A Theory of Business Transfers

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Abstract

In this paper, we study firm dynamics and the allocation of capital in the private business sector. Two restrictions are imposed on transfers of business capital: (i) indivisibility—all assets in a business are sold as a unit; and (ii) bilateral trades—the terms of trade are settled in pairwise meetings. While the equilibrium features dispersion in marginal products as well as transaction prices per unit of capital traded, we show that the allocation is efficient. Dispersion in per-unit prices arises because of time-to-build concerns for productive owners and variation in market thickness across the size distribution of firms. Firms grow over the life in two ways: through internal investment and through purchases of other firms. Capital is gradually traded upwards over the life of a business, from owners with a low marginal product of capital to owners with a high marginal product of capital. The model is used to estimate the impact of capital gains taxation and the degree of transferability of private business wealth.

Key words: optimal assignment, capital allocation, firm dynamics, capital taxation

1 Introduction

This paper starts with the premise that most of the assets transferred in private business sales are intangible in nature and accumulated over time by the business founders and successors. We are motivated by data on the typical business sale, which typically includes the transfer of intangibles such as customer bases, trademarks, and going concern value. These are assets that are not commonly leased and are usually sold all at once when the owners relocate, retire, or begin a new venture. Because such transfers are rare events, little is known about the investments that ongoing private businesses make, despite the fact that they generate more than half of all U.S. business income and significant wealth for their owners. To address this, we propose a new theory of business transfers that incorporates an indivisibility in bilateral trades, effectively assuming that businesses are sold as a unit in pairwise meetings. The theory is disciplined by administrative data on business transfers from the Internal Revenue Service and used to study firm dynamics and capital taxation.[1]

The main element of the theory is the technology of firms. Firms are collections of three factors: *nontransferable* capital that cannot be bought or sold, *transferable* capital that can be bought and sold, and external factors that are rented on spot markets. The nontransferable capital is in essence the owner's productivity or ability, which can change over time but is inalienable. The transferable capital is the intangible capital recorded on IRS tax forms. The external factors would in practice include employee time, physical capital, and materials. Firms in the model grow in two ways: through internal investment and through purchases of other businesses. Each period, owners have an opportunity to engage in a bilateral trades and buy or sell their business. Those that sell can restart another.

Despite the indivisibilities in capital exchange, we show that the allocation of capital is efficient. The model is effectively a neoclassical benchmark in the spirit of Lucas (1978) modified to include competitive factor markets for lumpy transferable capital with terms of trade settled in bilateral meetings. Capital is gradually traded upwards with owners that have a low marginal product of capital selling to those with a high marginal product. Per-unit prices vary across sales and depend on the quantity. The prices are highest for quantities that result in a relatively quick attainment of optimal size and decline for very large transactions because of decreasing returns to scale and thinner markets. Importantly, the price dispersion in this model is not indicative of misallocated resources.

Data from the IRS starting in tax year 2000 is used to discipline the model. Of particular relevance are business asset acquisitions reported on Forms 8594 and 8883 filed by both buyers and sellers. Taxpayers must allocate the business purchase price across different asset categories, including categories of marketable securities, fixed assets, and intangible assets.

[1] This work is part of a Joint Statistical Research Program Project of the Statistics of Income Division at the IRS investigating tax compliance of intangible-intensive businesses.

Included with intangibles are Section 197 assets—customer- and information-based intangibles; non-compete covenants; licenses and permits; franchises, trademarks, and trade names; workforce in place; business books and records; and processes, designs, and patterns—as well as goodwill and going concern value. This information is needed to assess capital gains for sellers and asset bases for buyers. The tax identification numbers from the filings are then linked to business tax forms and each owner's individual tax form. This allows us to construct longitudinal panels over the business and owner life cycles.[2]

We use the data to parameterize the model and then study its predictions for firm dynamics and business wealth. We find that roughly 4 percent of all transferable capital units are traded each period, indivisibly through business sales. The price per unit of transferable capital is in the range of 4 to 7 times sellers' income. Buyers tend to be more productive, with incomes that are on the order of 2 to 4 times that of the sellers they transact with. Since the reallocation process takes time, our baseline economy will appear to have significant dispersion in marginal products of capital across businesses. In fact, if we compare average capital holdings across firms with different productivity levels, we find a relatively flat profile when compared to an economy with divisible capital and a centralized asset market. The flatness in the profile reflects the fact that accumulating non-rentable, non-divisible intangible capital through own investment or purchases takes time. Time to build concerns are also evident in the dispersion of prices following trades, which can be as much as 50 percent higher per-unit of capital for medium-sized businesses relative to smaller businesses.

The model is then used to estimate business wealth and the degree of capital transferability, measured as the ratio of the value of transferable capital—say, if the business were sold today—to the total value of the ongoing concern that generates a flow of dividends to owners over the business life. The numerator of this ratio is the value typically asked of owners in surveys of consumer finances and the denominator is the standard notion of value in a finance textbook. The total value includes not only the value of the intangible assets that can be transferred, but also the value of non-transferable capital that reflects the owners' inalienable productivity. For our baseline parameterization, we find a range of one-third for the most productive owners to over one-half for the least productive. If we average over the population, we estimate the transferable share to be 43 percent.

We investigate the impact of taxing this wealth as it is done in the United States and most other countries, namely, through taxation of *realized* capital gains. When introducing this tax on capital gains in the model, we in effect have owners file the equivalent of the IRS Form 8594 when selling their businesses. Not surprisingly, the tax eliminates business

[2] This work is still in progress and unpublished IRS statistics have not yet been cleared for disclosure avoidance. As a result, this draft relies heavily on the Form 8594 purchase price allocations for brokered business transfers that are recorded in the Pratt's Stats database.

transfers that yield small gains to the seller. As a result, there is even greater dispersion in marginal products of capital in the model with a capital gains tax relative to the baseline without. What is more novel is the finding that tax incidence depends on the quantity of capital transferred. If the business is small- to medium-sized in terms of units of transferable capital, then there is greater incidence on buyers. This result follows from the fact that buyers are generally more productive and want to grow quickly to their optimal size by buying up the capital of smaller firms. If the capital stock transferred is large, the tax incidence is greater for the seller. The market for large businesses is very thin and, thus, for large quantities, the seller bears almost the full cost of the tax.

1.1 Related literature

This paper contributes to several strands of literature. First, our paper is related to the large body of work that studies firm dynamics, productivity, and the allocation of capital. In the seminal paper by Hopenhayn (1992), stochastic productivity drives the demand for capital that is perfectly divisible and competitively traded. Variations of Hopenhayan's framework have been brought to the data by, among others, Hsieh and Klenow (2009, 2014) and Sterk et al. (2021). The findings of these papers about size and productivity differentials across firms have in turn spawned a large literature that focuses on identifying the source of differences as "misallocation" due to regulatory, financial, or informational frictions.[3] We depart from this literature in our focus on intangibles and our modeling of business capital trades. In our model, measured dispersion is an artifact of the market structure whereby all assets in a business are sold as a unit with terms of trade settled in pairwise meetings and is not driven by a misallocation of resources due to surmountable frictions. Furthermore, our emphasis on intangibles in private business makes the existing empirical evidence on firm dynamics less portable since it is largely focused on physical capital in manufacturing. To fill the gap, we shed new light on the lifecycle dynamics of private business by using longitudinal data from annual tax filings along with intermittent business transactions.

A related literature has focused on the transfer of various forms of business capital, taking into account their indivisible nature (Holmes and Schmitz (1990)). These models necessarily make assumptions on the degree of input transferability by studying either the sale of some fixed factor (David (2021)), or capital that can be produced (Ottonello (2014)), or a combination of both, for example, the whole firm (Guntin and Kochen (2020), Gaillard and Kankanamge (2020)). One of the main contributions of our paper is using theory and detailed transaction data to estimate the share of business wealth that is transferable. From a technical perspective, modeling the demand and supply for heterogeneous, indivisible

[3] See, for example, David and Venkateswaran (2019) and others in the survey article by Restuccia and Rogerson (2017).

products—in this case, businesses—poses significant challenges in the absence of traditional assumptions adopted when modeling good markets (for example, demand functions with constant elasticity of of substitution). To ensure tractability, previous authors have typically developed models of random search with bargaining—which is well-known to generate inefficient allocations—or directed search with one-sided heterogeneity.[4] We take a different route and model business sales as transactions in a frictionless decentralized market. Building on tools from the matching literature (Choo and Siow (2006), Galichon et al. (2019)), we solve for the equilibrium set of prices and show that it implements an efficient allocation. To the best of our knowledge, we are the first to prove efficiency of a matching model in which the matches that are formed in each period determine the evolution of the distribution of agents' types. We find the efficiency property appealing as it allows us to isolate the dispersion in marginal products that is solely generated by the indivisible nature of capital in private business.

As an application of our framework, we recover measures of business value for both traded and non-traded private firms. Such estimates contribute to a growing literature on the measurement of private business wealth. Most papers in this literature rely either on structural models of entrepreneurship disciplined by survey data such as Cagetti and De Nardi (2006) or non-structural approaches that use administrative data, for instance, the capitalization method used by Saez and Zucman (2016) and Smith et al. (2019). Differently from these papers, our measure leverages theoretically-grounded valuation concepts and primitives that are disciplined by detailed data on business sales merged with the income statements of buyers and sellers.

Finally, we contribute to the public finance literature that studies the consequences of taxation of different sources of income. While a large literature studies the taxation of business income, we focus on the taxation of realized capital gains.[5] In neoclassical settings such as Hopenhayn (1992), if one were to introduce a capital gains tax on private business, there would be no effect unless applied to accrued gains. The reason is that the value of traded physical capital is constant in a stationary equilibrium, and gains in the value of non-traded capital (or owner productivity) are never realized. Our framework is particularly suited for the study of capital gains taxation for private businesses given our explicit modeling of self-created intangibles that are valued and taxed at the time of sale.

[4] Burdett and Mortensen (1998) introduce extensions to the canonical job ladder model to allow for one-sided heterogeneity. Extensions have also allowed for two-sided heterogeneity and random search (Postel-Vinay and Robin (2002) and Bagger and Lentz (2019)) or directed search (Schaal (2017)) but require additional assumptions for tractability. For example, while in Schaal (2017) firms hire a measure of workers, that assumption is not appropriate for modeling trades of indivisible units like businesses.

[5] For studies of business income taxation, see Kitao (2008), Meh (2005), Boar and Midrigan (2019), Bhandari and McGrattan (2021), Bruggemann (2021). Chari et al. (2003) is one of few papers that studies taxation of capital gains by analyzing the issue in the context of the Holmes and Schmitz (1990) model.

The rest of the paper is organized as follows. Section 2 sets up the theoretical environment, including timing of events, descriptions of problems solved by business owners, and a definition of a recursive equilibrium. A characterization of equilibrium and a proof of efficiency are provided in Section 3. In Section 4, we parameterize the model and put it to use to quantify the patterns of trade and resulting firm dynamics. We highlight two measures of business wealth and discuss impacts of taxing wealth that is transferred through business sale. Section 5 concludes.

2 Setup

Entrepreneurs are endowed with a technology that produces consumption goods using two factors: entrepreneurial productivity or skill (z) and capital (k). The factor z is non-transferable and the factor k is transferable. Every period, entrepreneurs have an opportunity to trade capital k with others. Then they produce, invest in capital, and consume. At the end of the period, they face the possibility of exit, which occurs exogenously at some constant rate. Details of these three stages are provided next, followed by the entrepreneurs dynamic program and the definition of recursive equilibrium we wish to characterize.

2.1 Timing

Before describing each stage, we first introduce some notation. Let Z be the set of productivity levels. We will assume that the productivities $z \in Z$ follow a Markov process with transition matrix T_z. We use the set $\mathcal{S} \equiv Z \times K$ to denote the space of productivity levels and potential capital. We let $s \in \mathcal{S}$ denote a pair (z, k) and use $z(s)$ and $k(s)$ to denote the first and the second component of s, respectively. We use $\Delta(\mathcal{S})$ to denote the set of measures over $\mathcal{S}$. For some $\phi \in \Delta(\mathcal{S})$, we use $\phi(ds)$ to denote its density at s.

Trading stage. At the beginning of the trading stage, the state vector for an entrepreneur is $s \equiv (z, k) \in \mathcal{S}$. An entrepreneur with state s faces a price-quantity menu denoted by $\{p^m(s, \tilde{s})\}_{\tilde{s} \in S}$ and $\{k^m(s, \tilde{s})\}_{\tilde{s} \in S}$. That is, an entrepreneur that has state s is matched with entrepreneur that has state $\tilde{s}$, pays $p^m(s, \tilde{s})$ to the trading partner, and exits the trading stage with capital level $k^m(s, \tilde{s})$. The functions $k^m : \mathcal{S}^2 \to K$ and $p^m : \mathcal{S}^2 \to \mathcal{R}$ are determined as a part of an equilibrium that we define later.[6]

We now introduce a few assumptions on the exchange of capital and payments within a match. For the allocation of capital, we impose that entrepreneurs can either sell their entire capital stock, buy the entire capital stock of their trading partner, or trade no capital at

[6]We omit the explicit dependence of individual choices and values on $\{p^m, k^m\}$ when it is clear from the context.

all. For payments, we allow any bilateral exchange as long as there is no outside funding to finance the trade. These assumptions amount to the following restrictions on the functions $\{k^m, p^m\}$. For all pairs $(s, \tilde{s}) \in \mathcal{S}^2$,

$$k^m(s, \tilde{s}) \in \{k(s) + k(\tilde{s}), k(s), 0\} \tag{1}$$

$$k^m(\tilde{s}, s) + k^m(s, \tilde{s}) \leq k(s) + k(\tilde{s}) \tag{2}$$

$$p^m(\tilde{s}, s) + p^m(s, \tilde{s}) \geq 0 \tag{3}$$

We refer to restriction (1) as *indivisibility* and (1) to (3) together as *feasibility of trades.* These restrictions capture the key friction in our model, namely, that the reallocation of capital across entrepreneurs in bilateral trades occurs in a "lumpy" fashion.

Production and investment stage. In this stage, decisions concerning goods production, capital investment, and consumption are made. Output is produced using a decreasing returns to scale technology:

$$y(s) = z(s)\, k(s)^{\alpha}.$$

The investment technology is modeled as a cost function $c(\theta)$, where c' and c'' are strictly positive. An entrepreneur incurs cost $c(\theta)$ to increase capital in the following period by 1 unit with probability θ.

Entry and exit stage. At the end of the production stage, entrepreneurs exit at rate δ. Surviving entrepreneurs draw a new z and increment their capital by one unit if their investment was successful. New entrants pay a cost c_e to draw a state $(z, k) \sim G(ds \in S)$. The entry decision $d_e \in \{0, 1\}$ is given by

$$\max_{d_e} \int W(s)\, G(ds \in S) - c_e. \tag{4}$$

2.2 Entrepreneurs Dynamic Program

Let $V : S \to \mathcal{R}^+$ denote the value of an entrepreneur at the beginning of the production stage. Let $W : \mathcal{S} \to \mathcal{R}^+$ be the value of the entrepreneur at the beginning of the trading stage. Given functions $\{p^m, k^m\}$, these value functions solve the following Bellman equation:

$$V(s) = z(s)\, k(s)^{\alpha} - c(\theta) + (1-\delta)\, \beta \mathbb{E}_{(s,\theta)} W(s'). \tag{5}$$

The distribution of the next period state (z', k') depends of the transition matrix T_z and investment choice $\theta(s)$ as follows. The shock $z(s')$ is drawn from the distribution $T_z(\cdot | z(s))$.

The level of capital is $k(s') = k(s)+1$ with probability $\theta(s)$ and $k(s') = k(s)$ with $1-\theta(s)$. Define $v(s,\tilde{s})$ as the value after trade for firm type s after trade with $\tilde{s}$:

$$v(s,\tilde{s}) \equiv V(z(s), k^m(s,\tilde{s})) - p^m(s,\tilde{s}). \tag{6}$$

Then, for all s', we have the continuation value given by

$$W(s') = \int \max_{\tilde{s}(\boldsymbol{\epsilon}) \in S \cup \{o\}} \left\{ v(s',\tilde{s}) + \sigma\epsilon(s',\tilde{s}), V(s') + \sigma\epsilon(s',o) \right\} F(d\epsilon). \tag{7}$$

The inner maximization represents the optimality for the entrepreneur, who has the price quantity menus $\{p^m(s,\tilde{s}), k^m(s,\tilde{s})\}_{\tilde{s}\in S}$ and, in addition, realizes non-pecuniary match-specific utilities $\{\epsilon(s,\tilde{s})\}_{\tilde{s}\in S} \sim F$. The parameter σ scales the relative importance of the pecuniary versus non-pecuniary benefits from trading. Given $\{p^m(s,\tilde{s}), k^m(s,\tilde{s}), \epsilon(s,\tilde{s})\}_{\tilde{s}\in S}$, the optimal choice of trading partner, $\tilde{s}(\boldsymbol{\epsilon}) \in S \cup \{o\}$ is described by maximization inside integral (7) with $\tilde{s}(\boldsymbol{\epsilon}) = o$ denoting the choice to remain unmatched.

The solution to this problem induces choice probabilities $\lambda : \mathcal{S} \to \Delta(\mathcal{S})$ and $\lambda_o : \mathcal{S} \to [0,1]$ given functions $\{p^m, k^m\}$, so that for all $A \subseteq \mathcal{S}$:

$$\lambda(s,A) \equiv \int \mathbb{I}(\tilde{s}(\boldsymbol{\epsilon}; s) \in A) F(d\boldsymbol{\epsilon})$$

is the probability measure over the event that type s is matched to agents of type $\tilde{s} \in A$, and

$$\lambda_o(s) \equiv \int \mathbb{I}(\tilde{s}(\boldsymbol{\epsilon}) = o; s) F(d\boldsymbol{\epsilon})$$

is the probability measure over the event that $s \in A$ are unmatched.

2.3 Equilibrium

Let $\phi \in \Delta(\mathcal{S})$ be the measure over entrepreneurs at the beginning of the period. Using the law of large numbers, we can define the aggregate measure of matches as follows. For all $A, \tilde{A} \subseteq \mathcal{S}$

$$\Lambda(A,\tilde{A}) = \int \phi(ds \in A)\, \lambda(s, d\tilde{s} \in \tilde{A}),$$

$$\Lambda_o(A) = \int \phi(ds \in A)\, \lambda_o(s).$$

Let $\phi_e \in \Delta(\mathcal{S})$ be the measure of new entrants. The measure over entrepreneurs in the next period, ϕ' is described by a function Γ as follows. For all $\hat{A} \equiv \hat{Z} \times \hat{K} \subseteq \mathcal{S}$

$$\phi'(\hat{A}) \equiv \Gamma(\phi; \lambda, \lambda_o, \theta, \phi_e, k^m)(\hat{A}) \tag{8}$$

where

$$\Gamma \equiv \Gamma_\lambda \times \Gamma_\delta \times \Gamma_\theta.$$

Each component is defined as follows:

$$\begin{aligned}
\phi_\theta(\hat{A}) &= \Gamma_\theta\left(\phi;\theta\right)(\hat{A}) \\
&= \int (1-\theta(s))\phi(ds \in \hat{A}) \int \mathbb{I}\left\{(z(s), k(s)+1) \in \hat{A}\right\} \theta(s)\phi(ds \in \mathcal{S}) \\
\phi_\delta(\hat{A}) &= \Gamma_\delta\left(\phi_\theta;\phi_e\right)(\hat{A}) \\
&= (1-\delta)\int \phi_\theta(ds \in Z \times \hat{K}) T_z(d\hat{z} \in \hat{Z}|z(s)) + \phi_e(ds \in \hat{A}) \\
\phi'(\hat{A}) &= \Gamma_\lambda(\phi_\delta;\lambda,\lambda_o,k^m)(\hat{A}) \\
&= \int \mathbb{I}\left\{z(s), k^m(s,\tilde{s}) \in \hat{A}\right\} \lambda\left(ds, d\tilde{s} \in \mathcal{S}\right) \phi_\delta(ds \in \mathcal{S}) + \int \lambda_o(s)\phi_\delta(ds \in \hat{A}).
\end{aligned}$$

We are now ready to define an equilibrium.

Definition 1. A *recursive equilibrium with matching* is given by (i) price quantity menus $p^m : \mathcal{S}^2 \to \mathcal{R}$ and $k^m : \mathcal{S}^2 \to K$; (ii) aggregate matching measures $\Lambda \in \Delta\left(\mathcal{S} \times \mathcal{S}\right)$ and $\Lambda_o \in \Delta\left(\mathcal{S}\right)$; (iii) mass and initial capital for entrants m; (iv) a measure $\phi^* \in \Delta\left(\mathcal{S}\right)$; (v) a pair of value functions $V : \mathcal{S} \to \mathcal{R}^+$and $W : \mathcal{S} \to \mathcal{R}^+$; and (vi) choice probabilities $\lambda : \mathcal{S} \to \Delta(\mathcal{S})$ and $\lambda_o : \mathcal{S} \to [0,1]$ such that:

1. The trading arrangements are feasible, that is, for all pairs $(s,\tilde{s}) \in \mathcal{S}^2$ with each having a positive density under ϕ^* the function k^m satisfies (*1*)-(*3*).

2. Given $\{k^m(\cdot), p^m(\cdot)\}$, the value functions for incumbent firms $\{V(\cdot), W(\cdot)\}$ solve the Bellman equations (5)-(7) with optimal choice probabilities $\{\lambda(\cdot), \lambda_o(\cdot)\}$ and the decision to enter for new entrants solves (4).

3. The aggregate measures of matches for existing entrepreneurs satisfy for all $A \subseteq \mathcal{S}$

$$\int \Lambda\left(ds \in A, d\tilde{s} \in \mathcal{S}\right) + \Lambda_o\left(ds \in A\right) = \phi^*\left(A\right). \tag{9a}$$

$$\int \Lambda\left(d\tilde{s} \in \mathcal{S}, ds \in A\right) + \Lambda_o\left(ds \in A\right) = \phi^*\left(A\right). \tag{9b}$$

   The mass of new entrants is given by

$$\int W\left(s\right) G\left(ds \in \mathcal{S}\right) - c_e \leq 0 \tag{10a}$$

$$m\left[\int W(s)\,G(ds\in\mathcal{S})-c_e\right]=0 \tag{10b}$$

4. The measure ϕ is stationary
$$\phi^*=\Gamma\phi^*. \tag{11}$$

Definition (1) summarizes individual optimality for all decision makers, market clearing, and stationary conditions that are standard in competitive environments. However, in our setup with bilateral matching, conditions 1–5 above are not sufficient to rule out pairwise deviations of groups of individuals. For instance, consider pair $(s,\tilde{s})$ that trades capital but ends up with $p^m(s,\tilde{s})+p^m(\tilde{s},s)>0$. Such a pair is on net lending to the rest of the economy. Since there is no punishment to default, this pair can come to a new agreement in which they keep the same allocation of capital but consume the extra resources. To rule this out along with other such deviations, we borrow the notion of pairwise stability from the matching literature and propose a refinement of the recursive competitive equilibrium in Defintion (1), which adds this notion of stability.

Definition 2. A *recursive equilibrium with matching is pairwise stable* if there does not exist an alternative price quantity $(\hat{k}^m,\hat{p}^m)\neq(k^m,p^m)$, and a pair $(s,\tilde{s})\in\mathcal{S}^2$ such that capital allocation $\left\{\hat{k}^m(s,\tilde{s}),\hat{k}^m(\tilde{s},s)\right\}$ satisfies (1)-(3), and

$$\begin{aligned}V\left(z(s),\hat{k}^m(s,\tilde{s})\right)-\hat{p}^m(s,\tilde{s})\geq&V\left(z(s),k^m(s,\tilde{s})\right)-p^m(s,\tilde{s})\\V\left(z(\tilde{s}),\hat{k}^m(\tilde{s},s)\right)-\hat{p}^m(\tilde{s},s)\geq&V\left(z(\tilde{s}),k^m(\tilde{s},s)\right)-p^m(\tilde{s},s)\end{aligned} \tag{12}$$

with at least one inequality being strict.

3 Characterizing the Equilibrium

In this section, we provide a characterization of the equilibrium with pairwise stability and discuss its efficiency properties. We compute the equilibrium in two steps. First, we take the post-trade value function V and the equilibrium measure of firms ϕ as given and characterize prices p^m and the allocation—that is, matching patterns (Λ,Λ_o) and capital k^m conditional on matching—that are consistent with market clearing and pairwise stability. Second, we solve for (V,ϕ) such that households optimize given the menu of prices and terms of trades and ϕ is, in turn, consistent with household decisions. We start with properties of the limiting case as $\sigma\to 0$ that turns off the nonpecuniary preference shocks. In section 3.2, we discuss the characterization of the more general setup with $\sigma>0$. The environment with preference shocks is motivated by our interest in studying capital gain taxes, which introduce a wedge in the transfer of utility between agents.

3.1 Equilibrium characterization without preference shocks

For the limit $\sigma \to 0$, we leverage Monge-Kantorovich duality to characterize the matching patterns and the terms of trade.[7]

Characterizing prices and allocations given (ϕ, V). As a first step, we introduce a version of the Monge-Kantorovich problem concerned with finding assignments that maximize the total surplus (as measured using V) by trading capital so as to preserve the measure ϕ. Define the largest (gross) surplus from matching for a pair $(s, \tilde{s})$ as follows:

$$X(s,\tilde{s}) = \max\left\{V(z,k+\tilde{k}), V(s)+V(\tilde{s}), V(\tilde{z},k+\tilde{k})\right\}.$$

The three arguments are possible outcomes in a match, namely, type s buys the capital from type $\tilde{s}$, no trade, and type s sells the capital to $\tilde{s}$. If we split the measure ϕ into two measures ϕ^a and ϕ^b such that for $A \subseteq \mathcal{S}$, then

$$\phi^a(A) = \phi^b(A) = \frac{\phi(A)}{2}.$$

For measures $\{\phi^a, \phi^b\}$, an assignment $\left(\pi, \pi_o^a, \pi_o^b\right)$ that maximizes surplus, solves the following maximization problem:

$$Q(\phi, V) = \max_{\pi,\pi_0^a,\pi_0^b \geq 0} \int X(s,\tilde{s})\pi(ds\in\mathcal{S}, d\tilde{s}\in\mathcal{S}) + \int V(s)\pi_o^a(ds\in\mathcal{S}) + \int V(\tilde{s})\pi_o^b(d\tilde{s}\in\mathcal{S}) \tag{13}$$

such that for $A \subseteq \mathcal{S}$

$$\int \pi\left(ds\in A, d\tilde{s}\in\mathcal{S}\right) + \pi_o^a\left(ds\in A\right) = \phi^a(A) \tag{14}$$

$$\int \pi(ds\in\mathcal{S}, d\tilde{s}\in A) + \pi_o^b(d\tilde{s}\in A) = \phi^b(A). \tag{15}$$

We label this problem as *P1.* The next theorem shows that we can back out $(p^m, k^m, \Lambda, \Lambda_o)$ from the solution of *P1.*

Theorem 1. *Let μ^a and μ^b be the Lagrange multipliers on (14) and (15), respectively, in problem P1. Let $\left(\pi, \pi_o^a, \pi_o^b\right)$ be the optimal assignment in problem P1. The functions*

$$k^m(s,\tilde{s}) \in \textit{argmax}\{V(z,k+\tilde{k}), V(s)+V(\tilde{s}), V(\tilde{z},k+\tilde{k})\} \tag{16}$$

$$p^m(s,\tilde{s}) = V(z,k^m(s,\tilde{s})) - \mu^a(s) \tag{17}$$

$$p^m(\tilde{s},s) = V(z,k^m(\tilde{s},s)) - \mu^b\left(\tilde{s}\right) \tag{18}$$

[7]See Galichon (2016) for details on the Monge-Kantorovich transportation problem.

and measures for all $A, \tilde{A} \subseteq \mathcal{S}$

$$\Lambda(A, \tilde{A}) = \pi(A, \tilde{A}) + \pi(\tilde{A}, A) \tag{19}$$

$$\Lambda_o(A) = \pi_o^a(ds \in A) + \pi_o^b(d\tilde{s} \in A) \tag{20}$$

satisfy (1)-(3), and (9a), and the pair (p^m, k^m) satisfies pairwise stability given V.

Theorem (1) states that the assignment from problem *P1* recovers the allocation of capital across entrepreneurs and the shadow prices on constraints (14) and (15) recover the terms of trade. More specifically, equation (16) implies that the allocation of capital maximizes the pairwise surplus. Then, the terms of trade are determined by exploiting the insight that social and private gains from trade are equal at the optimal assignment. To see this, consider a perturbation that changes the measure ϕ at some state s. Using the envelope theorem, the value of this perturbation to the interim planner who solves problem *P1* is given by $(\mu^a + \mu^b)/2$. In effect, this measures the social gains from having more entrepreneurs of type s. Given the symmetry of X, it is easy to verify that μ^a equals μ^b. Therefore, we can drop the superscript and denote the social value of type s as $\mu(s)$. The private value of a type s entrepreneur is given by $V\left(z\left(s\right), k^m\left(s, \tilde{s}\right)\right) - p^m\left(s, \tilde{s}\right)$. Optimality ensures that the private value is equalized across all trading partners $\tilde{s}$ with strictly positive probabilities. Equating the social and private values gives us equation (17) and (18) that pin down the pairwise terms of trade.

The next corollary further sharpens the characterization of the price function p^m.

Corollary 1. *With $\sigma \to 0$, there exists a function $\mathcal{P} : K \to R_+$ such that*

$$p^m\left(s, \tilde{s}\right) = \mathcal{P}\left(k\left(s\right)\right) \quad \textit{for all } k^m\left(s, \tilde{s}\right) < k(s).$$

This corollary says that the pairwise prices only depend on the quantity sold. The intuition for this result is straightforward. The sellers value from trade is equal to the price he extracts from the buyer plus the value of starting anew with zero capital and the current level of productivity. The second component is independent of the trading partner. Thus, conditional on selling to multiple buyers, a seller who maximizes the value from trading must necessarily charge the same price to all buyers. A similar argument from the perspective of the buyer shows that the prices will not depend on the sellers productivity. We can then conclude that the prices are only a function of the quantity traded, and summarize this dependence using the function $\mathcal{P}\left(\cdot\right)$ function. In Section 4, we will discuss the forces that determine the shape of $\mathcal{P}\left(\cdot\right)$.

Characterizing (ϕ, V) given $(p^m, k^m, \Lambda, \Lambda_o)$. In the second step, we use the outcomes of the first to update value functions and the invariant measure. The characterization in the first step gives us a handy way of solving the Bellman equation. In the appendix, we show that we can recover the values $V(s)$ and optimal investment rates $\theta(s)$ using the multipliers $\{\mu(\cdot)\}$,

$$V(s) = \max_{\theta} y(s) - c(\theta) + \beta(1-\delta)\mathbb{E}_{(s,\theta)}\mu(s').$$

The key to this result is to show that the continuation values $W(s) = \mu(s)$ where the function μ is the Lagrange multiplier in problem *P1*.

From equation (8), it is clear that $(\Lambda, \Lambda_o, \theta)$ fully characterize Γ given ϕ_e. Thus ϕ is given by condition (11). Together step 1 and step 2 characterize the recursive competitive equilibrium as a fixed point. This characterization naturally lends itself to a computational algorithm where we iterate between step 1 and step 2 until convergence.

Efficiency. We conclude this section by discussing the efficiency properties of our competitive equilibrium. Given ϕ_0, consider a planner that solves the following maximization problem.

$$P(\phi_0) = \max_{\{\lambda_t, \lambda_{o,t}, \theta_t, k_t^m, m_t\}} \sum_{t=0}^{\infty} \beta^t \int [y(s) - c(\theta_t(s))] \phi_t(ds \in S) - c_e m_t$$

such that k_t^m satisfies feasibility conditions (1) and (2), $\lambda_t, \lambda_{o,t}$ satisfy (9a) and (9b) given ϕ_t, $\phi_e(m) = mG$, and

$$\phi_{t+1} = \Gamma(\phi_t; \lambda_t, \lambda_{o,t}, \theta_t, \phi_e(m_t), k_t^m), \tag{21}$$

We label this problem as *P2*. Given linear preferences, maximizing discounted welfare is the same as maximizing discounted output. We denote a solution to *P2* as stationary if $\phi_t = \phi_0$ for all t. In the next theorem, we show that stationary recursive equilibrium is efficient.

Theorem 2. *A stationary recursive equilibrium with pairwise stability as defined in Definition (2) with the stationary measure ϕ^* achieves $P(\phi^*)$ in problem* P2. *Furthermore, any stationary solution to* P2 *constitutes a stationary recursive equilibrium with pairwise stability.*

The forces towards efficiency were foreshadowed in the formulation of the problem *P1*. Given (ϕ, V), the optimal assignment maximizes output. Beyond the static assignment, there are two additional features in problem *P2*—entry and investment—that need to be addressed. In the appendix, we show that the value of creating a new firm as well as the value of a new unit of capital to the planner coincides with the private value. Thus, zero profits for new entrants, and firm optimality with respect to θ are sufficient to ensure that the allocation is dynamically efficient.

3.2 Equilibrium characterization with preference shocks

For $\sigma > 0$, we leverage Choo and Siow (2006) and Galichon et al. (2019) framework to characterize the matching patterns and the terms of trade. Like with $\sigma = 0$ limit, the analysis proceeds in two steps.

Characterizing prices and allocations given (ϕ, V). As before, the first step involves recovering the allocations and prices. Recall that $v(s, \tilde{s})$ is the value after trade for firm type s with trading partner $\tilde{s}$ and given by

$$v\left(s, \tilde{s}\right) = V(z(s), k^m(s, \tilde{s})) - p^m(s, \tilde{s}). \tag{22}$$

When preference shocks ϵ are extreme value type 1 distributed, the choice probabilities λ have a familiar expression

$$\exp\left(\frac{v\left(s, \tilde{s}\right) - V\left(s\right)}{\sigma}\right) = \frac{\lambda\left(s, d\tilde{s}\right)}{\lambda_o\left(s\right)}. \tag{23}$$

Given (ϕ, V), the optimal assignment Λ, Λ_o is solution to following set of equations

$$\sigma \ln \frac{\Lambda\left(ds, d\tilde{s}\right)}{\Lambda_o\left(ds\right)} + \sigma \ln \frac{\Lambda\left(d\tilde{s}, ds\right)}{\Lambda_o\left(d\tilde{s}\right)} = V\left(z\left(s\right), k^m\left(s, \tilde{s}\right)\right) + V\left(z\left(\tilde{s}\right), k^m\left(\tilde{s}, s\right)\right) - V\left(s\right) - V\left(\tilde{s}\right)$$

$$\int \Lambda\left(ds, d\tilde{s} \in \mathcal{S}\right) + \Lambda_o\left(ds\right) = \phi\left(ds\right).$$

The allocation of capital given pair $(s, \tilde{s})$ is still given by (16), and payments $p^m\left(s, \tilde{s}\right)$ can be backed out from equation (22) and (23). This completes the counterpart of the first step from Section 3.1.

Characterizing (ϕ, V) given $(p^m, k^m, \Lambda, \Lambda_o)$. In the second step, we again use the outcomes of the first to update value functions and the invariant measure. As before, we can obtain a succinct expression for the net gains from trade and use that to update the value function V. In the appendix, we show that

$$W\left(s\right) = V\left(s\right) + \sigma \ln\left(\frac{\phi\left(s\right)}{\Lambda_o\left(s\right)}\right) \tag{24}$$

The ratio $\left(\frac{\Lambda_o(s)}{\phi(s)}\right)$ is the relative fraction of type s agents who do not trade and is larger when when gains from trade are smaller. Expression (24) and the Bellman equation (5) update

V. The update for ϕ is also same as before and uses the mapping Γ defined in equation (8).

4 Results

In this section, we parameterize the model and use it as a laboratory to study patterns of trade in our environment with capital indivisibilities and bilateral trades. The results are compared to an ideal analogue with divisible capital and centralized markets. We then estimate business wealth and the impact of taxing this wealth when the business is transferred.

4.1 Model Parameters

In Table 1, we report our baseline parameter estimates. The values are chosen so that the model generates realistic growth in profits by business age, investment rates, and relative sizes for businesses that buy transferable capital versus those that sell.

The first row in Table 1 reports the returns to scale parameter in production (once external factors have been optimized and substituted out). The value we use is $\alpha = 0.5$, primarily to capture the increasing variance in profits over time. The discount rate β is set to 0.95 to be consistent with U.S. returns to capital. The death rate δ is set to 20 percent, which is intended to capture both the rate of business exit and the rate of depreciation of transferable capital. Next in the table is the investment cost function, which is chosen to be quadratic with a coefficient A of 10 necessary to generate a plausible doubling in firm size within 5 years and a tripling within 10 years. The implied probability θ for this parameterization is roughly 20 percent for an entrant and declining after that.

The final two rows in Table 1 are parameter values that govern the productivity processes. The first is the probability distribution for entrants. This is given by Zipf with tail parameter equal to 1.2. This choice generates a realistic distribution in entrant output if we assume, as we do, that all entrants start with only 1 unit of capital k. After that, the productivity process is autoregressive with a serial correlation coefficient equal to 0.9 and a standard deviation of 0.3.

The productivity processes along with choices related to investment and entry and exit generate plausible growth rates in profits over the lifecycle when the model is compared to observations on private businesses. More specifically, we find that roughly 4 percent of capital units are traded each period. The sellers receive prices in the range of 4 to 7 times their pre-trade incomes and the buyers tend to be more productive, with incomes that are roughly 2 to 4 times that of the seller.

Table 1. Baseline Parameters

Parameters	**Notation**	**Values**
Returns to scale	α	0.50
Discount rate	β	0.95
Death rate	δ	0.20
Investment cost, $C(\theta) = A\theta^\rho$	(A, ρ)	$(10, 2)$
Entrant distribution, Zipf(z)	tail	1.20
Productivity, AR(1) $z'\|z$	(ρ_z, σ_z)	$(0.9, 0.3)$

4.2 Firm Dynamics

Next, we explore the patterns of trade that this model generates by comparing characteristics of buyers and sellers and by comparing the results of the economy with indivisible capital and bilateral trades to one with divisible capital and centralized markets.

In Figure 1, we plot a "bubble" map showing the frequency of trades between buyers and sellers according to their levels of productivity. On the x-axis, we report the sellers' productivity levels, which range from 1 to 6 in our baseline parameterization. On the y-axis, we report the buyers' productivity levels, which range from 2 to 10. The first pattern to notice is that capital is moving up in a marginal product of capital sense, that is, from sellers with low productivity to buyers with high productivity. The second pattern is the frequency: most of the sales are conducted between sellers with productivity in the range of 1 to 2 selling to buyers that have a productivity equal to 6. These multi-unit sales move them close to their optimal size quickly. As an example, consider business transfers occurring between sellers with $z = 1.3$ and buyers with $z = 6$. In these exchanges, buyers with $k = 1$ but high productivity acquire businesses with three times the capital that were in the hands of sellers with lower productivity before the trade.

Because building businesses takes time, whether owners invest or purchase, the marginal products of capital are not equated across firms. To see how the equilibrium in our baseline model compares to a "first-best" alternative, we compute the capital

$$k^{FB}(s) \in \text{argmax} \int z(s)[k^{FB}(s)]^\alpha \phi(s) ds$$

subject to

$$\int \phi(s)k^{FB}(s)ds = \int \phi(s)k(s)ds,$$

which equates all marginal products, effectively assuming capital is divisible and rental markets exist. In Figure 2, we plot averages of the solution against averages in our equilibrium economy for each productivity level. The equilibrium average capital profile is much flatter than the first-best implying that marginal products of capital are much more dispersed in our baseline. If we ignore businesses with no capital, we find MPKs at the 95$^{\text{th}}$ percentile of the distribution that are roughly five times larger than those at the 5$^{\text{th}}$ percentile.

In addition to a significant dispersion in marginal products of capital, we find significant dispersion in per-unit prices when the businesses are sold. Figure 3 plots these prices as a function of the quantity of capital in the businesses. As the figure shows, the prices rise quickly between $k = 2$ and $k = 6$. The higher price for medium-sized businesses reflects time to build concerns: owners would like to jump quickly to their optimal size and in our baseline are willing to pay 50 percent more to do just that. The prices fall off somewhat after $k = 10$ because there are decreasing returns to scale and few buyers looking for such a large quantity of capital.

4.3 Measures of Business Wealth

Before turning to our results for capital gains taxation, it is worth discussing what wealth or change in wealth is actually being taxed.

Our model has clear counterparts for two common measures of business wealth. The first is a measure often reported in standard finance textbooks, namely, the present discounted value of owner dividends. This measure in our model is $V(s)$ and captures returns to both the transferable capital k and non-transferable capital z. The second common measure of wealth is one often reported in surveys of consumer finances, namely, the price an owner would receive if the business were sold today. This measure in our model is $\mathcal{P}(k(s))$ and is equal to the price of the transferable capital only.

In Table 2, we report statistics related to these measures for owners with different levels of productivity z in the baseline model. The first, which is reported in the second column, is the ratio of the current value of transferable capital to the total value—what we call the *transferable share*. The second, which is reported in the the third column, is the ratio of owner income to total value—what we call the *income yield*. A notable feature of both statistics is the heterogeneity in range. Transferable shares range from 33 to 54 percent and income yields range from 13 to 23 percent. Also notable is the degree of transferability of private business wealth. If we use the aggregate measures $\phi(s)$ for weighting, we estimate the average transferable share is 43 percent.

Figure 1. Predicted Pattern of Trade

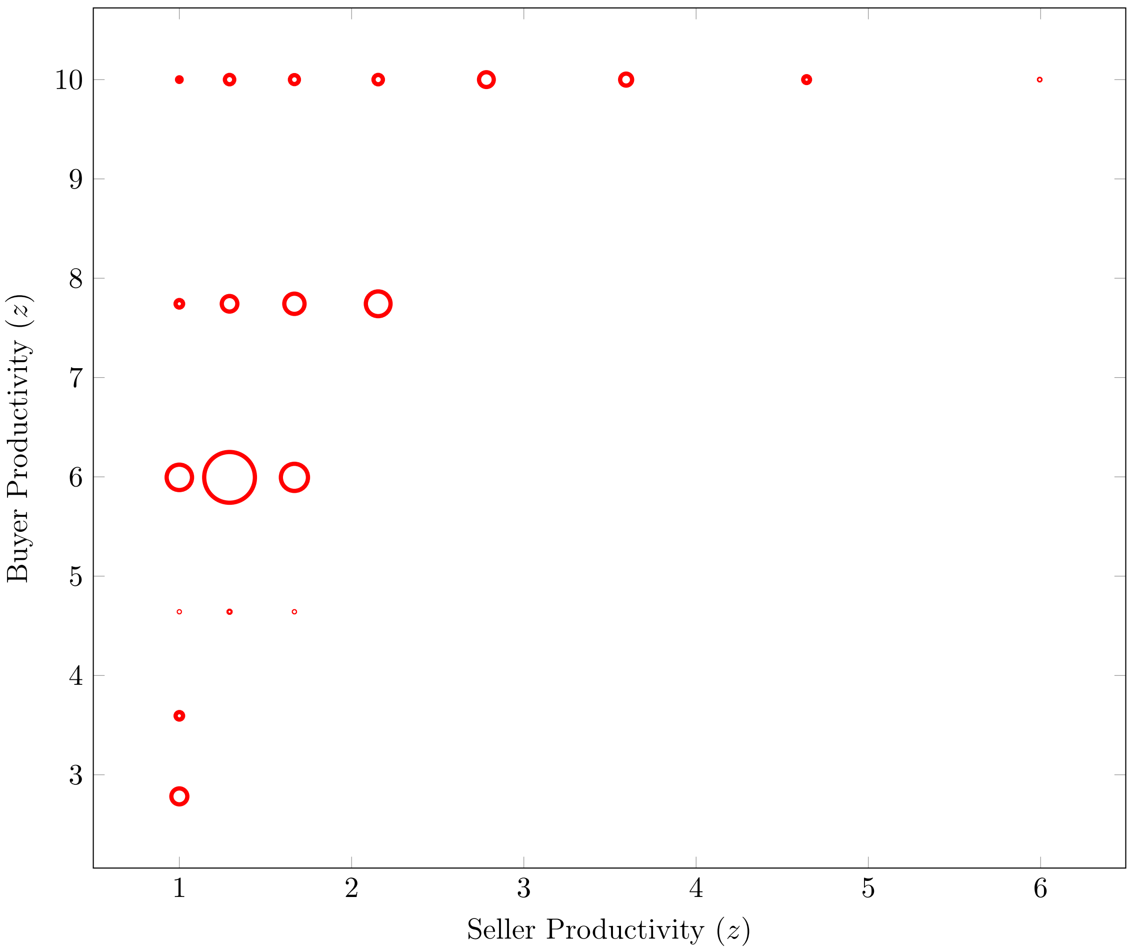


Figure 2. Predicted Dispersion in Allocations

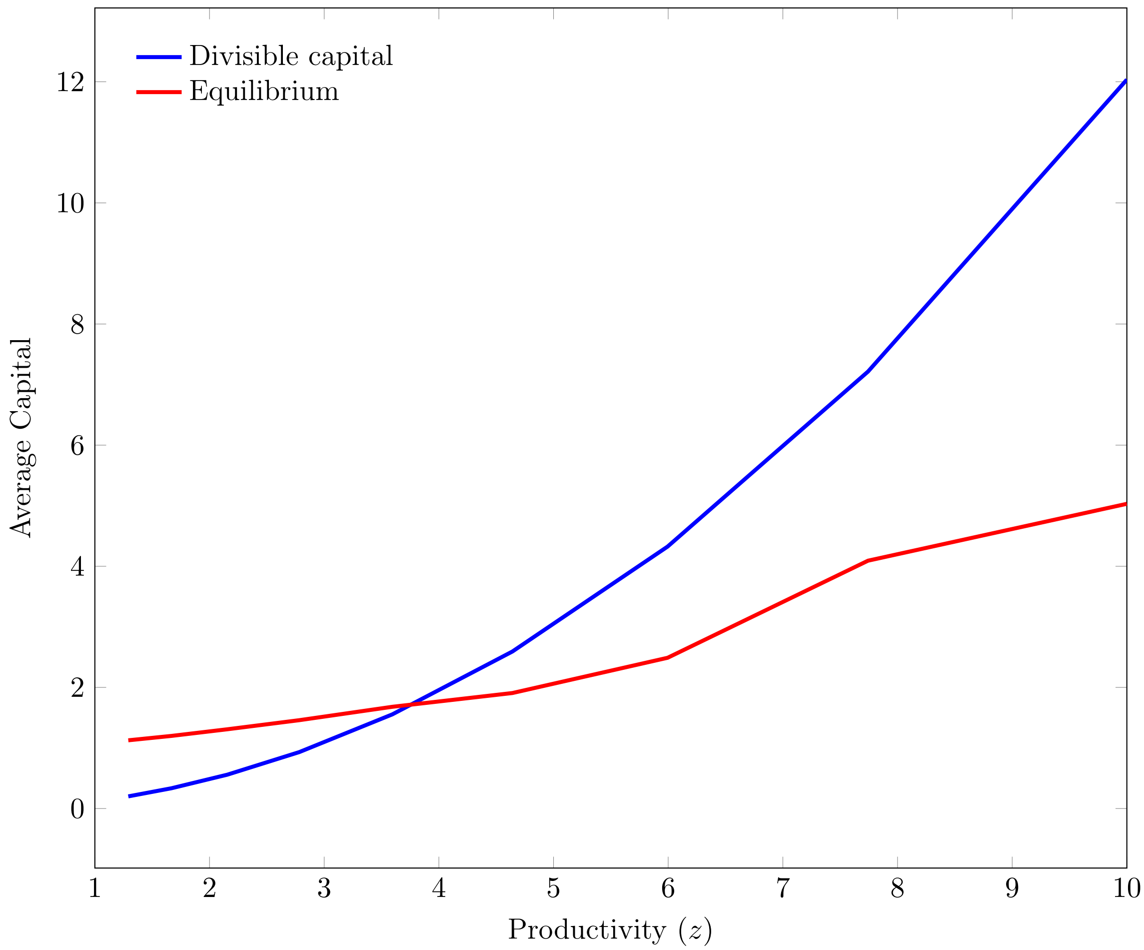


Figure 3. Predicted Dispersion in Prices

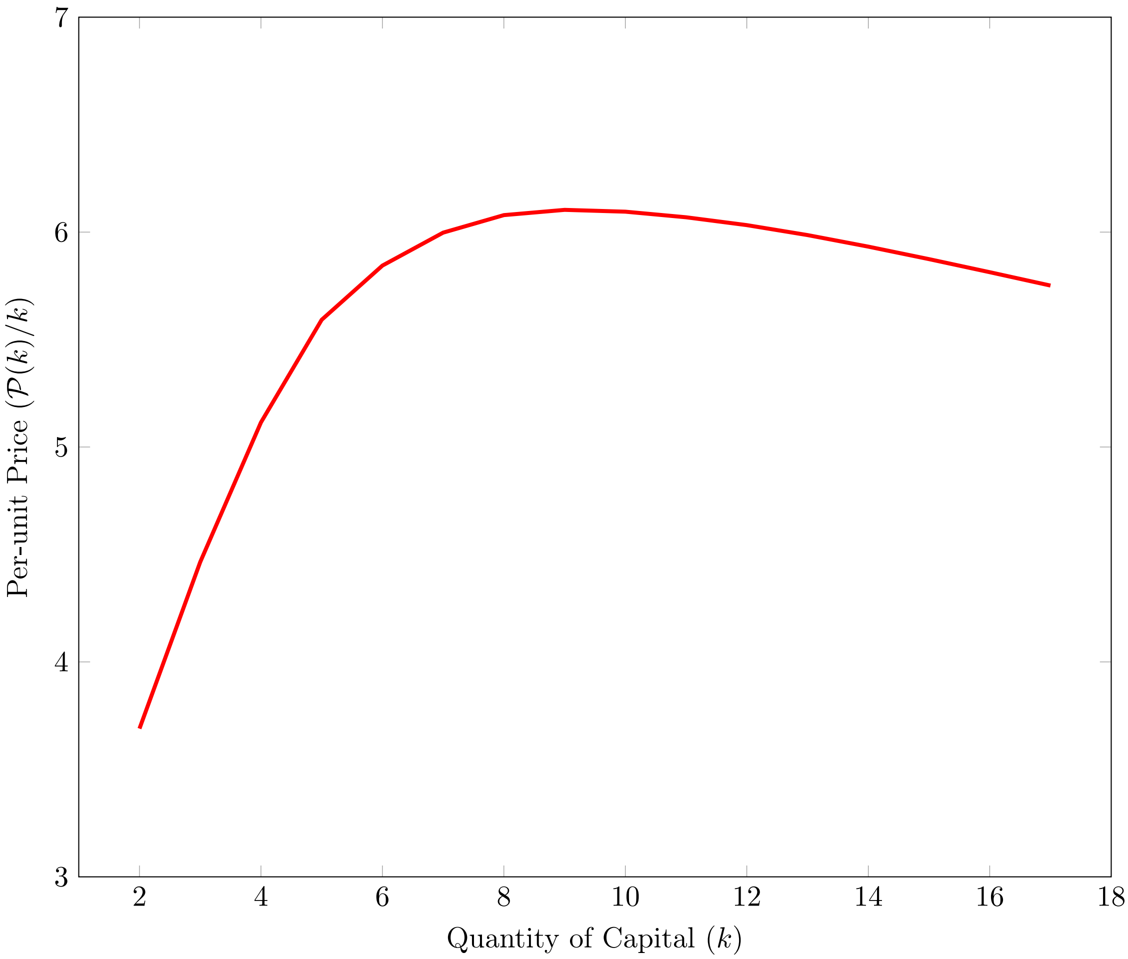


Table 2. Predicted Business Wealth

Productivity $z(s)$	**Transferable Share** $\frac{\mathcal{P}(k(s))}{V(s)}$	**Income Yield** $\frac{y(s)-C(\theta(s))}{V(s)}$
1.00	0.54	0.13
1.29	0.47	0.14
1.67	0.42	0.16
2.15	0.37	0.17
2.78	0.34	0.19
3.59	0.31	0.20
4.64	0.32	0.21
5.99	0.41	0.23
7.74	0.38	0.24
10.0	0.33	0.23
Average	0.43	0.17

We turn next to evaluating capital gains taxation, focusing in particular on tax incidence given the significant heterogeneity in our model's business population.

4.4 Capital Gains Taxation

Consistent with U.S. tax law, we introduce a tax τ on realized capital gains at the time of a business sale. This tax is assessed on the price paid to the seller, which is the same as the gain if the basis is zero.

Introducing capital gains takes us out of the transferable utility framework. The gains from trade depend on which firm is a buyer and which firm is the seller in a match. Relative to the model of labor income taxation in Dupuy et al. (2020), the requirements of pair-wise stability need to be augmented to include deviations with respect to which side of the market–buying or selling–any firm would optimally want to be. In the appendix, we show how we address this issue by applying the preference shock formulation in Galichon et al. (2019) to our setting.

Results An obvious effect of the tax is fewer sales of businesses. Any trade in the no-tax economy with small gains—for example, trades between owners that are similar—will be eliminated in the economy with positive τ. Without these marginal trades, we find a larger distance in marginal products of capital between buyers and sellers as shown in Table 3.

Table 3. Predicted Ratios of Buyer to Seller MPKs

Statistics	Tax Rate $\tau = 0$	$\tau = 23\%$
Mean	8.2	10.7
Standard deviation	1.8	1.7
Percentiles, 5^{th}	5.9	8.0
25^{th}	7.0	9.5
50^{th}	8.0	10.4
75^{th}	9.3	11.98
95^{th}	12	13.4

Statistics for the ratios of marginal products are reported for the two cases, without and with taxes on capital gains. Average and median ratios are on the order of 8 in the no-tax economy and close to 11 in the taxed economy.

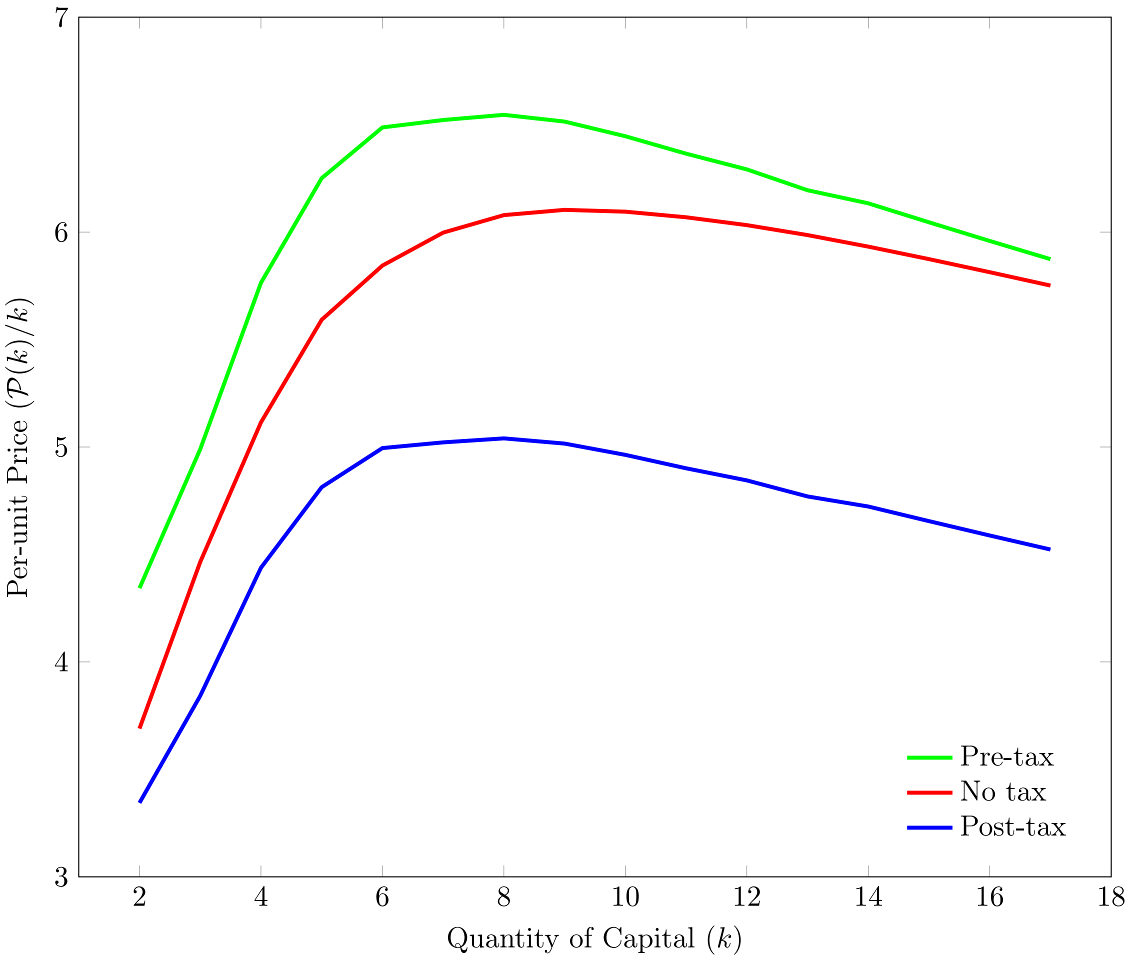
More interesting is the question of tax incidence. Figure 4 shows the price of transferred capital by quantity sold before and after the tax is assessed—which can be contrasted with the no-tax price discussed earlier. Although legal incidence is on the seller, buyers bear a significant brunt of the tax for sales of small- and medium-sized businesses. These smaller-sized businesses are highly valued by productive owners as a means of quickly attaining optimal size. For sellers with large businesses, thin markets translate into higher tax incidence with business sales.

5 Conclusion

Theory has been developed to study the reallocation of capital through business sales. The capital we modeled is neither divisible nor typically sold in centralized markets, but constitutes most capital transferred in private business sales in the United States.

In order to keep the mathematics and numerics as transparent as possible, we made certain assumptions that can be relaxed in future work. We used quasi-linear preferences to exploit tools from the matching literature and prove efficiency. We assumed capital is indivisible but otherwise homogeneous for tractability. These choices, among others, must ultimately be disciplined by the data.

Figure 4. Predicted Impact of Capital Gains Tax



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Appendix

Proof of Theorem 2

We prove efficiency under an assumption that productivity space is discrete. This implies a discrete space of agent types, $\mathcal{S} = \{s_1, ..., s_N\}$. We do so to keep notation simple, but the result extends naturally to a continuum of types. Accordingly, let $g(s)$ be the probability mass function of entrants of type s. Given ϕ_0, consider a planner that solves the following maximization problem.

$$P_t(\phi_0) = \max_{\{\lambda_t, \lambda_{o,t}, \theta_t, k_t^m, m_t\}} \sum_{t=0}^{\infty} \beta^t \left\{ \Sigma_{s \in S} \left[y(s) - c(\theta_t(s)) \right] \phi_t(s) - c_e m_t \right\}$$

subject to

$$\phi_{t+1}(s) = \Gamma(s, \phi_t; \lambda_t, \lambda_{o,t}, \theta_t, \phi_e, k_t^m) \quad \forall s \in \mathcal{S},$$

feasibility of k^m and $\phi_e(s, m) = m g(s)$ for all $s \in \mathcal{S}$.

Set-up

The recursive formulation of the planner's problem is

$$\begin{aligned} P_t(\phi_t) = & \max_{\{\Lambda_t, \Lambda_{o,t}, \theta_t, k_t^m, m_t\}} \quad \Sigma_s \left[y(s) - c(\theta_t(s)) \right] \phi_t(s) - c_e m_t + \beta P_{t+1}(\phi_{t+1}) \\ s.t. \quad & \phi_{t+1} = \Gamma(\phi_t; \Lambda_t, \Lambda_{o,t}, \theta_t, \phi_e(m_t), k_t^m). \end{aligned}$$

The arguments of the functions Γ_x, for $x = \{\Lambda, \theta, \delta\}$, are the same as in the main text so we drop them whenever there is no ambiguity. The first order condition with respect to investment and entry is

$$\begin{aligned} c'(\theta_t(s)) \phi_t(s) = \beta \Sigma_{\hat{s}} \frac{\partial P_{t+1}(\phi_{t+1})}{\partial \phi_{t+1}(\hat{s})} \frac{\partial \Gamma(\hat{s})}{\partial \theta_t(s)} \\ c_e = \Sigma_{\hat{s}} \frac{\partial P_{t+1}(\phi_{t+1})}{\partial \phi_{t+1}(\hat{s})} \frac{\partial \Gamma_\lambda(\hat{s})}{\partial \phi_{\delta,t}(s)} \frac{\partial \Gamma_\delta(\hat{s})}{\partial \phi_{e,t}(s)} \frac{\partial \phi_{e,t}(s)}{\partial m_t}. \end{aligned}$$

By the envelope theorem,

$$\frac{\partial P_t(\phi_t)}{\partial \phi_t(s)} = y(s) - c(\theta_t(s)) + \beta \Sigma_{\hat{s}} \frac{\partial P_{t+1}(\phi_{t+1})}{\partial \phi_{t+1}(\hat{s})} \frac{\partial \Gamma(\hat{s})}{\partial \phi_t(s)}.$$

We define the marginal value to the planner of an additional agents of type s at time t,

$$\tilde{V}_t(s;\phi_t) = \frac{\partial P_t(\phi_t)}{\partial \phi_t(s)}.$$

We also define the marginal value along the optimal trajectory

$$V_t(s) = \tilde{V}_t(s;\phi_t(\cdot)),$$

where $\phi_t(\cdot)$ is the distribution at time t at the optimum. We can formulate the envelope condition above as

$$V_t(s) = y(s) - c(\theta_t(s)) + \beta\Sigma_{\hat{s}} V_{t+1}(\hat{s}) \frac{\partial \Gamma(\hat{s})}{\partial \phi_t(s)}.$$

We focus on a stationary planner's problem, which allows us to drop the time subscript from the problem above,

$$V(s) = y(s) - c(\theta(s)) + \beta\Sigma_{\hat{s}} V(\hat{s}) \frac{\partial \Gamma(\hat{s})}{\partial \phi(s)} \tag{25}$$

where

$$\begin{aligned}
\Sigma_{\hat{s}} V(\hat{s}) \frac{\partial \Gamma(\hat{s})}{\partial \phi(s)} &= \Sigma_{\hat{s}} V(\hat{s}) \Sigma_{\tilde{z}} \frac{\partial \Gamma_{\lambda^*}(\hat{s})}{\partial \phi_\delta(\tilde{z}, k+1)} \frac{\partial \Gamma_\delta(\tilde{z}, k+1)}{\partial \phi_\theta(z, k+1)} \frac{\partial \Gamma_\theta(z, k+1)}{\partial \phi(s)} \\
&+ \Sigma_{\hat{s}} V(\hat{s}) \Sigma_{\tilde{z}} \frac{\partial \Gamma_{\lambda^*}(\hat{s})}{\partial \phi_\delta(\tilde{z}, k+1)} \frac{\partial \Gamma_\delta(\tilde{z}, k+1)}{\partial \phi_\theta(z, k)} \frac{\partial \Gamma_\theta(z, k)}{\partial \phi(s)} \\
&= (1-\delta) \Sigma_{\tilde{z}} T_z(\tilde{z}|z) \left[\theta(s) \Sigma_{\hat{s}} V(\hat{s}) \frac{\partial \Gamma_{\lambda^*}(\hat{s})}{\partial \phi_\delta(\tilde{z}, k+1)} + (1-\theta(s)) \Sigma_{\hat{s}} V(\hat{s}) \frac{\partial \Gamma_{\lambda^*}(\hat{s})}{\partial \phi_\delta(\tilde{z}, k)}\right]
\end{aligned}$$

and $\frac{\partial \Gamma_{\lambda^*}(\hat{s})}{\partial \phi_\delta(\tilde{z}, k)}$ is shorthand to indicate that the function Γ_λ is evaluated at the optimal (λ, λ_0). The FOCs with respect to investment and entry become

$$c'(\theta_t(s)) \phi_t(s) = \beta\Sigma_{\hat{s}} \frac{\partial P_{t+1}(\phi_{t+1})}{\partial \phi_{t+1}(\hat{s})} \frac{\partial \Gamma(\hat{s})}{\partial \theta_t(s)} = \beta\Sigma_{\hat{s}} V(\hat{s}) \frac{\partial \Gamma(\hat{s})}{\partial \theta(s)} \tag{26}$$

where

$$
\begin{aligned}
\Sigma_{\hat{s}} V(\hat{s}) \frac{\partial \Gamma(\hat{s})}{\partial \theta(s)} &= \Sigma_{\hat{s}} \frac{\partial P_{t+1}(\phi_{t+1})}{\partial \phi_{t+1}(\hat{s})} \frac{\partial \Gamma(\hat{s})}{\partial \theta_t(s)} = \beta \Sigma_{\hat{s}} V(\hat{s}) \frac{\partial \Gamma(\hat{s})}{\partial \theta(s)} \\
&= \Sigma_{\hat{s}} V(\hat{s}) \Sigma_{\tilde{z}} \frac{\partial \Gamma_{\lambda^*}(\hat{s})}{\partial \phi_\delta(\tilde{z}, k+1)} \frac{\partial \Gamma_\delta(\tilde{z}, k+1)}{\partial \phi_\theta(z, k+1)} \frac{\partial \Gamma_\theta(z, k+1)}{\partial \theta(s)} \\
&+ \Sigma_{\hat{s}} V(\hat{s}) \Sigma_{\tilde{z}} \frac{\partial \Gamma_{\lambda^*}(\hat{s})}{\partial \phi_\delta(\tilde{z}, k)} \frac{\partial \Gamma_\delta(\tilde{z}, k)}{\partial \phi_\theta(z, k)} \frac{\partial \Gamma_\theta(z, k)}{\partial \theta(s)} \\
&= (1-\delta) \Sigma_{\tilde{z}} T_z(\tilde{z}|z) \left(\Sigma_{\hat{s}} V(\hat{s}) \frac{\partial \Gamma_{\lambda^*}(\hat{s})}{\partial \phi_\delta(\tilde{z}, k+1)} - \Sigma_{\hat{s}} V(\hat{s}) \frac{\partial \Gamma_{\lambda^*}(\hat{s})}{\partial \phi_\delta(\tilde{z}, k)} \right) \phi(s)
\end{aligned}
$$

and

$$
c_e = \Sigma_s \left[\Sigma_{\hat{s}} V(\hat{s}) \frac{\partial \Gamma_{\lambda^*}(\hat{s})}{\partial \phi_\delta(s)} \right] g(s) . \tag{27}
$$

In equations (25), (26), and (27), solving the planner's problem requires solving for $\Sigma_{\hat{s}} V(\hat{s}) \frac{\partial \Gamma_{\lambda^*}(\hat{s})}{\partial \phi_\delta(s)}$ for each s. Next, we turn to a matching problem in which the object of interest has a convenient interpretation as a multiplier to a static resource constraint.

Optimal Matching

We set up the following linear problem

$$
\begin{aligned}
&\max_{\lambda \geq 0, \lambda_0 \geq 0, k^m} \Sigma_s V(s) \Gamma_\lambda(s, \phi; \lambda, \lambda_0, k^m) \\
s.t. \quad & \Sigma_{\tilde{s}} \lambda(s, \tilde{s}) + \lambda_0(s) = 1 \quad \forall s \\
& \Sigma_s \lambda(s, \tilde{s}) \phi(s) + \lambda_0(\tilde{s}) \phi(\tilde{s}) = \phi(\tilde{s}) \quad \forall \tilde{s}
\end{aligned}
$$

To make progress, we re-arrange the objective function using the definition of Γ_λ,

$$
\begin{aligned}
& \Sigma_s V(s) \left[\lambda_0(s) \phi(s) + \Sigma_{s',s''} \lambda(s', s'') \mathbb{I}\left\{ k^m(s', s'') = k(s), z(s') = z(s) \right\} \phi(s') \right] \\
=& \Sigma_s V(s) \left[\lambda_0(s) \phi(s) + \Sigma_{s',s''} \left(\frac{\lambda(s', s'')}{2} \mathbb{I}\left\{ k^m(s', s'') = k(s), z(s') = z(s) \right\} \phi(s') \right. \right. \\
+& \left. \left. \frac{\lambda(s'', s')}{2} \mathbb{I}\left\{ k^m(s', s'') = k(s), z(s'') = z(s) \right\} \phi(s'') \right) \right] \\
=& \Sigma_s V(s) \left[\lambda_0(s) \phi(s) + \Sigma_{s',s''} \left(\frac{\lambda(s', s'')}{2} \phi(s') \Sigma_s V(s) \mathbb{I}\left\{ k^m(s', s'') = k(s), z(s') = z(s) \right\} \right. \right. \\
+& \left. \left. \frac{\lambda(s'', s')}{2} \phi(s'') \Sigma_s V(s) \mathbb{I}\left\{ k^m(s', s'') = k(s), z(s'') = z(s) \right\} \right) \right]
\end{aligned}
$$

Imposing feasibility of k^m amounts to restricting the indicators above to be such that either s' is a buyer, or s'' is, or neither. The choice of k^m is equivalent to solving the problem

$$X\left(s',s''\right)=\max\left\{V\left(z',k'+k''\right)+V\left(z'',0\right),V\left(s'\right)+V\left(s''\right),V\left(z',0\right)+V\left(z'',k'+k''\right)\right\}.$$

The objective function thus simplifies to

$$\Sigma_s V\left(s\right)\lambda_0\left(s\right)\phi\left(s\right)+\Sigma_{s',s''}\frac{\lambda\left(s',s''\right)}{2}\phi\left(s'\right)X\left(s',s''\right)$$

Let $\pi\left(s,\tilde{s}\right)=\frac{\lambda(s,\tilde{s})}{2}\phi\left(s\right)$ and $\pi_0\left(s\right)=\frac{\lambda_0(s)}{2}\phi\left(s\right)$.

We label the value to the matching problem as Q.

$$\begin{aligned}Q\left(\phi\right)=\max_{\pi\geq 0,\pi_0\geq 0}&\Sigma_{s,\tilde{s}}\pi\left(s,\tilde{s}\right)X\left(s,\tilde{s}\right)+\Sigma_s V\left(s\right)\pi_0\left(s\right)+\Sigma_{\tilde{s}}V\left(\tilde{s}\right)\pi_0\left(\tilde{s}\right)\\ s.t.\Sigma_{\tilde{s}}\pi\left(s,\tilde{s}\right)+\pi_0\left(s\right)&=\frac{\phi\left(s\right)}{2}\\ s.t.\Sigma_{s}\pi\left(s,\tilde{s}\right)+\pi_0\left(\tilde{s}\right)&=\frac{\phi\left(\tilde{s}\right)}{2}\end{aligned}\tag{28}$$

Notice that this formulation of the matching problem is analogous to the one in the competitive equilibrium. Let $\mu^a\left(s\right)$ and $\mu^b\left(s\right)$ be the multipliers attached to the constraints of (28). From the envelope theorem,

$$\frac{\partial Q}{\partial\phi\left(s\right)}=\frac{\mu^a\left(s\right)+\mu^b\left(s\right)}{2}$$

and by the symmetry of $X\left(\cdot,\cdot\right)$, $\mu^a\left(s\right)=\mu^b\left(s\right)=\mu\left(s\right)$. Since at the solution,

$$Q\left(\phi\right)=\Sigma_s V\left(s\right)\Gamma_\lambda\left(s,\phi;\lambda^*,\lambda_0^*,k^{m,*}\right),$$

is satisfied at all ϕ, we can differentiate both sides to obtain

$$\Sigma_{\hat{s}}V\left(\hat{s}\right)\frac{\partial\Gamma_{\lambda^*}\left(\hat{s}\right)}{\partial\phi\left(s\right)}=\mu\left(s\right).$$

Characterization

Combining terms,

$$\begin{aligned}V\left(s\right)&=y\left(s\right)-c\left(\theta\left(s\right)\right)+\beta\left(1-\delta\right)\Sigma_{\hat{z}}T_z\left(\hat{z}|z\right)\left[\mu\left(\hat{z},k\right)+\theta\left(s\right)\left(\mu\left(\hat{z},k+1\right)-\mu\left(\hat{z},k\right)\right)\right]\\ c'\left(\theta\left(s\right)\right)&=\beta\left(1-\delta\right)\Sigma_{\hat{z}}T_z\left(\hat{z}|z\right)\left[\mu\left(\hat{z},k+1\right)-\mu\left(\hat{z},k\right)\right]\\ c_e&=\Sigma_s\mu\left(s\right)g\left(s\right)\end{aligned}$$

where $\mu(s)$ is the multiplier attached to the constraints of (28). Notice that the Bellman equation, the optimality condition for θ, and the static matching problem are identical to those in the competitive equilibrium. It immediately follows that the competitive equilibrium solves the planner's problem and the equilibrium value and policy functions are the same as the planner's. Last, let ϕ^* be the stationary distribution associated with the planner's problem. The condition $\phi_0 = \phi^*$ guarantees that the economy is stationary.