Random vs. Directed Search for Scarce Resources

Teddy Mekonnen*

June 28, 2022

Abstract

This paper studies how ex-ante information affects consumer welfare in a search market where buyers search and match to sellers of a vertically differentiated product. In a random search market, a buyer gets no informative signal about the quality of a seller's product prior to matching, whereas in a directed search market, a buyer observes a perfectly informative signal. I derive the unique equilibrium outcome in each type of market and show that consumers are worse off in a directed search market when sellers are scarce and prices are bilaterally ex-post efficient.

JEL Classifications: C78, D62, D83

Keywords: random search, directed search, congestion, market segmentation

***Contact**: mekonnen@brown.edu. Department of Economics, Brown University.

I would like to express my deepest appreciation to Eddie Dekel, Federico Echenique, Bruno Strulovici, and Asher Wolinsky for their guidance and mentorship. This work has also greatly benefited from conversations with Alessandro Bonatti, Odilon Câmara, Hector Chade, Yeon-Koo Che, Pedro Dal Bó, Laura Doval, Jack Fanning, Kyungmin Kim, Stelios Michalopoulos, João Ramos, Jesse Shapiro, Nikhil Vellodi, and numerous seminar participants. I thank Ricardo Fonseca for excellent research assistance.

1. Introduction

Agents in search markets often make use of publicly available information to aid in their search process. For example, firms seeking productive employees can narrow down their search by recruiting only from universities reputable for producing skilled workers. With the proliferation of online review aggregators, it has become ubiquitous practice for diners to direct their search towards four-star or five-star restaurants. Health insurance enrollees can now consult websites like *Vitals* and *Healthgrades* to direct their search towards highly rated primary care physicians.

In all these examples, agents are using ex-ante information to direct their search towards desirable goods and services. When there is a single agent in the market, such access to ex-ante information is beneficial; it helps the agent find what he is searching for quickly. Hence, the value of information can never be negative.

However, markets often have many agents simultaneously searching while capacity constraints limit the resources sought out by the agents. In reality, there are many firms and a limited number of recruits from a given university, many diners and a limited number of tables at a restaurant, and many insurance enrollees and a limited number of in-network doctors. The strategic choices agents make in such settings often give rise to search externalities, and the types of externalities agents impose on the market may depend on the ex-ante information they observe while they search. It is therefore not immediately clear if the non-negative value of information in single-agent search environments can be extended to the case with many agents. The goal of this paper is to study how ex-ante information affects the welfare of agents in a search market plagued with scarcity.

To that end, I consider a parsimonious model of a consumer search market.[1] The market is sub-divided into a continuum of two-sided queues that match buyers to sellers on a first-come-first-serve (FCFS) basis. The FCFS matching mechanism is frictionless in that the maximal number of matches that can be formed in each period within a queue are indeed formed. By focusing on a frictionless matching mechanism, I abstract away from any search inefficiencies due to miscoordination and instead

[1] It is important to note that the model can be adapted to a labor search market (for example, with firms in the role of consumers and workers in the role of sellers) as well as search with non-transferable utility (for example, health insurance enrollees in the role of consumers, primary care physicians in the role of sellers, and all prices set to zero).

highlight the externalities that arise from information alone.[2]

In each period, a mass of sellers (she), each with a unit supply of a vertically differentiated product, and a mass of buyers (he), each with a unit demand and homogenous preferences over product quality, enter the market. Each buyer and seller in the market joins some queue, waits for a match, and may perish with positive probability while waiting. Once a buyer and a seller are matched, the buyer observes the true quality of the seller's product, and the pair either trades and exits the market or searches for a different match by re-queueing.

I consider two types of markets: In a random search market, each queue contains a random and uniform sampling of sellers from the market. Thus, each queue contains the same distribution of quality and the buyers have no ex-ante information. In contrast, in a directed search market, each queue contains only sellers of a unique product quality. In other words, each queue contains a single quality and the buyers have full ex-ante information.

The main result of the paper shows that consumer welfare is higher in a random search market than in a directed search market under two key assumptions: First, there is scarcity—the market has (weakly) fewer sellers than buyers. Second, buyers and sellers avail themselves to ex-post efficient bilateral contracts. This allows me to nest non-transferable utility settings in which buyers can "purchase" a product of any quality for a price of zero as well as transferable utility settings with different pricing mechanisms such as take-it-or-leave-it offers and bargaining.

In order to derive the main result, I first characterize the equilibrium of both a random and a directed search market. The equilibrium of both types of markets is, generically, unique and ex-ante inefficient. The inefficiency arises because prices derived from ex-post efficient bilateral contracts do not fully internalize search externalities unless the well-known Hosios condition (Hosios, 1990) holds, which is satisfied in my model only in the generic case when sellers extract the match surplus.[3]

In the equilibrium of a random search market, buyers join queues uniformly at random (hence the name). Within each queue, buyers cream-skim—a matched buyer-

[2]Inefficiencies due to coordination frictions have been extensively studied in the search and matching literature. See Rogerson et al. (2005) and references therein.

[3]For example, suppose prices are determined via Nash bargaining. With scarcity and a frictionless matching mechanism, the elasticity of the matching function (with respect to buyers) will be zero implying that the Hosios condition holds only if buyers have no bargaining power. In this case, the ex-ante value of joining any queue for the buyers is zero. We can then construct an equilibrium with ex-ante efficient queueing strategy, regardless of the ex-ante information available to buyers.

seller pair trade if and only if the seller's product quality is above a cutoff—giving rise to two negative externalities. First, the distribution of quality within a queue is worse than what it would have been without cream-skimming because sellers with product quality below the cutoff remain in the market for a long time. Consequently, buyers are relatively more likely to randomly match with sellers whose quality falls below the cutoff. Second, as noted by Romanyuk and Smolin (2019), cream-skimming increases competition since buyers who reject a match will compete with other buyers for future matches. Together, the two externalities imply that a buyer may wait a long time before he matches to a seller whose quality falls above the cutoff.

In the equilibrium of a directed search market, I show that there are two cutoffs that divide the product type space into low-, intermediate-, and high-quality regions. The queues containing low-quality products are empty on the buyers' side, the queues containing intermediate-quality products are uncongested and prices are low enough to attract buyers despite the quality offerings, and the queues containing high-quality products are congested with high prices. As such, the main externality that arises in a directed search market is congestion. Once again, a buyer waits a long time before he matches to a seller of a high-quality product.

Despite the difference in ex-ante information across random and directed search markets, buyers' search behavior shares a similar structure—a buyer stops searching and trades once he matches to a seller whose product is of a "good-enough" quality. Perhaps counterintuitively, consumer welfare decreases as the cutoff for a good-enough quality increases. On the one hand, a high cutoff implies that trade occurs only when the match surplus is large, and a buyer benefits from getting a share of this large surplus. On the other hand, a high cutoff also implies that a buyer faces a high expected wait time before he matches to a seller whose product is of good-enough quality (either because of the two negative externalities in a random search market or because of congestion in a directed search market). In a market with scarcity, the latter effect dominates.

Unsurprisingly, what the buyers consider to be a good-enough quality in equilibrium depends on their ex-ante information. The main result shows that a directed search market yields a lower consumer welfare precisely because the cutoff for a good-enough quality is higher in a directed search market. In a random search market, the cutoff depends on a buyer's *inter-temporal* trade-off between instantaneous trade with his current match or searching for better matches by re-queueing in the *next period*.

The inter-temporality reflects the difference between the buyer's ex-ante information when he only knows the distribution of quality within a queue and his ex-post information when he observes a product's true quality. In a directed search market, the cutoff depends on a buyer's *contemporaneous* trade-off between instantaneous trade by joining an uncongested queue or searching for better matches by joining a congested queue in the *current period*. The contemporality reflects the fact that the buyer's ex-ante and ex-post information coincide in a directed search market. As a contemporaneous trade-off does not involve the cost of waiting until the next period to *search* for better matches, buyers in a directed search market can afford to be pickier, which makes the negative externalities they impose on the search market more severe. Overall, the queues containing products of good-enough quality in a directed search market involve not only a longer wait time for buyers because of congestion, but also higher prices (because congestion implies higher levels of buyer-competition), consequently leading to a lower consumer welfare.

The qualitative results of the paper continue to hold for various extensions of the model. First, I consider an endogenous entry model in which only sellers who expect to trade with positive probability choose to enter the market in the first place. Second, I consider a class of search markets with monotone partitional information structures, which are more informative than a random search market but less informative than a directed search market. Finally, I consider different frictionless matching mechanisms such as service-in-random-order (SIRO) and last-come-first-serve (LCFS). In all three extensions, I show that consumer welfare is higher in a random search market.

Related Literature

This paper is closely related to Menzio (2007), who studies a labor search market in which firms with differing productivity levels advertise vacancies through cheap talk messages. If a (partially) separating equilibrium of the cheap talk game exists, workers can direct their job search based the firms' credible messages. Otherwise, the workers search randomly. Menzio shows that directed search could be welfare dominated by random search for some model parameters. However, the result depends not just on congestion externalities that the workers generate when they direct their search towards more productive firms, but also on search frictions from miscoordination and signaling inefficiencies needed to sustain a separating equilibrium in the first place. In this paper, I abstract away from both frictional search as well as any signaling inefficiencies, yet I obtain an unambiguous welfare comparison.

A few papers have highlighted the benefits of restricting ex-ante information when agents can use it to direct search. For example, Hagiu and Jullien (2011) consider consumers who direct their search based on the recommendations of an information intermediary and they show that it may be profit maximizing for the intermediary to divert search by providing noisy information. Vellodi (2018) shows that when consumers direct their search based on past reviews, platforms like *Yelp!* can incentivize the entry of new firms and delay the exit of incumbent firms by suppressing reviews. While these papers are related to mine, my results are not a consequence of search diversion or inadequate incentives for entry.

A number of papers have also highlighted the benefits of restricting ex-post information, i.e., information between a matched pair. While I assume that a buyer observes the true quality of a product once he matches to a seller, Lauermann (2012) shows that such a symmetric information setting between a matched buyer-seller pair could be welfare dominated by its asymmetric information analog when search costs are small. Similarly, Lester et al. (2019) show that small reductions in information asymmetry between matched buyer-seller pairs can be detrimental depending on the market's competitiveness and search frictions. Romanyuk and Smolin (2019) echo these results by showing that an information designer can improve welfare by obfuscating what agents observe within a match.

A large literature on competitive search shows that equilibrium outcomes are ex-ante efficient when sellers post prices and buyers direct their search based on *both* the posted prices and ex-ante informative signals. Some of the papers in this literature include search for exchange goods, (Butters, 1977; Peters, 1991; Kim and Kircher, 2015), labor markets (Montgomery, 1991; Moen, 1997; Mortensen and Wright, 2002), and markets with two-sided heterogeneity (Shi, 2002; Shimer, 2005; Eeckhout and Kircher, 2010). The directed search market I consider is distinct from this literature because prices are not posted ex-ante in my model; they are determined ex-post conditional on a match, which implies that buyers cannot direct their search based on a price-dimension. One such application is the Economics Job Market in which salaries are rarely posted on vacancy advertisements but are instead negotiated ex-post. Directed search based on a price-dimension would also be infeasible when sellers cannot commit to the prices they post. For example, while restaurants post their prices on online menus, they do not offer any guarantees that their online menus are up-to-date. This would allow a restaurant to advertise a different price than the one it charges to a diner, making the posted price meaningless. Finally, directed search based on prices will clearly not be feasible in applications of non-transferable utility, such as the market for health-insurance enrollees and PCPs.

The distinction of directed search in my model from the competitive search literature is also important because it underscores the theoretical limitations of directed search when it is based only on the information-dimension. In both the random and directed search markets of my model, ex-ante efficiency obtains only when sellers can extract the match surplus (for example, when the Hosios condition is satisfied and sellers have all the bargaining power), in which case consumer welfare is zero in either type of market. In contrast, if buyers are able to retain a portion of the match surplus, then the equilibrium is ex-ante inefficient for both a random and a directed search market, and consumer welfare is strictly lower in a directed search market. Hence, the model highlights that ex-ante informative signals in a market with scarcity could be either (a) useless to consumers when the market features an ex-ante efficient selling mechanism, or (b) detrimental to consumers when the market lacks an ex-ante efficient selling mechanism.

Finally, this paper is related in spirit to the insight that more information could be detrimental in exchange economies (Hirshleifer, 1971; Schlee, 2001).

The remainder of the paper is structured as follows: In Section 2, I describe a simple model of a FCFS matching with a continuum of agents. I use the insights from this simple model to describe a consumer search market in Section 3. I then derive the equilibrium outcomes of a random and a directed search market in Section 4 and Section 5. Section 6 compares consumer welfare across the two markets, and Section 7 briefly discusses various extensions. All proofs are in the Appendix.

2. First-come-first-serve with a continuum of agents

This section describes FCFS matching in a queue with a continuum of agents. The steady state characterization in this setting has a simple form, which will prove useful to study a consumer search market in the subsequent sections.

The queue is comprised of two sides, Side A and B. Time is discrete with an infinite horizon ($t = \ldots, -1, 0, 1, \ldots$) and the sequence of events within each period is as follows: First, a mass $q_i \geq 0$ of new agents join Side $i = A, B$ of the queue with

$(q_A, q_B) \neq 0$. Agents on opposite sides are then matched on a FCFS basis according to a procedure I will describe shortly. Matched agents leave the queue and get a payoff of $v_i \geq 0$ for $i = A, B$. Unmatched agents get a payoff of zero. Finally, an unmatched agent on either side perishes with probability $\gamma \in (0, 1]$ and survives on to the next period with the complementary probability. The survival probability $1 - \gamma$ also serves as a discount factor for the agents; there is no additional waiting cost or discounting.

The FCFS matching mechanism rations all agents on the short-side of the queue to agents on the long-side, with priority given to agents who have waited the longest in the queue. As the maximal number of matches that can be formed in each period are indeed formed, FCFS is a frictionless matching mechanism.

Formally, let $q_A \leq q_B$ so that Side A agents are on the short-side. Hence, a Side A agent is matched with probability one upon joining the queue. On the other hand, Side B agents, who are on the long-side, may have to wait until a match is rationed to them. Let P_k^t denote the probability that a Side B agent is matched in period t conditional on having already waited for $k = 0, 1, \ldots$ periods. The conditional matching probabilities $(P_k^t)_{t \in \mathbb{Z}, k \in \mathbb{N}}$ satisfy a FCFS procedure if for each t,

$$P_k^t > 0 \implies P_{k'}^t = 1 \text{ for all } k' > k.$$

In words, a Side B agent is matched with a positive probability in a given period only if all Side B agents who have waited longer are guaranteed a match in that same period. The procedure implicitly gives equal chances to any two agents who have waited for the same duration because the conditional matching probabilities are anonymous.

A steady state of a FCFS matching mechanism is given by stationary conditional matching probabilities that depend only on how long an agent has waited in the queue, i.e., $P_k^t = P_k$. Let

$$n = \inf\{k \in \mathbb{N} : P_k > 0\}$$

so that a Side B agent waits at least n periods in a steady state to get matched. By the FCFS procedure, an agent is matched with positive probability after waiting n periods only if agents who have waited even longer are guaranteed a match, i.e., $P_n > 0$ only if $P_k = 1$ for all $k > n$. In particular, $P_{n+1} = 1$, so a Side B agent waits at most $n + 1$ periods for a match. The steady state of a FCFS matching mechanism is therefore characterized by a pair $(n, \beta) \in \mathbb{N} \times (0, 1]$ such that for any $k \in \mathbb{N}$,

$$P_k = \begin{cases} 0 & \text{if} \quad k < n \\ \beta & \text{if} \quad k = n \\ 1 & \text{if} \quad k > n. \end{cases}$$

The values (n, β) are pinned down from the parameters of the model, which proves existence and uniqueness of a steady state for any $\gamma \in (0, 1]$ and $q_B \geq q_A \geq 0$ with $(q_A, q_B) \neq 0$.[4] To that end, consider the steady state balance equations: Each period, there is a mass q_A of Side A agents who leave the queue via a match. On the other hand, there is a mass $q_B(1-\gamma)^n\beta$ Side B agents who leave the queue via a match after waiting for n periods and a mass $q_B(1-\gamma)^{n+1}(1-\beta)$ of agents that leave via a match after waiting for $n+1$ periods. Thus, the balance equation must satisfy

$$q_A = q_B(1-\gamma)^n\Big(\beta + (1-\beta)(1-\gamma)\Big).$$

By rearranging, the probability of a match for a Side B agent who has waited n periods is given by

$$\beta = \frac{q_A - q_B(1-\gamma)^{n+1}}{q_B(1-\gamma)^n\gamma},$$

which is well-defined, i.e., $\beta \in (0, 1]$, only if $q_B(1-\gamma)^{n+1} < q_A \leq q_B(1-\gamma)^n$. This chain of inequalities is satisfied at a unique and finite $n \in \mathbb{N}$ given by

$$n = \left\lfloor \frac{\ln(q_A) - \ln(q_B)}{\ln(1-\gamma)} \right\rfloor.$$

When the market is balanced with $q_A = q_B$, neither side has to wait for a match; in this case, $n = 0$ and $\beta = 1$. The more unbalanced the market is (a decrease in q_A/q_B), Side A agents become more scarce so that Side B agents have to wait longer for a match to arrive. Similarly, when $\gamma = 1$, the queue empties out each period since any unmatched agent perishes. In this case, $n = 0$ and $\beta = q_A/q_B$, i.e., each Side B agent is matched with a positive probability immediately upon joining the queue.

[4] The unique steady state for $q_A \geq q_B \geq 0$ with $(q_A, q_B) \neq 0$ is characterized symmetrically. When $q_A = q_B = 0$, the queue is empty on both sides, so waiting times and matching probabilities are not well-defined. There can be no steady state when $\gamma = 0$.

However, as γ decreases, more Side B agents survive each period, which makes the wait time longer.

Since Side A agents are matched with probability one upon joining the queue, the ex-ante payoff for a Side A agent is given by $V_A = v_A$. The ex-ante payoff for a Side B agent is given by

$$V_B = (1-\gamma)^n \Big(\beta + (1-\beta)(1-\gamma)\Big) v_B.$$

From the steady state balance equation, we can simplify the ex-ante payoff for Side B agents to $V_B = (q_A/q_B)v_B$. More generally, the ex-ante payoff for a Side i agent with $i = A, B$ can be expressed as

$$V_i = \underbrace{\min\left\{\frac{q_{-i}}{q_i}, 1\right\}}_{\substack{\text{Ex-ante matching} \\ \text{probability}}} \times \underbrace{v_i}_{\substack{\text{Payoff conditional} \\ \text{on match}}}. \tag{1}$$

This formulation of the ex-ante matching probability and the ex-ante payoff will prove useful in the upcoming analysis.

3. Search

3.1. Setup

I consider a market in a steady state that is populated by a continuum of buyers (he) and sellers (she). Each buyer has a unit demand for a product and each seller has a unit supply. The product is vertically differentiated with $\theta \in \Theta = [0, 1]$ denoting the quality of a seller's product, which I refer to as the seller's type.

Each period, a unit mass of new buyers and a mass $k > 0$ of new sellers enter the market. New seller types are distributed according to a cumulative distribution function (CDF) F with a positive and bounded density function f. A seller knows her type but a buyer observes it only after matching to the seller.

The market features a continuum of two-sided queues indexed by $\omega \in \Omega = [0, 1]$. Buyers and sellers within each queue are matched according to the FCFS protocol in Section 2. Once a buyer and seller are matched, each one chooses either to trade or to reject the match. If at least one of them rejects, the pair either perishes with probability $\gamma \in (0, 1]$ or searches for another match with probability $1 - \gamma$ by joining a queue at the "back of the line" in the following period. I refer to the latter group as *re-queuers*. If the buyer and seller instead mutually agree to trade at some price $p \in \mathbb{R}$, the pair exits the market with the buyer getting a payoff of $\theta - p$ and the seller getting a payoff of p.[5] All agents get a payoff of zero otherwise.

The timing of events in each period is as follows: First, new buyers and sellers enter the market. The new as well as the re-queueing buyers and sellers form a cohort. Each buyer and seller in a cohort joins some queue and waits to be matched. A matched buyer-seller pair either trades and exits the market, or rejects the match. Finally, a fraction γ of the buyers and sellers still waiting to match as well as those who rejected a match perish. The remaining $1 - \gamma$ continue on to the next period.

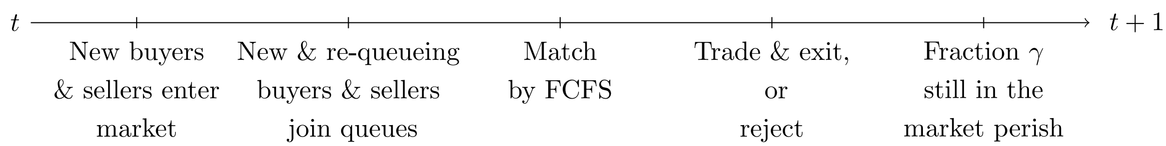

Figure 1: Sequence of events within each period.

Let $\psi \triangleq (M^B, M^S, G)$ represent the endogenous cohort composition in a steady state where M^B is the mass of new and re-queueing buyers, M^S is the mass of new and re-queueing sellers, and G is the type distribution of sellers within a cohort. A type $\theta \in \Theta$ seller in a cohort joins queues according to some exogenously given and commonly known CDF $\sigma(\cdot|\theta) : \Omega \to [0, 1]$. Let $\sigma(\omega) = \mathbb{E}_G[\sigma(\omega|\theta)]$, and let $G(\cdot|\omega) : \Theta \to [0, 1]$ be the type distribution in queue $\omega \in \Omega$, which is computed by Bayes rule whenever possible.

From the buyers' perspective, the queues serve a dual role: each queue facilitates matching and is also a source of ex-ante information about the seller types therein. Formally, G can be seen as a buyer's prior belief over seller types in the market, Ω as the signal space, and $\{\sigma(\cdot|\theta)\}_{\theta \in \Theta}$ as the information structure (or experiment). Under this interpretation, $G(\cdot|\omega)$ becomes the buyer's posterior belief conditional on observing signal realization $\omega \in \Omega$. By exogenously varying the seller-queueing CDF, we can consider various ex-ante information environments within the same consumer search market.

[5] The model can be enriched by considering common value payoffs $u(\theta) - p$ and $p - c(\theta)$ for the buyer and seller respectively, or by allowing for idiosyncratic taste shocks so that the surplus of a match is $\theta + \varepsilon$ for $\varepsilon \sim iid$. These formulations do not change the qualitative results.

Based on the ex-ante information, each buyer in a cohort strategically chooses which queues to join. Without loss of generality, I focus on symmetric and stationary strategies with each buyer joining queues according to some CDF $Q : \Omega \to [0, 1]$. I assume that

(i) the mappings $\theta \mapsto \sigma(\cdot|\theta)$ and $\omega \mapsto G(\cdot|\omega)$ are measurable, and

(ii) the buyer-queueing CDF Q is absolutely continuous with respect to σ, i.e., buyers do not join queues that contain no sellers.

Additionally, there are two substantive assumptions in this paper. First, I assume that fewer sellers than buyers enter the market in each period.

Assumption 1 (Scarcity) $k \leq 1$.

Prior to stating the second assumption, it is useful to introduce some notation. Let $u_B \in \mathbb{R}$ and $u_S : \Theta \to \mathbb{R}$ represent the buyers' and sellers' continuation values from re-queueing, and let $u \triangleq (u_B, u_S)$ denote the pair of buyer-seller continuation values. Conditional on matching to a type-θ seller, it is sequentially rational for a buyer to trade at a price p if and only if $\theta - p \geq (1-\gamma)u_B$. Similarly, conditional on matching to a buyer, it is sequentially rational for a type-θ seller to trade at a price p if and only if $p \geq (1-\gamma)u_S(\theta)$. Thus, a buyer and seller mutually agree to trade only if $\theta \geq (1-\gamma)\big(u_B + u_S(\theta)\big)$, i.e., the match is ex-post efficient. Let

$$\mathcal{E}(u) \triangleq \big\{\theta \in \Theta : \theta \geq (1-\gamma)\big(u_B + u_S(\theta)\big)\big\}$$

be the set of ex-post efficient types for a given pair of continuation values.

I assume that a matched buyer and seller avail themselves of a bilateral ex-post efficient contract.[6] Specifically, if a buyer matches with a seller of type $\theta \in \mathcal{E}(u)$, then the contract yields some price $p(\theta, u)$ that guarantees mutually agreeable trade, i.e.,

$$\theta - (1-\gamma)u_B \geq p(\theta, u) \geq (1-\gamma)u_S(\theta).$$

In other words, the price is assumed be some convex combination of $\theta - (1-\gamma)u_B$ and $(1-\gamma)u_S(\theta)$ when the match is ex-post efficient. Given that ex-post inefficient matches cannot lead to mutually agreeable trade for any price, I assume without loss

[6]See Section 8 for a brief discussion of ex-ante prices.

of generality that $p(\theta, u)$ continues to take the convex combination form for in-efficient matches.

Assumption 2 (Ex-post efficiency) *There exists a measurable "surplus splitting" function* $\lambda : \Theta \to (0, 1]$ *such that*

$$p(\theta, u) = \lambda(\theta)(1-\gamma)u_S(\theta) + \big(1-\lambda(\theta)\big)\big(\theta - (1-\gamma)u_B\big).$$

I refer to λ as the surplus splitting function as it represents the share of surplus captured by buyers. I take the surplus splitting function as exogenously given and fixed across random and directed search markets. Nonetheless, prices can still differ across the two markets based on the continuation values.

Assumption 2 nests several standard trading mechanisms. One foundation for $p(\theta, u)$ is as the outcome of a Rubinstein bargaining problem: a matched buyer and seller enter a "bargaining phase" in which the buyer has a discount factor of δ, the seller has a type-dependent discount factor of ρ_θ, and the pair make alternating offers on how to divide the surplus $\theta - (1-\gamma)(u_B + u_S(\theta))$. The unique solution to this bargaining phase is equivalent to a price $p(\theta, u)$ with $\lambda(\theta) = (1-\rho_\theta)/(1-\delta\rho_\theta)$. Additionally, $p(\theta, u)$ can also be rationalized as the solution to a Nash-bargaining problem

$$\max_{p\in\mathbb{R}} \big(\theta - p - (1-\gamma)u_B\big)^{\lambda(\theta)}\big(p - (1-\gamma)u_S(\theta)\big)^{1-\lambda(\theta)}$$

with $\lambda(\theta)$ capturing the buyer's bargaining power when matched to a type-θ seller. Finally, the formulation also incorporates buyers making take-it-or-leave-it offers with probability $\lambda(\theta)$ and sellers doing the same with the complementary probability. The case with $\lambda(\theta) = 1$ for all $\theta \in \Theta$ is equivalent to search with non-transferable utility because a buyer can trade with any seller he matches to at a price of zero.

Notice that buyers get none of the surplus from trade when $\lambda(\theta) = 0$ for all $\theta \in \Theta$, so consumer welfare is always zero regardless of the seller-queueing CDF. Thus, the problem is only interesting when $\lambda(\theta) > 0$ for a positive measure of types (with respect to F). I assume $\lambda(\theta) > 0$ for all $\theta \in \Theta$ only for expositional ease. Additionally, it is possible to generalize the surplus-splitting function to $\lambda(\theta, u)$ but an equilibrium may not exist without assuming λ is continuous and monotone in u. However, it is harder to motivate such a surplus-splitting function. For example, this would imply that in a Rubinstein bargaining model, the agents' discount rates in the alternating offers phase depend on their continuation values.

3.2. Steady state market composition

Let $\mu : \Omega \to [0,1]$ be a measurable function with $\mu(\omega)$ representing the ex-ante matching probability for a seller in queue $\omega \in \Omega$. Let $\pi : \Theta \to [0,1]$ be a measurable function with $\pi(\theta)$ representing the probability that a type-θ seller trades conditional on a match. The pair (μ, π) are determined as part of an equilibrium. However, for now, I take these functions as a given and characterize the associated steady state cohort composition.

The cohort composition $\psi \triangleq (M^B, M^S, G)$ is uniquely characterized by two sets of balance equations. First, the mass of sellers in a cohort must be made up of new and re-queueing sellers.[7] Hence, over any interval $[\theta', \theta''] \subseteq \Theta$, the balance equation for sellers satisfies

$$M^S \int_{\theta'}^{\theta''} dG(\theta) = \underbrace{k \int_{\theta'}^{\theta''} dF(\theta)}_{\text{New sellers}} + \underbrace{M^S \int_{\theta'}^{\theta''} \int_{\Omega} \mu(\omega)(1-\pi(\theta))(1-\gamma) d\sigma(\omega|\theta) dG(\theta)}_{\text{Re-queueing sellers}}. \tag{2}$$

Similarly, the mass of buyers in a cohort must be made up of new and re-queueing buyers. As a re-queueing buyer comes from a rejected buyer-seller pair, the mass of re-queueing buyers must equal the mass of re-queueing sellers. Hence, the balance equation for buyers satisfies

$$\begin{aligned} M^B = & \underbrace{1}_{\text{New buyers}} + \underbrace{M^S \int_{\Theta} \int_{\Omega} \mu(\omega)(1-\pi(\theta))(1-\gamma) d\sigma(\omega|\theta) dG(\theta)}_{\text{Re-queueing buyers}} \\ = & 1 + M^S - k, \end{aligned} \tag{3}$$

where the second equality follows from evaluating (2) at $\theta' = 0$ and $\theta'' = 1$ and substituting into (3). Given Assumption 1, we can conclude that $M^B \geq M^S$.

[7] A type-θ seller who joins queue $\omega \in \Omega$ has four possible outcomes: (i) perishes while waiting for a match with probability $1 - \mu(\omega)$, (ii) matches, trades, and exits with probability $\mu(\omega)\pi(\theta)$, (iii) matches, rejects, and perishes with probability $\mu(\omega)(1-\pi(\theta))\gamma$, or (iv) matches, rejects, and re-queues with probability $\mu(\omega)(1-\pi(\theta))(1-\gamma)$.

From (2),

$$k\big(F(\theta'') - F(\theta')\big) \leq M^S\big(G(\theta'') - G(\theta')\big) \leq \frac{k}{\gamma}\big(F(\theta'') - F(\theta')\big)$$

for any interval $[\theta', \theta''] \subseteq \Theta$. In particular, $k \leq M^S \leq k/\gamma$, and

$$\gamma\big(F(\theta'') - F(\theta')\big) \leq G(\theta'') - G(\theta') \leq \frac{F(\theta'') - F(\theta')}{\gamma}.$$

In other words, G is absolutely continuous with respect to F and vice versa. Hence, there exits a density function g which satisfies $\gamma f(\theta) \leq g(\theta) \leq f(\theta)/\gamma$ for all $\theta \in \Theta$. As f is positive and bounded, so is g. The following lemma summarizes these results.

Lemma 1 *Suppose Assumption 1 holds. Then for any pair (μ, π), the associated steady state cohort composition $\psi = (M^B, M^S, G)$ satisfies the following:*

1. $M^B \geq M^S$.

2. *G is absolutely continuous with a positive and bounded density function g.*

3.3. Equilibrium

Fix the seller-queueing CDF $\{\sigma(\cdot|\theta)\}_{\theta\in\Theta}$. Consider a tuple $\langle u, \psi, Q\rangle$ where $u \triangleq (u_B, u_S)$ is a pair of continuation values, $\psi \triangleq (M^B, M^S, G)$ is the cohort composition, and Q is the buyer-queueing CDF. Since Q is assumed to be absolutely continuous with respect to σ, let $q = dQ/d\sigma$ represent the Radon-Nikodym derivative. The ratio of buyers to sellers in queue $\omega \in \Omega$ is then given by $M^B q(\omega)/M^S$, often called the market tightness in the search literature.[8]

As shown in Section 2, a queue's steady state FCFS characterization depends only on its market tightness.[9] In particular, the ex-ante matching probability for a seller in queue $\omega \in \Omega$ is given by $\min\{M^B q(\omega)/M^S, 1\}$ while the ex-ante matching probability

[8]See Rogerson et al. (2005) and references therein.

[9]For example, if $M^B q(\omega) \geq M^S$, sellers are matched immediately and the steady state FCFS characterization for the buyers' side is given by (n_ω, β_ω) with

$$n_\omega = \left\lfloor \frac{\ln(M^S) - \ln(M^B q(\omega))}{\ln(1-\gamma)} \right\rfloor \quad \text{and} \quad \beta_\omega = \frac{M^S - M^B q(\omega)(1-\gamma)^{n_\omega+1}}{M^B q(\omega)(1-\gamma)^{n_\omega}\gamma}.$$

for a buyer is $\min\{M^S/(M^B q(\omega)), 1\}$. The payoff of a type-$\theta$ seller in queue ω is

$$V_S(\theta, \omega; u, \psi, Q) = \underbrace{\min\left\{\frac{M^B q(\omega)}{M^S}, 1\right\}}_{\substack{\text{Seller's ex-ante} \\ \text{matching probability}}} \underbrace{\max\{p(\theta, u), (1-\gamma)u_S(\theta)\}}_{\substack{\text{Seller's payoff} \\ \text{conditional on match}}}, \tag{4}$$

which is derived from (1) in Section 2. The ex-ante value of search is given by

$$V_S(\theta; u, \psi, Q) = \int_\Omega V_S(\theta, \omega; u, \psi, Q) d\sigma(\omega|\theta).$$

Similarly, a buyer's payoff in queue ω is given by

$$V_B(\omega; u, \psi, Q) = \underbrace{\min\left\{\frac{M^S}{M^B q(\omega)}, 1\right\}}_{\substack{\text{Buyer's ex-ante} \\ \text{matching probability}}} \underbrace{\int_\Theta \max\{\theta - p(\theta, u), (1-\gamma)u_B\} dG(\theta|\omega)}_{\substack{\text{Buyer's expected payoff} \\ \text{conditional on match}}} \tag{5}$$

and the ex-ante value of search is given by

$$V_B(u, \psi, Q) = \int_\Omega V_B(\omega; u, \psi, Q) dQ(\omega).$$

Definition 1 *An equilibrium of a steady state search market is given by a tuple $\langle u, \psi, Q\rangle$ such that*

(i) $V_S(\theta; u, \psi, Q) = u_S(\theta)$ *for all* $\theta \in \Theta$*,*

(ii) $V_B(u, \psi, Q) = u_B$*,*

(iii) $\text{supp}(Q) \subseteq \arg\max_{\omega\in\Omega} V_B(\omega; u, \psi, Q)$*, and*

(iv) ψ *is derived from* (2) *and* (3) *with* $\mu(\omega) = \min\left\{\frac{M^B q(\omega)}{M^S}, 1\right\}$ *and* $\pi(\theta) = \mathbb{1}_{\mathcal{E}(u)}(\theta)$.[10]

In words, a tuple $\langle u, \psi, Q\rangle$ constitutes an equilibrium of a steady state search market if (i) sellers have rational expectations such that their ex-ante value of search $V_S(\theta; u, \psi, Q)$ is consistent with their continuation value $u_S(\theta)$, (iii) buyers also have rational expectations such that their ex-ante value of search $V_B(u, \psi, Q)$ is consistent with their continuation value u_B, (iii) buyers only join queues that maximize their payoffs, and (iv) the steady state cohort composition is derived from the pair (μ, π)

[10] $\mathbb{1}_A(x)$ is the indicator function which equals 1 if $x \in A$ and 0 otherwise.

where μ is the sellers' ex-ante matching probability consistent with a FCFS procedure and π is the sequentially rational trading decision given ex-post efficient prices. Points (ii) and (iii) of Definition 1 can further be summarized as

$$V_B(\omega; u, \psi, Q) \leq u_B$$

for all $\omega \in \Omega$, with equality if $\omega \in \text{supp}(Q)$. This underscores the fact that buyers join multiple queues in equilibrium only if they are indifferent across them.

Notice that any buyer or seller can guarantee a payoff of zero by rejecting every match and eventually perishing. The best a buyer can hope for is to instantly match with the highest type and trade at price $p = 0$, and the best a type-θ seller can hope for is to instantly match with a buyer and trade at price $p = \theta$. Thus, in any equilibrium, $u_B \in [0, 1]$ and $u_S(\theta) \in [0, \theta]$ for all $\theta \in \Theta$.

Consider an ex-post inefficient type $\theta \notin \mathcal{E}(u)$, i.e., $\theta < (1-\gamma)\big(u_B + u_S(\theta)\big)$. Such an inefficient seller type never trades, and by construction, $p(\theta, u) \leq (1-\gamma)u_S(\theta)$. Since sellers have rational expectations in equilibrium, the continuation value for a seller of type $\theta \notin \mathcal{E}(u)$ must satisfy

$$\begin{aligned} u_S(\theta) &= V_S(\theta; u, \psi, Q) \\ &= \int_\Omega \min\left\{\frac{M^B q(\omega)}{M^S}, 1\right\} (1-\gamma)u_S(\theta) d\sigma(\omega|\theta) \\ &\leq (1-\gamma)u_S(\theta), \end{aligned}$$

which implies that $u_S(\theta) = 0$. Thus, $\theta < (1-\gamma)u_B$ is a necessary condition for $\theta \notin \mathcal{E}(u)$. It is also a sufficient condition: $\theta < (1-\gamma)u_B$ implies $\theta < (1-\gamma)(u_B + u_S(\theta))$ because $u_S(\theta) \geq 0$ for all $\theta \in \Theta$. Consequently, in any equilibrium $\langle u, \psi, Q\rangle$, there exists a cutoff type

$$\theta^E(u_B) \triangleq (1-\gamma)u_B$$

such that $\theta \in \mathcal{E}(u)$ if and only if $\theta \geq \theta^E(u_B)$.

4. Random Search

In a random search market, each seller type has a uniform probability of joining any queue. Formally, a random search market is characterized by a seller-queueing CDF

$\sigma(\omega|\theta) = \omega$ for all $\theta \in \Theta$, which implies that $G(\theta|\omega) = G(\theta)$ for all $\omega \in \Omega$. In other words, a queue's index is an uninformative signal about the seller types therein.

Since the type distribution within each queue is the same, a buyer's expected payoff conditional on a match in any queue is also the same. Thus, in equilibrium, each buyer joins queues uniformly at random.

Lemma 2 *Suppose Assumptions 1 and 2 hold. Then in any equilibrium $\langle u, \psi, Q\rangle$ of a random search market, the buyer-queueing CDF is the uniform distribution.*

From Lemma 2, the market tightness of any queue $\omega \in \Omega$ is $M^B q(\omega)/M^S = M^B/M^S$. As $M^B \geq M^S$ by Lemma 1, sellers are on the short-side of every queue while buyers are on the long-side. In any queue, a seller is matched to a buyer with probability $\min\{M^B/M^S, 1\} = 1$ while a buyer's ex-ante matching probability is $\min\{M^S/M^B, 1\} = M^S/M^B$. The sellers' payoff in (4) can be rewritten as

$$V_S(\theta, \omega; u, \psi, Q) = \begin{cases} (1-\gamma)u_S(\theta) & \text{if} \quad \theta < \theta^E(u_B) \\ p(\theta, u) & \text{if} \quad \theta \geq \theta^E(u_B) \end{cases}$$

and the buyers' payoff in (5) can be written as

$$V_B(\omega; u, \psi, Q) = \frac{M^S}{M^B}\left((1-\gamma)u_B + \int_{\theta^E(u_B)}^1 \theta - p(\theta, u) - (1-\gamma)u_B \, dG(\theta)\right).$$

Theorem 1 *Suppose Assumptions 1 and 2 hold. Then there exists a unique equilibrium $\langle u^r, \psi^r, Q^r\rangle$ in a random search market, and payoffs are given by*

$$u_S^r(\theta) = \begin{cases} 0 & \textit{if} \quad \theta < \theta^E(u_B^r) \\ \big(1-\hat{\lambda}(\theta)\big)\big(\theta - (1-\gamma)u_B^r\big) & \textit{if} \quad \theta \geq \theta^E(u_B^r) \end{cases} \tag{6}$$

and

$$u_B^r = \frac{k\displaystyle\int_{\theta^E(u_B^r)}^1 \hat{\lambda}(\theta)\theta dF(\theta)}{1 - k(1-\gamma)\displaystyle\int_{\theta^E(u_B^r)}^1 1 - \hat{\lambda}(\theta) dF(\theta)}, \tag{7}$$

where $\hat{\lambda} : \Theta \to (0, 1]$ is a re-weighted surplus splitting function given by

$$\hat{\lambda}(\theta) = \frac{\lambda(\theta)\gamma}{1 - \lambda(\theta) + \lambda(\theta)\gamma}.$$

While it was thus far necessary to keep track of the endogenous cohort composition, the equilibrium payoffs in (6) and (7) are entirely pinned down from the exogenous model parameters k, F, λ, and γ.

To gain some intuition for the equilibrium characterization, let us consider a setting in which the surplus-splitting rule is given by $\lambda(\theta) = 1$ for all $\theta \in \Theta$. As previously mentioned, this is equivalent to a non-transferable utility setting in which sellers get none of the surplus generated from trade and buyers are able to "trade" with any type θ seller at a price of $p(\theta, u) = 0$. In this case, the equilibrium payoff for buyers given in (7) simplifies to

$$u_B^r = k \int_{\theta^E(u_B^r)}^1 \theta dF(\theta)$$
$$\Leftrightarrow \theta^E(u_B^r) = (1-\gamma)k \int_{\theta^E(u_B^r)}^1 \theta dF(\theta). \tag{7'}$$

In the equilibrium of a random search market, buyers cream-skim—a buyer trades with a seller he is matched to if and only if the seller's type is above some cutoff. Cream-skimming arises from of the trade-off a buyer faces conditional on being matched to a seller. He can either *settle* for his current match by trading and exiting, or he can *search* for sellers with higher types by re-queueing. The trade-off between settling and searching is expressed in (7′): The left-hand-side is the lowest seller type the buyer is willing to settle for while the right-hand-side is the discounted value of searching for a better match by re-queueing in the following period. The marginal seller type that leaves the buyer indifferent between settling and searching is the ex-post efficiency cutoff type $\theta^E(u_B^r)$. The same intuition carries through for a general surplus-splitting rule albeit with a less tractable equilibrium payoff expression that accounts for non-trivial prices.

The buyers' cream-skimming behavior gives rise to two externalities. First, the competition for desirable matches is fiercer when buyers cream-skim. This externality arises because whenever a buyer rejects a match and re-queues in a market with scarcity, he increases the competition among the buyers for future matches. Second, the distribution of types within each queue worsens when the buyers cream-skim. In a market with scarcity, cream-skimming implies that every seller whose type falls above the cutoff exits the market (via trade) while a fraction of sellers whose types falls below the cutoff re-queue. Thus, among the sellers in a cohort, the seller types above the cutoff are comprised of only new sellers whereas the seller types below the cutoff are comprised of both new and re-queueing sellers. Consequently, the steady state distribution of types in a cohort is adversely shifted with G first-order stochastically dominated by F.

Since buyers lack ex-ante information in a random search market, they cannot direct their search towards sellers of any specific types. This implies that each buyer faces a positive probability of perishing before he matches to a seller whose type falls above $\theta^E(u_B^r)$. In fact, the two externalities make it even more likely that the buyer perishes before matching to a seller type above the cutoff.

In the following section, I consider a search market in which each seller type has a separate queue. This allows the buyers to direct their search only towards the seller types with whom they are willing to trade. Thus, buyers never re-queue in a directed search market. Additionally, since each queue is comprised of a single type of seller, the distribution of types within each queue is unaffected by the buyers' strategy. Hence, a directed search market alleviates the two negative externalities borne by a random search market.

5. Directed Search

In a directed search market, a type-θ seller joins queue ω if and only if $\theta = \omega$. Formally, a directed search market is characterized by a seller-queueing CDF $\sigma(\omega|\theta) = \mathbb{1}_{[\theta,1]}(\omega)$ for all $\theta \in \Theta$, which implies that $G(\theta|\omega) = \mathbb{1}_{[\omega,1]}(\theta)$ for all $\omega \in \Omega$. In other words, a queue's index is a fully informative signal about the seller types within the queue.

As contracts are bilaterally ex-post efficient, it remains sequentially rational for a matched buyer-seller pair to trade if and only if the match is ex-post efficient. Therefore, given a pair of continuation values $u \triangleq (u_B, u_S)$, a seller's probability of trade *conditional on a match* is $\pi(\theta) = \mathbb{1}_{[\theta^E(u_B),1]}(\theta)$. However, this does not imply that all ex-post efficient types trade because it could be ex-ante suboptimal for buyers to join the queue for some ex-post efficient types.

Given a tuple $\langle u, \psi, Q \rangle$, the payoff in (4) for a type-θ seller in queue $\omega = \theta$ can be rewritten as[11]

$$V_S(\theta,\theta;u,\psi,Q)=\begin{cases}\min\left\{\frac{M^B q(\theta)}{M^S},1\right\}(1-\gamma)u_S(\theta) & \text{if} \quad \theta<\theta^E(u_B)\\ \min\left\{\frac{M^B q(\theta)}{M^S},1\right\}p(\theta,u) & \text{if} \quad \theta\geq\theta^E(u_B),\end{cases} \tag{8}$$

and the payoff in (5) for a buyer who joins queue $\omega=\theta$ can be rewritten as

$$V_B(\theta;u,\psi,Q)=\begin{cases}\min\left\{\frac{M^S}{M^B q(\theta)},1\right\}(1-\gamma)u_B & \text{if} \quad \theta<\theta^E(u_B)\\ \min\left\{\frac{M^S}{M^B q(\theta)},1\right\}\big(\theta-p(\theta,u)\big) & \text{if} \quad \theta\geq\theta^E(u_B).\end{cases} \tag{9}$$

I start the equilibrium analysis by first considering the support of Q. The seller-queueing CDF in a directed search market satisfies $\sigma(\omega)=\mathbb{E}_G[\sigma(\omega|\theta)]=G(\omega)$. As G is absolutely continuous (Lemma 1), so is σ. Hence, the buyer-queueing CDF Q, which is assumed to be absolutely continuous with respect to σ, cannot have any atoms.

It is never optimal for a buyer to join the queue for a type-θ seller if it is also not sequentially rational to trade with that type conditional on a match; otherwise, the buyer could have done better by joining a different queue initially. Thus, in equilibrium, $\theta\in\operatorname{supp}(Q)$ only if $\theta\geq\theta^E(u_B)$.

Suppose $\theta\in\operatorname{supp}(Q)$ in equilibrium, which implies that $u_B=V_B(\theta;u,\psi,Q)$ by Definition 1. Using the expression in (9) for $V_B(\theta;u,\psi,Q)$,

$$\begin{aligned}u_B&=\min\left\{\frac{M^S}{M^B q(\theta)},1\right\}(\theta-p(\theta,u))\\&\leq\theta-p(\theta,u)\\&\leq\theta-\big(1-\lambda(\theta)\big)\big(\theta-(1-\gamma)u_B\big)\end{aligned}$$

where the last inequality follows from the definition of $p(\theta,u)$ in Assumption 2 and the fact that $u_S(\theta)\geq 0$ in equilibrium. Therefore, $\theta\in\operatorname{supp}(Q)$ in equilibrium only if

$$u_B\leq\frac{\theta\lambda(\theta)}{\lambda(\theta)(1-\gamma)+\gamma}.$$

Conversely, suppose $\theta\notin\operatorname{supp}(Q)$ in equilibrium so that $q(\theta)=0$, which implies

[11]There is no need to define $V_S(\theta,\omega;u,\psi,Q)$ for $\theta\neq\omega$ as such a type $\theta\notin\operatorname{supp}(\sigma(\cdot|\omega))$.

that a type-θ seller's ex-ante matching probability would be $\min\{M^B q(\theta)/M^S, 1\} = 0$. From (8) and Definition 1, $V_S(\theta, \theta; u.\psi, Q) = 0 = u_S(\theta)$. If a buyer were to deviate and join queue $\theta \notin \mathrm{supp}(Q)$, he would be immediately matched to a seller. However, such deviations cannot be profitable in an equilibrium. Formally, $u_B \geq V_B(\theta; u, \psi, Q)$ by Definition 1, which can be expressed as

$$\begin{aligned} u_B &\geq \underbrace{\min\left\{\frac{M^S}{M^B q(\theta)}, 1\right\}}_{=1} \max\{\theta - p(\theta, u), (1-\gamma)u_B\} \\ &\geq \theta - p(\theta, u) \\ &= \theta - \big(1 - \lambda(\theta)\big)\big(\theta - (1-\gamma)u_B\big) \end{aligned}$$

where the last equality follows from the definition of $p(\theta, u)$ in Assumption 2 and the fact that $u_S(\theta) = 0$ when the queue for such a type θ seller is empty on the buyer's side. Therefore, $\theta \notin \mathrm{supp}(Q)$ in equilibrium only if

$$u_B \geq \frac{\theta\lambda(\theta)}{\lambda(\theta)(1-\gamma) + \gamma}.$$

In other words, the support of Q is nested between two sets given by

$$\left\{\theta \in \Theta : u_B < \frac{\theta\lambda(\theta)}{\lambda(\theta)(1-\gamma) + \gamma}\right\} \subseteq \mathrm{supp}(Q) \subseteq \left\{\theta \in \Theta : u_B \leq \frac{\theta\lambda(\theta)}{\lambda(\theta)(1-\gamma) + \gamma}\right\}.$$

If λ is non-monotonic, the two sets nesting the support of Q could differ on a large subset of types, and could lead to non-existence of an equilibrium.[12] Therefore, for the remainder of the paper, I assume that λ is a weakly increasing function. I also assume that it is continuous for ease of exposition.

Assumption 3 (Regularity) $\lambda : \Theta \to (0, 1]$ *is continuous and weakly increasing.*

When Assumption 3 holds, the mapping $\theta \to \theta\lambda(\theta)/(\lambda(\theta)(1-\gamma) + \gamma)$ is strictly increasing and continuous. Thus, the two sets nesting the support of Q coincide on all but a σ-measure zero set of queues. In any equilibrium $\langle u, \psi, Q\rangle$, there exists a

[12]The correspondences $u_B \rightrightarrows \{\theta \in \Theta : u_B < \frac{\theta\lambda(\theta)}{\lambda(\theta)(1-\gamma)+\gamma}\}$ and $u_B \rightrightarrows \{\theta \in \Theta : u_B \leq \frac{\theta\lambda(\theta)}{\lambda(\theta)(1-\gamma)+\gamma}\}$ may be discontinuous, possibly leading to non-existence when solving for the fixed point of u_B.

cutoff type

$$\theta^{\dagger}(u_B) \triangleq \max\left\{\theta \in \Theta : \frac{\theta\lambda(\theta)}{\lambda(\theta)(1-\gamma)+\gamma} \leq u_B\right\}$$

such that $\theta \in \text{supp}(Q)$ if and only if $\theta \geq \theta^{\dagger}(u_B)$.

Each period, a mass M^B of buyers and a mass $M^S(1-G(\theta^{\dagger}(u_B)))$ of sellers join the queues for types above $\theta^{\dagger}(u_B)$. Yet, by Lemma 1, there are more buyers than sellers in a steady state. Therefore, buyers must be on the long-side of some (and possibly all) of the queues they join, i.e., $M^B q(\theta) \geq M^S$ for a positive measure of types $\theta \geq \theta^{\dagger}(u_B)$. I say that a queue is congested when $M^B q(\theta) > M^S$ and uncongested otherwise. The following lemma characterizes the set of queues that are congested in equilibrium.

Lemma 3 *Suppose Assumptions 1-3 hold. For any $u_B \in [0,1]$, define a cutoff*

$$\theta^{\dagger\dagger}(u_B) \triangleq \max\left\{\theta \in \Theta : \theta\lambda(\theta) \leq u_B\right\}.$$

Then in any equilibrium $\langle u, \psi, Q\rangle$ of a directed search market, the queue for type $\theta \in \Theta$ is congested if and only if $\theta > \theta^{\dagger\dagger}(u_B)$.

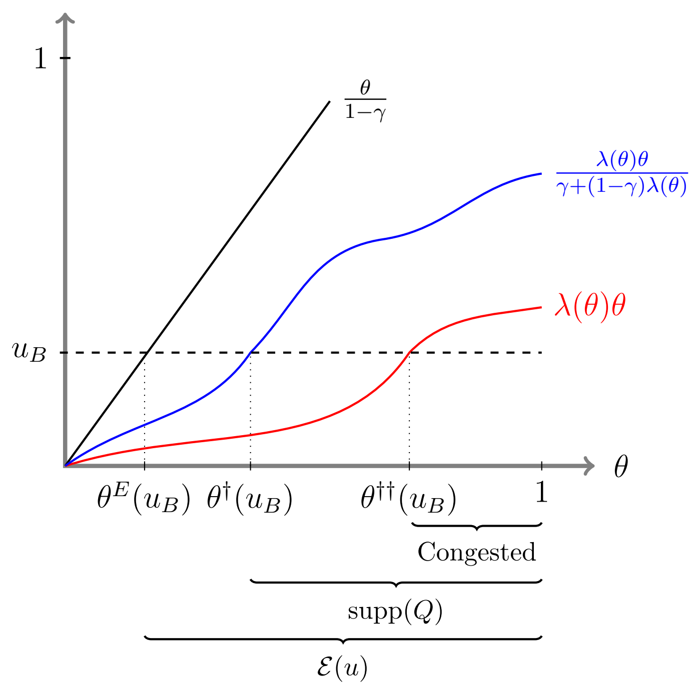

Figure 2

As illustrated in Figure 2, $\theta^E(u_B) \leq \theta^\dagger(u_B) \leq \theta^{\dagger\dagger}(u_B)$ for any $u_B \in [0,1]$, which implies that buyers in a directed search market may not trade with all ex-post efficient types and may be on the short-side of some queues. This is in contrast to a random search market in which buyers trade with all ex-post efficient types and are always on the long-side of every queue.

It may be surprising to have some uncongested non-empty queues in equilibrium. The overall market features scarcity, so it is natural to expect all queues in the support of Q to be congested. However, an uncongested queue for buyers means a congested queue for sellers. If queue $\theta \in \text{supp}(Q)$ is uncongested, then type-θ sellers have to compete against each other for matches. The fiercer competition among sellers reduces their continuation values in equilibrium, thereby also reducing the price that prevails in the queue. Hence, even if the quality offerings of these uncongested queues are lower, the prices are also sufficiently low enough to attract buyers.

Theorem 2 *Suppose Assumptions 1-3 hold. Then there exists a unique equilibrium* $\langle u^d, \psi^d, Q^d \rangle$ *in a directed search market. The equilibrium payoffs are given by*

$$u_S^d(\theta) = \begin{cases} 0 & \textit{if} \quad \theta < \theta^\dagger(u_B^d) \\ \dfrac{\theta\lambda(\theta) - u_B^d\left(\gamma + (1-\gamma)\lambda(\theta)\right)}{\lambda(\theta)(1-\gamma)} & \textit{if} \quad \theta^\dagger(u_B^d) < \theta \leq \theta^{\dagger\dagger}(u_B^d) \\ \left(1-\hat{\lambda}(\theta)\right)\left(\theta - (1-\gamma)u_B^d\right) & \textit{if} \quad \theta > \theta^{\dagger\dagger}(u_B^d) \end{cases} \tag{10}$$

and

$$u_B^d = \frac{k\displaystyle\int_{\theta^{\dagger\dagger}(u_B^d)}^1 \theta\hat{\lambda}(\theta)dF(\theta)}{1 - k(1-\gamma)\displaystyle\int_{\theta^{\dagger\dagger}(u_B^d)}^1 1-\hat{\lambda}(\theta)dF(\theta) - k\displaystyle\int_{\theta^\dagger(u_B^d)}^{\theta^{\dagger\dagger}(u_B^d)} \frac{\theta\lambda(\theta) - u_B^d\big(\gamma + \lambda(\theta)(1-\gamma)\big)}{\lambda(\theta)(1-\gamma)(\theta - u_B^d)}dF(\theta)}, \tag{11}$$

where $\hat{\lambda} : \Theta \to (0,1]$ *is the re-weighted surplus splitting function given by*

$$\hat{\lambda}(\theta) = \frac{\lambda(\theta)\gamma}{1-\lambda(\theta)+\lambda(\theta)\gamma}.$$

Once again, the equilibrium payoffs in (10) and (11) are entirely pinned down from the exogenous model parameters k, F, λ, and γ. To gain some intuition for the equilibrium characterization, let us reconsider the setting with $\lambda(\theta) = 1$ for all $\theta \in \Theta$. In this case, $\theta^{\dagger\dagger}(u_B) = \theta^\dagger(u_B)$ for all $u_B \in [0,1]$, i.e., almost every queue the buyers join is congested. This is because when utility is non-transferable, there is no mechanism by which fiercer competition amongst sellers leads to depressed prices. Hence, buyers are not attracted to the queues of low-type sellers even if these queues are uncongested.

Instead, each buyer uses his ex-ante information to join only the queues of sufficiently high-type sellers. The equilibrium payoff for buyers given in (11) simplifies to

$$
\begin{aligned}
u_B^d &= k \int_{\theta^{\dagger\dagger}(u_B^d)}^{1} \theta dF(\theta) \\
\Leftrightarrow \theta^{\dagger\dagger}(u_B^d) &= k \int_{\theta^{\dagger\dagger}(u_B^d)}^{1} \theta dF(\theta).
\end{aligned} \tag{11'}
$$

Similar to a random search market, the buyers in a directed search market face a trade-off between settling or searching: a buyer could *settle* by joining an uncongested queue and matching to a low-type seller without waiting, or he could *search* by joining a congested queue in the hopes of eventually matching to a high-type seller before he perishes. The trade-off between settling and searching is expressed in (11′): the left-hand-side is the highest seller type the buyer can match to without waiting while the right-hand-side is the buyer's expected value from entering one of the congested queues for high quality sellers. The marginal seller type that leaves buyers indifferent between settling and searching is the cutoff type $\theta^{\dagger\dagger}(u_B^d)$. The same intuition carries through for a general surplus-splitting rule albeit with a less tractable equilibrium payoff expression that accounts for the possibility of uncongested non-empty queues due to non-trivial prices.

As I already discussed at the end of Section 4, a directed search market, by providing full ex-ante information to buyers, eliminates the two negative externalities that arise in a random search market. However, a directed search market gives rise to a new externality—congestion. In particular, a directed search market features public information as all buyers share the same posterior beliefs conditional on observing a queue's index. Given the buyers' homogenous preferences, public information leads to too much coordination with all the buyers joining queues for types above $\theta^{\dagger\dagger}(u_B^d)$ and none below the cutoff. Effectively, this is another form of cream-skimming but the buyers cream-skim *across* queues in a directed search market as opposed to *within* queues like they do in a random search market. Hence, the effect of cream-skimming is not to adversely shift the distribution of types *within* a queue but to create varying levels of congestion *across* queues.

In the next section, I take a closer look at the differences between random and directed search markets in terms of the consumer welfare.

6. Consumer Welfare

In a steady state market with overlapping generations of long-lived buyers, the appropriate way to measure consumer welfare is by aggregating the equilibrium payoffs of the new buyers entering the market.[13] Since all buyers have the same preferences and there is a unit mass of new buyers in each period, consumer welfare is u_B^r in a random search market and u_B^d in a directed search market.

Theorem 3 *Suppose Assumptions 1-3 hold. Then consumer welfare is strictly higher in a random search market than a directed search market, i.e., $u_B^r > u_B^d$. Additionally, $\lim_{\gamma \to 0} u_B^r - u_B^d = 0$.*

Let us return to the setting in which utility is non-transferable, i.e., $\lambda(\theta) = 1$ for all $\theta \in \Theta$. Suppose buyers trade only with seller types above some arbitrary cutoff $x \in \Theta$. Each period, a unit mass of buyers enter the market demanding a product of quality $\theta \geq x$ while only a mass $k(1 - F(x))$ of new sellers enter the market supplying a product of the desired quality. Thus, in each period, there can only be a mass $k(1 - F(x))$ of buyers who trade, and conditional on trading, a buyer gets an expected value of $\mathbb{E}_F[\theta | \theta \geq x]$. Consumer welfare in this case can be expressed as

$$\begin{aligned} CW(x) &\triangleq k(1 - F(x)) \times \mathbb{E}_F[\theta | \theta \geq x] \\ &= k \int_x^1 \theta dF(\theta). \end{aligned}$$

As the cutoff x increases, there are two competing effects on welfare: the mass of buyers exiting via trade decreases but the expected payoff conditional on trading increases. In a market with scarcity, the former effect dominates. Hence, as x increases, $CW(x)$ decreases. As such, consumer welfare would be maximized when $x = 0$.

[13]See Olszewski and Wolinsky (2016) or Burdett and Menzio (2017) for a discussion.

However, the buyers' search behavior in equilibrium is not given by some exogenous cutoff $x \in \Theta$. Instead, the cutoff is endogenously determined based on a settle versus search trade-off, which depends on the buyers' ex-ante information. Therefore, a market that endogenously leads to lower cutoffs yields a higher consumer welfare.

Under this interpretation, consumer welfare is higher in a random search market than a directed search one because the endogenously determined random search cutoff $x = \theta^E(u_B^r)$ is lower than the directed search cutoff $x = \theta^{\dagger\dagger}(u_B^d)$. In a random search market, the cutoff is characterized by an inter-temporal trade-off given in (7′) between a buyer's current match surplus and his value from searching for better matches in the *next period*. In contrast, the cutoff in a directed search market is characterized by a contemporaneous trade-off given in (11′) between a guaranteed match surplus (by joining an uncongested queue) and his value of searching for better matches in the *current period* by entering a congested queue. As buyers facing the former trade-off have to discount their value of search for better matches by $1-\gamma$, they are less picky.

Figure 3 depicts the trade-offs in (7′) and (11′). The solid blue curve represents $CW(x)$ and the dashed red curve represents $(1-\gamma)CW(x)$ as the cutoff x varies between 0 and 1. As can be seen in the figure, the blue curve attains its maximum at $x = 0$. The cutoffs for a random and directed search market are the fixed points of the red and blue curves respectively. As the cutoff for a random search market is lower, the consumer welfare is higher.

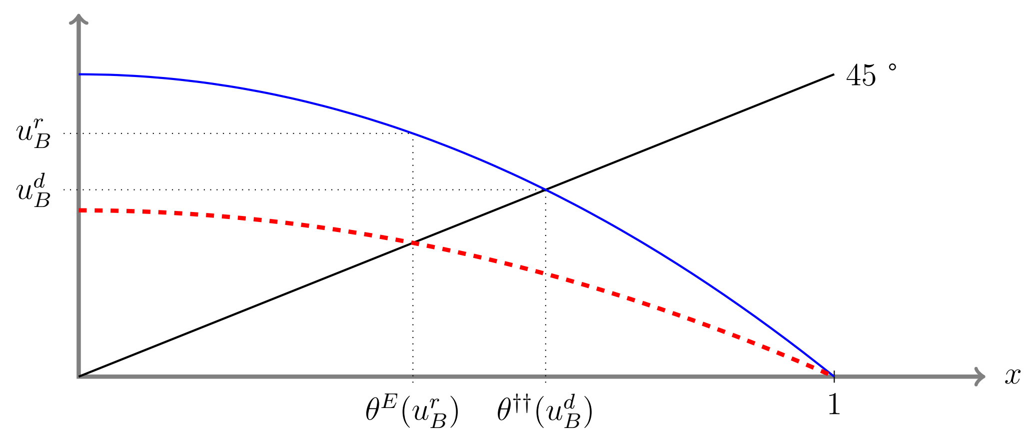

Figure 3: The solid blue curve is $CW(\cdot)$ and the dashed red curve is $(1-\gamma)CW(\cdot)$.

When utility is transferable, consumer welfare depends on the cutoffs as well as on the prices consumers pay. The higher level of congestion in a directed search market not only implies that buyers wait a long time to match with high-type sellers but also that the buyers pay a higher price when they trade (since higher congestion implies a fiercer competition amongst buyers), further lowering consumer welfare in a directed search market.

Finally, as the perishing probability γ goes to zero, any substantive difference in the buyers' strategies across the two types of markets disappears; even buyers in a random search market can effectively direct their search by continually re-queueing until they randomly match with sellers of their desired types. Thus, consumer welfare in random and directed markets converge to one another. In Figure 3, this can be seen by the dashed red curve converging to the blue line when $\gamma \to 0$, which implies a convergence in both the cutoff and the equilibrium payoff of a random search market to that of a directed search market.

7. Extensions

In this section, I show that the results of this paper hold for various extensions of the consumer search market.

7.1. Endogenous entry

In the equilibrium of both random and directed search markets, any seller whose type falls below some relevant cutoff never trades. If sellers face any entry cost into the market, such low-quality sellers would be absent from the steady state cohort. In this section, I consider an extension in which entry is endogenous—a seller enters the market if and only if she trades with a positive probability in equilibrium.[14]

Suppose a seller enters the market if and only if her type is $\theta \geq x$ for some cutoff $x \in (0,1)$. The original model is kept the same except in each period, a unit mass of new buyers and a mass $k_x \triangleq k(1-F(x))$ of new sellers with types distributed according to

$$F_x(\theta) \triangleq \begin{cases} 0 & \text{if } \theta < x \\ \frac{F(\theta)-F(x)}{1-F(x)} & \text{if } \theta \geq x \end{cases}$$

enter the market.

[14]This assumption can be micro-founded as follows: Let $\sup_{\theta\in\Theta} \lambda(\theta) < 1$ so that almost all types earn a positive payoff if they trade. Suppose a new seller can either enter the market by paying a fee $c > 0$ or can stay out and earn zero. In the limiting equilibrium as $c \to 0$, any type who does not trade prefers to stay out of the market, whereas any type who trades with a positive probability enters.

Given a cutoff x, let $\langle u^{r,x}, \psi^{r,x}, Q^{r,x}\rangle$ constitute the equilibrium of a random search market. Any seller of type $\theta > \theta^E(u_B^{r,x})$ has a positive probability of trading while any seller of type $\theta < \theta^E(u_B^{r,x})$ would never trade. Thus, the cutoff x would endogenously arise in a random search market if and only if $x = \theta^E(u_B^{r,x})$.

Similarly, given a cutoff x, let $\langle u^{d,x}, \psi^{d,x}, Q^{d,x}\rangle$ constitute the equilibrium of a directed search market. Any seller of type $\theta > \theta^\dagger(u_B^{d,x})$ has a positive probability of trading while any seller of type $\theta < \theta^\dagger(u_B^{d,x})$ would never trade. Thus, the cutoff x would endogenously arise in a directed search market if and only if $x = \theta^\dagger(u_B^{d,x})$.

Proposition 1 *Suppose Assumptions 1-3 hold, and entry is endogenous. Then*

(a) the cutoff for a random search market is $x = \theta^E(u_B^r)$ and $u^{r,x} = u^r$, and

(b) the cutoff for a directed search market is $x = \theta^\dagger(u_B^d)$ and $u^{d,x} = u^d$.

In other words, the payoffs in a market with and without endogenous entry are the same in both random and directed search markets. Consequently, the consumer welfare comparison of Theorem 3 continues to hold when entry is endogenous.

Corollary 1 *Suppose Assumptions 1-3 hold, and entry is endogenous. Then consumer welfare is strictly higher in a random search market than a directed search market.*

7.2. Monotone partitional Information

Thus far, I have considered two market types that represent extreme information structures: a random search market in which a queue's index is uninformative of the distribution of types therein, and a directed search market in which a queue's index is fully informative. In this section, I consider a partitional market in which a queue's index provides coarse information. I will restrict attention to the simpler case of non-transferable utility with $\lambda(\theta) = 1$ for all $\theta \in \Theta$.

For some integer $n > 1$, a monotone n-partitional search market is characterized by a sequence $\{x_k\}_{k=0}^n$ with $0 = x_0 < x_1 < \ldots < x_n = 1$ such that the seller-queueing CDF for $\theta \in [x_{k-1}, x_k]$ is given by

$$\sigma(\omega|\theta) = \begin{cases} 0 & \text{if} \quad \omega < x_{k-1} \\ \dfrac{\omega - x_{k-1}}{x_k - x_{k-1}} & \text{if} \quad \omega \in [x_{k-1}, x_k] \\ 1 & \text{if} \quad \omega > x_k \end{cases}.$$

Thus, the distribution of types in queue $\omega \in [x_{k-1}, x_k]$ is

$$G(\theta|\omega) = \begin{cases} 0 & \text{if} \quad \theta < x_{k-1} \\ \dfrac{G(\theta) - G(x_{k-1})}{G(x_k) - G(x_{k-1})} & \text{if} \quad \theta \in [x_{k-1}, x_k] \\ 1 & \text{if} \quad \theta > x_k \end{cases} .$$

In other words, buyers can be certain that each queue $\omega \in [x_{k-1}, x_k]$ only contains sellers whose type falls within $[x_{k-1}, x_k]$. Therefore, a monotone partitional search market is Blackwell more informative than a random search market but less informative than a directed search market. The main result of this section is that consumer welfare is (weakly) lower in *any* monotone partitional market than in a random search market.

Proposition 2 *Suppose Assumptions 1-3 hold, with $\lambda(\theta) = 1$ for all $\theta \in \Theta$. Then for any integer $n > 1$, consumer welfare is higher in a random search market than any monotone n-partitional search market.*

7.3. Other Frictionless Matching Mechanisms

The consumer search market in the main model is based on the FCFS matching mechanism described in Section 2. In this section, I consider other frictionless matching mechanisms such as service-in-random-order (SIRO) or last-come-first-serve (LCFS).

Let us revisit the simple model in Section 2 with $q_A \leq q_B$ so that Side A is the short-side of the queue. In any frictionless matching mechanism, a Side A agent is matched with probability one upon joining the queue, whereas a Side B agent is matched in period t conditional on having already waited for $k \geq 0$ periods with probability P_k^t. The conditional matching probabilities $(P_k^t)_{t \in \mathbb{Z}, k \in \mathbb{N}}$ depend on the matching mechanism under consideration.

A steady state of a frictionless matching mechanism (if it exists) is given by stationary conditional matching probabilities, i.e., $P_k^t = P_k$.[15] Each period, there is a mass q_A of Side A agents who leave the market via a match to a Side B agent.

[15]I show existence and uniqueness of a steady state for SIRO and LCFS in the Online Appendix.

Therefore, the stationary conditional probabilities $(P_k)_{k\in\mathbb{N}}$ must satisfy

$$q_A = q_B \left(P_0 + \sum_{k\geq 1} (1-\gamma)^k P_k \prod_{\ell=0}^{k-1} (1 - P_\ell) \right), \tag{12}$$

which generalizes the balance equation for FCFS in Section 2.

Since Side A agents are matched with probability one upon joining the queue, the ex-ante payoff for a Side A agent is given by $V_A = v_A$. The ex-ante payoff for a Side B agent is given by

$$V_B = \left(P_0 + \sum_{k\geq 1} (1-\gamma)^k P_k \prod_{\ell=0}^{k-1} (1 - P_\ell) \right) v_B,$$

which can be simplified to $V_B = (q_A/q_B)v_B$ using the balance equation in (12). Therefore, the ex-ante matching probability and the ex-ante payoff characterized in (1) for FCFS hold more generally for *any* frictionless matching mechanism.

Proposition 3 *For any frictionless matching mechanism in a steady state, the ex-ante matching probability for Side $i = A, B$ is given by*

$$\min\left\{ \frac{q_{-i}}{q_i}, 1 \right\}$$

and the ex-ante payoff is given by

$$V_i = \min\left\{ \frac{q_{-i}}{q_i}, 1 \right\} v_i.$$

From Proposition 3, we can conclude that none of the results in the consumer search market would change if we replace FCFS with a different frictionless matching mechanism, as long as the new mechanism converges to a steady state. However, it is important to note that the equivalence in ex-ante payoffs across the steady state of any frictionless matching mechanism will no longer hold if the agents face an additional cost of waiting or discounting beyond γ.

I conclude this section by showing that a steady state does indeed exist for SIRO and LCFS matching mechanisms. For SIRO, the conditional probabilities $(P_k^t)_{t\in\mathbb{Z},k\in\mathbb{N}}$ satisfy

$$P_k^t = P_{k'}^t, \text{ for all } k' \neq k \text{ and for all } t.$$

In words, SIRO gives equal chances of matching for any two Side B agents in the queue, regardless of how long they have waited. If a steady state exists for SIRO, the conditional matching probabilities would satisfy $P_k = P_{k'} = P$ for some $P \in (0, 1]$. The balance equation in (12) could then be simplified to

$$q_A = q_B P \sum_{k \geq 0} (1-\gamma)^k (1-P)^k,$$

which yields

$$P = \frac{q_A \gamma}{q_B - q_A(1-\gamma)}.$$

Notice that P is well-defined and uniquely pinned down from the parameters of the model.

For LCFS, the conditional probabilities $(P_k^t)_{t \in \mathbb{Z}, k \in \mathbb{N}}$ satisfy

$$P_k^t > 0 \implies P_{k'}^t = 1, \text{ for all } \; k' < k \text{ and for all } t.$$

In words, LCFS gives priority to Side B agents who have waited the least in the queue. If a steady state exists for LCFS, the conditional matching probabilities would satisfy $P_k > 0 \implies P_{k'} = 1$ for all $k' < k$. Suppose to the contrary that $P_n > 0$ for some $n \geq 1$. Then P_n is well-defined only if

$$q_A > q_B \left(P_0 + \sum_{k=1}^{n-1} (1-\gamma)^k P_k \prod_{\ell=0}^{k-1} (1 - P_\ell) \right).$$

In words, Side B agents who have waited for $n \geq 1$ periods can be matched with positive probability only if there are still Side A agents left over after matching every Side B agent that has waited less than n periods. However, in a LCFS matching mechanism, $P_n > 0 \implies P_k = 1$ for all $k = 0, \ldots, n-1$, which in turn would imply $q_A > q_B$; a contradiction. Thus, $P_n = 0$ for all $n \geq 1$. From the balance equation in (12), we have $q_A = q_B P_0$. Thus, the steady state of LCFS mechanism is given by

$$P_k = \begin{cases} \dfrac{q_A}{q_B} & \text{if} \quad k = 0 \\[2em] 0 & \text{if} \quad k \geq 1 \end{cases}$$

which is uniquely pinned down from the parameters of the model.

8. Conclusion

In this paper, I study the role of ex-ante information in a consumer search market for a vertically differentiated product. In the model, sellers are scarce, prices are ex-post efficient, and buyers are homogenous in both their preferences over quality and their access to information. The main result (Theorem 3) is that consumers are worse off when they observe an informative signal which allows them to direct their search towards sellers of high-quality products. The result is general in that it holds for all values of the model's parameters, such as distribution of quality and cost of searching.

I conclude by highlighting how the main result changes under a different set of assumptions. Using a continuity argument, the main result extends to the case when there are slightly more sellers than buyers. Intuitively, even if sellers are not scarce, the sellers of high-quality products would still be scarce when $k > 1$ but not too large. However, as k increases, the main result may hold only for a subset of parameter values. At the limit as $k \to \infty$, the main result completely breaks down because the market would approximate a single-buyer search problem in which an informative signal can never be detrimental. A possible comparative statics exercise would be to characterize how an increase in k changes the subset of parameter values for which Theorem 3 holds, although it is unclear if such an exercise would yield insightful results.

If ex-post efficient prices are replaced by ex-ante efficient prices, then ex-ante information would be neither detrimental nor beneficial to buyers in my model. For example, if sellers could post prices and buyers could direct their search based on the posted prices (as well as any ex-ante information available), then an equilibrium would involve each type-θ seller posting a price $p = \theta$. Buyers would be indifferent across all sellers as trade always leaves the buyers with no surplus. Hence, the buyers randomize across all sellers, and given scarcity, each seller would be guaranteed a match to a buyer, eliminating any competitive pressure on sellers Therefore, sellers would not need to compete for matches by posting lower prices. In fact, the posted-price equilibrium outcome is equivalent to the case when prices are determined ex-post but the Hosios condition is satisfied with $\lambda(\theta) = 0$ for almost all $\theta \in \Theta$, in which case consumer welfare is always zero regardless of the buyers' ex-ante information.

In this paper, congestion arises because a publicly informative signal creates too much coordination in the search behavior of buyers with homogenous preferences. This intuition can extend to the case when there is two-sided heterogeneity. For example, consider a horizontally differentiated product and buyers with heterogenous preferences. An informative signal could still lead to congestion if buyers preferences are not sufficiently diverse, which reduces consumer welfare. However, an informative signal could also improve consumer welfare by facilitating assortative matching. Thus, the overall effect of ex-ante information in a market with two-sided heterogeneity is unclear. Understanding the relationship between preference diversity, scarcity, and ex-ante information would be an interesting avenue for future work, which admittedly requires a richer model than the one in this paper.

Finally, it is possible to introduce heterogeneity in the buyers' access to ex-ante information by letting only a fraction of buyers observe an informative signal.[16] The model in this paper, with either none of the buyers (random search market) or all of the buyers (directed search market) observing an informative signal, can be seen as the limits of the enriched model. Therefore, a continuity argument to Theorem 3 would imply that consumer welfare is strictly higher when almost none of the buyers observe an informative signal than when almost all of the buyers observe an informative signal. However, a more general analysis away from the limits would allow for interesting interactions between informed and uninformed buyers. For example, such an analysis would need to account for *information leakage*—the informed buyers join queues differentially, which means that the wait times across queues differ. The uninformed buyers could then update their beliefs based on how long it takes to be matched. Such a thorough analysis would be an interesting and non-trivial extension of this paper.

9. Appendix

Proof of Lemma 2. Suppose the tuple $\langle u, \psi, Q \rangle$ constitutes an equilibrium. Since Q is absolutely continuous with respect to σ, which is the uniform distribution, Q does not have any mass points. Furthermore, since $\lambda(\theta) > 0$ for all $\theta \in \Theta$, buyers must capture some of the surplus from trade. Rational expectations would then entail that $u_B > 0$ in equilibrium.

[16] Lester (2011) considers such an environment in which a fraction of consumers observe an informative signal about a seller's posted price for a homogenous product.

In a random search market, $G(\theta|\omega) = G(\theta)$ for all $\omega \in \Omega$. Thus, the buyers' payoff $V_B(\omega; u, \psi, Q)$ given in (5) depends on ω only through the ex-ante matching probability. In equilibrium, the buyers must be indifferent across all the queues they join, which in this setting is equivalent to

$$\min\left\{\frac{M^S}{M^B q(\omega)}, 1\right\} = \min\left\{\frac{M^S}{M^B q(\omega')}, 1\right\}$$

for any $\omega, \omega' \in \operatorname{supp}(Q)$. Additionally, buyers join only queues that maximize their payoffs, i.e.,

$$\operatorname{supp}(Q) \subseteq \arg\max_{\omega \in \Omega} \min\left\{\frac{M^S}{M^B q(\omega)}, 1\right\}.$$

Suppose to the contrary that either the set $\{\omega \in \operatorname{supp}(Q) : M^S > M^B q(\omega)\}$ or the set $\Omega \setminus \operatorname{supp}(Q)$ has a positive measure (wrt σ). If the first set has positive measure, then indifference across the queues in $\operatorname{supp}(Q)$ would imply that the buyer's ex-ante matching probability in each $\omega \in \operatorname{supp}(Q)$ is $\min\{M^S/(M^B q(\omega)), 1\} = 1$.

If the second set has positive measure, then any $\hat{\omega} \notin \operatorname{supp}(Q)$ has $q(\hat{\omega}) = 0$ and $\min\{M^S/(M^B q(\hat{\omega})), 1\} = 1$, which would imply that $\hat{\omega}$ maximizes the ex-ante matching probability. Thus, for all $\omega \in \operatorname{supp}(Q)$, the ex-ante matching probability must also satisfy $\min\{M^S/(M^B q(\omega)), 1\} = 1$; otherwise, buyers could profitably deviate to queues off the support of Q.

In either case, we have $M^S \geq M^B q(\omega)$ for all $\omega \in \operatorname{supp}(Q)$. Integrating over $\operatorname{supp}(Q)$, we get

$$M^S \geq \int_{\operatorname{supp}(Q)} M^S d\sigma(\omega) \geq \int_{\operatorname{supp}(Q)} M^B q(\omega) d\sigma(\omega) = M^B,$$

with at least one of the inequalities strict since at least one of the two sets has positive measure. However, the conclusion $M^S > M^B$ contradicts Lemma 1.

In other words, in the equilibrium of a random search market, $M^S \leq M^B q(\omega)$ for almost all $\omega \in \Omega$. Furthermore, indifference across $\operatorname{supp}(Q)$ implies that the ex-ante matching probability must satisfy for almost all $\omega \in \Omega$,

$$\min\left\{\frac{M^S}{M^B q(\omega)}, 1\right\} = \frac{M^S}{M^B q}$$

for some $q > 0$. Finally, we must have $\int_{\mathrm{supp}(Q)} q(\omega)d\sigma(\omega) = q\int_{\Omega} d\sigma(\omega) = 1$. Thus, $q = 1$ and $Q(\omega) = \sigma(\omega) = \omega$, i.e., the buyer-queueing CDF is the uniform distribution. ■

Proof of Theorem 1. Let the tuple $\langle u, \psi, Q\rangle$ constitute an equilibrium. Then $V_S(\theta; u, \psi, Q) = u_S(\theta)$ for all $\theta \in \Theta$. Additionally, for a seller of type $\theta < \theta^E(u_B)$, $V_S(\theta, \omega; u, \psi, Q) = (1-\gamma)u_S(\theta)$ for all $\omega \in \Omega$, which implies $u_S(\theta) = 0$. After all, a seller of type $\theta < \theta^E(u_B)$ exits the market only by perishing as no buyer is willing to trade with such an ex-post inefficient type.

For a seller of type $\theta \geq \theta^E(u_B)$, $V_S(\theta, \omega; u, \psi, Q) = p(\theta, u)$ for all $\omega \in \Omega$, which implies

$$\begin{aligned} u_S(\theta) =& \lambda(\theta)(1-\gamma)u_S(\theta) + (1-\lambda(\theta))\big(\theta - (1-\gamma)u_B\big) \\ =& \underbrace{\frac{1-\lambda(\theta)}{1-\lambda(\theta)+\lambda(\theta)\gamma}}_{\triangleq 1-\hat{\lambda}(\theta)} \big(\theta - (1-\gamma)u_B\big), \end{aligned}$$

where the first equality is derived from the expression of $p(\theta, u)$ given in Assumption 2.

If a buyer matches to a seller of type $\theta \geq \theta^E(u_B)$, trade occurs and the buyer gets

$$\begin{aligned} \theta - p(\theta, u) =& \theta - \lambda(\theta)(1-\gamma)u_S(\theta) - (1-\lambda(\theta))\Big(\theta - (1-\gamma)u_B\Big) \\ =& (1-\gamma)u_B + \hat{\lambda}(\theta)\Big(\theta - (1-\gamma)u_B\Big). \end{aligned}$$

where the second equality follows from the expression of $u_S(\theta)$ given in (6). In equilibrium, $V_B(u, \psi, Q) = u_B$ and

$$V_B(\omega; u, \psi, Q) = \frac{M^S}{M^B}\left((1-\gamma)u_B + \int_{\theta^E(u_B)}^1 \hat{\lambda}(\theta)\big(\theta - (1-\gamma)u_B\big)dG(\theta)\right)$$

for all $\omega \in \Omega$. Hence, setting $u_B = \int_\Omega V_B(\omega; u, \psi, Q)dQ(\omega)$ and rearranging yields

$$u_B\left(M^B - M^S(1-\gamma)\left(1 - \int_{\theta^E(u_B)}^1 \hat{\lambda}(\theta)dG(\theta)\right)\right) = M^S\int_{\theta^E(u_B)}^1 \hat{\lambda}(\theta)\theta dG(\theta). \quad \text{(A1)}$$

From Definition 1 and the balance equations (2)-(3), the market composition satisfies $M^S g(\theta) = kf(\theta)$ for almost all $\theta \geq \theta^E(u_B)$, and

$$M^B = 1 + M^S(1-\gamma)G(\theta^E(u_B)).$$

Substituting these values into (A1) yields

$$u_B \left(1 - k(1-\gamma)\int_{\theta^E(u_B)}^1 1 - \hat{\lambda}(\theta)dF(\theta)\right) = k\int_{\theta^E(u_B)}^1 \hat{\lambda}(\theta)\theta dF(\theta),$$

which can be rearranged into (7).

The final step is to show that a unique solution exists for the fixed point problem in (7). Let $\mathcal{R} : [0,1] \to \mathbb{R}$ be a function given by

$$\mathcal{R}(u_B) = \frac{k\displaystyle\int_{\theta^E(u_B)}^1 \hat{\lambda}(\theta)\theta dF(\theta)}{1 - k(1-\gamma)\displaystyle\int_{\theta^E(u_B)}^1 1 - \hat{\lambda}(\theta)dF(\theta)}.$$

Note that $\mathcal{R}(\cdot)$ is continuous and decreasing, with $0 < \mathcal{R}(1) < \mathcal{R}(0) < 1$. Therefore, there exists a unique point $u_B^r \in (0,1)$ such that $\mathcal{R}(u_B^r) = u_B^r$. ■

Proof of Lemma 3. Let $\langle u, \psi, Q\rangle$ constitute an equilibrium of a directed search market. Since $\lambda(\theta) > 0$ for all $\theta \in \Theta$, buyers must capture some of the surplus from trade. Rational expectations would then entail that $u_B > 0$ in equilibrium, which implies $\theta^\dagger(u_B) > 0$ by construction.

For any $\theta \geq \theta^\dagger(u_B)$, we have $\theta \in \text{supp}(Q)$ and $\theta \geq \theta^E(u_B)$ by construction. In equilibrium, sellers must have rational expectations. From Definition 1 and (8),

$$u_S(\theta) = V_S(\theta; u, \psi, Q) = \underbrace{\min\left\{\frac{M^B q(\theta)}{M^S}, 1\right\}}_{\triangleq \mu(\theta)} p(\theta, u).$$

Using the definition of $p(\theta, u)$ in Assumption 2, a type θ seller's continuation payoff is given by

$$u_S(\theta) = \mu(\theta)\Big(\lambda(\theta)(1-\gamma)u_S(\theta) + \big(1-\lambda(\theta)\big)\big(\theta - (1-\gamma)u_B\big)\Big)$$

$$= \frac{\mu(\theta)\big(1-\lambda(\theta)\big)\big(\theta-(1-\gamma)u_B\big)}{1-\lambda(\theta)\mu(\theta)(1-\gamma)}. \tag{A2}$$

Additionally, a buyer's payoff conditional on a match in queue $\theta \geq \theta^\dagger(u_B)$ is given by

$$\theta - p(\theta, u) = \theta - \Big(\lambda(\theta)(1-\gamma)u_S(\theta) + \big(1-\lambda(\theta)\big)\big(\theta-(1-\gamma)u_B\big)\Big)$$

$$= \underbrace{\theta\lambda(\theta)\left(1 - \frac{\mu(\theta)(1-\lambda(\theta))(1-\gamma)}{1-\lambda(\theta)\mu(\theta)(1-\gamma)}\right) + u_B\frac{(1-\lambda(\theta))(1-\gamma)}{1-\lambda(\theta)\mu(\theta)(1-\gamma)}}_{>0}. \tag{A3}$$

In equilibrium, buyers must also have rational expectations. Thus, for any $\theta \geq \theta^\dagger(u_B)$,

$$u_B = V_B(\theta; u, \psi, Q) = \min\left\{\frac{M^S}{M^B q(\theta)}, 1\right\}(\theta - p(\theta, u))$$

from Definition 1 and (9). In other words, $u_B \leq \theta - p(\theta, u)$ for all $\theta \in \text{supp}(Q)$, with a strict inequality if only if $M^B q(\theta)/M^S > 1$.

Consider the queue for some type $\theta > \theta^{\dagger\dagger}(u_B) \geq \theta^\dagger(u_B)$. Then $\theta \in \text{supp}(Q)$ and $\theta\lambda(\theta) > u_B$ by Assumption 3. Hence, from (A3),

$$\theta - p(\theta, u) > u_B\left(1 + \frac{(1-\mu(\theta))(1-\lambda(\theta))(1-\gamma)}{1-\lambda(\theta)\mu(\theta)(1-\gamma)}\right) \geq u_B.$$

Since $\theta - p(\theta, u) > u_B$ only if $M^B q(\theta)/M^S > 1$, the queue for any type $\theta > \theta^{\dagger\dagger}(u_B)$ must be congested.

Next, consider the queue for some type $\theta \in [\theta^\dagger(u_B), \theta^{\dagger\dagger}(u_B)]$. Then $\theta \in \text{supp}(Q)$ and $\theta\lambda(\theta) \leq u_B$ by Assumption 3. Suppose, to the contrary that the queue is congested, i.e., $M^B q(\theta)/M^S > 1$. Then $\mu(\theta) = 1$. From (A3),

$$\theta - p(\theta, u) = \theta\lambda(\theta)\left(1 - \frac{(1-\lambda(\theta))(1-\gamma)}{1-\lambda(\theta)(1-\gamma)}\right) + u_B\frac{(1-\lambda(\theta))(1-\gamma)}{1-\lambda(\theta)(1-\gamma)} \leq u_B.$$

This however contradicts the fact that $\theta - p(\theta, u) > u_B$ if $M^B q(\theta)/M^S > 1$. Hence, the queue for any type $\theta \in [\theta^\dagger(u_B), \theta^{\dagger\dagger}(u_B)]$ must be uncongested.

Finally, consider the queue for some type $\theta < \theta^\dagger(u_B) \leq \theta^{\dagger\dagger}(u_B)$. Then $\theta \notin \text{supp}(Q)$ and $q(\theta) = 0$. This immediately implies the queue is uncongested. ■

Proof of Theorem 2. Let $\langle u, \psi, Q \rangle$ constitute an equilibrium of a directed search market. As already mentioned in the proof for Lemma 3, $u_B > 0$ in equilibrium, which implies $\theta^\dagger(u_B) > 0$. As $M^S \leq M^B$ by Lemma 1, and as buyers do not enter the queues for a positive measure of seller types $[0, \theta^\dagger(u_B))$, we have

$$\underbrace{M^B}_{\substack{\text{mass of} \\ \text{buyers joining} \\ \text{queues in} \\ \text{supp}(Q)}} \geq M^S > \underbrace{M^S \int_\Theta \int_{\theta^\dagger(u_B)}^1 d\sigma(\omega|\theta) dG(\theta)}_{\substack{\text{mass of} \\ \text{sellers joining} \\ \text{queues in} \\ \text{supp}(Q)}}.$$

Consequently, there must be a positive measure of types whose queues are congested, i.e., $\theta^{\dagger\dagger}(u_B) < 1$ in equilibrium, which in turn implies that $u_B < \lambda(1)$.

Consider the queue for a type $\theta \in (\theta^{\dagger\dagger}(u_B), 1]$. As the queue is congested with $M^B q(\theta)/M^S > 1$ by Lemma 3, the seller's payoff is given by evaluating (A2) with $\mu(\theta) = 1$ which yields

$$u_S(\theta) = \frac{1 - \lambda(\theta)}{\underbrace{1 - \lambda(\theta) + \lambda(\theta)\gamma}_{\triangleq 1 - \hat{\lambda}(\theta)}} \big(\theta - (1-\gamma)u_B\big).$$

The buyer's payoff satisfies

$$\begin{aligned} u_B &= V_B(\theta; u, \psi, Q) \\ &= \min\left\{\frac{M^S}{M^B q(\theta)}, 1\right\} (\theta - p(\theta, u)) \\ &= \frac{M^S}{M^B q(\theta)} \left(\theta\hat{\lambda}(\theta) + (1-\gamma)(1-\hat{\lambda}(\theta))u_B\right), \end{aligned}$$

where the last equality follows from evaluating (A3) with $\mu(\theta) = 1$. By rearranging the above expression and integrating over $(\theta^{\dagger\dagger}(u_B), 1]$ with respect to σ, we get[17]

$$u_B \int_{\theta^{\dagger\dagger}(u_B)}^1 \left(M^B q(\theta) - M^S(1-\gamma)(1-\hat{\lambda}(\theta))\right) \underbrace{d\sigma(\theta)}_{=dG(\theta)} = M^S \int_{\theta^{\dagger\dagger}(u_B)}^1 \theta\hat{\lambda}(\theta) \underbrace{d\sigma(\theta)}_{=dG(\theta)}. \quad \text{(A5)}$$

Next, consider the queues for types $\theta \leq \theta^{\dagger\dagger}(u_B)$. There are two cases to consider.

[17] $\sigma = G$ in a directed search market.

Case 1: If $\theta^{\dagger}(u_B) < \theta^{\dagger\dagger}(u_B)$, consider the queue for a type $\theta \in (\theta^{\dagger}(u_B), \theta^{\dagger\dagger}(u_B)]$. From Lemma 3, the queue is uncongested with $\mu(\theta) = M^B q(\theta)/M^S \leq 1$. The buyer's payoff satisfies

$$u_B = V_B(\theta; u, \psi, Q) = \underbrace{\left\{\frac{M^S}{M^B q(\theta)}, 1\right\}}_{=1}(\theta - p(\theta, u)).$$

Using (A3), the equality $\theta - p(\theta, u) = u_B$ can be expressed as

$$\theta\lambda(\theta) = u_B \left(\frac{\gamma + \lambda(\theta)(1-\gamma)\big(1-\mu(\theta)\big)}{1-\mu(\theta)(1-\gamma)}\right),$$

from which we can solve for $\mu(\theta)$ as

$$\mu(\theta) = \frac{\theta\lambda(\theta) - u_B\big(\gamma + \lambda(\theta)(1-\gamma)\big)}{\lambda(\theta)(1-\gamma)(\theta - u_B)}. \tag{A6}$$

Notice that (A6) is well-defined with $\mu(\theta) \in (0, 1]$ for all $\theta \in (\theta^{\dagger}(u_B), \theta^{\dagger\dagger}(u_B)]$. The seller's payoff can now be derived from (A2) and (A6) as

$$u_S(\theta) = \frac{\theta\lambda(\theta) - u_B^d\,(\gamma + (1-\gamma)\lambda(\theta))}{\lambda(\theta)(1-\gamma)}.$$

Furthermore, by equating $\mu(\theta) = M^B q(\theta)/M^S$ to the expression of $\mu(\theta)$ in (A6) and integrating over $(\theta^{\dagger}(u_B), \theta^{\dagger\dagger}(u_B)]$ with respect to σ, we have

$$M^B\left(Q(\theta^{\dagger\dagger}(u_B)) - Q(\theta^{\dagger}(u_B))\right) = M^S \int_{\theta^{\dagger}(u_B)}^{\theta^{\dagger\dagger}(u_B)} \frac{\theta\lambda(\theta) - u_B\big(\gamma + \lambda(\theta)(1-\gamma)\big)}{\lambda(\theta)(1-\gamma)(\theta - u_B)} dG(\theta), \tag{A7}$$

with $Q(\theta^{\dagger}(u_B)) = 0$ because Q is atomless and the buyers do not enter the queues for types that lie below $\theta^{\dagger}(u_B)$.

Case 2: If $\theta^{\dagger}(u_B) = \theta^{\dagger\dagger}(u_B)$, then $Q(\theta^{\dagger\dagger}(u_b)) = Q(\theta^{\dagger}(u_B)) = 0$. Thus, the equallity in (A7) still holds true.

Let us next consider the balance equations in a steady state. Recall that given an ex-ante matching probability μ and a trading probability π, a type-θ seller in queue $\omega = \theta$ re-queues with probability $\mu(\theta)(1 - \pi(\theta))(1 - \gamma)$. Sequential rationality implies that, conditional on a match, all ex-post efficient matches result in a trade, i.e., $\pi(\theta) = 1$ for all $\theta \geq \theta^E(u_B)$. Hence, the re-queueing probability for any type $\theta \geq \theta^\dagger(u_B) \geq \theta^E(u_B)$ is zero. Additionally, $\theta \notin \text{supp}(Q)$ for any type $\theta < \theta^\dagger(u_B)$, which implies that $\mu(\theta) = 0$ and the re-queueing probability for any such type is also zero.

The balance equations (2) and (3) imply that $M^S g(\theta) = kf(\theta)$ for almost all $\theta \in \Theta$ and $M^B = 1$. In other words, only new agents form cohorts as there is no re-queueing in equilibrium. This allows us to rewrite (A5) as

$$u_B \left(1 - Q(\theta^{\dagger\dagger}(u_B)) - k(1-\gamma)\int_{\theta^{\dagger\dagger}(u_B)}^{1} 1 - \hat{\lambda}(\theta)dF(\theta)\right) = k\int_{\theta^{\dagger\dagger}(u_B)}^{1} \theta\hat{\lambda}(\theta)dF(\theta)$$

and rewrite (A7) as

$$Q(\theta^{\dagger\dagger}(u_B)) = k\int_{\theta^{\dagger}(u_B)}^{\theta^{\dagger\dagger}(u_B)} \frac{\theta\lambda(\theta) - u_B\big(\gamma + \lambda(\theta)(1-\gamma)\big)}{\lambda(\theta)(1-\gamma)(\theta - u_B)}dF(\theta). \tag{A8}$$

We can combine these two expressions and rearrange to get (11).

The final step is to show that a unique solution exists for the fixed point problem in (11). Let $\mathcal{D} : [0,1] \to [0,1]$ be a function with $\mathcal{D}(u_B)$ given by

$$\frac{k\displaystyle\int_{\theta^{\dagger\dagger}(u_B)}^{1} \theta\hat{\lambda}(\theta)dF(\theta)}{1 - k(1-\gamma)\displaystyle\int_{\theta^{\dagger\dagger}(u_B)}^{1} 1 - \hat{\lambda}(\theta)dF(\theta) - k\int_{\theta^{\dagger}(u_B)}^{\theta^{\dagger\dagger}(u_B)} \frac{\theta\lambda(\theta) - u_B(\gamma + \lambda(\theta)(1-\gamma))}{\lambda(\theta)(1-\gamma)(\theta - u_B)}dF(\theta)}.$$

The function $\mathcal{D}(u_B)$ is continuous because the cutoffs $\theta^\dagger(u_B)$ and $\theta^{\dagger\dagger}(u_B)$ are continuous by Assumption 3. Additionally, $\mathcal{D}(0) = \mathcal{R}(0) > 0$ because $\theta^\dagger(0) = \theta^{\dagger\dagger}(0) = 0$, and $\mathcal{D}(u_B) = 0$ for $u_B \in [\lambda(1), 1]$ because $\theta^{\dagger\dagger}(u_B) = 1$. Therefore, there exists at least one point $u_B^d \in (0, \lambda(1))$ such that $\mathcal{D}(u_B^d) = u_B^d$.

Unfortunately, $\mathcal{D}(\cdot)$ may not be a monotone function, so proving uniqueness is not immediate. Assume that $\mathcal{D}(u_B)$ is differentiable on the relevant open interval $(0, \lambda(1))$.[18] The derivative is given by

$$\mathcal{D}'(u_B) = -\frac{k}{Z}f(\theta^{\dagger\dagger}(u_B))\frac{\partial\theta^{\dagger\dagger}(u_B)}{\partial u_B}\theta^{\dagger\dagger}(u_B)\hat{\lambda}(\theta^{\dagger\dagger}(u_B))$$

[18]Differentiability is not necessary for proving uniqueness but the argument is less cumbersome under differentiability.

$$- \mathcal{D}(u_B)\frac{k}{Z}\Bigg\{(1-\gamma)\Big(1-\hat{\lambda}\big(\theta^{\dagger\dagger}(u_B)\big)\Big)f(\theta^{\dagger\dagger}(u_B))\frac{\partial\theta^{\dagger\dagger}(u_B)}{\partial u_B}$$

$$- \underbrace{\frac{\theta^{\dagger\dagger}(u_B)\lambda(\theta^{\dagger\dagger}(u_B)) - u_B\big(\gamma+\lambda(\theta^{\dagger\dagger}(u_B))(1-\gamma)\big)}{\lambda(\theta^{\dagger\dagger}(u_B))(1-\gamma)(\theta^{\dagger\dagger}(u_B)-u_B)}}_{=1 \text{ by definition of } \theta^{\dagger\dagger}(u_B)} f(\theta^{\dagger\dagger}(u_B))\frac{\partial\theta^{\dagger\dagger}(u_B)}{\partial u_B}$$

$$+ \underbrace{\frac{\theta^{\dagger}(u_B)\lambda(\theta^{\dagger}(u_B)) - u_B\big(\gamma+\lambda(\theta^{\dagger}(u_B))(1-\gamma)\big)}{\lambda(\theta^{\dagger}(u_B))(1-\gamma)(\theta^{\dagger}(u_B)-u_B)}}_{=0 \text{ by definition of } \theta^{\dagger}(u_B)} f(\theta^{\dagger\dagger}(u_B))\frac{\partial\theta^{\dagger\dagger}(u_B)}{\partial u_B}$$

$$+ \int_{\theta^{\dagger}(u_B)}^{\theta^{\dagger\dagger}(u_B)} \frac{\theta\gamma(1-\lambda(\theta))}{(\theta-u_B)^2\lambda(\theta)(1-\gamma)}dF(\theta)\Bigg\}$$

$$= -\frac{k}{Z}f(\theta^{\dagger\dagger}(u_B))\frac{\partial\theta^{\dagger\dagger}(u_B)}{\partial u_B}\left(\theta^{\dagger\dagger}(u_B)\hat{\lambda}(\theta^{\dagger\dagger}(u_B)) - \mathcal{D}(u_B)\Big(1-(1-\gamma)(1-\hat{\lambda}(\theta^{\dagger\dagger}(u_B))\Big)\right)$$

$$- \mathcal{D}(u_B)\frac{k}{Z}\int_{\theta^{\dagger}(u_B)}^{\theta^{\dagger\dagger}(u_B)} \frac{\theta\gamma(1-\lambda(\theta))}{(\theta-u_B)^2\lambda(\theta)(1-\gamma)}dF(\theta),$$

where

$$Z = 1-k\int_{\theta^{\dagger}(u_B)}^{\theta^{\dagger\dagger}(u_B)} \frac{\theta\lambda(\theta)-u_B\big(\gamma+\lambda(\theta)(1-\gamma)\big)}{\lambda(\theta)(1-\gamma)(\theta-u_B)}dF(\theta) - k(1-\gamma)\int_{\theta^{\dagger\dagger}(u_B)}^{1} 1-\hat{\lambda}(\theta)dF(\theta).$$

Using the definition of $\hat{\lambda}$ and the fact that $\lambda(\theta^{\dagger\dagger}(u_B))\theta^{\dagger\dagger}(u_B) = u_B$ for $u_B \in (0, \lambda(1))$, we can simplify the derivative to

$$\mathcal{D}'(u_B) = -\frac{k}{Z}f(\theta^{\dagger\dagger}(u_B))\frac{\partial\theta^{\dagger\dagger}(u_B)}{\partial u_B}\left(\frac{\gamma}{1-\lambda(\theta^{\dagger\dagger}(u_B))+\lambda(\theta^{\dagger\dagger}(u_B))\gamma}\right)(u_B-\mathcal{D}(u_B))$$

$$- \mathcal{D}(u_B)\frac{k}{Z}\int_{\theta^{\dagger}(u_B)}^{\theta^{\dagger\dagger}(u_B)} \frac{\theta\gamma(1-\lambda(\theta))}{(\theta-u_B)^2\lambda(\theta)(1-\gamma)}dF(\theta).$$

By Assumption 3, $\theta^{\dagger\dagger}(\cdot)$ is strictly increasing on $(0, \lambda(1))$, so that $\partial\theta^{\dagger\dagger}(u_B)/\partial u_B > 0$. Hence, $\mathcal{D}'(u_B) \leq 0$ whenever $\mathcal{D}(u_B) \leq u_B < \lambda(1)$, which implies that $\mathcal{D}(u_B)$ can cross the 45° line only once from above. Thus, the fixed point u_B^d is unique. ■

Proof of Theorem 3. Let $\bar{\mathcal{D}} : [0,1] \to [0,1]$ be given by

$$\bar{\mathcal{D}}(u_B) = \frac{k \displaystyle\int_{\theta^{\dagger}(u_B)}^{1} \theta\hat{\lambda}(\theta)dF(\theta)}{1 - k(1-\gamma) \displaystyle\int_{\theta^{\dagger}(u_B)}^{1} 1 - \hat{\lambda}(\theta)dF(\theta)}.$$

Given Assumption 3, the cutoff $\theta^{\dagger}(u_B)$ is continuous and weakly increasing in u_B. Hence, the function $\bar{\mathcal{D}}(u_B)$ is continuous and weakly decreasing with $\bar{\mathcal{D}}(0) = \mathcal{R}(0) > 0$ and $\bar{\mathcal{D}}(1) = 0 < 1$. Therefore, there exists a unique point $\bar{u}_B^d \in (0,1)$ such that $\bar{\mathcal{D}}(\bar{u}_B^d) = \bar{u}_B^d$.

Consider a steady state equilibrium of a random search market $\langle u^r, \psi^r, Q^r, \rangle$. By Theorem 1, the buyer's equilibrium payoff is given by $u_B^r = \mathcal{R}(u_B^r)$. I aim to show that $u_B^r > \bar{u}_B^d$. To that end, notice that $\mathcal{R}(u_B) > \bar{\mathcal{D}}(u_B)$ for all $u_B \in (0,1)$ because $\theta^{\dagger}(u_B) > \theta^{E}(u_B)$ when the buyer's continuation value is positive. Assume to the contrary that $u_B^r \leq \bar{u}_B^d$. Since $\bar{\mathcal{D}}$ is a decreasing function, we would have $\bar{u}_B^d = \bar{\mathcal{D}}(\bar{u}_B^d) \leq \bar{\mathcal{D}}(u_B^r) < \mathcal{R}(u_B^r) = u_B^r$, which would yield a contradiction.

Next consider a steady state equilibrium of a directed search market $\langle u^d, \psi^d, Q^d, \rangle$. By Theorem 2, the buyer's equilibrium payoff is given by $u_B^d = \mathcal{D}(u_B^d)$. I aim to show that $u_B^d \leq \bar{u}_B^d$. To that end, from (A8),

$$\begin{aligned}
Q(\theta^{\dagger\dagger}(u_B)) &= k \int_{\theta^{\dagger}(u_B)}^{\theta^{\dagger\dagger}(u_B)} \frac{\theta\lambda(\theta) - u_B\big(\gamma + \lambda(\theta)(1-\gamma)\big)}{\lambda(\theta)(1-\gamma)(\theta - u_B)} dF(\theta) \\
&\leq k \int_{\theta^{\dagger}(u_B)}^{\theta^{\dagger\dagger}(u_B)} \frac{\theta\lambda(\theta) - u_B^d\big(\gamma + \lambda(\theta)(1-\gamma)\big) + \boldsymbol{u_B^d - \lambda(\theta)(1-\gamma)\theta}}{\lambda(\theta)(1-\gamma)(\theta - u_B^d) + \boldsymbol{u_B^d - \lambda(\theta)(1-\gamma)\theta}} dF(\theta) \\
&= k \int_{\theta^{\dagger}(u_B)}^{\theta^{\dagger\dagger}(u_B)} \frac{\theta\hat{\lambda}(\theta)}{u_B^d} + (1-\gamma)(1-\hat{\lambda}(\theta))dF(\theta),
\end{aligned}$$

where the inequality follows for the same reason that $a/b \leq (a + \mathbf{c})/(b + \mathbf{c})$ when a,

b, and c are constants with $a \leq b$ and $c \geq 0$. Thus,

$$\underbrace{\mathcal{D}(u_B^d)}_{=u_B^d} = \frac{k \displaystyle\int_{\theta^{\dagger\dagger}(u_B^d)}^{1} \theta\hat{\lambda}(\theta) dF(\theta)}{1 - k(1-\gamma) \displaystyle\int_{\theta^{\dagger\dagger}(u_B^d)}^{1} 1 - \hat{\lambda}(\theta) dF(\theta) - k \displaystyle\int_{\theta^{\dagger}(u_B^d)}^{\theta^{\dagger\dagger}(u_B^d)} \frac{\theta\lambda(\theta) - u_B^d\big(\gamma + \lambda(\theta)(1-\gamma)\big)}{\lambda(\theta)(1-\gamma)(\theta - u_B^d)} dF(\theta)}$$

$$\leq \frac{k \displaystyle\int_{\theta^{\dagger\dagger}(u_B^d)}^{1} \theta\hat{\lambda}(\theta) dF(\theta)}{1 - k(1-\gamma) \displaystyle\int_{\theta^{\dagger}(u_B^d)}^{1} 1 - \hat{\lambda}(\theta) dF(\theta) - k \displaystyle\int_{\theta^{\dagger}(u_B^d)}^{\theta^{\dagger\dagger}(u_B^d)} \frac{\theta\hat{\lambda}(\theta)}{u_B^d} dF(\theta)}$$

By rearranging, we get

$$u_B^d \leq \frac{k \displaystyle\int_{\theta^{\dagger}(u_B^d)}^{1} \theta\hat{\lambda}(\theta) dF(\theta)}{1 - k(1-\gamma) \displaystyle\int_{\theta^{\dagger}(u_B^d)}^{1} 1 - \hat{\lambda}(\theta) dF(\theta)} = \bar{\mathcal{D}}(u_B^d).$$

As $\bar{\mathcal{D}}(\cdot)$ is a decreasing function with a unique fixed point, we can conclude that $u_B^d \leq \bar{u}_B^d$. Therefore, $u_B^d < u_B^r$, which concludes the proof for the first statement.

To prove the second statement, let $\theta^* = \sup\{\theta \in \Theta : \lambda(\theta) < 1\}$. As λ is assumed to be monotone (Assumption 3), $\lambda(\theta) < 1$ for any $\theta < \theta^*$ and $\lambda(\theta) = 1$ for any $\theta > \theta^*$. For any $(\theta, \gamma) \in \Theta \times (0, 1]$, let

$$\hat{\lambda}(\theta, \gamma) \triangleq \frac{\lambda(\theta)\gamma}{(1 - \lambda(\theta) + \lambda(\theta)\gamma)},$$

and notice that $\hat{\lambda}(\theta, \gamma) < 1$ for any $\theta < \theta^*$ and $\hat{\lambda}(\theta, \gamma) = 1$ for any $\theta > \theta^*$.

For any $(u_B, \gamma) \in [0, 1] \times (0, 1]$, define

- $\theta^E(u_B, \gamma) \triangleq (1-\gamma)u_B$,
- $\theta^\dagger(u_B, \gamma) \triangleq \max\left\{\theta \in \Theta : \frac{\theta\lambda(\theta)}{\lambda(\theta)(1-\gamma)+\gamma} \leq u_B\right\}$, and
- $\theta^{\dagger\dagger}(u_B, \gamma) \triangleq \max\{\theta \in \Theta : \theta\lambda(\theta) \leq u_B\}$,

which are all continuous functions. Similarly define $\mathcal{R}(u_B, \gamma)$ and $\mathcal{D}(u_B, \gamma)$ based on the fixed-point operators in the proofs for Theorem 1 and Theorem 2. The two functions are continuous. Additionally, for any $\gamma \in (0,1]$, we have $0 = \mathcal{D}(1,\gamma) \leq \mathcal{R}(1,\gamma) < \mathcal{D}(0,\gamma) = \mathcal{R}(0,\gamma) < 1$, and the mappings $u_B \mapsto \mathcal{R}(u_B,\gamma)$ and $u_B \mapsto \mathcal{D}(u_B,\gamma)$ have a unique fixed point. Let $u_B^r(\gamma)$ and $u_B^d(\gamma)$ be the unique fixed points of $\mathcal{R}(\cdot,\gamma)$ and $\mathcal{D}(\cdot,\gamma)$ respectively. Since the fixed points are unique and each fixed-point operator is continuous, the mappings $\gamma \mapsto u_B^r(\gamma)$ and $\gamma \mapsto u_B^d(\gamma)$ are also continuous.

Note that $\lim_{\gamma\to 0} \hat{\lambda}(\theta,\gamma) = 0$ for all $\theta < \theta^*$ and $\lim_{\gamma\to 0} \hat{\lambda}(\theta,\gamma) = 1$ for all $\theta > \theta^*$. Additionally, for any $u_B \in [0,1]$, $\lim_{\gamma\to 0} \theta^E(u_B,\gamma) = \lim_{\gamma\to 0} \theta^\dagger(u_B,\gamma) = u_B$ while $\theta^{\dagger\dagger}(u_B,\gamma)$ is a constant function of γ. Finally,

$$\lim_{\gamma\to 0} \mathcal{R}(u_B,\gamma) = \lim_{\gamma\to 0} \mathcal{D}(u_B,\gamma) = \begin{cases} k\displaystyle\int_{u_B}^1 \theta dF(\theta) & \text{if} \quad \theta^* \leq u_B \\[2em] \dfrac{k\displaystyle\int_{\theta^*}^1 \theta dF(\theta)}{1 - k\left(F(\theta^*) - F(u_B)\right)} & \text{if} \quad \theta^* \geq u_B \end{cases}.$$

By continuity of the fixed-point operators and the fixed-points themselves, the fixed point of $\lim_{\gamma\to 0} \mathcal{R}(\cdot,\gamma)$ is $\lim_{\gamma\to 0} u_B^r(\gamma)$. Similarly, the fixed point of $\lim_{\gamma\to 0} \mathcal{D}(\cdot,\gamma)$ is $\lim_{\gamma\to 0} u_B^d(\gamma)$. Hence, $\lim_{\gamma\to 0} u_B^r(\gamma) = \lim_{\gamma\to 0} u_B^d(\gamma)$, which concludes the proof for the second statement. ∎

Proof of Proposition 1. Consider an arbitrary cutoff $x \in (0,1)$. The only change to the original model is that the parameters k and F are replaced by k_x and F_x. Let $\mathcal{R}^x : \mathbb{R} \to \mathbb{R}$ be given by

$$\mathcal{R}^x(u_B) = \frac{k_x \displaystyle\int_{\theta^E(u_B)}^1 \hat{\lambda}(\theta)\theta dF_x(\theta)}{1 - k_x(1-\gamma)\displaystyle\int_{\theta^E(u_B)}^1 1 - \hat{\lambda}(\theta) dF_x(\theta)}$$

$$= \frac{k \displaystyle\int_{\theta^E(u_B)\vee x}^1 \hat{\lambda}(\theta)\theta dF(\theta)}{1 - k(1-\gamma)\displaystyle\int_{\theta^E(u_B)\vee x}^1 1 - \hat{\lambda}(\theta) dF(\theta)},$$

where $\theta \vee x = \max\{\theta, x\}$. Notice that for any arbitrary $x \in (0,1)$, $\mathcal{R}^x(u_B)$ is continuous and weakly decreasing, with $\mathcal{R}^x(0) > 0$ and $\mathcal{R}^x(1) < 1$. Thus, it has a unique fixed point $u_B^{r,x}$. Additionally, $\mathcal{R}^x(u_B) \leq \mathcal{R}(u_B)$ for all $u_B \in [0,1]$ with a strict inequality if and only if $\theta^E(u_B) < x$. This implies that $u_B^{r,x} \leq u_B^r$ for all $x \in (0,1)$.

As the only change is a shift of parameter values, we can directly apply Theorem 1 to characterize the unique equilibrium $\langle u^{r,x}, \psi^{r,x}, Q^{r,x} \rangle$ of a random search market with entry cutoff x. In particular, the buyers' equilibrium payoff in a random search market with entry cutoff x is given by the fixed point $u_B^{r,x} = \mathcal{R}^x(u_B^{r,x})$. In equilibrium, a type-θ seller trades with a positive probability if and only if $\theta \geq \theta^E(u_B^{r,x})$. Therefore, the cutoff x is endogenous for a random search market if and only if $x = \theta^E(u_B^{r,x})$.

Case 1: Suppose $x = \theta^E(u_B^r)$, which implies $\mathcal{R}^x(u_B^r) = \mathcal{R}(u_B^r) = u_B^r$. Then $u_B^{r,x} = u_B^r$ and $\theta^E(u_B^{r,x}) = \theta^E(u_B^r) = x$. We can therefore conclude that $x = \theta^E(u_B^r)$ constitutes a cutoff for endogenous entry in a random search market.

Case 2: Suppose $x < \theta^E(u_B^r)$, which implies $\mathcal{R}^x(u_B^r) = \mathcal{R}(u_B^r) = u_B^r$. Then $u_B^{r,x} = u_B^r$. However, we now have $\theta^E(u_B^{r,x}) = \theta^E(u_B^r) > x$, so such an x does not constitute a cutoff for endogenous entry in a random search market.

Case 3: Suppose $x > \theta^E(u_B^r)$. Since $u_B^{r,x} \leq u_B^r$, we have $\theta^E(u_B^{r,x}) \leq \theta^E(u_B^r) < x$. However, such an x also does not constitute a cutoff for endogenous entry in a random search market.

Thus, $x = \theta^E(u_B^r)$ is the only cutoff that can arise in a random search market with endogenous entry. A similar argument establishes the result for directed search. ■

Proof of Proposition 2. Fix a monotone n-partitional search market. For each $\omega \in [x_{k-1}, x_k]$, the seller-queueing CDF satisfies

$$\sigma(\omega) = G(x_{k-1}) + \Big(G(x_k) - G(x_{k-1})\Big)\left(\frac{\omega - x_{k-1}}{x_k - x_{k-1}}\right),$$

which is absolutely continuous. Hence, Q, which is assumed to be absolutely continuous with respect to σ, does not have any atoms.

Let $\langle u, \psi, Q \rangle$ constitute an equilibrium. A similar argument to Lemma 2 establishes that Q is uniform over each subinterval. Specifically, there exists a non-negative sequence $\{q_k\}_{k=1}^{n}$ such that for each $k = 0, \ldots, n$ and each $\omega \in [x_{k-1}, x_k]$, $q(\omega) = q_k$, and

$$Q(\omega) = \int_0^{\omega} q(\omega) d\sigma(\omega) = \sum_{m<k} q_m \Big(G(x_m) - G(x_{m-1})\Big) + q_k \Big(G(x_k) - G(x_{k-1})\Big) \left(\frac{\omega - x_{k-1}}{x_k - x_{k-1}}\right).$$

Thus,

$$1 = \sum_{k=1}^{n} q_k \Big(G(x_k) - G(x_{k-1})\Big).$$

Let $k^* = \min\{k : q_k > 0\}$. Notice that $q_k > 0$ for all $k \geq k^*$; If $q_k = 0$ for some $k > k^*$, a buyer could profitably deviate by joining any queue $\omega \in [x_{k-1}, x_k]$. Additionally, $\theta^E(u_B) < x_{k^*}$; otherwise, the buyer would be better off not joining any queue $\omega \in [x_{k^*-1}, x_{k^*}]$.

Consider some $k > k^*$ (if any exist). Then $M^B q_k > M^S$; otherwise, buyers would never choose to join any $\omega \in [x_{k^*-1}, x_{k^*}]$ when they can join any $\omega \in [x_{k-1}, x_k]$ and match instantly with a higher type of seller. Thus, a buyer's value from joining some queue $\omega \in [x_{k-1}, x_k]$ is

$$\begin{aligned} u_B &= V_B(\omega; u, \psi, Q) \\ &= \frac{M^S}{M^B q_k} \left(\int_{\theta^E(u_B)}^{1} \theta dG(\theta|\omega) + (1-\gamma) u_B G(\theta^E(u_B)|\omega) \right) \\ &= \frac{M^S}{M^B q_k} \int_{\theta^E(u_B)}^{1} \theta dG(\theta|\omega), \end{aligned}$$

where the last equality follows from the fact that $\theta > \theta^E(u_B)$ for all $\theta \in \operatorname{supp}(G(\cdot|\omega))$.

Next, consider k^*. A buyer's value from joining some queue $\omega \in [x_{k^*-1}, x_{k^*}]$ is

$$\begin{aligned} u_B &= V_B(\omega; u, \psi, Q) \\ &= \min\left\{\frac{M^S}{M^B q_{k^*}}, 1\right\} \left(\int_{\theta^E(u_B)}^{1} \theta dG(\theta|\omega) + (1-\gamma) u_B G(\theta^E(u_B)|\omega) \right) \\ &\leq \frac{M^S}{M^B q_{k^*}} \left(\int_{\theta^E(u_B)}^{1} \theta dG(\theta|\omega) + (1-\gamma) u_B G(\theta^E(u_B)|\omega) \right). \end{aligned}$$

Integrating over Ω with respect to Q, we have

$$\begin{aligned}
u_B &= \int_\Omega V_B(\omega; u, \psi, Q) dQ(\omega) \\
&\leq \int_{x_{k^*-1}}^{x_{k^*}} \frac{M^S}{M^B q_{k^*}} \left(\int_{\theta^E(u_B)}^1 \theta dG(\theta|\omega) + (1-\gamma) u_B G(\theta^E(u_B)|\omega) \right) \\
&\quad + \sum_{k>k^*} \int_{x_{k-1}}^{x_k} \frac{M^S}{M^B q_k} \int_{\theta^E(u_B)}^1 \theta dG(\theta|\omega) dQ(\omega) \\
&= \frac{M^S}{M^B} \left(\int_{\theta^E(u_B) \vee x_{k^*-1}}^1 \theta dG(\theta) + (1-\gamma) u_B \Big[G\big(\theta^E(u_B) \vee x_{k^*-1}\big) - G(x_{k^*-1}) \Big] \right) \\
&\quad + \sum_{k>k^*} \frac{M^S}{M^B} \int_{x_{k-1}}^{x_k} \theta dG(\theta).
\end{aligned}$$

Since all seller types $\theta \geq \theta^E(u_B) \vee x_{k^*-1}$ leave the market via trade, they never re-queue. Hence, from the steady state balance equation (2), $M^S g(\theta) = kf(\theta)$ for all $\theta \geq \theta^E(u_B) \vee x_{k^*-1}$. We can therefore rewrite the above inequality as

$$u_B \leq \frac{k}{M^B} \int_{\theta^E(u_B) \vee x_{k^*-1}}^1 \theta dF(\theta) + \frac{M^S}{M^B}(1-\gamma) u_B \Big[G\big(\theta^E(u_B) \vee x_{k^*-1}\big) - G(x_{k^*-1}) \Big]$$

or

$$u_B \left(M^B - M^S(1-\gamma) \Big[G\big(\theta^E(u_B) \vee x_{k^*-1}\big) - G(x_{k^*-1}) \Big] \right) \leq \int_{\theta^E(u_B) \vee x_{k^*-1}}^1 \theta dF(\theta).$$

From the steady state balance equation (2), we have that

$$M_B = \underbrace{1}_{\text{new buyers}} + \underbrace{M^S(1-\gamma) \Big[G\big(\theta^E(u_B) \vee x_{k^*-1}\big) - G(x_{k^*-1}) \Big]}_{\text{re-queuers}}.$$

Thus, the equilibrium payoff u_B must satisfy

$$u_B \leq k \int_{\theta^E(u_B) \vee x_{k^*-1}}^1 \theta dF(\theta)$$

$$\leq k \int_{\theta^E(u_B)}^{1} \theta dF(\theta).$$

However, note that the equilibrium payoff u_B^r in a random search market satisfies the fixed point (see equation (7′))

$$u_B^r = k \int_{\theta^E(u_B^r)}^{1} \theta dF(\theta),$$

which implies that $u_B \leq u_B^r$. ■

References

Burdett, K. and Menzio, G. (2017), 'The (q, s, s) pricing rule', *The Review of Economic Studies* **85**(2), 892–928.

Butters, G. R. (1977), 'Equilibrium distributions of sales and advertising prices', *The Review of Economic Studies* **44**(3), 465–491.

Eeckhout, J. and Kircher, P. (2010), 'Sorting and decentralized price competition', *Econometrica* **78**(2), 539–574.

Hagiu, A. and Jullien, B. (2011), 'Why do intermediaries divert search?', *The RAND Journal of Economics* **42**(2), 337–362.

Hirshleifer, J. (1971), 'The private and social value of information and the reward to inventive activity', *The American Economic Review* **61**(4), 561–74.

Hosios, A. J. (1990), 'On the efficiency of matching and related models of search and unemployment', *The Review of Economic Studies* **57**(2), 279–298.

Kim, K. and Kircher, P. (2015), 'Efficient competition through cheap talk: the case of competing auctions', *Econometrica* **83**(5), 1849–1875.

Lauermann, S. (2012), 'Asymmetric information in bilateral trade and in markets: An inversion result', *Journal of Economic Theory* **147**(5), 1969–1997.

Lester, B. (2011), 'Information and prices with capacity constraints', *American Economic Review* **101**(4), 1591–1600.

Lester, B., Shourideh, A., Venkateswaran, V. and Zetlin-Jones, A. (2019), 'Screening and adverse selection in frictional markets', *Journal of Political Economy* **127**(1), 338–377.

Menzio, G. (2007), 'A theory of partially directed search', *Journal of political Economy* **115**(5), 748–769.

Moen, E. R. (1997), 'Competitive search equilibrium', *Journal of political Economy* **105**(2), 385–411.

Montgomery, J. D. (1991), 'Equilibrium wage dispersion and interindustry wage differentials', *The Quarterly Journal of Economics* **106**(1), 163–179.

Mortensen, D. T. and Wright, R. (2002), 'Competitive pricing and efficiency in search equilibrium', *International economic review* **43**(1), 1–20.

Olszewski, W. and Wolinsky, A. (2016), 'Search for an object with two attributes', *Journal of Economic Theory* **161**, 145–160.

Peters, M. (1991), 'Ex ante price offers in matching games non-steady states', *Econometrica: Journal of the Econometric Society* pp. 1425–1454.

Rogerson, R., Shimer, R. and Wright, R. (2005), 'Search-theoretic models of the labor market: A survey', *Journal of economic literature* **43**(4), 959–988.

Romanyuk, G. and Smolin, A. (2019), 'Cream skimming and information design in matching markets', *American Economic Journal: Microeconomics* **11**(2), 250–76.

Schlee, E. E. (2001), 'The value of information in efficient risk-sharing arrangements', *American Economic Review* **91**(3), 509–524.

Shi, S. (2002), 'A directed search model of inequality with heterogeneous skills and skill-biased technology', *The Review of Economic Studies* **69**(2), 467–491.

Shimer, R. (2005), 'The assignment of workers to jobs in an economy with coordination frictions', *Journal of Political Economy* **113**(5), 996–1025.

Vellodi, N. (2018), 'Ratings design and barriers to entry', *Available at SSRN 3267061* .