OLG with growth and endogenous fertility

Consider an overlapping generations economy in which a continuum of consumers is born in each period, $t = 1, 2, 3, \ldots$, and lives for two periods, child and adult. As you will see below, the size of the cohort is endogenous. Also, given the timing of choices in this model, it is easier to index a generation by the period when its members are adults. That is, a generation born in period $t-1$ is called generation t.

In the first period of life, a child i lives with its parents and does not make any relevant economic decision. In the second period, an adult i works, consumes, chooses how many children to have, n_{it}, and how much education, e_{it}, to provide for each of them (you can assume, throughout the question, that all adults make the same choices, although we still need to distinguish between individual and aggregate variables).

In terms of costs, each child requires a time commitment of τ_1. Educating children is also time-consuming. To deliver e_{it} for each child, parents must pay a total time cost of $n_{it}e_{it}\tau_2$. The level of education that children receive determines their level of human capital when they are adults: $h_{it+1} = e_{it}$.

Adults have a total time endowment of 1. Since they do not value leisure, they supply $1 - n_{it}(\tau_1 + \tau_2 e_{it})$ units of time to the labor market. The income that an adult receives per unit of labor depends on the unskilled and skilled wages, w_{it}^U and w_{it}^S, and its level of human capital, h_{it}. That is, for each unit of labor supplied, an adult receives income $y_{it} \equiv w_{it}^U + h_{it}w_{it}^S$ (you can think about an adult as supplying, jointly, unskilled and skilled labor, for example, because it represents a household with different members).

The preferences of adult consumers are:

$$\log c_{it} + \log n_{it} + \beta \log y_{i,t+1},$$

where y_{it+1} is the income each child will get as an adult. In other words, adult consumers get utility from their own consumption ($\log c_{it}$), the number of children ($\log n_{it}$), and their income level ($\beta \log y_{i,t+1}$). The choice of the number of children, n_{it}, determines the size of the new cohort $n_{t+1} = n_{it} * n_t$.

Finally, in period 1, there is an initial generation of adults with $n_1 = 1$ and $e_{i1} = 1$ for all i.

In terms of production, there is a representative firm with production function:

$$Y_t = U_t^{\alpha} S_t^{1-\alpha}$$

where aggregate final output, Y_t, is a function of aggregate unskilled labor (U_t) and aggregate skilled labor (S_t). Note that:

$$\begin{aligned}
U_t &= n_t\left(1 - n_{it}\left(\tau_1 + \tau_2 e_{it}\right)\right) \\
S_t &= n_t\left(1 - n_{it}\left(\tau_1 + \tau_2 e_{it}\right)\right) h_{it}
\end{aligned}$$

since variables with index i are the same for all adults. The wages are determined in competitive inputs markets.

Answer the following questions:

1. Write the budget constraints of an adult.

2. Define a sequential markets equilibrium for this economy.

3. Find and solve the first-order conditions for the consumption/fertility/education choice of an adult. You should get a closed-form solution for all three choices.

4. How does the skill-premium, i.e.,
$$\frac{w_{it}^S}{w_{it}^U}$$
matter for your answer above? And τ_1 and τ_2?

5. Characterize the sequential markets equilibrium of the economy. In particular, show how population, education, income per capita, and wages evolve over time and find the steady state of the model to which this evolution converges to.