ECON 897 - Waiver Exam
August 21, 2017

Important: This is a closed-book test. No books or lecture notes are permitted. You have **180** minutes to complete the test. Answer all questions. You can use all the results covered in class, but please make sure the conditions are satisfied. Write your name on each blue book and label each question clearly. Please write parts I, II, and III in separate blue books. Write legibly. Good luck!

Part I: Real Analysis

1. **(15 points)** Let (M, d) be a metric space.

   (a) (3 points) State the definition of a distance in a metric space M.

   (b) (4 points) Show that the function $\rho : M \to \mathbb{R}$ given by $\rho(x, y) = \frac{d(x,y)}{1+d(x,y)}$ is a distance in the metric space M. (Hint: Use that the function $f(\alpha) = \frac{\alpha}{1+\alpha}$ is strictly increasing).

   (c) (4 points) Show that the set U is open in the metric space (M, d) if and only it is open in (M, ρ).

   (d) (4 points) Argue that the identity mapping $id : (M, d) \to (M, \rho)$ is a homeomorphism. Is Boundedness a topological property? (Use the argument in this question to answer.)

2. **(10 points) Definition:** A collection of sets $\mathbb{B} = \{B_i\}_{i \in I}$ satisfies the Finite Intersection Property if the intersection over any finite subcollection of $\mathbb{B}$ is nonempty, that is, for all $\mathcal{C} \subset I$ with $|\mathcal{C}| < \infty$, $\cap_{i \in \mathcal{C}} B_i \neq \emptyset$

   Show that a subset A of a metric space M is compact if and only if for any collection of closed sets in A that satisifies the Finite Intersection Property, the intersection of all the sets in the collection is non-empty. (Hint: Use the covering definition of compactness.)

3. **(15 points)** Let (M, d) be a metric space.

   (a) (3 points) Let (x_n) be a sequence without a convergent subsequence. Argue that the set $\{x_n\}$ is closed.

   (b) (4 points) Define what a Cauchy sequence is in the metric space (M, d). Show that any Cauchy sequene is bounded.

   (c) (4 points) Show that if a subsequence of a Cauchy sequence converges then the sequence itself converges.

   (d) (4 points) Suppose that any bounded and closed set $A \subseteq M$ is compact. Conclude that M is a complete metric space.

Part II: Differentiation and Linear Algebra

4. **(20 points)** Define a *markov matrix* P as an $n \times n$ matrix that has non-negative entries where the entries of each column sum to one.

   (a) (3 points) Let $v \in \mathbb{R}^n$. Show that the entries of the vector Pv add up to $\sum_{j=1}^{n} v_j$.

   (b) (9 points) Let $v^* \in \mathbb{R}^n, v \neq 0$ be an eigenvector of P, with corresponding eigenvalue λ. Prove the following statements:

      i. (1 point) $P^s v^* = \lambda^s v^*, s \in \mathbb{N}$.

      ii. (4 points) Show that if $\sum_{j=1}^{n} v_j^* \neq 0$, then $\lambda = 1$. [Hint: show that P^s is also markov].

      iii. (4 points) Show that if $\sum_{j=1}^{n} v_j^* = 0, v^* \neq 0$, then $|\lambda| \leq 1$. [Hint: show that for any fixed $v \neq 0$ (not necessarily an eigenvector), $\sup_P ||Pv|| \leq M < \infty$, P markov].

   (c) (8 points) Define a jordan block $J(\lambda, m)$, as an $m \times m$ matrix with λ on the diagonal, 1 above the diagonal and zeros elsewhere. For example

$$J(\lambda, 1) = \begin{bmatrix} \lambda \end{bmatrix}, J(\lambda, 3) = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$$

      i. (2 points) Fix λ. Show that for $m = 3$, $[J(\lambda, 3)]^s = \begin{bmatrix} \lambda^s & s\lambda^{s-1} & \sum_{i=1}^{s-1} i\lambda^{s-2} \\ 0 & \lambda^s & s\lambda^{s-1} \\ 0 & 0 & \lambda^s \end{bmatrix}$.

      ii. (3 points) Consider $m = 1$ and $m = 3$, and three cases with λ: (i) $\lambda = 1$, (ii) $\lambda = -1$, (iii) $|\lambda| < 1$. For which cases does $\lim_{s\to\infty} J^s$ converge? (Hint: You can use the fact that $\lim_{s\to\infty} k\lambda^s = 0, |\lambda| < 1, k \leq s$).

      iii. (3 points) It can be shown that every square matrix (not necessarily markov) can be represented in a jordan canonical form $P = C^{-1}JC$ where C is an $n \times n$ invertible matrix whose vectors represent a (Jordan) basis. J is an $n \times n$ block diagonal matrix, made up of Jordan blocks:

$$J = \begin{bmatrix} J(\lambda_1, m_1) & 0 & \cdots & 0 \\ 0 & J(\lambda_2, m_2) & \cdots & 0 \\ \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & \cdots & J(\lambda_k, m_k) \end{bmatrix}_{n \times n}$$

      Show that $\lim_{s\to\infty} P^s$ exists if and only if $\lim_{s\to\infty} [J(\lambda_k, m_k)]^s$ exists $\forall k \in \{1, ..., K\}$.

5. **(20 points)** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a real-valued function. We say f has increasing differences (also known as supermodularity) if $\forall y_2 > y_1$, the difference

$$f(x, y_2) - f(x, y_1)$$

is increasing in x. Assume f is twice continuously differentiable.

(a) (5 points) Fix $(x, y) \in \mathbb{R}^2$. Define the function

$$g(s, h) = [f(x+s, y+h) - f(x+s, y)] - [f(x, y+h) - f(x, y)]$$

Show that if f has increasing differences, then $\forall s, h > 0, g(s, h) \geq 0$.

(b) (5 points) Show that $g(0, 0) = 0, (Dg)_{(0,0)} = [0, 0]$.

(c) (5 points) Compute the hessian of $g(s, h)$. What happens to the hessian as $(s, h) \to 0$? [Hint: Use continuity of the second partial derivatives of f]

(d) (5 points) Show that $\forall s, h > 0, g(s, h) \geq 0$ implies that $\frac{\partial^2 f(x,y)}{\partial x \partial y} \geq 0$ [Hint: Do a second order Taylor expansion of $g(s, h)$ around $g(0, 0)$]

Part III: Optimization and Probability

6. **(22 points)** Let $u_1, u_2 : \mathbb{R}^n_+ \to \mathbb{R}$ be strictly concave, continuously differentiable, and strictly increasing: $\frac{\partial u_i}{\partial x_j} > 0$ for all i and j. Let $X = \{x \in \mathbb{R}^n_+ | \ x \leq w\}$, and:

$$x^*(\bar{x}) = \arg\max_{x \in X} \ u_1(x)$$

$$\text{s.t.} \quad u_2(w - x) \geq u_2(w - \bar{x})$$

(Intuitively: u_i is person i's utility, $x \in \mathbb{R}^n_+$ is consumption, $\bar{x} \in \mathbb{R}^n_+$ is attainable by person 1, and $w \in \mathbb{R}^n_{++}$ is a fixed amount of wealth.) Carefully justify your following answers.

(a) (3 points) Must a solution exist? If it exists, must it be unique?

(b) (2 points) Must the constraint bind at any solution?

Consider the following alternative maximization problem, for any $\alpha \in [0, 1]$:

$$\hat{x}(\alpha) = \arg\max_{x \in X} \ \alpha u_1(x) + (1 - \alpha) u_2(w - x)$$

(c) (2 points) Must a solution exist? If it exists, must it be unique?

(d) (5 points) Show that $x^*(\hat{x}(\alpha)) = \hat{x}(\alpha)$ for any $\alpha \in (0, 1]$.

(e) (10 points) Fix an $\bar{x}$ where $0 << \bar{x} << w$, and suppose that $x^*(\bar{x}) = \bar{x}$. Use Kuhn–Tucker to prove that $\exists \alpha \in [0, 1]$ such that $\hat{x}(\alpha) = \bar{x}$.

7. **(13 points)** Let X_1 and X_2 be nonnegative real valued random variables. For a fixed income $m \in \mathbb{R}_{++}$, and prices $p_1, p_2 \in \mathbb{R}_{++}$, we have the relationship:

$$p_1 X_1 + p_2 X_2 = m$$

Let X_1 be a Uniform$(0, m/p_1)$ random variable.

(a) (3 points) Find the pdf of X_2.

(b) (10 points) Compute the expectation and variance of X_1 and X_2, i.e.:

$$E[X_1],\ E[X_2],\ Var(X_1),\ \text{and } Var(X_2)$$

In addition, compute:

$$\frac{\partial E[X_1]}{\partial p_1},\ \frac{\partial E[X_1]}{\partial p_2},\ \text{and}\ \frac{\partial E[X_1]}{\partial m}$$

Give a brief economic interpretation of the signs of these derivatives.

8. **(5 points)** Let U be Uniform(0,1), and $\hat{F}$ be an arbitrary cumulative density function (cdf). Let $X = \hat{F}^{-1}(U)$, compute the cdf $F_X(x)$ in terms of $\hat{F}$.