**University of Pennsylvania** **Economics 705, August 2019**

# Prelim Examination

## Friday, August 9, 2019. Time limit: 150 minutes

**Instructions**:

(i) The total number of points is 75. The number of points for each question is given below.

(ii) The exam is closed book and closed notes.

(iii) To receive full credit for your answers you have to explain your calculations. **You may state additional assumptions.**

**Question 1:** Concepts (18 Points)

(i) (3 Points) What is the *Neyman-Pearson Lemma*?

(ii) (3 Points) What does it mean for an estimator to be not *admissible*?

(iii) (3 Points) What is a *sufficient statistic*?

(iv) (3 Points) What is the difference between *Ridge Regression* and *LASSO*?

(v) (3 Points) What is a *generalized method of moments* (GMM) estimator?

(vi) (3 Points) How would you formulate a regression model for a *binary* dependent variable?

**Question 2:** Asymptotics in a Location-Shift Model (20 Points).

Consider the following location-shift model

$$Y_i = \theta_0 + U_i, \quad U_i \sim (0,1), \quad \mathbb{E}[|U_i|^4] < \infty, \quad i = 1, \ldots, n. \tag{1}$$

In addition, we impose the constraint

$$\theta_0 \geq 0. \tag{2}$$

(i) (2 Points) Derive a quasi-maximum likelihood estimate (QMLE) of $\theta_0$ under the assumption that the $U_i$'s are normally distributed. Denote the quasi-likelihood function by $p(Y_{1:n}|\theta)$ (Note: It's called a "quasi" MLE because according to the DGP in (1) the $U_i$'s do not have to be normally distributed.)

(ii) (2 Points) Derive the probability limit of the QMLE $\hat{\theta}$.

(iii) (2 Points) Derive the asymptotic distribution of $\hat{\theta}$. Distinguish between the cases $\theta_0 = 0$ and $\theta_0 > 0$.

(iv) (6 Points) Instead of deriving the asymptotic distribution under a fixed true parameter $\theta_0$, now derive it under a sequence of true parameters $\theta_{0,n} = c/\sqrt{n}$.

(v) (3 Points) Now derive a posterior distribution $p(\theta|Y_{1:n})$ for $\theta$ under the assumption that the $U_i$'s are normally distributed. Use the prior $\theta \sim N(0,1)$, truncated at $\theta = 0$ so that the prior has support on $\theta \geq 0$.

(vi) (5 Points) Show that the posterior distribution is consistent in the following sense: for any $\delta > 0$

$$\mathbb{P}\big[|\theta - \theta_0| > \delta \,\big|\, Y_{1:n}\big] \longrightarrow 0$$

as $n \longrightarrow \infty$. Assume the true parameter is fixed. If necessary, distinguish between the cases $\theta_0 = 0$ and $\theta_0 > 0$. In plain English, what does this result imply?

**Question 3:** Inference for Variance Parameters (20 Points)

Consider the model $Y_i \sim iidN(0, \theta)$, $i = 1, \dots, n$. The goal is to make inference about $\theta$. We denote the "true" value by $\theta_0$.

(i) (2 Points) Derive the maximum likelihood estimator (MLE) $\hat{\theta}$ for $\theta$.

(ii) (2 Points) Derive the score $s(\theta) = \partial \ln p(Y_{1:n}|\theta)/\partial\theta$.

(iii) (4 Points) Assume that the "true" value is $\theta_0$ and derive the limit distribution of the (properly normalized) score evaluated at $\theta = \theta_0$.

(iv) (3 Points) Construct the Lagrange multiplier/score test for the hypothesis $H_0 : \theta = \theta_0$ and state 5% critical value as well as the acceptance and rejection region.

(v) (4 Points) Show that the power of the LM test against any fixed alternative $\theta_1 \neq \theta_0$ converges to one as $n \longrightarrow \infty$.

(vi) (5 Points) Derive the Cramer-Rao lower bound for estimators of $\theta$. Is the MLE unbiased? Does the MLE achieve this bound?

**Question 4:** (17 Points) Consider the following regression model:

$$\begin{aligned} y_i &= x_i\beta + u_{1i}, \quad |\beta| < M \\ x_i &= z_i'\gamma + u_{2i}, \end{aligned} \tag{3}$$

where $x_i$ is an endogenous regressor, $z_i$ a $k \times 1$ vector of instruments that is independent of $u_{1i}$ and $u_{2i}$ and

$$\left[\begin{array}{c} u_{1i} \\ u_{2i} \end{array}\right] \sim iidN\left( \left[\begin{array}{c} 0 \\ 0 \end{array}\right], \underbrace{\left[\begin{array}{cc} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{array}\right]}_{\Sigma} \right).$$

We assume that the covariance matrix $\Sigma$ of $u_i = [u_{1i}, u_{2i}]'$ is known. You can use matrix notation and define $Y$, $X$, and $Z$ as the vectors/matrices that stack $y_i$, $x_i$, and $z_i'$, respectively.

(i) (5 Points) Derive the conditional distribution of $u_{1i}|u_{2i}$ and the marginal distribution of $u_{2i}$. Note that

$$\left[\begin{array}{cc} a & b \\ c & d \end{array}\right]^{-1} = \frac{1}{ad-bc}\left[\begin{array}{cc} d & -b \\ -c & a \end{array}\right].$$

(ii) (2 Points) Derive the likelihood function $p(Y, X|Z, \beta, \gamma, \Sigma)$. Use the factorization

$$p(Y, X|Z, \beta, \gamma, \Sigma) = p(Y|X, Z, \beta, \gamma, \Sigma)p(X|Z, \gamma, \Sigma).$$

(iii) (5 Points) Suppose that both $\gamma$ and $\Sigma$ are known and that $\beta$ is unknown. Derive the maximum likelihood estimator for $\beta$, show that it is consistent, and derive its asymptotic distribution.

(iv) (5 Points) Suppose that only $\Sigma$ is known. Rather than estimating $(\beta, \gamma)$ jointly, suppose we proceed in two steps:

1. estimate $\gamma$ by $\hat{\gamma} = \text{argmax } \ln p(X|Z, \gamma, \Sigma)$;
2. estimate $\beta$ by $\hat{\beta} = \text{argmax } \ln p(Y|X, Z, \beta, \hat{\gamma}, \Sigma)$.

Show that $\hat{\beta}$ is consistent. Note that your proof should account for the fact that in the second step $\gamma$ is replaced by $\hat{\gamma}$.