value that is based on common equilibrium prices. To reflect this, Income Anonymity requires anonymity with respect to income only when all people face the same price vectors. In this case, the value of each person's consumption bundle is given by that person's income.

Like all axioms, the normative appeal of our axioms is not universal. It is possible to come up with examples in which each of our axioms contradicts basic moral intuitions. The Pareto condition is unappealing in circumstances involving addiction, as people's choices go against their own best interests. Similarly, Income Anonymity is problematic in situations in which people have very different needs (see Sen [1980]). Nonetheless, there are many situations in which these two moral principals seem like reasonable starting points.

3 Uniqueness

On their own, both Pareto and Income Anonymity are consistent with various degrees of inequality aversion. For example, the Pareto condition is consistent with any social welfare function of the form

$$W(\mathbf{m}, \mathbf{p}) = \sum_{i=1}^{I} \phi_i(v_i(m_i, p_i))$$

where ϕ_i is strictly increasing. In this class of social welfare functions, the concavities of the functions $\{\phi_i\}_{i=1}^{I}$ determine aversion to inequalities in utilities, and hence, indirectly, aversion to income inequality.

Similarly, Income Anonymity it is consistent with any social welfare function of the form

$$W(\mathbf{m}, \mathbf{p}) = \sum_{i=1}^{I} \frac{m_i^{1-\eta}}{1-\eta}$$

here, η determines the degree of inequality aversion. This functional form is often identified as the Atkinson index (Atkinson [1970]). In this class of social welfare functions, Income Anonymity is satisfied because the social ranking is symmetric with respect to all incomes.

In addition, when all individuals have the same preferences, there are many social preference relations that are consistent with both axioms. For example, when $v_i = v$ for all i, then both axioms are satisfied by any social preference relation that is represented by

$$W(\mathbf{m},\mathbf{p}) = \sum_{i=1}^{I} \phi(v(m_i, p_i))$$

where ϕ is any strictly increasing function. Here, the concavity of ϕ determines the amount of inequality aversion.

The following theorem establishes that, when preferences are heterogeneous, the combination of Pareto and Income Anonymity uniquely characterizes the entire social preference relation. In particular, at most one level of inequality aversion is consistent with both axioms.

To state the theorem, it is necessary to introduce the following notation. Let $c_i^j(m,p)$ denote individual i's consumption of good j, given the budget (m,p):

$$c_i(m,p) = (c_i^1(m,p),...,c_i^J(m,p)) = \arg\max_{c^1,...,c^J} u_i(c^1,...,c^J) \text{ s.t. } \sum_{j=1}^{J} p^j c^j \le m$$

and let $e_i(m,p,p')$ denote the solution to the indifference condition,

$$(m,p) \sim_i (e_i(m,p,p'),p')$$

The quantity $e_i(m,p,p')$ is individual i's equivalent income at prices p', given a budget (m,p).[6]

We are now ready to state our theorem.

Theorem 1. *Assume that $c_1^j(m_0,p_0) > c_2^j(m_0,p_0)$. If there exists a social preference relation that satisfies Pareto and Income Anonymity, then it is unique and represented by*

$$W(\mathbf{m},\mathbf{p}) = \sum_{i=1}^{I} \int_1^{e_i(m_i,p_i,p_0)} \frac{1}{c_1^j(m',p_0) - c_2^j(m',p_0)} dm' \qquad (1)$$

This theorem establishes that, when preferences are heterogeneous, then there is at most one social preference relation that satisfies both of our axioms. When it exists, it is represented by (1); it particular, it must be (a) continuous and (b) additively separable in the

[6] To see that $e_i(m,p,p')$ is well-defined, note that, for $m' > 0$ sufficiently large, the consumption bundle $c_i(m,p)$ is affordable given the budget (m',p'); it follows that, in this case, $(m,p) \preceq_i (m',p')$. In addition, for $m' > 0$ sufficiently small, it holds that $(m/J)/p^j > m'/p'^j$: that is, with the budget $(m/J,p)$, the individual can buy the entire amount of each good that is affordable by the budget (m',p'). Because $c_i^j(m',p') \le m'/p'^j$ for every j, it follows that $u_i(c_i(m',p')) \le u_i(m'/p'^1,...,m'/p'^J) \le u_i((m/J)/p^1,...,(m/J)/p^J) \le u_i(c_i(m,p))$. Hence, for this m', it holds that $(m',p') \preceq_i (m,p)$. By the continuity of individual preferences, it follows that there exists some $m' = e_i(m,p,p')$ such that $(m',p') \sim (m,p)$. Because indirect preferences are strictly increasing income, m' is unique and hence $e_i(m,p,p')$ is uniquely defined.

individual budgets, $(m_1, p_1), ..., (m_I, p_I)$. These are desirable properties which we did not assume, but rather obtained as an implication of combining Pareto and Income Anonymity.

In addition, the theorem provides a formula for computing the social preference relation based on the distribution of ordinal preferences. As we illustrate in section 3.1, this formula can be used to obtain analytical characterizations of the social welfare function in some important special cases. In section 5, we show how this formula can be used for estimating a social welfare function based on consumer expenditure data.

The complete proof of the theorem is in the appendix, together with other omitted proofs. Below we sketch the key steps of the proof of uniqueness. Consider a simple case in which there are two individuals and two goods ($I = J = 2$). Figure 1 presents the two individuals' indifference curves over combinations of income and prices (where the prices p_0 and p differ only in the price of good 2). Their indifference curves are upward sloping, because higher prices can be compensated with higher incomes. In this figure, the two individuals have different preferences over the two goods, so their indifference curves are not the same.

In this figure, individual 1 is indifferent between (m, p_0) and (m'_1, p), and individual 2 is indifferent between (m, p_0) and (m'_2, p). Pareto requires that, when all individuals are indifferent, then the social preference relation is indifferent as well. Consequently, Pareto implies

$$((m, m), p_0) \sim ((m'_1, m'_2), p) \tag{2}$$

Income Anonymity requires that, at any given prices, income distributions can be ranked anonymously. In particular, social preferences must be indifferent with respect to switching the incomes of the two individuals at the prices p:

$$((m'_1, m'_2), p) \sim ((m'_2, m'_1), p) \tag{3}$$

Furthermore, note that, in this figure, individual 1 is indifferent between (m'_2, p) and (m_2, p_0), and individual 2 is indifferent between (m'_1, p) and (m_1, p_0):

$$(m'_1, p) \sim_2 (m_1, p_0) \text{ and } (m'_2, p) \sim_1 (m_2, p_0) \tag{4}$$

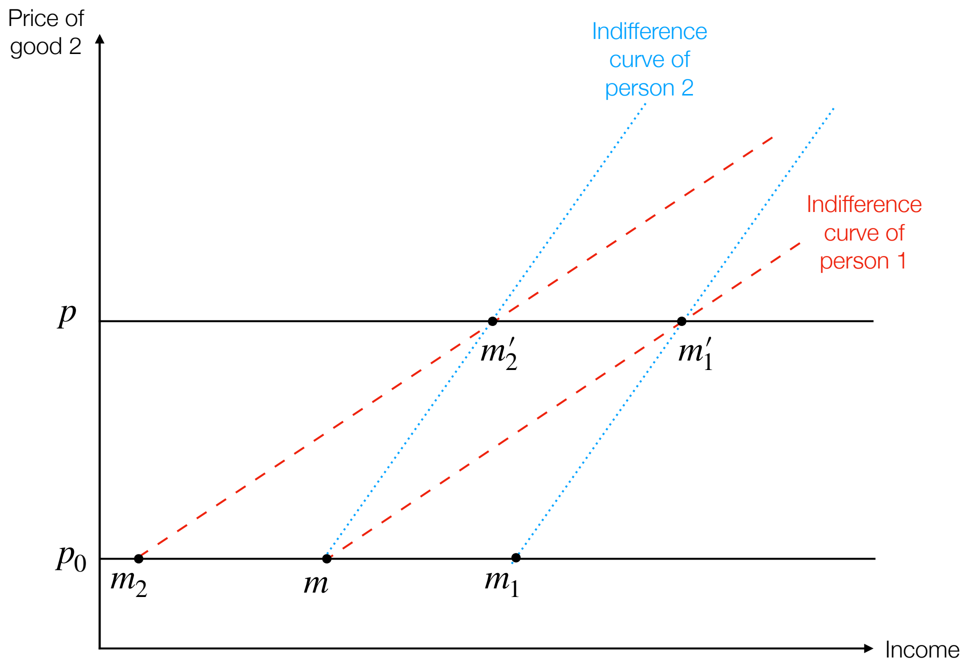
By Pareto indifference, it follows that

$$((m'_2, m'_1), p) \sim ((m_2, m_1), p_0) \tag{5}$$

By the transitivity of the indifference relation, (2), (3) and (5) imply

$$((m, m), p_0) \sim ((m'_1, m'_2), p) \sim ((m'_2, m'_1), p) \sim ((m_2, m_1), p_0) \tag{6}$$

Figure 1: Sketch of the proof of uniqueness



Consequently, given a price level of p_0, any social preferences that satisfy Pareto and Income Anonymity must be indifferent between the equal allocation of income, (m, m), and the unequal allocation, (m_2, m_1). This is a restriction on the amount of inequality aversion. A sufficiently inequality-averse social preference relation would strictly prefer the equal allocation over the unequal one; a sufficiently inequality-tolerant relation would prefer the unequal allocation instead. Neither of these preferences would be consistent with the combination of Pareto and Income Anonymity. The uniqueness of the social preference relation follows from the uniqueness of the level of inequality aversion.

3.1 Examples

Theorem 1 provides a formula for computing a social welfare function that represents the unique social preference relation that satisfies our axioms. In this section, we use this formula to characterize the social ranking of income distributions for three classes of individual preferences: homothetic preferences, Stone-Geary preferences, and quasilinear preferences. Here, we focus on illustrating how to compute the social welfare function in (1), and studying its implications. In Appendix E we establish that, for these preference profiles, the two axioms are, in fact, consistent, and satisfied by the social preference relation (1).

For simplicity, we restrict attention to the social ranking of income distributions given $p = p_0$ (note that, as the social preference ranking is unique, it will be the same regardless of how we specify p_0, provided that consumption patterns are heterogeneous given p_0). Given this specification,

$$W(\mathbf{m}, p) = \sum_{i=1}^{I} \int_{1}^{m_i} \frac{1}{c_1^j(m', p) - c_2^j(m', p)} dm' \tag{7}$$

Consider the following special cases.

Homothetic preferences. When preferences are homothetic, each individuals' consumption bundle changes proportionately with his income: that is, for each individual, i, and good, j, there exists $\alpha_i^j(p) \in [0, 1]$ such that

$$c_i^j(m, p) = \alpha_i^j(p)m$$

Substituting into (7), we have that

$$W(\mathbf{m}, p) = \sum_{i=1}^{I} \int_{1}^{m_i} \frac{1}{\alpha_1^j(p)m' - \alpha_2^j(p)m'} dm' = \sum_{i=1}^{I} \int_{1}^{m_i} \frac{1}{(\alpha_1^j(p) - \alpha_2^j(p))m'} dm'$$

$$= \frac{1}{\alpha_1^j(p) - \alpha_2^j(p)} \sum_{i=1}^{I} \int_{1}^{m_i} \frac{1}{m'} dm' = \frac{1}{\alpha_1^j(p) - \alpha_2^j(p)} \sum_{i=1}^{I} (\ln(m_i) - \ln(1)) =$$

$$\frac{1}{\alpha_1^j(p) - \alpha_2^j(p)} \sum_{i=1}^{I} \ln(m_i)$$

As $\alpha_1^j(p) > \alpha_2^j(p)$, this social preference relation ranks income distributions according to the sum of log incomes. Note that we did not assume anywhere that individual preferences are "log" - individuals may have arbitrary constant-elasticity-of-substitution preferences, or other homothetic preferences. Regardless, the social ranking will rank income distributions according to their geometric means.

Stone-Geary preferences. Consider a modification of the homothetic case, in which preferences are non-homothetic due to the presence of a common "subsistence bundle". To survive, each individual must consume at least $\underline{c}^j \geq 0$ goods of type j. However, after consuming the subsistence bundle, individuals' preferences over their remaining consumption are homothetic; formally, there exist $\{\alpha_i^j(p) \in [0, 1]\}$ such that

$$c_i^j(m, p) = \underline{c}^j + \frac{\alpha_i^j(p)}{p^j} \left(m - \sum_{k=1}^{J} p^k \underline{c}^k \right)$$

In this case,

$$c_1^j(m,p) - c_2^j(m,p) = \frac{\alpha_1^j(p) - \alpha_2^j(p)}{p^j}\left(m - \sum_{k=1}^{J} p^k \underline{c}^k\right)$$

Following similar steps as in the homothetic case (replacing m with $m - \sum_{k=1}^{J} p^k \underline{c}^k$), we obtain

$$W(\mathbf{m},p) = \frac{p^j}{\alpha_1^j(p) - \alpha_2^j(p)} \sum_{i=1}^{I} \ln\left(m_i - \sum_{k=1}^{J} p^k \underline{c}^k\right)$$

Note that, in this case, the social ranking of income distributions depends on the cost of the subsistence bundles. The social preference relation is more averse to income inequality when the subsistence bundle is more expensive.

Quasilinear preferences. Assume that consumption patterns are as follows:

$$c_i^j(m,p) = c_i^j(1,p)\ \forall j > 1$$

$$c_i^1(m,p) = \frac{m - \sum_{j=2}^{J} c_i^j(1,p)}{p^1}$$

These consumption patterns arise when preferences are quasilinear, that is, when $u_i(c^1, ..., c^J) = c^1 + g_i(c^2, ..., c^J)$, where g_i is a strictly concave function. Note that the functions $\{g_i\}_{i=1}^{I}$ may be heterogeneous, and take arbitrary functional forms.

Assume that $c_1^2(m,p) > c_2^2(m,p)$ for some m. Given these consumption patterns, it holds that

$$c_1^2(m,p) - c_2^2(m,p) = c_1^2(1,p) - c_2^2(1,p)$$

We specify $j = 2$ and substitute into (7):

$$W(\mathbf{m},p) = \sum_{i=1}^{I} \int_1^{m_i} \frac{1}{c_1^j(1,p) - c_2^j(1,p)} dm' = \frac{1}{c_1^j(1,p) - c_2^j(1,p)} \sum_{i=1}^{I} \int_1^{m_i} 1 dm'$$

$$= \frac{1}{c_1^j(1,p) - c_2^j(1,p)} \sum_{i=1}^{I} (m_i - 1)$$

This social welfare function ranks income distributions according to the sum of individual incomes – that, is, the simple measure of per-capita income. Unlike the previous examples, this preference ranking exhibits no aversion to income inequality.

4 Existence

Theorem 1 establishes that *if* there exists a social preference relation that is consistent with our axioms, then there is only one, and it is represented by (1). However, there is also a possibility that there are no social preference relations that satisfy both of the axioms.

The following theorem establishes two equivalent conditions on the profile of individual preferences which are necessary and sufficient for the consistency of our axioms.

Theorem 2. *The following conditions are equivalent.*

(a) There exists a social preference relation that satisfies Pareto and Income Anonymity.

(b) The following two conditions hold:

(i) For every i, i', j, p_0 and m_0,

$$c_i^j(m_0, p_0) > c_{i'}^j(m_0, p_0) \Rightarrow c_i^j(m, p_0) > c_{i'}^j(m, p_0) \ \forall m$$

(ii) If $c_1^j(m_0, p_0) > c_{i'}^j(m_0, p_0)$ for some m_0, p_0, j and i', then, for every m, i and p,

$$\int_{e_1(m,p,p_0)}^{e_i(m,p,p_0)} \frac{1}{c_1^j(m', p_0) - c_{i'}^j(m', p_0)} dm' = \int_{e_1(1,p,p_0)}^{e_i(1,p,p_0)} \frac{1}{c_1^j(m', p_0) - c_{i'}^j(m', p_0)} dm'$$

(c) There exists a function $\mu : \mathbb{R}_{++} \times \mathbb{R}_{++}^J \mapsto \mathbb{R}$ and functions $\{\gamma_i : \mathbb{R}_{++}^J \mapsto \mathbb{R}\}_{i=1}^I$ such that, for every i, the function $(m, p) \mapsto \mu(m, p)+\gamma_i(p)$ is a representation of the indirect preferences $\preceq_i$.

The equivalence between (a) and (b) is useful for verifying whether our two axioms are consistent given a certain profile of individual preferences. Note that both conditions (b.i) and (b.ii) do not depend on a particular representation of individual preferences. Thus, they can be easily verified based on the ordinal preference relations. In section 4.1, we provide graphical illustrations of what it means to violate conditions (b.i) and (b.ii), and explain why this implies the inconsistency of our axioms.

The equivalence between (a) and (c) is more useful for developing an economic intuition for the conditions that a preference profile must satisfy in order to avoid a conflict between our axioms. This condition requires that the profile of individual preferences is consistent with a model in which individuals' objective is to maximize *cardinal* utilities of the form $\{\mu + \gamma_i\}$. Note that it does not require that this representation actually captures any meaningful notion of cardinal utility; rather, it requires only that if we hypothesized this to be the case, we could not reject our hypothesis based solely on people's ordinal preference relations.

In this hypothetical environment, if two people face the same budget, (m,p), then they also have the same marginal utility of income, $\frac{\partial\mu(m,p)}{\partial m}$ – at the margin, all individuals are equally productive in transforming consumption into utils. In this case, a utilitarian criterion of maximizing the sum of individuals' cardinal utilities is the only social ranking that is consistent with both Pareto and Income Anonymity:

Corollary 1. *When there exists a social preference relation that satisfies Pareto and Income Anonymity, then it is represented by*

$$W(\mathbf{m},\mathbf{p}) = \sum_{i=1}^{I} \left(\mu(m_i, p_i) + \gamma_i(p_i)\right)$$

where, for each i, $\mu + \gamma_i$ is a representation of individual i's preferences.

This corollary is, in fact, established as a step in the proof of Theorem 1.

4.1 What can go wrong

The second clause of Theorem 2 establishes two necessary conditions for the consistency of our axioms. In this section, we illustrate why our axioms are in conflict when each one of them is violated.

A violation of condition (b.i) is illustrated in Figure 2. To see that these preferences violate condition (b.i), note that, by Roy's identity,

$$-\frac{\frac{\partial v_i(m,p)}{\partial p^2}}{\frac{\partial v_i(m,p)}{\partial m}} = c_i^2(m,p)$$

The left hand side is the marginal rate of substitution between m and the price of good 2, which is the inverse of the slope of the indifference curve at the point (m,p) in the $m \times p^2$ space (holding constant the prices p^1 and $p^3,...,p^I$). In Figure 2, it holds that:

- At the point (m_1', p), the slope of individual 1's indifference curve is larger than the slope of individual 2's indifferent curve; hence, $c_1^2(m_1', p) < c_2^2(m_1', p)$.

- At the point (m_1', p), the opposite is true: the slope of individual 2's indifference curve is larger than the slope of individual 1's indifferent curve; hence, $c_1^2(m_1', p) > c_2^2(m_1', p)$.

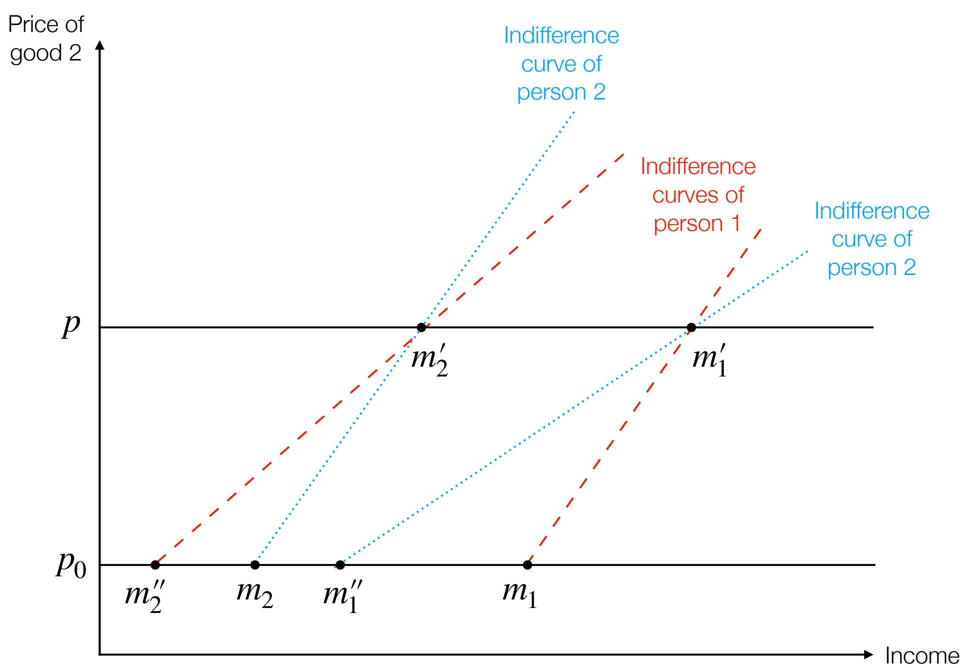
Hence, in this case, we have a "preference flip": at low level of incomes, person 1 is more sensitive to the price of some good, but at higher incomes, person 2 becomes more sensitive. This is a violation of condition (b.i).

To see why our axioms are inconsistent in this scenario, observe that

$$
\begin{aligned}
((m_1, m_2), p_0) &\sim ((m_1', m_2'), p) && \text{(Pareto)} \\
&\sim ((m_2', m_1'), p) && \text{(Income Anonymity)} \\
&\sim ((m_2'', m_1''), p_0) && \text{(Pareto)} \\
&\sim ((m_1'', m_2''), p_0). && \text{(Income Anonymity)}
\end{aligned}
$$

This is a contradiction to the Pareto condition, as $m_1'' < m_1$ and $m_2'' < m_2$.

Figure 2: A violation of condition (b.i)



A violation of condition (b.ii) is illustrated in Figure 3. It may not be immediately obvious why these preferences violate condition (b.ii); this is proven in Appendix D. In what follows, we show that, given the preferences depicted in Figure 3, our axioms are in conflict. Following the steps laid out in expressions 2-6 (with a final additional application of Income Anonymity), we obtain

$$((m, m), p_0) \sim ((m_1, m_2), p_0) \tag{8}$$

Similarly, these axioms imply that

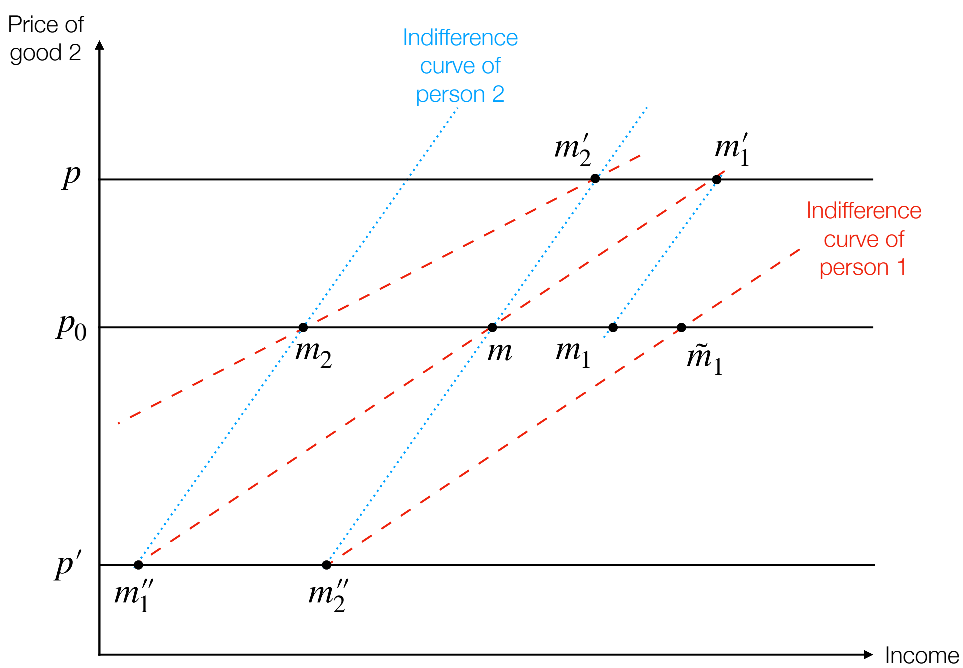
$$\begin{aligned}((m, m), p_0) &\sim ((m_1'', m_2''), p') && \text{(Pareto)} \\ &\sim ((m_2'', m_1''), p') && \text{(Income Anonymity)} \\ &\sim ((\tilde{m}_1, m_2), p_0) && \text{(Pareto)}\end{aligned}$$

Thus, by the transitivity of the indifference relation, (8) implies that

$$((m_1, m_2), p_0) \sim ((m, m), p_0) \sim ((\tilde{m}_1, m_2), p_0)$$

But this would be a violation of the Pareto condition, as individual 2 is indifferent between the right hand side and the left hand side, and individual 1 strictly prefers the right hand side (as $\tilde{m}_1 > m_1$). We have thus established that, given this particular preference profile, there exists no social preference relation that is consistent with both Pareto and Income Anonymity. To guarantee existence, the profile of individual preferences must be such that $\tilde{m}_1$ and m_1 always line up perfectly; this property is guaranteed by condition (b.ii).

Figure 3: A violation of condition (b.ii)



These examples suggest that our two axioms are unlikely to be consistent in practice, especially for large populations. When there are very many people, it becomes increasingly likely that there is at least one pair of individuals that exhibits a "preference flip" as in

Figure 2. One such pair is sufficient for rendering the two axioms incompatible. On top of that, Figure 3 illustrates that, even if we happen to stumble upon a preference profile for which the two axioms *are* consistent, a small perturbation of a single indifference curve may render them incompatible.

This fragility prompts the question: is Theorem 1 of any use, given that the two axioms are consistent only in knife-edge cases? One way to think about this more formally is to consider a multi-profile setting, in which the social preference relation is a function of the individual preferences ($\preceq = F(\preceq_1, ..., \preceq_I)$). In this setting, we might require the social preference relation to be consistent with both axioms whenever this is possible. Our uniqueness result can then be used to characterize the function F for a "measure-zero" set of individual preference profiles (we put "measure-zero" in quotation marks because we did not formally define a topology on the set of preference profiles). If we are willing to make some continuity assumptions on F, then this also tells us something about what the social preference relation looks like for individual preference profiles that are in the "neighborhoods" of these preference profiles.

5 Application

We now turn to a calibration of the social welfare function identified in (1) based on consumption expenditure data. This exercise serves two purposes. Our main aim is to provide a practical theory-grounded approach for decisionmaking based on our results. We present a calibration procedure that can be used to estimate the social welfare function, and illustrate this procedure using US consumer expenditure data. We find that, for US consumers, our social welfare function ranks income distributions approximately according to the sum of log incomes.

A secondary goal is to assess the degree to which consumption patterns are consistent with the conditions of Theorem 2. As we explain below, our separability condition implies certain regularities in consumption patterns. In our application, we find that the consumption patterns in the US are roughly consistent with these regularities.

5.1 Calibration procedure

For the purpose of this calibration, it is useful to set $p_0 = p$, where p are the prices that the surveyed consumers face. Given the choice $p_0 = p$, we have that that $e_i(m, p, p) = m$, so the social welfare function can be rewritten as in expression (7).

To calculate this expression, we must know the demand functions, $c_1^j(\cdot, p)$ and $c_2^j(\cdot, p)$. Unfortunately, surveys typically tell us only how much of each good individuals consume given their actual budgets, and not how much they would consume given hypothetical budgets. To proceed, we need to relate demand functions to the cross-sectional distribution of expenditures.

Let $C_x^j(m, p)$ denote the x-th percentile of the distribution of expenditures on good j, given income m and prices p. Consider the following assumption.

Assumption 1. *If, for some individual i with income m, it holds that $c_i^j(m, p) = C_x^j(m, p)$, then, for every m', it holds that $c_i^j(m', p) = C_x^j(m', p)$.*

This assumption says that if Anne's consumption of asparagus is at the 75th percentile among all people of her income level, then this would be true regardless of what their income level happens to be. Furthermore, it implies that rich people have the same distribution of preferences as poor people. Of course, differences in income may produce differences in expenditure patterns. But, if rich people were poor, their consumption choices would be the same as the consumption choices of the other poor. Similarly, if poor people were rich, their consumption patterns would be similar to the consumption patterns of the other rich. This assumption neglects the possibility of historical or other factors that affect both income and tastes.

Using this assumption, we can identify "individual 1" as an individual whose consumption of good j is at the $\bar{x}$-th percentile among others with the same income level, and "individual 2" as an individual whose consumption is at the $\underline{x}$-th percentile among others with the same income level. We can empirically estimate the functions $c_2^j(\cdot, p) = C_{\underline{x}}^j(\cdot, p)$ and $c_1^j(\cdot, p) = C_{\bar{x}}^j(\cdot, p)$, based on how the relevant percentiles of the consumption of good j change with income. We then compute a discrete approximation of the integral

$$\mu(m, p) = \int_{\hat{m}}^{m} \frac{1}{C_{\bar{x}}^j(m', p) - C_{\underline{x}}^j(m', p)} dm',$$

where $\hat{m}$ is the lowest income level in the dataset. Our estimated social welfare function is then

$$W(\mathbf{m}, p) = \sum_{i=1}^{I} \mu(m, p)$$

Note that, in this procedure, we make two arbitrary choices. First, we need to choose the expenditure category j (food, housing, etc). Second, we need to choose which two percentiles to compare ($\underline{x}$ and $\bar{x}$). In principle, different choices may yield very different estimates of μ. In this case, we have a violation of our existence condition: because the social ranking of income distributions is unique, it must be the same regardless of how we make these choices.

However, if different specifications result in similar estimates of μ, then, under Assumption 1, preferences are "close" to satisfying our separability condition.

In our framework, Assumption 1 effectively rules out the problem illustrated in Figure 2. Hence, varying the expenditure category j and the percentiles $\underline{x}$ and $\bar{x}$ allows us to assess the extent to which the problem outlined in Figure 3 arises.

5.2 Empirical implementation

The data used in our estimates comes from the Consumer Expenditure Survey's Quarterly Interview. It has been conducted quarterly since 1980, and each survey contains a (weighted) representative sample of approximately 6,000 US households. Households are asked to report their expenditures on 14 expenditure categories over the previous quarter.

We use data for the four most recent quarters available for this survey (2021-Q4 to 2022-Q3), and we perform two adjustments to the sample. First, given the well-known difficulties in comparing consumption across different family structures, we restrict our sample to single consumers. Second, as we are interested in the cross-sectional distribution of regular expenditures, we drop the bottom 5% and the top 10% of consumption expenditures. If a typical household purchases a house or a car, then its expenditure during that quarter will vastly exceed its typical quarterly consumption. Similarly, there may be some quarters with unusually low expenditures.[7] Figure 6 in the appendix shows the distribution of total consumption expenditure without removing the extreme values.

The number of observations in our final sample is 5549 consumption units. Descriptive statistics for each expenditure category for this sample are provided in Table 1 in the appendix.

In our theoretical framework, given that consumers exhaust their budgets, m represents both income and consumption expenditure. In practice, income varies over the lifecycle, and people can smooth these fluctuations through borrowing and saving. Consequently, consumption and income may not be the same in any given period. Here, we identify m with total consumption expenditure rather than with income, echoing the view that people's consumption reflects their permanent income.[8]

For the same reason, we chose not to supplement consumption expenditure with an imputed value of leisure or home production. A person's leisure at a given quarter is a

[7] For example, because insurance reimbursements are recorded as negative expenses, total expenditures may be abnormally small or even negative in quarters in which the household is reimbursed for an expensive medical procedure.

[8] Household expenditure includes some categories which are not obviously "consumption" categories, such as education, cash contributions and insurance and pension contributions. Our results are robust to the exclusion of these categories from our measure of m.

poor measure of his lifetime leisure consumption. As income varies over the lifecycle, so does leisure. Most notably, people consume vastly more leisure after retirement. From a theoretical perspective, we can ignore the value of leisure if we are willing to assume that labor supply is inelastic. This assumption is very problematic for married couples that may choose to have one of the spouses specialize in home production (for example, a stay-at-home parent). However, as our sample contains only single consumer units, the avenues for substituting home production for market production seem more limited.

To test the consistency of the data with our axioms, we estimate μ for multiple consumption categories, as well as multiple combinations of high and low consumption percentiles. We use five different consumption categories: housing, food, transport, health, and entertainment. These are the consumption categories with the largest mean in the dataset, and the only ones with a majority of nonzero observations.[9] Our choice of consumption percentiles is arbitrary, and aimed to reflect a range of possibilities. The (low/high) percentile pairs we chose for our estimations were: 25/75, 10/90, 10/60, and 40/90.

For estimating the expenditure functions $C_x^j(\cdot, p)$, we proceed as follows. Recall that $C_x^j(m, p)$ is the x-th percentile of the expenditure on good j, conditional on a budget (m, p). To estimate this function, we re-weigh our sample so that weights are distributed equally across the distribution of m, and estimate the coefficients of the polynomial approximation $c_x^j = \sum_{k=0}^{5} \beta_k m_i^k$, where c_x^j is the consumption of an individual at the xth percentile on good j. This estimation is done through a quantile regression for each of the percentiles we use. We choose a polynomial of degree 5, which provides a good fit for the data (see Figures 7 to 11 in the appendix). However, our results are robust to variations in the polynomial degree.

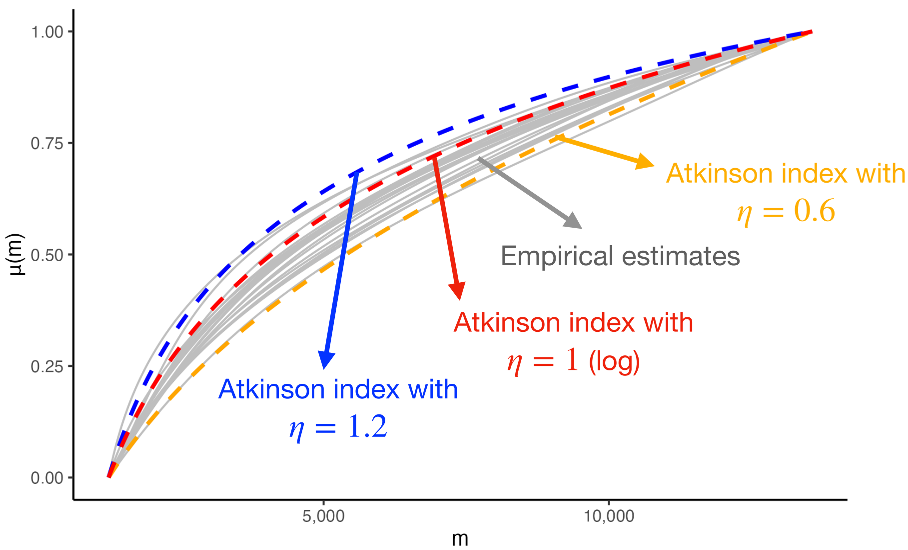
5.3 Results

Figure 4 presents the estimates obtained from our various specifications. As expected, different specifications of $\underline{x}$, $\bar{x}$ and j yield different estimates of μ. In principle, this is a violation of the additive separability condition that is necessary for the consistency of our axioms. Nevertheless, the results illustrate a surprising uniformity of the estimates across the various specifications. This uniformity lends support to the hypothesis that the empirical distribution of preferences is "close" to a preference profile for which our separability condition holds.

A logarithmic μ, as in the homothetic case, is a good approximation for most estimated curves, although a slightly lower degree of inequality aversion (such as $\eta = 0.9$) would be a

[9]Even though "insurance and pensions" is a larger category than entertainment, we exclude it because it is more appropriately classified as savings than consumption.

Figure 4: Empirical estimates of μ



Note: Each of the 20 gray lines represents the estimated μ for a combination of expenditure category and a pair of expenditure quantiles. Thicker lines represent the Atkinson index $(m^{1-\eta}/(1-\eta))$ with different inequality aversion coefficients. All lines are normalized so that their range is between zero and one in the given domain of total expenditure values.

better fit for the observed distribution. As the dotted and dashed lines suggest, the plausible range of values of η for our data is between 0.6 and 1.2.

6 Conclusion

Income Anonymity is the idea that income distributions should be evaluated independently of how they covary with individual preferences, or individual abilities to transform consumption into "utils". It turns out that this idea has powerful implications. Combined with the Pareto principle, it implies concrete guidelines for how income distributions should be ranked.

This welfare criterion can be evaluated based on the ordinal properties of individuals' preferences. In fact, under certain assumptions, data about the cross-sectional distribution of consumption expenditures suffices for calibrating the social welfare function. We find that, in the United States, the two axioms imply a social objective that is approximately the maximization of the sum of log incomes.

However, it is important to emphasize that the social welfare criterion may vary with the distribution of individual preferences, and may depend on the particular prices that consumers face. It would be erroneous to conclude, based on our analysis, that the social welfare criterion should always and everywhere be the maximization of the sum of log incomes. Instead, different circumstances may warrant a re-calibration of the social welfare criterion, based on the procedure laid out in this paper.

References

Anthony B. Atkinson. On the measurement of inequality. *Journal of Economic Theory*, 2 (3):244–263, 1970.

Charles Blackorby, François Laisney, and Rolf Schmachtenberg. Reference-price-independent welfare prescriptions. *Journal of Public Economics*, 50(1):63–76, 1993.

Charles Blackorby, Daniel Primont, and R. Robert Russell. Separability: A survey. *Handbook of utility theory*, 1:51–92, 1998.

Angus Deaton and John Muellbauer. *Economics and consumer behavior*. Cambridge University Press, 1980.

David Dollar, Tatjana Kleineberg, Aart Kraay, and Sergei Guriev. Growth, inequality and social welfare. *Economic Policy*, 30(82):335–377, April 2015.

Ronald Dworkin. What is equality? part 2: Equality of resources. *Philosophy & Public Affairs*, 10(4):283–345, Autumn 1981.

Marc Fleurbaey and Didier Blanchet. *Beyond GDP: Measuring Welfare and Assessing Sustainability*. Oxford University Press, 2013.

Marc Fleurbaey and François Maniquet. Fair income tax. *Review of Economic Studies*, 73: 55–83, 2006.

Marc Fleurbaey and François Maniquet. *A theory of fairness and social welfare*. Cambridge University Press, 2011.

Marc Fleurbaey and François Maniquet. Optimal income taxation theory and principles of fairness. *Journal of Economic Literature*, 56(3):1029–1079, 2018.

Marc Fleurbaey and Alain Trannoy. The impossibility of a paretian egalitarian. *Social Choice and Welfare*, 21(2):243–263, 2003.

W. M. Gorman. Conditions for additive separability. *Econometrica*, 36(3/4), Jul. – Oct. 1968.

Louis Kaplow and Steven Shavell. Any non-welfarist method of policy assessment violates the pareto principle. *Journal of Political Economy*, 109(2):281–286, 2001.

Simon Keller. Expensive tastes and distributive justice. *Social Theory and Practice*, 28(4): 529–552, 2002.

Aart Kraay, Christoph Lakner, Berk Özler, Benoit Decerf, Dean Jolliffe, Olivier Sterck, and Nishant Yonzan. A new distribution sensitive index for measuring welfare, poverty, and inequality. World Bank Policy Research Working Paper No. 10470, 2023.

Wei-Zhi Qin and Hendrik Rommeswinkel. Quasi-separable preferences. 2022.

Kevin Roberts. Price-independent welfare prescriptions. *Journal of Public Economics*, 13 (3):277–297, 1980.

Amartya Sen. The impossibility of a paretian liberal. *Journal of Political Economy*, 78(1): 152–157, 1970a.

Amartya Sen. *Collective Choice and Social Welfare*. Holden-Day, San-Fransisco, 1970b.

Amartya Sen. Equality of what? In S. McMurrin, editor, *The Tanner Lectures on Human Values: Volume 1*. Salt Lake City: University of Utah Press, 1980.

Itai Sher. Generalized social marginal welfare weights imply inconsistent comparisons of tax policies. 2021.

Daniel T. Slesnick. Aggregate deadweight loss and money metric social welfare. *International Economic Review*, 32(1):123–146, 1991.

Kotaro Suzumura. On pareto-efficiency and the no-envy concept of equity. *Journal of Economic Theory*, 25(3):367–379, 1981a.

Kotaro Suzumura. On the possibility of "fair" collective choice rule. *International Economic Review*, 22(2):351–364, 1981b.

Kotaro Suzumura. *Rational choice, collective decisions, and social welfare*. Cambridge University Press, 1983.

Koichi Tadenuma. Efficiency first or equity first? two principles and rationality of social choice. *Journal of Economic Theory*, 104(2):462–472, 2002.

Hal R. Varian. Two problems in the theory of fairness. *Journal of Public Economics*, 5(3-4): 249–260, 1976.

Peter P. Wakker. *Additive Representations of Preferences: A New Foundation of Decision Analysis*. Dordrecht: Springer Netherlands, 1989.

A Proofs

B Proof of Theorem 1

It will be convenient to use the notation $\preceq$ to denote various partial orderings of $\mathbb{R}^I_{++} \times \mathbb{R}^J_{++}$.

Fix some p_0. For two income distributions $\mathbf{m}, \mathbf{m}'$, define

$$\mathbf{m} \preceq \mathbf{m}' \Leftrightarrow (\mathbf{m}, p_0) \preceq (\mathbf{m}', p_0)$$

For each $1 \leq i < I$, define the partial ordering $\preceq$ on $\mathbb{R}^i_{++}$ as

$$(m_1, ..., m_i) \preceq (m'_1, ..., m'_i) \Leftrightarrow (\forall m_{i+1}, ..., m_I)(m_1, ..., m_I) \preceq (m'_1, ..., m'_i, m_{i+1}, ..., m_I)$$

B.1 Defining S_m

We begin by defining a sequence of "jumps". There are two kinds of jumps, which we will call "jumps to the right" and "jumps to the left" (we will draw pictures consistent with this terminology, but it is in principle possible that a jump to the right brings us somewhere to the left of where we started, and a jump to the left brings us to the right of where we started; this is not going to matter for our construction).

Without loss of generality, assume that individuals 1 and 2 have different preferences. Fix some p_0. For a price vector, p, a *jump to the right through* p is a function denoted $[p] : \mathbb{R}_{++} \mapsto \mathbb{R}_{++}$, defined as

$$[p](\cdot) = e_2(e_1(\cdot, p_0, p), p, p_0)$$

Figure 1 illustrates this construction. Starting from the point (m, p_0), we follow the indifference curve of individual 1 up until the point of intersection with the price vector p. At that point, we switch to the indifference curve of person 2, and travel back to the point in which it intersects with the price vector p. The resulting income level is m_1. We say that m_1 is obtained from m through jumping to the right through $[p]$.

A *jump to the left through* p is defined as the inverse of a jump to the right through p, as

$$[p]^{-1}(\cdot) = e_1(e_2(\cdot, p_0, p), p, p_0)$$

A sequence of n jumps to the right through p is denoted $[p]^n$ (which is the function $[p]$ composed n times with itself). A sequence of n jumps to the left through p is denoted $[p]^{-n}$, which is the composition of the function $[p]^{-1}$ n times with itself. For completeness of notation, let $[p]^0$ denote the identity function ($[p]^0(m) = m$ for all m).

A sequence of jumps is a composition of the form

$$S = [p_k] \circ \cdots \circ [p_1]$$

Note that, as a composition of strictly increasing and invertible functions, any sequence of jumps is strictly increasing and invertible.

To proceed, consider the following claims.

Claim 1. *For any two incomes m, m', and a jump, $[p]$, it holds that*

$$(m, m') \sim ([p]^{-1}(m), [p](m')).$$

Proof. Consider a permutation σ such that $\sigma(1) = 2$ and $\sigma(2) = 1$. By applying Income Anoymity with permutation σ and Pareto indifference, we get

$$\begin{aligned}
((m, m'), p_0) &\sim ((m', m), p_0) && \text{(Income Anonymity)}\\
&\sim ((e_1(m', p_0, p), e_2(m, p_0, p)), p) && \text{(Pareto)}\\
&\sim ((e_2(m, p_0, p), e_1(m', p_0, p)), p) && \text{(Income Anonymity)}\\
&\sim ((e_1(e_2(m, p_0, p), p, p_0), e_2(e_1(m', p_0, p), p, p_0)), p_0) && \text{(Pareto)}\\
&= (([p]^{-1}(m), [p](m')), p_0).
\end{aligned}$$

□

Claim 2. *Let m, m' and m'' be such that $(m, m') \preceq (m, m'')$. Then, $m' \leq m''$.*

Proof. Note that person 1 is always indifferent between $((m, m'), p_0)$ and $((m, m''), p_0)$. If $m' > m''$, then person 2 strictly prefers the former. In this case, the Pareto condition demands that $((m, m'), p_0) \succ ((m, m''), p_0)$, in contradiction to the assumption that $((m, m'), p_0) \preceq ((m, m''), p_0)$. □

Claim 3. *For every two sequences of jumps, S and S', it holds that $S \circ S' = S' \circ S$.*

Proof. It suffices to show that, for any p, p', it holds that $[p] \circ [p'] = [p'] \circ [p]$, since it is then possible to apply pairwise permutations to jumps in a sequence $S \circ S'$ of arbitrary length and obtain that $S \circ S' = S' \circ S$. We thus show that $[p] \circ [p'] = [p'] \circ [p]$. We have

$$(([p](m), [p'](m)), p_0) \sim ((m, [p]([p'](m))), p_0). \qquad \text{(Claim 1)}$$

Similarly,

$$\begin{aligned}
(([p](m),[p'](m)),p_0) &\sim (([p']([p](m)),m),p_0) && \text{(Claim 1)}\\
&\sim ((m,[p']([p](m))),p_0). && \text{(Income Anonymity)}
\end{aligned}$$

Thus, by transitivity,

$$((m,[p]([p'](m))),p_0) \sim ((m,[p']([p](m))),p_0)$$

Hence, by Claim 2, it holds that $[p]([p'](m)) = [p']([p](m))$.

□

Claim 4. *For any two incomes, m, m', and a sequence of jumps, S, it holds that*

$$(m,m') \sim (S^{-1}(m), S(m')).$$

Proof. We use induction on the number of jumps in the sequence S, which we denote by n. For the case $n = 0$, the claim follows from the reflexivity of the indifference relation.

Assume that the claim holds for a sequence S with n jumps. For an arbitrary p, we have

$$\begin{aligned}
(m,m') &\sim ((S^{-1}(m), S(m')) && \text{(Inductive hypothesis)}\\
&\sim ([p]^{-1}(S^{-1}(m)), [p](S(m'))) && \text{(Claim 1)}\\
&= (S^{-1}([p]^{-1}(m)), [p](S(m'))) && \text{(Claim 3)}\\
&= (([p]\circ S)^{-1}(m), ([p]\circ S)(m')).
\end{aligned}$$

This completes the proof. □

Because individuals 1 and 2 have different preferences, we can choose p_0 such that there exists some p_1 satisfying

$$m_0 < [p_1](m_0).$$

Claim 5. *The sequence $\{[p_1]^k(m_0)\}_{k=-\infty}^{\infty}$ is strictly increasing.*

Proof. First, note that for any p and any k, $[p]^k$ is strictly increasing in m, since equivalent income functions are strictly increasing in income.

Suppose that $[p_1]^k(m_0) < [p_1]^{k+1}(m_0)$ for some k (we assumed this for $k = 0$). We now show that this implies that $[p_1]^{k+1}(m_0) < [p_1]^{k+2}(m_0)$, and that $[p_1]^{k-1}(m_0) < [p]^k(m_0)$, from which we can conclude that $\{[p_1]^k(m_0)\}_{k=-\infty}^{\infty}$ is strictly increasing. We have that

$$\begin{aligned}[p_1]^{k+1}(m_0) &= [p_1]([p_1]^k(m_0)) \\ &< [p_1]([p_1]^{k+1}(m_0)) \\ &= [p_1]^{k+2}(m_0),\end{aligned}$$

where the inequality follows from the hypothesis and fact that $[p_1]$ is strictly increasing in m. Similarly,

$$\begin{aligned}[p_1]^k(m_0) &= [p_1]^{-1}([p]^{k+1}(m_0)) \\ &> [p_1]^{-1}([p]^k(m_0)) \\ &= [p_1]^{k-1}(m_0).\end{aligned}$$

We have thus established that $[p_1]^k(m_0) < [p_1]^{k+1}(m_0)$ for all k.

□

Claim 6. *The sequence* $\{[p_1]^k(m_0)\}_{k=-\infty}^{\infty}$ *is unbounded in* $\mathbb{R}_{++}$.

Proof. Assume by way of contradiction that there exists $m' > 0$ for which either $[p_1]^k(m_0) < m'$ for every k, or $[p_1]^k(m_0) > m'$ for every k. Consider the case where $[p_1]^k(m_0) < m'$ for all k (the proof for the other case is similar, and hence omitted). This implies that $\{[p_1]^k(m_0)\}_{k=-\infty}^{\infty}$ is bounded from above. As $\{[p_1]^k(m_0)\}_{k=-\infty}^{\infty}$ is an increasing and bounded sequence, it converges—let m^* be its limit. By continuity of $[p_1]$, it follows that

$$\begin{aligned}m^* &= \lim_{k\to\infty}[p_1]^k(m_0) \\ &= \lim_{k\to\infty}[p_1]\left([p_1]^{k-1}(m_0)\right) \\ &= [p_1]\left(\lim_{k\to\infty}[p_1]^{k-1}(m_0)\right) \\ &= [p_1](m^*),\end{aligned}$$

(where the third equality uses the continuity of $[p_1]$).

By Claim 4, we have that

$$([p](m_0), m^*) \sim ([p]^{-1}([p](m_0)), [p](m^*)) = (m_0, m^*)$$

But this is a contradiction to Claim 2, as $[p](m_0) > m_0$.

□

Claim 7. *For every* m*, there exists a sequence of jumps,* S*, for which* $m = S(m_0)$.

Proof. Fix some m. Note that, by Claims 5 and 6, the sequence $\{[p_1]^k(m_0)\}_{k=-\infty}^{\infty}$ is unbounded and strictly increasing. Hence, there exists some k such that

$$[p_1]^k(m_0) \leq m < [p_1]^{k+1}(m_0)$$

We show that there exists a price p for which

$$[p]([p_1]^k(m_0)) = m$$

To see this, note that

1. For $p = p_0$, the function $[p]$ is the identity function, and hence, $[p]([p_1]^k(m_0)) = [p_1]^k(m_0) \leq m$.

2. For $p = p_1$, we have $[p]([p_1]^k(m_0)) = [p_1]^{k+1}(m_0) > m$.

Further, note that $[p](m_0)$ is a continuous function of p. Using an intermediate value argument, it follows that there exists a linear combination of p_0 and p_1 which satisfies the above condition. □

We thus define S_m as a sequence of jumps for which $S_m(m_0) = m$. Note that, as a sequence of jumps, S_m is strictly increasing and invertible.

B.2 Uniqueness

We start with the current proof of uniqueness (Theorem 1). By construction of S_m, we have that

$$(m_1, ..., m_I) = (S_{m_1}(m_0), ..., S_{m_I}(m_0))$$

By Claim 4,

$$(S_{m_1}(m_0), ..., S_{m_I}(m_0)) \sim (S_{m_2}(S_{m_1}(m_0)), S_{m_2}^{-1}(S_{m_2}(m_0)), S_{m_3}(m_0)..., S_{m_I}(m_0))$$

$$= ((S_{m_2} \circ S_{m_1})(m_0), m_0, S_{m_3}(m_0)..., S_{m_I}(m_0))$$

Using similar steps, we obtain that

$$(m_1, ..., m_I) \sim ((S_{m_I} \circ \cdots \circ S_{m_1})(m_0), m_0, ..., m_0)$$

Thus, we have that

$$(m_1, ..., m_I) \preceq (m'_1, ..., m'_I) \Leftrightarrow$$

$$((S_{m_I} \circ \cdots \circ S_{m_1})(m_0), m_0, ..., m_0) \preceq ((S_{m'_I} \circ \cdots \circ S_{m'_1})(m_0), m_0, ..., m_0)$$

which, by Claim 2, holds if and only if

$$(S_{m_I} \circ \cdots \circ S_{m_1})(m_0) \leq (S_{m'_I} \circ \cdots \circ S_{m'_1})(m_0)$$

We have thus established that our preference relation must be representable by $(S_{m_I} \circ \cdots \circ S_{m_1})(m_0)$, and hence it is unique.

B.3 Additive separability

Wakker [1989, p. 70] says that $\succeq$ satisfies *generalized triple cancellation* if, for all i and all income vectors m^1, m^2, m^3, m^4,

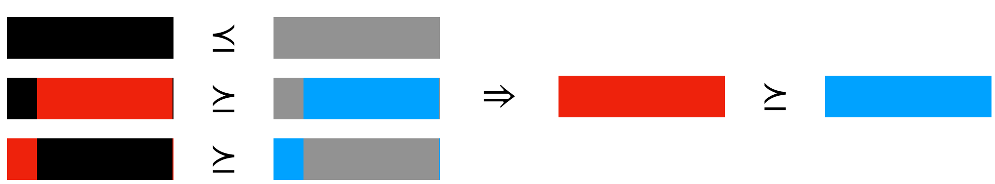
$$m^1 \preceq m^2,$$
$$(m_i^3, m_{-i}^1) \succeq (m_i^4, m_{-i}^2), \text{ and}$$
$$(m_i^1, m_{-i}^3) \succeq (m_i^2, m_{-i}^4)$$

imply that

$$m^3 \succeq m^4.$$

This condition is illustrated in Figure 5.

Figure 5: Illustration of Wakker's condition



Note: Consider a situation in which (a) the gray vector is better than the black vector, and (b) there are mixtures of the red vector and the black vector and mixtures of the gray vector and the blue vector such that the black-red mixtures are better than the corresponding gray-blue mixtures. Wakker's condition requires that, in this case, the red vector must be better than the blue vector.

Theorem 3 (Wakker, 1989, p. 70)**.** *Suppose a relation $\succeq$ on $\mathbb{R}^I_{++}$ is a continuous weak order satisfying generalized triple cancellation. Then, there exists a continuous additive representation for $\succeq$.*

We now show that preferences $\succeq$ satisfy the necessary conditions for this theorem.

Claim 8. *Preferences $\succeq$ satisfy generalized triple cancellation.*

Proof. For an income vector m_{-i}, let $S_{m_{-i}}$ denote a composition of the sequences of jumps S_{m_j}, for all $j \neq i$. By the representation shown in the previous section, we have

$$\begin{aligned} m^1 \preceq m^2 &\Rightarrow S_{m^1}(m_0) \leq S_{m^2}(m_0) \\ (m_i^3, m_{-i}^1) \succeq (m_i^4, m_{-i}^2) &\Rightarrow S_{m_i^3} \circ S_{m_{-i}^1}(m_0) \geq S_{m_i^4} \circ S_{m_{-i}^2}(m_0) \\ (m_i^1, m_{-i}^3) \succeq (m_i^2, m_{-i}^4) &\Rightarrow S_{m_i^1} \circ S_{m_{-i}^3}(m_0) \geq S_{m_i^2} \circ S_{m_{-i}^4}(m_0) \end{aligned}$$

Given that sequences of jumps are strictly increasing in their initial arguments, the inequalities above imply that

$$S_{m^2} \circ S_{m_i^3} \circ S_{m_{-i}^1} \circ S_{m_i^1} \circ S_{m_{-i}^3}(m_0) \geq S_{m^1} \circ S_{m_i^4} \circ S_{m_{-i}^2} \circ S_{m_i^2} \circ S_{m_{-i}^4}(m_0).$$

Combining Claim 3 and the fact that sequences of jumps are strictly increasing in their initial arguments, we can then cancel out sequences of jumps that appear on both sides of the inequality. It then follows that

$$S_{m^3}(m_0) \geq S_{m^4}(m_0).$$

This implies that $m^3 \succeq m^4$, as we wanted to show.

□

Claim 9. *The preference relation represented by* $(m_1, ..., m_I) \mapsto S_{m_I} \circ \cdots \circ S_{m_1}(m_0)$ *is continuous.*

Proof. It is sufficient to show that $S_{m_I} \circ \cdots \circ S_{m_1}(m_0)$ is continuous in $m_1, ..., m_I$ (because a preference relation that is represented by a continuous function is continuous). We begin by showing that $S_m(\tilde{m})$ is continuous in m. Note that

$$\begin{aligned} S_m(\tilde{m}) &= S_m(S_{\tilde{m}}(m_0)) && \text{(Definition of } S_{\tilde{m}}) \\ &= S_{\tilde{m}}(S_m(m_0)) && \text{(Claim 3)} \\ &= S_{\tilde{m}}(m). && \text{(Definition of } S_m) \end{aligned}$$

As $S_{\tilde{m}}$ is a composition of continuous functions, it is continuous. It follows that, for every $\epsilon > 0$, there exists $\delta > 0$ such that, if $|m' - m| < \delta$, then $|S_{\tilde{m}}(m') - S_{\tilde{m}}(m)| < \epsilon$, and hence

$$|S_{m'}(\tilde{m}) - S_m(\tilde{m})| < \epsilon,$$

establishing the continuity of $S_m(\tilde{m})$ with respect to m. Because $S_m(\tilde{m})$ is also continuous in $\tilde{m}$ (as a composition of continuous functions), the mapping $f(m, \tilde{m}) = S_m(\tilde{m})$ is continuous in both arguments. Note that

$$S_{m_1}(m_0) = f(m_1, m_0)$$
$$S_{m_2}(S_{m_1}(m_0)) = f(m_2, S_{m_1}(m_0)) = f(m_2, f(m_1, m_0)),$$

and so on. Hence, by induction, it holds that $S_{m_I} \circ \cdots \circ S_{m_1}(m_0)$ is a composition of continuous functions, and is therefore continuous. □

Theorem 3 and Claims 8 and 9 allow us to conclude that social preferences are represented by

$$\sum_{i=1}^{I} \phi_i(m_i).$$

Because these preferences satisfy Income Anonymity, it follows that $\phi := \phi_1 = \phi_i$ for all i. Hence, our social preference relation is represented simply by

$$\sum_{i=1}^{I} \phi(m_i).$$

We now go back to characterizing our social preference relation over the entire domain of income distributions and prices (not just incomes given p_0). As the social preference relation must satisfy the Pareto indifference condition, it must be represented by

$$W(\mathbf{m}, \mathbf{p}) = \sum_{i=1}^{I} \phi(e_i(m_i, p_i, p_0)).$$

By Income Anonymity, for any m, p and i, it holds that

$$\sum_{k \neq 1,i} \phi(e_k(m_0, p, p_0)) + \phi(e_1(m_0, p, p_0)) + \phi(e_i(m, p, p_0)) =$$
$$\sum_{k \neq 1,i} \phi(e_k(m_0, p, p_0)) + \phi(e_1(m, p, p_0)) + \phi(e_i(m_0, p, p_0))$$

$$\Rightarrow \phi(e_i(m, p, p_0)) = \underbrace{\phi(e_1(m, p, p_0))}_{\mu(m,p)} + \underbrace{(\phi(e_i(m_0, p, p_0)) - \phi(e_1(m_0, p, p_0)))}_{\gamma_i(p)}, \tag{9}$$

and thus

$$W(\mathbf{m}, \mathbf{p}) = \sum_{i=1}^{I} \mu(m_i, p_i) + \gamma_i(p_i). \tag{10}$$

B.4 Establishing the representation (1)

To establish the representation in (1), note that, by Pareto, ϕ must be strictly increasing. By Lebesgue's Theorem for the differentiability of monotone functions, it follows that ϕ is differentiable almost everywhere.[10]

Further, as individuals' indirect preferences are assumed to be representable by differentiable functions, $\{v_i\}$, it follows that the functions $\{e_i(\cdot,\cdot,p_0)\}$ are differentiable:

$$v_i(m,p) = v_i(e_i(m,p,p_0),p_0) \Rightarrow \frac{\partial v_i(m,p)}{\partial m} = \frac{\partial v_i(e_i(m,p,p_0),p_0)}{\partial e_i}\frac{\partial e_i(m,p,p_0)}{\partial m}$$

$$\Rightarrow \frac{\partial e_i(m,p,p_0)}{\partial m} = \frac{\frac{\partial v_i(m,p)}{\partial m}}{\frac{\partial v_i(e_i(m,p,p_0),p_0)}{\partial e_i}}$$

(note that the denominator is not zero, as preferences are strictly increasing in income). Similar steps establish that $e_i(m,p,p_0)$ is differentiable with respect to p.

Hence, by (9), the functions μ and γ_i are differentiable almost everywhere (as compositions of $\phi(\cdot)$ which is differentiable almost everywhere and $\{e_i\}$ which are differentiable). It follows that γ_i is differentiable: to see this, observe that the set $\{e_i(m_0,\lambda p,p_0)|\lambda \in \mathbb{R}_{++}\}$ is open (as $e_i(m_0,\cdot,p_0)$ is continuous). Hence, we can choose λ so that $\phi(e_i(m_0,\lambda p,p_0))$ and $\phi(e_1(m_0,\lambda p,p_0)$ are both differentiable at λp. By (9), it follows that γ_i is a linear combination of two functions that are differentiable at λp, and is thus differentiable at λp. But, as the indirect utility function is homogeneous of degree 0, it holds that

$$\mu(\lambda m,\lambda p) + \gamma_i(\lambda p) = \mu(m,p) + \gamma_i(p)$$

as $\gamma_1 \equiv 0$ (by 9), it holds that $\mu(\lambda m,\lambda p) = \mu(m,p)$, and hence the above implies that $\gamma_i(\lambda p) = \gamma_i(p)$. As γ_i is differentiable at λp, it follows that it is also differentiable at p.

In addition, note that $\mu(\cdot,p_0)$ is differentiable almost everywhere, as, by (9),

$$\mu(m,p_0) = \phi(e_1(m,p_0,p_0)) = \phi(m)$$

which is differentiable almost everywhere (as ϕ is continuous and strictly monotone).

By Roy's identity, for each (m,p) such that $\mu(m,p)$ is differentiable, it holds that

$$-\frac{\frac{\partial \mu(m,p)}{\partial p^j}}{\frac{\partial \mu(m,p)}{\partial m}} = c_1^j(m,p) \text{ and } -\frac{\frac{\partial \mu(m,p)}{\partial p^j}+\frac{\partial \gamma_2(p)}{\partial p^j}}{\frac{\partial \mu(m,p)}{\partial m}} = c_2^j(m,p)$$

Hence,

$$\frac{\frac{\partial \gamma_2(p)}{\partial p^j}}{\frac{\partial \mu(m,p)}{\partial m}} = c_1^j(m,p) - c_2^j(m,p)$$

$$\Rightarrow \frac{\partial \mu(m,p)}{\partial m} = \frac{\frac{\partial \gamma_2(p)}{\partial p^j}}{c_1^j(m,p) - c_2^j(m,p)}$$

[10]See http://mathonline.wikidot.com/lebesgue-s-theorem-for-the-differentiability-of-monotone-fun

Because $\mu(\cdot, p_0)$ is differentiable almost everywhere, it holds that

$$\mu(m, p_0) = \mu(1, p_0) + \int_1^m \frac{\partial \mu(m', p_0)}{\partial m'} dm' = \mu(1, p_0) + \int_1^m \frac{\frac{\partial \gamma_2(p_0)}{\partial p^j}}{c_1^j(m', p_0) - c_2^j(m', p_0)} dm'$$

$$\Rightarrow \mu(m, p_0) = \mu(1, p_0) + \frac{\partial \gamma_2(p_0)}{\partial p^j} \int_1^m \frac{1}{c_1^j(m', p_0) - c_2^j(m', p_0)} dm'$$

As $\{\mu+\gamma_i\}$ represent individual preferences, it holds that $\mu(m_i, p_i)+\gamma_i(p_i) = \mu(e_i(m_i, p_i, p_0), p_0)$. Thus, (10) can be rewritten as

$$W(\mathbf{m}, \mathbf{p}) = \sum_{i=1}^{I} \mu(e_i(m_i, p_i, p_0), p_0) =$$

$$= \sum_{i=1}^{I} \left(\mu(1, p_0) + \frac{\partial \gamma_2(p_0)}{\partial p^j} \int_1^{e_i(m_i, p_i, p_0)} \frac{1}{c_1^j(m', p_0) - c_2^j(m', p_0)} dm' \right)$$

$$= I\mu(1, p_0) + \frac{\partial \gamma_2(p_0)}{\partial p^j} \sum_{i=1}^{I} \int_1^{e_i(m_i, p_i, p_0)} \frac{1}{c_1^j(m', p_0) - c_2^j(m', p_0)} dm'$$

As $c_1^j(m_0, p_0) > c_2^j(m_0, p_0)$ and $c_i^j(\cdot, p_0)$ is continuous, it must hold that $c_1^j(m', p_0) > c_2^j(m', p_0)$ for every m' in an open neighborhood of m_0. Because $\mu(\cdot, p_0)$ is strictly increasing at m_0, it must hold that $\frac{\partial \gamma_2(p_0)}{\partial p^j} > 0$. Hence, this social welfare function is an affine transformation of (1), and hence these social preferences are also represented by (1).

C Proof of Theorem 2

We begin by establishing that (a) implies (b). Assume that $c_1^j(m_0, p_0) > c_2^j(m_0, p_0)$. By Theorem 1, when a social preference relation that satisfies the axioms exists, then it must be represented by (1). For this social preference relation to satisfy Pareto, its ranking of individual i's budget sets, (m_i, p_i) (holding the other individuals' budgets fixed), must coincide with the preferences of individual i. Note that $e_i(\cdot, \cdot, p_0)$ is a representation of individual i's preferences (it is the money-metric utility function, with reference prices p_0). Thus, for our social welfare function to be Paretian, it must hold that the integral

$$\int_1^{e_i(m_i, p_i, p_0)} \frac{1}{c_1^j(m', p_0) - c_2^j(m', p_0)} dm'$$

is a strictly monotone transformation of $e_i(m_i, p_i, p_0)$. This holds if and only if condition (b.i) holds for $i = 1$ and $i' = 2$. Note that there is nothing in this argument that relies on individuals 1 and 2, the good j or the budget (m_0, p_0) specifically: the only requirement is that they jointly satisfy $c_1^j(m_0,p_0) > c_2^j(m_0,p_0)$. A similar argument thus establishes that, when our axioms are consistent, then, for every i, i', j, m_0 and p_0,

$$c_i^j(m_0,p_0) > c_{i'}^j(m_0,p_0) \Rightarrow c_i^j(m,p_0) > c_{i'}^j(m,p_0)\ \forall m$$

We have thus established that (1) represents a Paretian social preference relation only if condition (b.i) holds.

To satisfy Income Anonymity, the social welfare function in (1) must be symmetric in $m_1,...,m_I$ whenever $p = p_1 = ... = p_I$. To see when this is the case, it is useful to rewrite the social welfare function as

$$W(\mathbf{m},p) = \left(\sum_{i=1}^{I}\int_1^{e_1(m_i,p,p_0)} \frac{1}{c_1^j(m',p_0)-c_2^j(m',p_0)}dm'\right) + \left(\sum_{i=1}^{I}\int_{e_1(m_i,p,p_0)}^{e_i(m_i,p,p_0)} \frac{1}{c_1^j(m',p_0)-c_2^j(m',p_0)}dm'\right) \tag{11}$$

The first component is symmetric in $\{m_i\}$. Thus, for the function to be symmetric in $\{m_i\}$, the second component must be symmetric in $\{m_i\}$ as well. In particular, the value of the function should remain the same if we switch m_1 and m_i. This requires that

$$\int_{e_1(m_1,p,p_0)}^{e_1(m_1,p,p_0)} \frac{1}{c_1^j(m',p_0)-c_2^j(m',p_0)}dm' + \int_{e_1(m_i,p,p_0)}^{e_i(m_i,p,p_0)} \frac{1}{c_1^j(m',p_0)-c_2^j(m',p_0)}dm' =$$

$$\int_{e_1(m_i,p,p_0)}^{e_1(m_i,p,p_0)} \frac{1}{c_1^j(m',p_0)-c_2^j(m',p_0)}dm' + \int_{e_1(m_1,p,p_0)}^{e_i(m_1,p,p_0)} \frac{1}{c_1^j(m',p_0)-c_2^j(m',p_0)}dm'$$

As the first terms in the summations are both zero, this condition holds only if

$$\int_{e_1(m_i,p,p_0)}^{e_i(m_i,p,p_0)} \frac{1}{c_1^j(m',p_0)-c_2^j(m',p_0)}dm' = \int_{e_1(m_1,p,p_0)}^{e_i(m_1,p,p_0)} \frac{1}{c_1^j(m',p_0)-c_2^j(m',p_0)}dm'$$

As this must hold for any m_i, it follows that the term on the left hand side must be independent of m_i. This implies condition (b.ii).

We now show that (b) implies (c). Observe that, when (b.i) and (b.ii) hold, individual i's preferences are represented by

$$\int_1^{e_i(m,p,p_0)} \frac{1}{c_1^j(m',p_0)-c_2^j(m',p_0)}dm' =$$

$$\int_1^{e_1(m,p,p_0)} \frac{1}{c_1^j(m',p_0)-c_2^j(m',p_0)}dm' + \int_{e_1(m,p,p_0)}^{e_i(m,p,p_0)} \frac{1}{c_1^j(m',p_0)-c_2^j(m',p_0)}dm'$$

$$= \underbrace{\int_1^{e_1(m,p,p_0)} \frac{1}{c_1^j(m',p_0) - c_2^j(m',p_0)} dm'}_{\mu(m,p)} + \underbrace{\int_{e_1(1,p,p_0)}^{e_i(1,p,p_0)} \frac{1}{c_1^j(m',p_0) - c_2^j(m',p_0)} dm'}_{\gamma_i(p)}$$

Finally, observe that (c) implies (a): when individual preferences are represented by $\mu(m_i, p_i) + \gamma_i(p_i)$, then it is straightforward to verify that the social preference relation represented by $\sum_{i=1}^{I} \mu(m_i, p_i) + \gamma_i(p_i)$ satisfies Pareto and Income Anonymity, and hence such a social preference relation exists.

D Proof that the preferences in Figure 3 violate condition (b.ii)

Assume that condition (b.i) holds, and assume by way of contradiction that condition (b.ii) is satisfied. Then,

$$\int_{e_1(m_2',p,p_0)}^{e_2(m_2',p,p_0)} \frac{1}{c_1^j(m',p_0) - c_2^j(m',p_0)} dm' = \int_{e_1(m_1',p,p_0)}^{e_2(m_1',p,p_0)} \frac{1}{c_1^j(m',p_0) - c_2^j(m',p_0)} dm'$$

and

$$\int_{e_1(m_1'',p',p_0)}^{e_2(m_1'',p',p_0)} \frac{1}{c_1^j(m',p_0) - c_2^j(m',p_0)} dm' = \int_{e_1(m_2'',p',p_0)}^{e_2(m_2'',p',p_0)} \frac{1}{c_1^j(m',p_0) - c_2^j(m',p_0)} dm'.$$

Substituting in the relevant equivalent incomes yield

$$\int_{m_2}^{m} \frac{1}{c_1^j(m',p_0) - c_2^j(m',p_0)} dm' = \int_{m}^{m_1} \frac{1}{c_1^j(m',p_0) - c_2^j(m',p_0)} dm' \tag{12}$$

and

$$\int_{m}^{m_2} \frac{1}{c_1^j(m',p_0) - c_2^j(m',p_0)} dm' = \int_{\tilde{m}_1}^{m} \frac{1}{c_1^j(m',p_0) - c_2^j(m',p_0)} dm'. \tag{13}$$

Combining (12) and (13), we get

$$\int_{m}^{m_1} \frac{1}{c_1^j(m',p_0) - c_2^j(m',p_0)} dm' = \int_{m}^{\tilde{m}_1} \frac{1}{c_1^j(m',p_0) - c_2^j(m',p_0)} dm'.$$

But this is a false statement. To see this, note that, by Roy's identity, it holds that $c_1^2(m,p_0) > c_2^2(m,p_0)$ (this is explained in the discussion around Figure 2). By condition (b.i), it follows that $c_1^j(m',p_0) > c_2^j(m',p_0)$ for every m'. As $\tilde{m}_1 > m_1$, the integral on the right hand side is strictly greater than the integral on the left hand side, and hence the above equality cannot hold.

E The consistency of the axioms in the examples of section 3.1

The proof of existence uses the Gorman polar form for the indirect utility functions. For homothetic preferences, the Gorman form is $v_i(m, p) = f_i(p)m$, where $f_i > 0$. Applying the log transformation, these preferences are also represented by $\ln(v_i(m, p)) = \ln(m) + \ln(f_i(p))$. Specifying $\mu(m, p) := \ln(m)$ and $\gamma_i(p) = \ln(f_i(p))$ provides a representation that is consistent with the third clause of Theorem 2, thus proving existence.

For the Stone-Geary case, the indirect utility function is represented by $v_i(m, p) = f_i(p)(m - \sum_{j=1}^{J} p^j \underline{c}^j)$ (to see this, note that these preferences are homothetic after the cost of the subsistence bundle is deducted from income). Applying the log transformation, these preferences are also represented by $\ln(f_i(p)) + \ln(m - \sum_{j=1}^{J} p^j \underline{c}^j)$. Specifying $\mu(m, p) := \ln(m - \sum_{j=1}^{J} p^j \underline{c}^j)$ and $\gamma_i(p) = \ln(f_i(p))$ provides a representation consistent with the third clause of Theorem 2.

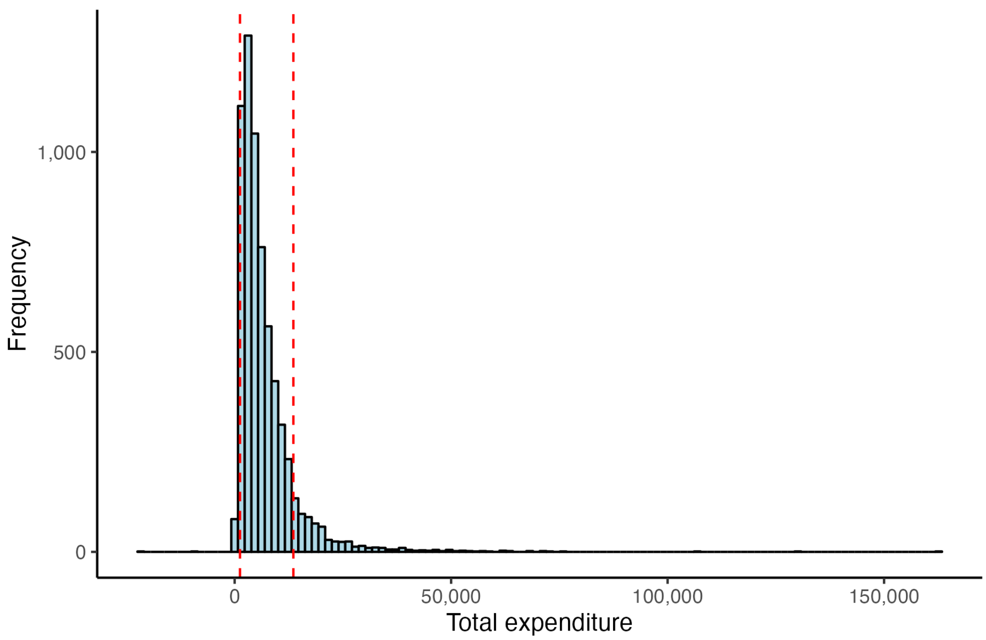
Finally, for quasilinear preferences, the Gorman form is $v_i(m, p) = m/p^1 + f_i(p)$. In this case, specifying $\mu(m, p) = m/p^1$ and $\gamma_i(p) = f_i(p)$ provides a representation consistent with the third clause of Theorem 2.

F Empirical estimates

Table 1: Descriptive statistics for expenditure categories in the final sample

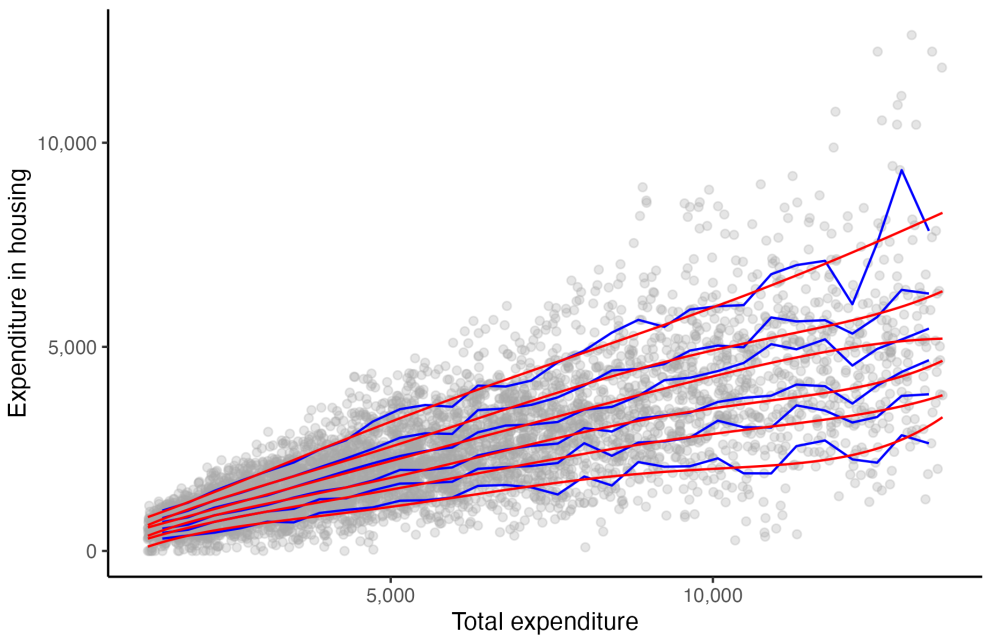
Expenditure category	Mean	Median	Min	Max
Housing	2,240	1,840	0	12,638
Food	968	780	0	6,175
Transport	577	378	0	10,085
Health	524	352	-353	8,988
Insurance and pensions	490	153	0	9,697
Entertainment	211	105	0	5,282
Cash contributions	158	0	0	7,910
Alcohol	57	0	0	3,500
Apparel	54	0	0	3,000
Tobacco	43	0	0	3,900
Personal care	36	0	0	1,175
Education	16	0	0	7,778
Reading	9	0	0	900
Misc.	56	0	0	8,000

Figure 6: Distribution of total expenditures before removing extreme values



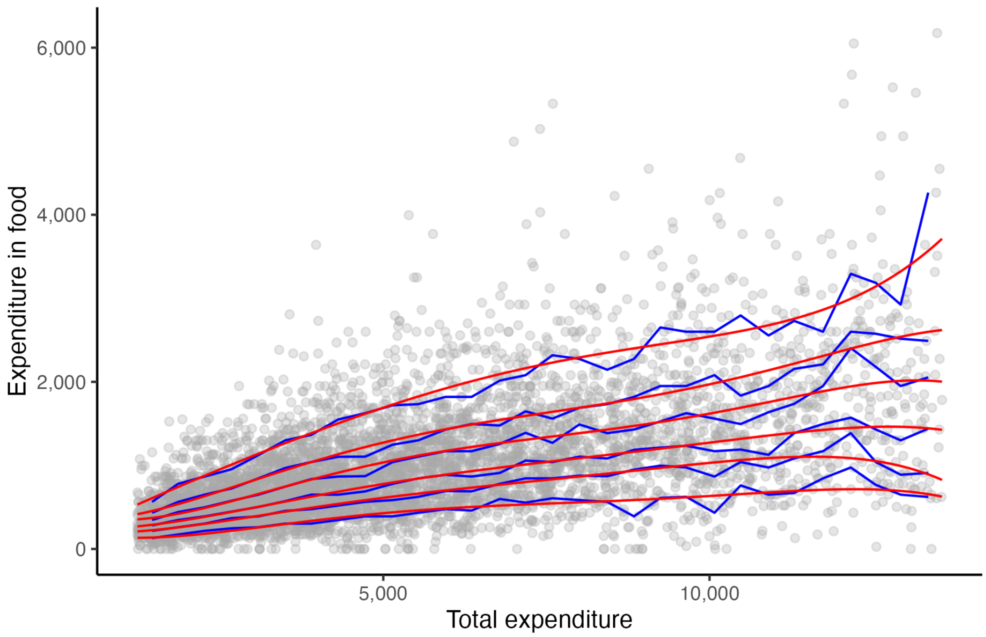
Note: Observations are restricted to single consumers. Dashed lines represent expenditure cutoffs in our data.

Figure 7: Empirical and estimated quantiles for housing expenditure



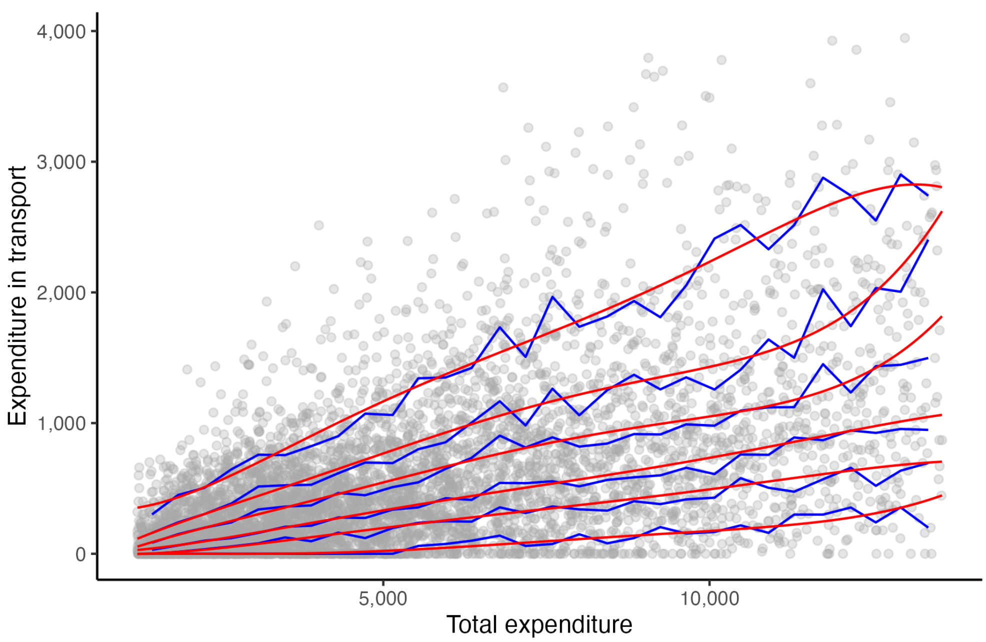
Note: Gray dots represent observations in our dataset. Red lines represent the estimated housing expenditure as a polynomial function of total expenditure for the 90th, 75th, 60th, 40th, 25th, and 10th percentiles. Polynomal functions are estimated through quantile regressions. Blue lines represent the empirically observed expenditure quantiles for each of 30 total expenditure groups, for the percentiles indicated above. These empirical quantiles are estimated by a polynomial regression of housing consumption on a quantile, for each of the 30 bins.

Figure 8: Empirical and estimated quantiles for food expenditure



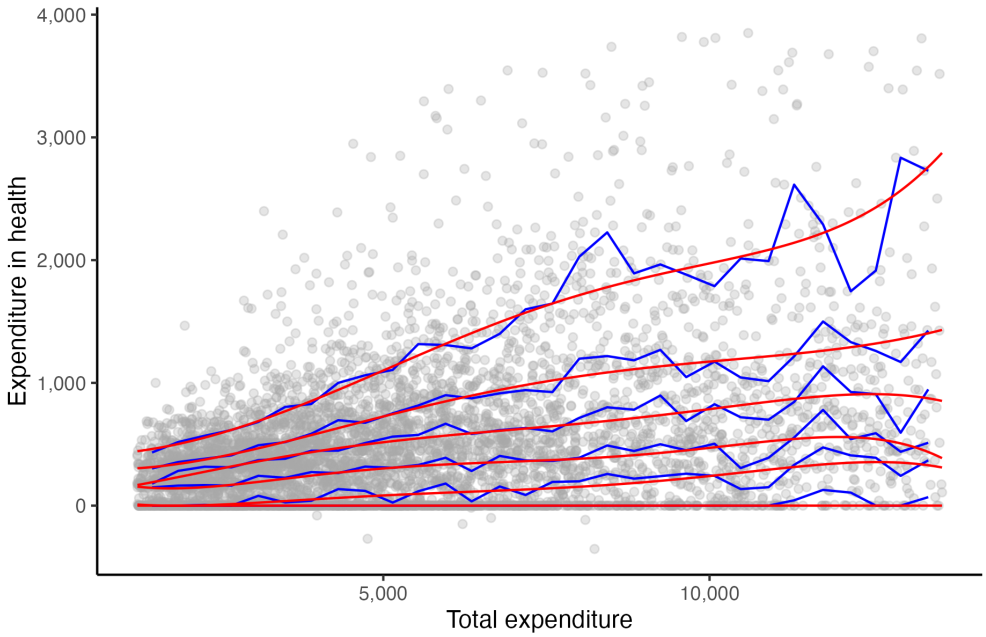
Note: See notes for Figure 7.

Figure 9: Empirical and estimated quantiles for transport expenditure



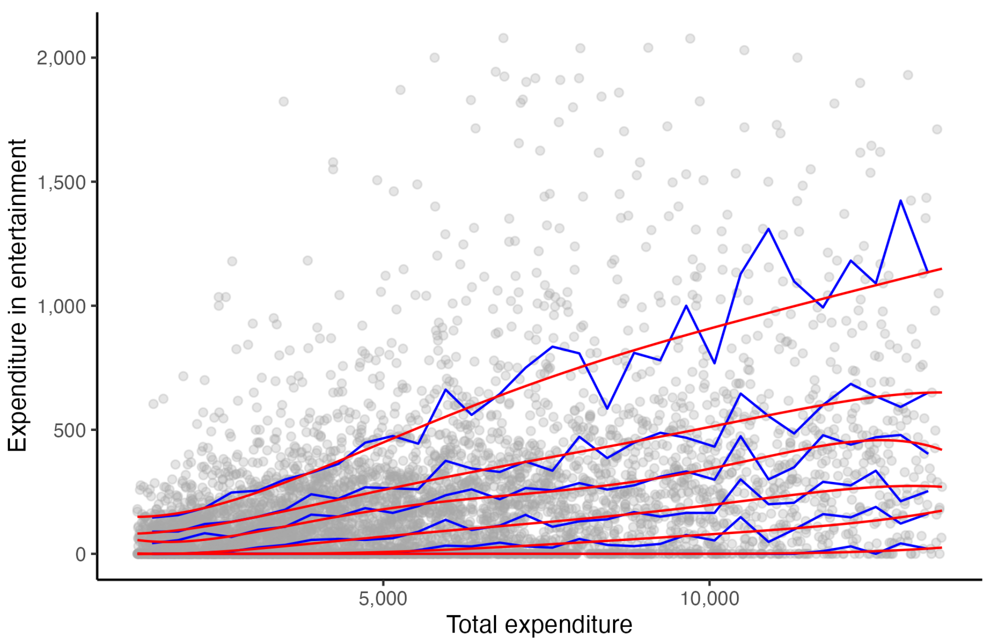
Note: See notes for Figure 7. For ease of visualization, observations in the highest .5% of transport expenditure were dropped from the scatterplot.

Figure 10: Empirical and estimated quantiles for health expenditure



Note: See notes for Figure 7. For ease of visualization, observations in the highest .5% of health expenditure were dropped from the scatterplot.

Figure 11: Empirical and estimated quantiles for entertainment expenditure



Note: See notes for Figure 7. For ease of visualization, observations in the highest .5% of entertainment expenditure were dropped from the scatterplot.