Repeated bargaining with imperfect information about previous transactions

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This paper studies repeated bargaining with noisy information about previous transactions. A buyer has private information about his willingness to pay, which is either low or high, and buys goods from different sellers over time. Each seller observes a noisy history of signals about the buyer's previous purchases and sets a price. We compare the cases where previous prices are observable to sellers with the case where they are not. We show that more signal precision is counterbalanced by two equilibrium mechanisms that slow learning and keep incentives in balance: (1) sellers offer discounted prices more often, and (2) the buyer rejects high prices with a higher probability. The effect of making prices observable depends on the signal precision: When the signal is imprecise, making prices public strengthens the discounting mechanism, improving efficiency and buyer welfare; when the signal is precise, making prices public activates the rejection mechanism, and efficiency and buyer welfare may decrease. Independently of the price observability, the buyer tends to benefit from a more precise signal about previous purchases.

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1 Introduction

Internet cookies record individual user information and are used to tailor the ads, prices, and offered products.[1] Such individualized monitoring and offers may make users cautious about which ads to click on or which goods to purchase. In turn, from the information contained in a buyer's cookies, sellers can infer the buyer's willingness to pay and tailor the prices or goods they offer accordingly. Similar considerations are important in financial markets, where past transactions convey the traders' private information, and in the context of repeated purchases by firms and government agencies, where each provider can condition its offer on the information available about the terms of trade in previous transactions.

The current understanding of how information about previous transactions affects bargaining in theoretical models is limited and based on two extreme cases. The first extreme case is when information is perfect. In this case, Hart and Tirole (1988) show that Coasian forces favor low equilibrium prices. The basic logic is clear. If the buyer accepts a high-price offer, this informs future sellers that he has a high valuation, increasing future prices. Hence the buyer rejects high prices even when he has a high valuation; this means he loses a small surplus now but secures low prices in the future. To avoid rejection, sellers offer a low price in equilibrium. The second extreme case is when there is no information about previous transactions. Here the buyer's signaling motive disappears, and the repeated static monopolistic outcome results in equilibrium.

In practice, these extreme cases are rare, whereas intermediate cases with imperfect and noisy information are common. Current models based on the extremes therefore give little guidance on the possible implications of regulatory or technological changes affecting the information available to sellers.

This paper's first contribution is to provide a tractable model to study how noisy information about previous transactions affects repeated bargaining. A long-lived buyer meets a short-lived seller at each instant in continuous time. The buyer has a permanent type, corresponding to his valuation of the seller's good, which is either low (ℓ) or high (h), with $0<\ell<h$. (The buyer's

[1] Cookies are small text files stored in browser directories, created by the websites a user visits and by the ads and widgets run on these websites. Cookies help developers make website navigation more efficient, but they also track the user's online activity. The information collected is often sold to third parties, who use it to tailor ads, goods, and prices offered to individual consumers.

type may also be interpreted as his wealth or marginal value for money.) Each seller sets a price, and the buyer decides to accept it or reject it. We initially assume that each seller observes the previous price offers, together with a history of noisy signals about the buyer's previous acceptance decisions. We study Markov perfect equilibria with the sellers' posterior belief about the high type as the Markov variable where ℓ is always accepted.

We find that imperfect information on acceptance decisions averts Coasian dynamics. Intuitively, if ℓ were offered and accepted for sure in an interval of high posteriors, the signal would be uninformative and the buyer's continuation value would be flat in this region. A seller could then benefit from offering a price slightly below h: such a price would be accepted by the high-valuation buyer (the h-buyer), because it would give him a positive surplus while not changing much the sellers' belief about his type and hence keeping his continuation value. In our model, we show that Coasian dynamics do not happen: in the unique equilibrium, while only low prices are offered for low posteriors, high prices are offered with positive probability for medium and high posteriors.

The paper's second main contribution is to uncover two equilibrium mechanisms that keep the incentives of the buyer and sellers in balance. When the signal is not very informative, making it more informative induces a *discount mechanism* for intermediate-low posteriors: sellers offer the discounted price ℓ with a positive probability, always accepted by the buyer. Offering such a low price is necessary to slow learning and preserve incentives. As the signal becomes more informative, the discount mechanism pushes the prices down. When the signal is informative enough, the *rejection mechanism* arises for intermediate-high posteriors, where learning is further slowed because sellers charge high prices that the h-buyer rejects with positive probability. Our mechanisms correspond to the two possible equilibrium mechanisms that lower the informativeness of the acceptance decision required to balance the incentives in the dynamic model: through a very low acceptance probability (high price offers which are likely rejected) or a very high acceptance probability (low price offers which are likely accepted).[2] In either case, a contrarian signal is deemed as noise, while a conformist signal is not informative. We believe that these mechanisms extend beyond the particular assumptions of our model.

The paper's third main contribution is its analysis of the welfare effects of policies affecting the precision of the information available to sellers—for example, internet privacy regulations,

[2] Note that, if α is the unconditional probability that a given price is accepted, both the variance and the entropy of the implied Bernoulli random variable are single-peaked in α.

or transparency laws for financial markets and government purchases.[3] For the extremes of signal informativeness, we recover the existing results: the price tends to ℓ as the signal becomes arbitrarily precise and to the static monopolistic price as the signal tends toward pure noise. When the signal is not precise, only the discount mechanism is present, thus lowering the signal precision lowers buyer welfare and efficiency. When the signal precision is high, reducing it weakens both the discount and rejection mechanisms, so the total effect on efficiency and welfare is unclear. In this case, we prove that the total effect of reducing the signal precision is detrimental to efficiency and buyer welfare in a large region of small-intermediate posteriors.

We also analyze how the effects of signal precision on welfare depend on the observability of prices. For this we consider a modification of our main model in which sellers observe the previous acceptance signals but not the prices offered. In this model, deviations by sellers are unobservable, which implies that all equilibrium offers are accepted for sure; thus, the rejection mechanism is absent. We show that, nevertheless, both buyer welfare and efficiency are lower than in the model where prices are observable to sellers when the signal precision is low. The reason is that the equilibrium information carried by the signal is higher if sellers know the prices offered to the buyer in past transactions: When prices are observable, the buyer's acceptance of a high price is very informative; when prices are unobservable, sellers cannot distinguish between acceptance of high prices and acceptance of low prices. Thus, a stronger discounting mechanism is needed when prices are observable, and buyer welfare and efficiency are higher as a result. However, when the signal precision is high, this result may be reversed: observability of prices may be detrimental to the buyer. With high signal precision, when prices are observable, the rejection mechanism keeps the posterior high for a longer time, lowering the value to the h-buyer of mimicking the ℓ-buyer. Additionally, observability of prices may lower market efficiency in this case, by causing the h-buyer to trade less often.

[3] The pioneer regulation governing internet privacy was the European Union's General Data Protection Regulation (GDPR), implemented in 2018 and commonly referred to as the Cookie Law. Other countries and states, such as India, Australia, and California, have also established regulations on cookies. Conversely, government purchases are often subject to "transparency" regulations. For instance, several articles in the Treaty on the Functioning of the European Union (TFEU) concern transparency and openness in decision-making, which are seen as foundational values of the EU. In US financial markets, the effect of the introduction of the Trade Reporting and Compliance Engine (TRACE) in 2002 and the Financial Industry Regulatory Authority (FINRA) in 2017 on post-trade transparency has been the object of extensive academic study, although from a different angle than ours. See Bessembinder and Maxwell (2008) and Duffie et al. (2017), and the literature discussed therein.

1.1 Literature review

Most of the literature on bargaining with one-sided offers studies the purchase of a single good by a buyer facing one or more sellers. Its most influential result is the so-called *Coase conjecture* (Coase, 1972; see Gul et al., 1986, for a formal proof), which states that the price offered by a monopolist converges to the competitive price as offers become more frequent. Kaya and Liu (2015) verify that this result extends to the case where a buyer receives offers from a sequence of short-lived sellers, independently of the observability of the previous price offers. Some work has shown that the Coase conjecture fails under other assumptions, such as adverse selection (Deneckere and Liang, 2006, Hörner and Vieille, 2009, and Daley and Green, 2020), capacity choice (McAfee and Wiseman, 2008), or outside options (Board and Pycia, 2014).

The literature on repeated bargaining is more limited. As explained above, Hart and Tirole (1988) show that Coasian forces favor an equilibrium where prices are equal to the buyer's lowest valuation, as the acceptance of a high price results in a permanent price increase (there is one seller in their setting). Kaya and Roy (2020, 2022) show that, in the presence of adverse selection, an upper bound on the buyer's payoff when offers are private is lower than his payoff in some equilibria when offers are public (they consider a sequence of sellers), and also analyze the effect of increasing competition. Our assumption of imperfect observability of acceptance decisions averts Coasian dynamics, shedding light on the interplay between information and bargaining and providing unique predictions and rich trade dynamics.[4] Bonatti and Cisternas (2020) analyze the effects of third-party exponential scores on the previous purchasing decisions of a consumer whose willingness to pay stochastically evolves over time. They find that, in linear Markov equilibria, sellers tend to offer lower prices when scores are less persistent. Our analysis focuses on the case where types are permanent and information is disaggregated (e.g., because it is not transmitted through third parties or because they cannot commit to disregarding some information). We show that more informative signals tend to be compensated with less informative equilibrium purchasing decisions, which may be induced by either low (welfare-enhancing) prices or high (welfare-diminishing) prices.

The paper most closely related to ours is Lee and Liu (2013), which studies a repeated

[4] Villas-Boas (2004) studies the case where the monopolist faces overlapping generations of two-period-lived buyers, hence avoiding Coasian dynamics. He shows that equilibrium involves cycles in the prices offered to new consumers.

bargaining model where a type-dependent random outside option is publicly drawn if players fail to agree, which generates adverse selection. The observability of the acceptance decision makes the predictions similar to those for the analogous model with one trade (Daley and Green, 2012): all offers are rejected for intermediate beliefs and accepted either by both types or only by one of the types for extreme beliefs. In contrast, our model's outside option is fixed, and the acceptance decision is observed with noise (so it depends on the buyer's type only through his equilibrium behavior). The implied dynamics and equilibrium mechanisms are significantly different, and also differ from the equilibrium outcome of the analogous model with one trade.[5] We argue that random discounts and offer rejections for intermediate posteriors play an important role in determining the welfare effects of privacy, secrecy, and transparency regulations.

Our model is also related to the reputations literature. The paper closest to ours in this literature is Faingold and Sannikov (2011), in which a firm sells goods at a fixed price to a continuum of buyers, and information about the firm's previous quality choices is revealed through a diffusion process. The firm's type is either "behavioral", meaning it produces only high-quality goods, or "normal", meaning it can produce high-quality goods at a greater cost. In our model, the ℓ-buyer resembles a firm of behavioral type, and the h-buyer wants to build a reputation for having a low valuation. Nevertheless, paralleling the approach in the bargaining literature, our buyer interacts with only one seller at a time, whose observable offer affects the buyer's decision and determines the informativeness of that time's signal. Our analysis permits analyzing how information affects pricing and welfare in markets with repeated bargaining.[6]

The rest of the paper is organized as follows. We present the model with observable prices in Section 2, and we analyze it in Section 3. In Section 4, we study the welfare effects of reducing the information observed by sellers. In Section 5, we discuss some policy implications and conclude. The appendix contains the proofs of the results. An online appendix provides results for discrete-time versions of our model as the length of the period vanishes.

[5] It is not difficult to see that the Coase conjecture holds in the one-trade version of our model.

[6] Our stage game is similar to the chain-store paradox game (studied by Kreps and Wilson, 1982, and Milgrom and Roberts, 1982) instead of the usual product-choice game.

2 The model

Time is continuous. There is a buyer. At each instant $t \in \mathbb{R}_+$, the buyer meets a short-lived seller, the "t-seller", who offers price p_t. The buyer decides either to purchase from the t-seller ($a_t = 1$) or not ($a_t = 0$). The buyer values all sellers' goods equally. His valuation, also referred to as his type, is private, and it is either ℓ or h with $0 < \ell < h$. A natural interpretation is that the buyer's type is his willingness to pay, which correlates with his wealth or access to alternative purchasing options. The initial probability that the type is h is $\phi_0 \in (0, 1)$.

There is a public signal about the buyer's previous purchasing decisions. More concretely, at each instant t, the t-seller observes $(X_{t'})_{t' \in [0,t)}$, with

$$X_t \equiv \mu \int_0^t a_{t'} \, \mathrm{d}t' + B_t \ , \tag{1}$$

where B_t is a normalized Wiener process and $\mu > 0$ is a parameter capturing the precision of the signal. Throughout Sections 2 and 3, we will consider the case where the t-seller also observes the history of price offers made by previous sellers. Note that, unlike in models with a single transaction, the game continues after the buyer makes a purchase (i.e. after $a_t = 1$ for some t).

We will use ϕ_t to denote the public belief at time t about the buyer's type being h (given the signal and price histories). We will focus on Markov strategies. For the buyer with type $\theta \in \{\ell, h\}$, an *acceptance strategy* associates to each belief $\phi \in [0, 1]$ and (on- or off-path) price $\hat{p}$, a probability of acceptance $\alpha_\theta(\phi, \hat{p}) \in [0, 1]$. An *offer strategy* associates to each belief ϕ a price distribution $\tilde{\pi}(\phi) \in \Delta(\mathbb{R})$.

Consistent strategy profiles

We will now state standard regularity conditions, which we will term "consistency", that permit the use of continuous-time techniques to analyze diffusion processes. In particular, for a given strategy profile $(\alpha_\ell, \alpha_h, \tilde{\pi})$ and acceptance strategy $\hat{\alpha}$, this condition will guarantee that there is a unique belief process satisfying

$$\mathrm{d}\phi_t = \tilde{\mu}(\phi_t; \hat{\alpha}, \alpha_\ell, \alpha_h, \tilde{\pi}) \mathrm{d}t + \tilde{\sigma}(\phi_t; \alpha_\ell, \alpha_h, \tilde{\pi}) \mathrm{d}B_t \ , \tag{2}$$

where

$$\tilde{\mu}(\phi,\hat{p};\hat{\alpha},\alpha_\ell,\alpha_h)\equiv\mu(1-\phi)\phi\,(\alpha_h(\phi,\hat{p})-\alpha_\ell(\phi,\hat{p}))$$
$$(\hat{\alpha}(\phi,\hat{p})-\phi\,\alpha_h(\phi,\hat{p})-(1-\phi)\,\alpha_\ell(\phi,\hat{p}))\,, \tag{3}$$
$$\tilde{\sigma}(\phi,\hat{p};\alpha_\ell,\alpha_h)\equiv\mu(1-\phi)\phi\,|\alpha_h(\phi,\hat{p})-\alpha_\ell(\phi,\hat{p})|\,. \tag{4}$$

are the drift and the diffusion parameters of the belief process, respectively. Equation (2), as well as equation (6) below, can be obtained as the limit of analogous equations in discrete time as the length of the time increments vanishes; see the Online Appendix.

We say that $\hat{\alpha}$ is *consistent* with $(\alpha_\ell,\alpha_h,\tilde{\pi})$ if both

$$\overbrace{\mathbb{E}_{\tilde{p}}\Big[\tilde{\mu}(\phi,\tilde{p};\hat{\alpha},\alpha_\ell,\alpha_h)\Big|\tilde{\pi}(\phi)\Big]}^{\equiv\tilde{\mu}(\phi;\hat{\alpha},\alpha_\ell,\alpha_h,\tilde{\pi})} \text{ and } \overbrace{\mathbb{E}_{\tilde{p}}\Big[\tilde{\sigma}(\phi,\tilde{p};\alpha_\ell,\alpha_h)^2\Big|\tilde{\pi}(\phi)\Big]}^{\equiv\tilde{\sigma}^2(\phi;\alpha_\ell,\alpha_h,\tilde{\pi})} \tag{5}$$

are piecewise Lipschitz continuous (as functions of ϕ), where $\mathbb{E}_{\tilde{p}}[\cdot|\tilde{\pi}(\phi)]$ is the expectation operator with respect to the variable $\tilde{p}$, which is distributed according to $\tilde{\pi}(\phi)$. We say that $(\alpha_\ell,\alpha_h,\tilde{\pi})$ is *consistent* if both α_ℓ and α_h are consistent with $(\alpha_\ell,\alpha_h,\tilde{\pi})$.

Continuation payoff and equilibrium concept

For a given strategy profile $(\alpha_\ell,\alpha_h,\tilde{\pi})$ and acceptance strategy $\hat{\alpha}$ consistent with $(\alpha_\ell,\alpha_h,\tilde{\pi})$, the θ-buyer's payoff is given by

$$\mathbb{E}\left[\int_0^\infty \mathbb{E}_{\tilde{p}}\Big[\hat{\alpha}(\phi_t,\tilde{p})(\theta-\tilde{p})\Big|\tilde{\pi}(\phi_t)\Big]e^{-rt}\,r\,\mathrm{d}t\,\middle|\,\hat{\alpha},\alpha_\ell,\alpha_h,\tilde{\pi},\phi_0=\phi\right],$$

where $r>0$ is the buyer's discount rate and ϕ_t evolves according to (2).[7] Given a strategy profile $(\alpha_\ell,\alpha_h,\tilde{\pi})$ and posterior $\phi\in[0,1]$, the buyer's continuation value $V_\theta(\phi)$ is the payoff he obtains by maximizing the right-hand side of the previous expression with respect to $\hat{\alpha}$. Standard arguments imply that the continuation value is continuously differentiable at all ϕ where

[7] It is not difficult to see that only two parameters are relevant in determining the equilibrium behavior: ℓ/h and μ/r. However, instead of normalizing some parameters away (e.g., by setting $h=r=1$), we will keep all of them for clarity.

$\tilde{\sigma}^2(\phi;\alpha_\ell,\alpha_h,\tilde{\pi})>0$. The corresponding Bellman equation is

$$\begin{aligned} r\,V_\theta(\phi;\hat{\alpha},\alpha_\ell,\alpha_h,\tilde{\pi}) =& \mathbb{E}_{\tilde{p}}\Big[\,r\,\hat{\alpha}(\phi,\tilde{p})(\theta-\tilde{p})+\tilde{\mu}(\phi,\tilde{p};\hat{\alpha},\alpha_\ell,\alpha_h)V_\theta'(\phi;\hat{\alpha},\alpha_\ell,\alpha_h,\tilde{\pi}) \\ &+\tfrac{1}{2}\tilde{\sigma}^2(\phi;\alpha_\ell,\alpha_h,\tilde{\pi})V_\theta''(\phi;\hat{\alpha},\alpha_\ell,\alpha_h,\tilde{\pi})\Big|\tilde{\pi}(\phi)\Big] \\ =& \mathbb{E}_{\tilde{p}}[\,r\,\hat{\alpha}(\phi,\tilde{p})(\theta-\tilde{p})|\tilde{\pi}(\phi)]+\tilde{\mu}(\phi;\hat{\alpha},\alpha_\ell,\alpha_h,\tilde{\pi})V_\theta'(\phi;\hat{\alpha},\alpha_\ell,\alpha_h,\tilde{\pi}) \\ &+\tfrac{1}{2}\tilde{\sigma}^2(\phi;\alpha_\ell,\alpha_h,\tilde{\pi})V_\theta''(\phi;\hat{\alpha},\alpha_\ell,\alpha_h,\tilde{\pi})\,. \end{aligned} \tag{6}$$

Definition 2.1. A *(regular Markov perfect) equilibrium* is a consistent $(\alpha_\ell,\alpha_h,\tilde{\pi})$, with corresponding value functions (V_ℓ,V_h) which are continuous, piecewise twice differentiable, and differentiable at all ϕ such that $\alpha_\ell(\phi,\hat{p})\neq\alpha_h(\phi,\hat{p})$ for some $\hat{p}$,[8] satisfying the following conditions:

1. For all ϕ and $\hat{p}$, $\alpha_\theta(\phi,\hat{p})$ belongs to[9]

$$\arg\max_{\hat{\alpha}}\Big(r\,\hat{\alpha}(\theta-\hat{p})+\tilde{\mu}(\phi,\hat{p};\hat{\alpha},\alpha_\ell,\alpha_h)V_\theta'(\phi)\Big)\,. \tag{7}$$

2. For all ϕ, $\tilde{\pi}(\phi)$ belongs to

$$\arg\max_{\pi}\mathbb{E}_{\tilde{p}}\Big[\big((1-\phi)\alpha_\ell(\phi,\tilde{p})+\phi\,\alpha_h(\phi,\tilde{p})\big)\tilde{p}\,\Big|\,\pi\Big]\,. \tag{8}$$

The first condition in Definition 2.1 says that the buyer acts optimally. Equation (7) indicates the tradeoff he faces. If a price $\hat{p}$ is such that $\alpha_\ell(\phi,\hat{p})=\alpha_h(\phi,\hat{p})$ (i.e., if the signal is uninformative and so $\tilde{\mu}(\phi,\hat{p};\hat{\alpha},\alpha_\ell,\alpha_h)=0$), then the buyer accepts for sure any price below his valuation (i.e., any $p<\theta$). If, instead, $\hat{p}\in(\ell,h)$ and $\alpha_\ell(\phi,\hat{p})<\alpha_h(\phi,\hat{p})$, then acceptance gives the h-buyer an instantaneous payoff equal to $h-\tilde{p}>0$, but reveals information about his type to future sellers, which affects the continuation payoff (as the drift of the posterior is positive). The second condition in Definition 2.1 says that sellers behave optimally. When the posterior is ϕ, the seller chooses the price to maximize the expected revenue, that is, the price multiplied by the probability that it is accepted.

[8] Note that if $\alpha_\ell(\phi,\hat{p})=\alpha_h(\phi,\hat{p})$ for some ϕ and all $\hat{p}$, then, for any continuation play, the belief remains equal to ϕ. Conversely, if $\alpha_\ell(\phi,\hat{p})\neq\alpha_h(\phi,\hat{p})$ for some $\hat{p}$, then the differentiability of the continuation values allows us to compute the buyer's incentive to accept the price offer using equation (7).

[9] Note that if $\alpha_\ell(\phi,\hat{p})=\alpha_h(\phi,\hat{p})$, then $\tilde{\mu}(\phi,\hat{p};\hat{\alpha},\alpha_\ell,\alpha_h)=0$ for all $\hat{\alpha}$; hence the second term in the argument of arg max in equation (7) is 0 (even if V_θ is not differentiable at ϕ).

We now present a condition on the equilibrium behavior.

Condition 1. The buyer accepts for sure any offer less than or equal to ℓ.

From now on, we focus on equilibria satisfying Condition 1, which we call simply equilibria. Condition 1 is intuitive, and it is a convenient way to make the analysis tractable. It is analogous to a result obtained in most bargaining models with one purchase: the *Diamond paradox* establishes that, if a buyer receives one offer at a time, the lowest equilibrium offer is no lower than the lowest buyer valuation.[10,11]

Note that Condition 1 effectively transforms the ℓ-buyer into a "behavior" or "action" type who accepts an offer if and only if it is weakly lower than ℓ. It is then suboptimal for sellers to offer prices below ℓ in equilibrium. As we will see, equilibria under Condition 1 have the property that it is optimal for both types of buyer to behave as prescribed by the condition. Hence, Condition 1 can be seen as either an equilibrium refinement or a behavioral assumption.

3 Equilibrium analysis

3.1 Preliminary results

We begin by presenting some preliminary results that will help build intuition for our main results. These will establish some necessary conditions that strategy profiles have to satisfy to be an equilibrium.

Offered prices: We first note that when the signal is uninformative (i.e., when $\mu = 0$) the buyer behaves myopically, as he would do in a one-shot version of the game. In the one-shot game, a seller offers ℓ if $\phi < \phi^*$ and h if $\phi > \phi^*$, where $\phi^* \equiv \ell/h$. As the following result establishes, the threshold ϕ^* also plays an important role when the signal is informative.

[10] In a repeated-trade setting without noise, prices lower than ℓ can be sustained in equilibrium, for example by "punishing" the buyer with prices equal to h if he accepts a higher price. In our model, the noise in the acceptance signal rules out this possibility.

[11] Other authors make other assumptions with similar effects on the equilibrium play. For example, Lee and Liu (2013) require the value functions to be monotone, and the reputation literature assumes that all types except one are behavioral.

Lemma 3.1. *The following hold in any equilibrium:*

1. *For each $\phi \le \phi^*$, the support of $\tilde{\pi}(\phi)$ is $\{\ell\}$.*
2. *For each $\phi \in (\phi^*, 1)$, there is some $p(\phi) \in (\ell, h)$ such that the support of $\tilde{\pi}(\phi)$ is either $\{p(\phi)\}$ or $\{\ell, p(\phi)\}$.*

Lemma 3.1 implies that price offers are always smaller than h. This result would be trivial in a model with a unique transaction, since no buyer type would accept an offer larger than h; hence each seller would be strictly better off offering ℓ than offering a price higher than h. In our model with repeated trade, the result is not obvious, because the h-buyer has signaling motives. The proof of Lemma 3.1 shows that the continuation value of the h-buyer is a decreasing function of the posterior. As a result, the h-buyer never accepts an offer higher than h, as doing so both gives him a negative payoff and decreases his continuation value on expectation.

The previous observation implies that, as in the static model, ℓ is offered and accepted for sure in equilibrium when $\phi < \phi^*$. Indeed, an immediate implication of Condition 1 is that no seller offers a price strictly below ℓ, and hence the equilibrium payoff of the ℓ-buyer is 0. This implies that the ℓ-buyer rejects all offers above ℓ. As a result, when $\phi < \phi^*$, offering ℓ (which is accepted by both types of buyer) gives the seller a larger payoff than offering any price in $(\ell, h]$ (which is rejected by the ℓ-buyer).

The fact that, when $\phi > \phi^*$, the support of the price distribution is either $\{p(\phi)\}$ or $\{\ell, p(\phi)\}$ for some $p(\phi) \in (\ell, h)$ is obtained as follows. Assume a seller offers $\hat{p} \in (\ell, h)$ (on or off path). The h-buyer cannot be strictly willing to reject such a price: if he were, the signal would be deemed uninformative by future sellers, but then the buyer would have the incentive to accept the price (recall equation (7)). Hence, either the h-buyer is indifferent between accepting the price $\hat{p}$ or not, or he has a strict incentive to accept it. He is indifferent if and only if

$$\overbrace{r(h-\hat{p})}^{\text{surplus from trade}} = \overbrace{\mu(1-\phi)\phi\,\alpha_h(\phi,\hat{p})(-V'(\phi))}^{\text{reputation loss}} \tag{9}$$

(here and from now on, to save notation, we denote the h-buyer's continuation value by V instead of V_h). The term on the left-hand side of equation (9) is the buyer's instantaneous gain from accepting the offer; it equals his valuation minus the price. The term on the right-hand side is the implied loss in terms of continuation value, which can be interpreted as a reputation loss.[12]

[12] In our model, the buyer would want to build a reputation on having low willingness to pay. This implies that V

Keeping all else equal, this term is larger when the signal is more informative, the equilibrium acceptance probability is larger, or the continuation value is more sensitive to changes in the posterior.

Acceptance decisions: The h-buyer is strictly willing to accept $\hat{p}$ when the left-hand side of equation (9) is strictly bigger than the right-hand side. This implies that, in equilibrium,

$$\alpha_h(\phi,\hat{p}) = \min\left\{ -\frac{r(h-\hat{p})}{\mu(1-\phi)\phi V'(\phi)}, 1 \right\} . \tag{10}$$

This acceptance probability acts as a *downward-sloping demand*: it is 1 if $\hat{p}$ is low enough and decreases linearly as $\hat{p}$ increases until it reaches 0, when $\hat{p}=h$. We can then compute the price $\hat{p}\in(\ell,h)$ that maximizes the seller's payoff $\hat{p}\,\alpha_h(\hat{p};\phi)$. The price $p(\phi)$ in Lemma 3.1 is the unique maximizer of $\hat{p}\,\alpha_h(\hat{p};\phi)$, which is given by

$$p(\phi) \equiv \max\Big\{h/2, \overbrace{h-\mu/r\,(1-\phi)\,\phi\,(-V'(\phi))}^{(*)}\Big\} . \tag{11}$$

The expression $(*)$ in equation (11) represents the highest price accepted with probability one by the h-buyer, which corresponds to the kink of $\alpha_h(\hat{p};\phi)$. The seller's optimal offer is then either the maximizer of the linear part of $\alpha_h(\hat{p};\phi)$—that is, $h/2$, which is rejected by the h-buyer with positive probability—or the corner solution $(*)$, if that price is above $h/2$—which is accepted by the h-buyer for sure. In particular, we have

$$\alpha(\phi)<1 \;\Rightarrow\; p(\phi)=h/2\;, \tag{12}$$

where from now on $\alpha(\phi)\equiv\alpha_h(p(\phi);\phi)$ is the equilibrium probability that the h-buyer accepts $p(\phi)$ (recall that such an offer is rejected for sure by the ℓ-buyer).

It is important to note that, in any equilibrium, an optimal strategy for the h-buyer is to mimic the ℓ-buyer, that is, to accept an offer if and only if it is equal to or lower than ℓ. This follows from the observation that either the high price $p(\phi)$ is the largest price which the buyer accepts with probability one, in which case he is indifferent between acceptance and rejection, or $p(\phi)$ equals $h/2$ and the buyer randomizes between acceptance and rejection. Thus, (9) holds for $\hat{p}=p(\phi)$ and $\alpha_h(\phi,\hat{p})=\alpha(\phi)$.

is decreasing; hence $-V'(\phi)$ is positive.

Optimal high prices: The gain a seller obtains from offering $p(\phi)$ is $\phi\,\alpha(\phi)p(\phi)$. By Lemma 3.1, it is weakly optimal for a seller to offer $p(\phi)>\ell$ for all $\phi>\phi^*$. Also, offering ℓ gives a seller a payoff equal to ℓ (since she sells for sure). Hence, we have that

$$\phi\,\alpha(\phi)p(\phi)\geq\ell \tag{13}$$

for all $\phi>\phi^*$. The previous expression holds with equality when ℓ is offered with positive probability. Since, by equation (12), either $\alpha(\phi)=1$ or $p(\phi)=h/2$ (or both), we have the following condition for sellers to offer h with positive probability for $\phi>\phi^*$:

$$\pi(\phi)\in(0,1) \;\;\Rightarrow\;\; p(\phi)=\max\{h/2,\ell/\phi\}\ , \tag{14}$$

where, by another abuse of notation, $\pi(\phi)$ indicates the probability that the offered price equals $p(\phi)$ (in this case, the term $(*)$ in equation (11) is equal to ℓ/ϕ). Hence, ℓ is offered with probability $1-\pi(\phi)$ (by Lemma 3.1).

3.2 Equilibrium characterization

This section characterizes the equilibrium behavior for the situation in which the price offers are observable. We divide the analysis into two cases based on how informative the signal is.

Less informative signal

We first focus on the case where the signal is relatively uninformative—that is, the case where μ is small (in a sense that will be made precise). Equivalent results can be obtained when the buyer is relatively impatient, that is, in the case where r is large. Note that, from the previous section, a strategy profile is fully determined by $p(\phi)$ (the high price intended only for h-buyers), $\pi(\phi)$ (the probability with which a seller offers $p(\phi)$), and $\alpha(\phi)$ the probability with which a h-buyer accepts $p(\phi)$), for each $\phi\in[0,1]$.

The following result provides important properties of the unique equilibrium. Below, we provide an intuition of why these properties must hold and we argue that a discount mechanism is necessary to balance the buyer's and sellers' incentives.

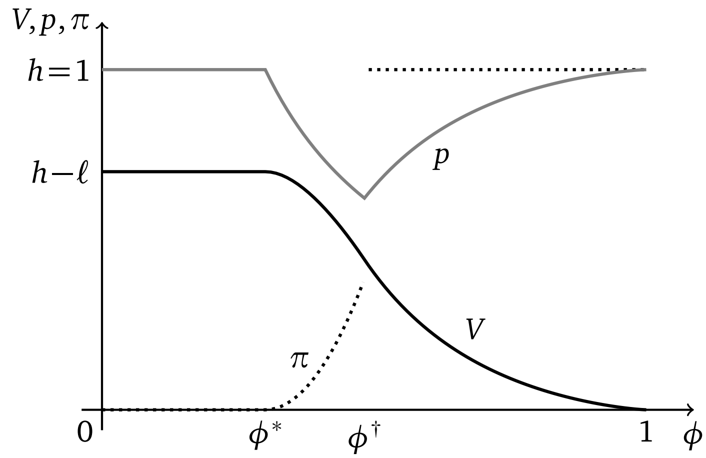

Figure 1: Various equilibrium objects for $h=r=1$, $\ell=0.3$, and $\mu=0.8$.[14]

Proposition 3.1. *There is some largest $\overline{\mu}\in(0,+\infty]$ such that, for all $\mu<\overline{\mu}$, there is an essentially unique equilibrium.*[13] *In such an equilibrium, there is some $\phi^\dagger\in(\phi^*,2\phi^*]$ such that the following hold:*

1. *On $(0,\phi^*]$, π is equal to 0.*
2. *On $(\phi^*,\phi^\dagger)$, π is strictly increasing, α is equal to 1, and p is strictly decreasing.*
3. *On $(\phi^\dagger,1)$, π and α are equal to 1, and p is strictly increasing.*

Figure 1 illustrates Proposition 3.1. For $\phi<\phi^*$, the sellers offer ℓ, the signal is uninformative in equilibrium, and the payoff of the h-buyer is $h-\ell$. As in the static game, the sellers are pessimistic enough about the buyer's valuation that they offer a price equal to ℓ, even though the h-buyer is willing to accept any price below h.

Now consider a posterior ϕ higher than, but close to, the threshold ϕ^*. The lowest price above ℓ that a seller is willing to offer is ℓ/ϕ, which is strictly larger than $h/2$. Hence, $p(\phi)\geq\ell/\phi$ is accepted for sure by the h-buyer (i.e., $\alpha(\phi)=1$ by (12)). It may seem contradictory that, when the posterior is close to ϕ^*, the buyer is willing to accept high prices—which give him little surplus—even though rejection would be highly informative and would bring the posterior

[13] "Essentially unique" here means that other equilibria differ from it in a zero-measure set of posteriors that do not affect the outcome of the game.

[14] It is natural to set $p(\phi)=h$ and $\alpha(\phi)=1$ for all $\phi\in(0,\phi^*)$, even though only ℓ is offered in equilibrium in this region. The reason is that when $\phi\in(0,\phi^*)$, it is optimal for the h-buyer to reject all prices above h and to accept with probability one all prices below h, as the latter give him a positive surplus from trade and no loss of reputation (since $V'(\phi)=0$; see equation (9)).

close to ϕ^*, where his continuation value is maximal. The apparent contradiction is explained by a *discount mechanism*: sellers offer ℓ with a high probability. As we see in Figure 1, such offers flatten the buyer's continuation value for posteriors close to ϕ^*. Consequently, the signaling gain from rejection is small enough that the ℓ-buyer is willing to accept high prices.

As ϕ increases, the high price $p(\phi)=\ell/\phi$ decreases, which makes it easier to incentivize the h-buyer to accept the high price. The discount mechanism thus weakens as the posterior increases (i.e., π is increasing). Equivalently, the price decreases because it becomes more attractive for the h-buyer to reject offers, as there are more discounts at lower posteriors. Proposition 3.1 establishes that there is a threshold $\phi^\dagger \in (\phi^*, 1)$ at which sellers stop offering discounts, obtained using that V is differentiable (i.e., requiring the standard smooth-pasting condition at $\phi^\dagger$). Since $p(\phi)=\ell/\phi > h/2$ for $\phi < \phi^\dagger$, we have

$$\ell/\phi^\dagger = p(\phi^\dagger) \geq h/2 \quad \Rightarrow \quad \phi^\dagger \leq 2\phi^* \, .$$

In fact, $\phi^\dagger$ is increasing in μ (see the proof of Proposition 4.1) and reaches $2\phi^*$ when $\mu=\overline{\mu}$. It then follows that $\overline{\mu}=+\infty$ if and only if $\phi^* \geq 1/2$ (i.e., $h/2 \leq \ell$).

It is easy to see that the price increases toward h as the posterior increases toward 1. Indeed, as explained above, the h-buyer is indifferent between mimicking the ℓ-buyer (by rejecting the high price) and not (by accepting the high price). This implies that, for a high posterior, it may take an arbitrarily long time for the posterior to reach $\phi^\dagger$—that is, for a seller to offer the low price. Thus, the h-buyer is more willing to accept higher prices for high posteriors.

Remark 3.1. We can see in Figure 1 that $\pi(\cdot)$ is discontinuous at $\phi^\dagger$. To see why, recall that, at each posterior $\phi \in (0,1)$, the h-buyer is indifferent between accepting and rejecting $p(\phi)$. As we argued earlier, when the posterior is only slightly higher than ϕ^*, the discount mechanism makes the h-buyer's willingness to reject high prices increasing in the posterior; hence p is decreasing and π is increasing.[15] On the other hand, when the posterior is high, there are no discounts and so the value of rejection decreases; hence p is increasing and π is equal to 1. The discontinuity in π then arises from the necessary discontinuous change in the monotonicity of p: while π and p complement each other to provide the buyer's incentive on $(\phi^*, \phi^\dagger)$, only p provides such an incentive on $(\phi^\dagger, 1)$.

[15] Note that π does not directly appear in the h-buyer's indifference condition (11). Nevertheless, it indirectly affects the price by affecting the continuation value (see equation (6), for example).

More informative signal

We now study the case where the signal is relatively informative, that is, where $\mu > \overline{\mu}$. This is equivalent to studying the case where the buyer is relatively patient. As discussed above, this case occurs only if $\phi^* < 1/2$ (since $\overline{\mu} = +\infty$ otherwise).

As explained in the introduction, as the signal becomes more informative, the price for intermediate-high posterior decreases, as the strengthening of the discount mechanism makes it more attractive for the h-buyer to imitate the ℓ-buyer. Nevertheless, we argued that if the price that guarantees acceptance by the h-buyer is too low, sellers prefer offering a higher price (recall equation (11)). The following result establishes that, when the signal is informative, two more equilibrium regions are added to the three regions described in Proposition 3.1. These regions are described below and depicted in Figure 2.

Proposition 3.2. *Let $\overline{\mu}$ be as defined in Proposition 3.1. Then, for all $\mu > \overline{\mu}$, there is an essentially unique equilibrium. For each $\mu > \overline{\mu}$ there are thresholds $\hat{\phi}^{\dagger} \in (2\phi^*, 1)$ and $\hat{\phi}^{\dagger\dagger} \in (\hat{\phi}^{\dagger}, 1)$ such that, in the unique equilibrium, the following hold:*

1. *On $(0, \phi^*)$, π is equal to 0.*
2. *On $(\phi^*, 2\phi^*)$, π is strictly increasing, α is equal to 1, and p is strictly decreasing.*
3. *On $(2\phi^*, \hat{\phi}^{\dagger})$, π is strictly increasing, α is strictly decreasing, and p is equal to $h/2$.*
4. *On $(\hat{\phi}^{\dagger}, \hat{\phi}^{\dagger\dagger})$, π is equal to 1, α is strictly increasing, and p is equal to $h/2$.*
5. *On $(\hat{\phi}^{\dagger\dagger}, 1)$, π and α are equal to 1, and p is strictly increasing.*

As in the case where the signal is relatively uninformative, there is a lower region where ℓ is offered and accepted for sure. By the same logic as before, for low-intermediate posteriors (posteriors close to but above ϕ^*), sellers randomize between offering ℓ, which both types of buyer accept for sure, and offering $p(\phi) = \ell/\phi$, which the h-buyer accepts for sure (recall (14)).

As the posterior increases, the high price $p(\phi) = \ell/\phi$ decreases. Eventually, when the posterior reaches $2\phi^*$, the price is equal to $h/2$, which is the lowest equilibrium price above ℓ (by (11)). Then, for intermediate posteriors (i.e., $\phi \in (2\phi^*, \hat{\phi}^{\dagger})$), additional increases in the posterior do not decrease the price $p(\phi)$ further, as it remains equal to $h/2$. As ϕ increases, the sellers' indifference between offering ℓ and $h/2$ is maintained by a *rejection mechanism*: an increase in the posterior lowers the probability with which $h/2$ is accepted. The sellers' indifference

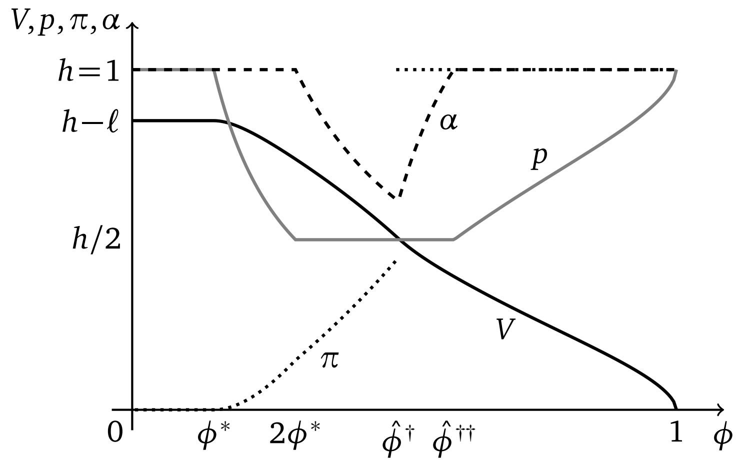

Figure 2: Various equilibrium objects for $h=r=1$, $\ell=0.15$, and $\mu=1.5$.

condition requires that

$$\phi\,\alpha(\phi)h/2=\ell \quad \text{(i.e., } \alpha(\phi)=2\phi^*/\phi\text{)}$$

for beliefs in this region. As the posterior increases, the probability with which sellers offer ℓ decreases. The positive probability of rejection of $h/2$ slows the sellers' learning, so that the buyer remains indifferent between accepting and rejecting $h/2$.[16]

As when the signal is less informative, sellers never offer ℓ at higher posteriors (here higher than $\hat{\phi}^\dagger$). The high price must then remain equal to $h/2$ for some range of posteriors above $\hat{\phi}^\dagger$. Indeed, if the high price were to increase immediately after $\hat{\phi}^\dagger$, the h-buyer would accept such a price for sure (by (12)), but this would be incompatible with the fact that the h-buyer would have a strict incentive to reject the high price (because the right-hand side of (9) with $\hat{p}=p(\phi)$ would jump up). It follows that $p(\phi)=h/2$ for intermediate-high posteriors (i.e., on $(\hat{\phi}^\dagger, \hat{\phi}^{\dagger\dagger})$, where both thresholds are obtained from the corresponding smooth-pasting conditions), and the probability that the h-buyer accepts the high price is less than one. As the posterior increases, the value of mimicking the ℓ-buyer decreases, and so the h-buyer becomes more willing to accept a given high price. The equilibrium indifference is maintained, however, because the acceptance probability is increasing, so accepting the high price entails a larger reputation loss.

When α reaches 1, it stays equal to 1. For high posteriors (i.e., on $(\hat{\phi}^{\dagger\dagger}, 1)$), the high price p increases in the posterior and is accepted for sure by the h-buyer.

[16] The proof of Proposition 3.1 shows that $\phi^\dagger$ increases as μ increases when the signal is less informative—indicating that it becomes easier for the buyer to lower the seller's posterior—and $\overline{\mu}$ is such that $\phi^\dagger=2\phi^*$. It then follows that the rejection mechanism only can occur when the signal is informative enough, that is, when $\mu>\overline{\mu}$.

4 Effects of privacy policies

We now study the effects of reducing the amount of information each seller has about the previous history. The results give insight into the possible impact of policies regulating or banning cookies, which we discuss further in Section 5.

In our model, there are two ways of reducing the information available to each seller. We will first study the effect of making the acceptance signal less precise (i.e., of reducing μ) in Section 4.1. We will then analyze a model in which sellers can observe the acceptance signal but not the prices offered by previous sellers in Section 4.2. We will then study he effect of making the price offers unobservable in Section 4.3.

4.1 Limiting signal precision

We first study the effect of reducing signal informativeness. Reductions in μ could correspond to policies limiting the data stored in cookies, while increases in μ could be attributed to improvements in tracking technology or to regulations requiring transparency in the transactions of government agencies.

Proposition 4.1. *1. For each $\phi \in (\phi^*, 1)$, $V(\phi)$ is strictly increasing in μ on $(0, \overline{\mu})$.*

2. If $\mu > \overline{\mu}$ and $\phi \in (\phi^*, \hat{\phi}^\dagger)$, then $\frac{\mathrm{d}}{\mathrm{d}\mu} V(\phi) > 0$.

3. For each $\phi \in [0,1)$, $\lim_{\mu\to\infty} V(\phi) = h - \ell$.

4. For each $\phi \in [0,1]$, $\lim_{\mu\to 0} V(\phi) = (h-\ell)\mathbb{I}_{[0,\phi^*]}(\phi)$.

The first claim in Proposition 4.1 establishes that when the signal is not very informative, the h-buyer's payoff is increasing in the signal precision. The intuition for the result is the following. When the signal is more informative, the buyer's acceptance of a high price is more informative about his high valuation. Consequently, the buyer should accept a high price only if his expected loss (in terms of the continuation value) is small. Greater signal informativeness is offset, in equilibrium, by more frequent discounts, which increase—and therefore flatten—the h-buyer's continuation value. When the signal informativeness is low, only this discount effect takes place, thus increasing the signal informativeness benefits the buyer.

When the signal is already informative, it is less clear whether informativeness translates into a higher buyer payoff or not. Proposition 4.1 establishes that small increases in μ increase

$V(\phi)$ when ϕ is in $(\phi^*, \hat{\phi}^\dagger]$, so the strengthening of the discount mechanism dominates the strengthening of the rejection mechanism in this region. Since only the rejection mechanism takes place in $(\hat{\phi}^\dagger, \hat{\phi}^{\dagger\dagger})$, higher signal precision makes it more costly for the h-buyer to mimic the ℓ-buyer and reach $\hat{\phi}^\dagger$ in this region (where discounts begin). Still, since the continuation value upon reaching $\hat{\phi}^\dagger$ increases, the change in $V(\phi)$ for $\phi \in (\hat{\phi}^\dagger, 1)$ is unclear. While the proof of Proposition 4.1 provides explicit equations for $\hat{\phi}^\dagger$, $\hat{\phi}^{\dagger\dagger}$, and the continuation value for the different regions, analytically determining the effect of changes in μ has proved not possible. Numerical simulations seem to indicate that increases in μ do increase $V(\phi)$ for all $\phi \in (\phi^*, 1)$.

When the signal is very imprecise ($\mu \to 0$), the h-buyer obtains a very low payoff for all $\phi > \phi^*$: as we discussed before, the equilibrium outcome converges to the repetition of the static equilibrium outcome where the price is equal to h. By contrast, when the signal is very precise ($\mu \to \infty$), the h-buyer obtains a high payoff for all $\phi > \phi^*$. In this case, the outcome converges to an equilibrium of a game where offers and acceptance decisions are perfectly observable. In this equilibrium, all sellers offer ℓ, as the buyer rejects all prices above ℓ/ϕ, since their acceptance would be interpreted as his type being high.

4.2 The private offers case

We now study the case where offers are unobservable to future sellers, which is referred to in the literature as the "private offers case" (the model presented and studied in Sections 2 and 3 corresponds to the "public offers case").

In practice, price offers may be unobservable because of regulations restricting the information available to sellers. For example, cookies may be allowed to collect metrics about a user's previous activity but not data on his actual transactions. A search engine may be able to track a user up to the point where he opens a webpage, and even to collect some information about his behavior on the webpage, but not to identify the actual offers made to him.

We now construct a version of the model described in Section 2 in which price offers are unobservable to other sellers. Now, the t-seller observes only the public signal $(X_{t'})_{t'<t}$ defined in (1). Markov strategies are defined in exactly the same way as in Section 2. Given a strategy profile, for each $\theta \in \{\ell, h\}$ we define the θ-buyer's expected acceptance probability for the belief

ϕ as

$$\overline{\alpha}_\theta(\phi) = \mathbb{E}_{\tilde{p}}[\alpha_\theta(\tilde{p}|\phi)|\tilde{\pi}(\phi)] .$$

The value of $\overline{\alpha}_\theta(\phi)$ indicates the equilibrium probability with which a θ-buyer accepts the price offer when the posterior is ϕ. Hence, the Bellman equation is now

$$r V_\theta(\phi) = \mathbb{E}_{\tilde{p}}\Big[\max_{\hat{\alpha}}\Big(r\,\hat{\alpha}(\theta - \tilde{p}) + \tilde{\mu}(\phi, \tilde{p}; \hat{\alpha}, \overline{\alpha}_\ell, \overline{\alpha}_h) V_\theta'(\phi)\Big) + \tfrac{1}{2}\tilde{\sigma}(\phi, \tilde{p}; \overline{\alpha}_\ell, \overline{\alpha}_h)^2 V_\theta''(\phi)\Big|\tilde{\pi}(\phi)\Big] , \tag{15}$$

where $\tilde{\mu}$ and $\tilde{\sigma}$ are defined in (3) and (4). The crucial difference between the analysis of the private offers case and that of the public offers case is that now the drift and variance of the belief process (for a given strategy of the buyer) are independent of the price offer. Thus, while the drift in equation (15) does not depend on the actual offer received by the buyer, the drift in the analogous equation when offers are observable (equation (6)) does depend on it.

The equilibrium concept is analogous to that of Definition 2.1, with the following differences. First, since now seller deviations are not observable to future sellers, V_θ is differentiable at all ϕ such that $\overline{\alpha}_h(\phi) \neq \overline{\alpha}_\ell(\phi)$. Second, instead of the condition (7), $\alpha_\theta(\phi, \hat{p})$ belongs to the following set for all ϕ where V_θ is differentiable:

$$\arg\max_{\hat{\alpha}}\Big(r\,\hat{\alpha}(\theta - \hat{p}) + \tilde{\mu}(\phi; \hat{\alpha}, \overline{\alpha}_\ell, \overline{\alpha}_h) V_\theta'(\phi)\Big) . \tag{16}$$

Equilibrium analysis

We begin by stating that Lemma 3.1 also holds for the unobservable offers model.

Lemma 4.1. *Lemma 3.1 holds in the private offers case.*

As in the public offers case, if $\phi_t \le \phi^*$ then the t-seller offers ℓ, while if $\phi_t > \phi^*$ she may randomize between offering some price $p(\phi_t)$ (with some probability again denoted by $\pi(\phi_t)$) and offering ℓ (with probability $1 - \pi(\phi_t)$). Again, $\alpha(\phi_t)$ denotes the probability with which the h-buyer accepts the price $p(\phi_t)$ when the t-seller offers it.

Although, just as in the public offers case, the support of prices consists of either one or two points, the logic for this is quite different in the private offers case. When prices are observable, each price is accepted with a different probability in equilibrium (that is, α_h depends on both $\hat{p}$ and ϕ in equation (9)). This probability affects the informativeness of the signal so that the buyer is indifferent between accepting and rejecting the price (recall equation (9)). As we saw, there is a unique price above ℓ that maximizes the seller's payoff (i.e., the acceptance probability multiplied by the price). When instead prices are unobservable, the buyer's reputation loss from accepting an offer $\hat{p}$ is independent of the price offered. Indeed, the h-buyer is indifferent between accepting and rejecting $\hat{p} \in (\ell, h)$ if

$$\overbrace{r(h-\hat{p})}^{\text{surplus from trade}} = \overbrace{\mu(1-\phi)\phi(\overline{\alpha}_h(\phi)-\overline{\alpha}_\ell(\phi))(-V'(\phi))}^{\text{reputation loss}}, \tag{17}$$

and he is strictly willing to accept (reject) the offer if the right-hand side of (17) is strictly higher (lower) than the left-hand side.

Note that $\overline{\alpha}_\ell(\phi) = 1-\pi(\phi)$; that is, the ℓ-buyer's acceptance probability coincides with the probability with which ℓ is offered. By the standard take-it-or-leave-it offer argument, a seller offers a price higher than ℓ in equilibrium only if the h-buyer is indifferent between accepting or rejecting it and he accepts it for sure. We then have that $\overline{\alpha}_h(\phi) = 1$. Hence, in equilibrium, sellers offer either ℓ or

$$p(\phi) = h - \mu r^{-1}(1-\phi)\phi\, \pi(\phi)(-V'(\phi)) \,. \tag{18}$$

Any off-path offer in $(\ell, p(\phi))$ is accepted for sure by the h-buyer, while any price strictly above $p(\phi)$ is rejected for sure. Unlike in the public offers case, the equilibrium probability that the h-buyer accepts $p(\phi)$ (again denoted by $\alpha(\phi)$) is always 1. Hence, the rejection mechanism is not present in the private offers case.

Proposition 4.2. *Assume price offers are unobservable. Then there is a unique equilibrium. In such an equilibrium, there is some $\phi^\ddagger \in (\phi^*, 1)$ such that the following hold:*

1. *On $(0, \phi^*]$, π is equal to 0.*
2. *On $(\phi^*, \phi^\ddagger)$, π is strictly increasing, α is equal to 1, and p is strictly decreasing.*
3. *On $(\phi^\ddagger, 1)$, π and α are equal to 1, and p is strictly increasing.*

Proposition 4.2 resembles Proposition 3.1, but applies to all values of μ. As in the public offers case, for low posteriors sellers offer ℓ with probability one. Again, when ϕ is close to (but above) ϕ^*, $\pi(\phi)\in(0,1)$. (To see this, recall that the t-seller is only willing to offer $p(\phi_t)$ if $p(\phi_t)\geq\ell/\phi_t$. The buyer, in turn, is willing to accept a high price only if his continuation value is not very sensitive to the posterior. As a result, equilibrium learning should be slow when the posterior is higher but close to ϕ^*. This occurs because sellers randomize between offering $p(\phi)$ and offering ℓ, deeming the signal less informative.) For high posteriors, the buyer again accepts the high price for sure.

We finalize the section with a result analogous to Proposition 4.1, establishing that a more informative signal is always beneficial for the buyer when prices are not observable.

Proposition 4.3. *When prices are not observable:*

1. *For each $\phi\in(\phi^*,1)$, $V(\phi)$ is strictly increasing in μ.*
2. *For each $\phi\in[0,1)$, $\lim_{\mu\to\infty}V(\phi)=h-\ell$.*
3. *For each $\phi\in(0,1)$, $\lim_{\mu\to 0}V(\phi)=(h-\ell)\mathbb{I}_{[0,\phi^*]}(\phi)$.*

4.3 Welfare analysis

In this section, we compare the buyer welfare and efficiency in the public and private offers cases analyzed in Sections 3 and 4.2. We will also consider the benchmark case where the signal is uninformative, interpreted as an online market where cookies are banned and thus buyers are fully anonymous.[17]

We denote the three cases as follows: "$x=$no" refers to the case where the sellers observe neither the prices nor the signal, "$x=$ob" refers to the case where the sellers observe both the prices and the signal, and "$x=$un" refers to the case where the sellers observe the signal but not the prices. As discussed above, in the case where nothing is observable, the sellers offer ℓ for sure when $\phi<\phi^*$ and offer h for sure when $\phi>\phi^*$.

The buyer's surplus

We first analyze how the buyer's surplus is affected by cookies. Given that the ℓ-buyer obtains no surplus in any of the cases, we focus on the payoff of the h-buyer.

[17] There is a movement aimed at banning the practices that allow advertisers and political organizations to track individuals with tailored messages, a practice called "microtargeting". See, for example, https://www.politico.eu/article/targeted-advertising-tech-privacy.

Proposition 4.4. *1. If $\mu \leq \overline{\mu}$, then $V^{\text{ob}}(\phi) > V^{\text{un}}(\phi) > V^{\text{no}}(\phi)$ for all $\phi \in (\phi^*, 1)$.*

2. If $\mu > \overline{\mu}$, then $\min\{V^{\text{ob}}(\phi), V^{\text{un}}(\phi)\} > V^{\text{no}}(\phi)$ for all $\phi \in (\phi^*, 1)$.

Comparing the three cases is easier when the informativeness of the signal is low. The equilibrium structure is then similar in the observable case (Proposition 3.1) and the unobservable case (Proposition 4.2). For low posteriors, the sellers offer low prices; for intermediate posteriors, they randomize between low and high prices; and for high posteriors, they offer only high prices. The h-buyer accepts all offers.

For low and high posteriors, the incentives of the buyer and the sellers do not qualitatively depend on the observability of the price offers. Sellers offer the discounted price ℓ when ϕ is low and the highest price which is accepted for sure by the h-buyer when ϕ is high. Let us now fix some intermediate posterior ϕ. In the public offers case, accepting the high price ℓ/ϕ leads to a significant expected increase in the posterior. The buyer nevertheless accepts the high price, because his continuation value is not sensitive to the posterior. On the other hand, in the private offers case, other sellers do not know whether the high price ℓ/ϕ or the low price ℓ was offered. Hence, the buyer's acceptance of ℓ/ϕ leads to a lower expected increase in the posterior. The continuation value is therefore steeper in the private offers case. Formally, equations (11) and (18) can be written as

$$-\frac{\mathrm{d}}{\mathrm{d}\phi} V^x(\phi) = \gamma^x(\phi) \frac{r(h-\ell/\phi)}{\mu(1-\phi)\phi}\,, \tag{19}$$

where $\gamma^{\text{ob}}(\phi) = 1$ and $\gamma^{\text{un}}(\phi) = \frac{1}{\pi^{\text{un}}(\phi)} > 1$. For higher ϕ, V^x solves the same equation for both $x =$ ob and $x =$ un, since in both cases the sellers make only high offers, and the h-buyer is indifferent between acceptance and rejection. The proof of Proposition 4.4 shows that this implies that the buyer is indeed better off in the public offers case.

When the signal is informative (i.e., $\mu > \overline{\mu}$), the relative order between V^{ob} and V^{un} may be reversed at some posteriors. The reason lies in the different mechanisms that slow learning in equilibrium. When prices are unobservable, only the discount mechanism is present: the signal becomes less informative only when the low price is offered with positive probability. When prices are observable, learning is slowed down by the rejection mechanism as well as the discount mechanism. That is, as described in Section 3.2, the h-buyer sometimes rejects high-price offers, which slows learning for intermediate posteriors. This means that, if the initial prior

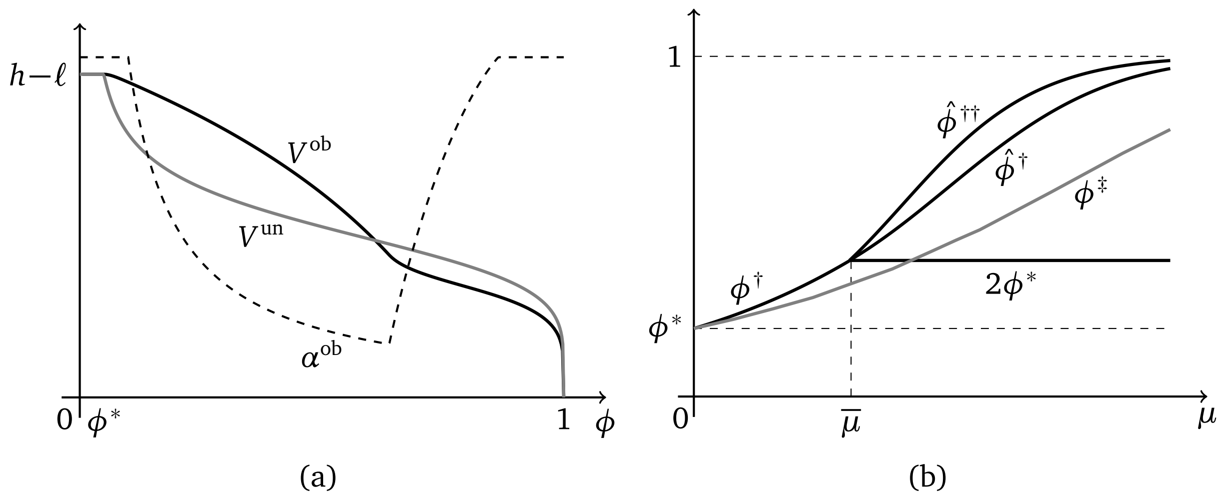

Figure 3: (a) V^{ob} and V^{un} for $h=r=1$, $\ell=0.05$, and $\mu=3$. (b) Thresholds for the public offers case ($\phi^{\dagger}$, $\hat{\phi}^{\dagger}$, and $\hat{\phi}^{\dagger\dagger}$) and the private offers case ($\phi^{\ddagger}$) as a function of μ, for $h=r=1$ and $\ell=0.2$.

is high, it takes a long time for the posterior to reach low values when the h-buyer mimics the ℓ-buyer. The implication is that when the signal is informative enough, at high posteriors the buyer is worse off in the public offers case than in the private offers case.

Figure 3(a) depicts V^{ob} and V^{un} for parameters under which, when the signal is very informative, at some posteriors the buyer is worse off in the public offers case than in the private offers case. For intermediate posteriors, sellers offer the low price more often in the public offers case, and so the buyer's payoff is higher there. However, for higher posteriors, sellers offer higher prices in the public offers case. We can see this from the figure: the low acceptance probability at intermediate posteriors implies that V^{ob} decreases quickly to preserve the buyer's indifference between accepting high prices or rejecting them. This means that if the posterior is initially very high, it takes a long time for it to become low, even if the h-buyer mimics the ℓ-buyer. As a result, at high posteriors, sellers offer higher prices in the public offers case, which means the buyer's payoff is lower.

Figure 3(b) depicts the equilibrium thresholds in both the public and private offers cases as functions of the signal precision μ. When the signal is not very informative (i.e., $\mu<\overline{\mu}$), the range where sellers offer discounts in the public offers case, $(0,\phi^{\dagger})$, is larger than its counterpart in the private offers case, $(0,\phi^{\ddagger})$. While this is also true when the signal is very informative (with discounts offered on $(0,\hat{\phi}^{\dagger})$ in the public offers case and on $(0,\phi^{\ddagger})$ in the private offers case), for these signals there is also, in the public offers case, a large region $(\hat{\phi}^{\dagger},\hat{\phi}^{\dagger\dagger})$ where the buyer rejects high prices in equilibrium, making him worse off than in the private offers case.

Remark 4.1. Proposition 4.4 establishes that the buyer tends to prefer public offers over private offers in repeated bargaining, as rejecting a high offer then sends a stronger signal of low type. The logic would be reversed if there were only one transaction (see Kaya and Liu, 2015), as in that case unobservability of offers would enhance the Coasian forces, since sellers could not update their beliefs according to previous sellers' deviations. However, our result also shows that unobservability is sometimes beneficial to the buyer in repeated bargaining, since it prevents the rejection mechanism from slowing equilibrium learning.

Efficiency

We now take the perspective of a social planner who has the same discount rate as the buyer. The social planner values each transaction of the θ-buyer at θ, independently of the transaction price, for both $\theta \in \{\ell, h\}$.

We use $W^x(\phi)$ to denote the social welfare (or efficiency) for each case $x \in \{\text{no}, \text{un}, \text{ob}\}$. The social welfare is given by

$$W^x(\phi) = (1-\phi)\overbrace{\mathbb{E}^{x,\ell}\Big[\int_0^\infty \ell(1-\pi^x(\phi_t))e^{-rt}\, r\, \mathrm{d}t \,\Big|\, \phi_0 = \phi\Big]}^{(*)} + \phi\, \underbrace{\mathbb{E}^{x,h}\Big[\int_0^\infty h(1-\pi^x(\phi_t)+\pi^x(\phi_t)\alpha^x(\phi_t))e^{-rt}\, r\, \mathrm{d}t \,\Big|\, \phi_0 = \phi\Big]}_{(**)} \tag{20}$$

for all $\phi \in (\phi^*, 1)$, where $\mathbb{E}^{x,\theta}$ is the expectation in the equilibrium for the x-model conditional on the strategy of the θ-buyer.[18] The term $(*)$ is equal to the social welfare generated by the transactions of the ℓ-buyer, who purchases only when the price is ℓ. The term $(**)$ is equal to the social welfare generated by the transactions of the h-buyer, who purchases for sure when the price is ℓ and with probability $\alpha^x(\phi)$ otherwise.

Proposition 4.5. *1. If $\mu \leq \overline{\mu}$, then $W^{\text{ob}}(\phi) > W^{\text{un}}(\phi) > W^{\text{no}}(\phi)$ for all $\phi \in (\phi^*, 1)$.*

2. If $\mu > \overline{\mu}$, then $W^{\text{un}}(\phi) > W^{\text{no}}(\phi)$ for all $\phi \in (\phi^*, 1)$.

We observe that the term $(*)$ in equation (20) is 0 when $x = \text{no}$ and $\phi > \phi^*$. Hence, the social welfare gain from the presence of a signal (with or without information about prices) comes

[18] Note that $\alpha^{\text{no}}(\phi) = \pi^{\text{no}}(\phi) = 1$ and $p(\phi) = h$ for $\phi > \phi^*$; that is, the unique outcome of the game with no information when $\phi_0 > \phi^*$ is one where all sellers offer h and only the h-buyer accepts such offers.

from the fact that sellers then offer ℓ more frequently than if there is no signal, which implies that the ℓ-buyer purchases more often. Recall that the h-buyer's payoff from mimicking the ℓ-buyer coincides with his equilibrium payoff, since he is always indifferent between accepting and rejecting $p(\phi)$. Note also that, if the h-buyer mimics the ℓ-buyer, his payoff equals $h-\ell$ multiplied by the discounted times the price ℓ is offered. Since $(*)$ is equal to ℓ multiplied by the discounted measure of the times the price ℓ is offered, we have that $(*)$ is equal to $\frac{\ell}{h-\ell}V^x(\phi)$ for all $x \in \{\text{no}, \text{un}, \text{ob}\}$.

The term $(**)$ in equation (20) is equal to h for both $x \in \{\text{no}, \text{un}\}$, since the h-buyer buys with probability one at all times. It is then clear that $W^{\text{un}}(\phi) > W^{\text{no}}(\phi)$ for all $\phi \in (\phi^*, 1)$, since cookies increase the probability of trade for the ℓ-buyer while leaving it unchanged for the h-buyer. When $x = \text{ob}$, on the other hand, for $\mu > \overline{\mu}$ there is a wide range of posteriors where the h-buyer purchases with probability less than one. Additionally, given the slowness of learning due to the rejection mechanism, the ℓ-buyer purchases less often in the public offers case than in the private offers case if μ is large enough. This implies that some transactions that occur in the other cases are not realized in the public offers case. Thus, making prices unobservable may improve social welfare when the signal is very informative.

5 Discussion

There is an intense debate over privacy regulations for internet browsing, with internet cookies at its center. Similarly, new secrecy or transparency regulations are often subject to political considerations. While perfect transparency or complete obfuscation can be rarely achieved in these markets, debates focus on the desirability of policies that make information about previous transactions more or less accessible. Our analysis has abstracted from certain considerations relevant to these debates, focusing on the dynamic implications of information on pricing and trade efficiency.[19] We now describe some of these considerations and how our model can shed light on them. We focus on online markets, but similar arguments can be provided in financial or procurement markets.

Right to privacy: An important objective of some regulations is to guarantee users the so-called right to privacy. For example, a major outcome of the EU's Cookie Law is the requirement that

[19] See Acquisti et al. (2016) for a survey on theoretical and empirical research on the economics of privacy.

websites allow users to opt out of cookies. We now discuss how allowing the buyer to hide his purchase history would affect our results.

Consider a new model in which the buyer can decide whether or not to hide his history from each seller. Naturally, the buyer's decision to hide or reveal his history conveys information about his type. Standard unraveling arguments (à la Milgrom, 1981) would favor the existence of equilibria satisfying our characterization (where the buyer is always believed to be of type h unless he reveals his history). This would lower the value of a privacy regulation enabling buyers to hide their histories. Nevertheless, there would be inefficient equilibria in which disclosure of the history would be perceived as a sign of high type, so that in equilibrium the buyer would not disclose his history.[20]

Efficiency: A point often raised in defense of cookies is that they can improve the user's experience by making internet browsing more efficient. For example, cookies can be used to tailor advertisements to the user's interests, reducing frictions in the market for advertising. Such individualized advertisement is also sometimes perceived as harmful, as it may enable price discrimination (by tailoring prices or products), and also (political) manipulation by third parties.

To capture this, our model could be adapted to allow for some horizontal differentiation, for example, by allowing each seller to offer the buyer one of two types of products. The usual tradeoff would then arise: the more information the seller has, the more efficient trade becomes (as the buyer is offered the goods he prefers) and the more easily the seller can extract surplus from the buyer. It is then plausible that, as in our model, the buyer's signaling motives would induce sellers to offer low prices.[21]

Other sources of information: Our model focuses on transactions as the sole source of information about the buyer's type (beyond the prior). In practice, however, cookies often contain information about all websites visited by a user, including social media, news websites, blogs,

[20] It has also been noted that the requirement for users to report their cookie preferences at every website they visit is costly to them, generating "opt-out fatigue", which diminishes the quality of their browsing experience. Johnson et al. (2020) find that, in the US, even though users express strong privacy concerns, only a very small fraction opt out of targeted online advertising.

[21] A simple generalization of our model consists in assuming that sellers offer one of two types of products. Product 1 is valued at ℓ by ℓ-buyers and at h' by h-buyers. Product 2 is valued at $\ell' \leq \ell$ by ℓ-buyers and $h \geq h'$ by h-buyers (our model would be the case where $h' = h$). Each seller would offer either product 1 at price ℓ (accepted by both types of buyers), or product 2 at a higher price (accepted only by h-buyers).

and entertainment platforms. They may therefore reveal additional characteristics such as the user's income, age, gender, or occupation, which sellers may use to determine their price offers.

To study the effect of such additional information, one could add to our model an exogenous signal about the buyer's type, à la Daley and Green (2012).[22] It is reasonable to expect that, if such a signal were very informative, the buyer's surplus would be close to 0, while if the signal were very imprecise, the outcome would be close to our predictions. We leave to future research the study of the interaction between exogenous and endogenous learning in the intermediate case.

Observability of cookies. In the model of this paper, we assume that the buyer also observes the signal history. This assumption keeps the model tractable, as it ensures that first-order beliefs are the only relevant state variable. A model where the buyer could not observe the history would most likely not be tractable, as the buyer would learn part of this information over time from the prices offered by sellers.[23]

Overall, we argue that incorporating dynamic considerations helps to understand the effect of policies on internet privacy and financial transparency. Such policies affect the information available and distort the incentives for agents in the market to undertake actions that reveal their private information. We show that allowing more informative signals (e.g., deregulating cookies or limiting transparency) tends to enhance efficiency by limiting the bargaining power of uninformed agents. Nevertheless, some caution is needed: when prices are made accessible to sellers, changes in behavior may overturn the direct effect, implying less information transmission and lower welfare in equilibrium.

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[22] One could further allow for buyers to distort their behavior to affect the information provided by additional browsing activity (similarly to our model). We could model such a possibility as in Dilmé (2019), which studies a dynamic signaling game where the signal depends on effort instead of type.

[23] Cripps et al. (2007) show that, in a model with private reputation, the reputation of the long-run player eventually vanishes. More recently, Cisternas and Kolb (2021) study signaling with private monitoring in a linear–quadratic model with a Gaussian information structure.

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A Proofs of the results

A.1 Proofs of results in Section 3

Proof of Lemma 3.1

Proof. We divide the proof into 3 steps:

Step 1: Preliminary observations. Note that the price is never lower than ℓ, since offering ℓ ensures trade. Then, it is without loss of generality to focus on equilibria where the ℓ-seller accepts a price offer if and only if it is equal to ℓ, and where each seller never offers a price lower than ℓ. As a result, $V_\ell(\phi)=0$ for all $\phi\in(0,1)$.

Note also that, if the h-buyer accepts a price $\hat{p}>\ell$ for sure, then a seller is willing to offer such a price if and only if $\hat{p}\geq \ell/\phi$. Hence, no price in $(\ell,\ell/\phi)$ is offered in equilibrium.

Finally, it will sometimes be convenient to use the log-likelihood instead of the posterior. For each $\phi\in(0,1)$ and $z\in\mathbb{R}$, we define

$$\check{z}(\phi)=\log\left(\tfrac{\phi}{1-\phi}\right) \text{ and } \check{\phi}(z)=\tfrac{e^z}{1+e^z} \ . \tag{21}$$

Note that $\check{\phi}(\cdot)$ is the inverse of $\check{z}(\cdot)$. Abusing notation, for some function f of ϕ, we will sometimes use $f(z)$ and $f'(z)$ to denote $f(\check{\phi}(z))$ and $\frac{\mathrm{d}}{\mathrm{d}z}f(\check{\phi}(z))$, respectively. Note that, for example, $f'(\check{z}(\phi))=\phi(1-\phi)f'(\phi)$.

Step 2: Monotonicity of V_h. We continue the proof by stating and proving the following result:

Lemma A.1. *$V_h(\phi)=h-\ell$ for all $\phi\in(0,\phi^*]$ and V_h is strictly decreasing on $(\phi^*,1)$.*

Proof.

1. **Proof that $V_h(\phi)=h-\ell$ for all $\phi\in(0,\phi^*)$.** If only ℓ is offered in $(0,\phi^*)$ then the result follows. Assume then, for the sake of contradiction, that it is optimal for a seller to offer $\hat{p}>\ell$ when the posterior is $\phi\in(0,\phi^*]$. Since $\hat{p}$ has to be at least ℓ/ϕ by Step 1—hence strictly higher than h—, it must be that $\hat{p}$ is accepted for sure (by the argument laid out in the main text after Lemma 3.1). Then, from equation (11), such price should then satisfy:

$$\ell/\phi\leq\hat{p}=h+\mu/r\, V_h'(\check{z}(\phi)) \ . \tag{22}$$

Note that, if $|V_h'(\check{z}(\phi))|$ is small enough, the right-hand side of the previous expression is close to h. This implies that, if $|V_h'(\check{z}(\phi))|$ is small enough, sellers offer ℓ for sure (since no price above ℓ is optimal), and $V_h(\phi)=h-\ell$. We conclude that either $V_h'(\phi)=0$ for all $\phi\in(0,\phi^*)$, in which case $V_h(\phi)=h-\ell$ for all $\phi\in(0,\phi^*]$, or $V_h'(\phi)\neq 0$ for all $\phi\in(0,\phi^*)$.[24] In other words, when restricted to $(0,\phi^*)$, either V_h is strictly decreasing, or strictly increasing, or equal to $h-\ell$. It is clear V_h cannot be strictly decreasing, since in this case the h-buyer rejects all prices above h, hence it is strictly optimal for each seller to offer ℓ for all $\phi\in(0,\phi^*)$ and $V_h(\phi)=h-\ell$. Assume, for the sake of contradiction, that V_h is strictly increasing. Equation (22) implies that, in this case, $\lim_{z\to-\infty} V_h'(z)=+\infty$ and hence $\lim_{z\to-\infty} V_h(z)=-\infty$, which is a contradiction. Then, the only possibility is that $V_h(\phi)$ is equal to $h-\ell$ for all $\phi\in(0,\phi^*]$.

2. **Proof that V_h is strictly decreasing on** $(\phi^*,1)$**.** Assume first $V_h'(\phi_1)=0$ for some $\phi_1\in(\phi^*,1)$. This implies that $\alpha_h(\phi_1,\hat{p})=1$ for all $\hat{p}<h$, and $\alpha_h(\phi_1,\hat{p})=0$ for all $\hat{p}>h$. Hence, at posterior ϕ_1, the seller offers h for sure in equilibrium, since $\phi\, h>\ell$ when $\phi>\phi^*$. Equation (6) then becomes

$$r\, V_h(\phi_1)=\tfrac{1}{2}\mu(1-\phi_1)^2\phi_1^2 V_h''(\phi_t)\,.$$

We then have that ϕ_1 it is a minimizer of V_h. This implies that V_h is strictly increasing on $(\phi_1,1)$, so all prices offered on $(\phi_1,1)$ are higher than h (by the argument used to obtain equation (11)). But then this implies that $\lim_{\phi\nearrow 1} V_h(\phi)\leq 0$, and therefore $V_h(\phi)<0$ for some ϕ, which is a contradiction. We conclude that $V_h'(\phi)\neq 0$ for all $\phi\in(\phi^*,1)$.

Since V_h cannot be strictly increasing on $(\phi^*,1)$ (because $V_h(\phi^*)=h-\ell$ and equilibrium offers are never lower than ℓ) and $V_h'(\phi)\neq 0$ for all $\phi\in(\phi^*,1)$, we have that the V_h must be strictly decreasing on $(\phi^*,1)$.

□

(End of the proof of Lemma A.1. Proof of Lemma 3.1 continues.)

Step 3: Proof of the result. The argument in the main text implies that either the seller offers ℓ, or $p(\phi)$ satisfying equation (11), or randomizes between them. An immediate implication of

[24] The observation implies that V_h' cannot continuously "approach" 0, since a small value of $V_h'(\phi)$ implies $V_h(\phi)=h-\ell$ when $\phi<\phi^*$.

Lemma A.1 is that ℓ is offered with probability one when $\phi \in (0, \phi^*]$. Also, since V_h is strictly decreasing on $(\phi^*, 1)$, we have that ℓ is never offered with probability one at posteriors in $(\phi^*, 1)$.

□

Proof of Propositions 3.1 and 3.2

Proof. We prove Propositions 3.1 and 3.2 together. We divide the proof into ten steps:

Step 1: Preliminary derivations. We begin the proof by providing a useful equation. The arguments in the main text (following Lemma 3.1) show that, for each posterior $\phi \in (\phi^*, 1)$, the buyer is indifferent between accepting $p(\phi)$ or not (that is, equation (9) holds for $\hat{p} = p(\phi)$). This implies that the buyer's continuation value can be computed *as if* he did reject $p(\phi)$ for all posteriors ϕ. As a result, the continuation value of the buyer satisfies the following equation:

$$r V(\phi) = (1-\pi(\phi))(h-\ell) - \mu(1-\phi)\phi^2 \pi(\phi)^2 \alpha(\phi)^2 V'(\phi) \\ + \tfrac{1}{2}\mu(1-\phi)^2 \phi^2 \pi(\phi)^2 \alpha(\phi)^2 V''(\phi) \,. \tag{23}$$

The previous expression is convenient as it does not depend on $p(\phi)$.

Step 2: Preliminary results on continuity. We continue by providing a result on the continuity of $\alpha(\cdot)$ and $p(\cdot)$, and the limits of $V(\phi)$ as ϕ tends to ϕ^* and 1.

Lemma A.2. *Both $\alpha(\cdot)$ and $p(\cdot)$ are continuous on $(\phi^*, 1)$. Furthermore,* $\lim_{\phi \searrow \phi^*} V(\phi) = h - \ell$ *and* $\lim_{\phi \nearrow 1} V(\phi) = 0$.

Proof. The continuity of $\alpha(\cdot)$ and $p(\cdot)$ on $(\phi^*, 1)$ is immediately implied by equations (10) and (11).[25] That $\lim_{\phi \searrow \phi^*} V(\phi) = h - \ell$ follows from continuity of the continuation value and Lemma A.1.

We finally prove that $\lim_{\phi \nearrow 1} V(\phi) = 0$. Recall that, by Lemma A.1, V is strictly decreasing on $(\phi^*, 1)$. We assume, for the sake of contradiction, that $\lim_{\phi \nearrow 1} V(\phi) > 0$. Since it is optimal for the h-buyer to follow the ℓ-buyer's strategy (that is, only accepting offers equal to ℓ), there must exist some increasing sequence $(\phi_n)_n$ converging to 1 such that $(\pi(\phi_n))_n$ is convergent and

[25] Recall that we defined $\alpha(\phi)$ to be equal to $\alpha_h(\phi, p(\phi))$. Also, note that $V'(\cdot)$ is continuous when $\tilde{\sigma}^2(\phi; \alpha_\ell, \alpha_h, \tilde{\pi}) > 0$, which is guaranteed for all $\phi \in (0,1)$ because $\pi(\phi), \alpha(\phi) > 0$.

$\lim_{n\to\infty}\pi(\phi_n)<1$.[26] We can write equation (11) using log-likelihoods (see equation (21)) as

$$p(\phi)\equiv\max\left\{h/2,h+\mu/r\,V'(\check{z}(\phi))\right\}\ .$$

Hence, since $\lim_{\phi\nearrow 1}V'(\check{z}(\phi))=0$ (because V is strictly decreasing and bounded below by 0), we have that $\lim_{\phi\nearrow 1}p(\phi)=h$. This implies that, if ϕ is close enough to 1, $p(\phi)>\max\{\ell/\phi,h/2\}$, hence $\pi(\phi)=1$ (by (14)). As a result, $\lim_{\phi\nearrow 1}V(\phi)=0$. □

(End of the proof of Lemma A.2. Proof of Propositions 3.1 and 3.2 continues.)

Step 3: Preliminary results on regimes. We now present a result providing the equations for each of four possible types of regimes on $(\phi^*,1)$.

Lemma A.3. *The following statements follow for all $\phi_1,\phi_2\in[\phi^*,1]$ with $\phi_1<\phi_2$:*

1. *If $p(\phi)=\phi/\ell$ for all $\phi\in(\phi_1,\phi_2)$ then $\phi_2\leq 2\phi^*$, $\alpha(\phi)=1$, $\pi(\phi)\in(0,1)$,*

$$\pi(\phi)=\frac{2\phi(h-\ell-V(\phi))}{\phi(h-\ell)+2\ell(1-\phi)}\ , \tag{24}$$

   and

$$\pi'(\phi)=\frac{2r(\phi-\phi^*)+2(1-\phi)\phi^*\mu\pi(\phi)}{(1-\phi)\phi\mu(\phi(1-\phi^*)+2(1-\phi)\phi^*)}>0\ . \tag{25}$$

2. *If $p(\phi)=h/2$ for all $\phi\in(\phi_1,\phi_2)$ then $\phi_1\geq 2\phi^*$ (hence $\phi^*<1/2$), $\alpha(\phi)\in[2\phi^*/\phi,1)$, and $V(\phi)>h/4$. Also,*

   (a) If $\alpha(\phi)=2\phi^*/\phi$ for all $\phi\in(\phi_1,\phi_2)$, then

$$\pi(\phi)=\frac{2\phi(h-\ell-V(\phi))}{2h-3\ell} \tag{26}$$

   and

$$\pi'(\phi)=\frac{r}{2\mu(1-\phi)(2-3\phi^*)\phi^*}>0\ . \tag{27}$$

[26] If no such sequence exists, then there is a region $(\overline{\phi},1)$ where ℓ is never offered. This implies that, by mimicking the ℓ-seller, the h-seller obtains a continuation value converging to 0 as $\phi\to 1$.

(b) *If* $\alpha(\phi)\in(2\phi^*/\phi,1)$ *for all* $\phi\in(\phi_1,\phi_2)$*, then* $\pi(\phi)=1$*,*

$$\alpha(\hat{p})=-\tfrac{rh}{2\mu(1-\phi)\phi V'(\phi)}\ , \tag{28}$$

and

$$\alpha'(\phi)=\tfrac{4V(\phi)-h\alpha(\phi)}{(1-\phi)\phi h}>0\ . \tag{29}$$

3. *If* $p(\phi)>\max\{h/2,\ell/\phi\}$ *for all* $\phi\in(\phi_1,\phi_2)$ *then* $\alpha(\phi)=1$*,* $\pi(\phi)=1$*, and*

$$p'(\phi)=\tfrac{2V(\phi)-h+p(\phi)}{(1-\phi)\phi}>0\ . \tag{30}$$

Proof. 1. The fact that $\phi_2\leq 2\phi^*$ follows from the fact that $p(\phi)\geq h/2$ for all ϕ (by equation (11)). The fact that $\alpha(\phi)=1$ follows from equation (12). Equations (24) and (25) follow from equations (11) (with $p(\phi)=\ell/\phi$) and (23) (with $\alpha(\phi)=1$). Finally, $\pi(\phi)\in(0,1)$ because $\pi(\phi)\in[0,1]$ and π is strictly increasing by equation (25).

2. That $\phi_1\geq 2\phi^*$ follows from equation (13). That $\alpha(\phi)\geq 2\phi^*/\phi$ follows from the definition of ϕ^* and the optimality condition for the sellers requiring that $\phi\,\alpha(\phi)p(\phi)\geq\ell$.

   That $\alpha(\phi)<1$ follows from the fact that, if $\alpha(\phi)>2\phi^*/\phi$, then the first equality in equation (29) holds (from equations (23) and (28) with $\pi(\phi)=1$), so $\alpha(\phi)=1$ and $p(\phi)=h/2$ only if $V(\phi)=h/4$. That $\alpha(\phi)<1$ for all $\phi\in(\phi_1,\phi_2)$ follows from the fact that V is strictly increasing on $(\phi^*,1)$ (by Lemma A.1). Also:

   (a) Equation (27) follows from equations (23) and equation (9) (with $\alpha(\phi)=2\phi^*/\phi$).

   (b) That $\pi(\phi)=1$ follows because the seller strictly prefers offering $h/2$ than offering ℓ (since $\ell<\phi\,\alpha(\phi)p(\phi)$). Equation (28) follows from equation (10), and equation (29) follows from equations (23) and (28) (with $\pi(\phi)=1$).

   To prove that $V(\phi)>h/4$, recall the end of the proof of Lemma A.2. It shows that there exists some $\overline{\phi}<1$ such that $\alpha(\phi)=1$ for all $\phi\in[\overline{\phi},1)$. This implies that, if $\alpha(\phi)=1$ for all $\phi\in(\phi_1,\phi_2)$, then there must be some $\overline{\phi}<1$ such that $\alpha'(\overline{\phi})\geq 0$ and so $V(\overline{\phi})\geq h/4$. Since V is strictly decreasing on $(\phi^*,1)$ by Lemma A.1, we have $V(\phi)>h/4$ for all $\phi\in(\phi_1,\phi_2)$.

3. That $\alpha(\phi)=1$ follows from equation (12). That $\pi(\phi)=1$ follows from equation (14). Equation (30) follows from differentiating equation (11) (since the max operator on its right-hand side is larger than $h/2$) and from using equation (23). That $p'(\phi)>0$ follows from the fact that, from equations (11) and (30), we have that $p''(\phi)<0$ when $p'(\phi)=0$, but we know that $p(\phi)<h$ and $\lim_{\phi'\nearrow 1} p(\phi')=1$. □

(End of the proof of Lemma A.3. Proof of Propositions 3.1 and 3.2 continues.)

Step 4: Proof that $p(\phi)=\ell/\phi$ and $\alpha(\phi)=1$ for ϕ close to ϕ^*. Take a sequence $(\phi_n)_n$ strictly decreasing toward ϕ^* such that $\phi_n\in(\phi^*,\min\{1,2\phi^*\})$ for all n. From equation (13), we have that $p(\phi_n)\geq\ell/\phi_n>h/2$ for all n, and so $p(\phi_n)\to h$. By Lemma A.3, we have that $\alpha(\phi_n)=1$ for all n. Recall that, from Lemma A.2, $V(\phi_n)$ converges to $h-\ell$. Assume, for the sake of contradiction and taking a subsequence if necessary, that $p(\phi_n)>\ell/\phi_n>h/2$ for all n. Lemma A.3 implies that $p(\phi)$ is increasing when it is strictly larger $\max\{\ell/\phi_n,h/2\}$, hence this implies that $p(\phi)>\max\{\ell/\phi_n,h/2\}$ for all $\phi\in(\phi^*,1)$. From equation (30) it is clear that there must be some posterior $\phi'>\phi^*$ such that $p(\phi')>h$, which contradicts Lemma 3.1.

Step 5: Definition of $\phi^\dagger$. From the previous step, there must be some maximal $\phi^\dagger$ such that $p(\phi)=\ell/\phi$ for all $\phi\in(\phi^*,\phi^\dagger)$. Take some $\phi\in(\phi^*,\phi^\dagger)$; that is, we have that $p(\phi)=\ell/\phi$ and $\alpha(\phi)=1$. From equation (11) we have

$$\ell/\phi=p(\phi)=h+r^{-1}\mu(1-\phi)\phi V'(\phi)\,. \tag{31}$$

Additionally, by Lemma A.3, we have $\pi(\phi)\in(0,1)$. We can solve (31) for V (with boundary condition $V(\phi^*)=h-\ell$), and obtain

$$V(\phi)=h-\ell+\tfrac{r(\phi h-\ell)}{\mu\phi}+\tfrac{r(h-\ell)}{\mu}\log\left(\tfrac{1-\phi}{\phi}\Big/\tfrac{1-\phi^*}{\phi^*}\right)\,. \tag{32}$$

Note that we have $V'(\phi^*)=0$, so V' is continuous at ϕ^*. Then, from equation (23), we have

$$\pi(\phi)=\tfrac{2r}{(\phi(h-3\ell)+2\ell)\mu}\left(\ell-\phi h-\phi(h-\ell)\log\left(\tfrac{1-\phi}{\phi}\Big/\tfrac{1-\phi^*}{\phi^*}\right)\right)\,. \tag{33}$$

Since the right-hand side of equation (32) tends to $-\infty$ as $\phi\to 1$, it must be that $\phi^\dagger<1$.

Step 6: Preliminaries for less informative signal. We first focus on the case $\phi^\dagger<2\phi^*$. We want to show that, for all $\phi > \phi^\dagger$, we have $p(\phi) > \max\{h/2, \ell/\phi\}$, and hence $\alpha(\phi)=1$, $\pi(\phi)=1$, and equation (30) holds (by Lemma A.3).

We first argue that there is no $\phi \in (\phi^\dagger, \min\{2\phi^*, 1\})$ where $p(\phi) = \ell/\phi$. Assume, for the sake of contradiction, that there is an interval (ϕ_1, ϕ_2) with $\phi^\dagger \le \phi_1 < \phi_2 \le 2\phi^*$ such that $p(\phi) > \ell/\phi$ for all $\phi \in (\phi_1, \phi_2)$ and $p(\phi_2) = \ell/\phi_2$. Since $p(\phi)$ approaches ℓ/ϕ_2 as $\phi \to \phi_2$ from above the curve ℓ/ϕ, it must be that $\lim_{\phi \nearrow \phi_2} p'(\phi) \le -\ell/\phi_2^2$ which, by equation (30), implies

$$V(\phi_2) \le \tfrac{1}{2}(h + \ell - 2\ell/\phi_2) < \tfrac{1}{2}(\ell - h) < 0 \ ,$$

which is a contradiction. Hence, there is no $\phi \in (\phi^\dagger, \min\{2\phi^*, 1\})$ where $p(\phi) = \ell/\phi$.

We now argue that there is no $\phi \in (\min\{1, 2\phi^*\}, 1)$ such that $p(\phi) = h/2$. If $\phi^* \ge 1/2$ the result is clear, so assume that $\phi^* < 1/2$. Assume, for the sake of contradiction, that there is an interval (ϕ_1, ϕ_2) with $2\phi^* \le \phi_1 < \phi_2 < 1$ such that $p(\phi) > h/2$ for all $\phi \in (\phi_1, \phi_2)$ and $p(\phi_2) = h/2$. It is clear that $p(\phi) > \max\{\ell/\phi, h/2\}$ for all $\phi \in (\phi_1, \phi_2)$ (since $h/2 > \ell/\phi$ for $\phi > 2\phi^*$). This implies, by Lemma A.3, that p is increasing on (ϕ_1, ϕ_2), hence it is not possible that $p(\phi_2) = h/2$ (p is continuous by Lemma A.2).

We then have proven that $p(\phi) > \max\{h/2, \ell/\phi\}$ and $\alpha(\phi) = \pi(\phi) = 1$ for all $\phi > \phi^\dagger$; hence, by Lemma A.3, equation (30) holds. This implies that equation (23) can be written, for all $\phi \in (\phi^\dagger, 1)$, as

$$\begin{aligned} r\,V(\phi) = {} & (1-\pi(\phi))(h-\ell) - \mu(1-\phi)\,\phi^2\,\pi(\phi)^2\,\alpha(\phi)^2\,V'(\phi) \\ & + \tfrac{1}{2}\mu(1-\phi)^2\,\phi^2\,\pi(\phi)^2\,\alpha(\phi)^2\,V''(\phi) \ . \end{aligned} \tag{34}$$

The solution to this equation is

$$V(\phi) = C_1\left(\tfrac{1-\phi}{\phi}\right)^{\kappa} + C_2\left(\tfrac{1-\phi}{\phi}\right)^{-1-\kappa} , \tag{35}$$

where

$$\kappa \equiv \tfrac{1}{2}(\sqrt{1 + 8r/\mu} - 1) > 0 \ , \tag{36}$$

and C_1 and C_2 are integration constants. Using that, by Lemma A.2, we have that $\lim_{\phi \nearrow 1} V(\phi) = 0$, and so $C_2 = 0$. We can use the continuity of V' at $\phi^\dagger$ and equations (32) and (35) with $C_2 = 0$

to obtain the value of C_1 as a function of $\phi^\dagger$, so we have that

$$V(\phi)=\tfrac{r}{\mu}\tfrac{\phi^\dagger-\phi^*}{\phi^\dagger}\Big(\tfrac{\phi}{1-\phi}\Big/\tfrac{\phi^\dagger}{1-\phi^\dagger}\Big)^{-\kappa}h \tag{37}$$

for all $\phi\in(\phi^\dagger,1)$. Finally, the value of $\phi^\dagger$ is obtained using that V is continuous at $\phi^\dagger$. This requirement can be written as

$$-\tfrac{(\kappa-1)r(\phi^\dagger-\phi^*)}{\phi^\dagger\kappa}+r(1-\phi^*)\log\Big(\tfrac{\phi^\dagger}{1-\phi^\dagger}\Big/\tfrac{\phi^*}{1-\phi^*}\Big)-\mu(1-\phi^*)=0\ . \tag{38}$$

The left-hand side of the previous expression is $-\mu(1-\phi^*)$ when $\phi^\dagger=\phi^*$, and tends to $+\infty$ when $\phi^\dagger\to 1$, so a solution exists. Furthermore, the derivative of the left-hand side of the previous equation with respect to $\phi^\dagger$ is

$$\tfrac{r(\kappa(\phi^\dagger-\phi^*)+(1-\phi^\dagger)\phi^*)}{\kappa(1-\phi^\dagger)(\phi^\dagger)^2}>0\ .$$

Hence, there is exactly one value of $\phi^\dagger$ solving equation (38). The derivative of the left-hand side of expression (37) with respect to μ is

$$-\tfrac{2r^2(\phi^\dagger-\phi^*)}{\mu^2\phi^\dagger(1+2\kappa)\kappa^2}-(1-\phi^*)<0\ .$$

It is then clear that $\phi^\dagger$ is increasing in μ.

Step 7: Definition of $\overline{\mu}$. We now claim that there is some value $\overline{\mu}\in(0,+\infty]$ such that a solution $\phi^\dagger$ strictly smaller than $2\phi^*$ for equation (38) exists if and only if $\mu<\overline{\mu}$. The result is obviously true if $\phi^*\geq 1/2$, since then $\overline{\mu}=+\infty$ (by the arguments in Step 6). Assume then that $\phi^*<1/2$.

Note that, differentiating the left-hand side of expression (38) two times with respect to μ (recall that κ depends on μ, see equation (36)), we obtain

$$-\tfrac{7}{8}+\phi^*+\tfrac{\mu+4r}{8\mu(1+2\kappa)}\ .$$

Using the value of κ (from (36)) it is easily seen that the previous expression is negative for all $\phi^*<1/2$. Furthermore, the left-hand side of expression (38) is negative when $\phi^\dagger=\phi^*$, and tends to $+\infty$ when $\phi^\dagger\to 1$. It is then clear that there is only one value of μ for which $\phi^\dagger=2\phi^*$. That is, using Step 6, we have proven that, for all $\mu\leq\overline{\mu}$, the unique equilibrium is as described in the state of Proposition 3.1, while for all $\mu > \overline{\mu}$ there is no equilibrium of this form.[27]

Step 8: Preliminaries for the more informative signal. Assume for the rest of the proof that $\mu > \overline{\mu}$. In this case, $\phi^{\dagger} = 2\phi^*$ (where $\phi^{\dagger}$ is defined in Step 5 as the maximal such that $p(\phi) = \ell/\phi$ for all $\phi \in (\phi^*, \phi^{\dagger})$). Since $\alpha(\cdot)$ is continuous on $(\phi^*, 1)$ (by Lemma A.2), an implication of Lemma A.3 is that all equilibria satisfy the characterization provided in Proposition 3.2.[28]

Then, to show existence and uniqueness of equilibria, we have prove the existence and uniqueness thresholds $\hat{\phi}^{\dagger}$ and $\hat{\phi}^{\dagger\dagger}$ such that the implied continuation value satisfies the smooth pasting conditions and such that $\alpha(\hat{\phi}^{\dagger\dagger}) = 1$.

To prove existence of an equilibrium, we construct the continuation value of the h-buyer. To do so, we use Lemma A.3 to determine the equations governing V for the different regions of beliefs, for some given values of $\hat{\phi}^{\dagger}$ and $\hat{\phi}^{\dagger\dagger}$.

1. Region $(0, \phi^*)$: In this region we have $V(\phi) = h - \ell$ (by Lemma 3.1).
2. Region $(\phi^*, 2\phi^*)$: In this region, equation (32) holds.
3. Region $(2\phi^*, \hat{\phi}^{\dagger})$: Imposing the smooth pasting conditions at $2\phi^*$, we obtain that, in this region,

$$V(\phi) = h - \ell + \left(\frac{h}{2} + \frac{\ell}{4} \log\left(\frac{1-\phi}{1-2\phi^*}\right) - (h-\ell)\log\left(\frac{2-2\phi^*}{1-2\phi^*}\right)\right)\frac{r}{\mu} \,. \tag{39}$$

4. Region $(\hat{\phi}^{\dagger}, \hat{\phi}^{\dagger\dagger})$: In this region, V follows equation (23) with $\pi(\phi) = 1$. Using equation (28), we obtain

$$V(\phi) = \frac{1}{2\sqrt{\mu/r}} h H\left((1-\phi)^{-1} c_1 + c_2\right)$$

for some $c_1 < 0$ (so V is decreasing) and $c_2 \in \mathbb{R}$; where H is the inverse of the integral of the Gaussian distribution multiplied by 2, that is,

$$H^{-1}(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x e^{-x'^2} \mathrm{d}x' \,.$$

[27] Indeed, as $\mu \to 0$ we have $\phi^{\dagger} \to \phi^*$, so the value of $\phi^{\dagger}$ is increasing in μ at $\phi^{\dagger} = 2\phi^*$. (Proposition 4.1 sows that the unique $\phi^{\dagger}$ solving equation (38) is increasing in μ.)

[28] Indeed, recall that α is continuous by Lemma A.2. By Lemma A.3, it is decreasing only if it is equal to $2\phi^*/\phi$. Hence, if an equilibrium exists for $\mu > \overline{\mu}$, there must be some $\hat{\phi}^{\dagger}$ and $\hat{\phi}^{\dagger\dagger}$ such that $\alpha(\phi) = 2\phi^*/\phi$ for all $\phi \in (2\phi^*, \hat{\phi}^{\dagger})$ (case 2(a) in Lemma A.3), then satisfies equation (28) in $(\hat{\phi}^{\dagger}, \hat{\phi}^{\dagger\dagger})$ (where it is strictly increasing), and it is equal to 1 in $(\hat{\phi}^{\dagger\dagger}, 1)$.

Note that the domain and codomain of H are, respectively, $(-1,1)$ and $\mathbb{R}$, and also that H is strictly increasing and strictly convex on $(0,1)$.

5. Region $(\hat{\phi}^{\dagger\dagger},1)$: In this region, V follows equation (35) for some $C_1>0$ and $C_2=0$.

We have 5 unknown variables to be determined: $\hat{\phi}^{\dagger}$, $\hat{\phi}^{\dagger\dagger}$, c_1, c_2 and C_1. To do so, we have two smooth pasting conditions at $\hat{\phi}^{\dagger}$, two other smooth pasting conditions at $\hat{\phi}^{\dagger\dagger}$, and the requirement that $\alpha(\hat{\phi}^{\dagger\dagger})=1$. We have then as many unknown variables as conditions.

Using the smooth pasting conditions at $\hat{\phi}^{\dagger\dagger}$ and that $\alpha(\hat{\phi}^{\dagger\dagger})=1$, we obtain

$$C_1=\tfrac{rh}{2\kappa\mu}\Big(\tfrac{\hat{\phi}^{\dagger\dagger}}{1-\hat{\phi}^{\dagger\dagger}}\Big)^{\kappa}$$

(where recall that κ is defined in (36)) and also

$$H\big((1-\hat{\phi}^{\dagger\dagger})^{-1}c_1+c_2\big)=\big(\kappa\sqrt{\mu/r}\big)^{-1} \tag{40}$$

and

$$H'\big((1-\hat{\phi}^{\dagger\dagger})^{-1}c_1+c_2\big)=-\tfrac{1-\hat{\phi}^{\dagger\dagger}}{\hat{\phi}^{\dagger\dagger}c_1\sqrt{\mu/r}}\ . \tag{41}$$

We define the variables $c^{\dagger}\equiv(1-\hat{\phi}^{\dagger})^{-1}c_1+c_2$ and $c^{\dagger\dagger}\equiv(1-\hat{\phi}^{\dagger\dagger})^{-1}c_1+c_2$. Since V is positive and decreasing and H is increasing, it must be that $c^{\dagger}>c^{\dagger\dagger}>0$. From the two equations (40) and (41), and from the boundary conditions at $\hat{\phi}^{\dagger}$ we obtain the following four equations:

1. The first equation determines the value of $c^{\dagger\dagger}$:

$$c^{\dagger\dagger}=H^{-1}\Big(\tfrac{1}{\kappa\sqrt{\mu/r}}\Big)=\tfrac{2}{\sqrt{\pi}}\int_0^{\frac{r^{1/2}}{\kappa\mu^{1/2}}}e^{-x^2}\,\mathrm{d}x\ . \tag{42}$$

2. The second and third equations express $\hat{\phi}^{\dagger}$ and $\hat{\phi}^{\dagger\dagger}$ as functions of $c^{\dagger}$ and $c^{\dagger\dagger}$:

$$\hat{\phi}^{\dagger}=2\phi^*\frac{H'(c^{\dagger})}{H'(c^{\dagger\dagger})}\big(1-\sqrt{\mu/r}\,(c^{\dagger}-c^{\dagger\dagger})H'(c^{\dagger\dagger})\big)\ ,$$
$$\hat{\phi}^{\dagger\dagger}=\frac{2\phi^*H'(c^{\dagger})}{H'(c^{\dagger\dagger})(1+2\phi^*\sqrt{\mu/r}\,(c^{\dagger}-c^{\dagger\dagger})H'(c^{\dagger}))}\ .$$

It is not difficult to see that, as long as $c^{\dagger}>c^{\dagger\dagger}$, we have $\hat{\phi}^{\dagger}<\hat{\phi}^{\dagger\dagger}<1$.

3. The only value left to determine is $c^\dagger$. This is obtained by solving the fourth equation

$$0 = r\log\Big((1-2\phi^*)\Big(1-2\phi^* H'(c^\dagger)\big(\overbrace{\tfrac{1}{H'(c^{\dagger\dagger})}-\tfrac{\sqrt{\mu}(c^\dagger-c^{\dagger\dagger})}{\sqrt{r}}}^{(*)}\big)\Big)\Big) + 2\phi^*\big(2(1-\phi^*)\mu - 2(1-\phi^*)r\log\big(1+\tfrac{1}{1-2\phi^*}\big) - \sqrt{\mu}\sqrt{r}H(c^\dagger)+r\big)\,. \tag{43}$$

To prove existence of an equilibrium, we have to prove that equation (43) has a solution for $c^\dagger$ in $(c^{\dagger\dagger},1)$, and to prove uniqueness of an equilibrium that such solution is unique.

Step 9: Existence of an equilibrium for the more informative signal. We first note that if $c^\dagger$ is replaced by $c^{\dagger\dagger}$ in equation (43), the resulting equation is equivalent to equation (38) with $\phi^\dagger = 2\phi^*$. This result is intuitive: when $\mu = \overline{\mu}$ (hence equation (38) with $\phi^\dagger = 2\phi^*$ holds), we have $c^\dagger = c^{\dagger\dagger}$, which implies that $\hat{\phi}^\dagger = \hat{\phi}^{\dagger\dagger} = 2\phi^*$. Since in this part of the proof we assume that $\mu > \overline{\mu}$ implies that the right-hand side of equation (43) is positive when $c^\dagger$ is replaced by $c^{\dagger\dagger}$.

It is easy to see that the term $(*)$ in equation (43) is positive when $c^\dagger$ is replaced by $c^{\dagger\dagger}$. Hence, since $H'(c^\dagger)$ tends to $+\infty$ when $c^\dagger$ tends to 1, we have that there exists a value $\overline{c}^\dagger$ such that the right-hand side of equation (43) tends to $-\infty$ as $c^\dagger \nearrow \overline{c}^\dagger$.[29] Then, continuity proves the existence of an equilibrium.

Step 10: Uniqueness of an equilibrium for the more informative signal. It is only left to prove that the right-hand side of equation (43) is strictly decreasing on $(c^{\dagger\dagger}, \overline{c}^\dagger)$. To do so, define $\tilde{c}^\dagger \equiv H(c^\dagger)$. Then, the derivative of the right-hand side of equation (43) with respect to $\tilde{c}^\dagger$ is

$$0 = -2\phi^* r H''(c^\dagger)\frac{\overbrace{\frac{1}{H'(c^{\dagger\dagger})}-\frac{\sqrt{\mu}(c^\dagger-c^{\dagger\dagger})}{\sqrt{r}}+\frac{\sqrt{\mu}}{\sqrt{r}}\frac{H'(c^\dagger)}{H''(c^\dagger)}}^{(**)}}{1-2\phi^* H'(c^\dagger)\big(\frac{1}{H'(c^{\dagger\dagger})}-\frac{\sqrt{\mu}(c^\dagger-c^{\dagger\dagger})}{\sqrt{r}}\big)} - 2\phi^*\sqrt{\mu}\sqrt{r}\,. \tag{44}$$

The derivative of $(**)$ with respect to $c^\dagger$ is $\sqrt{\mu}/(2\sqrt{r}H(c^\dagger)) > 0$, and its value at $c^\dagger = c^{\dagger\dagger}$ is

$$\frac{e^{-\frac{r}{\kappa^2\mu}}\big(4r+\mu-\sqrt{\mu}\sqrt{8r+\mu}\big)}{2\sqrt{\pi}r}\,,$$

which is positive. Hence, the term $(**)$ is positive. As a result, the derivative of the right-hand side of equation (43) with respect to $c^\dagger$ is negative. We conclude that there is a unique pair

[29] Indeed, the term inside the logarithm on the first line of equation (43) tends to 0 as $c^\dagger \nearrow \overline{c}^\dagger$.

of thresholds $(\hat{\phi}^{\dagger},\hat{\phi}^{\dagger\dagger})$, with $2\phi^* < \hat{\phi}^{\dagger} < \hat{\phi}^{\dagger\dagger} < 1$ such that V satisfies all boundary conditions,. Then, a unique equilibrium (which satisfies the characterization of Proposition 3.2) exists. □

A.2 Proofs of results in Section 4

Proof of Proposition 4.1

Proof. The proof is divided into four steps.

Step 1. We first show that V is increasing in μ on $(0,\overline{\mu})$. We divide this step of the proof into two sub-steps:

1. We first prove that $\phi^{\dagger}$ is increasing in μ. To do so, recall from the proof of Propositions 3.1 and 3.2 that $\phi^{\dagger}$ is the unique solution to (38), which we denote $\phi^{\dagger}_{\mu}$ in this proof.

   We note that the derivative of the left-hand side of expression (38) with respect to μ (recall that κ depends on μ, see equation (36)) is equal to

   $$\frac{\mu(1+\kappa)+2r}{2\mu(1+2\kappa)}\,\frac{\phi^{\dagger}-\phi^*}{\phi^*} - (1-\phi^*)\ .$$

   Such expression is strictly increasing in $\phi^{\dagger}$ and negative for all $\phi^{\dagger} \in (\phi^*, \overline{\phi}^{\dagger})$, where

   $$\overline{\phi}^{\dagger} \equiv \min\left\{1, \phi^*\left(1 - \frac{2(1+2\kappa)\mu(1-\phi^*)}{2r+(1+\kappa)\mu}\right)^{-1}\right\}\ .$$

   If $\overline{\phi}^{\dagger} = 1$ then we have that the left-hand side of expression (38) is decreasing in μ, and hence $\phi^{\dagger}$ is increasing in μ. Assume then that $\overline{\phi}^{\dagger} < 1$. When we evaluate the left-hand side of expression (38) for $\phi^{\dagger} = \overline{\phi}^{\dagger}$ we obtain

   $$-(1-\phi^{\dagger})\left(\mu + \frac{2(\kappa-1)(1+2\kappa)r\mu}{2\kappa r + \kappa(1+\kappa)\mu} + r\log\left(\frac{(1+3\kappa)\mu - 2r}{(1+\kappa)\mu + 2r}\right)\right)\ .$$

   Using the definition of κ (see equation (36)), it is easy to see that the previous expression is positive. This implies that $\overline{\phi}^{\dagger} > \phi^{\dagger}$. As a result, we have that the derivative of the left-hand side of equation (38) at $\phi^{\dagger}_{\mu}$ is negative. Recalling that the left-hand side of equation (38) is strictly increasing in $\phi^{\dagger}$ (see the proof of Propositions 3.1 and 3.2), we have that $\phi^{\dagger}_{\mu}$ is increasing in μ on $(0,\overline{\mu})$.

2. Let $z^* \equiv \check{z}(\phi^*)$ and $z^\dagger \equiv \check{z}(\phi^\dagger)$ (recall the definition of $\check{z}$ in (21)). Note that $\phi^\dagger$ for all $z \in [z^\dagger, \infty)$ we have[30]

$$\frac{V(z^\dagger)}{-V'(z^\dagger)} \begin{cases} > \kappa^{-1} & \text{if } z \in (z^*, z^\dagger). \\ = \kappa^{-1} & \text{if } z \in [z^\dagger, +\infty). \end{cases} \tag{45}$$

Take now two values $\mu_1 < \mu_2 < \overline{\mu}$. By part 1, we have that $z^\dagger_{\mu_1} < z^\dagger_{\mu_2}$, where the subindexes μ_1 and μ_2 indicate equilibrium values for each of the values for μ. From equations (11) (with $p(\phi) = \ell/\phi$) and (32), we have $V_{\mu_1}(z) < V_{\mu_2}(z)$ and $-V'_{\mu_1}(z) > -V'_{\mu_2}(z)$ for all $z \in (z^*, z^\dagger_{\mu_1}]$. Assume, for a contradiction, that there is some $z > z^\dagger_{\mu_1}$ such that $V_{\mu_1}(z) = V_{\mu_2}(z)$. There must then be some $\hat{z} > z^\dagger_{\mu_1}$ such that $V_{\mu_1}(\hat{z}) = V_{\mu_2}(\hat{z})$ and $-V'_{\mu_1}(\hat{z}) \le -V'_{\mu_2}(\hat{z})$. But then,

$$\frac{V_{\mu_2}(\hat{z})}{-V'_{\mu_2}(\hat{z})} \le \frac{V_{\mu_1}(\hat{z})}{-V'_{\mu_1}(\hat{z})} = \kappa^{-1}_{\mu_1} < \kappa^{-1}_{\mu_2} \le \frac{V_{\mu_2}(\hat{z})}{-V'_{\mu_2}(\hat{z})} .$$

This is a contradiction. We then showed that $V_{\mu_1}(\phi) < V_{\mu_2}(\phi)$ for all $\phi \in (\phi^*, 1)$.

Step 2. That $\frac{\mathrm{d}}{\mathrm{d}\mu} V(\phi) > 0$ when $\mu > \overline{\mu}$ and $\phi \in (\phi^*, \hat{\phi}^\dagger)$ follows immediately from differentiating the right-hand side of expressions (32) (which holds when $\phi \in (\phi^*, 2\phi^*)$) and (39) (which holds when $\phi \in (2\phi^*, \hat{\phi}^\dagger)$).

Step 3. We now show that $\lim_{\mu \to \infty} V(\phi) = h - \ell$ for all $\phi \in (\phi^*, 1)$. There are two cases:

1. Assume first $\overline{\mu} = +\infty$ (that is, if $\phi^* \ge 1/2$). Take a sequence $(\mu_n)_n$ tending to $+\infty$. Let $(\phi^\dagger_n)_n$ be the sequence of the corresponding equilibrium thresholds, which is increasing by Step 1, and let $\phi^\dagger_\infty \in (\phi^*, 1]$ be its limit. From equation (32), we have that $\lim_{n \to \infty} V'_n(\phi) = 0$ for all $\phi \in (\phi^*, \phi^\dagger_\infty)$. Hence, $\lim_{n \to \infty} V_n(\phi) = h - \ell$ for all $\phi \in (\phi^*, \phi^\dagger_\infty)$.

   If $\phi^\dagger_\infty = 1$ then the result holds. If $\phi^\dagger_\infty < 1$ then, given that the drift of the belief process from rejecting offers at each $\phi \in (\phi^\dagger_\infty, 1)$ becomes arbitrarily large as $\mu \to \infty$, we have that $\lim_{n \to \infty} V_n(\phi) = h - \ell$ for all $\phi \in (\phi^\dagger_\infty, \phi^*)$, hence the result holds.

2. Assume that, instead, $\overline{\mu} < +\infty$ (that is, $\phi^* < 1/2$). Then, if μ is large enough, the equilibrium characterization in Proposition 3.2 applies. By the same argument as when $\overline{\mu} = +\infty$,

[30] Indeed, if there is a solution V of (32) and a value of $\phi^\dagger$ such that equation (45) is satisfied, there is an equilibrium as in Proposition 3.1 (since the smooth pasting conditions hold at $\phi^\dagger$ for dome continuation value from the right given in (35) for some C_1 and with $C_2 = 0$).

we now have $\lim_{\mu \nearrow +\infty} V'(\phi) = 0$ for all $\phi \in (\phi^*, 2\phi^*)$.

For each $\phi \in (2\phi^*, \hat{\phi}^{\dagger\dagger})$, we have (from equation (10))

$$V'(\phi) = -\frac{rh/2}{\mu(1-\phi)\phi\,\alpha(\phi)} \geq -\frac{rh}{4\mu(1-\phi)\,\phi^*} \,,$$

where we used $\alpha(\phi) \geq 2\phi^*/\phi$ by the optimality of the seller's strategy.

We can now take a sequence $(\mu_n)_n$ tending to $+\infty$ and let $(\hat{\phi}_n^{\dagger\dagger})_n$ be the sequence of the corresponding equilibrium thresholds. Taking a subsequence if necessary, assume that $(\hat{\phi}_n^{\dagger\dagger})_n$ tends to some value $\hat{\phi}_\infty^{\dagger\dagger} \in [\phi^*, 1]$. Then, we can then use the same argument as for the case where $\overline{\mu} = +\infty$.

Step 4. We now want to show that $\lim_{\mu \to 0} V(\phi) = 0$ for all $\phi \in (\phi^*, 1)$. When μ is small enough, we can use the equilibrium characterization in Proposition 3.1. Take a sequence $(\mu_n)_n$ tending to 0. Let $(\phi_n^\dagger)_n$ be the sequence of the corresponding equilibrium thresholds. Taking a subsequence if necessary, assume that $(\phi_n^\dagger)_n$ tends to some value $\phi_\infty^\dagger \in [\phi^*, 1]$.

Now, for all $\phi \in (\phi^*, \phi_\infty^\dagger)$, we have $\lim_{n \to \infty} V_n'(\phi) = +\infty$ (from equation (32)). This has the implication that, $\phi_\infty^\dagger = \phi^*$. Because, for each n, the buyer is willing to reject all offers for all $\phi > \phi_n^\dagger$ for all n, and because the learning speed tends to 0 for all ϕ, the result holds. □

Proof of Lemma 4.1

Proof. That Lemma A.1 holds in the case where offers are not observable can be proven analogously. Additionally, by the usual take-it-or-leave-it offer argument, each seller either offers ℓ (which is accepted for sure by both types of the buyer) or the price $p(\phi) > \ell$ given in equation (18) (which is accepted for sure by the h-buyer and rejected for sure by the ℓ-buyer). □

Proof of Proposition 4.2

Proof. The result follows trivially from Lemma 4.1 when $\phi \leq \phi^* \equiv \ell/h$. The rest of the proof is divided into six steps.

Step 1: Preliminary results on regimes. Recall that, when prices are unobservable, the continuation value satisfies

$$V(\phi)=r\left(\pi(\phi)(h-p(\phi))+(1-\pi(\phi))(h-\ell)\right)$$
$$+\tilde{\mu}(\phi,1,(1,1-\pi(\phi)))V'(\phi)+\tfrac{1}{2}\sigma^2(\phi,(1,1-\pi(\phi)))V''(\phi)\ , \tag{46}$$

where $\tilde{\mu}$ and σ^2 are defined in (3) and (4), respectively (note that the ℓ-buyer accepts an offer with probability $1-\pi(\phi)$). Additionally, the h-buyer's indifference between accepting $p(\phi)$ or not implies that equation (18) holds as well.

We begin with a result characterizing some of the characteristics of the different possible regimes. The following result is analogous to Lemma A.3 in the proofs of Propositions 3.1 and 3.2.

Lemma A.4. *Assume prices are not observable. The following statements follow for all $\phi^*\leq\phi_1<\phi_2\leq 1$:*

1. *If $\pi(\phi)\in(0,1)$ for all $\phi\in(\phi_1,\phi_2)$ then $\phi_2<1$, $p(\phi)=\ell/\phi$, and*

$$\pi(\phi)=-\tfrac{r(h-\ell/\phi)}{(1-\phi)\phi\,\mu V'(\phi)}\ . \tag{47}$$

2. *If $\pi(\phi)=1$ for all $\phi\in(\phi_1,\phi_2)$ then*

$$p(\phi)=h+\tfrac{\mu}{r}(1-\phi)\phi\,V'(\phi)\ . \tag{48}$$

Proof. 1. The indifference of the sellers implies that $p(\phi)=\ell/\phi$. As a result, from equations (18) and (46), we have

$$0=r(\phi-\phi^*)\tfrac{V'(\phi)}{h}+\tfrac{r}{2}(\phi-\phi^*)^2\tfrac{V''(\phi)}{h}+\phi^2\mu\left(1-\phi^*-\tfrac{V(\phi)}{h}\right)\tfrac{V'(\phi)^2}{h^2} \tag{49}$$

holds. Equation (47) follows from equations (46) (with $p(\phi)=\ell/\phi$) and (49). Note that equation (47) can be written using log-likelihoods as

$$\pi(z)=-\tfrac{r(h-(1+e^{-z})\ell)}{\mu V'(z)}\ .$$

Since it must be that $V'(z)\to 0$ as $z\to\infty$ (because V is bounded below by 0), we have that $\phi_2 < 1$.

2. **Case $\pi(\phi) = 1$ for all $\phi \in (\phi_1, \phi_2)$:** Equation (48) follows from the indifference of the h-buyer on accepting $p(\phi)$ or not, that is, equation (18). Additionally, we have the equation (46) becomes

$$r\,V(\phi) = \mu(1-\phi)^2\,\phi\,V'(\phi) + \tfrac{1}{2}\mu(1-\phi)^2\,\phi^2\,V''(\phi)\;. \tag{50}$$

□

(End of the proof of Lemma A.4. Proof of Proposition 4.2 continues.)

Step 2: Continuity of p and π. In this step, we prove that p and π are continuous. Note that, since equation (18) holds for all ϕ, the continuity of π implies the continuity of p. Hence, we prove that π is continuous by contradiction. There are then two cases:

1. Assume first that $\pi(\phi) \in (0,1)$ for some $\phi \in (0,1)$ and there is a sequence $(\phi_n)_n$ converging to ϕ such that $\pi(\phi_n) = 1$ for all k. Note that $\pi(\phi) < 1$ only if $p(\phi) = \ell/\phi$. Then, we have that

$$\lim_{n\to\infty} p(\phi_n) = h + \tfrac{\mu}{r}(1-\phi)\,\phi\,V'(\phi) = \tfrac{\ell}{\phi} - \tfrac{1-\pi(\phi)}{\pi(\phi)}(h - \ell/\phi) < \ell/\phi\;.$$

   This is a contradiction.

2. The alternative case is that $\pi(\phi) \in [0,1]$ and there is a sequence $(\phi_n)_n$ converging to ϕ such that $\pi(\phi_n) < 1$ for all k and $(\pi(\phi_n))_n$ converges to some $\overline{\pi} \neq \pi(\phi)$. Note that $p(\phi_n) = \ell/\phi_n$ for all n. Then, we have that

$$\ell/\phi = \lim_{n\to\infty} p(\phi_n) = h + \tfrac{\mu}{r}(1-\phi)\,\phi\,\overline{\pi}\,V'(\phi) = h - (h - p(\phi))\,\tfrac{\overline{\pi}}{\pi(\phi)}\;.$$

   If $\pi(\phi) = 1$ then, since $p(\phi) \geq \ell/\phi$, the right-hand side of the rightmost equality is strictly bigger than ℓ/ϕ, a contradiction. If instead $\pi(\phi) < 1$ then $p(\phi) = \ell/\phi$, but then the previous equation implies $\pi(\phi) = \overline{\pi}$, again a contradiction.

Step 3: Determination of the equilibrium structure. Let (ϕ_1, ϕ_2) be a maximal region with $\pi(\phi) \in (0,1)$ for all $\phi \in (\phi_1, \phi_2)$ (that is, $\pi(\phi_1) \in \{0,1\}$ and $\pi(\phi_2) = 1$). We aim at proving that there is at most one such interval, and is such that $\phi_1 = \phi^*$. Note that $\phi_2 < 1$ by Lemma A.4. We have that $p(\phi_2) = \ell/\phi_2$. Then, using $\pi(\phi_2) = 1$, we have

$$\lim_{\phi \nearrow \phi_2} \pi'(\phi_2) = \frac{2\ell + \phi_2(2V(\phi_2) - h - \ell)}{h\phi_2(1-\phi_2)(\phi_2 - \phi^*)} . \tag{51}$$

We first argue that there is a unique value $\overline{\phi}_2$ such that the right-hand side of the previous equation is positive if $\phi_2 > \overline{\phi}_2$ and negative if $\phi_2 > \overline{\phi}_2$. To see this, recall V is decreasing. Hence, if the numerator is 0 for some value $\overline{\phi}_2$, it must be that $2V(\overline{\phi}_2) - h - \ell < 0$. If $\phi_2 > \overline{\phi}_2$, the numerator of the right-hand side of (51) is negative, while if $\phi_2 > \bar{\phi}_2$ the numerator is positive. Now, note that since $\pi'(\phi_2) \geq 0$ and $\phi_2 < 1$, we have that $\phi_2 \leq \overline{\phi}_2$. Also, if $\pi(\phi_1) = 1$, we have $\phi_1 \geq \bar{\phi}_2$ since $\pi'(\phi_1) \leq 0$. It then follows that there is at most one interval (ϕ_1, ϕ_2) where $\pi(\phi) \in (0,1)$ for all $\phi \in (\phi_1, \phi_2)$ and $\pi(\phi_1), \pi(\phi_2) \in \{0,1\}$. Furthermore, if such an interval exists, we have that $\phi_1 = \phi^*$ (and so $\pi(\phi_1) = 0$) and $\pi(\phi_2) = 1$.

We now prove that there must be some $\hat{\phi}^{\ddagger} \in (\phi^*, 1)$ such that $\pi(\phi) \in (0,1)$ for all $\phi \in (\phi^*, \hat{\phi}^{\ddagger})$. To show this, assume not; that is, assume that $\pi(\phi) = 1$ for all $\phi \in (\phi^*, 1)$. In this case, equation (50) holds for all $\phi \in (\phi^*, 1)$. The general solution to this equation is (35) for some $C_1, C_2 \in \mathbb{R}$ and where $\kappa > 0$ is defined in equation (36). Since V is bounded, it must be that $C_2 = 0$. The value-matching condition imposes that $V(\phi^*) = h - \ell$; that is,

$$V(\phi) = \left(\tfrac{1-\phi}{\phi} \Big/ \tfrac{1-\phi^*}{\phi^*}\right)^{\kappa} (h - \ell) .$$

This implies that

$$\lim_{\phi \searrow \phi^*} p(\phi) = h - \tfrac{\mu}{r} \kappa (h - \ell) < h .$$

This is a contradiction, since $p(\phi) \geq \ell/\phi$.

We let $\phi^{\ddagger} \in (\phi^*, 1)$ be the supremum value such that $\pi(\phi) \in (0,1)$ for all $\phi \in (\phi^*, \phi^{\ddagger})$.

Step 4. Equations for existence and uniqueness. From Step 3, we have that there is some

$C_1 \in \mathbb{R}_{++}$ such that, for all $\phi \in (\phi^{\ddagger}, 1)$, we have[31]

$$V(\phi; C_1) = C_1 \left(\tfrac{1-\phi}{\phi}\right)^{\kappa} . \tag{52}$$

For each C_1, we let $\hat{\phi}(C_1)$ be determined by setting $\pi(\hat{\phi}(C_1)) = 1$ in equation (47) replacing $V'(\phi)$ by $V'(\hat{\phi}(C_1); C_1)$. We then have

$$C_1 = \frac{r\left(\tfrac{1-\hat{\phi}(C_1)}{\hat{\phi}(C_1)}\right)^{-\kappa}(\hat{\phi}(C_1) - \phi^*)h}{\mu\kappa\hat{\phi}(C_1)} . \tag{53}$$

The derivative of the right-hand side of the previous expression with respect to $\hat{\phi}(C_1)$ is

$$\frac{r\left(\tfrac{1-\hat{\phi}(C_1)}{\hat{\phi}(C_1)}\right)^{-\kappa}(\phi^*(1-\phi^*) - \overbrace{(\phi^* - \kappa)(\hat{\phi}(C_1) - \phi^*)}^{(*)})h}{\mu\kappa(1-\hat{\phi}(C_1))\hat{\phi}(C_1)^2} .$$

Given that $\hat{\phi}(C_1) > \phi^*$ and $\kappa > 0$, the term $(*)$ is bounded above by $\phi^*(1-\phi^*)$. Then, for each $C_1 > 0$, $\hat{\phi}(C_1)$ is uniquely defined, and $\hat{\phi} : \mathbb{R}_{++} \to (\phi^*, 1)$ is a bijective function. Hence, from now on, abusing notation, we use $\hat{\phi} \in (\phi^*, 1)$ instead of C_1 as the free variable coming from the differential equation (50). We also use $V(\cdot; \hat{\phi})$ instead of $V(\cdot; C_1)$.

For any given $\hat{\phi}$, the pair of values of $V(\hat{\phi}; \hat{\phi})$ and $V'(\hat{\phi}; \hat{\phi})$ can be used as boundary conditions to obtain a unique solution to equation (49) on $(\phi^*, \hat{\phi}]$, which we denote $V(\phi; \hat{\phi})$ without risk of confusion.[32] From the previous expressions we have that

$$V(\hat{\phi}; \hat{\phi}) = \frac{(\hat{\phi} - \phi^*) r h}{\hat{\phi}\kappa\mu} . \tag{54}$$

The right hand side of equation (54) is increasing in $\hat{\phi}$ on $(\phi^*, 1)$, and 0 at $\hat{\phi} = \phi^*$. Additionally, the previous expressions imply

$$V'(\hat{\phi}; \hat{\phi}) = -\frac{(\hat{\phi} - \phi^*) r h}{(1-\hat{\phi})\hat{\phi}^2\mu} . \tag{55}$$

[31] Note that this coincides with expression (35) with $C_2 = 0$, satisfied by the continuation value on the upper belief regions of beliefs in the observable case.

[32] Note that, by the smooth pasting condition, $V(\cdot; \hat{\phi})$ is continuous and differentiable at $\hat{\phi}$. It satisfies equation (49) on $(\phi^*, \hat{\phi}]$ and equation (50) on $[\hat{\phi}, 1)$.

It is easy to see that this is decreasing in $\hat{\phi}$ for all $\hat{\phi}\in(\phi^*,1)$.

Hence, there is an equilibrium (which must be as specified in the statement of the proposition) for a given value $\phi^\ddagger\in(\phi^*,1)$ only if there is a solution of equation (49) on $(\phi^*,\phi^\ddagger)$, denoted $V(\cdot;\phi^\ddagger)$, with boundary conditions (54) and (55), and the lower boundary condition holds, that is, if $\lim_{\phi\searrow\phi^*}V(\phi;\phi^\ddagger)=h-\ell$.

Step 5. Change of variables. From now on, we assume an equilibrium with continuation value V exists, and we will establish the necessary and sufficient conditions that it satisfies, and we will finally establish the existence of a unique equilibrium in Step 6. We change variables defining, for each $\hat{\phi}\in(\phi^*,1)$,

$$W(y)\equiv\frac{2^{1/2}\sqrt{\mu/r}\,\hat{\phi}}{(\hat{\phi}-\phi^*)h}\Big(V\big(\tfrac{\hat{\phi}\phi^* y}{\hat{\phi}(y-1)+\phi^*}\big)-(h-\ell)\Big)$$

for all $y\in\big[1,+\infty\big)$. Note that W is negative and increasing. Note also that the limit $\phi\searrow\phi^*$ corresponds to the limit $y\to\infty$, while $\phi=\hat{\phi}$ corresponds to $y=1$. Using our definition, equation (49) takes the following simpler form:

$$W''(y)=y^2W(y)W'(y)^2\;. \tag{56}$$

Since $W(\cdot)$ is negative, it is also concave. The requirement that $\lim_{\phi\searrow\phi^*}V(\phi)=h-\ell$ corresponds to $\lim_{y\to\infty}W(y)=0$. We will now analyze solutions to equation (56).

We note that the boundary conditions (54) and (55) can be written as boundary conditions on W at $y=1$ as follows:

$$W(1;\hat{\phi})=2^{1/2}\sqrt{\mu/r}\big(\tfrac{r}{\kappa\mu}-\tfrac{\hat{\phi}(1-\phi^*)}{\hat{\phi}-\phi^*}\big)\;, \tag{57}$$

$$W'(1;\hat{\phi})=\frac{2^{1/2}}{\sqrt{\mu/r}}\frac{\hat{\phi}-\phi^*}{(1-\hat{\phi})\phi^*}\;. \tag{58}$$

The right-hand side of equation (57) is strictly negative if and only if $\hat{\phi}\in(\phi^*,\hat{\phi}^+)$, where

$$\hat{\phi}^+\equiv\min\big\{1,\phi^*\big(1-(1-\phi^*)\kappa\mu/r\big)^{-1}\big\}>\phi^*\;.$$

Also, in this range, $W(1;\hat{\phi})$ increases from $-\infty$ to either 0 (if $\hat{\phi}^+<1$) or $2^{1/2}\big(\kappa^{-1}-\sqrt{\mu/r}\big)$ (if $\hat{\phi}^+=1$). The right-hand side of equation (58) is strictly positive and strictly increasing in $\hat{\phi}$ on

$(\phi^*, 1)$, and increases from 0 to $+\infty$.

Step 6. Existence and uniqueness. We now aim to show that there is a unique $\hat{\phi} \in (\phi^*, 1)$ such that

$$\lim_{y \nearrow \infty} W(y; \hat{\phi}) = 0 ,$$

where $W(\cdot; \hat{\phi})$ is the solution to (56) with boundary conditions given by equations (57) and (58). We do it in two parts:

1. **Uniqueness:** Take two different values $\hat{\phi}_1$ and $\hat{\phi}_2$ satisfying $\phi^* < \hat{\phi}_1 < \hat{\phi}_2 < \hat{\phi}^+$. By the previous results, $W(1; \hat{\phi}_1) < W(1; \hat{\phi}_2) < 0$ and $0 < W'(1; \hat{\phi}_1) < W'(1; \hat{\phi}_2)$. We want to show that $W(y; \hat{\phi}_2) - W(y; \hat{\phi}_1)$ is increasing in y, and so $W(\cdot; \hat{\phi})$ tends to 0 for at most one value of $\hat{\phi} \in (\phi^*, \hat{\phi}^+)$. Assume, for the sake of contradiction, there is some value y' such that $W'(y'; \hat{\phi}_1) = W'(y'; \hat{\phi}_2)$, and let y be the infimum with this property. It then has to be that $W(y; \hat{\phi}_1) < W(y; \hat{\phi}_2)$. Then we have

$$W''(y; \hat{\phi}_1) = y^2 W(y; \hat{\phi}_1) W'(y; \hat{\phi}_1)^2 < y^2 W(y; \hat{\phi}_2) W'(y; \hat{\phi}_2)^2 = W''(y; \hat{\phi}_2) .$$

   This is a contradiction, since $W'(\cdot; \hat{\phi}_1) < W'(\cdot; \hat{\phi}_2)$ on $(1, y)$. Hence, we have that $W'(y; \hat{\phi}_1) < W'(y; \hat{\phi}_2)$ for all $y > 1$. Therefore, if $W(y; \hat{\phi}_1)$ and $W(y; \hat{\phi}_2)$ are convergent as $y \to \infty$, they converge to different values. A similar argument implies that two solutions of equation (56) cross at most once.

2. **Existence:** Note that a particular solution of (56) is $\hat{W}(y) \equiv -2^{1/2}/y$.[33] Note also that, as $\hat{\phi} \to \phi^*$,

$$W(1; \hat{\phi}) \to -\infty < \hat{W}(1) \ \text{ and } \ W'(1; \hat{\phi}) \to 0 < \hat{W}'(1) .$$

   Hence, if $\hat{\phi}$ is close enough to ϕ^*, $\lim_{y \to \infty} W(y; \hat{\phi})$ is strictly lower than $\lim_{y \to \infty} \hat{W}(y)$, which is equal to 0. We assume, for the sake of contradiction, that $\lim_{y \to \infty} W(y; \hat{\phi}^+) = w^+$, for some $w^+ < 0$. Since $W(1; \hat{\phi}^+) = 0$ if $\hat{\phi}^+ < 1$, then it must be that $\hat{\phi}^+ = 1$ and so $W'(1; \hat{\phi}^+) = +\infty$. Then, any solution of equation (56) with $W(1) < W(1; \hat{\phi}^+)$ is such that

[33] Such solution is an equilibrium when $\mu = r$, in which case $\hat{\phi}^\dagger = 2\phi^*/(1+\phi^*)$.

$\lim_{y\to\infty} W(y) < w^+$. We let $W(y;1)$ be defined as $\lim_{\hat{\phi}\to 1} W(y;\hat{\phi})$, which by assumption satisfies $\lim_{y\to\infty} W(y;1) < 0$. For each ε, let $\tilde{W}_\varepsilon(\cdot)$ be defined as the solution to equation (56) satisfying $\tilde{W}_\varepsilon(2) = W(2;1)$ and $\tilde{W}'_\varepsilon(2) = W'(2;1) + \varepsilon$. It is clear that if $\varepsilon > 0$ is chosen strictly larger than 0, then $\tilde{W}_\varepsilon(1) < W(1,1)$. Since solutions to (56) only cross once, we have $\lim_{y\to\infty} \tilde{W}_\varepsilon(y) > w^+$, but this is a contradiction. Hence, there exists a unique value $\hat{\phi}^\dagger$ such that $W(y;\hat{\phi}^\dagger) = 0$, and hence a unique equilibrium exists.

□

Proof of Proposition 4.3

Proof. Recall that, as we proved in the proof of Proposition 4.2, the equilibrium is divided into three regions. In the region $(0,\phi^*)$, $V = h - \ell$. In the region $(\phi^*, \phi^\ddagger)$, V follows equation (49) (with $\lim_{\phi\searrow\phi^*} V(\phi) = h - \ell$) and π satisfies equation (47). In the region $(\phi^*, \phi^\ddagger)$, V satisfies equation (35) with $C_2 = 0$, and $\pi(\phi) = 1$. Recall also that $\lim_{\phi\nearrow\phi^\ddagger} \pi(\phi) = 1$.

We first note that we can define

$$U(\phi) = \sqrt{\mu/r}\left(\tfrac{V(\phi)}{h} - 1 + \phi^*\right). \tag{59}$$

so that equation (49) becomes

$$U''(b) = -\frac{\phi - \phi^* + \phi^2 U(\phi) U'(\phi)}{(\phi - \phi^*)^2}\ ,$$

with the condition now that $U(\phi^*) = 0$. Note that U is negative and does not depend on μ. Note now that we can rewrite equation (47) as

$$\pi(\phi) = \frac{\phi - \phi^*}{\sqrt{\mu/r}(1-\phi)\phi^2(-U'(\phi))}\ .$$

Since π is increasing, $\pi(\phi^\ddagger) = 1$, and the right-hand side of the previous equation is decreasing in μ, it follows that $\phi^\ddagger$ is increasing in μ (recall Figure 3(b)).

We define

$$\hat{V}_\mu(\phi) = h - \ell + \frac{1}{\sqrt{\mu/r}} h U(\phi)\ . \tag{60}$$

(Note that $V(\phi) = \hat{V}_\mu(\phi)$ for $\phi \in (\phi^*, \phi^\ddagger]$. Note that $\hat{V}_\mu(\phi)$ is increasing in μ (since $U(\phi)$ is negative). Define also

$$\tilde{V}_\mu(\phi; C_\mu) = C_\mu \left(\tfrac{1-\phi}{\phi}\right)^{\kappa(\mu)} ,$$

where $\kappa(\mu)$ is the right-hand side of (36). (Note that $V(\phi) = \tilde{V}_\mu(\phi; C_\mu)$ when $\phi \in [\phi^\ddagger, 1)$, where C_μ is the value of C_1 in equation (35) for the unique equilibrium.) By letting $\phi^\ddagger_\mu$ be the value of $\phi^\ddagger$ for μ, note also that the smooth pasting condition implies $\hat{V}_\mu(\phi^\ddagger_\mu) = \tilde{V}_\mu(\phi^\ddagger_\mu; C_\mu)$ and $\hat{V}'_\mu(\phi^\ddagger_\mu) = \tilde{V}'_\mu(\phi^\ddagger_\mu; C_\mu)$.

Take $\mu_1, \mu_2 \in \mathbb{R}_{++}$ with $\mu_1 < \mu_2$. Using subindexes to denote the variables of the corresponding (unique) equilibria, the previous observations imply that $\phi^\ddagger_{\mu_1} < \phi^\ddagger_{\mu_2}$ and that $V_{\mu_1}(\phi) < V_{\mu_2}(\phi)$ for all $\phi \in (\phi^*, \phi^\ddagger_{\mu_1}]$. Assume, for the sake of contradiction, that there is some $\hat{\phi} \in (\phi^\ddagger_{\mu_1}, 1)$ such that $V_{\mu_1}(\phi) = V_{\mu_2}(\phi)$. There are two cases:

1. Consider first the case $\hat{\phi} \in (\phi^\ddagger_{\mu_1}, \phi^\ddagger_{\mu_2})$. We then have $\tilde{V}_{\mu_1}(\hat{\phi}, C_{\mu_1}) = \hat{V}_{\mu_2}(\hat{\phi})$ and $\tilde{V}'_{\mu_1}(\hat{\phi}, C_{\mu_1}) \geq \hat{V}'_{\mu_2}(\hat{\phi})$. Let $\hat{C}_{\mu_2,\mu_1}$ be the value such that $\tilde{V}_{\mu_2}(\hat{\phi}, \hat{C}_{\mu_2,\mu_1}) = \tilde{V}_{\mu_1}(\hat{\phi}, C_{\mu_1})$, that is,

   $$\hat{C}_{\mu_2,\mu_1} = C_\mu \left(\tfrac{1-\hat{\phi}}{\hat{\phi}}\right)^{\kappa(\mu_1)-\kappa(\mu_2)} .$$

   Simple algebra shows that

   $$\tilde{V}'_{\mu_2}(\hat{\phi}, \hat{C}_{\mu_2,\mu_1}) = \frac{\kappa(\mu_2)}{\kappa(\mu_1)} \tilde{V}'_{\mu_1}(\hat{\phi}, C_{\mu_1}) .$$

   Since $\kappa(\cdot)$ is a decreasing function, we have that $\tilde{V}'_{\mu_2}(\hat{\phi}, \hat{C}_{\mu_2,\mu_1})$ is smaller in absolute value than $\tilde{V}'_{\mu_1}(\hat{\phi}, C_{\mu_1})$, hence it is higher because both are negative. Therefore, using that $\tilde{V}'_{\mu_1}(\hat{\phi}, C_{\mu_1}) \geq \hat{V}'_{\mu_2}(\hat{\phi})$, we have $\tilde{V}'_{\mu_2}(\hat{\phi}, \hat{C}_{\mu_2,\mu_1}) > \hat{V}'_{\mu_2}(\hat{\phi})$. It then follows that there exists some $\hat{C}'_{\mu_2} < \hat{C}_{\mu_2,\mu_1}$ and $\hat{\phi}' < \hat{\phi}$ such that $\hat{V}_{\mu_2}(\hat{\phi}') = \tilde{V}_{\mu_2}(\hat{\phi}', \hat{C}'_{\mu_2})$ and $\hat{V}'_{\mu_2}(\hat{\phi}') = \tilde{V}'_{\mu_2}(\hat{\phi}', \hat{C}'_{\mu_2})$. Nevertheless, this implies that there exists an equilibrium for μ_2 with $\phi^\ddagger = \hat{\phi}' < \phi^\ddagger_{\mu_1}$, which contradicts the uniqueness of the equilibrium established in Proposition 4.2.

2. Consider now the case $\hat{\phi} \in [\phi^\ddagger_{\mu_2}, 1)$. In this case we have $\tilde{V}_{\mu_1}(\phi; C_{\mu_1}) = \tilde{V}'_{\mu_2}(\phi, C_{\mu_2})$ and $\tilde{V}'_{\mu_1}(\phi; C_{\mu_2}) \geq \tilde{V}'_{\mu_2}(\phi, C_{\mu_2})$. This implies that $C_{\mu_2} = \hat{C}_{\mu_2,\mu_1}$ defined above. As we argued, we have that $\tilde{V}_{\mu_1}(\phi; C_{\mu_1}) < \tilde{V}'_{\mu_2}(\phi, C_{\mu_2})$, which is a contradiction.

□

Proof of Proposition 4.4

Proof. When $\mu \geq \overline{\mu}$ the result is trivial, so we focus on the case $\mu < \overline{\mu}$. We define

$$\hat{V}^{\text{ob}}(\phi) = h - \ell + \tfrac{r(\phi h - \ell)}{\mu \phi} + \tfrac{r(h-\ell)}{\mu} \log\Big(\tfrac{1-\phi}{\phi} \Big/ \tfrac{1-\phi^*}{\phi^*}\Big)$$

for all $\phi \in [\phi^*, 1)$. Note that $\hat{V}^{\text{ob}}(\phi) = V^{\text{ob}}(\phi)$ for all $\phi \in [\phi^*, \phi^\dagger]$ (recall equation (32)), but $\hat{V}^{\text{ob}}(\phi) > V^{\text{ob}}(\phi)$ for all $\phi \in (\phi^\dagger, 1]$. As the proof of Propositions 3.1 and 3.2 argues, $\frac{\hat{V}^{\text{ob}}(\phi)}{-\hat{V}^{\text{ob}\prime}(\phi)} > \kappa^{-1}$ for all $\phi < \phi^\dagger$ and $\frac{\hat{V}^{\text{ob}}(\phi^\ddagger)}{-\hat{V}^{\text{ob}\prime}(\phi^\ddagger)} < \kappa^{-1}$ for all $\phi > \phi^\dagger$.

We also define $z^\dagger = \check{z}(\phi^\dagger)$ and $z^\ddagger = \check{z}(\phi^\ddagger)$ (recall the definition of $\check{z}$ in (21)). Recall that equation (45) holds for V^{ob}, and we also have

$$\frac{V^{\text{un}}(z)}{-V^{\text{un}\prime}(z)} \begin{cases} > \kappa^{-1} & \text{if } z \in (z^*, z^\ddagger), \\ = \kappa^{-1} & \text{if } z \in [z^\ddagger, +\infty). \end{cases} \tag{61}$$

Note that $\hat{V}^{\text{ob}\prime}(\phi^\ddagger) = V^{\text{un}\prime}(\phi^\ddagger)$ (because $\pi^{\text{un}}(\phi^\ddagger) = 1$, and so $\gamma^{\text{ob}}(\phi^\ddagger) = \gamma^{\text{un}}(\phi^\ddagger) = 1$ in equation (19)), and so $\hat{V}^{\text{ob}}(\phi^\ddagger) > V^{\text{un}}(\phi^\ddagger)$. As a result, $\frac{\hat{V}^{\text{ob}}(\phi^\ddagger)}{-\hat{V}^{\text{ob}\prime}(\phi^\ddagger)} > \kappa^{-1}$, hence it must be that $\phi^\ddagger < \phi^\dagger$.

The argument proceeds as in the proof of Proposition 4.1. Assume, for the sake of contradiction, that there is some $z > z^\ddagger$ such that $V^{\text{un}}(z) = V^{\text{ob}}(z)$. Since $V^{\text{un}}(z^\ddagger) < V^{\text{ob}}(z^\ddagger)$, there must then be some $\hat{z} > z^\ddagger$ such that $V^{\text{un}}(\hat{z}) = V^{\text{ob}}(\hat{z})$ and $-V^{\text{un}\prime}(\hat{z}) \leq -V^{\text{ob}\prime}(\hat{z})$. But then, this implies,

$$\frac{V^{\text{ob}}(\hat{z})}{-V^{\text{ob}\prime}(\hat{z})} \leq \frac{V^{\text{un}}(\hat{z})}{-V^{\text{un}\prime}(\hat{z})} = \kappa^{-1} ,$$

that is, $\frac{V^{\text{ob}}(\hat{z})}{-V^{\text{ob}\prime}(\hat{z})} = \kappa^{-1}$. This implies that $\hat{z} \geq z^\dagger$. Nevertheless, we then have that $V^{\text{un}}(z^\dagger) < V^{\text{ob}}(z^\dagger)$. Since V^{un} and $V^{\text{ob}}(z)$ follow the same equation for $z > z^\dagger$, this implies that $V^{\text{un}}(z) < V^{\text{ob}}(z)$ for all $z > z^\dagger$, which is a contradiction. This concludes the proof of the proposition. □

Proof of Proposition 4.5

Proof. We prove each part separately:

1. We first prove that, if $\mu \leq \overline{\mu}$, then $W^{\text{ob}}(\phi_0) > W^{\text{un}}(\phi_0) > W^{\text{no}}(\phi_0)$ for all $\phi_0 \in (\phi^*, 1)$. The last inequality is trivial for the reasons laid out in the main text after the proposition. The first inequality is obtained as follows. Note that, when $\mu \leq \overline{\mu}$, we have that $\alpha^x(\phi) = 1$ for all $x \in \{\text{un}, \text{ob}\}$. Hence, the term $(**)$ in equation (20) is equal to h for all x. As explained after

the proposition, the term $(*)$ in equation (20) is equal to $\frac{\ell}{h-\ell} V^x(\phi_0)$ for all $x \in \{\text{un}, \text{ob}\}$. Then, applying Proposition 4.4, the result follows.

2. We now prove that, if $\mu > \overline{\mu}$, then $W^{\text{un}}(\phi_0) > W^{\text{no}}(\phi_0)$ for all $\phi_0 \in (\phi^*, 1)$. This result holds trivially by the arguments after the proposition. □