Payoff Continuity in Games of Incomplete Information: An Equivalence Result *

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Abstract

Monderer and Samet (1996) and Kajii and Morris (1998) define notions of proximity for countable, common prior information structures that preserve equilibrium payoff continuity. Monderer and Samet (1996) fix a common prior and perturb lists of partitions, while Kajii and Morris (1998) fix a type space and perturb common priors. Due to these differences, the precise relationship between the two papers has remained an open question. We establish an equivalence between them by mapping pairs of partition lists to pairs of common priors, and vice-versa. The key condition of the mapping ensures that belief types are changed independently of payoff types in the Kajii and Morris (1998) perturbation.

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1 Introduction

A game of incomplete information consists of a set of players, a set of actions and payoff functions for each player, and an information structure. How does the set of equilibrium payoffs change as the information structure changes? Rubinstein (1989)'s Email Game illustrates a striking discontinuity; equilibrium payoffs under a common knowledge information structure may not be approximated under an information structure in which there are arbitrarily many, but finite, levels of mutual knowledge. As higher-order beliefs are in principle unobservable, the existence of the discontinuity casts doubt on the robustness of equilibrium predictions.

A natural approach to this problem is to identify what type of precision is required for a modeler to make robust predictions. In the context of countable, common prior games of incomplete information, Monderer and Samet (1996) and Kajii and Morris (1998) define topologies on information structures under which no equilibrium payoff discontinuities arise. While both topologies are based on the same notion of approximate common knowledge, that of common-p belief (Monderer and Samet (1989)), Monderer and Samet (1996) and Kajii and Morris (1998) model the proximity of incomplete information differently. Monderer and Samet (1996) fix a state space and a common prior over it, and consider the differences in beliefs induced by a change in partitions over the state space. Kajii and Morris (1998) fix parameter and type spaces, and consider the differences in beliefs induced by a change in the common prior over its product. Until now, the precise relationship between the two has remained an open question. By mapping pairs of partition lists to pairs of common priors, and vice-versa, we reconcile the two topologies. Consequently, we clarify how a modeler's assessment of the robustness of her model might depend on her specification of incomplete information.

We begin by fixing the following primitives: a countable state space S with a common prior; a countable set of payoff parameters Θ; and a *parameter function* ϕ mapping states to payoff parameters. Following Monderer and Samet (1996), we define their topology, hereafter the *MS topology*, on the space of all partition lists of S. To define Kajii and Morris (1998)'s topology, hereafter the *KM topology*, we consider the space of *type functions* mapping S to an ambient, countably infinite set of types T. Observing that a common prior $P \in \Delta(S)$, a parameter function $\phi : S \to \Theta$, and a type function $\tau : S \to T$ pin down a unique measure μ over an induced state space $\Theta \times T$, we then define the KM topology on the space of all type functions, placing conditions on the measures the type functions identify.

Given this setup, we define a *partition labeling* as a function from pairs of partition lists to pairs of type functions. We define two key properties of this labeling. The first, called *consistency*, ensures that the labeling agrees with the canonical mapping from partitions to types defined by Werlang and Tan (1992). In particular, a consistent partition labeling sends any partition list to a type function mapping any state contained in the same partition element to the same type. The second, called the *common support condition*, places restrictions on the pairs of type functions to which each pair of partition lists is mapped. Theorem 1 shows that if Π and Π' are close in the MS topology and mapped to a pair of type functions τ and τ' by a consistent labeling satisfying the common support condition, then τ and τ' are close in the KM topology.[1] Furthermore, as long as the elements of $\Theta \times T$ can be mapped back to individual states in S, Theorem 2 shows a converse holds.[2]

At first glance, it is surprising that both consistency and the common support condition are necessary to prove Theorem 1. Indeed, as τ-consistent type functions agree with the canonical mapping from partitions to types, any event that is common p-belief in some partition list is common p-belief under a type function consistent with that list (Lemma 1). Why, then, is consistency alone insufficient to prove Theorem 1? The reason relates to a breakdown in the equivalence of Kajii and Morris (1998)'s and Monderer and Samet (1996)'s payoff environments under their perturbations.

Monderer and Samet (1996)'s payoff functions depend on S, while Kajii and Morris (1998)'s depend on $\Theta \times T$. Fixing a single model of incomplete information, one may interpret S as a state space implicitly containing both Θ and T (Aumann (1987), for example, argues for this interpretation). Furthermore, we may think of each player's type (implicit in the

[1]By imposing another condition on partition labelings called *invariance*, we state the Theorems in terms of limiting partition lists. Proposition 1 shows that a partition labeling satisfying consistency, invariance, and the common support condition exists.

[2]Under this condition, called *invertibility*, the inverse of any consistent partition labeling coheres with the canonical mapping from types to partitions defined in Brandenburger and Dekel (1993). See Section 4.5.

Partition Model and explicit in the Type Model) as encoding both their psychological payoffs (i.e. their "payoff type") and their beliefs (i.e. their "belief type").[3] If we change partitions over S, however, each player's payoff type at each state remains the same, but their belief type may change. The common support condition is required so that belief types are changed independently of payoff types in the Kajii and Morris (1998) perturbation.[4]

We illustrate the role of the common support condition in the context of a familiar example (Section 2), before introducing our general notation (Section 3), the formal definitions of partition labelings and their properties (Section 4), and our main results (Section 5). We conclude with a brief discussion of our findings in the context of a conjecture left by Kajii and Morris (1998) at the end of their paper (Section 6).

2 An Example: Cournot Competition with Private Costs

Firm 1 and Firm 2 choose quantities $q_1 \in \mathbb{R}_+ := A_1$ and $q_2 \in \mathbb{R}_+ := A_2$. The market clearing price is $p(Q) = \max\{y - Q, 0\}$, where $Q = q_1 + q_2$ and $y > 0$. Each firm has a linear cost function $C_i(q_i) = c_i q_i$. The marginal cost, c_i, is either high, H, or low, L, and $y > H > L > 0$.

We model incomplete information over marginal costs. First, define a state space

$$S := \{s_{HL}, s_{HH}, s_{LH}, s_{LL}\},$$

where we interpret s_{ij} as a state in which Firm 1's marginal cost is i and Firm 2's marginal cost is j. Next, define a set of payoff-relevant parameters,

$$\Theta := \{\theta_{HL}, \theta_{HH}, \theta_{LH}, \theta_{LL}\},$$

along with a *parameter function* $\phi : S \to \Theta$ mapping states to parameters with the same subscripts, i.e. $\phi(s_{ij}) := \theta_{ij}$ for $i, j \in \{H, L\}$. Finally, fixing a small, positive number ϵ, define

[3] We borrow the terminology in quotations from Bergemann and Morris (2005).

[4] In the absence of the condition, two consistent type functions, τ and τ', may not map S to any common types in the payoff type space T. Consequently, the induced common priors over $\Theta \times T$ may have non-overlapping support. It follows that there exists a utility function depending on types under which equilibrium payoffs under the type function τ are not approximated by those under the type function τ'.

a common prior $P \in \Delta(S)$, where $P(s_{HH}) = P(s_{LL}) = 0.5 - \epsilon$ and $P(s_{HL}) = P(s_{LH}) = \epsilon$ so that it is unlikely for firms to have different marginal costs.

2.1 Partition Model to Type Model

Consider a situation in which each firm knows its own marginal cost, but not the marginal cost of the other firm. A Partition Model of this setting specifies a partition of S for each firm,

$$\Pi_1 = \{\{s_{HL}, s_{HH}\}, \{s_{LH}, s_{LL}\}\} \quad \Pi_2 = \{\{s_{LH}, s_{HH}\}, \{s_{HL}, s_{LL}\}\}. \tag{1}$$

A Type Model of this setting specifies a type space $T := T_1 \times T_2$, with types corresponding to partition elements, and a common prior over over parameters and types $\mu \in \Delta(\Theta \times T)$.

We seek to define a common prior over parameters and types that is "consistent" with the partitions in Equation 1. Defining $T_1 = T_2 := \{H, L\}$, consider the following *type function* from states to types,

$$\tau(s) := \begin{cases} (H,H) & \text{if } s = s_{HH} \\ (H,L) & \text{if } s = s_{HL} \\ (L,L) & \text{if } s = s_{LL} \\ (L,H) & \text{if } s = s_{LH} \end{cases}. \tag{2}$$

τ is Π-*consistent* in the following sense;

$$\tau_i(s) = \tau_i(s') \text{ if and only if } s, s' \in \pi \in \Pi_i, \tag{3}$$

where $\tau_i(s)$ is Firm i's type at state $s \in S$. Further, having fixed a prior over states P, a parameter function ϕ, and a type space T, τ identifies a unique probability measure over the parameter and type space,

$$\mu(\theta, t) = P(\{s \in S : \phi(s) = \theta \text{ and } \tau(s) = t\}) = \begin{cases} 0.5 - \epsilon & \text{if } (\theta, t) \in \{(\theta_{HH}, H, H), (\theta_{LL}, L, L)\} \\ \epsilon & \text{if } (\theta, t) \in \{(\theta_{HL}, H, L), (\theta_{LH}, L, H)\} \\ 0 & \text{otherwise.} \end{cases}$$

The probability measure μ coheres with the partitions in Equation 1; given its type, each firm believes the other firm has the same marginal cost as itself with conditional probability $1-2\epsilon$, the same belief it would have if it were to condition on the partition element corresponding to its type.

2.2 Perturbations Across Models

To compare perturbations of information structures across models of incomplete information, we map pairs of partition lists to pairs of type functions. Consider another situation in which Firm 2's marginal cost is public so that partitions are

$$\Pi_1' = \{\{s_{HL}\}, \{s_{HH}\}, \{s_{LH}\}, \{s_{LL}\}\} \quad \Pi_2' = \{\{s_{LH}, s_{HH}\}\{s_{HL}, s_{LL}\}\}. \tag{4}$$

Enlarging Firm 1's type space to include as many types as partition elements, i.e. $T_1 := \{H, H', L, L'\}$, the following type function is Π'-consistent,

$$\tau'(s) := \begin{cases} (H', H) & \text{if } s = s_{HH} \\ (H, L) & \text{if } s = s_{HL} \\ (L', L) & \text{if } s = s_{LL} \\ (L, H) & \text{if } s = s_{LH} \end{cases}. \tag{5}$$

It turns out that consistency of this sort is not enough to ensure that two nearby partition lists are mapped to two nearby type functions. It can be verified that Π and Π' are close in the MS topology, while the common priors identified by τ and τ' are *not* close in the KM topology.[5] The reason, as we next demonstrate, is that Kajii and Morris (1998) quantify over a different class of payoff functions than those in Monderer and Samet (1996).

2.3 Quantifiers over Payoff Functions

Monderer and Samet (1996) assume that payoff functions map states and actions to real numbers. As a consequence of their result, fixing any payoff functions $u_i : \Theta \times A \to \mathbb{R}$, for every equilibrium under the information structure Π there is an equilibrium under the information structure Π' in which ex-ante payoffs are close. Consider, for example, payoff

[5] See Examples 1 and 2.

functions

$$u_1(\theta_{HH}, q_1, q_2) = u_1(\theta_{HL}, q_1, q_2) = p(Q)Q - Hq_1$$

$$u_1(\theta_{LL}, q_1, q_2) = u_1(\theta_{LH}, q_1, q_2) = p(Q)Q - Lq_1,$$

and symmetrically for Firm 2. Then, at s_{HH} and s_{LL}, the states that occur with high probability, under both Π and Π', each firm chooses an equilibrium quantity that approximates their equilibrium quantity chosen under complete information. It follows that ex-ante payoffs are similar across the two partition lists.

In contrast, Kajii and Morris (1998) assume that payoff functions map parameters, types, and actions to real numbers. Consider payoff functions similar to those previously specified, but which depend on types,

$$u_1(\theta_{HH}, H, H, q_1, q_2) = u_1(\theta_{HH}, H', H, q_1, q_2) = u_1(\theta_{HL}, H, L, q_1, q_2) = p(Q)Q - Hq_1$$

$$u_1(\theta_{LL}, L, L, q_1, q_2) = u_1(\theta_{LH}, L, H, q_1, q_2) = p(Q)Q - Lq_1,$$

and symmetrically for Firm 2. But now, suppose Firm 1 perceives itself to be a monopolist when its type is L',

$$u_1(\theta_{LL}, L', L, q_1, q_2) = u_1(\theta_{LH}, L', H, q_1, q_2) = (a - q_1)q_1 - Lq_1.$$

Then, Firm 1 produces at the monopoly quantity when its type is L', causing Firm 2 to produce almost zero when its type is L. Since the type profile (L', L) is realized with high probability, it follows that ex-ante payoffs under τ do not approximate those under τ'.

2.4 The Common Support Condition

The previous counterexample is possible because, even though the conditional beliefs of Firm 1 under τ, when its type is H, are similar to those under τ', when its type is H', its payoffs are significantly different. After introducing the general notation for the paper, we define a property on mappings from partition lists to type functions called the *common support condition*, which requires types with similar beliefs to have the same payoffs. Indeed, given that Π is mapped to τ, any consistent labeling satisfying the common support condition sends (Π, Π) to $(\tau, \hat{\tau})$, where

$$\hat{\tau}(s) := \begin{cases} (H, H) & \text{if } s = s_{HH} \\ (H', L) & \text{if } s = s_{HL} \\ (L, L) & \text{if } s = s_{LL} \\ (L', H) & \text{if } s = s_{LH} \end{cases}. \tag{6}$$

We verify this labeling satisfies the common support condition in Section 4.2.

3 Preliminaries

Fix the following primitives: (i) a set of two or more players $\mathcal{N} := \{1, 2..., N\}$, (ii) a countable set of states S, (iii) a countable set of payoff parameters Θ, (iv) a parameter function mapping states to payoff parameters $\phi : S \to \Theta$, and (v) a full-support common prior $P \in \Delta(S)$.

3.1 The MS Topology on Partition Models

Denote by $\mathcal{P}$ the set of all partitions of S and a partition list by $\Pi := (\Pi_1, ..., \Pi_N) \in \mathcal{P}^N$. We define the Monderer and Samet (1996) topology on the set of partition lists $\mathcal{P}^N$.

Denote Player i's partition element containing a state $s \in S$ by $\Pi_i(s)$. At a state $s \in S$, Player i assigns probability $P(E|\Pi_i(s))$ to the event $E \subseteq S$. Player i **p-believes** an event $E \subseteq S$ at a state $s \in S$ if $P(E|\Pi_i(s)) \geq p$. Denote $B^p_{\Pi_i}(E)$ as the set of states at which i p-believes E under the partition Π_i. The set of states at which E is **mutual p-belief** is $B^p_\Pi(E) := \cap B^p_{\Pi_i}(E)$. The set of states at which E is **m-level mutual p-belief** is $(B^p_\Pi)^m(E)$, the m-th iteration of $B^p_\Pi(\cdot)$ over the set E. Finally, the set of states at which E is **common p-belief** is $C^p_\Pi(E) := \cap_{m \geq 1}(B^p_\Pi(E))^m$.

Define $I_{\Pi,\Pi'}(\epsilon)$ as the set of states at which the conditional symmetric difference between each player's partition elements containing that state is less than ϵ,

$$I_{\Pi,\Pi'}(\epsilon) := \bigcap_{i \in \mathcal{N}} \{s \in S : \max\{P(\Pi_i(s) \backslash \Pi'_i(s)|\Pi_i(s)), P(\Pi'_i(s) \backslash \Pi_i(s)|\Pi'_i(s))\} < \epsilon\}.$$

Define

$$d^{MS}(\Pi, \Pi') := \max\{d^{MS}_1(\Pi, \Pi'), d^{MS}_1(\Pi', \Pi)\},$$

where $d_1^{MS}(\Pi,\Pi')$ is an ex-ante measure of states for which the event $I_{\Pi,\Pi'}(\epsilon)$ is common $(1-\epsilon)$-belief,

$$d_1^{MS}(\Pi,\Pi') := \inf\{\epsilon | P(C_{\Pi'}^{1-\epsilon}(I_{\Pi,\Pi'}(\epsilon))) \geq 1-\epsilon\}.$$

While d^{MS} is not the metric defined in Monderer and Samet (1996) (indeed, it does not satisfy the triangle inequality), it follows from Theorem 5.2 in their paper that a sequence converges in their topology if and only if $d^{MS}(\Pi,\Pi_n) \to 0$. Hereafter, we call the topology generated by these convergent sequences the **MS topology**.

Example 1 (Cournot Partitions) *Re-visit the state space in the Cournot Model with private costs: $S = \{s_{HH}, s_{HL}, s_{LH}, s_{LL}\}$, $P(s_{HH}) = P(s_{LL}) = 0.5-\epsilon$, and $P(s_{HL}) = P(s_{LH}) = \epsilon$, where $\epsilon > 0$ is small. Consider the partition lists $\Pi := (\Pi_1,\Pi_2)$ and $\Pi' := (\Pi'_1,\Pi'_2)$ where, as defined in Equation 1 and 4,*

$$\Pi_1 = \{\{s_{HL}, s_{HH}\},\{s_{LH}, s_{LL}\}\} \qquad \Pi_2 = \{\{s_{LH}, s_{HH}\},\{s_{HL}, s_{LL}\}\}$$
$$\Pi'_1 = \{\{s_{HL}\},\{s_{HH}\},\{s_{LH}\},\{s_{LL}\}\} \qquad \Pi'_2 = \{\{s_{LH}, s_{HH}\}\{s_{HL}, s_{LL}\}\}.$$

We verify that $d^{MS}(\Pi,\Pi') \leq 2\epsilon$. First, observe that $\Pi_1(s_{HH})\backslash\Pi'_1(s_{HH}) = \{s_{HL}\}$ and $\Pi_1(s_{LL})\backslash\Pi'_1(s_{LL}) = \{s_{LH}\}$, while $\Pi_2(s_{HH})\backslash\Pi'_2(s_{HH}) = \Pi_2(s_{HH})\backslash\Pi'_2(s_{HH}) = \emptyset$. Since $P(s_{HH}) = P(s_{LL}) = \epsilon$,

$$\{s_{HH}, s_{LL}\} \subseteq I_{\Pi,\Pi'}(2\epsilon).$$

Next, observe that $B_{\Pi}^{1-2\epsilon}(\{s_{HH}, s_{LL}\}) = B_{\Pi'}^{1-2\epsilon}(\{s_{HH}, s_{LL}\}) = \{s_{HH}, s_{LL}\}$ so $\{s_{HH}, s_{LL}\}$ is a fixed point of the belief operators $B_{\Pi}^{p}(\cdot)$ and $B_{\Pi'}^{p}(\cdot)$. It follows that $\{s_{HH}, s_{LL}\} \subseteq C_{\Pi}^{1-2\epsilon}(I_{\Pi,\Pi'}(2\epsilon)) \cap C_{\Pi'}^{1-2\epsilon}(I_{\Pi,\Pi'}(2\epsilon))$. Finally,

$$P(C_{\Pi}^{1-2\epsilon}(I_{\Pi,\Pi'}(2\epsilon))) \geq P(\{s_{HH}, s_{LL}\}) = 1-2\epsilon,$$

and,

$$P(C_{\Pi'}^{1-2\epsilon}(I_{\Pi,\Pi'}(2\epsilon))) \geq P(\{s_{HH}, s_{LL}\}) = 1-2\epsilon.$$

3.2 The KM Topology on Type Models

Fix a countably infinite type space $T := T_1 \times ... \times T_N$, where T_i is a countable set of types for player $i \in \mathcal{N}$. Denote T^S as the set of *type functions* from S to T. Since each function

$\tau \in T^S$ induces a measure $\mu \in \Delta(\Theta \times T)$, we define the Kajii and Morris (1998) topology on the function space T^S, placing conditions on the measures they identify. For convenience, call the elements of $\Theta \times T$ **induced states**.

Denote $\mu(t_i)$ as the marginal distribution of μ on T_i evaluated at t_i. If $\mu(t_i) > 0$, the probability of an event $E \subseteq \Theta \times T$ conditional on Player i's type t_i is given by $\mu(E|t_i) := \sum_{(\theta,(t_i,t_{-i}))\in E} \mu(s, t_{-i}|t_i)$. Player i p**-believes** an event $E \subseteq \Theta \times T$ at an induced state $(\theta, t) \in \Theta \times T$ if $\mu(E|\tau_i(s)) \geq p$, where $\tau_i(s)$ is the i-th component of τ. Denote $B^p_{\mu_i}(E)$ as the set of all induced states at which i p-believes E. The set of induced states at which E is **mutual** p**-belief** is $B^p_\mu(E) := \underset{i\in\mathcal{N}}{\cap} B^p_{\mu_i}(E)$. The set of induced states at which E is m**-level mutual** p**-belief** is $(B^p_\mu(E))^m$, the m-th iteration of $B^p_\mu(\cdot)$ over the set E. The set of induced states at which E is **common** p**-belief** is $C^p_\mu(E) := \cap_{m\geq 1}(B^p_\mu(E))^m$.

Define $A_{\mu,\mu'}(\epsilon)$ as the set of induced states at which each player has conditional beliefs that differ by at most ϵ over any event given their type,

$$A_{\mu,\mu'}(\epsilon) := \{(\theta, t) \in \Theta \times T : \text{for all } i \in \mathcal{N},\ \mu(t_i) > 0, \mu'(t_i) > 0, \text{ and}$$

$$|\mu(E|t_i) - \mu'(E|t_i)| \leq \epsilon \text{ for all } E \subseteq \Theta \times T\}.$$

Define a function mapping pairs of common priors to real numbers:

$$d^{KM}(\mu, \mu') := \max\{d^{KM}_1(\mu, \mu'), d^{KM}_1(\mu', \mu), d^{KM}_0(\mu, \mu')\},$$

where $d^{KM}_1(\mu, \mu')$ is an ex-ante measure of states for which the event $A_{\mu,\mu'}(\epsilon)$ is common $(1-\epsilon)$-belief,

$$d^{KM}_1(\mu, \mu') := \inf\{\epsilon > 0 : \mu'(C^{1-\epsilon}_{\mu'}(A_{\mu,\mu'}(\epsilon)) \geq 1 - \epsilon\},$$

and $d^{KM}_0(\mu, \mu')$ is the distance between μ and μ' in the weak topology,

$$d^{KM}_0(\mu, \mu') := \underset{E\subseteq\Theta\times T}{sup} |\mu(E) - \mu'(E)|.$$

d^{KM} generates a topology on T^S by specifying which nets converge. A net[6] $\{\tau^k : k \in K\}$, where K is a set with partial order $\succ$, converges to τ if for any $\epsilon > 0$, there is a $\bar{k} \in K$

[6] In metric spaces, (or, more generally, in first-countable spaces) one can define a topology by specifying which sequences converge. Since Monderer and Samet (1996) prove their topology is metrizable, we thus define theirs by specifying convergent sequences. Since we have not proved that Kajii and Morris (1998)'s topology is metrizable, we maintain their original definition.

such that $k \succ \bar{k}$ implies that $d^{KM}(\mu^k, \mu) < \epsilon$, where μ^k is the unique probability measure identified by τ^k and μ is the unique probability measure identified by τ. Hereafter, we call the topology generated by these nets the **KM topology**.

Example 2 (Cournot Type Functions) *Consider the type function τ, defined in Equation 2, and the type function τ', defined in Equation 5. The common priors they identify are,*

$$\mu(\theta, t) = \begin{cases} 0.5-\epsilon & \text{if } \theta = \theta_{HH} \text{ and } t = (H,H) \\ 0.5-\epsilon & \text{if } \theta = \theta_{LL} \text{ and } t = (L,L) \\ \epsilon & \text{if } \theta = \theta_{HL} \text{ and } t = (H,L) \\ \epsilon & \text{if } \theta = \theta_{LH} \text{ and } t = (L,H) \\ 0 & \text{otherwise} \end{cases} \qquad \mu'(\theta, t) = \begin{cases} 0.5-\epsilon & \text{if } \theta = \theta_{HH} \text{ and } t = (H',H) \\ 0.5-\epsilon & \text{if } \theta = \theta_{LL} \text{ and } t = (L',L) \\ \epsilon & \text{if } \theta = \theta_{HL} \text{ and } t = (H,L) \\ \epsilon & \text{if } \theta = \theta_{LH} \text{ and } t = (L,H) \\ 0 & \text{otherwise.} \end{cases}$$

We show that $d^{KM}(\mu, \mu') = 1 - 2\epsilon$. Notice that the only type profiles in the common support of both μ and μ' are (H, L) and (L, H). Further,

$$\max_{E \subseteq \Theta \times T} |\mu(E|t_2 = L) - \mu'(E|t_2 = L)| = 1 - 2\epsilon,$$

where the maximum is attained setting $E = (\theta_{LL}, L, L) \in \Theta \times T$. Similarly,

$$\max_{E \subseteq \Theta \times T} |\mu(E|t_1 = H) - \mu'(E|t_1 = H)| = 1 - 2\epsilon,$$

where the maximum is attained setting $E = (\theta_{HH}, H, H) \in \Theta \times T$. Hence, $\{(\theta_{HL}, H, L)\} \subseteq A_{\mu,\mu'}(1 - 2\epsilon)$. By a symmetric argument,

$$\{(\theta_{HL}, H, L), (\theta_{LH}, L, H)\} = A_{\mu,\mu'}(1 - 2\epsilon).$$

Under μ, Firm 1 (Firm 2) believes (θ_{LH}, L, H) $((\theta_{HL}, H, L))$ occurs with probability 2ϵ, when its type is L, and believes (θ_{HL}, H, L) $((\theta_{LH}, L, H))$ occurs with probability 2ϵ, when its type is H. It is then easy to verify that (θ_{HL}, H, L) and (θ_{LH}, L, H) are common 2ϵ-belief at those states, i.e. $C^{2\epsilon}_{\mu}(A_{\mu,\mu'}(1 - 2\epsilon)) = \{(\theta_{HL}, H, L), (\theta_{LH}, L, H)\}$. While Firm 1 knows the state at (θ_{HL}, H, L) and (θ_{LH}, L, H) under μ', Firm 2 has the same beliefs as under μ. Hence, $C^{2\epsilon}_{\mu'}(A_{\mu,\mu'}(1 - 2\epsilon)) = \{(\theta_{HL}, H, L), (\theta_{LH}, L, H)\}$. It follows that,

$$\mu'(C^{2\epsilon}_{\mu'}(A_{\mu,\mu'}(1 - 2\epsilon))) = \mu(C^{2\epsilon}_{\mu}(A_{\mu,\mu'}(1 - 2\epsilon))) = 2\epsilon.$$

4 Labelings

A **partition labeling** is a function from pairs of partition lists to pairs of type functions $L : \mathcal{P}^N \times \mathcal{P}^N \to T^S \times T^S$. We first define and illustrate three properties of partition labelings–consistency, the common support condition, and invariance. We then define the converse labeling from type functions to partition lists. We conclude by discussing the relationship between our labelings and the canonical mappings defined in Werlang and Tan (1992) and Brandenburger and Dekel (1993).

4.1 Consistency

We define Π-consistency as in Section 2.

Definition 1 (Π-Consistency) *A type function τ is Π-**consistent** if,*

$$\tau_i(s) = \tau_i(s') \text{ if and only if } s, s' \in \pi \in \Pi_i.$$

A consistent partition labeling maps pairs of partition lists to pairs of consistent type functions.

Definition 2 (Consistency) *A partition labeling L is **consistent**, or is said to satisfy **COS**, if $L(\Pi, \Pi') = (\tau, \tau')$ implies τ is Π-consistent and τ' is Π'-consistent.*

Notice, consistency places no restrictions on the relationship between the type functions in the co-domain. Hence, as seen in Section 2, there may be many consistent partition labelings.

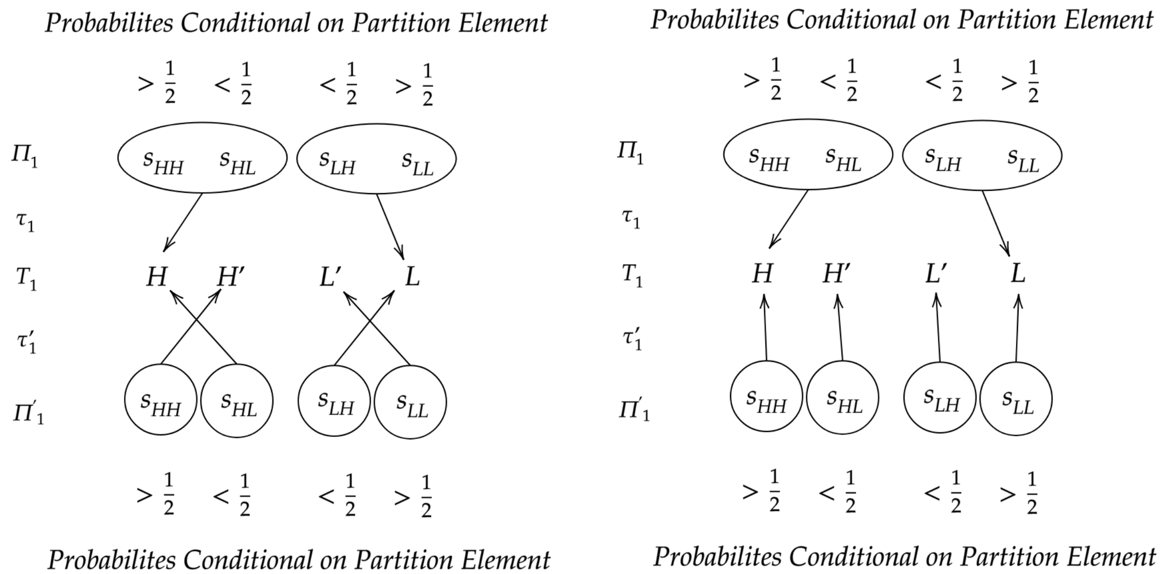
4.2 The Common Support Condition

The common support condition *does* place restrictions on the pairs of type functions in the co-domain.

Definition 3 (Common Support Condition) *A partition labeling L satisfies the **common support condition (CSC)** if $L(\Pi, \Pi') = (\tau, \tau')$ and $s \in I_{\Pi,\Pi'}(1/2)$ implies*

$$\tau(s) = \tau'(s).$$

Fixing partition lists Π and Π', any partition labeling satisfying the common support condition must send states to the same type if they are contained in similar enough partition elements across lists, as measured by the conditional symmetric difference.



(a) COS, but not CSC.

(b) COS and CSC.

Figure 1: Cournot Labelings at (Π, Π').

Figure 1 re-visits the two labelings defined at (Π, Π') in Section 2. While the labeling in Figure 1a does not violate consistency, it does violate the common support condition. To see why, notice that

$$I_{\Pi,\Pi'}(1/2) = \{s_{HH}, s_{LL}\},$$

because

$$P(\Pi_1(s_{HH}) \backslash \Pi'_1(s_{HH}) | \Pi_1(s_{HH})) = P(s_{HL}) < 1/2,$$

and

$$P(\Pi_1(s_{LL}) \backslash \Pi'_1(s_{LL}) | \Pi_1(s_{LL})) = P(s_{LH}) < 1/2.$$

Nevertheless, the type function τ does not coincide with τ' at either s_{HH} or s_{LL}; $\tau_1(s_{HH}) = H \neq H' = \tau'_1(s_{HH})$ and $\tau_1(s_{LL}) = L \neq L' = \tau'_1(s_{LL})$. On the other hand, the labeling in Figure 1b satisfies both consistency and the common support condition. This follows because states in $I_{\Pi,\Pi'}(1/2)$ are mapped to the same type ($\tau_1(s_{HH}) = H = \hat{\tau}_1(s_{HH})$ and $\tau_1(s_{LL}) = L = \hat{\tau}_1(s_{LL})$).

4.3 Invariance

Define $L_i(\Pi, \Pi')$ as the i-th component of the co-domain of L. Our final property, called invariance, requires partition lists in the first coordinate of the domain to be mapped to the same type function independently of the second coordinate.

Definition 4 (Invariance) *A partition labeling L is **invariant**, or is said to satisfy **INV**, if for all partition lists $\Pi \in \mathcal{P}^N$ there exists a type function $\tau \in T^S$ such that,*

$$L_1(\Pi, \Pi') = \tau \quad \textit{for any } \Pi' \in \mathcal{P}^N.$$

If a partition labeling L satisfies INV, $L_2(\Pi, \cdot)$ maps the entire space of partition lists $\mathcal{P}^N$ to a subset of the space of type functions T^S. If L also satisfies CSC, so that restrictions are made on pairs of type functions in the co-domain of L, then $L_2(\Pi, \cdot)$ may be regarded as the space of type functions relative to a fixed interpretation τ of a partition list Π. For example, this subset must contain τ, as depicted in Figure 2.

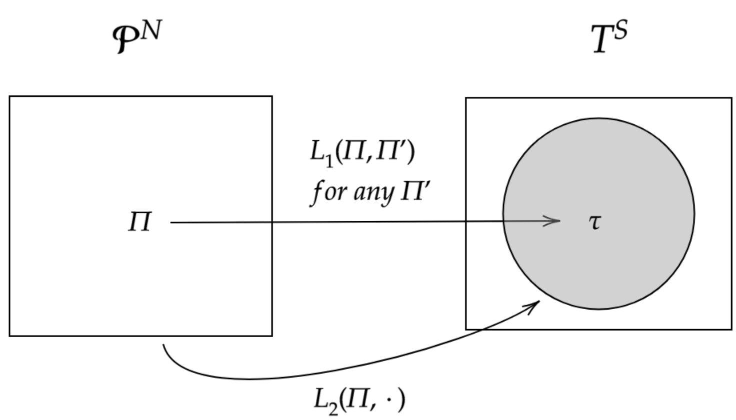


Figure 2: Illustration of INV when CSC is satisfied.

4.4 The Converse Labeling

Given any pair of type functions (τ, τ') we can identify the following pair of partition lists:

$$\Pi := (\Pi_1, ..., \Pi_N) \text{ where } \Pi_i := (\Pi_i(s))_{s \in S} \text{ and } \Pi_i(s) := \{s' \in S : \tau_i(s') = \tau_i(s)\}, \tag{7}$$

and,

$$\Pi' := (\Pi'_1, ..., \Pi'_N) \text{ where } \Pi'_i := (\Pi'_i(s))_{s \in S} \text{ and } \Pi'_i(s) := \{s' \in S : \tau'_i(s') = \tau'_i(s)\}. \tag{8}$$

Say that Π is **τ-consistent** if it is obtained from Equation 7 and that the pair of partition lists (Π, Π') is **consistent** with (τ, τ') if they are obtained from Equation 7 and 8.

4.5 Relationship to Canonical Mappings and Invertibility

Werlang and Tan (1992) define the canonical mapping from a single Partition Model to a single Type Model. In their construction, as in ours, each player's types are their partition elements. Their mapping identifies each player's entire hierarchy of beliefs at their partitions, i.e. their Universal types. In contrast, we construct a common prior over an ambient set of types, and derive higher-order beliefs from this common prior. Despite this difference, the common prior obtained from any Π-consistent type function τ agrees with the canonical mapping in the sense that each player's beliefs over *induced* states at their partition in the Partition Model coincide with their conditional beliefs at the types to which they are mapped to in the Type Model.

Brandenburger and Dekel (1993) define the canonical mapping from a single Type Model to a single Partition Model. Fixing a space of parameters and types $\Theta \times T$, they define partitions over a state space $S := \Theta \times T$. In our construction, distinct states in S may be mapped by the parameter function ϕ and type function τ to identical elements of $\Theta \times T$.[7] Hence, we may not be able to recover a unique partition of S.

We introduce a new property of type functions called invertibility to address this issue.

[7] To see the problems this causes, suppose that $S = \{s_1, s_2\}$, $P = 1/2 \circ s_1 + 1/2 \circ s_2$, $\phi(s_1) = \phi(s_2) = s$ and $\tau(s_1) = \tau(s_2) = t$. Consider the partition $\Pi = \{\{s_1, s_2\}\}$ and a Π-consistent Type Model τ. If $E = \{s_1\}$, then $\tilde{E}_\tau = \{(s, t)\}$ implying $P(E|\Pi(s_1)) = 1/2 \neq 1 = \mu(\tilde{E}|\tau(s_1))$, where μ is identified by τ.

Definition 5 (Invertibility) *Fixing a parameter function ϕ, a type function τ is **invertible** if $(\theta, t) = (\phi(s), \tau(s)) = (\phi(s'), \tau(s'))$ implies $s = s'$.*

Under invertibility, any τ-consistent partition list Π agrees with the one obtained under the canonical mapping. We do not view it as a strong condition for the following reason. Given two states s and s', suppose $(\phi(s), \tau(s)) = (\phi(s'), \tau(s'))$ and τ is Π-consistent. Then, s and s' could have been collapsed into a single state without affecting beliefs over induced states.

5 Main Results

In the Appendix, we construct a labeling satisfying COS, CSC, and INV. Consequently, we obtain the following proposition.

Proposition 1 *There exists a partition labeling satisfying COS, CSC, and INV.*

We now show that any labeling satisfying COS, CSC, and INV sends convergent sequences of partition lists in the MS topology to convergent sequences of type functions in the KM topology.

5.1 Theorem 1: Partition Lists to Type Functions

The set of states induced by an event $E \subseteq S$ is defined by,

$$(\tilde{E})_\tau := \{(\theta, t) \in \Theta \times T : \exists\ s \in E \text{ for which } (\theta, t) = (\phi(s), \tau(s))\}.$$

We first show that if $E \subseteq S$ is common p-belief under the partition list Π, then the set of induced states $(\tilde{E})_\tau$ is common p-belief under any Π-consistent type function τ.[8]

Lemma 1 *Suppose τ is Π-consistent and $E \subseteq S$. If $s \in C^p_\Pi(E)$, then $(\phi(s), \tau(s)) \in C^p_\mu((\tilde{E})_\tau)$, where μ is identified by τ.*

[8]The proof of the lemma is similar to the proof of Theorem 5.3 in Werlang and Tan (1992). Werlang and Tan (1992) show that common knowledge operations are preserved under the canonical mapping from Partition Models to Type Models. As Π-consistent type functions induce a Type Model that coincides with one obtained under the canonical mapping, the lemma may be viewed as a generalization of their theorem to the case of common p-belief.

Our next lemma shows that the restrictions placed on conditional beliefs at partitions in the MS topology bound the conditional beliefs of the types to which they are mapped. We require the common support condition so that partitions at which players have similar conditional beliefs are mapped to the same type.

Lemma 2 *Fix $0 < \epsilon < 1/2$. Suppose L satisfies COS and CSC. If $L(\Pi, \Pi') = (\tau, \tau')$, then $(\tilde{I}_{\Pi,\Pi'}(\epsilon))_\tau \subseteq A_{\mu,\mu'}(2\epsilon)$, where μ is identified by τ and μ' is identified by τ'.*

Using the two lemmas, we may then show that if two partition lists are close in the MS topology and are mapped to a pair of type fuctions by a labeling satisfying consistency and the common support condition, then the type functions are close in the KM topology. If the labeling also satisfies invariance, we can fix a limit partition list and the unique limit type function it identifies. As a corollary of the pairwise result, it follows that sequences of partition lists approaching the limit partition list are sent to sequences of type functions approaching the limit type function. We thus obtain Theorem 1, illustrated in Figure 3.

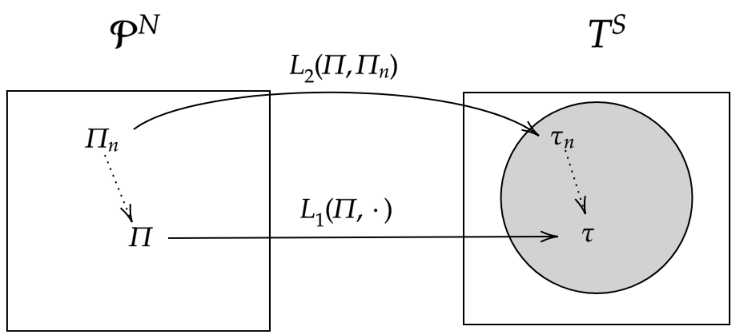


Figure 3: Illustration of Theorem 1.

Theorem 1 *Fix a partition labeling L satisfying COS, CSC, and INV. If a sequence of partition lists (Π_n) converges to Π in the MS topology and $L(\Pi, \Pi_n) = (\tau, \tau_n)$ for all n, then the sequence of type functions (τ_n) converges to τ in the KM topology.*

5.2 Theorem 2: Type Functions to Partition Lists

Given invertibility, the proof of the converse mirrors the proof of Theorem 1. For any event $E \subseteq \Theta \times T$, define the set of states sent to some element of E,

$$(\tilde{E})_\Pi := \{s \in S : \exists(\theta, t) \in E \text{ for which } (\phi(s), \tau(s)) = (\theta, t)\}.$$

We first show that if E is common p-belief under an invertible type function τ, then $(\tilde{E})_\Pi$ is common p-belief under a τ-consistent partition list.

Lemma 3 *If $(\theta, t) \in C_\mu^{1-\epsilon}(E)$ and τ is invertible, then $s \in C_\Pi^{1-\epsilon}((\tilde{E})_\Pi)$ if Π is τ-consistent.*

We next show that if conditional beliefs are close under τ and τ' at some induced state (θ, t), the state s mapped to (θ, t) must be contained in partitions having a small conditional symmetric difference.

Lemma 4 *Fix $0 < \epsilon < 1/2$ and suppose the partition lists (Π, Π') are consistent with invertible type functions (τ, τ'). Then, $(\tilde{A}_{\mu,\mu'}(\epsilon))_\Pi \subseteq I_{\Pi,\Pi'}(\epsilon)$, where μ is identified by τ and μ' is identified by τ'.*

Finally, fixing an invertible type function τ and a τ-consistent partition list Π, we prove that if a net of invertible type functions converges to τ and is sent to a consistent net of partition lists, then the net of partition lists must converge to Π.

Theorem 2 *Suppose an invertible net of type functions (τ_n) converges to an invertible type function τ in the KM topology. If (Π_n, Π) is consistent with (τ_n, τ) for all n, then the net of partition lists (Π_n) converges to Π in the MS topology.*

Consider a type function τ and a partition list $\Pi = L_1^{-1}(\tau, \cdot)$, where L satisfies COS, CSC, and INV. An immediate corollary of Theorem 2 is that any net of invertible type functions contained in the range of $L_2(\Pi, \cdot)$ converging to τ, must be sent by $L_2^{-1}(\tau, \cdot)$ to a net of partition lists converging to Π. Figure 4 illustrates.

6 Discussion

At the end of their paper, Kajii and Morris (1998) opine,

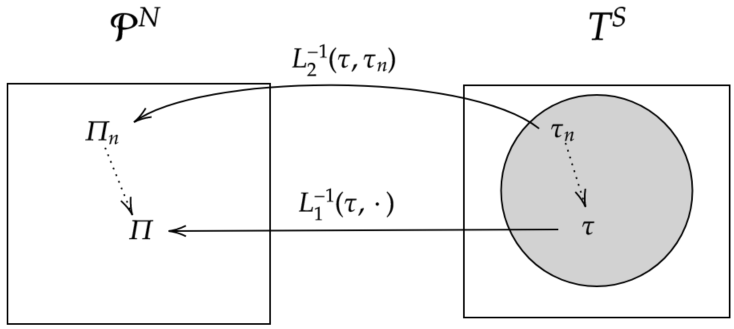


Figure 4: Illustration of Theorem 2.

> Our characterization of the proximity of information has a similar flavor to Monderer and Samet's, but we have not been able to establish a direct comparison. By considering a fixed type space, we exogenously determine which types in the information systems correspond to each other. In the Monderer and Samet approach, it is necessary to work out how to identify types in the two information systems. Thus we conjecture that two information systems are close in Monderer and Samet's sense if and only if the types in their construction can be labelled in such a way that the information systems are close in our sense.

We exhibit such a labeling and prove Kajii and Morris (1998)'s conjecture true. As illustrated in Section 2, however, the primary role of our labeling is to reconcile a fundamental difference in the payoff environments of the two papers. In particular, the common support condition ensures that belief types are changed independently of payoff types in the Kajii and Morris (1998) perturbation. Consequently, we suggest an amendment to the above quote to say, "it is necessary to work out how to identify *payoff types* across the two information systems". To avoid issues of translation in applications, we thus recommend separately specifying payoffs and beliefs, as is now common in the robust mechanism design literature (i.e. Bergemann and Morris (2005)).

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Proofs

Proposition 1

We construct an invariant, consistent partition labeling L satisfying the common support condition. Begin by indexing S by the natural numbers. Denote its elements $s_1, s_2, ...$[9]

[9]Indexing S is without loss of generality. We may think of S as an arbitrary countable set together with an injective function $f : S \to \mathbb{N}$. By the well-ordering principle, we may order the elements of S so that s_1

We first define $\tau = L_1(\Pi, \cdot)$ for all $\Pi \in \mathcal{P}^N$ so that L satisfies invariance. Given $\Pi := \Pi_1 \times ... \times \Pi_N$, order the partition elements of Π_i so that $\pi_i^1 \in \Pi_i$ contains s_1 and $\pi_i^k \in \Pi_i$ contains the smallest $s \in S \backslash (\bigcup_{j<k} \pi_i^j)$ for $k > 1$. Define $\tau_i(s) := t_i^j$ if $s \in \pi_i^j$. Notice, by construction, τ is consistent with Π.

Next, we define $L(\Pi, \Pi') = (\tau, \tau')$ for arbitrary partitions $\Pi, \Pi' \in \mathcal{P}^N$. τ is pinned down by the previous step so we need only choose τ'. Define $\tau'(s) := \tau(s)$ for all $s \in I_{\Pi,\Pi'}(1/2)$ so that L satisfies CSC.

If $s \in I_{\Pi,\Pi'}(1/2)$ and $s' \in \Pi_i(s)$, then either $\tau'(s') = \tau'(s)$ or $\tau'(s')$ has not yet been specified. In other words, two states in the same partition element cannot have been mapped to distinct types. Hence, if $s' \in \Pi_i'(s)$ for some $s \in I_{\Pi,\Pi'}(1/2)$, we may define $\tau_i'(s') := \tau_i(s)$ so that τ' is consistent with all partition elements containing a state in $I_{\Pi,\Pi'}(1/2)$. Denoting $\tilde{\Pi}_i := \{\pi \in \Pi_i \colon \pi \cap I_{\Pi,\Pi'}(1/2) \neq \emptyset\}$, this property follows as a consequence of the following claim.

Claim 1 *The mapping $b_i : \tilde{\Pi}_i \to \tilde{\Pi}_i'$ where $b_i(\Pi_i(s)) = \Pi_i'(s)$ is a bijection.*

Proof To prove b_i is one-to-one take two distinct partition elements $\pi_1, \pi_2 \in \tilde{\Pi}_i$. Suppose towards contradiction that $b_i(\pi_1) = b_i(\pi_2) := \pi' \in \tilde{\Pi}_i'$. Since $P(\pi' \backslash \pi_1 | \pi') + P(\pi' \cap \pi_1 | \pi') = 1$ and $\pi' \in \tilde{\Pi}_i'$ implies $P(\pi' \backslash \pi_1 | \pi') < 1/2$, $P(\pi' \cap \pi_1 | \pi') > 1/2$. Similarly, $P(\pi' \cap \pi_2 | \pi') > 1/2$. But then we have disjoint events $E_1 = \pi' \cap \pi_1$ and $E_2 = \pi' \cap \pi_2$ such that $P(E_1|\pi') + P(E_2|\pi') > 1$, a contradiction.

To prove b_i is onto take a partition element $\pi' \in \tilde{\Pi}_i'$. We claim $b_i(\Pi_i(s)) = \pi'$ for some $s \in \pi'$. By the definition of $\tilde{\Pi}_i'$, there exists an $s \in \pi' \cap I_{\Pi,\Pi'}$. If $s \in \pi' \cap I_{\Pi,\Pi'}$, then $s \in \Pi_i(s) \cap I_{\Pi,\Pi'}$ and so $b_i(\Pi_i(s)) = \pi'$. $\blacksquare$

Finally, for all i, we define τ_i' on the domain $S \backslash B_i$, where $B_i := \{s' \in S : s' \in \pi_i' \cap I_{\Pi,\Pi'}$ for some $\pi_i' \in \Pi_i\}$. If for all i, (i) $\tau_i'(s) = \tau_i'(s')$ if and only if $s, s' \in \pi_i' \subset S \backslash B_i$ and (ii) for every $s \in S \backslash B_i$, $\tau_i'(s) \neq \tau_i'(s')$ for any $s' \in B_i$, τ' is Π'-consistent. There is at least one, and may be many, functions satisfying these conditions.

is the element of S mapped to the smallest natural number in the range of f, s_2 is the element of S mapped to the second smallest natural number in the range of f, and so on.

Lemma 1

Suppose τ is Π-consistent and $E \subseteq S$. We show that, for any $m \geq 1$ and $i \in \mathcal{N}$, if $s \in (B^p_{\Pi_i})^m(E)$, then $(\phi(s), \tau(s)) \in (B^p_{\mu_i})^m((\tilde{E})_\tau)$. If this is true, then $s \in \cap_{m\geq 1}(B^p_\Pi)^m(E) = C^p_\Pi(E)$ implies $(\phi(s), \tau(s)) \in \cap_{m\geq 1}(B^p_\mu)^m((\tilde{E})_\tau) = C^p_\mu((\tilde{E})_\tau)$.

The proof is by induction. For the base case, take $s \in B^p_{\Pi_i}(E)$ and consider $(\phi(s), \tau(s))$. Then,

$$\mu((\tilde{E})_\tau | \tau_i(s)) \geq P(E|\Pi_i(s)) \geq p,$$

where the first inequality follows because μ is identified by τ and τ is Π-consistent, and the second because $s \in B^p_{\Pi_i}(E)$.[10]

The induction hypothesis is that if $s \in (B^p_{\Pi_i})^m(E)$, then $(\phi(s), \tau(s)) \in (B^p_{\mu_i})^m((\tilde{E})_\tau)$. Take $s \in (B^p_{\Pi_i})^{m+1}(E)$. Then,

$$P((B^p_\Pi)^m(E)|\Pi_i(s)) \geq p.$$

By the induction hypothesis, if $s \in (B^p_\Pi)^m(E)$, then $(\phi(s), \tau(s)) \in (B^p_\mu)^m((\tilde{E})_\tau)$. Since μ is identified by τ and τ is Π-consistent,

$$\mu((B^p_\Pi)^m((\tilde{E})_\tau)|\tau_i(s)) \geq P((B^p_\Pi)^m(E)|\Pi_i(s)) \geq p.$$

Lemma 2

Since $0 < \epsilon < 1/2$, by the common support condition, if $(\theta, t) \in (\tilde{I}_{\Pi,\Pi'}(\epsilon))_\tau$, then for some $s \in I_{\Pi,\Pi'}(\epsilon)$, $\tau(s) = \tau'(s) = t := (t_1, ..., t_N)$ and $\phi(s) = \theta$. Suppose, towards contradiction and without loss of generality, that $\mu(E|t_i) - \mu'(E|t_i) > 2\epsilon$ for some $E \subseteq \Theta \times T$ and some player i. By consistency, then, there must be a collection of states $F \subseteq \Pi_i(s)$ for which $(\tilde{F})_\tau \subseteq E$ and for which $P(F|\Pi_i(s)) - P(F|\Pi'_i(s)) > 2\epsilon$. We show that this cannot occur and hence $(\theta, t) \in A_{\mu,\mu'}(2\epsilon)$.

By the triangle inequality, for any $F \subseteq S$, $|P(F|\Pi_i(s)) - P(F|\Pi'_i(s))|$ is less than or equal to the sum of the following three terms:

[10] Without invertibility, the first inequality cannot be made an equality because distinct states may be mapped to the same induced state.

1. $|P((\Pi_i'(s)\backslash\Pi_i(s)) \cap F)|\Pi_i(s)) - P((\Pi_i(s)\backslash\Pi_i'(s)) \cap F)|\Pi_i'(s))|$.

2. $|P((\Pi_i(s)\backslash\Pi_i'(s)) \cap F|\Pi_i(s)) - P(\Pi_i'(s)\backslash\Pi_i(s)) \cap F|\Pi_i'(s))|$.

3. $|P((\Pi_i'(s) \cup \Pi_i(s)) \cap F|\Pi_i(s)) - P((\Pi_i'(s) \cup \Pi_i(s)) \cap F|\Pi_i'(s))|$.

The first term is zero because any states out of one's partition element are believed to have occurred with probability zero. The second term is less than ϵ since, by the definition of $I_{\Pi,\Pi'}(\epsilon)$,

$$\max\{P(\Pi_i(s)\backslash\Pi_i'(s)|\Pi_i(s)), P(\Pi_i'(s)\backslash\Pi_i(s)|\Pi_i'(s))\} \leq \epsilon.$$

We bound the third term as follows:

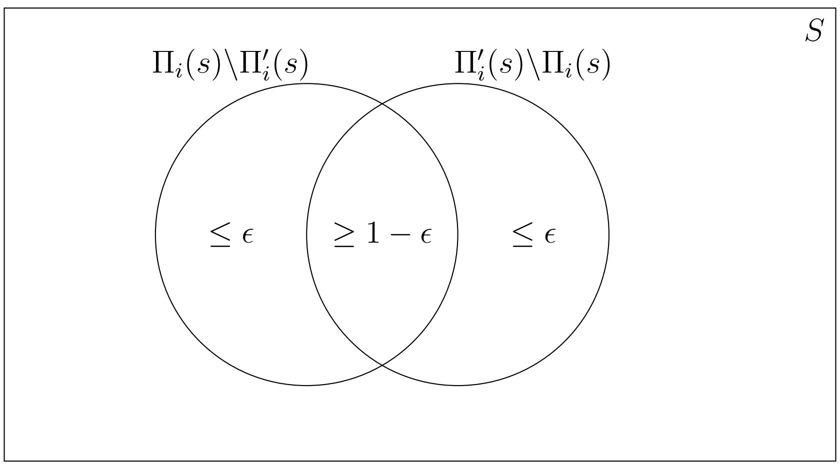

Figure 5: Visual Aid for the Proof of Lemma 2.

$$|P((\Pi_i'(s) \cap \Pi_i(s)) \cap F|\Pi_i(s)) - P((\Pi_i'(s) \cap \Pi_i(s)) \cap F|\Pi_i'(s))| =$$

$$|\frac{(P(\Pi_i(s)) - P(\Pi_i'(s)))P(\Pi_i(s) \cap \Pi_i'(s) \cap F)}{P(\Pi_i(s))P(\Pi_i'(s))}| \leq |\frac{(P(\Pi_i(s)) - P(\Pi_i'(s)))P(\Pi_i(s) \cap \Pi_i'(s))}{P(\Pi_i(s))P(\Pi_i'(s))}| =$$

$$|P(\Pi_i'(s) \cap \Pi_i(s)|\Pi_i(s)) - P(\Pi_i'(s) \cap \Pi_i(s)|\Pi_i'(s))|.$$

Since $P(\Pi_i(s)\backslash\Pi_i'(s)|\Pi_i(s)) + P(\Pi_i'(s) \cap \Pi_i(s)|\Pi_i(s)) = 1$ and $P(\Pi_i(s)\backslash\Pi_i'(s)|\Pi_i(s)) \leq \epsilon$, $P(\Pi_i'(s) \cap \Pi_i(s)|\Pi_i(s)) \geq 1-\epsilon$. Similarly, $P(\Pi_i'(s) \cap \Pi_i(s)|\Pi_i'(s)) \geq 1-\epsilon$. Hence, $|P(\Pi_i'(s) \cap \Pi_i(s)|\Pi_i(s)) - P(\Pi_i'(s) \cap \Pi_i(s)|\Pi_i'(s))| \leq \epsilon$.

Theorem 1

If (Π_n) converges to Π in the MS topology, then as $n \to \infty$, $d^{MS}(\Pi, \Pi_n) \to 0$. Suppose $L(\Pi, \Pi_n) = (\tau, \tau_n)$ for all n, where τ is the invariant function to which L maps Π. Fixing $0 < \epsilon < 1/2$, we show that if $L(\Pi, \Pi') = (\tau, \tau')$ and $d^{MS}(\Pi, \Pi') \leq \epsilon$, then $d^{KM}(\mu, \mu') \leq 2\epsilon$, where μ is identified by τ and μ' is identified by τ'. Hence, for the sequence of measures (μ_n) identified by the sequence of Type Models (τ_n), $d^{KM}(\mu, \mu_n) \to 0$. As any sequence is a net, the result follows.

We now prove that if $L(\Pi, \Pi') = (\tau, \tau')$ and $d^{MS}(\Pi, \Pi') \leq \epsilon$, then for μ identified by τ and μ' identified by τ', $d^{MS}(\mu, \mu') \leq 2\epsilon$. We first show that $\max\{d_1^{KM}(\mu', \mu), d_1^{KM}(\mu, \mu')\} < 2\epsilon$. If $d^{MS}(\Pi, \Pi') \leq \epsilon$, then $P(C_\Pi^{1-\epsilon}(I_{\Pi,\Pi'}(\epsilon))) \geq 1 - \epsilon$. We claim that if $s \in C_\Pi^{1-\epsilon}(I_{\Pi,\Pi'}(\epsilon))$, then $(\phi(s), \tau(s)) \in C_\mu^{1-\epsilon}(A_{\mu,\mu'}(2\epsilon))$. Take $s \in C_\Pi^{1-\epsilon}(I_{\Pi,\Pi'}(\epsilon))$. Then, by Lemma 1, $(\phi(s), \tau(s)) \in C_\mu^{1-\epsilon}((\tilde{I}_{\Pi,\Pi'}(\epsilon))_\tau)$. As L satisfies CSC, by Lemma 2, $(\tilde{I}_{\Pi,\Pi'}(\epsilon))_\tau \subseteq A_{\mu,\mu'}(2\epsilon)$ and hence $(\phi(s), \tau(s)) \in C_\mu^{1-\epsilon}((\tilde{I}_{\Pi,\Pi'}(\epsilon))_\tau) \subseteq C_\mu^{1-\epsilon}(A_{\mu,\mu'}(2\epsilon))$. It follows that,

$$1 - 2\epsilon \leq 1 - \epsilon \leq P(C_\Pi^{1-\epsilon}(I_{\Pi,\Pi'}(\epsilon))) \leq \mu(C_\mu^{1-\epsilon}(A_{\mu,\mu'}(2\epsilon))) \leq \mu(C_\mu^{1-2\epsilon}(A_{\mu,\mu'}(2\epsilon))).$$

Then, by definition, $d_1^{KM}(\mu', \mu) \leq 2\epsilon$. Since $d^{MS}(\Pi, \Pi') \leq \epsilon$ also implies $P(C_{\Pi'}^{1-\epsilon}(I_{\Pi,\Pi'}(\epsilon))) \geq 1 - \epsilon$, a symmetric argument ensures $d_1^{KM}(\mu, \mu') \leq 2\epsilon$.

Finally, we show that $d_0^{KM}(\mu, \mu') < 2\epsilon$. For an arbitrary event $E \subseteq \Theta \times T$,

$$\begin{aligned}
|\mu(E) - \mu'(E)| \leq & |\mu(E \cap (\tilde{I}_{\Pi,\Pi'}(\epsilon))_\tau) - \mu'(E \cap (\tilde{I}_{\Pi,\Pi'}(\epsilon))_\tau)| + \\
& \quad |\mu(E \backslash (\tilde{I}_{\Pi,\Pi'}(\epsilon))_\tau) - \mu'(E \backslash (\tilde{I}_{\Pi,\Pi'}(\epsilon))_\tau)| \\
\leq & |\mu(E \cap (\tilde{I}_{\Pi,\Pi'}(\epsilon))_\tau) - \mu'(E \cap (\tilde{I}_{\Pi,\Pi'}(\epsilon))_\tau)| + \epsilon \\
= & |\sum_{t \in \tilde{I}_{\Pi,\Pi'}} \left(\mu(E \cap (\tilde{I}_{\Pi,\Pi'}(\epsilon))_\tau | t) - \mu'(E \cap (\tilde{I}_{\Pi,\Pi'}(\epsilon))_\tau | t)\right) \mu(t)| + \epsilon \\
\leq & \epsilon \sum_{t \in \tilde{I}_{\Pi,\Pi'}} \mu(t) + \epsilon \\
\leq & 2\epsilon,
\end{aligned}$$

where the first inequality follows from the triangle inequality, the second inequality because $P(I_{\Pi,\Pi'}(\epsilon)) > 1 - \epsilon$, the equality from the law of total probability, the third inequality from CSC and COS, and the final inequality because μ is a probability measure.

Lemma 3

Suppose τ is Π-consistent and $E \subseteq \Theta \times T$. We show that, for any $m \geq 1$ and $i \in \mathcal{N}$, if $(\theta, t) \in (B^p_{\mu_i})^m(E)$ and $(\phi(s), \tau(s)) = (\theta, t)$, then $s \in (B^p_{\Pi_i})^m((\tilde{E})_\Pi)$. If this is true, then $(\theta, t) \in \cap_{m\geq 1}(B^p_\mu)^m(E) = C^p_\mu(E)$ implies $s \in \cap_{m\geq 1}(B^p_\Pi)^m(E) = C^p_\Pi((\tilde{E})_\Pi)$.

The proof is by induction. For the base case, take $(\theta, t) \in B^p_{\mu_i}(E)$ and any s for which $(\phi(s), \tau(s)) = (\theta, t)$. Then,

$$P((\tilde{E})_\Pi | \Pi_i(s)) = \mu(E|\tau_i(s)) \geq p,$$

where the equality follows because μ is identified by τ and τ is Π-consistent, and the second because of invertibility.

The induction hypothesis is that if $(\theta, t) \in (B^p_{\mu_i})^m(E)$, then $s \in (B^p_{\Pi_i})^m((\tilde{E})_\Pi)$. Take $s \in (B^p_{\Pi_i})^{m+1}(E)$. Then,

$$\mu((B^p_\mu)^m(E)|\tau_i(s)) \geq p.$$

By the induction hypothesis, if $(\theta, t) \in (B^p_\mu)^m(E)$, then $s \in (B^p_\Pi)^m((\tilde{E})_\Pi)$. Since μ is identified by τ and τ is Π-consistent,

$$P((B^p_\Pi)^m((\tilde{E})_\Pi)|\Pi_i(s)) \geq \mu((B^p_\mu)^m(E)|\tau_i(s)) \geq p.$$

Lemma 4

If $s \in (\tilde{A}_{\mu,\mu'}(\epsilon))_\Pi$, then there is an induced state $(\theta, t) \in A_{\mu,\mu'}(\epsilon)$ for which $(\phi(s), \tau(s)) = (\theta, t)$. If $(\theta, t) \in A_{\mu,\mu'}(\epsilon)$, then, for all $i \in \mathcal{N}$, $\mu(t_i) > 0$ and $\mu'(t_i) > 0$. Hence, there is a state s for which $\tau(s) = \tau'(s) = t := (t_1, ..., t_N)$. Further, defining $E := \Pi(s)\backslash\Pi'_i(s)$, for all $i \in \mathcal{N}$,

$$\epsilon \geq |\mu((\tilde{E})_\Pi|t_i) - \mu'((\tilde{E})_\Pi|t_i)| = |P(E|\Pi_i(s)) - P(E|\Pi'_i(s)|)| = P(E|\Pi_i(s)),$$

where the inequality follows from the definition of $A_{\mu,\mu'}(\epsilon)$, the first equality follows from invertibility, and the second equality because states outside of one's partition are believed to have occurred with probability zero. Setting $F := \Pi'(s)\backslash\Pi_i(s)$ and repeating the previous steps shows $s \in I_{\Pi,\Pi'}(\epsilon)$.

Theorem 2

If (τ_n) converges to τ in the KM topology, then $d^{KM}(\mu, \mu_n) \to 0$, where μ is identified by τ and μ_n by τ_n. We show that if $d^{KM}(\mu, \mu') \leq \epsilon$, where μ is identified by τ and μ' by τ', then $d^{MS}(\Pi, \Pi') \leq \epsilon$, where (Π, Π') are consistent with τ and τ'. The result follows.

If $d^{KM}(\mu, \mu') \leq \epsilon$, then $\mu(C_\mu^{1-\epsilon}(A_{\mu,\mu'}(\epsilon))) \geq 1 - \epsilon$. Suppose $(\theta, t) \in C_\mu^{1-\epsilon}(A_{\mu,\mu'}(\epsilon))$ and consider any s for which $(\phi(s), \tau(s)) = (\theta, t)$ for some $(\theta, t) \in A_{\mu,\mu'}(\epsilon)$. Then, from Lemma 3, $s \in C_\Pi^{1-\epsilon}((\tilde{A}_{\mu,\mu'}(\epsilon))_\Pi)$. By Lemma 4, $(\tilde{A}_{\mu,\mu'}(\epsilon))_\Pi \subseteq I_{\Pi,\Pi'}(\epsilon)$. Hence, $s \in C_\Pi^{1-\epsilon}((\tilde{A}_{\mu,\mu'}(\epsilon))_\Pi) \subseteq C_\Pi^{1-\epsilon}(I_{\Pi,\Pi'}(\epsilon))$. It follows from invertibility that,

$$1 - \epsilon \leq \mu(C_\mu^{1-\epsilon}(A_{\mu,\mu'}(\epsilon))) = P(C_\Pi^{1-\epsilon}(I_{\Pi,\Pi'}(\epsilon))).$$

By definition, then, $d_1^{MS}(\Pi', \Pi) \leq \epsilon$. A symmetric argument ensures $d_1^{MS}(\Pi, \Pi') \leq \epsilon$.