Instructor: Artem Kuriksha

Penn Economics Math Camp
Final Exam. Part I

You should work on the exam independently and not using the Internet. Violating these two rules will lead to a penalty.

Problem 1. Which of the following sets are finite, denumerable, or uncountable? Explain.

(1) $2^{\{1,2,3,4\}} \times \mathbb{R}$
(2) $\mathbb{Q} \setminus \mathbb{N}$
(3) $\Big\{(x, y) \in \mathbb{R}^2 \mid x^2 - y^2 = 4\Big\}$
(4)
$$\prod_{n=1}^{100} \{0, 1\}$$
(5)
$$\prod_{n=1}^{\infty} \{0, 1\}$$

Problem 2. Show that a topological space X is Hausdorff if and only if the set

$$\Big\{(x, x) \in X \times X | x \in X\Big\}$$

is closed in $X \times X$.

Problem 3. Which of the following sets are open? Closed? Compact? Connected? Totally bounded? Complete with the metric induced from the Euclidean space? For each of the sets, answer all six questions. Your explanation might be brief but should show your understanding.

(1) $\Big\{\frac{1}{n} \mid n \in \mathbb{N}\Big\} \subset \mathbb{R}$
(2) $\Big\{(x, y) \mid y^2 - x^2 = 4\Big\} \subset \mathbb{R}^2$
(3) $\Big\{(x, y) \mid x^2 - \sin(y)^2 = 1\Big\} \subset \mathbb{R}^2$
(4) $\Big\{(x, y) \mid 9x^2 + y^2 < 1\Big\} \subset \mathbb{R}^2$
(5) $\Big\{(x, y, z) \mid x^2 + y^2 = 1, z = 3\Big\} \subset \mathbb{R}^3$

Problem 4. Consider correspondence $\phi : \mathbb{R} \rightrightarrows \mathbb{R}$ given by $\phi(x) = [0, x^2]$. While the domain has the standard topology on $\mathbb{R}$, the codomain has the cocountable topology. Is ϕ upper-hemicontinuous? Is ϕ lower-hemicontinuous?

Penn Math Camp Part II

Final Exam

August 23, 2021

This part has 40 points. Indicate your reasoning and write legibly. If you skip a subquestion, you can continue to the next one assuming that the result in the previous one holds.

1. (18 points) Consider a stochastic matrix (entries are non-negative and the sum of the entries in each column adds up to one)

$$P = \begin{pmatrix} 1-f & s \\ f & 1-s \end{pmatrix}.$$

   1. (4 points) Under what conditions for f and s is P a projection matrix?
   2. (3 points) Under what conditions for f and s is $P^t P$ full rank?
   3. (3 points) Under what conditions for f and s is P positive definite?
   4. (4 points) A vector x_0 is a probability distribution if its entries are non-negative and add up to one. A probability distribution vector x^* is a stationary distribution of the stochastic matrix P if $Px^* = x^*$. Find the stationary distributions of P.
   5. (4 points) Suppose all entries of P are strictly positive. Prove that, for an arbitrary probability distribution vector x_0,

$$\lim_{n\to\infty} P^n x_0 = x^*,$$

   where x^* is the stationary distribution vector.

2. (17 points) Consider a risky asset, whose rate of return $\tilde{r}$ is a random variable with $n \geq 2$ possible realizations: with probability p_i, the rate of return is r_i, $\forall i = 1, 2, ..., n$. The expected return is positive, i.e.,

$$\sum_{i=1}^{n} p_i r_i > 0. \tag{1}$$

Suppose the agent's initial wealth is $w > 0$. Therefore the ex post wealth when x is invested in this asset becomes $\tilde{y} = w - x + (1 + \tilde{r})x = w + \tilde{r}x$. The agent chooses $x \geq 0$ in order to maximize the expected utility

$$U(x) := \sum_{i=1}^{n} p_i u(w + r_i x), \tag{2}$$

where $u' > 0$, $u'' < 0$. Denote the optimal choice by $x^*(w)$. Define

$$A(x) = -\frac{u''(x)}{u'(x)}. \tag{3}$$

We assume $A(x)$ is strictly decreasing.

1. (3 points) Is $u' > 0$ equivalent to u being strictly increasing (assuming u differentiable)? Is $u'' < 0$ equivalent to u being strictly concave (assuming u twice differentiable)? Comment on both directions of the equivalence. Provide a counterexample if your answer is no.
2. (4 points) Write down the interior first order condition for $x^*(w)$. Prove that the optimal solution is indeed interior, i.e., show that $x^*(w) \neq 0$.
3. (4 points) Show that $\sum_{i=1}^{n} p_i r_i u''(w + r_i x^*(w)) > 0$.
4. (3 points) Show that $x^*(w)$ is increasing in w.
5. (3 points) Define $g = u' \circ u^{-1}$. Show that g is a convex function.

3. (5 points)

   1. (1 point) State a weak separating hyperplane theorem that separates a set and a point.
   2. (2 point) Using the theorem stated in the previous question, show that there exists a non-zero vector $p \in \mathbb{R}^n$ such that

      $$p^t x \geq 0, \forall x \geq 0,$$

      where $x \in \mathbb{R}^n$.
   3. (2 point) Furthermore, prove that $p > 0$ (all coordinates are non-negative but at least one is strictly positive).

Math Camp Part III University of Pennsylvania

Final Exam

Instructions

1. Please submit this exam via Canvas before August 24 at 10:00 am Eastern Time.
2. Scan your solutions and upload them as a single PDF document.
3. This exam is open book.
4. Please restraint yourself from commenting the solutions with other students.
5. In all True or False questions, if the statement is True, provide a sketch proof. Else, provide a counterexample.
6. This exam consists of five questions. Each one is worth 10 points. The first question (The Pareto Distribution) and the second question (True or False) are MANDATORY. Then, you must choose two out of the three remaining questions. Please write down the title of the questions you are answering, so I know what to grade.

The Pareto Distribution

We say that X has a Pareto distribution if:

$$p_X(x) = \frac{\alpha\theta^\alpha}{x^{\alpha+1}} \quad \text{if } x \in [\theta, \infty).$$

Consider that we have a sample of n random variables $\{X_1, X_2, ..., X_n\}$ and we know these are independent from each other and each of them follow the same Pareto distribution. We wish to estimate the value of α considering that we know $\theta > 0$ (and henceforth the value of θ is considered to be fixed). In order to estimate α, we follow the **Maximum Log-Likelihood Method**, that is, we consider our estimator of α, denoted $\hat{\alpha}$, to be the solution to the following problem:

$$\max \sum_{i=1}^{n} log\left(p_X(x_i)\right) \quad \text{subject to:}$$

$$\alpha \geq 0.$$

1. Briefly argue the following things about this optimization problem:
   - That this problem has a unique solution.
   - That the solution is not at the boundary of the constraint set.
2. Justifying your steps, compute $\hat{\alpha}$.
3. Justifying your answer, compute the probability limit of $\hat{\alpha}$.
4. Do changes in θ vary $\hat{\alpha}$ in a lower-hemi continuous way?

True or False Questions

Please answer two out of these four questions.

1. Let $f : \mathcal{D} \to \mathbb{R}$ be strictly concave and $\mathcal{D} \subseteq \mathbb{R}^n$ be a compact and convex set. Then the problem:

$$\max_{x \in \mathcal{D}} \quad f(x),$$

has a unique solution.

2. Consider $X \neq \emptyset$. Let S_1, S_2 be two σ-algebras on X such that $S_1 \subseteq S_2$. If $f : X \to \mathbb{R}$ is not measurable within the measurable space (X, S_1), then it is also not measurable within the measurable space (X, S_2).

3. σ-algebra are closed under unions, i.e., if S_1, S_2 are both σ-algebras, then $S_1 \cup S_2$ is also a σ-algebra.

4. The following problem is always a concave problem for any $M \in \mathbb{R}^{n \times n}$ such that $M \neq 0$:

$$\max \quad x^T M^T M x + n^T x \quad \text{subject to:}$$

$$b^T x \leq c.$$

Norm of Matrices

Let us consider $A \in \mathbb{R}^{n\times n}$ to be a non-singular matrix. Given $C \in \mathbb{R}^{n\times n}$ positive definite, we define the $C-$norm of A, denoted as $||A||_C$ to be:

$$||A||_C = \max \ \ ||Ax||_2 \quad \text{subjetct to:}$$

$$x^T C x \leq 1,$$

where $||y||_2 = y^T y$ for $y \in \mathbb{R}^n$.

1. Show that this is a concave optimization problem that satisfies the conditions required for strong duality to hold.

2. For the rest of the question consider $C = I$ (the identity matrix). Using the first order conditions, characterize the solutions $(x^\star, \mu^\star)$.[1]

3. Given $M \in \mathbb{R}^{n\times n}$ let $\sigma(M) = \{\lambda_1, ...\lambda_n\}$ denote the set of the eigenvalues of M. With this in mind, show that, when considering $C = I$:

$$||A||_C = \max \ \ \lambda^2 \quad \text{subject to:}$$

$$\lambda \in \sigma(A^T A).$$

[1]Hint: You won't be able to come up with explicit solutions, but you can say something about the solution of this problem.

Dominated Convergence Theorem

Consider (X, S, μ) to be a measure space where $X = \mathbb{N}$, $S = \mathcal{P}(\mathbb{N})$ and $\mu : S \to \mathbb{R}$ given by:

$$\mu(E) = \begin{cases} 0 & \text{if } E = \emptyset, \\ \sum_{n \in E} \frac{1}{2^n} & \text{if } E \neq \emptyset. \end{cases}$$

1. Let $E_n = \{n + k | k \in \mathbb{N}\} = \{n+1, n+2, ...\}$. Compute $\mu(E_n)$.

2. For the rest of the question, consider the following sequence of functions (f_n) given by:

$$f_n(m) = 2^n \mathcal{X}_{E_n}(m).$$

   Make a drawing of f_n for $n = 1, 2, 3$.

3. Show that:

$$\int \lim_{n \to \infty} f_n \, d\mu = 0.$$

4. Compute:

$$\lim_{n \to \infty} \int f_n \, d\mu.$$

5. What is going on with this example?[2]

[2]Hint: See the title of the question.

Linear Programming

Consider the following linear program:

$$\max \;\; 5x_1 + 6x_2$$
$$x_1 + x_2 \leq 10$$
$$5x_1 + 4x_2 \leq 43$$
$$x_1 \leq 7$$
$$x_1, x_2 \geq 0.$$

1. Find the optimal solution of this linear program using the Simplex method. Please write down all dictionaries that you attain during the algorithm.

2. Make a drawing of the feasible region and indicate the edges that the Simplex method visits during the algorithm.

3. At the optimal solution, which dual variables are we certain that are equal to zero? Relate your answer with the graph you made at the previous question.

4. Now consider that we are solving the following problem:

$$\max \;\; ax_1 + x_2$$
$$x_1 + x_2 \leq 10$$
$$5x_1 + 4x_2 \leq 43$$
$$x_1 \leq 7$$
$$x_1, x_2 \geq 0,$$

   where $a > 0$. Give conditions on a such that the optimal solution of this problem is $(x_1^\star, x_2^\star) = (7, 2)$.[3]

[3] Hint: Use a graph.