# Nonparametric Sample Splitting<sup>\*</sup>

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#### Abstract

This paper develops a threshold regression model, where the threshold is determined by an unknown relation between two variables. The threshold function is estimated fully nonparametrically. The observations are allowed to be cross-sectionally dependent and the model can be applied to determine an unknown spatial border for sample splitting over a random field. The uniform rate of convergence and the nonstandard limiting distribution of the nonparametric threshold estimator are derived. The root-n consistency and the asymptotic normality of the regression slope parameter estimator are also obtained. Empirical relevance is illustrated by estimating an economic border induced by the housing price difference between Queens and Brooklyn in New York City, where the economic border deviates substantially from the administrative one.

*Keywords*: sample splitting, threshold, nonparametric, random field. *JEL Classifications*: C14, C21, C24.

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### 1 Introduction

This paper develops a regression model whose coefficients can vary over different regimes or subsamples (i.e., threshold regression model), where the subsample classes are determined by some unknown relation between two variables. More precisely, we consider a model given by

$$y_i = x_i^\top \beta_0 + x_i^\top \delta_0 \mathbf{1} \left[ q_i \le \gamma_0 \left( s_i \right) \right] + u_i$$

for i = 1, ..., n, in which the marginal effect of  $x_i$  to  $y_i$  can be different as  $\beta_0$  or  $(\beta_0 + \delta_0)$ depending on whether  $q_i \leq \gamma_0(s_i)$  or not. The threshold function  $\gamma_0(\cdot)$  is unknown and the main parameters of interest are  $\beta_0$ ,  $\delta_0$ , and  $\gamma_0(\cdot)$ . The novel feature of this model is that the sample splitting is determined by an unknown relation between two variables  $q_i$  and  $s_i$ , and their relation is characterized by the nonparametric threshold function  $\gamma_0(\cdot)$ .

In contrast, the classical threshold regression or structural break models assume  $\gamma_0(\cdot)$  to be a constant and consider the sample splitting induced by whether a scalar running variable exceeds a certain constant threshold. Examples include, among others, Andrews (1993), Andrews and Ploberger (1994), Bai (1997), Bai and Perron (1998), Bai, Lumsdaine, and Stock (1998), Qu and Perron (2007), Elliott and Müller (2007), Chen and Hong (2012), Elliott and Müller (2014), Elliott, Müller, and Watson (2015), and Eo and Morley (2015) for structural break models; and Hansen (2000), Caner and Hansen (2004), Seo and Linton (2007), Lee, Seo, and Shin (2011), Li and Ling (2012), Yu (2012), Lee, Liao, Seo, and Shin (2018), and Yu and Fan (2019) for threshold regression models.<sup>1</sup>

This paper contributes to the literature in two folds. First, we formulate the threshold by some unknown relation among variables. Existing literature on sample splitting typically assumes  $\gamma_0(\cdot)$  to be a fixed constant or a linear combination of variables. In contrast, we leave the threshold function  $\gamma_0(\cdot)$  to be fully nonparametric as long as it is a smooth function. This specification can cover interesting cases that have not been studied. For example, we can consider the case that the threshold is heterogeneous and specific to each observation *i* if we see  $\gamma_0(s_i) = \gamma_{0i}$ ; or the case that the threshold is

<sup>&</sup>lt;sup>1</sup>Seo and Linton (2007), Lee, Liao, Seo, and Shin (2018), and Yu and Fan (2019) allow multiple variables to define the threshold. However, they consider the index form  $\gamma_0(s_i) = s_i^{\top} \gamma_0$  for some parameter vector  $\gamma_0$ , where Lee, Liao, Seo, and Shin (2018) use principal components for  $s_i$ .

determined by the direction of some moment conditions  $\gamma_0(s_i) = \mathbb{E}[q_i|s_i]$ . Apparently, when  $\gamma_0(s) = \gamma_0$  or  $\gamma_0(s) = \gamma_0 s$  for some parameter  $\gamma_0$  and  $s \neq 0$ , it reduces to the standard threshold regression model, where the threshold is determined by the ratio  $q_i/s_i$  for the latter case.

Second, we let the variables be cross-sectionally dependent, which has not been considered in the threshold model literature. More precisely, we consider strong-mixing random fields as Bolthausen (1982). This generalization allows us to study sample splitting over a random field. For instance, if we let  $(q_i, s_i)$  correspond to the the geographical location (i.e., latitude and longitude on the map), then the threshold  $\mathbf{1} [q_i \leq \gamma_0 (s_i)]$  identifies the unknown border yielding a two-dimensional sample splitting. In more general contexts, the model can be applied to identify social or economic segregation over interacting agents.

The main results of this paper can be summarized as follows. First, we develop a two-step estimator of this semiparametric model, where we estimate  $\gamma_0(\cdot)$  using local constant estimation. Second, under the shrinking threshold setup (e.g., Bai (1997), Bai and Perron (1998), and Hansen (2000)) with  $\delta_0 = c_0 n^{-\epsilon}$  for some  $c_0 \neq 0$  and  $\epsilon \in (0, 1/2)$ , we show that the nonparametric estimator  $\widehat{\gamma}(\cdot)$  is uniformly consistent and  $(\hat{\beta}^{\top}, \hat{\delta}^{\top})^{\top}$  satisfies the root-n-consistency. The uniform rate of convergence and the pointwise limiting distribution of  $\widehat{\gamma}(\cdot)$  are also derived. In particular, we find that  $\widehat{\gamma}(\cdot)$  is asymptotically unbiased even if the optimal bandwidth is used. This feature is novel in comparison with the existing literature on kernel estimation. Since the nonparametric function  $\gamma_0(\cdot)$  is in the indicator function, it causes additional technical challenges and the proofs are nonstandard. We also develop a pointwise specification test of  $\gamma_0(s)$  for given s (i.e., a test for the null hypothesis  $H_0: \gamma_0(s) = \gamma_*(s)$ ). Simulation studies show its good finite sample performance. Third, we extend the basic threshold regression model to estimate threshold contours or sample splitting circles by combining estimates of  $\gamma_0(\cdot)$  over the artificially rotated coordinates. Fourth, as an empirical illustration, we consider  $(q_i, s_i)$  as geographic location indices (i.e., latitude and longitude) and examine the border between Brooklyn and Queens boroughs in New York City. In particular, we estimate an unknown economic border that splits these two boroughs by different levels of elasticity to the housing price. The economic border turns out to be substantially different from the current administrative one.

The rest of the paper is organized as follows. Section 2 summarizes the model and our estimation procedure. Section 3 derives asymptotic properties of the estimators and develops a likelihood ratio test of the threshold function. Section 4 describes how to extend the main model to estimate a threshold contour. Section 5 studies small sample properties of the proposed statistics by Monte Carlo simulations. Section 6 applies the results to the housing price data to identify unknown economic border. Section 7 concludes this paper with some remarks. The main proofs are in the Appendix and all the omitted proofs are collected in the supplementary material.

We use the following notations. Let  $\rightarrow_p$  denote convergence in probability,  $\rightarrow_d$  convergence in distribution, and  $\Rightarrow$  weak convergence of the underlying probability measure as  $n \rightarrow \infty$ . Let  $\lfloor r \rfloor$  denote the biggest integer smaller than or equal to r and  $\mathbf{1}[A]$  the indicator function of a generic event A. Let  $\|B\|$  denote the Euclidean norm of a vector or matrix B, and C a generic constant that may vary over different lines.

## 2 Nonparametric Threshold Regression

We consider a threshold regression model given by

$$y_i = x_i^\top \beta_0 + x_i^\top \delta_0 \mathbf{1} \left[ q_i \le \gamma_0 \left( s_i \right) \right] + u_i \tag{1}$$

for i = 1, ..., n, where  $(y_i, x_i^{\top}, q_i, s_i)^{\top} \in \mathbb{R}^{1+p+1+1}$  are observed but the threshold function  $\gamma_0 : \mathbb{R} \to \mathbb{R}$  as well as the structural parameters  $\theta_0 = (\beta_0^{\top}, \delta_0^{\top})^{\top} \in \mathbb{R}^{2p}$  are unknown.<sup>2</sup> The parameters of interest are  $\theta_0$  and  $\gamma_0(\cdot)$ .

We estimate this semiparametric model in two steps. First, for given  $s \in S$ , where S is a compact subset of the support of  $s_i$ , we fix  $\gamma_0(s) = \gamma$  and obtain  $\hat{\beta}(\gamma; s)$  and  $\hat{\delta}(\gamma; s)$  by local least squares conditional on  $\gamma$ :

$$\left(\widehat{\beta}\left(\gamma;s\right),\widehat{\delta}\left(\gamma;s\right)\right) = \arg\min_{\beta,\delta} Q_n\left(\beta,\delta,\gamma;s\right),\tag{2}$$

where

$$Q_n\left(\beta,\delta,\gamma;s\right) = \sum_{i=1}^n K\left(\frac{s_i - s}{b_n}\right) \left(y_i - x_i^\top \beta - x_i^\top \delta \mathbf{1} \left[q_i \le \gamma\right]\right)^2 \tag{3}$$

for some kernel function  $K(\cdot)$  and a bandwidth parameter  $b_n$ . If we suppose that the

<sup>&</sup>lt;sup>2</sup>The main results of this paper can be extended to consider multi-dimensional  $s_i$  using multivariate kernels. However, we only consider the scalar case for the expositional simplicity. Furthermore, the results are readily generalized to the case where only a subset of parameters differ between regimes.

space of  $\gamma_0(s)$  for any s is a compact set  $\Gamma \subset \mathbb{R}^3$  then  $\gamma_0(s)$  is estimated by

$$\widehat{\gamma}(s) = \arg\min_{\gamma\in\Gamma} Q_n(\gamma;s)$$

for given s, where  $Q_n(\gamma; s)$  is the concentrated sum of squares defined as

$$Q_n(\gamma; s) = Q_n\left(\widehat{\beta}(\gamma; s), \widehat{\delta}(\gamma; s), \gamma; s\right).$$
(4)

Note that it is basically a constant threshold estimator in the neighborhood of the given point s. Therefore,  $\hat{\gamma}(s)$  can be naturally seen as a local version of the standard (constant) threshold regression estimator.

Second, to estimate the parametric components  $\beta_0$  and  $\delta_0$ , we estimate  $\beta_0$  and  $\delta_0^* = \beta_0 + \delta_0$  by

$$\widehat{\beta} = \arg\min_{\beta} \sum_{i=1}^{n} \left( y_i - x_i^{\top} \beta \right)^2 \mathbf{1} \left[ q_i > \widehat{\gamma}_{-i} \left( s_i \right) + \Delta_n \right] \mathbf{1} [s_i \in \mathcal{S}],$$
(5)

$$\widehat{\delta}^* = \arg\min_{\delta^*} \sum_{i=1}^n \left( y_i - x_i^{\mathsf{T}} \delta^* \right)^2 \mathbf{1} \left[ q_i < \widehat{\gamma}_{-i} \left( s_i \right) - \Delta_n \right] \mathbf{1} \left[ s_i \in \mathcal{S} \right]$$
(6)

for some constant  $\Delta_n > 0$  satisfying  $\Delta_n \to 0$  as  $n \to \infty$ , which is defined later. We use the leave-one-out estimator  $\widehat{\gamma}_{-i}(s)$  as the first step estimation. The change size  $\delta$  can be estimated as  $\widehat{\delta} = \widehat{\delta}^* - \widehat{\beta}$ .

We first assume the conditions for identification. Let  $\mathcal{Q}$  be the support of  $q_i$ .

#### Assumption ID

- (i)  $\mathbb{E}[u_i x_i | q_i, s_i] = 0.$
- (*ii*)  $\mathbb{E}\left[x_i x_i^{\top}\right] > \mathbb{E}\left[x_i x_i^{\top} \mathbf{1} \left[q_i \leq \gamma\right]\right] > 0 \text{ for any } \gamma \in \Gamma.$
- (iii) For any  $s \in \mathcal{S}$ , there exists  $\varepsilon(s) > 0$  such that  $\varepsilon(s) < \mathbb{P}(q_i \le \gamma_0(s_i)|s_i = s) < 1 \varepsilon(s)$  and  $\delta_0^\top \mathbb{E}[x_i x_i^\top | q_i = q, s_i = s] \delta_0 > 0$  for all  $(q, s) \in \mathcal{Q} \times \mathcal{S}$ .
- (iv)  $q_i$  is continuously distributed and its conditional density f(q|s) is bounded away from zero for all  $(q, s) \in \mathcal{Q} \times \mathcal{S}$ .

<sup>&</sup>lt;sup>3</sup>When  $\gamma_0(s) \in \Gamma_s \subset \mathbb{R}$  for each s, we let  $\Gamma$  be a smallest compact set that includes  $\cup_{s \in S} \Gamma_s$ .

Assumption ID-(i) excludes endogeneity. Assumption ID-(ii) is the rank condition that yields the identification of the parameters  $\beta_0$  and  $\delta_0$ . Assumption ID-(iii) restricts that the threshold  $\gamma_0(s)$  lies in the interior of the support of  $q_i$  for any  $s \in S$  and  $\delta_0 \neq 0$ , which yields that the nonparametric threshold function  $\gamma_0(\cdot)$  is identified pointwisely under Assumption ID-(iv).<sup>4</sup>

**Theorem 1** Under Assumption ID, the threshold function  $\gamma_0(\cdot)$  and the parameters  $(\beta_0^{\top}, \delta_0^{\top})^{\top}$  are uniquely identified.

We allow for cross-sectional dependence in  $(x_i^{\top}, q_i, s_i, u_i)^{\top}$  in this paper so that we can apply the main results to the spatial (or two-dimensional) sampling splitting. More precisely, we suppose  $\alpha$ -mixing over a random field similarly as Bolthausen (1982) and Jenish and Prucha (2009). We consider the samples over a random expanding lattice  $N_n \subset \mathbb{R}^2$  endowed with a metric  $\lambda(i, j) = \max_{1 \leq \ell \leq 2} |i_\ell - j_\ell|$  and the corresponding norm  $\max_{1 \leq \ell \leq 2} |i_\ell|$ , where  $i_\ell$  denotes the  $\ell$ th component of i. We denote  $|N_n|$  as the cardinality of  $N_n$  and  $\partial N_n = \{i \in N_n:$  there exists  $j \notin N_n$  with  $\lambda(i, j) = 1\}$ . We let  $|N_n| = n$  and then the summation in (3) can be written as  $\sum_{i \in N_n}$ . We also define a mixing coefficient:

$$\alpha(m) = \sup \left\{ \left| \mathbb{P} \left( A_i \cap A_j \right) - \mathbb{P} \left( A_i \right) \mathbb{P} \left( A_j \right) \right| : A_i \in \mathcal{F}_i \text{ and } A_j \in \mathcal{F}_j \text{ with } \lambda\left(i, j\right) \ge m \right\},$$
(7)

where  $\mathcal{F}_i$  is the  $\sigma$ -algebra generated by  $(x_i^{\top}, q_i, s_i, u_i)^{\top}$ .

We suppose additional conditions for deriving asymptotic properties of the semiparametric estimators. Let f(q, s) be the joint density function of  $(q_i, s_i)$  and

$$D(q,s) = \mathbb{E}[x_i x_i^\top | (q_i, s_i) = (q, s)],$$
(8)

$$V(q,s) = \mathbb{E}[x_i x_i^\top u_i^2 | (q_i, s_i) = (q, s)]$$
(9)

for  $(q, s) \in \mathcal{Q} \times \mathcal{S} \subset \mathbb{R}^2$ .

<sup>&</sup>lt;sup>4</sup>Since the last condition in Assumption ID-(iii) does not require the strict positive definiteness of  $\mathbb{E}\left[x_ix_i^{\top}|q_i=q, s_i=s\right], q_i \text{ or } s_i \text{ can be one of the elements of } x_i \text{ (e.g., threshold autoregressive model,} Tong (1983)) or a linear combination of <math>x_i$ , even when  $x_i$  includes a constant term.

#### Assumption A

- (i) The lattice  $N_n \subset \mathbb{R}^2$  is infinite countable; all the elements in  $N_n$  are located at distances at least  $\lambda_0 > 1$  from each other, i.e., for any  $i, j \in N_n : \lambda(i, j) \geq \lambda_0$ ; and  $\lim_{n \to \infty} |\partial N_n| / n = 0$ .
- (ii)  $\delta_0 = c_0 n^{-\epsilon}$  for some  $c_0 \neq 0$  and  $\epsilon \in (0, 1/2)$ ;  $(c_0^{\top}, \beta_0^{\top})^{\top}$  belongs to some compact subset of  $\mathbb{R}^{2p}$ .
- (iii)  $(x_i^{\top}, q_i, s_i, u_i)^{\top}$  is stationary and  $\alpha$ -mixing with bounded  $(2 + \varphi)$ th moments for some  $\varphi > 0$ ; the mixing coefficient  $\alpha(m)$  defined in (7) satisfies  $\sum_{m=1}^{\infty} m\alpha(m) < \infty$  and  $\sum_{m=1}^{\infty} m^2 \alpha(m)^{\varphi/(2+\varphi)} < \infty$  for some  $\varphi \in (0, 2)$ .
- (iv)  $0 < \mathbb{E}[u_i^2 | x_i, q_i, s_i] < \infty$  almost surely.
- (v) Uniformly in (q, s), there exists some constant  $C < \infty$  such that  $\mathbb{E}[||x_i||^8 | (q_i, s_i) = (q, s)] < C$  and  $\mathbb{E}[||x_iu_i||^8 | (q_i, s_i) = (q, s)] < C$ .
- (vi)  $\gamma_0 : \mathcal{S} \mapsto \Gamma$  is a twice continuously differentiable function with bounded derivatives.
- (vii) D(q, s), V(q, s), and f(q, s) are bounded, continuous in q, and twice continuously differentiable in s with bounded derivatives.
- (viii)  $c_0^{\top} D(\gamma_0(s), s) c_0 > 0$ ,  $c_0^{\top} V(\gamma_0(s), s) c_0 > 0$ , and  $f(\gamma_0(s), s) > 0$  for all  $s \in \mathcal{S}$ .

(ix) As 
$$n \to \infty$$
,  $b_n \to 0$  and  $n^{1-2\epsilon}b_n \to \infty$ .

(x)  $K(\cdot)$  is uniformly bounded, continuous, symmetric around zero, and satisfies  $\int K(v) dv = 0, \int v^2 K(v) dv \in (0, \infty), \int K^2(v) dv \in (0, \infty), and \lim_{v \to \infty} |v| K(v) = 0.$ 

Many of these conditions are similar to Assumption 1 of Hansen (2000). Note that  $\lambda_0$  in Assumption A-(i) can be any strictly positive value, but we can impose  $\lambda_0 > 1$  without loss of generality. It is well known that a constant change size leads to a complicated asymptotic distribution of the threshold estimator, which depends on nuisance parameters (e.g., Chan (1993)). In Assumption A-(ii), we adopt the widely used shrinking change size setup as in Bai (1997), Bai and Perron (1998), and Hansen (2000) to obtain a simpler limiting distribution. The conditions in Assumption A-(iii) are required to establish the central limit theorem (CLT) for spatially dependent random field. The condition on the mixing coefficient is slightly stronger than that of Bolthausen (1982). This is because we need to control for the dependence within the local neighborhood in kernel estimation. When  $\alpha(m)$  decays at an exponential rate, these conditions are readily satisfied. When  $\alpha(m)$  decays at a polynomial rate (i.e.,  $\alpha(m) \leq C_{\alpha}m^{-k}$  for some k > 0), we need some restrictions on k and  $\varphi$  to satisfy these conditions, such as  $k > 3(2 + \varphi)/\varphi$ . Note that  $f(\gamma_0(s), s)$  and the marginal density  $f_s(s)$  are both strictly positive for all  $s \in S$  from Assumption A-(viii). In practice, we choose S such that their estimates are bounded away from zero in finite samples. Assumptions A-(ix) and (x) are standard in the kernel estimation literature, except that the magnitude of the bandwidth  $b_n$  depends on not only n but also  $\epsilon$ . The conditions in A-(x) holds for the most of the kernel functions including the Gaussian kernel and the kernels with bounded supports.

It is important to note that we suppose  $\gamma_0$  as a function from S to  $\Gamma$  in Assumption A-(vi), which is not necessarily one-to-one. For this reason, sample splitting based on  $\mathbf{1} [q_i \leq \gamma_0 (s_i)]$  can be different from that based on  $\mathbf{1} [s_i \geq \check{\gamma}_0 (q_i)]$  for some function  $\check{\gamma}_0$ . Instead of restricting  $\gamma_0$  be one-to-one in this paper, for the identification purpose, we presume that we know which variables should be respectively assigned as  $q_i$  and  $s_i$  from the context. In Section 4, however, we discuss how to relax this point and to identify a convex threshold contour as an extreme case.

## **3** Asymptotic Results

We first obtain the asymptotic properties of the nonparametric estimator  $\hat{\gamma}(s)$ . The following theorem derives the pointwise consistency and the pointwise rate of convergence.

**Theorem 2** For a given  $s \in S$ , under Assumptions ID and A,  $\widehat{\gamma}(s) \rightarrow_p \gamma_0(s)$  as  $n \rightarrow \infty$ . Furthermore,

$$\widehat{\gamma}(s) - \gamma_0(s) = O_p\left(\frac{1}{n^{1-2\epsilon}b_n}\right)$$

provided that  $n^{1-2\epsilon}b_n^2$  does not diverge.

The pointwise rate of convergence of  $\widehat{\gamma}(s)$  depends on two parameters,  $\epsilon$  and  $b_n$ . It is decreasing in  $\epsilon$  like the parametric (constant) threshold case: a larger  $\epsilon$  reduces the threshold effect  $\delta_0 = c_0 n^{-\epsilon}$  and hence decreases effective sampling information on the threshold. Since we estimate  $\gamma_0(\cdot)$  using the kernel estimation method, the rate of convergence depends on the bandwidth  $b_n$  as well. As in the standard kernel estimator case, a smaller bandwidth decreases the effective local sample size, which reduces the precision of the estimator  $\hat{\gamma}(s)$ . Therefore, in order to have a sufficient level of rate of convergence, we need to choose  $b_n$  large enough when the threshold effect  $\delta_0$  is expected to be small (i.e., when  $\epsilon$  is close to 1/2).

Unlike the standard kernel estimator, there appears no bias-variance trade-off in  $\widehat{\gamma}(s)$  as we further discuss after Theorem 3. It thus seems like that we can improve the rate of convergence by choosing a larger bandwidth  $b_n$ . However,  $b_n$  cannot be chosen too large to result in  $n^{1-2\epsilon}b_n^2 \to \infty$ , because  $n^{1-2\epsilon}b_n(\widehat{\gamma}(s) - \gamma_0(s))$  is no longer  $O_p(1)$  in that case. Therefore, we can use the restriction  $n^{1-2\epsilon}b_n^2 \to \varrho$  for some  $\varrho \in (0,\infty)$  to obtain the optimal bandwidth.

Under the choice that  $n^{1-2\epsilon}b_n^2 \to \varrho \in (0,\infty)$ , the optimal bandwidth can be chosen such that  $b_n^* = n^{-(1-2\epsilon)/2}c^*$  for some constant  $0 < c^* < \infty$ . This  $b_n^*$  provides the fastest convergence rate. Using this optimal bandwidth, the optimal pointwise rate of convergence of  $\widehat{\gamma}(s)$  is then given as  $n^{-(1-2\epsilon)/2}$ . However, such a bandwidth choice is not feasible in practice since the constant term  $c^*$  is unknown, which also depends on the nuisance parameter  $\epsilon$  that is not estimable. In practice, we suggest cross-validation as we implement in Section 6, although its statistical properties need to be studied further.<sup>5</sup>

The next theorem derives the limiting distribution of  $\widehat{\gamma}(s)$ . We let  $W(\cdot)$  be a two-sided Brownian motion defined as in Hansen (2000):

$$W(r) = W_1(-r)\mathbf{1} [r < 0] + W_2(r)\mathbf{1} [r > 0], \qquad (10)$$

where  $W_1(\cdot)$  and  $W_2(\cdot)$  are independent standard Brownian motions on  $[0,\infty)$ .

**Theorem 3** Under Assumptions ID and A, for a given  $s \in S$ , if  $n^{1-2\epsilon}b_n^2 \to \varrho \in (0,\infty)$ ,

$$n^{1-2\epsilon}b_{n}\left(\widehat{\gamma}\left(s\right)-\gamma_{0}\left(s\right)\right)\rightarrow_{d}\xi\left(s\right)\arg\max_{r\in\mathbb{R}}\left(W\left(r\right)+\mu\left(r,\varrho;s\right)\right)$$
(11)

<sup>&</sup>lt;sup>5</sup>If  $\epsilon$  is close to zero, the rate of convergence of  $\widehat{\gamma}(s)$  is close to  $n^{-1/2}$ . Such fast convergence rate requires infinite order of smoothness in the standard kernel regressions. In contrast, we only require the second-order differentiability in this nonparametric threshold model.

as  $n \to \infty$ , where

$$\begin{split} \mu \left( r, \varrho; s \right) &= - |r| \psi_1 \left( r, \varrho; s \right) + \varrho \psi_2 \left( r, \varrho; s \right), \\ \psi_1 \left( r, \varrho; s \right) &= \int_0^{\xi(s)|r|/(\varrho|\dot{\gamma}_0(s)|)} K \left( t \right) dt, \\ \psi_2 \left( r, \varrho; s \right) &= \xi(s) \left| \dot{\gamma}_0(s) \right| \int_0^{\xi(s)|r|/(\varrho|\dot{\gamma}_0(s)|)} t K \left( t \right) dt \end{split}$$

and

$$\xi\left(s\right) = \frac{\kappa_{2}c_{0}^{\top}V\left(\gamma_{0}\left(s\right),s\right)c_{0}}{\left(c_{0}^{\top}D\left(\gamma_{0}\left(s\right),s\right)c_{0}\right)^{2}f\left(\gamma_{0}\left(s\right),s\right)}$$

with  $\kappa_2 = \int K(v)^2 dv$  and  $\dot{\gamma}_0(s)$  is the first derivative of  $\gamma_0$  at s. Furthermore,  $\mathbb{E}\left[\arg\max_{r\in\mathbb{R}} \left(W\left(r\right) + \mu\left(r, \varrho; s\right)\right)\right] = 0.$ 

The drift term  $\mu(r, \varrho; s)$  in (11) depends on  $\varrho$ , the limit of  $n^{1-2\epsilon}b_n^2 = (n^{1-2\epsilon}b_n)b_n$ , and  $|\dot{\gamma}_0(s)|$ , the steepness of  $\gamma_0(\cdot)$  at s. Interestingly, it resembles the typical  $O(b_n)$ boundary bias of the standard local constant estimator even when s belongs to the interior of the support of  $s_i$ , which is from the inequality restriction in the indicator function of the threshold regression.

However, having this non-zero drift term in the limiting expression does not mean that the limiting distribution of  $\hat{\gamma}(s)$  itself has a non-zero mean even when we use the optimal bandwidth  $b_n^* = O(n^{-(1-2\epsilon)/2})$  satisfying  $n^{1-2\epsilon}b_n^{*2} \to \varrho \in (0,\infty)$ . This is mainly because the drift function  $\mu(r,\varrho;s)$  is symmetric about zero and hence the limiting random variable  $\arg \max_{r \in \mathbb{R}} (W(r) + \mu(r,\varrho;s))$  is mean zero. In particular, we can show that the random variable  $\arg \max_{r \in \mathbb{R}} (W(r) + \mu(r,\varrho;s))$  always has zero mean if  $\mu(r,\varrho;s)$  is a non-random function that is symmetric about zero and monotonically decreasing fast enough. This result might be of independent research interest and is summarized in Lemma A.9 in the Appendix. Figure 1 depicts the drift function  $\mu(r,\varrho;s)$  for various kernels when  $|\dot{\gamma}_0(s)| = 1$  and  $\varrho = 1$ .

Since the limiting distribution in (11) depends on unknown components, like  $\rho$  and  $\dot{\gamma}_0(s)$ , it is hard to use this result for further inference. We instead suggest undersmoothing for practical use. More precisely, if we suppose  $n^{1-2\epsilon}b_n^2 \to 0$  as  $n \to \infty$ ,

Figure 1: Drift Function



then the limiting distribution in (11) simplifies to<sup>6</sup>

$$n^{1-2\epsilon}b_n\left(\widehat{\gamma}\left(s\right) - \gamma_0\left(s\right)\right) \to_d \xi\left(s\right) \arg\max_{r \in \mathbb{R}} \left(W\left(r\right) - \frac{|r|}{2}\right)$$
(12)

as  $n \to \infty$ , which appears the same as in the parametric case in Hansen (2000) except for the scaling factor  $n^{1-2\epsilon}b_n$ . The distribution of  $\arg \max_{r\in\mathbb{R}} (W(r) - |r|/2)$  is known (e.g., Bhattacharya and Brockwell (1976) and Bai (1997)), which is also described in Hansen (2000, p.581). The term  $\xi(s)$  determines the scale of the distribution at given s: it increases in the conditional variance  $\mathbb{E}[u_i^2|x_i, q_i, s_i]$ ; and decreases in the size of the threshold constant  $c_0$  and the density of  $(q_i, s_i)$  near the threshold.

Even when  $n^{1-2\epsilon}b_n^2 \to 0$  as  $n \to \infty$ , the asymptotic distribution in (12) still depends on the unknown parameter  $\epsilon$  (or equivalently  $c_0$ ) in  $\xi(s)$  that is not estimable. Thus, this result cannot be directly used for inference of  $\gamma_0(s)$ . Alternatively, given any  $s \in S$ , we can consider a pointwise likelihood ratio test statistic for

$$H_0: \gamma_0(s) = \gamma_*(s) \quad \text{against} \quad H_1: \gamma_0(s) \neq \gamma_*(s) \tag{13}$$

<sup>&</sup>lt;sup>6</sup>We let  $\psi_1(r, 0; s) = \int_0^\infty K(t) dt = 1/2.$ 

for a fixed  $s \in \mathcal{S}$ , which is given as

$$LR_n(s) = \sum_{i=1}^n K\left(\frac{s_i - s}{b_n}\right) \frac{Q_n\left(\gamma_*\left(s\right), s\right) - Q_n\left(\widehat{\gamma}\left(s\right), s\right)}{Q_n\left(\widehat{\gamma}\left(s\right), s\right)}.$$
(14)

The following corollary obtains the limiting null distribution of this test statistic that is free of nuisance parameters. Using the likelihood ratio statistic inversion approach, we can form a pointwise asymptotic confidence interval of  $\gamma_0(s)$ .

**Corollary 1** Suppose  $n^{1-2\epsilon}b_n^2 \to 0$  as  $n \to \infty$ . Under the same condition in Theorem 3, for any fixed  $s \in S$ , the test statistic in (14) satisfies

$$LR_n(s) \to_d \xi_{LR}(s) \max_{r \in \mathbb{R}} \left( 2W(r) - |r| \right)$$
(15)

as  $n \to \infty$  under the hull hypothesis (13), where

$$\xi_{LR}\left(s\right) = \frac{\kappa_{2}c_{0}^{\top}V\left(\gamma_{0}\left(s\right),s\right)c_{0}}{\sigma^{2}(s)c_{0}^{\top}D\left(\gamma_{0}\left(s\right),s\right)c_{0}}$$

with  $\sigma^2(s) = \mathbb{E}[u_i^2|s_i = s]$  and  $\kappa_2 = \int K(v)^2 dv$ .

When  $\mathbb{E}[u_i^2|x_i, q_i, s_i] = \mathbb{E}[u_i^2|s_i]$ , which is the case of local conditional homoskedasticity, the scale parameter  $\xi_{LR}(s)$  is simplified as  $\kappa_2$ , and hence the limiting null distribution of  $LR_n(s)$  becomes free of nuisance parameters and the same for all  $s \in S$ . Though this limiting distribution is still nonstandard, the critical values in this case can be obtained using the same method as Hansen (2000, p.582) with the scale adjusted by  $\kappa_2$ . More precisely, since the distribution function of  $\zeta = \max_{r \in \mathbb{R}} (2W(r) - |r|)$ is given as  $\mathbb{P}(\zeta \leq z) = (1 - e^{-z/2})^2 \mathbf{1} [z \geq 0]$ , the distribution function of  $\zeta^* = \kappa_2 \zeta$ is  $\mathbb{P}(\zeta^* \leq z) = (1 - e^{-z/2\kappa_2})^2 \mathbf{1} [z \geq 0]$ , where  $\zeta^*$  is the limiting random variable of  $LR_n(s)$  given in (15) under the local conditional homoskedasticity. By inverting it, we can obtain the asymptotic critical values given a choice of  $K(\cdot)$ . For instance, the asymptotic critical values for the Gaussian kernel is reported in Table 1, where  $\kappa_2 = (2\sqrt{\pi})^{-1} \simeq 0.2821$  in this case.

Table 1: Simulated Critical Values of the LR Test (Gaussian Kernel)

$\mathbb{P}(\zeta^* > cv)$	0.800	0.850	0.900	0.925	0.950	0.975	0.990
cv	1.268	1.439	1.675	1.842	2.074	2.469	2.988

Note:  $\zeta^*$  is the limiting distribution of  $LR_n(s)$  under the local conditional homoskedasticity. The Gaussian kernel is used.

In general, we can estimate  $\xi_{LR}(s)$  by

$$\widehat{\xi}_{LR}\left(s\right) = \frac{\kappa_{2}\widehat{\delta}^{\top}\widehat{V}\left(\widehat{\gamma}\left(s\right),s\right)\widehat{\delta}}{\widehat{\sigma}^{2}\left(s\right)\widehat{\delta}^{\top}\widehat{D}\left(\widehat{\gamma}\left(s\right),s\right)\widehat{\delta}}$$

where  $\hat{\delta}$  is from (5) and (6), and  $\hat{\sigma}^2(s)$ ,  $\hat{D}(\hat{\gamma}(s), s)$ , and  $\hat{V}(\hat{\gamma}(s), s)$  are the standard Nadaraya-Watson estimators. In particular, we let  $\hat{\sigma}^2(s) = \sum_{i=1}^n \omega_{1i}(s)\hat{u}_i^2$  with  $\hat{u}_i = y_i - x_i^{\mathsf{T}}\hat{\beta} - x_i^{\mathsf{T}}\hat{\delta}\mathbf{1} [q_i \leq \hat{\gamma}(s_i)],$ 

$$\widehat{D}\left(\widehat{\gamma}\left(s\right),s\right) = \sum_{i=1}^{n} \omega_{2i}(s) x_{i} x_{i}^{\top}, \text{ and } \widehat{V}\left(\widehat{\gamma}\left(s\right),s\right) = \sum_{i=1}^{n} \omega_{2i}(s) x_{i} x_{i}^{\top} \widehat{u}_{i}^{2}$$

where

$$\omega_{1i}(s) = \frac{K\left((s_i - s)/b_n\right)}{\sum_{j=1}^n K\left((s_j - s)/b_n\right)} \text{ and } \omega_{2i}(s) = \frac{\mathbb{K}\left((q_i - \widehat{\gamma}(s))/b'_n, (s_i - s)/b''_n\right)}{\sum_{j=1}^n \mathbb{K}\left((q_j - \widehat{\gamma}(s))/b'_n, (s_j - s)/b''_n\right)}$$

for some bivariate kernel function  $\mathbb{K}(\cdot,\cdot)$  and bandwidth parameters  $(b'_n,b''_n).$ 

Finally, we show the  $\sqrt{n}$ -consistency of the semiparametric estimators  $\hat{\beta}$  and  $\hat{\delta}^*$  in (5) and (6). For this purpose, we first obtain the uniform rate of convergence of  $\hat{\gamma}(s)$ .

**Theorem 4** Under Assumptions ID and A,

$$\sup_{s \in \mathcal{S}} \left| \widehat{\gamma} \left( s \right) - \gamma_0 \left( s \right) \right| = O_p \left( \frac{\log n}{n^{1 - 2\epsilon} b_n} \right)$$

provided that  $n^{1-2\epsilon}b_n^2$  does not diverge.

Apparently, the uniform consistency of  $\widehat{\gamma}(s)$  follows provided  $\log n/(n^{1-2\epsilon}b_n) \rightarrow 0$ .

Based on this uniform convergence, the following theorem derives the joint limiting distribution of  $\widehat{\beta}$  and  $\widehat{\delta}^*$ . We let  $\widehat{\theta}^* = (\widehat{\beta}^\top, \widehat{\delta}^{*\top})^\top$  and  $\theta_0^* = (\beta_0^\top, \delta_0^{*\top})^\top$ .

**Theorem 5** Suppose the conditions in Theorem 4 hold and  $\log n/(n^{1-2\epsilon}b_n) \to 0$  as  $n \to \infty$ . If we let  $\Delta_n > 0$  such that  $\Delta_n \to 0$ ,  $\{\log n/(n^{1-2\epsilon}b_n)\}/\Delta_n \to 0$  as  $n \to \infty$ , we have

$$\sqrt{n}\left(\widehat{\theta}^* - \theta_0^*\right) \to_d \mathcal{N}\left(0, \Lambda^{*-1}\Omega^*\Lambda^{*-1}\right)$$
(16)

as  $n \to \infty$ , where

$$\Lambda^* = \begin{bmatrix} \mathbb{E}\left[x_i x_i^\top \mathbf{1}_i^+\right] & 0\\ 0 & \mathbb{E}\left[x_i x_i^\top \mathbf{1}_i^-\right] \end{bmatrix} \text{ and } \Omega^* = \lim_{n \to \infty} n^{-1} Var \begin{bmatrix} \sum_{i=1}^n x_i u_i \mathbf{1}_i^+\\ \sum_{i=1}^n x_i u_i \mathbf{1}_i^- \end{bmatrix}$$

with  $\mathbf{1}_{i}^{+} = \mathbf{1}[q_{i} > \gamma_{0}(s_{i})]\mathbf{1}[s_{i} \in \mathcal{S}]$  and  $\mathbf{1}_{i}^{-} = \mathbf{1}[q_{i} < \gamma_{0}(s_{i})]\mathbf{1}[s_{i} \in \mathcal{S}].$ 

Note that we do not use the conventional plug-in semiparametric least squares estimators,  $\arg \min_{\beta,\delta} \sum_{i=1}^{n} (y_i - x_i^{\top}\beta - x_i^{\top}\delta \mathbf{1} [q_i \leq \hat{\gamma}_{-i}(s_i)])^2 \mathbf{1}[s_i \in S]$ . The reason why we propose an alternative estimation approach here is that this conventional semiparametric least square estimators may not be asymptotically orthogonal to the first-step nonparametric estimator when  $n^{1-2\epsilon}b_n^2 \to \varrho \in (0,\infty)$  as  $n \to \infty$ , though they are still consistent. This is because the first-step nonparametric estimator  $\hat{\gamma}(s)$  could have very slow rate of convergence, and the estimation error will affect the limiting distribution of the second stage parametric estimators. The new estimation idea above, however, only uses the observations that are not affected by the estimation error in the first-step nonparametric estimator. This is done by choosing a large enough  $\Delta_n$  in (5) and (6) such that the observations are outside the uniform convergence bound of  $|\hat{\gamma}(s) - \gamma_0(s)|$ . Thanks to the threshold regression structure, we then can estimate the parameters on each side of the threshold even using these subsamples. However, we also want  $\Delta_n \to 0$  fast enough so that more observations are included in estimation.

The estimator  $(\hat{\beta}^{\top}, \hat{\delta}^{*\top})^{\top}$  or equivalently  $(\hat{\beta}^{\top}, \hat{\delta}^{\top})^{\top}$  thus satisfies the Neyman orthogonality condition (e.g., Assumption N(c) in Andrews (1994)), that is, replacing  $\hat{\gamma}$ by the true  $\gamma_0$  in estimating the parametric component has an effect at most  $o_p(n^{-1/2})$ in their limiting distribution. Though we lose some efficiency in finite samples, we can derive the asymptotic normality of  $(\widehat{\boldsymbol{\beta}}^{\top}, \widehat{\boldsymbol{\delta}}^{\top})^{\top}$  that has mean zero and achieves the same asymptotic variance as if  $\gamma_0(\cdot)$  was known.

Using the delta method, we can readily obtain the limiting distribution of  $\hat{\theta} = (\hat{\beta}^{\top}, \hat{\delta}^{\top})^{\top}$  as

$$\sqrt{n}\left(\widehat{\theta}-\theta_{0}\right)\rightarrow_{d}\mathcal{N}\left(0,\Lambda^{-1}\Omega\Lambda^{-1}\right) \text{ as } n\to\infty,$$
(17)

where  $\theta_0 = (\beta_0^{\top}, \delta_0^{\top})^{\top}$ ,  $\Lambda = \mathbb{E}\left[z_i z_i^{\top} \mathbf{1} [s_i \in \mathcal{S}]\right]$ , and  $\Omega = \lim_{n \to \infty} n^{-1} Var[\sum_{i=1}^n z_i u_i \mathbf{1} [s_i \in \mathcal{S}]]$ with  $z_i = \left[x_i^{\top}, x_i^{\top} \mathbf{1} [q_i \leq \gamma_0(s_i)]\right]^{\top}$ . The asymptotic variance expressions in (16) and (17) allow for cross-sectional dependence as they have the long-run variance forms  $\Omega^*$ and  $\Omega$ . They can be consistently estimated by the spatial HAC estimator of Conley and Molinari (2007) using  $\hat{u}_i = \{y_i - x_i^{\top} \hat{\beta} - x_i^{\top} \hat{\delta} \mathbf{1} [q_i \leq \hat{\gamma}_{-i}(s_i)]\} \mathbf{1} [s_i \in \mathcal{S}]$ . The terms  $\Lambda^*$  and  $\Lambda$  can be estimated by their sample analogues.

## 4 Threshold Contour

When we consider sample splitting over a two-dimensional space (i.e.,  $q_i$  and  $s_i$  respectively correspond to the latitude and longitude on the map), the threshold model (1) can be generalized to estimate a nonparametric contour threshold model:

$$y_i = x_i^{\top} \beta_0 + x_i^{\top} \delta_0 \mathbf{1} \left[ m_0 \left( q_i, s_i \right) \le 0 \right] + u_i, \tag{18}$$

where the unknown function  $m_0 : \mathcal{Q} \times \mathcal{S} \mapsto \mathbb{R}$  determines the contour on a random field. An interesting example includes identifying an unknown closed boundary over the map, such as a city boundary relative to some city center, and an area of a disease outbreak or airborne pollution. In social science, it can identify a group boundary or a region in which the agents share common demographic, political, or economic characteristics.

To relate this generalized form to the original threshold model (1), we suppose there exists a known center at  $(q_i^*, s_i^*)$  such that  $m_0(q_i^*, s_i^*) < 0$ . Without loss of generality, we can normalize  $(q_i^*, s_i^*)$  to be (0,0) and re-center all other observations  $\{q_i, s_i\}_{i=1}^n$  accordingly. In addition, we define the radius distance  $l_i$  and angle  $a_i^\circ$  of the

#### Figure 2: Illustration of Rotation



*i*th observation relative to the origin as

$$l_{i} = \sqrt{q_{i}^{2} + s_{i}^{2}},$$
  

$$a_{i}^{\circ} = \bar{a}_{i}^{\circ} \mathbf{I}_{i} + (180^{\circ} - \bar{a}_{i}^{\circ}) \mathbf{I} \mathbf{I}_{i} + (180^{\circ} + \bar{a}_{i}^{\circ}) \mathbf{I} \mathbf{I}_{i} + (360^{\circ} - \bar{a}_{i}^{\circ}) \mathbf{I} \mathbf{V}_{i},$$

where  $\bar{a}_i^{\circ} = \arctan(|q_i/s_i|)$ , and each of  $(\mathbf{I}_i, \mathbf{II}_i, \mathbf{III}_i, \mathbf{IV}_i)$  respectively denotes the indicator that the *i*th observation locates in the first, second, third, and forth quadrant.

We suppose that there is only one breakpoint at any angle and the threshold contour is convex. For each fixed  $a^{\circ} \in [0^{\circ}, 360^{\circ})$ , we rotate the original coordinate counterclockwise and implement the least squares estimation (4) only using the observations in the first two quadrants after rotation. Note that using the observations in the first two quadrants ensures that the threhold mapping after rotation is a well-defined function.

In particular, the angle relative to the origin is  $a_i^{\circ} - a^{\circ}$  after rotating the coordinate by  $a^{\circ}$  degrees counterclockwise, and the new location (after the rotation) is given as  $(q_i(a^{\circ}), s_i(a^{\circ}))$ , where

$$\begin{pmatrix} q_i(a^\circ) \\ s_i(a^\circ) \end{pmatrix} = \begin{pmatrix} q_i\cos(a^\circ) - s_i\sin(a^\circ) \\ s_i\cos(a^\circ) + q_i\sin(a^\circ) \end{pmatrix}$$

After this rotation, we estimate the following nonparametric threshold model:

$$y_i = x_i^{\top} \beta_0 + x_i^{\top} \delta_0 \mathbf{1} \left[ q_i \left( a^{\circ} \right) \le \gamma_{a^{\circ}} \left( s_i \left( a^{\circ} \right) \right) \right] + u_i \tag{19}$$

using only the observations satisfying  $q_i(a^\circ) \geq 0$ , where  $\gamma_{a^\circ}(\cdot)$  serves as the un-

known threshold line as in the model (1) in the  $a^{\circ}$ -degree-rotated coordinate. Such reparametrization guarantees that  $\gamma_{a^{\circ}}(\cdot)$  is always positive and we estimate its value pointwisely at 0. Figure 2 illustrates the idea of such rotation and pointwise estimation over a bounded support so that only the red cross points are included for estimation at different angles. Thus, the estimation and inference procedure developed before is directly applicable, though we expect efficiency loss as we only use a subsample in estimation at each rotated coordinate.

This rotating coordinate idea can be a quick solution when we do not know which variables should be assigned as  $q_i$  versus  $s_i$ , in the original model (1). As an extreme example, if  $\gamma_0$  is the vertical line, the original model does not work. In this case, we can check if  $\gamma_0$  is (near) the vertical line by investigating the estimates among different rotations; when  $\gamma_0$  is suspected as the vertical line or has a very steep slope, we can switch  $q_i$  and  $s_i$  in the original model (1) to improve the local constant fitting. In addition, this idea can be also used as a robustness check of a threshold function estimate. As we demonstrate in Section 6, if  $\gamma_0$  is a well-behaving function as in Assumption A, we should have similar estimates even after rotations to some angles when we use the entire sample for each rotation. In the robustness check, the rotation angle has to be within  $\pm 90^{\circ}$  unless the mapping from  $s_i$  to  $q_i$  is deemed to be one-to-one.

### 5 Monte Carlo Experiments

We examine the small sample performance of the semiparametric threshold regression estimator by Monte Carlo simulations. We generate n draws from

$$y_i = X_i^{\top} \beta_0 + X_i^{\top} \delta_0 \mathbf{1} \left[ q_i \le \gamma_0 \left( s_i \right) \right] + u_i, \tag{20}$$

where  $X_i = (1, x_i)^{\top}$  and  $x_i \in \mathbb{R}$ . We let  $\beta_0 = (\beta_{01}, \beta_{02})^{\top} = 0\iota_2$  and consider three different values of  $\delta_0 = (\delta_{01}, \delta_{02})^{\top} = 1\iota_2$ ,  $2\iota_2$ , and  $3\iota_2$ , where  $\iota_2 = (1, 1)^{\top}$ . For the threshold function, we let  $\gamma_0(s) = \sin(s)/2$ . We consider the cross-sectional dependence structure in  $(x_i, q_i, s_i, u_i)^{\top}$  as follows:

$$\begin{cases} (q_i, s_i)^{\top} \sim iid\mathcal{N}(0, I_2); \\ x_i | (q_i, s_i) \sim iid\mathcal{N}(0, (1 + \rho (s_i^2 + q_i^2))^{-1}); \\ \underline{\mathbf{u}} | \{(x_i, q_i, s_i)\}_{i=1}^n \sim \mathcal{N}(0, \Sigma), \end{cases}$$
(21)

s = 0.0					s = 0.5					s = 1.0				
n	$\delta_{02}=1$	2	3	4		1	2	3	4	-	1	2	3	4
100	0.14	0.06	0.05	0.05		0.16	0.07	0.05	0.05		0.25	0.18	0.14	0.13
200	0.08	0.03	0.02	0.02		0.08	0.04	0.02	0.02		0.15	0.10	0.06	0.06
500	0.05	0.01	0.02	0.02		0.05	0.02	0.02	0.02		0.09	0.05	0.03	0.01

Table 2: Rej. Prob. of the LR Test with i.i.d. Data

Note: Entries are rejection probabilities of the LR test (14) when data are generated from (20) with  $\gamma_0(s) = \sin(s)/2$ . The dependence structure is given in (21) with  $\rho = 0$ . The significance level is 5% and the results are based on 1000 simulations.

s = 0.0						s = 0.5						s = 1.0				
n	$\delta_{02}=1$	2	3	4	1	2	3	4		1	2	3	4			
100	0.19	0.10	0.07	0.03	0.20	0.10	0.08	0.07		0.28	0.19	0.17	0.11			
200	0.10	0.04	0.03	0.03	0.12	0.07	0.04	0.04		0.21	0.11	0.08	0.04			
500	0.05	0.02	0.02	0.02	0.06	0.03	0.02	0.02		0.14	0.05	0.03	0.03			

Table 3: Rej. Prob. of the LR Test with Cross-sectionally Correlated Data

Note: Entries are rejection probabilities of the LR test (14) when data are generated from (20) with  $\gamma_0(s) = \sin(s)/2$ . The dependence structure is given in (21) with  $\rho = 1$  and m = 10. The significance level is 5% and the results are based on 1000 simulations.

where  $\mathbf{\underline{u}} = (u_1, \ldots, u_n)^{\top}$ . The (i, j)-th element of  $\Sigma$  is  $\Sigma_{ij} = \rho^{\lfloor \ell_{ij}n \rfloor} \mathbf{1}[\ell_{ij} < m/n]$ , where  $\ell_{ij} = \{(s_i - s_j)^2 + (q_i - q_j)^2\}^{1/2}$  is the  $L^2$ -distance between the *i*th and *j*th observations. The diagonal elements of  $\Sigma$  are normalized as  $\Sigma_{ii} = 1$ . This *m*-dependent setup follows from the Monte Carlo experiment in Conley and Molinari (2007) in the sense that there are roughly at most  $2m^2$  observations that are correlated with each observation. Within the *m* distance, the dependence decays at a polynomial rate as indicated by  $\rho^{\lfloor \ell_{ij}n \rfloor}$ . The parameter  $\rho$  describes the strength of cross-sectional dependence in the way that a larger  $\rho$  leads to stronger dependence relative to the unit standard deviation. In particular, we consider the cases with  $\rho = 0$  (i.e., i.i.d. observations), 0.5, and 1. We consider the sample size n = 100, 200, and 500.

First, Tables 2 and 3 report the small sample rejection probabilities of the LR test in (14) for  $H_0$ :  $\gamma_0(s) = \sin(s)/2$  against  $H_1$ :  $\gamma_0(s) \neq \sin(s)/2$  at 5% nominal level at three different locations s = 0, 0.5, and 1. In particular, Table 2 examines the case with no cross-sectional dependence ( $\rho = 0$ ), while Table 3 examines the case with cross-sectional dependence whose dependence decays slowly with  $\rho = 1$  and m = 10.

$\beta_{20}$						$\beta_{20}+\delta_{20}$					$\delta_{20}$				
n	$\delta_{02}=1$	2	3	4	1	2	3	4	-	1	2	3	4		
100	0.85	0.89	0.91	0.87	0.87	0.87	0.89	0.90		0.85	0.87	0.93	0.91		
200	0.86	0.90	0.93	0.93	0.89	0.92	0.94	0.93		0.85	0.90	0.93	0.92		
500	0.83	0.92	0.95	0.96	0.84	0.90	0.93	0.94		0.78	0.88	0.93	0.95		

Table 4: Coverage Prob. of the Plug-in Confidence Interval

Note: Entries are coverage probabilities of 95% confidence intervals for  $\beta_{02}$  and  $\delta_{02}$  based on asymptotic normality and plugging in  $\hat{\gamma}(s_i)$  for  $\gamma_0(s_i)$ . Data are generated from (20) with  $\gamma_0(s) = \sin(s)/2$ , where the dependence structure is given in (21) with  $\rho = 0.5$  and m = 3. The results are based on 1000 simulations.

Table 5: Coverage Prob. of the Plug-in Confidence Interval (w/ LRV adj.)

$\beta_{20}$						$\beta_{20}+\delta_{20}$					$\delta_{20}$				
n	$\delta_{02}=1$	2	3	4		1	2	3	4		1	2	3	4	
100	0.92	0.95	0.94	0.95	(	0.91	0.95	0.94	0.95		0.93	0.95	0.95	0.95	
200	0.93	0.95	0.97	0.96	(	0.94	0.94	0.95	0.96		0.90	0.93	0.97	0.94	
500	0.89	0.95	0.97	0.97	(	0.89	0.96	0.97	0.97		0.84	0.92	0.95	0.97	

Note: Entries are coverage probabilities of 95% confidence intervals for  $\beta_{02}$  and  $\delta_{02}$  based on asymptotic normality, plugging in  $\hat{\gamma}(s_i)$  for  $\gamma_0(s_i)$ , and a small sample adjustment of the LRV estimator. Data are generated from (20) with  $\gamma_0(s) = \sin(s)/2$ , where the dependence structure is given in (21) with  $\rho = 0.5$  and m = 3. The results are based on 1000 simulations.

For the bandwidth parameter, we normalize  $s_i$  and  $q_i$  to have mean zero and unit standard deviation and choose  $b_n = 0.5n^{-1/2}$  in the main regression. This choice is for undersmoothing as  $n^{1-2\epsilon}b_n^2 = n^{-2\epsilon} \to 0$ . To estimate  $D(\gamma_0(s), s)$  and  $V(\gamma_0(s), s)$ , we use the rule-of-thumb bandwidths from the standard kernel regression satisfying  $b'_n = O(n^{-1/5})$  and  $b''_n = O(n^{-1/6})$ . All the results are based on 1000 simulations. In general, the test for  $\gamma_0$  performs better as (i) the sample size gets larger; (ii) the coefficient change gets more significant; (iii) the cross-sectional dependence gets weaker; and (iv) the target gets closer to the mid-support of s. When  $\delta_0$  and n are large, the LR test is conservative, which is also found in the classic threshold regression case (Hansen (2000)).

Second, Table 4 shows the finite sample coverage properties of the 95% confidence intervals for the parametric components  $\beta_{02}$ ,  $\delta_{02}^* = \beta_{02} + \delta_{02}$ , and  $\delta_{02}$ . The results are based on the same simulation design as above with  $\rho = 0.5$  and m = 3. Regarding the tuning parameters, we use the same bandwidth choice  $b_n = 0.5n^{-1/2}$  as before and set the truncation parameter  $\Delta_n = (nb_n)^{-1/2}$ . Unreported results suggest that choice of the constant in the bandwidth matters particularly with small samples like n = 100, but such effect quickly decays as the sample size gets larger. For the lag number required for the HAC estimator, we use the spatial lag order of 5 following Conley and Molinari (2007). Results with other lag choices are similar and hence omitted. The result suggests that the asymptotic normality is better approximated with larger samples and larger change sizes. Table 5 shows the same results with a small sample adjustment of the LRV estimator for  $\Omega^*$  by dividing it by the sample truncation fraction  $\sum_{i=1}^{n} (\mathbf{1}[q_i > \hat{\gamma}(s_i)] + \mathbf{1}[q_i < \hat{\gamma}(s_i)])\mathbf{1}[s_i \in \mathcal{S}] / \sum_{i=1}^{n} \mathbf{1}[s_i \in \mathcal{S}]$ . This ratio enlarges the LRV estimator and hence the coverage probabilities, especially when the change size is small. It only affects the finite sample performance as it approaches one in probability as  $n \to \infty$ .

## 6 Empirical illustration

As an illustration of the nonparametric threshold, we study the economic border between the Queens and the Brooklyn boroughs in New York City. The current administrative border is determined in 1931 using coordinates suggested by multiple federal agencies but ignores the rapid development in the city. Some part of it now even runs through houses, causing troubles for policy maker and local residents.<sup>7</sup> We collect the single family house sales data in the year 2017 and examine an economic border induced by a nonparametric threshold regression model.<sup>8</sup> In particular, we consider the model (1) with the following variables:<sup>9</sup>

 $<sup>^7\</sup>mathrm{Pictures}$  of the confusing border are available at https://urbanomnibus.net/2015/01/borderlands-traveling-the-brooklyn-queens-divide/

<sup>&</sup>lt;sup>8</sup>The data set (*Rolling Sales Data*) is available at http://www1.nyc.gov/site/finance/taxes/property-rolling-sales-data.page.

 $<sup>^9</sup>$  "Gross Square Footage" is the total area of all the floors of a building as measured from the exterior surfaces of the outside walls of the building, including the land area and space within any building or structure on the property. (Source: http://www1.nyc.gov/assets/finance/downloads/pdf/07pdf/glossary\_rsf071607.pdf)

#### Figure 3: Border Estimation in NYC



$y_i$ :	log house price (in \$)
$x_i$ :	constant
	log of Gross Square Footage (in $\mathrm{ft}^2)$
	dummy for built before 1945, WWII
$q_i$ :	(rotated) latitude
$s_i$ :	(rotated) longitude

For the pair  $(q_i, s_i)$ , we consider two cases: the original latitude-longitude on the map; and the "rotated" latitude-longitude relative to the middle point of the administrative border. The rotation method is described in Section 4, where we choose the rotation angle as the slope of the linear regression line approximating the administrative border. We focus on single family houses under property tax Class 1, accounting for 57.9% of the original sample, and drop duplicate observations. The sample size is n = 8121, including 5966 observations in Queens and 2155 observations in Brooklyn.

Figure 3 depicts the nonparametric threshold function estimates  $\hat{\gamma}$  based on the rotated coordinate, which is the "unknown" economic border that splits the Queens and the Brooklyn boroughs in New York City based on the threshold in housing price. The estimated border (black solid line) is found to be substantively different from the administrative border between these two boroughs (orange dot line). Somewhat

surprisingly, the 95% pointwise confidence interval (blue dash lines) contains the Forest Park and the Long Island Rail Road (LIRR) route to the east of Jamaica Center Station. As a robustness check, we also estimate the model by setting  $(q_i, s_i)$  as the original (unrotated) latitude and longitude on the map. The estimated border is very close to the depicted results.<sup>10</sup>

We choose the bandwidth  $b_n$  in the main regression as  $cn^{-1/2}$  and we obtain the constant c by the cross validation. In particular, we choose c that minimizes  $\sum_{i=1}^{n} (y_i - \hat{y}_i)^2 \mathbf{1} [s_i \in S]$ , where  $\hat{y}_i = x_i^{\top} \hat{\beta}_{-i} + x_i^{\top} \hat{\delta}_{-i} \mathbf{1} [q_i \leq \hat{\gamma}_{-i} (s_i)]$  and  $(\hat{\beta}_{-i}^{\top}, \hat{\delta}_{-i}^{\top}, \hat{\gamma}_{-i} (\cdot))^{\top}$ are obtained using the leave-one-out observations as described in Section 2. Here, Sincludes the observations between 15th and 85th percentiles of the sample  $\{s_i\}_{i=1}^{n}$ .

Table 6 summarizes the coefficient estimates for the parametric components,  $\hat{\beta}$  and  $\hat{\delta}$ . The standard errors reported in the parentheses are computed using the spatial HAC estimator with 5 spatial lags (e.g., Conley and Molinari (2007)). The average housing price elasticity on the southern side of the economic border is lower than that on the northern side. The (semi-)elasticity of the Gross Square Footage and the effect of the house age are slightly larger on the southern side. These patterns are quite robust to whether we use the rotated coordinate or not. As a comparison, we also run the experiment using the current administrative border as  $\gamma_0(\cdot)$ . In particular, the last two columns in Table 6 suggest that there does not exist a significant coefficient change if the sample splitting is based on the current existing administrative border.

## 7 Concluding Remarks

In this paper, we propose a general approach of sample splitting, where multiple variables can jointly determine the unknown separation boundary. We develop a semiparametric threshold regression model over a random field, in which the threshold is determined by a nonparametric function between two variables. Our approach can be easily generalized so that the sample splitting depends on more than two variables, though such extension is subject to curse of dimensionality as usually observed in the kernel regression literature. The main interest is in identifying the threshold function resulting in sample splitting, and thus the model developed in this paper should be

<sup>&</sup>lt;sup>10</sup>Since  $\gamma_0(\cdot)$  is inside of the indicator function, the local constant estimator  $\hat{\gamma}(s)$  is not smooth in finite samples. It is also related with the well known phenomenon that the threshold estimate is not unique in finite samples even in the standard constant threshold model. In particular, we use the lower bound value of the estimated set as the local constant estimate  $\hat{\gamma}(s)$ .

	Estimat	ed Border	Estima	ted Border	Admin	Border
	(rotated	coordinate)	(original	coordinate)		
	$\widehat{eta}$	$\widehat{\delta}$	$\widehat{\beta}$	$\widehat{\delta}$	$\widehat{\beta}$	$\widehat{\delta}$
constant	9.68	-3.24	9.91	-1.08	8.19	-2.95
	$(0.24)^{**}$	$(0.35)^{**}$	$(0.01)^{**}$	$(0.01)^{**}$	$(1.24)^{**}$	(2.96)
$\log of Gross ft^2$	0.53	0.38	0.40	-0.01	0.71	0.39
	$(0.03)^{**}$	$(0.05)^{**}$	$(0.05)^{**}$	(0.04)	$(0.17)^{**}$	(0.40)
built before 1945	-0.06	0.10	-0.07	-0.06	-0.19	0.44
	$(0.02)^{**}$	$(0.03)^{**}$	$(0.01)^{**}$	$(0.01)^{**}$	$(0.09)^*$	(0.26)

Table 6: Estimation Results

Note: Entries are estimates and standard errors of coefficients in the economic border example. Columns of "Estimated Border" are based on the nonparametric threshold estimates; columns of "Admin Border" are based on the current administrative border as the threshold function. \*\* and \* are significant at 1% and 5%, respectively.

distinguished from the smoothed threshold regression model or the random coefficient regression model.

This new model has high applicability in broad areas studying sample splitting (e.g., segregations and group-formation) and heterogeneous effects over different subsamples. The potential areas include economics, political science, sociology, and marketing science, where the agent-specific heterogeneity and social segregation are important; and regional science and urban economics, where the identification of unobserved/unknown boundaries is of interest using satellite data.

In practice, we may need a testing procedure to check whether or not the classic constant threshold model is sufficient to describe a sample splitting phenomenon. In a companion project, the authors are developing a test for a constant threshold, based on which the nonparametric threshold developed in this paper can be supported. Unlike the existing studies that focus on testing no change (i.e.,  $\delta_0 = 0$  in (1)) against one change, or testing on a fixed number of changes (e.g., Bai and Perron (1998)), we are developing a test that works for a general null hypothesis of any number of changes versus nonparametric alternatives.

## A Appendix

#### A.1 Proof of Theorem 1

**Proof of Theorem 1** First, any given  $\gamma_0(\cdot) = \gamma \in \Gamma$ , the parameters  $\beta_0$  and  $\delta_0$  are well identified as the unique minimizer of

$$\mathbb{E}\left[\left(y_i - x_i^{\top}\beta_0 - x_i^{\top}\delta_0 \mathbf{1} \left[q_i \leq \gamma\right]\right)^2\right]$$

since  $\mathbb{E}\left[z_i(\gamma)z_i(\gamma)^{\top}\right]$  is positive definite under Assumptions ID-(i) and (ii), where  $z_i(\gamma) = [x_i^{\top}, x_i^{\top} \mathbf{1} [q_i \leq \gamma]]^{\top}$ . Second, the function  $\gamma_0(\cdot)$  is pointwisely identified as the minimizer of

$$\mathbb{E}\left[\left.\left(y_i - x_i^\top \beta_0 - x_i^\top \delta_0 \mathbf{1} \left[q_i \le \gamma(s_i)\right]\right)^2\right| s_i = s\right]$$

for each  $s \in \mathcal{S}$ . This is because for any  $\gamma(s) \neq \gamma_0(s)$  at  $s_i = s$  and given  $(\beta_0^{\top}, \delta_0^{\top})^{\top}$ ,

$$\begin{aligned} R(\beta_0, \delta_0, \gamma(s); s) \\ &= \mathbb{E}\left[\left(y_i - x_i^\top \beta_0 - x_i^\top \delta_0 \mathbf{1} \left[q_i \le \gamma(s_i)\right]\right)^2 \middle| s_i = s\right] \\ &- \mathbb{E}\left[\left(y_i - x_i^\top \beta_0 - x_i^\top \delta_0 \mathbf{1} \left[q_i \le \gamma_0(s_i)\right]\right)^2 \middle| s_i = s\right] \\ &= \delta_0^\top \mathbb{E}\left[x_i x_i^\top \left(\mathbf{1} \left[q_i \le \gamma(s_i)\right] - \mathbf{1} \left[q_i \le \gamma_0(s_i)\right]\right)^2 \middle| s_i = s\right] \delta_0 \\ &= \delta_0^\top \mathbb{E}\left[x_i x_i^\top \mathbf{1} \left[\min\{\gamma(s_i), \gamma_0(s_i)\} < q_i \le \max\{\gamma(s_i), \gamma_0(s_i)\}\right] \middle| s_i = s\right] \delta_0 \\ &= \int_{\min\{\gamma(s), \gamma_0(s)\}}^{\max\{\gamma(s), \gamma_0(s)\}} \delta_0^\top \mathbb{E}\left[x_i x_i^\top \middle| q_i = q, s_i = s\right] \delta_0 f(q|s) dq \\ &\ge C(s) \mathbb{P}\left(\min\{\gamma(s_i), \gamma_0(s_i)\} < q_i \le \max\{\gamma(s_i), \gamma_0(s_i)\} \middle| s_i = s\right) \\ &> 0 \end{aligned}$$

from Assumptions ID-(i) and (iii), where  $C(s) = \inf_{q \in \mathcal{Q}} \delta_0^\top \mathbb{E} \left[ x_i x_i^\top | q_i = q, s_i = s \right] \delta_0 > 0$ . Note that the last probability is strictly positive because we assume f(q|s) > 0 for any  $(q, s) \in \mathcal{Q} \times \mathcal{S}$  and  $\gamma_0(s)$  is not located on the boundary of  $\mathcal{Q}$  as  $\varepsilon(s) < \mathbb{P} \left( q_i \leq \gamma_0(s_i) | s_i = s \right) < 1 - \varepsilon(s)$  for some  $\varepsilon(s) > 0$ . The identification follows since  $R(\beta_0, \delta_0, \gamma(s); s)$  is continuous at  $\gamma(s) = \gamma_0(s)$  from Assumption ID-(iv).

### A.2 Proof of Theorem 2

Throughout the proof, we denote  $K_i(s) = K((s_i - s)/b_n)$  and  $\mathbf{1}_i(\gamma) = \mathbf{1} [q_i \leq \gamma]$ . We let  $C \in (0, \infty)$  stand for a generic constant term that may vary, which can depend on the location s. We also let  $a_n = n^{1-2\epsilon}b_n$ . All the lemmas in the proof assume the conditions in Assumptions ID and A hold. Omitted proofs for some lemmas are all collected in the supplementary material.

For a given  $s \in \mathcal{S}$ , we define

$$M_{n}(\gamma; s) = \frac{1}{nb_{n}} \sum_{i=1}^{n} x_{i} x_{i}^{\top} \mathbf{1}_{i}(\gamma) K_{i}(s),$$
  
$$J_{n}(\gamma; s) = \frac{1}{\sqrt{nb_{n}}} \sum_{i=1}^{n} x_{i} u_{i} \mathbf{1}_{i}(\gamma) K_{i}(s).$$

Lemma A.1

$$\sup_{\gamma \in \Gamma} \|M_n(\gamma; s) - M(\gamma; s)\| \to_p 0,$$
$$\sup_{\gamma \in \Gamma} \|n^{-1/2} b_n^{-1/2} J_n(\gamma; s)\| \to_p 0$$

as  $n \to \infty$ , where

$$M(\gamma; s) = \int_{-\infty}^{\gamma} D(q, s) f(q, s) \, dq$$

and

$$J_n\left(\gamma;s\right) \Rightarrow J\left(\gamma;s\right)$$

a mean-zero Gaussian process indexed by  $\gamma$ .

**Proof of Lemma A.1** For expositional simplicity, we only present the case of scalar  $x_i$ . We first prove the pointwise convergence of  $M_n(\gamma; s)$ . By stationarity, Assumptions A-(vii), (x), and Taylor expansion, we have

$$\mathbb{E}\left[M_{n}\left(\gamma;s\right)\right] = \frac{1}{b_{n}} \iint \mathbb{E}[x_{i}^{2}|q,v]\mathbf{1}[q \leq \gamma]K\left(\frac{v-s}{b_{n}}\right)f\left(q,v\right)dqdv \qquad (A.1)$$
$$= \iint D(q,s+b_{n}t)\mathbf{1}[q \leq \gamma]K\left(t\right)f\left(q,s+b_{n}t\right)dqdt$$
$$= \int_{-\infty}^{\gamma} D(q,s)f\left(q,s\right)dq + O\left(b_{n}^{2}\right),$$

where D(q, s) is defined in (8). For the variance, we have

$$Var\left[M_{n}\left(\gamma;s\right)\right] = \frac{1}{n^{2}b_{n}^{2}}\mathbb{E}\left[\left(\sum_{i=1}^{n}\left\{x_{i}^{2}\mathbf{1}_{i}\left(\gamma\right)K_{i}\left(s\right)-\mathbb{E}\left[x_{i}^{2}\mathbf{1}_{i}\left(\gamma\right)K_{i}\left(s\right)\right]\right\}\right)^{2}\right](A.2)$$
$$= \frac{1}{nb_{n}^{2}}\mathbb{E}\left[\left\{x_{i}^{2}\mathbf{1}_{i}\left(\gamma\right)K_{i}\left(s\right)-\mathbb{E}\left[x_{i}^{2}\mathbf{1}_{i}\left(\gamma\right)K_{i}\left(s\right)\right]\right\}^{2}\right]$$
$$+\frac{2}{n^{2}b_{n}^{2}}\sum_{i
$$= O\left(\frac{1}{nb_{n}}\right)+O\left(\frac{1}{n}+b_{n}^{2}\right)\to 0,$$$$

where the order of the first term is from the standard kernel estimation result. For the second term, we use Assumptions A-(v), (vii), (x), and Lemma 1 of Bolthausen (1982) to obtain that

$$\left| \frac{1}{n} \sum_{i < j}^{n} Cov \left[ x_{i}^{2} \mathbf{1}_{i} \left( \gamma \right) K_{i} \left( s \right), x_{j}^{2} \mathbf{1}_{j} \left( \gamma \right) K_{j} \left( s \right) \right] \right|$$

$$\leq \frac{1}{n} \sum_{i < j}^{n} \left| Cov \left[ x_{i}^{2} \mathbf{1}_{i} \left( \gamma \right) K \left( \frac{s_{i} - s}{b_{n}} \right), x_{j}^{2} \mathbf{1}_{j} \left( \gamma \right) K \left( \frac{s_{j} - s}{b_{n}} \right) \right] \right|$$

$$= \frac{b_{n}^{2}}{n} \sum_{i < j}^{n} \left| Cov \left[ x_{i}^{2} \mathbf{1}_{i} \left( \gamma \right) K \left( t_{i} \right), x_{j}^{2} \mathbf{1}_{j} \left( \gamma \right) K \left( t_{j} \right) \right] + O \left( b_{n}^{2} \right) \right|$$

$$\leq Cb_{n}^{2} \sum_{m=1}^{\infty} m\alpha \left( m \right)^{\varphi/(2+\varphi)} \left( \mathbb{E} \left[ x_{i}^{4+2\varphi} \mathbf{1}_{i} \left( \gamma \right) K \left( t_{i} \right)^{2+\varphi} \right] \right)^{2/(2+\varphi)} + O \left( nb_{n}^{4} \right)$$

$$= O \left( b_{n}^{2} + nb_{n}^{4} \right)$$
(A.3)

for some finite  $\varphi > 0$ , where  $\alpha(m)$  is the mixing coefficient defined in (7) and the first equality is by the change of variables  $t_i = (s_i - s)/b_n$  in the covariance operator. Hence, the pointwise convergence is established. For given s, the uniform tightness of  $M_n(\gamma; s)$ in  $\gamma$  follows similarly as (and even simpler than) that of  $J_n(\gamma; s)$  below, and the uniform convergence follows from standard argument. For  $J_n(\gamma; s)$ , since  $\mathbb{E}[u_i x_i | q_i, s_i] = 0$ , the proof for  $\sup_{\gamma \in \Gamma} |(nb_n)^{-1/2} J_n(\gamma, s)| \xrightarrow{p} 0$  is identical as  $M_n(\gamma; s)$  and hence omitted.

Next, we derive the weak convergence of  $J_n(\gamma; s)$ . For any fixed s and  $\gamma$ , the Theorem of Bolthausen (1982) implies that  $J_n(\gamma; s) \Rightarrow J(\gamma; s)$  under Assumption A-(iii). Because  $\gamma$  is in the indicator function, such pointwise convergence in  $\gamma$  can be generalized into any finite collection of  $\gamma$  to yield the finite dimensional convergence in distribution. By theorem 15.5 of Billingsley (1968), it remains to show that, for each positive  $\eta(s)$  and  $\varepsilon(s)$  at given s, there exist  $\varpi > 0$  such that if n is large enough,

$$\mathbb{P}\left(\sup_{\gamma\in[\gamma_{1},\gamma_{1}+\varpi]}\left|J_{n}\left(\gamma;s\right)-J_{n}\left(\gamma_{1};s\right)\right|>\eta(s)\right)\leq\varepsilon(s)\varpi$$

for any  $\gamma_1$ . To this end, we consider a fine enough grid over  $[\gamma_1, \gamma_1 + \varpi]$  such that  $\gamma_g = \gamma_1 + (g-1)\varpi/\overline{g}$  for  $g = 1, \ldots, \overline{g}+1$ , where  $nb_n\varpi/2 \leq \overline{g} \leq nb_n\varpi$  and  $\max_{1\leq g\leq \overline{g}} \left(\gamma_g - \gamma_{g-1}\right) \leq \varpi/\overline{g}$ . We define  $h_{ig}(s) = x_i u_i K_i(s) \mathbf{1} \left[\gamma_g < q_i \leq \gamma_{g+1}\right]$  and  $H_{ng}(s) = n^{-1}b_n^{-1}\sum_{i=1}^n |h_{ig}(s)|$  for  $1 \leq g \leq \overline{g}$ . Then for any  $\gamma \in [\gamma_g, \gamma_{g+1}]$ ,

$$\begin{aligned} \left| J_n\left(\gamma;s\right) - J_n\left(\gamma_g;s\right) \right| &\leq \sqrt{nb_n} H_{ng}(s) \\ &\leq \sqrt{nb_n} \left| H_{ng}(s) - \mathbb{E}\left[ H_{ng}(s) \right] \right| + \sqrt{nb_n} \mathbb{E}\left[ H_{ng}(s) \right] \end{aligned}$$

and hence

$$\begin{split} \sup_{\substack{\gamma \in [\gamma_1, \gamma_1 + \varpi]}} & \left| J_n\left(\gamma; s\right) - J_n\left(\gamma_1; s\right) \right| \\ \leq & \max_{\substack{2 \le g \le \overline{g} + 1 \\ 1 \le g \le \overline{g}}} \sqrt{nb_n} \left| J_n\left(\gamma_g; s\right) - J_n\left(\gamma_1; s\right) \right| \\ & + \max_{\substack{1 \le g \le \overline{g}}} \sqrt{nb_n} \left| H_{ng}(s) - \mathbb{E}\left[ H_{ng}(s) \right] \right| + \max_{\substack{1 \le g \le \overline{g}}} \sqrt{nb_n} \mathbb{E}\left[ H_{ng}(s) \right] \\ \equiv & \Psi_1(s) + \Psi_2(s) + \Psi_3(s). \end{split}$$

In what follows, we simply denote  $h_i(s) = x_i u_i K_i(s) \mathbf{1} [\gamma_g < q_i \le \gamma_k]$  for any given  $1 \le g < k \le \overline{g}$  and for fixed s. First, for  $\Psi_1(s)$ , we have

$$\mathbb{E}\left[\left|J_{n}\left(\gamma_{g};s\right) - J_{n}\left(\gamma_{k};s\right)\right|^{4}\right]$$

$$= \frac{1}{n^{2}b_{n}^{2}}\sum_{i=1}^{n}\mathbb{E}\left[h_{i}^{4}(s)\right] + \frac{1}{n^{2}b_{n}^{2}}\sum_{i\neq j}^{n}\mathbb{E}\left[h_{i}^{2}(s)h_{j}^{2}(s)\right] + \frac{1}{n^{2}b_{n}^{2}}\sum_{i\neq j}^{n}\mathbb{E}\left[h_{i}^{3}(s)h_{j}(s)\right]$$

$$+ \frac{1}{n^{2}b_{n}^{2}}\sum_{i\neq j\neq k\neq l}^{n}\mathbb{E}\left[h_{i}(s)h_{j}(s)h_{k}(s)h_{l}(s)\right] + \frac{1}{n^{2}b_{n}^{2}}\sum_{i\neq j\neq k}^{n}\mathbb{E}\left[h_{i}^{2}(s)h_{j}(s)h_{k}(s)\right]$$

$$\equiv \Psi_{11}(s) + \Psi_{12}(s) + \Psi_{13}(s) + \Psi_{14}(s) + \Psi_{15}(s),$$

where each term's bound is obtained as follows. For  $\Psi_{11}(s)$ , a straightforward calculation and Assumptions A-(v) and (x) yield  $\Psi_{11}(s) \leq C_1(s)n^{-1}b_n^{-1} + O(b_n/n) = O(n^{-1}b_n^{-1})$  for some constant  $0 < C_1(s) < \infty$ . For  $\Psi_{12}(s)$ , similarly as (A.3),

$$\Psi_{12}(s) \leq \frac{2}{n^2 b_n^2} \sum_{i < j}^n \left( \mathbb{E} \left[ h_i^2(s) \right] \mathbb{E} \left[ h_j^2(s) \right] + \left| Cov \left[ h_i^2(s), h_j^2(s) \right] \right| \right)$$

$$\leq 2 \left( \mathbb{E} \left[ \widetilde{h}_i^2 \right] \right)^2 + \frac{2}{n b_n^2} \left\{ C b_n^2 \sum_{m=1}^\infty m \alpha \left( m \right)^{\varphi/(2+\varphi)} \left( \mathbb{E} \left[ \widetilde{h}_i^{4+2\varphi} \right] \right)^{2/(2+\varphi)} + O \left( n b_n^4 \right) \right\}$$

$$\approx 1 + O \left( n b_n^4 \right)$$

for some  $\varphi > 0$  that depends on s, where we let  $\tilde{h}_i = x_i u_i K(t_i) \mathbf{1} \left[ \gamma_g < q_i \leq \gamma_k \right]$  from the change of variables  $t_i = (s_i - s)/b_n$ . Then, by the stationarity, Cauchy-Schwarz inequality, and Lemma 1 of Bolthausen (1982), we have

$$\Psi_{12}(s) \le C' \left(\gamma_k - \gamma_g\right)^2 + O(n^{-1}) + O(b_n^2).$$

for some constant  $0 < C' < \infty$ . Using the same argument as the second component in

(A.4), we can also show that  $\Psi_{13}(s) = O(n^{-1}) + O(b_n^2)$ . For  $\Psi_{14}(s)$ , by stationarity,

$$\Psi_{14}(s) \leq \frac{4!n}{n^2 b_n^2} \sum_{1 < i < j < k}^n |\mathbb{E} \left[ h_1(s)h_i(s)h_j(s)h_k(s) \right] |$$

$$\leq \frac{4!}{nb_n^2} \sum_{i=1}^n \sum_{j,k \leq i} |Cov \left[ h_1(s), h_{i+1}(s)h_{i+j+1}(s)h_{i+j+k+1}(s) \right] |$$

$$+ \frac{4!}{nb_n^2} \sum_{j=1}^n \sum_{i,k \leq j} |Cov \left[ h_1(s)h_{i+1}(s), h_{i+j+1}(s)h_{i+j+k+1}(s) \right] |$$

$$+ \frac{4!}{nb_n^2} \sum_{k=1}^n \sum_{i,j \leq k} |Cov \left[ h_1(s), h_{i+1}(s)h_{i+j+1}(s), h_{i+j+k+1}(s) \right] |$$

similarly as Billingsley (1968), p.173. By Assumptions A-(v), (vii), (x), and Lemma 1 of Bolthausen (1982),

$$\begin{aligned} &|Cov\left[h_{1}(s),h_{i+1}(s)h_{i+j+1}(s)h_{i+j+k+1}(s)\right]| \\ &\leq C\alpha\left(i\right)^{\varphi/(2+\varphi)} \\ &\times \left(\mathbb{E}\left[h_{1}(s)^{2+\varphi}\right]\right)^{1/(2+\varphi)} \left(\mathbb{E}\left[\left(h_{i+1}(s)h_{i+j+1}(s)h_{i+j+k+1}(s)\right)^{2+\varphi}\right]\right)^{1/(2+\varphi)} \\ &= C\alpha\left(i\right)^{\varphi/(2+\varphi)} \\ &\times \left(b_{n}\left\{\mathbb{E}\left[\widetilde{h}_{1}^{2+\varphi}\right] + O\left(b_{n}^{2}\right)\right\}\right)^{1/(2+\varphi)} \left(b_{n}^{3}\left\{\mathbb{E}\left[\left(\widetilde{h}_{i+1}\widetilde{h}_{i+j+1}\widetilde{h}_{i+j+1}\widetilde{h}_{i+j+k+1}\right)^{2+\varphi}\right] + O\left(b_{n}^{2}\right)\right\}\right)^{1/(2+\varphi)} \\ &= Cb_{n}^{4/(2+\varphi)}\alpha\left(i\right)^{\varphi/(2+\varphi)} \\ &\times \left\{\left(\mathbb{E}\left[\widetilde{h}_{1}^{2+\varphi}\right]\right)^{1/(2+\varphi)} \left(\mathbb{E}\left[\left(\widetilde{h}_{i+1}\widetilde{h}_{i+j+1}\widetilde{h}_{i+j+k+1}\right)^{2+\varphi}\right]\right)^{1/(2+\varphi)} + O\left(b_{n}^{2}\right)\right\},\end{aligned}$$

where the first equality is by the change of variables  $t_i = (s_i - s)/b_n$ . It follows that the first term in (A.5) satisfies

$$\frac{4!}{nb_n^2} \sum_{i=1}^n \sum_{j,k \le i} |Cov[h_1(s), h_{i+1}(s)h_{i+j+1}(s)h_{i+j+k+1}(s)]| \\
\le \frac{C4!}{nb_n^{2^{-(4/(2+\varphi))}}} \sum_{i=1}^\infty i^2 \alpha(i)^{\varphi/(2+\varphi)} \\
\times \left\{ \left( \mathbb{E}\left[\tilde{h}_1^{2+\varphi}\right] \right)^{1/(2+\varphi)} \left( \mathbb{E}\left[\left(\tilde{h}_{i+1}\tilde{h}_{i+j+1}\tilde{h}_{i+j+k+1}\right)^{2+\varphi}\right] \right)^{1/(2+\varphi)} + O\left(b_n^2\right) \right\} \\
= O\left(\frac{1}{nb_n^{2\varphi/(2+\varphi)}}\right) + O\left(\frac{b_n^{4/(2+\varphi)}}{n}\right) \tag{A.6}$$

by Assumption A-(iii). However, we select  $\varphi$  small enough such that

$$\frac{2\varphi}{2+\varphi} \le \frac{1}{1-2\epsilon},\tag{A.7}$$

which holds for  $\varphi \in (0, 2)$  in Assumption A-(iii). Then (A.6) becomes o(1) because  $nb_n^{2\varphi/(2+\varphi)} = (n^{1-2\epsilon}b_n^{(2\varphi/(2+\varphi))(1-2\epsilon)})^{1/(1-2\epsilon)} \to \infty$  by Assumption A-(ix). Using the same argument, we can also verify that the rest of terms in (A.5) are all o(1) and hence  $\Psi_{14}(s) = o(1)$ . For  $\Psi_{15}(s)$ , we can similarly show that it is o(1) as well because

$$\Psi_{15}(s) \leq \frac{3!}{nb_n^2} \sum_{i=1}^n \sum_{j \leq i} \left| Cov \left[ h_1^2(s), h_{i+1}(s) h_{i+j+1}(s) \right] \right| \\ + \frac{3!}{nb_n^2} \sum_{j=1}^n \sum_{i \leq j} \left| Cov \left[ h_1^2(s) h_{i+1}(s), h_{i+j+1}(s) \right] \right|.$$

By combining these results for  $\Psi_{11}(s)$  to  $\Psi_{15}(s)$ , we thus have

$$\mathbb{E}\left[\left|J_{n}\left(\gamma_{g};s\right)-J_{n}\left(\gamma_{k};s\right)\right|^{4}\right] \leq C_{1}(s)\left(\gamma_{k}-\gamma_{g}\right)^{2}$$

for some constant  $0 < C_1(s) < \infty$  given s, and Theorem 12.2 of Billingsley (1968) yields

$$\mathbb{P}\left(\max_{1\leq g\leq \overline{g}}\left|J_n\left(\gamma_g;s\right) - J_n\left(\gamma_1;s\right)\right| > \eta(s)\right) \leq \frac{C_1(s)\varpi^2}{\eta^4(s)b_n},\tag{A.8}$$

which bounds  $\Psi_1(s)$ .

To bound  $\Psi_2(s)$ , the standard result of kernel estimation yields that  $\mathbb{E}[h_{ik}^2] \leq C_2(s)b_n$  by Assumption A-(x) for some constant  $0 < C_2(s) < \infty$  given s. Then by Lemma 1 of Bolthausen (1982), we have

$$\mathbb{E}\left[\left(\sqrt{nb_n} \left|H_{ng}(s) - \mathbb{E}\left[H_{ng}(s)\right]\right|\right)^2\right] = \frac{1}{nb_n} Var\left[\sum_{i=1}^n |h_{ig}(s)|\right]$$
$$\leq \frac{1}{b_n} \mathbb{E}\left[h_{ig}^2(s)\right] + \frac{2}{nb_n} \sum_{i < j} |Cov\left(|h_{ig}(s)|, |h_{jg}(s)|\right)$$
$$\leq C_2(s) \varpi/\overline{g}$$

and hence by Markov's inequality,

$$\mathbb{P}\left(\max_{1 \le g \le \overline{g}} \sqrt{nb_n} \left| H_{ng}(s) - \mathbb{E}\left[ H_{ng}(s) \right] \right| > \eta(s) \right) \le \frac{C_2(s)\varpi}{\eta^2(s)}.$$
(A.9)

Finally, to bound  $\Psi_3(s)$ , note that

$$\sqrt{nb_n} \mathbb{E}\left[H_{ng}(s)\right] = \sqrt{nb_n} C_3(s) \overline{\omega} / \overline{g} \le 2C_3(s) / \sqrt{nb_n}$$
(A.10)

for some constant  $0 < C_3(s) < \infty$  given s, where  $\varpi/\overline{g} \leq 2/nb_n$ . So tightness is proved

by combining (A.8), (A.9), and (A.10), and hence the weak convergence follows from Theorem 15.5 of Billingsley (1968).  $\blacksquare$ 

**Lemma A.2** Uniformly over  $s \in S$ ,

$$\Delta M_n(s) \equiv \frac{1}{nb_n} \sum_{i=1}^n x_i x_i^{\top} \{ \mathbf{1}_i \left( \gamma_0(s_i) \right) - \mathbf{1}_i \left( \gamma_0(s) \right) \} K_i(s) = O_p(b_n) .$$
 (A.11)

**Lemma A.3** For a given  $s \in S$ ,  $\widehat{\gamma}(s) \to_p \gamma_0(s)$  as  $n \to \infty$ .

**Proof of Lemma A.3** For given  $s \in S$ , we let  $\tilde{y}_i(s) = K_i(s)^{1/2}y_i$ ,  $\tilde{x}_i(s) = K_i(s)^{1/2}x_i$ ,  $\tilde{u}_i(s) = K_i(s)^{1/2}u_i$ ,  $\tilde{x}_i(\gamma; s) = K_i(s)^{1/2}x_i\mathbf{1}_i(\gamma)$ , and  $\tilde{x}_i(\gamma_0(s_i); s) = K_i(s)^{1/2}x_i\mathbf{1}_i(\gamma_0(s_i))$ ; we denote  $\tilde{y}(s)$ ,  $\tilde{X}(s)$ ,  $\tilde{u}(s)$ ,  $\tilde{X}(\gamma; s)$ , and  $\tilde{X}(\gamma_0(s_i); s)$  as their corresponding matrices of *n*-stacks. Then  $\hat{\theta}(\gamma; s) = (\hat{\beta}(\gamma; s)^{\top}, \hat{\delta}(\gamma; s)^{\top})^{\top}$  in (2) is given as

$$\widehat{\theta}(\gamma;s) = (\widetilde{Z}(\gamma;s)^{\top}\widetilde{Z}(\gamma;s))^{-1}\widetilde{Z}(\gamma;s)^{\top}\widetilde{y}(s), \qquad (A.12)$$

where  $\widetilde{Z}(\gamma; s) = [\widetilde{X}(s), \widetilde{X}(\gamma; s)]$ . Therefore, since  $\widetilde{y}(s) = \widetilde{X}(s)\beta_0 + \widetilde{X}(\gamma_0(s_i); s)\delta_0 + \widetilde{u}(s)$ and  $\widetilde{X}(s)$  lies in the space spanned by  $\widetilde{Z}(\gamma; s)$ , we have

$$Q_{n}(\gamma;s) - \widetilde{u}(s)^{\top}\widetilde{u}(s) = \widetilde{y}(s)^{\top} \left(I_{n} - P_{\widetilde{Z}}(\gamma;s)\right) \widetilde{y}(s) - \widetilde{u}(s)^{\top}\widetilde{u}(s) = -\widetilde{u}(s)^{\top} P_{\widetilde{Z}}(\gamma;s)\widetilde{u}(s) + 2\delta_{0}^{\top}\widetilde{X}(\gamma_{0}(s_{i});s)^{\top} \left(I_{n} - P_{\widetilde{Z}}(\gamma;s)\right) \widetilde{u}(s) + \delta_{0}^{\top}\widetilde{X}(\gamma_{0}(s_{i});s)^{\top} \left(I_{n} - P_{\widetilde{Z}}(\gamma;s)\right) \widetilde{X}(\gamma_{0}(s_{i});s)\delta_{0},$$

where  $P_{\widetilde{Z}}(\gamma; s) = \widetilde{Z}(\gamma; s)(\widetilde{Z}(\gamma; s)^{\top} \widetilde{Z}(\gamma; s))^{-1} \widetilde{Z}(\gamma; s)^{\top}$  and  $I_n$  is the identity matrix of rank n. Note that  $P_{\widetilde{Z}}(\gamma; s)$  is the same as the projection onto  $[\widetilde{X}(s) - \widetilde{X}(\gamma; s), \widetilde{X}(\gamma; s)]$ , where  $\widetilde{X}(\gamma; s)^{\top}(\widetilde{X}(s) - \widetilde{X}(\gamma; s)) = 0$ . Furthermore, for  $\gamma \geq \gamma_0(s_i), \widetilde{x}_i(\gamma_0(s_i); s)^{\top}(\widetilde{x}_i(s) - \widetilde{x}_i(\gamma; s))) = 0$  and hence  $\widetilde{X}(\gamma_0(s_i); s)^{\top} \widetilde{X}(\gamma; s) = \widetilde{X}(\gamma_0(s_i); s)^{\top} \widetilde{X}(\gamma_0(s_i); s)$ . Since

$$M_n(\gamma; s) = \frac{1}{nb_n} \sum_{i=1}^n \widetilde{x}_i(\gamma; s) \widetilde{x}_i(\gamma; s)^\top \text{ and}$$
$$J_n(\gamma; s) = \frac{1}{\sqrt{nb_n}} \sum_{i=1}^n \widetilde{x}_i(\gamma; s) \widetilde{u}_i(s),$$

Lemma A.1 yields that

$$\begin{split} \widetilde{Z}(\gamma;s)^{\top}\widetilde{u}(s) &= [\widetilde{X}(s)^{\top}\widetilde{u}(s), \widetilde{X}(\gamma;s)^{\top}\widetilde{u}(s)] = O_p\left((nb_n)^{1/2}\right) \\ \widetilde{Z}(\gamma;s)^{\top}\widetilde{X}(\gamma_0(s_i);s) &= [\widetilde{X}(s)^{\top}\widetilde{X}(\gamma_0(s_i);s), \widetilde{X}(\gamma;s)^{\top}\widetilde{X}(\gamma_0(s_i);s)] \\ &= [\widetilde{X}(s)^{\top}\widetilde{X}(\gamma_0(s_i);s), \widetilde{X}(\gamma_0(s_i);s)^{\top}\widetilde{X}(\gamma_0(s_i);s)] = O_p\left(nb_n\right) \end{split}$$

for given s. It follows that

$$\frac{1}{a_n} \left( Q_n\left(\gamma;s\right) - \widetilde{u}(s)^\top \widetilde{u}(s) \right) \tag{A.13}$$

$$= O_p\left(\frac{1}{a_n}\right) + O_p\left(\frac{1}{a_n^{1/2}}\right) + \frac{1}{nb_n} c_0^\top \widetilde{X}(\gamma_0(s_i);s)^\top \left(I_n - P_{\widetilde{Z}}(\gamma;s)\right) \widetilde{X}(\gamma_0(s_i);s)c_0$$

$$= \frac{1}{nb_n} c_0^\top \widetilde{X}(\gamma_0(s_i);s)^\top \left(I - P_{\widetilde{Z}}(\gamma;s)\right) \widetilde{X}(\gamma_0(s_i);s)c_0 + o_p(1)$$

for  $a_n = n^{1-2\epsilon} b_n \to \infty$  as  $n \to \infty$ . Moreover, we have

$$M_n(\gamma_0(s_i);s) = \frac{1}{nb_n} \sum_{i=1}^n \widetilde{x}_i(\gamma_0(s_i);s) \widetilde{x}_i(\gamma_0(s_i);s)^\top$$

$$= M_n(\gamma_0(s);s) + \Delta M_n(s)$$

$$= M_n(\gamma_0(s);s) + O_p(b_n)$$
(A.14)

from Lemma A.2, where  $\Delta M_n(s)$  is defined in (A.11). It follows that

$$\frac{1}{nb_n} c_0^{\top} \widetilde{X}(\gamma_0(s_i); s)^{\top} \left( I_n - P_{\widetilde{Z}}(\gamma; s) \right) \widetilde{X}(\gamma_0(s_i); s) c_0 \tag{A.15}$$

$$\rightarrow_p c_0^{\top} M(\gamma_0(s); s) c_0 - c_0^{\top} M(\gamma_0(s); s)^{\top} M(\gamma; s)^{-1} M(\gamma_0(s); s) c_0 \equiv \Upsilon(\gamma; s) < \infty$$

uniformly over  $\gamma \in \Gamma \cap [\gamma_0(s), \infty)$ , from Lemma A.1 and Assumptions ID-(ii) and A-(viii), as  $b_n \to 0$  as  $n \to \infty$ . However,

$$d\Upsilon(\gamma; s)/d\gamma = c_0^{\top} M(\gamma_0(s); s)^{\top} M(\gamma; s)^{-1} D(\gamma, s) f(\gamma, s) M(\gamma; s)^{-1} M(\gamma_0(s); s) c_0 \ge 0$$

and

$$d\Upsilon(\gamma_0(s);s)/d\gamma = c_0^\top D(\gamma_0(s),s)f(\gamma_0(s),s)c_0 > 0$$

from Assumption A-(viii), which implies that  $\Upsilon(\gamma; s)$  is continuous, non-decreasing, and uniquely minimized at  $\gamma_0(s)$  given  $s \in \mathcal{S}$ .

We can symmetrically show that the probability limit of (A.15) for  $\gamma \in \Gamma \cap (-\infty, \gamma_0(s)]$  is continuous, non-increasing, and uniquely minimized at  $\gamma_0(s)$  as well. Therefore, given  $s \in \mathcal{S}$ , uniformly over  $\Gamma$ , the probability limit of  $a_n^{-1} \left(Q_n\left(\gamma;s\right) - \widetilde{u}(s)^\top \widetilde{u}(s)\right)$  in (A.13) is continuous and uniquely minimized at  $\gamma_0(s)$ . Since  $\widehat{\gamma}(s)$  is the minimizer of  $a_n^{-1} \left(Q_n\left(\gamma;s\right) - \widetilde{u}(s)^\top \widetilde{u}(s)\right)$ , the pointwise consistency follows as the proof of Lemma A.5 of Hansen (2000).

We let  $\phi_{1n} = a_n^{-1}$ , where  $a_n = n^{1-2\epsilon} b_n$  and  $\epsilon$  is given in Assumption A-(ii). For a

given  $s \in \mathcal{S}$ , we define

$$T_{n}(\gamma; s) = \frac{1}{nb_{n}} \sum_{i=1}^{n} (c_{0}^{\top} x_{i})^{2} |\mathbf{1}_{i}(\gamma(s)) - \mathbf{1}_{i}(\gamma_{0}(s))| K_{i}(s),$$
  

$$\overline{T}_{n}(\gamma, s) = \frac{1}{nb_{n}} \sum_{i=1}^{n} ||x_{i}||^{2} |\mathbf{1}_{i}(\gamma(s)) - \mathbf{1}_{i}(\gamma_{0}(s))| K_{i}(s),$$
  

$$L_{n}(\gamma; s) = \frac{1}{\sqrt{nb_{n}}} \sum_{i=1}^{n} c_{0}^{\top} x_{i} u_{i} \{\mathbf{1}_{i}(\gamma(s)) - \mathbf{1}_{i}(\gamma_{0}(s))\} K_{i}(s),$$
  

$$\overline{L}_{n}(\gamma; s) = \frac{1}{\sqrt{nb_{n}}} \sum_{i=1}^{n} ||x_{i}u_{i}|| \{\mathbf{1}_{i}(\gamma(s)) - \mathbf{1}_{i}(\gamma_{0}(s))\} K_{i}(s).$$

**Lemma A.4** For a given  $s \in S$ , for any  $\eta(s) > 0$  and  $\varepsilon(s) > 0$ , there exist constants  $0 < C_T(s), C_{\overline{T}}(s), \overline{C}(s), \overline{r}(s) < \infty$  such that for all n,

$$\mathbb{P}\left(\inf_{\overline{r}(s)\phi_{1n}<|\gamma(s)-\gamma_{0}(s)|<\overline{C}(s)}\frac{T_{n}\left(\gamma;s\right)}{|\gamma\left(s\right)-\gamma_{0}\left(s\right)|} < C_{T}(1-\eta(s))\right) \leq \varepsilon(s), \quad (A.16)$$

$$\mathbb{P}\left(\sup_{\overline{r}(s)\phi_{1n}<|\gamma(s)-\gamma_{0}(s)|<\overline{C}(s)}\frac{T_{n}\left(\gamma;s\right)}{|\gamma\left(s\right)-\gamma_{0}\left(s\right)|}>C_{\overline{T}}(1+\eta(s))\right) \leq \varepsilon(s), \quad (A.17)$$

$$\mathbb{P}\left(\sup_{\overline{r}(s)\phi_{1n} < |\gamma(s) - \gamma_0(s)| < \overline{C}(s)} \frac{L_n(\gamma; s)}{\sqrt{a_n} |\gamma(s) - \gamma_0(s)|} > \eta(s)\right) \leq \varepsilon(s), \quad (A.18)$$

$$\mathbb{P}\left(\sup_{\overline{r}(s)\phi_{1n}<|\gamma(s)-\gamma_{0}(s)|<\overline{C}(s)}\frac{\overline{L}_{n}(\gamma;s)}{\sqrt{a_{n}}|\gamma(s)-\gamma_{0}(s)|}>\eta(s)\right) \leq \varepsilon(s), \quad (A.19)$$

 $\text{ if } n^{1-2\epsilon}b_n^2 \to \varrho < \infty.$ 

For a given  $s \in \mathcal{S}$ , we let  $\widehat{\theta}(\widehat{\gamma}(s)) = (\widehat{\beta}(\widehat{\gamma}(s))^{\top}, \widehat{\delta}(\widehat{\gamma}(s))^{\top})^{\top}$  and  $\theta_0 = (\beta_0^{\top}, \delta_0^{\top})^{\top}$ .

**Lemma A.5** For a given  $s \in S$ ,  $n^{\epsilon}(\widehat{\theta}(\widehat{\gamma}(s)) - \theta_0) = o_p(1)$ .

**Proof of Theorem 2** The consistency is proved in Lemma A.3 above. For given  $s \in S$ , we let

$$Q_{n}^{*}(\gamma(s);s) = Q_{n}(\widehat{\beta}(\widehat{\gamma}(s)), \widehat{\delta}(\widehat{\gamma}(s)), \gamma(s);s)$$

$$= \sum_{i=1}^{n} \left\{ y_{i} - x_{i}^{\top}\widehat{\beta}(\widehat{\gamma}(s)) - x_{i}^{\top}\widehat{\delta}(\widehat{\gamma}(s)) \mathbf{1}_{i}(\gamma(s)) \right\}^{2} K_{i}(s)$$
(A.20)

for any  $\gamma(\cdot)$ , where  $Q_n(\beta, \delta, \gamma; s)$  is the sum of squared errors function in (3). Consider  $\gamma(s)$  such that  $\gamma(s) \in \left[\gamma_0(s) + \overline{r}(s)\phi_{1n}, \gamma_0(s) + \overline{C}(s)\right]$  for some  $0 < \overline{r}(s), \overline{C}(s) < \infty$ 

that are chosen in Lemma A.4. We let  $\Delta_i(\gamma; s) = \mathbf{1}_i(\gamma(s)) - \mathbf{1}_i(\gamma_0(s)); \ \hat{c}_j(\widehat{\gamma}(s))$ and  $c_{0j}$  be the *j*th element of  $\widehat{c}(\widehat{\gamma}(s)) \in \mathbb{R}^p$  and  $c_0 \in \mathbb{R}^p$ , respectively. Then, since  $y_i = \beta_0^\top x_i + \delta_0^\top x_i \mathbf{1}_i(\gamma_0(s_i)) + u_i$ ,

$$Q_{n}^{*}(\gamma(s);s) - Q_{n}^{*}(\gamma_{0}(s);s)$$

$$= \sum_{i=1}^{n} \left(\widehat{\delta}\left(\widehat{\gamma}\left(s\right)\right)^{\top} x_{i}\right)^{2} \Delta_{i}(\gamma;s) K_{i}\left(s\right)$$

$$-2\sum_{i=1}^{n} \left(y_{i} - \widehat{\beta}\left(\widehat{\gamma}\left(s\right)\right)^{\top} x_{i} - \widehat{\delta}\left(\widehat{\gamma}\left(s\right)\right)^{\top} x_{i} \mathbf{1}_{i}\left(\gamma_{0}\left(s\right)\right)\right) \left(\widehat{\delta}\left(\widehat{\gamma}\left(s\right)\right)^{\top} x_{i}\right) \Delta_{i}(\gamma;s) K_{i}\left(s\right)$$

$$= \sum_{i=1}^{n} \left(\delta_{0}^{\top} x_{i}\right)^{2} \Delta_{i}(\gamma;s) K_{i}\left(s\right) + \sum_{i=1}^{n} \left\{\left(\widehat{\delta}\left(\widehat{\gamma}\left(s\right)\right)^{\top} x_{i}\right)^{2} - \left(\delta_{0}^{\top} x_{i}\right)^{2}\right\} \Delta_{i}(\gamma;s) K_{i}\left(s\right)$$

$$-2\sum_{i=1}^{n} \delta_{0}^{\top} x_{i} u_{i} \Delta_{i}(\gamma;s) K_{i}\left(s\right) - 2\sum_{i=1}^{n} \left(\widehat{\delta}\left(\widehat{\gamma}\left(s\right)\right) - \delta_{0}\right)^{\top} x_{i} u_{i} \Delta_{i}(\gamma;s) K_{i}\left(s\right)$$

$$-2\sum_{i=1}^{n} \left(\widehat{\beta}\left(\widehat{\gamma}\left(s\right)\right) - \beta_{0}\right)^{\top} x_{i} x_{i}^{\top} \widehat{\delta}\left(\widehat{\gamma}\left(s\right)\right) \Delta_{i}(\gamma;s) K_{i}\left(s\right)$$

$$-2\sum_{i=1}^{n} \delta_{0}^{\top} x_{i} x_{i}^{\top} \left(\widehat{\delta}\left(\widehat{\gamma}\left(s\right)\right) - \delta_{0}\right) \left\{\mathbf{1}_{i}\left(\gamma_{0}\left(s_{i}\right)\right) - \mathbf{1}_{i}\left(\gamma_{0}\left(s_{i}\right)\right)\right\} \Delta_{i}(\gamma;s) K_{i}\left(s\right)$$

$$(A.21)$$

$$-2\sum_{i=1}^{n} \delta_{0}^{\top} x_{i} x_{i}^{\top} \left( \delta\left(\widehat{\gamma}\left(s\right)\right) - \delta_{0} \right) \left\{ \mathbf{1}_{i}\left(\gamma_{0}\left(s_{i}\right)\right) - \mathbf{1}_{i}\left(\gamma_{0}\left(s\right)\right) \right\} \Delta_{i}(\gamma; s) K_{i}\left(s\right) \qquad (A.22)$$

$$-2\sum_{i=1}^{n} \left(\widehat{\delta}\left(\widehat{\gamma}\left(s\right)\right) - \delta_{0}\right)^{\top} x_{i} x_{i}^{\top} \widehat{\delta}\left(\widehat{\gamma}\left(s\right)\right) \mathbf{1}_{i}\left(\gamma_{0}\left(s\right)\right) \Delta_{i}(\gamma; s) K_{i}\left(s\right), \qquad (A.23)$$

where the absolute values of the last two summations (A.22) and (A.23) are bounded by

$$\sum_{i=1}^{n} \delta_{0}^{\top} x_{i} x_{i}^{\top} \left( \widehat{\delta} \left( \widehat{\gamma} \left( s \right) \right) - \delta_{0} \right) \left| \Delta_{i}(\gamma; s) \right| K_{i}(s) \text{ and}$$
$$\sum_{i=1}^{n} \left( \widehat{\delta} \left( \widehat{\gamma} \left( s \right) \right) - \delta_{0} \right)^{\top} x_{i} x_{i}^{\top} \widehat{\delta} \left( \widehat{\gamma} \left( s \right) \right) \left| \Delta_{i}(\gamma; s) \right| K_{i}(s),$$

respectively, since  $|\mathbf{1}_{i}(\gamma_{0}(s))| \leq 1$  and  $|\mathbf{1}_{i}(\gamma_{0}(s_{i})) - \mathbf{1}_{i}(\gamma_{0}(s))| \leq 1$ . Moreover, for the term in (A.21), we have

$$\frac{1}{a_n} \sum_{i=1}^n \delta_0^\top x_i x_i^\top \delta_0 \left\{ \mathbf{1}_i \left( \gamma_0 \left( s_i \right) \right) - \mathbf{1}_i \left( \gamma_0 \left( s \right) \right) \right\} \Delta_i(\gamma; s) K_i(s)$$

$$\leq \frac{1}{a_n} \sum_{i=1}^n \delta_0^\top x_i x_i^\top \delta_0 \left| \mathbf{1}_i \left( \gamma_0 \left( s_i \right) \right) - \mathbf{1}_i \left( \gamma_0 \left( s \right) \right) \right| K_i(s) = C^*(s) b_n$$

for some  $C^*(s) = O_p(1)$  as in (A.14). It follows that

$$\frac{Q_n^*(\gamma(s);s) - Q_n^*(\gamma_0(s);s)}{a_n(\gamma(s) - \gamma_0(s))} \tag{A.24}$$

$$\geq \frac{T_n(\gamma;s)}{\gamma(s) - \gamma_0(s)} - \|\widehat{c}(\widehat{\gamma}(s)) - c_0\| \|\widehat{c}(\widehat{\gamma}(s)) + c_0\| \frac{\overline{T}_n(\gamma,s)}{\gamma(s) - \gamma_0(s)} \\
-2\frac{L_n(\gamma;s)}{\sqrt{a_n}(\gamma(s) - \gamma_0(s))} - 2\max_{1 \le j \le p} |\widehat{c}_j(\widehat{\gamma}(s)) - c_{0j}| \frac{\overline{L}_n(\gamma;s)}{\sqrt{a_n}(\gamma(s) - \gamma_0(s))} \\
-2\left\|n^{\epsilon}(\widehat{\beta}(\widehat{\gamma}(s)) - \beta_0)\right\| \|\widehat{c}(\widehat{\gamma}(s))\| \frac{\overline{T}_n(\gamma,s)}{\gamma(s) - \gamma_0(s)} \\
-2\left\|c_0\| \|\widehat{c}(\widehat{\gamma}(s)) - c_0\| \frac{\overline{T}_n(\gamma,s)}{\gamma(s) - \gamma_0(s)} \\
-2\left\|n^{\epsilon}(\widehat{\delta}(\widehat{\gamma}(s)) - \delta_0)\right\| \|\widehat{c}(\widehat{\gamma}(s))\| \frac{\overline{T}_n(\gamma,s)}{\gamma(s) - \gamma_0(s)} \\
= \frac{T_n(\gamma;s)}{\gamma(s) - \gamma_0(s)} - \frac{2L_n(\gamma;s)}{\sqrt{a_n}(\gamma(s) - \gamma_0(s))} - \frac{2C^*(s)b_n}{\gamma(s) - \gamma_0(s)} + o_p(1),$$

where the last line follows from Lemma A.5. Then given Lemma A.4 and the Markov's inequality, there exist  $0 < C(s), \overline{C}(s), \overline{r}(s), \eta(s), \varepsilon(s) < \infty$  such that

$$\mathbb{P}\left(\inf_{\overline{r}(s)\phi_{1n}<|\gamma(s)-\gamma_{0}(s)|<\overline{C}(s)}\frac{T_{n}(\gamma;s)}{|\gamma(s)-\gamma_{0}(s)|}<(1-\eta(s))C(s)\right) \leq \frac{\varepsilon(s)}{3}, \\
\mathbb{P}\left(\sup_{\overline{r}(s)\phi_{1n}<|\gamma(s)-\gamma_{0}(s)|<\overline{C}(s)}\frac{2L_{n}(\gamma;s)}{\sqrt{a_{n}}|\gamma(s)-\gamma_{0}(s)|}>\eta(s)\right) \leq \frac{\varepsilon(s)}{3}.$$

In addition, for  $\gamma(s) \in [\gamma_0(s) + \overline{r}(s)\phi_{1n}, \gamma_0(s) + \overline{C}(s)]$ , since

$$\sup_{\overline{r}(s)\phi_{1n} < |\gamma(s) - \gamma_0(s)| < \overline{C}(s)} \frac{C^*(s)b_n}{\gamma(s) - \gamma_0(s)} < \frac{C^*(s)b_n}{\overline{r}(s)\phi_{1n}} = a_n b_n \frac{C^*(s)}{\overline{r}(s)} < \infty$$

provided  $n^{1-2\epsilon}b_n^2 \to \rho < \infty$ , we also have

$$\mathbb{P}\left(\sup_{\overline{r}(s)\phi_{1n}<|\gamma(s)-\gamma_0(s)|<\overline{C}(s)}\frac{2C^*(s)b_n}{|\gamma(s)-\gamma_0(s)|}>\eta(s)\right)\leq\frac{\varepsilon(s)}{3}$$

by choosing  $\overline{r}(s)$  large enough. Thus for any  $\varepsilon(s) > 0$  and  $\eta(s) > 0$ , we have

$$\mathbb{P}\left(\inf_{\overline{r}(s)\phi_{1n}<|\gamma(s)-\gamma_0(s)|<\overline{C}(s)}\left\{Q_n^*(\gamma(s);s)-Q_n^*(\gamma_0(s);s)\right\}>\eta(s)\right)\geq 1-\varepsilon(s),$$

which yields  $\mathbb{P}(Q_n^*(\gamma(s);s) - Q_n^*(\gamma_0(s);s) > 0) \to 1 \text{ as } n \to \infty$ . We can similarly show the same result when  $\gamma(s) \in [\gamma_0(s) - \overline{C}(s), \gamma_0(s) - \overline{r}(s)\phi_{1n}]$ . Therefore, with probability approaching to one, it should hold that  $|\widehat{\gamma}(s) - \gamma_0(s)| \leq r(s)\phi_{1n}$  since  $Q_n^*(\widehat{\gamma}(s);s) - Q_n^*(\gamma_0(s);s) \leq 0$  for any  $s \in \mathcal{S}$  by construction.

#### A.3 Proof of Theorem 3 and Corollary 1

For a given  $s \in S$ , we let  $\gamma_n(s) = \gamma_0(s) + r/a_n$  with some  $|r| < \infty$ , where  $a_n = n^{1-2\epsilon}b_n$ and  $\epsilon$  is given in Assumption A-(ii). We define

$$\begin{aligned} A_{n}^{*}\left(r,s\right) &= \sum_{i=1}^{n} \left(\delta_{0}^{\top} x_{i}\right)^{2} \left|\mathbf{1}_{i}\left(\gamma_{n}\left(s\right)\right) - \mathbf{1}_{i}\left(\gamma_{0}\left(s\right)\right)\right| K_{i}\left(s\right), \\ B_{n}^{*}\left(r,s\right) &= \sum_{i=1}^{n} \delta_{0}^{\top} x_{i} u_{i} \left\{\mathbf{1}_{i}\left(\gamma_{n}\left(s\right)\right) - \mathbf{1}_{i}\left(\gamma_{0}\left(s\right)\right)\right\} K_{i}\left(s\right). \end{aligned}$$

Lemma A.6 If  $n^{1-2\epsilon}b_n^2 \to \varrho < \infty$ ,

$$A_{n}^{*}(r,s) \rightarrow_{p} |r| c_{0}^{\top} D(\gamma_{0}(s),s) c_{0} f(\gamma_{0}(s),s)$$

and

$$B_{n}^{*}(r,s) \Rightarrow W(r) \sqrt{c_{0}^{\top}V(\gamma_{0}(s),s) c_{0}f(\gamma_{0}(s),s) \kappa_{2}}$$

as  $n \to \infty$ , where  $\kappa_2 = \int K(v)^2 dv$  and W(r) is the two-sided Brownian Motion defined in (10).

**Proof of Lemma A.6** Let  $\Delta_i(\gamma_n; s) = \mathbf{1}_i(\gamma_n(s)) - \mathbf{1}_i(\gamma_0(s))$ . First, for  $A_n^*(r, s)$ , consider the case with r > 0. Note that  $\delta_0 = c_0 n^{-\epsilon} = c_0 (a_n/(nb_n))^{1/2}$ . By change of variables and Taylor expansion, Assumptions A-(v), (viii), and (x) imply that

$$\mathbb{E}\left[A_{n}^{*}\left(r,s\right)\right] = \frac{a_{n}}{nb_{n}} \sum_{i=1}^{n} \mathbb{E}\left[\left(c_{0}^{\top}x_{i}\right)^{2} \Delta_{i}(\gamma_{n};s)K_{i}\left(s\right)\right] \qquad (A.25)$$

$$= a_{n} \iint_{\gamma_{0}(s)}^{\gamma_{0}(s)+r/a_{n}} \mathbb{E}\left[\left(c_{0}^{\top}x_{i}\right)^{2} | q, s+b_{n}t\right] K\left(t\right) f\left(q, s+b_{n}t\right) dqdt$$

$$= rc_{0}^{\top} D\left(\gamma_{0}\left(s\right), s\right) c_{0}f\left(\gamma_{0}\left(s\right), s\right) + O\left(\frac{1}{a_{n}} + b_{n}^{2}\right),$$

where the third equality holds under Assumption A-(vi). Next, we have

$$Var [A_{n}^{*}(r,s)] = \frac{a_{n}^{2}}{n^{2}b_{n}^{2}} Var \left[ \sum_{i=1}^{n} (c_{0}^{\top}x_{i})^{2} \Delta_{i}(\gamma_{n};s)K_{i}(s) \right]$$

$$= \frac{a_{n}^{2}}{nb_{n}^{2}} Var \left[ (c_{0}^{\top}x_{i})^{2} \Delta_{i}(\gamma_{n};s)K_{i}(s) \right]$$

$$+ \frac{2a_{n}^{2}}{n^{2}b_{n}^{2}} \sum_{i < j}^{n} Cov \left[ (c_{0}^{\top}x_{i})^{2} \Delta_{i}(\gamma_{n};s)K_{i}(s), (c_{0}^{\top}x_{j})^{2} \Delta_{j}(\gamma_{n};s)K_{j}(s) \right]$$

$$\equiv \Psi_{A1}(r,s) + \Psi_{A2}(r,s).$$
(A.26)

Similarly as (A.25), Taylor expansion and Assumptions A-(vii), (viii), and (x) lead to

$$\Psi_{A1}(r,s) = \frac{a_n}{nb_n} \left( \frac{a_n}{b_n} \mathbb{E} \left[ \left( c_0^{\mathsf{T}} x_i \right)^4 \Delta_i(\gamma_n;s) K_i^2(s) \right] \right) - \frac{1}{n} \left( \frac{a_n}{b_n} \mathbb{E} \left[ \left( c_0^{\mathsf{T}} x_i \right)^2 \Delta_i(\gamma_n;s) K_i(s) \right] \right)^2 = O\left( n^{-2\epsilon} + \frac{1}{n} \right)$$

since  $\{\Delta_i(\gamma_n; s)\}^2 = \Delta_i(\gamma_n; s)$  for r > 0. Furthermore, by change of variables  $t_i = (s_i - s)/b_n$  in the covariance operator and Lemma 1 of Bolthausen (1982),

$$\begin{split} \Psi_{A2}(r,s) &\leq \frac{2a_n^2}{n^2} \sum_{i$$

where the last line follows from the conditions that  $\varphi \in (0, 2)$  in Assumption A-(iii) and  $n^{1-2\epsilon}b_n^2 \to \varrho < \infty$ . Hence, the pointwise convergence of  $A_n^*(r, s)$  is obtained. Since  $rc_0^{\top}D(\gamma_0(s), s)c_0f(\gamma_0(s), s)$  is strictly increasing and continuous in r, the convergence holds uniformly on any compact set. Symmetrically, we can show that  $\mathbb{E}[A_n^*(r, s)] =$  $-rc_0^{\top}D(\gamma_0(s), s)c_0f(\gamma_0(s), s) + O(a_n^{-1} + b_n^2)$  when r < 0. The uniform convergence also holds in this case using the same argument as above, which completes the proof for  $A_n^*(r, s)$ .

For  $B_n^*(r, s)$ , Assumption ID-(i) leads to  $\mathbb{E}[B_n^*(r, s)] = 0$ . Then, similarly as for  $A_n^*(r, s)$ , for any  $i \neq j$ , we have

$$Cov\left[c_0^{\top} x_i u_i \Delta_i(\gamma_n; s) K_i(s), c_0^{\top} x_j u_j \Delta_j(\gamma_n; s) K_j(s)\right] \le C b_n^2 a_n^{-1}$$
(A.27)

for some positive constant  $C < \infty$ , by the change of variables in the covariance operator

and Lemma 1 of Bolthausen (1982). It follows that, similarly as (A.25),

$$Var[B_{n}^{*}(r,s)] = \frac{a_{n}}{b_{n}} Var[c_{0}^{\top}x_{i}u_{i}\Delta_{i}(\gamma_{n};s)K_{i}(s)] + O(b_{n})$$
  
=  $|r|c_{0}^{\top}V(\gamma_{0}(s),s)c_{0}f(\gamma_{0}(s),s)\kappa_{2} + o(1),$ 

where  $\kappa_2 = \int K(v)^2 dv$ . Then by the CLT for stationary and mixing random field (e.g. Bolthausen (1982); Jenish and Prucha (2009)), we have

$$B_{n}^{*}(r,s) \Rightarrow W(r) \sqrt{c_{0}^{\top} V(\gamma_{0}(s),s) c_{0} f(\gamma_{0}(s),s) \kappa_{2}}$$

as  $n \to \infty$ , where W(r) is the two-sided Brownian Motion defined in (10). This pointwise convergence in r can be extended to any finite-dimensional convergence in rby the fact that for any  $r_1 < r_2$ ,  $Cov [B_n^*(r_1, s), B_n^*(r_2, s)] = Var [B_n^*(r_1, s)] + o(1)$ , which is because  $(\mathbf{1}_i (\gamma_0 + r_2/a_n) - \mathbf{1}_i (\gamma_0 + r_1/a_n)) \mathbf{1}_i (\gamma_0 + r_1/a_n) = 0$  and (A.27). The tightness follows from a similar argument as  $J_n(\gamma; s)$  in Lemma A.1 and the desired result follows by Theorem 15.5 in Billingsley (1968).

For a given  $s \in \mathcal{S}$ , we let  $\widehat{\theta}(\gamma_0(s)) = (\widehat{\beta}(\gamma_0(s))^\top, \widehat{\delta}(\gamma_0(s))^\top)^\top$ . Recall that  $\theta_0 = (\beta_0^\top, \delta_0^\top)^\top$  and  $\widehat{\theta}(\widehat{\gamma}(s)) = (\widehat{\beta}(\widehat{\gamma}(s))^\top, \widehat{\delta}(\widehat{\gamma}(s))^\top)^\top$ .

**Lemma A.7** For a given  $s \in S$ ,  $\sqrt{nb_n}(\widehat{\theta}(\widehat{\gamma}(s)) - \theta_0) = O_p(1)$ , if  $n^{1-2\epsilon}b_n^2 \to \varrho < \infty$  as  $n \to \infty$ .

**Proof of Theorem 3** From Theorem 2, we define a random variable  $r^*(s)$  such that

$$r^{*}(s) = a_{n}(\widehat{\gamma}(s) - \gamma_{0}(s)) = \arg\max_{r \in \mathbb{R}} \left\{ Q_{n}^{*}(\gamma_{0}(s); s) - Q_{n}^{*}\left(\gamma_{0}(s) + \frac{r}{a_{n}}; s\right) \right\},$$

where  $Q_n^*(\gamma(s); s)$  is defined in (A.20). We let  $\Delta_i(s) = \mathbf{1}_i (\gamma_0(s) + (r/a_n)) - \mathbf{1}_i (\gamma_0(s))$ . We then have

$$\Delta Q_n^*(r;s) \tag{A.28}$$

$$= Q_n^*(\gamma_0(s);s) - Q_n^*\left(\gamma_0(s) + \frac{r}{a_n};s\right)$$

$$= -\sum_{i=1}^n \left(\widehat{\delta}\left(\widehat{\gamma}\left(s\right)\right)^\top x_i\right)^2 |\Delta_i(s)| K_i(s)$$

$$+2\sum_{i=1}^n \left(y_i - \widehat{\beta}\left(\widehat{\gamma}\left(s\right)\right)^\top x_i - \widehat{\delta}\left(\widehat{\gamma}\left(s\right)\right)^\top x_i \mathbf{1}_i\left(\gamma_0\left(s\right)\right)\right) \left(\widehat{\delta}\left(\widehat{\gamma}\left(s\right)\right)^\top x_i\right) \Delta_i(s) K_i(s)$$

$$\equiv -A_n(r;s) + 2B_n(r;s).$$

For  $A_n(r; s)$ , Lemmas A.6 and A.7 yield

$$A_{n}(r;s)$$

$$= \sum_{i=1}^{n} \left( \left( \delta_{0} + n^{-1/2} b_{n}^{-1/2} C_{\delta}(s) + o_{p} (n^{-1/2} b_{n}^{-1/2}) \right)^{\top} x_{i} \right)^{2} |\Delta_{i}(s)| K_{i}(s)$$

$$= A_{n}^{*}(r,s) + \frac{1}{n^{1-2\epsilon} b_{n}} \sum_{i=1}^{n} \left( n^{-\epsilon} C_{\delta}(s) \right)^{\top} x_{i} x_{i}^{\top} \left( n^{-\epsilon} C_{\delta}(s) \right) |\Delta_{i}(s)| K_{i}(s) + o_{p} \left( a_{n}^{-1} \right)$$

$$= A_{n}^{*}(r,s) + O_{p}(a_{n}^{-1})$$
(A.29)

for some  $p \times 1$  vector  $C_{\delta}(s) = O_p(1)$ , since  $\sum_{i=1}^n n^{-2\epsilon} C_{\delta}^{\top}(s) x_i x_i^{\top} C_{\delta}(s) |\Delta_i(s)| K_i(s) = O_p(1)$  from Lemma A.6 and  $a_n = n^{1-2\epsilon} b_n \to \infty$ . Note that  $\widehat{\delta}(\widehat{\gamma}(s)) - \delta_0 = O_p((nb_n)^{-1/2})$  from Lemma A.7. Similarly, for  $B_n(r;s)$ , since  $y_i = \beta_0^{\top} x_i + \delta_0^{\top} x_i \mathbf{1}_i(\gamma_0(s_i)) + u_i$ , we have for some  $p \times 1$  vector  $C_{\beta}(s) = O_p(1)$ ,

$$\begin{split} B_{n}(r;s) & (A.30) \\ = \sum_{i=1}^{n} \left( u_{i} + \delta_{0}^{\top} x_{i} \left\{ \mathbf{1}_{i} \left( \gamma_{0} \left( s_{i} \right) \right) - \mathbf{1}_{i} \left( \gamma_{0} \left( s \right) \right) \right\} - \left( \widehat{\beta} \left( \widehat{\gamma} \left( s \right) \right) - \beta_{0} \right)^{\top} x_{i} \\ & - \left( \widehat{\delta} \left( \widehat{\gamma} \left( s \right) \right) - \delta_{0} \right)^{\top} x_{i} \mathbf{1}_{i} \left( \gamma_{0} \left( s \right) \right) \right) \widehat{\delta} \left( \gamma_{0} \left( s \right) \right)^{\top} x_{i} \Delta_{i} \left( s \right) K_{i} \left( s \right) \\ = \sum_{i=1}^{n} \left( u_{i} + \delta_{0}^{\top} x_{i} \left\{ \mathbf{1}_{i} \left( \gamma_{0} \left( s \right) \right) \right) - \mathbf{1}_{i} \left( \gamma_{0} \left( s \right) \right) \right) \right\} - n^{-1/2} b_{n}^{-1/2} C_{\beta}^{\top} \left( s \right) x_{i} \\ & - n^{-1/2} b_{n}^{-1/2} C_{\delta}^{\top} \left( s \right) x_{i} \mathbf{1}_{i} \left( \gamma_{0} \left( s \right) \right) \right) \left( \delta_{0} + n^{-1/2} b_{n}^{-1/2} C_{\delta} \left( s \right) \right)^{\top} x_{i} \Delta_{i} \left( s \right) K_{i} \left( s \right) \\ & + \frac{1}{\sqrt{n^{1-2\epsilon} b_{n}}} \sum_{i=1}^{n} u_{i} x_{i} \left( n^{-\epsilon} C_{\delta} \left( s \right) \right) \Delta_{i} \left( s \right) K_{i} \left( s \right) \\ & + \frac{1}{\sqrt{n^{1-2\epsilon} b_{n}}} \sum_{i=1}^{n} \delta_{0}^{\top} x_{i} x_{i}^{\top} \left( n^{-\epsilon} C_{\beta} \left( s \right) \right) \Delta_{i} \left( s \right) K_{i} \left( s \right) \\ & + \frac{1}{\sqrt{n^{1-2\epsilon} b_{n}}} \sum_{i=1}^{n} \left( n^{-\epsilon} C_{\beta} \left( s \right) \right)^{\top} x_{i} x_{i}^{\top} \left( n^{-\epsilon} C_{\delta} \left( s \right) \right) \Delta_{i} \left( s \right) K_{i} \left( s \right) \\ & + \frac{1}{\sqrt{n^{1-2\epsilon} b_{n}}} \sum_{i=1}^{n} \left( n^{-\epsilon} C_{\beta} \left( s \right) \right)^{\top} x_{i} x_{i}^{\top} \left( n^{-\epsilon} C_{\delta} \left( s \right) \right) \Delta_{i} \left( s \right) K_{i} \left( s \right) \\ & + \frac{1}{\sqrt{n^{1-2\epsilon} b_{n}}} \sum_{i=1}^{n} \left( n^{-\epsilon} C_{\beta} \left( s \right) \right)^{\top} x_{i} x_{i}^{\top} \left( n^{-\epsilon} C_{\delta} \left( s \right) \right) \Delta_{i} \left( s \right) K_{i} \left( s \right) \\ & + \frac{1}{\sqrt{n^{1-2\epsilon} b_{n}}} \sum_{i=1}^{n} \left( n^{-\epsilon} C_{\beta} \left( s \right) \right)^{\top} x_{i} x_{i}^{\top} \left( n^{-\epsilon} C_{\delta} \left( s \right) \right) \Delta_{i} \left( s \right) K_{i} \left( s \right) \\ & + \frac{1}{\sqrt{n^{1-2\epsilon} b_{n}}} \sum_{i=1}^{n} \left( n^{-\epsilon} C_{\beta} \left( s \right) \right)^{\top} x_{i} x_{i}^{\top} \left( n^{-\epsilon} C_{\delta} \left( s \right) \right) \left\{ \Delta_{i} \left( s \right) \mathbf{1}_{i} \left( \gamma_{0} \left( s \right) \right) \right\} K_{i} \left( s \right) \\ & + \frac{1}{\sqrt{n^{1-2\epsilon} b_{n}}} \sum_{i=1}^{n} \left( n^{-\epsilon} C_{\beta} \left( s \right) \right)^{\top} \left\{ \Delta_{i} \left( s \right) \mathbf{1}_{i} \left( \gamma_{0} \left( s \right) \right) \right\} K_{i} \left( s \right) \\ & + \frac{1}{\sqrt{n^{1-2\epsilon} b_{n}}} \sum_{i=1}^{n} \left( n^{-\epsilon} C_{\beta} \left( s \right) \right) \left\{ \Delta_{i} \left( s \right) \mathbf{1}_{i} \left( \gamma_{0} \left( s \right) \right) \right\} K_{i} \left( s \right) \\ & + \frac{1}{\sqrt{n^{1-2\epsilon} b_{n}}} \sum_{i=1}^{n} \left( n^{-\epsilon} C_{\beta} \left( s \right) \right) \left\{ \Delta_{i} \left( s \right) \mathbf{1}_{i} \left( \gamma_{0} \left( s \right) \right) \right\} K_$$

$$+\frac{1}{n^{1-2\epsilon}b_n}\sum_{i=1}^n \left(n^{-\epsilon}C_{\delta}(s)\right)^{\top} x_i x_i^{\top} \left(n^{-\epsilon}C_{\delta}(s)\right) \left\{\Delta_i(s)\mathbf{1}_i\left(\gamma_0(s)\right)\right\} K_i(s) +o_p\left(\left(n^{1-2\epsilon}b_n\right)^{-1/2}\right),$$

where all the terms are  $O_p\left(\left(n^{1-2\epsilon}b_n\right)^{-1/2}\right) = O_p(a_n^{-1/2})$  except for the first term  $B_n^*(r,s)$ and the third term in the line of (A.31) that we denote  $B_{n3}^*(r,s)$ . In Lemma A.8 below, we show that, if  $n^{1-2\epsilon}b_n^2 \to \varrho \in (0,\infty)$ ,

$$B_{n3}^{*}(r,s) \rightarrow_{p} |r| c_{0}^{\top} D(\gamma_{0}(s),s) c_{0} f(\gamma_{0}(s),s) \left\{ \frac{1}{2} - \mathcal{K}_{0}(r,\varrho;s) \right\} \\ + \varrho c_{0}^{\top} D(\gamma_{0}(s),s) c_{0} f(\gamma_{0}(s),s) |\dot{\gamma}_{0}(s)| \mathcal{K}_{1}(r,\varrho;s)$$

as  $n \to \infty$ , where  $\dot{\gamma}_0(\cdot)$  is the first derivatives of  $\gamma_0(\cdot)$  and  $\mathcal{K}_j(r,\varrho;s) = \int_0^{|r|/(\varrho|\dot{\gamma}_0(s)|)} t^j K(t) dt$  for j = 0, 1.

From Lemma A.6, it follows that

$$\begin{split} \Delta Q_n^*(r;s) &= -A_n^*\left(r,s\right) + 2B_{n3}^*(r,s) + 2B_n^*\left(r,s\right) \\ &= -\left|r\right|c_0^\top D\left(\gamma_0\left(s\right),s\right)c_0 f\left(\gamma_0\left(s\right),s\right) \\ &+ \left|r\right|c_0^\top D\left(\gamma_0\left(s\right),s\right)c_0 f\left(\gamma_0\left(s\right),s\right)\left\{1 - 2\mathcal{K}_0\left(r,\varrho;s\right)\right\} \\ &+ 2\varrho c_0^\top D\left(\gamma_0\left(s\right),s\right)c_0 f\left(\gamma_0\left(s\right),s\right)\left|\dot{\gamma}_0(s)\right|\mathcal{K}_1\left(r,\varrho;s\right) \\ &+ 2W\left(r\right)\sqrt{c_0^\top V\left(\gamma_0\left(s\right),s\right)c_0 f\left(\gamma_0\left(s\right),s\right)\kappa_2} + O_p(a_n^{-1/2} + b_n), \end{split}$$
$$= -2\left|r\right|\ell_D(s)\widetilde{\psi}_1\left(r,\varrho;s\right) + 2\varrho\ell_D(s)\widetilde{\psi}_2\left(r,\varrho;s\right) \\ &+ 2W\left(r\right)\sqrt{\ell_V(s)} + O_p(a_n^{-1/2} + b_n), \end{split}$$

where

$$\begin{split} \ell_D(s) &= c_0^\top D\left(\gamma_0\left(s\right), s\right) c_0 f\left(\gamma_0\left(s\right), s\right), \\ \ell_V(s) &= c_0^\top V\left(\gamma_0\left(s\right), s\right) c_0 f\left(\gamma_0\left(s\right), s\right) \kappa_2, \\ \widetilde{\psi}_1\left(r, \varrho; s\right) &= \mathcal{K}_0\left(r, \varrho; s\right), \\ \widetilde{\psi}_2\left(r, \varrho; s\right) &= \left|\dot{\gamma}_0(s)\right| \mathcal{K}_1\left(r, \varrho; s\right). \end{split}$$

However, if we let  $\xi(s) = \ell_V(s)/\ell_D^2(s)$  and  $r = \xi(s)\nu$ , we have

$$\underset{r \in \mathbb{R}}{\operatorname{arg\,max}} \left( 2W\left(r\right)\sqrt{\ell_{V}(s)} - 2\left|r\right|\ell_{D}(s)\widetilde{\psi}_{1}\left(r,\varrho;s\right) + 2\varrho\ell_{D}(s)\widetilde{\psi}_{2}\left(r,\varrho;s\right) \right)$$

$$= \xi(s) \operatorname{arg\,max}_{\nu \in \mathbb{R}} \left( W\left(\xi(s)\nu\right)\sqrt{\ell_{V}(s)} - \left|\xi(s)\nu\right|\ell_{D}(s)\widetilde{\psi}_{1}\left(\xi(s)\nu,\varrho;s\right) + \varrho\ell_{D}(s)\widetilde{\psi}_{2}\left(\nu,\varrho;s\right) \right)$$

$$= \xi(s) \operatorname{arg\,max}_{\nu \in \mathbb{R}} \left( W\left(\nu\right)\frac{\ell_{V}(s)}{\ell_{D}(s)} - \left|\nu\right|\frac{\ell_{V}(s)}{\ell_{D}(s)}\widetilde{\psi}_{1}\left(\xi(s)\nu,\varrho;s\right) + \varrho\frac{\ell_{V}(s)}{\ell_{D}(s)}\xi(s)\widetilde{\psi}_{2}\left(\xi(s)\nu,\varrho;s\right) \right)$$

$$= \xi(s) \operatorname{arg\,max}_{\nu \in \mathbb{R}} \left( W\left(\nu\right) - \left|\nu\right|\widetilde{\psi}_{1}\left(\xi(s)\nu,\varrho;s\right) + \varrho\xi(s)\widetilde{\psi}_{2}\left(\xi(s)\nu,\varrho;s\right) \right)$$

similar to the proof of Theorem 1 in Hansen (2000). By Theorem 2.7 of Kim and Pollard (1990), it follows that (rewriting  $\nu$  as r)

$$n^{1-2\epsilon}b_{n}\left(\widehat{\gamma}\left(s\right)-\gamma_{0}\left(s\right)\right) \rightarrow_{d} \xi\left(s\right) \arg\max_{r\in\mathbb{R}}\left(W\left(r\right)-\left|r\right|\psi_{1}\left(r,\varrho;s\right)+\varrho\psi_{2}\left(r,\varrho;s\right)\right)$$

as  $n \to \infty$ , where

$$\begin{split} \psi_1(r,\varrho;s) &= \widetilde{\psi}_1\left(\xi(s)r,\varrho;s\right) = \int_0^{\xi(s)|r|/(\varrho|\dot{\gamma}_0(s)|)} K\left(t\right) dt, \\ \psi_2(r,\varrho;s) &= \xi(s)\widetilde{\psi}_2\left(\xi(s)r,\varrho;s\right) = \xi(s)\left|\dot{\gamma}_0(s)\right| \int_0^{\xi(s)|r|/(\varrho|\dot{\gamma}_0(s)|)} tK\left(t\right) dt. \end{split}$$

Note that when  $\rho = 0$ , we let  $\psi_1(r, 0; s) = \int_0^\infty K(t) dt = 1/2$ . Finally, letting

$$\mu(r,\varrho;s) = -|r|\psi_1(r,\varrho;s) + \varrho\psi_2(r,\varrho;s), \qquad (A.32)$$

 $\mathbb{E}\left[\arg\max_{r\in\mathbb{R}}\left(W\left(r\right)+\mu\left(r,\varrho;s\right)\right)\right]=0 \text{ follows from Lemmas A.9 and A.10 below.} \blacksquare$ 

**Lemma A.8** For a given  $s \in S$ , let r be the same term used in Lemma A.6. If  $n^{1-2\epsilon}b_n^2 \to \varrho \in (0,\infty)$ ,

$$B_{n3}^{*}(r,s) \equiv \sum_{i=1}^{n} \left( \delta_{0}^{\top} x_{i} \right)^{2} \left\{ \mathbf{1}_{i} \left( \gamma_{0} \left( s \right) + r/a_{n} \right) - \mathbf{1}_{i} \left( \gamma_{0} \left( s \right) \right) \right\} \left\{ \mathbf{1}_{i} \left( \gamma_{0} \left( s \right) \right) - \mathbf{1}_{i} \left( \gamma_{0} \left( s \right) \right) \right\} K_{i} \left( s \right)$$
$$\rightarrow_{p} |r| c_{0}^{\top} D \left( \gamma_{0} \left( s \right), s \right) c_{0} f \left( \gamma_{0} \left( s \right), s \right) \left\{ \frac{1}{2} - \mathcal{K}_{0} \left( r, \varrho; s \right) \right\}$$
$$+ \varrho c_{0}^{\top} D \left( \gamma_{0} \left( s \right), s \right) c_{0} f \left( \gamma_{0} \left( s \right), s \right) |\dot{\gamma}_{0}(s)| \mathcal{K}_{1} \left( r, \varrho; s \right)$$

as  $n \to \infty$ , where  $\dot{\gamma}_0(\cdot)$  is the first derivatives of  $\gamma_0(\cdot)$  and

$$\mathcal{K}_{j}(r,\varrho;s) = \int_{0}^{|r|/(\varrho|\dot{\gamma}_{0}(s)|)} t^{j} K(t) dt$$

for j = 0, 1.

**Lemma A.9** Let  $\tau = \arg \max_{r \in \mathbb{R}} (W(r) + \mu(r))$ , where W(r) is a two-sided Brownian motion in (10) and  $\mu(r)$  is a continuous drift function satisfying:  $\mu(0) = 0$ ,  $\mu(-r) = \mu(r), \ \mu(r)$  is monotonically decreasing on  $\mathbb{R} \setminus [-\underline{r}, \underline{r}]$  for some  $\underline{r} > 0$ , and  $\lim_{|r|\to\infty} |r|^{-((1/2)+\varepsilon)} \mu(r) = -\infty$  for some  $\varepsilon > 0$ . Then,  $\mathbb{E}[\tau] = 0$ .

**Lemma A.10** For given  $(\varrho, s)$ ,  $\mu(r, \varrho; s)$  in (A.32) satisfies conditions in Lemma A.9

**Proof of Corollary 1** From (A.13) and (A.15), we have

$$\frac{1}{nb_n}Q_n\left(\widehat{\gamma}\left(s\right),s\right) = \frac{1}{nb_n}\sum_{i=1}^n u_i^2 K_i\left(s\right) + o_p(1) \to_p \mathbb{E}\left[u_i^2|s_i=s\right] f_s\left(s\right),$$

where  $f_s(s)$  is the marginal density of  $s_i$ . In addition, from Theorem 3 and the proof of Lemma A.7, we have

$$Q_{n}(\gamma_{0}(s), s) - Q_{n}(\widehat{\gamma}(s), s) = Q_{n}^{*}(\gamma_{0}(s), s) - Q_{n}^{*}(\widehat{\gamma}(s), s) + o_{p}(1)$$

since  $\hat{\theta}(\hat{\gamma}(s)) - \hat{\theta}(\gamma_0(s)) = o_p((nb_n)^{-1/2})$ . Similar to Theorem 2 of Hansen (2000), the rest of the proof follows from the change of variables and the continuous mapping theorem because  $(nb_n)^{-1} \sum_{i=1}^n K_i(s) \to_p f_s(s)$  by the standard result of the kernel density estimator.

#### A.4 Proof of Theorem 4

We let  $\phi_{2n} = \log n/a_n$ , where  $a_n = n^{1-2\epsilon}b_n$  and  $\epsilon$  is given in Assumption A-(ii).

**Lemma A.11** For a given  $s \in S$ , let  $\gamma(s) = \gamma_0(s) + r(s)\phi_{2n}$  for some continuously differentiable r(s) satisfying  $0 < \underline{r} = \inf_{s \in S} r(s) \leq \sup_{s \in S} r(s) = \overline{r} < \infty$ . Then there exist constants  $0 < C_T, C_{\overline{T}} < \infty$  such that for any  $\eta > 0$ ,

$$\mathbb{P}\left(\sup_{s\in\mathcal{S}}\left|T_{n}\left(\gamma;s\right)-\mathbb{E}\left[T_{n}\left(\gamma;s\right)\right]\right| > \eta\right) \leq \frac{C_{T}}{\eta}\left(\phi_{2n}\frac{\log n}{nb_{n}}\right)^{1/2},\\
\mathbb{P}\left(\sup_{s\in\mathcal{S}}\left|\overline{T}_{n}\left(\gamma;s\right)-\mathbb{E}\left[\overline{T}_{n}\left(\gamma;s\right)\right]\right| > \eta\right) \leq \frac{C_{T}}{\eta}\left(\phi_{2n}\frac{\log n}{nb_{n}}\right)^{1/2},$$

if n is large enough.

**Lemma A.12** For a given  $s \in S$ , let  $\gamma(s) = \gamma_0(s) + r(s)\phi_{2n}$ , where r(s) is defined in Lemma A.11. Then there exists a constant  $0 < C_L, C_{\overline{L}} < \infty$  such that for any  $\eta > 0$ ,

$$\mathbb{P}\left(\sup_{s\in\mathcal{S}} \|L_n(\gamma;s)\| > \eta\right) \leq \frac{C_L}{\eta} (\phi_{2n}\log n)^{1/2} \\
\mathbb{P}\left(\sup_{s\in\mathcal{S}} \|\overline{L}_n(\gamma;s)\| > \eta\right) \leq \frac{C_{\overline{L}}}{\eta} (\phi_{2n}\log n)^{1/2}$$

if n is large enough.

**Lemma A.13** For any  $\eta > 0$  and  $\varepsilon > 0$ , there exist constants  $0 < \overline{C}, \overline{r}, C_T, C_{\overline{T}} < \infty$  such that

$$\mathbb{P}\left(\inf_{\overline{r}\phi_{2n}<\sup_{s\in\mathcal{S}}|\gamma(s)-\gamma_{0}(s)|<\overline{C}}\frac{\sup_{s\in\mathcal{S}}T_{n}\left(\gamma;s\right)}{\sup_{s\in\mathcal{S}}|\gamma\left(s\right)-\gamma_{0}\left(s\right)|} < C_{T}(1-\eta)\right) \leq \varepsilon, \quad (A.33)$$

$$\mathbb{P}\left(\sup_{\overline{r}\phi_{2n}<\sup_{s\in\mathcal{S}}|\gamma(s)-\gamma_{0}(s)|<\overline{C}}\frac{\sup_{s\in\mathcal{S}}\overline{T}_{n}\left(\gamma;s\right)}{\sup_{s\in\mathcal{S}}|\gamma\left(s\right)-\gamma_{0}\left(s\right)|} > C_{\overline{T}}(1+\eta)\right) \leq \varepsilon, \quad (A.34)$$

$$\mathbb{P}\left(\sup_{\bar{r}\phi_{2n}<\sup_{s\in\mathcal{S}}|\gamma(s)-\gamma_{0}(s)|<\bar{C}}\frac{\sup_{s\in\mathcal{S}}|\Gamma(r)-\gamma_{0}(s)|}{\sqrt{a_{n}}\sup_{s\in\mathcal{S}}|\gamma(s)-\gamma_{0}(s)|}>\eta\right) \leq \varepsilon, \quad (A.35)$$

$$\mathbb{P}\left(\sup_{\overline{r}\phi_{2n}<\sup_{s\in\mathcal{S}}|\gamma(s)-\gamma_{0}(s)|<\overline{C}}\frac{\sup_{s\in\mathcal{S}}\left\|\overline{L}_{n}\left(\gamma;s\right)\right\|}{\sqrt{a_{n}}\sup_{s\in\mathcal{S}}\left|\gamma\left(s\right)-\gamma_{0}\left(s\right)\right|}>\eta\right)\leq\varepsilon,\tag{A.36}$$
  
*if*  $n^{1-2\epsilon}b_{n}^{2}\to\varrho<\infty.$ 

Lemma A.14  $n^{\epsilon} \sup_{s \in S} \left\| \widehat{\theta}(\widehat{\gamma}(s)) - \theta_0 \right\| = o_p(1).$ 

**Proof of Theorem 4** Since  $\sup_{s \in S} (Q_n^*(\widehat{\gamma}(s); s) - Q_n^*(\gamma_0(s); s)) \leq 0$  by construction, where  $Q_n^*(\gamma(s); s)$  is defined in (A.20), it suffices to show that as  $n \to \infty$ ,

$$\mathbb{P}\left(\sup_{s\in\mathcal{S}}\left\{Q_n^*(\gamma(s);s) - Q_n^*(\gamma_0(s);s)\right\} > 0\right) \to 1$$

for any  $\gamma(s)$  such that  $\sup_{s \in \mathcal{S}} |\gamma(s) - \gamma_0(s)| > \overline{r}\phi_{2n}$  where  $\overline{r}$  is chosen in Lemma A.13.

To this end, consider  $\gamma$  such that  $\overline{r}\phi_{2n} \leq \sup_{s\in\mathcal{S}} |\gamma(s) - \gamma_0(s)| \leq \overline{C}$  for some  $0 < \overline{r}, \overline{C} < \infty$ . Then, using (A.24) and Lemma A.14, we have

$$\frac{Q_n^*(\gamma(s);s) - Q_n^*(\gamma_0(s);s)}{a_n \sup_{s \in \mathcal{S}} |\gamma(s) - \gamma_0(s)|} \\
\geq \frac{T_n(\gamma;s)}{\sup_{s \in \mathcal{S}} |\gamma(s) - \gamma_0(s)|} - 2\frac{2L_n(\gamma;s)}{\sqrt{a_n} \sup_{s \in \mathcal{S}} |\gamma(s) - \gamma_0(s)|} - \frac{2C^*(s)b_n}{\sup_{s \in \mathcal{S}} |\gamma(s) - \gamma_0(s)|} + o_p(1)$$

for some  $C^*(s) = O_p(1)$ . Furthermore, Lemma A.2 gives that  $\sup_{s \in \mathcal{S}} C^*(s)$  is also  $O_p(1)$ , and hence

$$\sup_{\overline{r}\phi_{2n} < |\gamma(s) - \gamma_0(s)| < \overline{C}} \frac{\sup_{s \in \mathcal{S}} C^*(s)b_n}{\sup_{s \in \mathcal{S}} |\gamma(s) - \gamma_0(s)|} < \frac{\sup_{s \in \mathcal{S}} C^*(s)b_n}{\overline{r}\phi_{2n}}$$
$$= \frac{\sup_{s \in \mathcal{S}} C^*(s)}{\overline{r}} \left(\frac{a_n b_n}{\log n}\right)$$
$$= O_p(1)$$

given  $a_n b_n \to \rho < \infty$ . Thus, we have

$$\mathbb{P}\left(\sup_{\bar{r}\phi_{2n}<|\gamma(s)-\gamma_{0}(s)|<\overline{C}}\frac{2\sup_{s\in\mathcal{S}}C^{*}(s)b_{n}}{\sup_{s\in\mathcal{S}}|\gamma(s)-\gamma_{0}(s)|}>\eta(s)\right)\leq\frac{\varepsilon}{3}$$

when n is sufficiently large. Therefore, Lemma A.13 yields that, for  $\varepsilon > 0$  and  $\eta > 0$ ,

$$\mathbb{P}\left(\inf_{\bar{r}\phi_{2n}<\sup_{s\in\mathcal{S}}|\gamma(s)-\gamma_0(s)|<\overline{C}}\sup_{s\in\mathcal{S}}\left\{Q_n^*(\gamma(s);s)-Q_n^*(\gamma_0(s);s)\right\}>\eta\right)\geq 1-\varepsilon,$$

which completes the proof by the same argument as Theorem 2.  $\blacksquare$ 

### A.5 Proof of Theorem 5

**Proof of Theorem 5** We simply denote the leave-one-out estimator  $\widehat{\gamma}_{-i}(s_i)$  as  $\widehat{\gamma}(s_i)$  in this proof. We let  $\mathbf{1}_{\mathcal{S}} = \mathbf{1}[s_i \in \mathcal{S}]$  and consider a sequence  $\Delta_n > 0$  such that  $\Delta_n \to 0$  as  $n \to \infty$ . Then,

$$\sqrt{n} \left( \widehat{\beta} - \beta_0 \right) = \left( \frac{1}{n} \sum_{i=1}^n x_i x_i^{\mathsf{T}} \mathbf{1} \left[ q_i > \widehat{\gamma} \left( s_i \right) + \Delta_n \right] \mathbf{1}_S \right)^{-1} \\
\times \left\{ \frac{1}{\sqrt{n}} \sum_{i=1}^n x_i u_i \mathbf{1} \left[ q_i > \gamma_0 \left( s_i \right) + \Delta_n \right] \mathbf{1}_S \\
+ \frac{1}{\sqrt{n}} \sum_{i=1}^n x_i u_i \left\{ \mathbf{1} \left[ q_i > \widehat{\gamma} \left( s_i \right) + \Delta_n \right] - \mathbf{1} \left[ q_i > \gamma_0 \left( s_i \right) + \Delta_n \right] \right\} \mathbf{1}_S \right\} \\
+ \frac{1}{\sqrt{n}} \sum_{i=1}^n x_i x_i^{\mathsf{T}} \delta_0 \mathbf{1} \left[ q_i \le \gamma_0 \left( s_i \right) \right] \mathbf{1} \left[ q_i > \widehat{\gamma} \left( s_i \right) + \Delta_n \right] \mathbf{1}_S \right\} \\
\equiv \Xi_{n00}^{-1} \left\{ \Xi_{n01} + \Xi_{n02} + \Xi_{n03} \right\} \tag{A.37}$$

and

$$\sqrt{n} \left( \widehat{\delta}^{*} - \delta_{0}^{*} \right) = \left( \frac{1}{n} \sum_{i=1}^{n} x_{i} x_{i}^{\top} \mathbf{1} \left[ q_{i} < \widehat{\gamma} \left( s_{i} \right) - \Delta_{n} \right] \mathbf{1}_{S} \right)^{-1} \\
\times \left\{ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} x_{i} u_{i} \mathbf{1} \left[ q_{i} < \gamma_{0} \left( s_{i} \right) - \Delta_{n} \right] \mathbf{1}_{S} \\
+ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} x_{i} u_{i} \left\{ \mathbf{1} \left[ q_{i} < \widehat{\gamma} \left( s_{i} \right) - \Delta_{n} \right] - \mathbf{1} \left[ q_{i} < \gamma_{0} \left( s_{i} \right) - \Delta_{n} \right] \right\} \mathbf{1}_{S} \right\} \\
+ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} x_{i} x_{i}^{\top} \delta_{0} \mathbf{1} \left[ q_{i} \le \gamma_{0} \left( s_{i} \right) \right] \mathbf{1} \left[ q_{i} < \widehat{\gamma} \left( s_{i} \right) - \Delta_{n} \right] \mathbf{1}_{S} \right\} \\
\equiv \left[ \Xi_{n10}^{-1} \left\{ \Xi_{n11} + \Xi_{n12} + \Xi_{n13} \right\}, \quad (A.38)$$

where  $\Xi_{n02}$ ,  $\Xi_{n03}$ ,  $\Xi_{n12}$ , and  $\Xi_{n13}$  are all  $o_p(1)$  from Lemma A.15 below. Therefore,

$$\sqrt{n}\left(\widehat{\theta}^{*} - \theta_{0}^{*}\right) = \left(\begin{array}{c} \Xi_{n00}^{-1} \Xi_{n01} \\ \Xi_{n10}^{-1} \Xi_{n11} \end{array}\right) + o_{p}\left(1\right) = \left(\begin{array}{c} \Xi_{n00} & 0 \\ 0 & \Xi_{n10} \end{array}\right)^{-1} \left(\begin{array}{c} \Xi_{n01} \\ \Xi_{n11} \end{array}\right) + o_{p}\left(1\right)$$

and the desired result follows since

$$\Xi_{n00} \to_p \mathbb{E} \left[ x_i x_i^\top \mathbf{1} \left[ q_i > \gamma_0 \left( s_i \right) \right] \mathbf{1}_S \right], \tag{A.39}$$

$$\Xi_{n10} \to_p \mathbb{E} \left[ x_i x_i^\top \mathbf{1} \left[ q_i < \gamma_0 \left( s_i \right) \right] \mathbf{1}_S \right], \tag{A.40}$$

and

$$\left(\begin{array}{c} \Xi_{n01} \\ \Xi_{n11} \end{array}\right) \rightarrow_{d} \mathcal{N}\left(0, \lim_{n \to \infty} \frac{1}{n} Var\left[\sum_{i=1}^{n} \left(\begin{array}{c} x_{i} u_{i} \mathbf{1} \left[q_{i} > \gamma_{0}\left(s_{i}\right)\right] \mathbf{1}_{S} \\ x_{i} u_{i} \mathbf{1} \left[q_{i} < \gamma_{0}\left(s_{i}\right)\right] \mathbf{1}_{S} \end{array}\right)\right]\right)$$
(A.41)

as  $n \to \infty$ .

First, by Assumptions A-(v) and (ix), (A.39) can be readily verified since we have

$$\frac{1}{n} \sum_{i=1}^{n} x_i x_i^{\mathsf{T}} \mathbf{1} \left[ q_i > \widehat{\gamma} \left( s_i \right) + \Delta_n \right] \mathbf{1}_S$$

$$= \frac{1}{n} \sum_{i=1}^{n} x_i x_i^{\mathsf{T}} \mathbf{1} \left[ q_i > \gamma_0 \left( s_i \right) + \Delta_n \right] \mathbf{1}_S$$

$$+ \frac{1}{n} \sum_{i=1}^{n} x_i x_i^{\mathsf{T}} \left\{ \mathbf{1} \left[ q_i > \widehat{\gamma} \left( s_i \right) + \Delta_n \right] - \mathbf{1} \left[ q_i > \gamma_0 \left( s_i \right) + \Delta_n \right] \right\} \mathbf{1}_S$$

$$= \frac{1}{n} \sum_{i=1}^{n} x_i x_i^{\mathsf{T}} \mathbf{1} \left[ q_i > \gamma_0 \left( s_i \right) + \Delta_n \right] \mathbf{1}_S + O_p \left( \phi_{2n} \right)$$

with  $\Delta_n \to 0$  as  $n \to \infty$ . More precisely, given Theorem 4, we consider  $\widehat{\gamma}(s)$  in a neighborhood of  $\gamma_0(s)$  with distance at most  $\overline{r}\phi_{2n}$  for some large enough constant  $\overline{r}$ . We define a non-random function  $\widetilde{\gamma}(s) = \gamma_0(s) + \overline{r}\phi_{2n}$  and  $\widetilde{\Delta}_i(s_i) = \mathbf{1} [q_i > \widetilde{\gamma}(s_i) + \Delta_n] - \mathbf{1} [q_i > \gamma_0(s_i) + \Delta_n]$ . Then, on the event  $E_n^* = \{\sup_{s \in \mathcal{S}} |\widehat{\gamma}(s) - \gamma_0(s)| \le \overline{r}\phi_{2n}\},$ 

$$\mathbb{E}\left[x_{i}x_{i}^{\mathsf{T}}\widehat{\Delta}_{i}\left(s_{i}\right)\mathbf{1}_{\mathcal{S}}\right] \leq \mathbb{E}\left[x_{i}x_{i}^{\mathsf{T}}\widetilde{\Delta}_{i}\left(s_{i}\right)\mathbf{1}_{\mathcal{S}}\right] \qquad (A.42)$$

$$= \int_{\mathcal{S}}\int_{\gamma_{0}(v)+\Delta_{n}}^{\widetilde{\gamma}(v)+\Delta_{n}} D\left(q,v\right)f\left(q,v\right)dqdv$$

$$= \int_{\mathcal{S}}\left\{D\left(\gamma_{0}\left(v\right),v\right)f\left(\gamma_{0}\left(v\right),v\right)\left(\widetilde{\gamma}\left(v\right)-\gamma_{0}\left(v\right)\right)+o_{p}\left(\phi_{2n}\right)\right\}dv$$

$$\leq \overline{r}\phi_{2n}\int D\left(\gamma_{0}\left(v\right),v\right)f\left(\gamma_{0}\left(v\right),v\right)dv$$

$$= O_{p}\left(\phi_{2n}\right)=o_{p}\left(1\right)$$

from Theorem 4, Assumptions A-(v), (vii), and (ix). (A.40) can be verified symmetrically. Using a similar argument, since  $\mathbb{E}[x_i u_i \mathbf{1} [q_i > \gamma_0 (s_i)] \mathbf{1}_S] = \mathbb{E}[x_i u_i \mathbf{1} [q_i < \gamma_0 (s_i)] \mathbf{1}_S] = 0$  from Assumption ID-(i), asymptotic normality in (A.41) follows by the Theorem of Bolthausen (1982) under Assumption A-(iii), which completes the proof.

**Lemma A.15** When  $\phi_{2n} \to 0$  as  $n \to \infty$ , if we let  $\Delta_n > 0$  such that  $\Delta_n \to 0$  and  $\phi_{2n}/\Delta_n \to 0$  as  $n \to \infty$ , then it holds that  $\Xi_{n02}$ ,  $\Xi_{n03}$ ,  $\Xi_{n12}$ , and  $\Xi_{n13}$  in (A.37) and (A.38) are all  $o_p(1)$ .

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## Supplement to "Nonparametric Sample Splitting"

#### BY YOONSEOK LEE AND YULONG WANG

This supplementary material contains omitted proofs of some lemmas.

**Proof of Lemma A.2** We first show the pointwise convergence. For expositional simplicity, we only present the case of scalar  $x_i$ . Similarly as (A.1), we have

$$\mathbb{E}\left[\Delta M_{n}\left(s\right)\right]$$

$$=\iint D(q,s+b_{n}t)f(q,s+b_{n}t)\left\{\mathbf{1}\left[q<\gamma_{0}\left(s+b_{n}t\right)\right]-\mathbf{1}\left[q<\gamma_{0}\left(s\right)\right]\right\}K(t)dqdt,$$

which is non-zero only when (i)  $\gamma_0(s) < q < \gamma_0(s + b_n t)$  if  $\gamma_0(s) < \gamma_0(s + b_n t)$ ; or (ii)  $\gamma_0(s + b_n t) < q < \gamma_0(s)$  if  $\gamma_0(s) > \gamma_0(s + b_n t)$ . We suppose  $\gamma_0(\cdot)$  is increasing around s. Then, for the case (i), since  $0 < \gamma_0(s + b_n t) - \gamma_0(s)$ , it restricts t > 0. For the case (ii), however, it restricts t < 0. Therefore, if we let  $m(q, s) = D(q, s)f(q, s) < \infty$ , by Taylor expansion,

$$\mathbb{E} \left[ \Delta M_n \left( s \right) \right]$$

$$= \int_0^\infty \int_{\gamma_0(s)}^{\gamma_0(s+b_nt)} m(q,s+b_nt)K(t)dqdt + \int_{-\infty}^0 \int_{\gamma_0(s+b_nt)}^{\gamma_0(s)} m(q,s+b_nt)K(t)dqdt$$

$$= m(\gamma_0(s),s)\dot{\gamma}_0(s) b_n \int_0^\infty tK(t)dt - m(\gamma_0(s),s)\dot{\gamma}_0(s) b_n \int_{-\infty}^0 tK(t)dt + O\left(b_n^2\right)$$

$$= m(\gamma_0(s),s)\dot{\gamma}_0(s) b_n + O\left(b_n^2\right),$$

where  $\int_0^\infty tK(t)dt = -\int_{-\infty}^0 tK(t)dt$  and  $\dot{\gamma}_0(s) = d\gamma_0(s)/ds > 0$  in this case.

Symmetrically, we can also derive  $\mathbb{E}[\Delta M_n(s)] = -m(\gamma_0(s), s)\dot{\gamma}_0(s)b_n + O(b_n^2)$ when  $\gamma_0(\cdot)$  is decreasing around s. Therefore,  $\mathbb{E}[\Delta M_n(s)] = m(\gamma_0(s), s)|\dot{\gamma}_0(s)|b_n = O(b_n)$  because  $m(\gamma_0(s), s)|\dot{\gamma}_0(s)| < \infty$  from Assumptions A-(vi) and (vii). The desired result follows since  $Var[\Delta M_n(s)] \leq 2Var[M_n(\gamma_0(s_i); s)] + 2Var[M_n(\gamma_0(s); s)] = o(1)$  from (A.2).

Given the pointwise rate, it suffices to show  $\Delta M_n(s)$  is uniformly tight. This is implied by the tightness of  $M_n(s)$  in Lemma A.1 since  $\gamma_0(\cdot)$  is continuous. The proof is complete.

**Proof of Lemma A.4** We first show (A.16). We consider the case with  $\gamma(s) > \gamma_0(s)$ , and the other direction can be shown symmetrically. In this case, since  $T_n(\gamma; s) = c_0^{\top}(M_n(\gamma(s); s) - M_n(\gamma_0(s); s))c_0$  where  $\partial \mathbb{E}[T_n(\gamma; s)] / \partial \gamma(s) = c_0^{\top} D(\gamma(s), s)c_0 f(\gamma(s), s)$ 

is continuous at  $\gamma_0(s)$  and  $c_0^\top D(\gamma_0(s), s)c_0 f(\gamma_0(s), s) > 0$  from Assumptions A-(vii) and (viii), there exists a sufficiently small  $\overline{C}(s) > 0$  such that

$$\underline{\ell}_D(s) = \inf_{|\gamma(s) - \gamma_0(s)| < \overline{C}(s)} c_0^\top D(\gamma(s), s) c_0 f(\gamma(s), s) > 0.$$

By Taylor expansion, we have

$$\mathbb{E}\left[T_n\left(\gamma;s\right)\right] = \int \int_{\gamma_0(s)}^{\gamma(s)} \mathbb{E}\left[\left(c_0^{\top} x_i\right)^2 | q, s + b_n t\right] f(q, s + b_n t) K(t) \, dq dt$$
$$= \left\{\gamma(s) - \gamma_0(s)\right\} \left\{c_0^{\top} D(\gamma, s) c_0 f(\gamma, s) + C_1(s) b_n^2\right\}$$

for some  $C_1(s) < \infty$ , which yields

$$\mathbb{E}\left[T_n\left(\gamma;s\right)\right] \ge \left\{\gamma\left(s\right) - \gamma_0\left(s\right)\right\} \left(\underline{\ell}_D(s) + C_1(s)b_n^2\right),\tag{B.1}$$

since  $\mathbb{E}[T_n(\gamma_0;s)] = 0$ . Furthermore, if we let  $\Delta_i(\gamma;s) = \mathbf{1}_i(\gamma(s)) - \mathbf{1}_i(\gamma_0(s))$  and  $Z_{n,i}(s) = (c_0^{\top}x_i)^2 \Delta_i(\gamma;s) K_i(s) - \mathbb{E}[(c_0^{\top}x_i)^2 \Delta_i(\gamma;s) K_i(s)]$ , using a similar argument as (A.2), we have

$$\mathbb{E}\left[\left(T_{n}\left(\gamma;s\right) - \mathbb{E}\left[T_{n}\left(\gamma;s\right)\right]\right)^{2}\right]$$

$$= \frac{1}{n^{2}b_{n}^{2}}\sum_{i=1}^{n}\mathbb{E}\left[Z_{n,i}^{2}(s)\right] + \frac{1}{n^{2}b_{n}^{2}}\sum_{i\neq j}Cov[Z_{n,i}(s), Z_{n,j}(s)]$$

$$\leq \frac{C_{2}(s)}{nb_{n}}\left\{\gamma\left(s\right) - \gamma_{0}\left(s\right)\right\}$$
(B.2)

for some  $C_2(s) \in (0, \infty)$  since  $\varphi \in (0, 2)$  in Assumption A-(iii).

We suppose *n* is large enough so that  $\overline{r}(s)\phi_{1n} \leq \overline{C}(s)$ . Similarly as Lemma A.7 in Hansen (2000), we set  $\gamma_g$  for  $g = 1, 2, ..., \overline{g} + 1$  such that, for any  $s \in \mathcal{S}$ ,  $\gamma_g(s) = \gamma_0(s) + 2^{g-1}\overline{r}(s)\phi_{1n}$ , where  $\overline{g}$  is the integer satisfying  $\gamma_{\overline{g}}(s) - \gamma_0(s) = 2^{\overline{g}-1}\overline{r}(s)\phi_{1n} \leq \overline{C}(s)$  and  $\gamma_{\overline{g}+1}(s) - \gamma_0(s) = 2^{\overline{g}}\phi_{1n} > \overline{C}(s)$ . Then Markov's inequality and (B.2) yield that for any fixed  $\eta(s) > 0$ ,

$$\mathbb{P}\left(\max_{1\leq g\leq \overline{g}} \left| \frac{T_n\left(\gamma_g; s\right)}{\mathbb{E}\left[T_n\left(\gamma_g; s\right)\right]} - 1 \right| > \eta(s) \right) \tag{B.3}$$

$$\leq \mathbb{P}\left(\max_{1\leq g\leq \overline{g}} \left| \frac{T_n\left(\gamma_g; s\right) - \mathbb{E}\left[T_n\left(\gamma_g; s\right)\right]}{\mathbb{E}\left[T_n\left(\gamma_g; s\right)\right]} \right| > \eta(s) \right)$$

$$\leq \frac{1}{\eta^2(s)} \sum_{g=1}^{\overline{g}} \frac{\mathbb{E}\left[ \left(T_n\left(\gamma_g; s\right) - \mathbb{E}\left[T_n\left(\gamma_g; s\right)\right]\right)^2 \right]}{\left|\mathbb{E}\left[T_n\left(\gamma_g; s\right)\right]\right|^2}$$

$$\leq \frac{1}{\eta^2(s)} \sum_{g=1}^{\overline{g}} \frac{C_2(s)\overline{r}(s)\phi_{1n}\left(nb_n\right)^{-1}}{\left|\overline{r}(s)\phi_{1n}(\underline{\ell}_D(s) + C_1(s)b_n^2)\right|^2}$$

$$\leq \frac{1}{\eta^2(s)} \sum_{g=1}^{\overline{g}} \frac{C_2(s) (nb_n)^{-1}}{2^{g-1} \underline{\ell}_D^2(s) \overline{r}(s) \phi_{1n}}$$
$$\leq \frac{C_2(s)}{\eta^2(s) \overline{r}(s) \underline{\ell}_D^2(s)} \sum_{g=1}^{\infty} \frac{1}{2^{g-1}} \times \frac{1}{n^{2\epsilon}}$$
$$\leq \varepsilon(s)$$

for any  $\varepsilon(s) > 0$ . From eq. (33) of Hansen (2000), for any  $\gamma(s)$  such that  $\overline{r}(s)\phi_{1n} \leq \gamma(s) - \gamma_0(s) \leq \overline{C}(s)$ , there exists some g satisfying  $\gamma_g(s) - \gamma_0(s) < \gamma(s) - \gamma_0(s) < \gamma_{g+1}(s) - \gamma_0(s)$ , and then

$$\frac{T_n(\gamma;s)}{|\gamma(s) - \gamma_0(s)|} \geq \frac{T_n(\gamma_g;s)}{\mathbb{E}\left[T_n(\gamma_g;s)\right]} \times \frac{\mathbb{E}\left[T_n(\gamma_g;s)\right]}{|\gamma_{g+1}(s) - \gamma_0(s)|}$$
$$\geq \left\{ 1 - \max_{1 \le g \le \overline{g}} \left| \frac{T_n(\gamma_g;s)}{\mathbb{E}\left[T_n(\gamma_g;s)\right]} - 1 \right| \right\} \frac{\mathbb{E}\left[T_n(\gamma_g;s)\right]}{|\gamma_{g+1}(s) - \gamma_0(s)|}.$$

Hence, we can find  $C_T(s) < \infty$  such that

$$\mathbb{P}\left(\inf_{\overline{r}(s)\phi_{1n} < |\gamma(s) - \gamma_{0}(s)| < \overline{C}(s)} \frac{T_{n}(\gamma; s)}{|\gamma(s) - \gamma_{0}(s)|} < C_{T}(s)(1 - \eta(s))\right)$$

$$\leq \mathbb{P}\left(\frac{T_{n}(\gamma_{g}; s)}{\mathbb{E}\left[T_{n}(\gamma_{g}; s)\right]} \times \frac{\mathbb{E}\left[T_{n}(\gamma_{g}; s)\right]}{|\gamma_{g+1}(s) - \gamma_{0}(s)|} < C_{T}(s)(1 - \eta(s))\right)$$

$$\leq \mathbb{P}\left(\left\{1 - \max_{1 \le g \le \overline{g}} \left|\frac{T_{n}(\gamma_{g}; s)}{\mathbb{E}\left[T_{n}(\gamma_{g}; s)\right]} - 1\right|\right\} \frac{\mathbb{E}\left[T_{n}(\gamma_{g}; s)\right]}{|\gamma_{g+1}(s) - \gamma_{0}(s)|} < C_{T}(s)(1 - \eta(s))\right)$$

$$\leq \varepsilon(s),$$

where the last line follows from (B.1) and (B.3). The proof for (A.17) is similar to that for (A.16) and hence omitted.

For (A.18),  $\mathbb{E}[L_n(\gamma; s)] = 0$  and we have

$$\mathbb{E}\left[\left|L_{n}\left(\gamma;s\right)\right|^{2}\right] \leq \phi_{1n}C_{3}(s) \tag{B.4}$$

for some  $C_3(s) \in (0,\infty)$  similarly as (B.2). By defining  $\gamma_g$  in the same way as above,

the Markov's inequality and (B.4) get us that for any fixed  $\eta(s) > 0$ ,

$$\mathbb{P}\left(\max_{1\leq g\leq \overline{g}} \frac{\left|L_{n}\left(\gamma_{g};s\right)\right|}{\sqrt{a_{n}}\left(\gamma_{g}\left(s\right)-\gamma_{0}\left(s\right)\right)} > \eta(s)\right) \tag{B.5}$$

$$\leq \frac{1}{\eta^{2}(s)} \sum_{g=1}^{\infty} \frac{\mathbb{E}\left[L_{n}\left(\gamma_{g},s\right)^{2}\right]}{a_{n}\left|\gamma_{g}\left(s\right)-\gamma_{0}\left(s\right)\right|^{2}}$$

$$\leq \frac{1}{\eta^{2}(s)} \sum_{g=1}^{\infty} \frac{\phi_{1n}C_{3}(s)}{a_{n}\left|\gamma_{g}\left(s\right)-\gamma_{0}\left(s\right)\right|^{2}}$$

$$\leq \frac{C_{3}(s)}{\eta^{2}(s)\overline{r}(s)} \sum_{g=1}^{\infty} \frac{1}{2^{g-1}}.$$

This probability is arbitrarily close to zero if  $\overline{r}(s)$  is chosen large enough. It is worth to note that (B.5) provides the maximal (or sharp) rate of  $\phi_{1n}$  as  $a_n^{-1}$  because we need  $\phi_{1n}/a_n |\gamma_g(s) - \gamma_0(s)|^2 = O(\phi_{1n}a_n) = O(1)$  but  $\phi_{1n} \to 0$  as  $n \to \infty$ . This  $\phi_{1n}a_n = O(1)$  condition also satisfies (B.3).

Finally, for a given g, we define  $\Gamma_g(s)$  as the collection of  $\gamma(s)$  satisfying  $\overline{r}(s)2^{g-1}\phi_{1n} < \gamma(s) - \gamma_0(s) < \overline{r}(s)2^g\phi_{1n}$  for each  $s \in S$ . Then,

$$\mathbb{P}\left(\sup_{\overline{r}(s)\phi_{1n} < |\gamma(s) - \gamma_{0}(s)| < \overline{C}(s)} \frac{|L_{n}(\gamma; s)|}{\sqrt{a_{n}} |\gamma(s) - \gamma_{0}(s)|} > \eta(s)\right) \tag{B.6}$$

$$= \mathbb{P}\left(\max_{1 \le g \le \overline{g}} \sup_{\gamma \in \Gamma_{g}(s)} \frac{|L_{n}(\gamma; s)|}{\sqrt{a_{n}} (\gamma(s) - \gamma_{0}(s))} > \eta(s)\right)$$

$$\le \mathbb{P}\left(\max_{1 \le g \le \overline{g}} \frac{|L_{n}(\gamma_{g}; s)|}{\sqrt{a_{n}} (\gamma_{g+1}(s) - \gamma_{0}(s))} > \eta(s)\right)$$

$$\le \frac{C_{4}(s)}{\eta^{2}(s) \overline{r}(s)}$$

for some  $C_4(s) \in (0, \infty)$ . Combining (B.5) and (B.6), we thus have

$$\mathbb{P}\left(\sup_{\overline{r}(s)\phi_{1n} < |\gamma(s) - \gamma_{0}(s)| < \overline{C}(s)} \frac{\left|L_{n}\left(\gamma_{g}; s\right)\right|}{\sqrt{a_{n}}\left(\gamma\left(s\right) - \gamma_{0}\left(s\right)\right)} > \eta\left(s\right)\right) \\
\leq 2\mathbb{P}\left(\max_{1 \le g \le \overline{g}} \frac{\left|L_{n}\left(\gamma_{g}; s\right)\right|}{\sqrt{a_{n}}\left(\gamma_{g}\left(s\right) - \gamma_{0}\left(s\right)\right)} > \eta\left(s\right)\right) \\
+ 2\mathbb{P}\left(\max_{1 \le g \le \overline{g}} \sup_{\gamma \in \Gamma_{g}(s)} \frac{\left|L_{n}\left(\gamma; s\right)\right|}{\sqrt{a_{n}}\left(\gamma\left(s\right) - \gamma_{0}\left(s\right)\right)} > \eta\left(s\right)\right) \\
\leq \varepsilon(s)$$

for any  $\varepsilon(s) > 0$  if we pick  $\overline{r}(s)$  sufficiently large. The proof for (A.19) is similar to that for (A.18) and hence omitted.

**Proof of Lemma A.5** Using the same notations in Lemma A.3, (A.12) yields

$$n^{\epsilon} \left( \widehat{\theta}(\widehat{\gamma}(s)) - \theta_{0} \right)$$
(B.7)  
$$= \left\{ \frac{1}{nb_{n}} \widetilde{Z}(\widehat{\gamma}(s); s)^{\top} \widetilde{Z}(\widehat{\gamma}(s); s) \right\}^{-1} \times \left\{ \frac{n^{\epsilon}}{nb_{n}} \widetilde{Z}(\widehat{\gamma}(s); s)^{\top} \widetilde{u}(s) - \frac{n^{\epsilon}}{nb_{n}} \widetilde{Z}(\widehat{\gamma}(s); s)^{\top} \left( \widetilde{Z}(\widehat{\gamma}(s); s) - \widetilde{Z}(\gamma_{0}(s_{i}); s) \right) \theta_{0} \right\}$$
$$\equiv \Theta_{A1}^{-1}(s) \left\{ \Theta_{A2}(s) - \Theta_{A3}(s) \right\}.$$

For the denominator  $\Theta_{A1}(s)$ , we have

$$\Theta_{A1}(s) = \begin{pmatrix} (nb_n)^{-1} \sum_{i=1}^n x_i x_i^\top K_i(s) & M_n\left(\widehat{\gamma}(s);s\right) \\ M_n\left(\widehat{\gamma}(s);s\right) & M_n\left(\widehat{\gamma}(s);s\right) \end{pmatrix}$$

$$\rightarrow_p \begin{pmatrix} M(s) & M\left(\gamma_0(s);s\right) \\ M\left(\gamma_0(s);s\right) & M\left(\gamma_0(s);s\right) \end{pmatrix},$$
(B.8)

where  $M_n(\widehat{\gamma}(s); s) \to_p M(\gamma_0(s); s) < \infty$  from Lemma A.1 and the pointwise consistency of  $\widehat{\gamma}(s)$  in Lemma A.3. In addition,  $(nb_n)^{-1} \sum_{i=1}^n x_i x_i^\top K_i(s) \to_p M(s) = \int_{-\infty}^{\infty} D(q, s) f(q, s) dq < \infty$  from the standard kernel estimation result. Note that the probability limit is positive definite since both M(s) and  $M(\gamma_0(s); s)$  are positive definite and

$$M(s) - M(\gamma_0(s); s) = \int_{\gamma_0(s)}^{\infty} D(q, s) f(q, s) \, dq > 0$$

for any  $\gamma_0(s) \in \Gamma$  from Assumption A-(viii).

For the numerator part  $\Theta_{A2}(s)$ , we have  $\Theta_{A2}(s) = O_p(a_n^{-1/2}) = o_p(1)$  because

$$\frac{1}{\sqrt{nb_n}}\widetilde{Z}(\widehat{\gamma}(s);s)^{\top}\widetilde{u}(s) = \begin{pmatrix} (nb_n)^{-1/2}\sum_{i=1}^n x_i u_i K_i(s) \\ J_n\left(\widehat{\gamma}(s);s\right) \end{pmatrix} = O_p\left(1\right)$$
(B.9)

from from Lemma A.1 and the pointwise consistency of  $\widehat{\gamma}(s)$  in Lemma A.3. Note that the standard kernel estimation result gives  $(nb_n)^{-1/2} \sum_{i=1}^n x_i u_i K_i(s) = O_p(1)$ . Moreover, we have

$$\Theta_{A3}(s) = \left(\begin{array}{c} (nb_n)^{-1} \sum_{i=1}^n c_0^\top x_i x_i^\top \left\{ \mathbf{1}_i\left(\widehat{\gamma}(s)\right) - \mathbf{1}_i\left(\gamma_0\left(s_i\right)\right) \right\} K_i\left(s\right) \\ (nb_n)^{-1} \sum_{i=1}^n c_0^\top x_i x_i^\top \mathbf{1}_i\left(\widehat{\gamma}(s)\right) \left\{ \mathbf{1}_i\left(\widehat{\gamma}(s)\right) - \mathbf{1}_i\left(\gamma_0\left(s_i\right)\right) \right\} K_i\left(s\right) \end{array}\right)$$
(B.10)

and

$$\frac{1}{nb_n} \sum_{i=1}^n c_0^\top x_i x_i^\top \{ \mathbf{1}_i(\widehat{\gamma}(s)) - \mathbf{1}_i(\gamma_0(s_i)) \} K_i(s) \quad (B.11)$$

$$\leq \|c_0\| \| M_n(\widehat{\gamma}(s); s) - M_n(\gamma_0(s_i); s) \|$$

$$\leq \|c_0\| \{ \| M_n(\widehat{\gamma}(s); s) - M_n(\gamma_0(s); s) \| + O_p(b_n) \}$$

$$= o_p(1),$$

where the second inequality is from (A.14) and the last equality is because  $M_n(\gamma; s) \rightarrow_p M(\gamma; s)$  is continuous in  $\gamma$  and  $\widehat{\gamma}(s) \rightarrow_p \gamma_0(s)$  in Lemma A.3. Since

$$\frac{1}{nb_n} \sum_{i=1}^n c_0^\top x_i x_i^\top \mathbf{1}_i(\widehat{\gamma}(s)) \{ \mathbf{1}_i(\widehat{\gamma}(s)) - \mathbf{1}_i(\gamma_0(s_i)) \} K_i(s)$$

$$\leq \|c_0\| \|M_n(\widehat{\gamma}(s); s) - M_n(\gamma_0(s_i); s)\| = o_p(1)$$
(B.12)

from (B.11), we have  $\Theta_{A3}(s) = o_p(1)$  as well, which completes the proof.

Proof of Lemma A.7 Using the same notations in Lemma A.3, we write

$$\begin{split} &\sqrt{nb_n} \left( \widehat{\theta}\left(\widehat{\gamma}\left(s\right)\right) - \theta_0 \right) \\ &= \left\{ \frac{1}{nb_n} \widetilde{Z}(\widehat{\gamma}(s);s)^\top \widetilde{Z}(\widehat{\gamma}(s);s) \right\}^{-1} \\ &\times \left\{ \frac{1}{\sqrt{nb_n}} \widetilde{Z}(\widehat{\gamma}(s);s)^\top \widetilde{u}(s) - \frac{1}{\sqrt{nb_n}} \widetilde{Z}(\widehat{\gamma}(s);s)^\top \left( \widetilde{Z}(\widehat{\gamma}(s);s) - \widetilde{Z}(\gamma_0(s_i);s) \right) \theta_0 \right\} \\ &\equiv \Theta_{B1}^{-1}(s) \left\{ \Theta_{B2}(s) - \Theta_{B3}(s) \right\} \end{split}$$

similarly as (B.7). For the denominator, since  $\Theta_{B1}(s) = \Theta_{A1}(s)$  in (B.7), then  $\Theta_{B1}^{-1}(s) = O_p(1)$  from (B.8). For the numerator, we first have  $\Theta_{B2}(s) = O_p(1)$  from (B.9). For  $\Theta_{B3}(s)$ , similarly as (B.10),

$$\Theta_{B3}(s) = \left(\begin{array}{c}a_n^{-1/2} \sum_{i=1}^n n^{-\epsilon} \delta_0^{\top} x_i x_i^{\top} \left\{ \mathbf{1}_i\left(\widehat{\gamma}(s)\right) - \mathbf{1}_i\left(\gamma_0\left(s_i\right)\right) \right\} K_i\left(s\right)\\a_n^{-1/2} \sum_{i=1}^n n^{-\epsilon} \delta_0^{\top} x_i x_i^{\top} \mathbf{1}_i\left(\widehat{\gamma}(s)\right) \left\{ \mathbf{1}_i\left(\widehat{\gamma}(s)\right) - \mathbf{1}_i\left(\gamma_0\left(s_i\right)\right) \right\} K_i\left(s\right)\end{array}\right).$$

However, since  $\hat{\gamma}(s) = \gamma_0(s) + r(s)\phi_{1n}$  for some  $r(s) < \infty$  from Theorem 2, similarly as (A.25), we have

$$\mathbb{E}\left[\sum_{i=1}^{n} n^{-\epsilon} \delta_{0}^{\top} x_{i} x_{i}^{\top} \left\{\mathbf{1}_{i}\left(\widehat{\gamma}(s)\right) - \mathbf{1}_{i}\left(\gamma_{0}\left(s_{i}\right)\right)\right\} K_{i}\left(s\right)\right] \\
\leq a_{n} \left| \iint_{\gamma_{0}(s)+r(s)\phi_{1n}}^{\gamma_{0}(s)+r(s)\phi_{1n}} c_{0}^{\top} \mathbb{E}\left[x_{i} x_{i}^{\top} | q, s+b_{n}t\right] K\left(t\right) f\left(q, s+b_{n}t\right) dq dt \\
\leq a_{n} \left| \iint_{\gamma_{0}(s)}^{\gamma_{0}(s)+r(s)\phi_{1n}} c_{0}^{\top} \mathbb{E}\left[x_{i} x_{i}^{\top} | q, s+b_{n}t\right] K\left(t\right) f\left(q, s+b_{n}t\right) dq dt \\
+a_{n} \left| \iint_{\gamma_{0}(s)}^{\gamma_{0}(s+b_{n}t)} c_{0}^{\top} \mathbb{E}\left[x_{i} x_{i}^{\top} | q, s+b_{n}t\right] K\left(t\right) f\left(q, s+b_{n}t\right) dq dt \right| \\
= a_{n}\phi_{1n} |r(s)| \left| c_{0}^{\top} D\left(\gamma_{0}\left(s\right), s\right) \right| f\left(\gamma_{0}\left(s\right), s\right) + O(a_{n}b_{n}) \\
= O(1)$$

as  $a_n\phi_{1n} = 1$  and  $a_nb_n = n^{1-2\epsilon}b_n^2 \to \varrho < \infty$ . We also have

$$Var\left[\sum_{i=1}^{n} n^{-\epsilon} \delta_0^{\top} x_i x_i^{\top} \left\{ \mathbf{1}_i\left(\widehat{\gamma}(s)\right) - \mathbf{1}_i\left(\gamma_0\left(s_i\right)\right) \right\} K_i\left(s\right) \right] = O(n^{-2\epsilon}) = o(1),$$

similarly as (A.26). Therefore, from the same reason as (B.12), we have  $\Theta_{B3}(s) = O_p(a_n^{-1/2}) = o_p(1)$ , which completes the proof.

**Proof of Lemma A.8** First consider the case with r > 0. In this case, we have

$$\{ \mathbf{1}[q \le \gamma_0(s) + r/a_n] - \mathbf{1}[q \le \gamma_0(s)] \} \{ \mathbf{1}[q \le \gamma_0(s + b_n t)] - \mathbf{1}[q \le \gamma_0(s)] \}$$
  
=  $\mathbf{1}[\gamma_0(s) < q \le \gamma_0(s + b_n t) < \gamma_0(s) + r/a_n]$   
+  $\mathbf{1}[\gamma_0(s) < q \le \gamma_0(s) + r/a_n < \gamma_0(s + b_n t)].$ 

Therefore, if we denote  $g(q,s) = c_0^{\top} D(q,s) c_0 f(q,s)$ ,

$$\begin{split} & \mathbb{E}\left[B_{n3}^{*}(r,s)\right] \\ &= a_{n} \iint c_{0}^{\top} D(q,s+b_{n}t)c_{0}\left\{\mathbf{1}[q \leq \gamma_{0}\left(s\right)+r/a_{n}]-\mathbf{1}[q \leq \gamma_{0}\left(s\right)]\right\} \\ &\quad \times \left\{\mathbf{1}[q \leq \gamma_{0}\left(s+b_{n}t\right)]-\mathbf{1}[q \leq \gamma_{0}\left(s\right)]\right\} K\left(t\right) f\left(q,s+b_{n}t\right) dq dt \\ &= a_{n} \int_{\mathcal{T}_{1}(r;s)} \int_{\gamma_{0}(s)}^{\gamma_{0}(s+b_{n}t)} g(q,s+b_{n}t) K\left(t\right) dq dt \\ &\quad +a_{n} \int_{\mathcal{T}_{2}(r;s)} \int_{\gamma_{0}(s)}^{\gamma_{0}(s)+r/a_{n}} g(q,s+b_{n}t) K\left(t\right) dq dt \\ &\equiv B_{n31}(r,s) + B_{n32}(r,s), \end{split}$$

where

$$\begin{aligned} \mathcal{T}_{1}(r;s) &= & \{\gamma_{0}\left(s\right) < \gamma_{0}\left(s+b_{n}t\right)\} \cap \{\gamma_{0}\left(s+b_{n}t\right) < \gamma_{0}\left(s\right)+r/a_{n}\}, \\ \mathcal{T}_{2}(r;s) &= & \{\gamma_{0}\left(s\right) < \gamma_{0}\left(s+b_{n}t\right)\} \cap \{\gamma_{0}\left(s\right)+r/a_{n} < \gamma_{0}\left(s+b_{n}t\right)\}. \end{aligned}$$

Note that  $\gamma_0(s) < \gamma_0(s) + r/a_n$  always holds for r > 0. However, similarly as in the proof of Lemma A.2, when  $\gamma_0(\cdot)$  is increasing around s,  $\gamma_0(s) < \gamma_0(s + b_n t)$  restricts that t > 0. Furthermore,  $\gamma_0(s + b_n t) < \gamma_0(s) + r/a_n$  implies that  $t < r/(a_n b_n \dot{\gamma}_0(s))$ , where  $0 < r/(a_n b_n \dot{\gamma}_0(s)) < \infty$ . Therefore,  $\mathcal{T}_1(r; s) = \{t : t > 0 \text{ and } t < r/(a_n b_n \dot{\gamma}_0(s))\}$ . Similarly, since  $\gamma_0(s) + r/a_n < \gamma_0(s + b_n t)$  implies  $t > r/(a_n b_n \dot{\gamma}_0(s))$ , we have  $\mathcal{T}_2(r; s) = \{t : t > 0 \text{ and } t > r/(a_n b_n \dot{\gamma}_0(s))\}$ . It follows that, by Taylor expansion,

$$B_{n31}(r,s) = a_n \int_0^{r/(a_n b_n \dot{\gamma}_0(s))} \int_{\gamma_0(s)}^{\gamma_0(s+b_n t)} g(q,s+b_n t) K(t) \, dq dt$$
  
=  $a_n b_n g(\gamma_0(s),s) \dot{\gamma}_0(s) \int_0^{r/(a_n b_n \dot{\gamma}_0(s))} t K(t) \, dt + a_n b_n O(b_n)$   
=  $\varrho g(\gamma_0(s),s) \dot{\gamma}_0(s) \mathcal{K}_1(r,\varrho;s) + O(b_n)$ 

as  $a_n b_n = n^{1-2\epsilon} b_n^2 \to \varrho \in (0,\infty)$ , and

$$B_{n32}(r,s) = a_n \int_{r/a_n b_n \dot{\gamma}_0(s)}^{\infty} \int_{\gamma_0(s)}^{\gamma_0(s)+r/a_n} g(q,s+b_n t) K(t) \, dq dt$$
  
=  $rg(\gamma_0(s),s) \int_{r/(a_n b_n \dot{\gamma}_0(s))}^{\infty} K(t) \, dt + O(b_n)$   
=  $rg(\gamma_0(s),s) \left\{ \frac{1}{2} - \mathcal{K}_0(r,\varrho;s) \right\} + O(b_n)$ 

as  $|\mathcal{K}_0(r,\varrho;s)| \leq 1/2$  and  $|\mathcal{K}_1(r,\varrho;s)| \leq 1/2$ .

When  $\gamma_0(\cdot)$  is decreasing around  $s, -\infty < r/(a_n b_n \dot{\gamma}_0(s)) < 0$  and we can also derive

$$B_{n31}(r,s) = a_n \int_{r/(a_n b_n \dot{\gamma}_0(s))}^{0} \int_{\gamma_0(s)}^{\gamma_0(s+b_n t)} g(q,s+b_n t) K(t) \, dq dt$$
  

$$= -\varrho g(\gamma_0(s),s) \dot{\gamma}_0(s) \mathcal{K}_1(r,\varrho;s) + O(b_n),$$
  

$$B_{n32}(r,s) = a_n \int_{-\infty}^{r/(a_n b_n \dot{\gamma}_0(s))} \int_{\gamma_0(s)}^{\gamma_0(s)+r/a_n} g(q,s+b_n t) K(t) \, dq dt$$
  

$$= rg(\gamma_0(s),s) \{(1/2) - \mathcal{K}_0(r,\varrho;s)\} + O(b_n),$$

because, when  $\dot{\gamma}_0(s) < 0$ , we have  $\int_{r/(a_n b_n \dot{\gamma}_0(s))}^0 tK(t) dt = -\int_0^{r/(a_n b_n (-\dot{\gamma}_0(s)))} tK(t) dt$ and  $\int_{-\infty}^{r/(a_n b_n \dot{\gamma}_0(s))} K(t) dt = \int_{r/(a_n b_n (-\dot{\gamma}_0(s)))}^\infty K(t) dt$  with  $\dot{\gamma}_0(s) < 0$ . It follows that, by combining these results, we have

$$\mathbb{E}\left[B_{n3}^{*}(r,s)\right] = |r| g(\gamma_{0}(s),s) \left\{\frac{1}{2} - \mathcal{K}_{0}\left(r,\varrho;s\right)\right\} + \varrho g(\gamma_{0}(s),s) \left|\dot{\gamma}_{0}(s)\right| \mathcal{K}_{1}\left(r,\varrho;s\right) + O\left(b_{n}\right).$$

Furthermore, since  $|B_{n3}^*(r,s)| \leq \sum_{i=1}^n (\delta_0^\top x_i)^2 |\mathbf{1}_i (\gamma_0(s) + r/a_n) - \mathbf{1}_i (\gamma_0(s))| K_i(s)$ , we have  $Var [B_{n3}^*(r,s)] = O(n^{-2\epsilon}) = o(1)$  from (A.26) in Lemma A.6, which completes the proof.  $\blacksquare$ 

**Proof of Lemma A.9** Define  $W_{\mu}(r) = W(r) + \mu(r)$ ,  $\tau^+ = \arg \max_{r \in \mathbb{R}^+} W_{\mu}(r)$ , and  $\tau^- = \arg \max_{r \in \mathbb{R}^-} W_{\mu}(r)$ . The process  $W_{\mu}(\cdot)$  is a Gaussian process, and hence Lemma 2.6 of Kim and Pollard (1990) implies that  $\tau^+$  and  $\tau^-$  are unique almost surely. Recall that we define  $W(r) = W_1(-r)\mathbf{1}[r < 0] + W_2(r)\mathbf{1}[r > 0]$ , where  $W_1(\cdot)$  and  $W_2(\cdot)$  are two independent standard Wiener processes defined on  $\mathbb{R}^+$ . We claim that

$$\mathbb{E}[\tau^+] = -\mathbb{E}[\tau^-] < \infty, \tag{B.13}$$

which gives the desired result.

The equality in (B.13) follows directly from the symmetry (i.e.,  $\mathbb{P}(W_{\mu}(\tau^{+}) > W_{\mu}(\tau^{-})) = 1/2$ ) and the fact that  $W_1$  is independent of  $W_2$ . Now, we focus on r > 0 and show that  $\mathbb{E}[\tau^{+}] < \infty$ . First, for any r > 0,

$$\mathbb{P}\left(W_{\mu}(r) \ge 0\right) = \mathbb{P}\left(W_{2}(r) \ge -\mu(r)\right) = \mathbb{P}\left(\frac{W_{2}(r)}{\sqrt{r}} \ge -\frac{\mu(r)}{\sqrt{r}}\right) = 1 - \Phi\left(-\frac{\mu(r)}{\sqrt{r}}\right),$$

where  $\Phi(\cdot)$  denotes the standard normal distribution function. Since the sample path of  $W_{\mu}(\cdot)$  is continuous, for some  $\underline{r} > 0$ , we then have

$$\mathbb{E}[\tau^{+}] = \int_{0}^{\infty} \left\{ 1 - \mathbb{P}\left(\tau^{+} \leq r\right) \right\} dr$$

$$= \int_{0}^{\underline{r}} \mathbb{P}\left(\tau^{+} > r\right) dr + \int_{\underline{r}}^{\infty} \mathbb{P}\left(\tau^{+} > r\right) dr$$

$$\leq C_{1} + \int_{\underline{r}}^{\infty} \mathbb{P}\left(W_{\mu}(\tau^{+}) \geq 0 \text{ and } \tau^{+} > r\right) dr$$

$$\leq C_{1} + \int_{\underline{r}}^{\infty} \mathbb{P}\left(W_{\mu}(r) \geq 0\right) dr$$

$$= C_{1} + \int_{\underline{r}}^{\infty} \left(1 - \Phi\left(-\frac{\mu(r)}{\sqrt{r}}\right)\right) dr \qquad (B.14)$$

for some  $C_1 < \infty$ , where the first inequality is because  $W_{\mu}(\tau^+) = \max_{r \in \mathbb{R}^+} W_{\mu}(r) \ge 0$ given  $W_{\mu}(0) = 0$ , and the second inequality is because  $\mathbb{P}(W_{\mu}(r) \ge 0)$  is monotonically decreasing to zero on  $\mathbb{R}^+$ . The second term in (B.14) can be shown bounded as follows. Using change of variables  $t = r^{\varepsilon}$ , integral by parts, and the condition that  $\lim_{r\to\infty} r^{-((1/2)+\varepsilon)}\mu(r) = -\infty$  for some  $\varepsilon > 0$  in turn, we have

$$\begin{split} \int_{\underline{r}}^{\infty} \left( 1 - \Phi\left(-\frac{\mu(r)}{\sqrt{r}}\right) \right) dr &\leq C_2 \int_{\underline{r}}^{\infty} \left(1 - \Phi\left(r^{\varepsilon}\right)\right) dr \\ &= C_2 \int_{\underline{r}^{1/\varepsilon}}^{\infty} \left(1 - \Phi\left(t\right)\right) dt^{1/\varepsilon} \\ &= C_2 + C_3 \int_{r^{1/\varepsilon}}^{\infty} t^{1/\varepsilon} \phi(t) dt < \infty \end{split}$$

for some  $C_2, C_3 < \infty$  if <u>r</u> is large enough, where  $\phi(\cdot)$  denotes the standard normal density function and we use  $\lim_{t\to\infty} t^{1/\varepsilon} (1 - \Phi(t)) = 0$ . The same result can be obtained for r < 0 symmetrically, which completes the proof.

**Proof of Lemma A.10** For given  $(\varrho, s)$ , we simply let  $\mu(r) = \mu(r, \varrho; s)$ . Then, for the kernel functions satisfying Assumption A-(x), it is readily verified that  $\mu(0) = 0$ ,  $\mu(r)$  is continuous in r, and  $\mu(r)$  is symmetric about zero. To check other conditions, for r > 0, we first write

$$\mu(r) = -r \int_0^{rC_1} K(t)dt + C_2 \int_0^{rC_1} tK(t)dt,$$

where  $C_1$  and  $C_2$  are some positive constants depending on  $(\varrho, |\dot{\gamma}_0(s)|, \xi(s))$ . We consider the two possible cases.

First, if  $K(\cdot)$  has a bounded support, say  $[-\underline{r}, \underline{r}]$ , then  $\mu(r) = -rC_3 + C_4$  for  $r > \underline{r}$ and some  $0 < C_3, C_4 < \infty$ . Thus,  $\mu(r)$  is monotonically decreasing on  $\mathbb{R} \setminus [-\underline{r}, \underline{r}]$  and  $\lim_{r \to \infty} r^{-((1/2)+\varepsilon)} \mu(r) = -\infty$  for any  $\varepsilon > 0$ .

Second, if  $K(\cdot)$  has an unbounded support, we have

$$\frac{\partial \mu(r)}{\partial r} = -\int_0^{rC_1} K(t)dt - rC_1K(C_1r) + rC_1^2C_2K(C_1r)$$

by the Leibniz integral rule. However, for  $r > \underline{r}$  for some large enough  $\underline{r}$ , it is strictly negative because  $\int_0^{rC_1} K(t) dt > 0$  and  $\lim_{r\to\infty} rK(r) = 0$ . This proves  $\mu(r)$  is monotonically decreasing on  $\mathbb{R} \setminus [-\underline{r}, \underline{r}]$ . In addition,  $\lim_{r\to\infty} r^{-((1/2)+\varepsilon)}\mu(r) = -\infty$  for any  $\varepsilon > 0$  because  $\int_0^{rC_1} K(t) dt < \int_0^\infty K(t) dt < \infty$ ,  $\int_0^{rC_1} tK(t) dt < \int_0^\infty tK(t) dt < \infty$ . The r < 0 case follows symmetrically using the identical argument.

**Proof of Lemma A.11** We only present the argument for  $T_n(\gamma; s)$  as the proof for  $\overline{T}_n(\gamma; s)$  is identical. Let  $\tau_n$  be some large truncation parameter to be chosen later, satisfying  $\tau_n \to \infty$  as  $n \to \infty$ . Define  $\mathbf{1}_{\tau_n} = \mathbf{1}[(c_0^{\top} x_i)^2 < \tau_n]$  and

$$T_n^{\tau}(\gamma, s) = \frac{1}{nb_n} \sum_{i=1}^n \left( c_0^{\top} x_i \right)^2 \left| \Delta_i(\gamma; s) \right| K_i(s) \mathbf{1}_{\tau_n},$$

where  $\Delta_i(\gamma; s) = \mathbf{1}_i(\gamma(s)) - \mathbf{1}_i(\gamma_0(s))$ . The triangular inequality gives that, for any  $\eta$ ,

$$\mathbb{P}\left(\sup_{s\in\mathcal{S}} |T_{n}(\gamma;s) - \mathbb{E}[T_{n}(\gamma;s)]| > \eta\right) \tag{B.15}$$

$$\leq \mathbb{P}\left(\sup_{s\in\mathcal{S}} |T_{n}^{\tau}(\gamma;s) - T_{n}(\gamma;s)| > \eta/3\right) +\mathbb{P}\left(\sup_{s\in\mathcal{S}} |\mathbb{E}[T_{n}^{\tau}(\gamma;s)] - \mathbb{E}[T_{n}(\gamma;s)]| > \eta/3\right) +\mathbb{P}\left(\sup_{s\in\mathcal{S}} |T_{n}^{\tau}(\gamma;s) - \mathbb{E}[T_{n}^{\tau}(\gamma;s)]| > \eta/3\right) = P_{T1n} + P_{T2n} + P_{T3n}.$$

For the first one, since r(s) > 0 for all  $s, \gamma(s) > \gamma_0(s)$  and

$$\mathbb{E}\left[\sup_{s\in\mathcal{S}}\left|T_{n}^{\tau}\left(\gamma;s\right)-T_{n}(\gamma;s)\right|\right]$$

$$\leq \mathbb{E}\left[\left|\frac{1}{nb_{n}}\sum_{i=1}^{n}\left(c_{0}^{\top}x_{i}\right)^{2}\mathbf{1}\left[\inf_{s\in\mathcal{S}}\gamma_{0}(s)\leq q_{i}\leq \sup_{s\in\mathcal{S}}\gamma_{0}(s)+\overline{r}\phi_{2n}\right]K_{i}\left(s\right)\left(1-\mathbf{1}_{\tau_{n}}\right)\right|\right]$$

$$\leq \frac{1}{b_{n}}\mathbb{E}\left[\left|\left(c_{0}^{\top}x_{i}\right)^{2}\mathbf{1}\left[\inf_{s\in\mathcal{S}}\gamma_{0}(s)\leq q_{i}\leq \sup_{s\in\mathcal{S}}\gamma_{0}(s)+\overline{r}\phi_{2n}\right]K_{i}\left(s\right)\left(1-\mathbf{1}_{\tau_{n}}\right)\right|\right]$$

$$=\tau_{n}^{-1}\int\int_{\inf_{s\in\mathcal{S}}\gamma_{0}(s)}^{\sup_{s\in\mathcal{S}}\gamma_{0}\left(s\right)}\mathbb{E}\left[\left(c_{0}^{\top}x_{i}\right)^{4}|q,s+b_{n}t\right]f\left(q,s+b_{n}t\right)K\left(t\right)dqdt$$

$$\leq C_{1}\phi_{2n}\tau_{n}^{-1}$$

for some  $C_1 \in (0, \infty)$ , where we use the fact that

$$\int_{|a| > \tau_n} |a| f_A(a) da \le \tau_n^{-1} \int_{|a| > \tau_n} |a|^2 f_A(a) da \le \tau_n^{-1} \mathbb{E}[A^2]$$

for a generic random variable A. Hence, Markov's inequality yields that  $P_{T1n} \leq C\phi_{2n}/(\eta\tau_n)$ .

Next, to bound  $P_{T2n}$ , note that

$$\mathbb{E}\left[T_{n}^{\tau}\left(\gamma;s\right)\right] - \mathbb{E}\left[T_{n}\left(\gamma;s\right)\right]$$

$$= b_{n}^{-1}\mathbb{E}\left[\left|\left(c_{0}^{\top}x_{i}\right)^{2}\mathbf{1}\left[\gamma_{0}(s)\leq q_{i}\leq\gamma(s)\right]K_{i}\left(s\right)\left(1-\mathbf{1}_{\tau_{n}}\right)\right|\right]$$

$$\leq \tau_{n}^{-1}\int\int_{\gamma_{0}(s)}^{\gamma(s)}\mathbb{E}\left[\left(c_{0}^{\top}x_{i}\right)^{4}|q,s+b_{n}t\right]f(q,s+b_{n}t)K\left(t\right)dqdt$$

$$\leq C_{2}\phi_{2n}\tau_{n}^{-1}$$

for some  $C_2 \in (0,\infty)$ . By Assumptions A-(v), (vii), and (viii), the above bound is

uniform in s. Hence Markov's inequality yields that  $P_{T2n} \leq C_2 \phi_{2n}/(\eta \tau_n)$  as well.

Now we bound  $P_{T3n}$  and then specify the choice of  $\tau_n$ . Since S is compact, we can find  $m_n$  intervals centered at  $s_1, \ldots, s_{m_n}$  with length  $C_S/m_n$  that cover S for some  $C_S \in (0, \infty)$ . We denote these intervals as  $\mathcal{I}_k$  for  $k = 1, \ldots, m_n$  and choose  $m_n$  later. The triangular inequality yields

$$\sup_{s \in \mathcal{S}} \left| T_n^{\tau} \left( \gamma; s \right) - \mathbb{E} \left[ T_n^{\tau} \left( \gamma; s \right) \right] \right| \le T_{1n}^* + T_{2n}^* + T_{3n}^*,$$

where

$$T_{1n}^{*} = \max_{1 \le k \le m_{n}} \sup_{s \in \mathcal{I}_{k}} |T_{n}^{\tau}(\gamma; s) - T_{n}^{\tau}(\gamma; s_{k})|$$
  

$$T_{2n}^{*} = \max_{1 \le k \le m_{n}} \sup_{s \in \mathcal{I}_{k}} |\mathbb{E} [T_{n}^{\tau}(\gamma; s)] - \mathbb{E} [T_{n}^{\tau}(\gamma; s_{k})]|$$
  

$$T_{3n}^{*} = \max_{1 \le k \le m_{n}} |T_{n}^{\tau}(\gamma; s_{k}) - \mathbb{E} [T_{n}^{\tau}(\gamma; s_{k})]|.$$

We first bound  $T_{3n}^*$ . Let

$$Z_{n,i}^{\tau}(s) = (nb_n)^{-1} \left\{ (c_0^{\top} x_i)^2 \Delta_i(\gamma; s) K_i(s) \mathbf{1}_{\tau_n} - \mathbb{E}[(c_0^{\top} x_i)^2 \Delta_i(\gamma; s) K_i(s) \mathbf{1}_{\tau_n}] \right\}$$

and

$$U_n(s) = T_n^{\tau}(\gamma; s) - \mathbb{E}\left[T_n^{\tau}(\gamma; s)\right] = \sum_{i=1}^n Z_{n,i}^{\tau}(s)$$

Note that  $\sup_{s\in\mathcal{S}} |(c_0^{\top}x_i)^2 \Delta_i(\gamma; s)K_i(s) \mathbf{1}_{\tau_n}|$  is bounded by  $C_3\tau_n$  for some constant  $C_3 \in (0,\infty)$  and hence  $|Z_{n,i}^{\tau}(s)| \leq 2C_3\tau_n/(nb_n)$  for all  $i = 1,\ldots,n$ . Define  $\lambda_n = (nb_n \log n)^{1/2}/\tau_n$ . Then  $\lambda_n |Z_{n,i}^{\tau}(s)| \leq 2C_3(\log n/(nb_n))^{1/2} \leq 1/2$  for all  $i = 1,\ldots,n$  when n is sufficiently large. Using the inequality  $\exp(v) \leq 1 + v + v^2$  for  $|v| \leq 1/2$ , we have  $\exp(\lambda_n |Z_{n,i}^{\tau}(s)|) \leq 1 + \lambda_n |Z_{n,i}^{\tau}(s)| + \lambda_n^2 |Z_{n,i}^{\tau}(s)|^2$ . Hence

$$\mathbb{E}[\exp(\lambda_n \left| Z_{n,i}^{\tau}(s) \right|)] \le 1 + \lambda_n^2 \mathbb{E}\left[ (Z_{n,i}^{\tau}(s))^2 \right] \le \exp\left(\lambda_n^2 \mathbb{E}\left[ (Z_{n,i}^{\tau}(s))^2 \right] \right)$$
(B.16)

since  $\mathbb{E}\left[Z_{n,i}^{\tau}(s)\right] = 0$  and  $1 + v \leq \exp(v)$  for  $v \geq 0$ . Using the fact that  $\mathbb{P}(X > c) \leq \mathbb{E}[\exp(Xa)]/\exp(ac)$  for any random variable X and nonrandom constants a and c, we

have that

$$\mathbb{P}\left(|U_{n}(s)| > \phi_{2n}^{1/2}\eta_{n}\right) = \mathbb{P}\left(\phi_{2n}^{-1/2}U_{n}(s) > \eta_{n}\right) + \mathbb{P}\left(-\phi_{2n}^{-1/2}U_{n}(s) > \eta_{n}\right) \\
\leq \frac{\mathbb{E}\left[\exp\left(\lambda_{n}\phi_{2n}^{-1/2}\sum_{i=1}^{n}Z_{n,i}^{\tau}(s)\right)\right] + \mathbb{E}\left[\exp\left(-\lambda_{n}\phi_{2n}^{-1/2}\sum_{i=1}^{n}Z_{n,i}^{\tau}(s)\right)\right]}{\exp(\lambda_{n}\eta_{n})} \\
\leq 2\exp(-\lambda_{n}\eta_{n})\exp\left(\lambda_{n}^{2}\phi_{2n}^{-1}\sum_{i=1}^{n}\mathbb{E}\left[(Z_{n,i}^{\tau}(s))^{2}\right]\right) \quad (by (B.16)) \\
\leq 2\exp(-\lambda_{n}\eta_{n})\exp\left(\lambda_{n}^{2}C_{4}\tau_{n}^{2}/(nb_{n})\right)$$

for some sequence  $\eta_n \to 0$  as  $n \to \infty$ , where the last inequality is from

$$\mathbb{E}\left[ (Z_{n,i}^{\tau}(s))^2 \right] \le (nb_n)^{-2} \mathbb{E}\left[ \left( c_0^{\top} x_i \right)^4 \Delta_i(\gamma; s)^2 K_i^2(s) \mathbf{1}_{\tau_n} \right] \le C_4 \tau_n^2 (n^2 b_n)^{-1} \phi_{2n}(1+o(1))$$

for some  $C_4 \in (0, \infty)$ . However, this bound is independent of s given Assumptions A-(v) and (x), and hence it is also the uniform bound, i.e.,

$$\sup_{s\in\mathcal{S}} \mathbb{P}\left(\left|U_n(s)\right| > \phi_{2n}^{1/2}\eta_n\right) \le 2\exp\left(-\lambda_n\eta_n + \lambda_n^2 C_4 \tau_n^2/\left(nb_n\right)\right).$$
(B.17)

Now given  $\tau_n$ , we need to choose  $\eta_n \to 0$  as fast as possible, and at the same time we let  $\lambda_n \eta_n \to \infty$  at a rate that ensures (B.17) is summable and  $\lambda_n \eta_n > \lambda_n^2 \tau_n^2 / (nb_n)$ . This is done by choosing  $\lambda_n = (nb_n \log n)^{1/2} / \tau_n$  and  $\eta_n = C^* \lambda_n^{-1} \log n = C^* \tau_n ((\log n) / (nb_n))^{1/2}$  for some finite constant  $C^*$ . This choice yields

$$-\lambda_n \eta_n + \lambda_n^2 C_4 \tau_n^2 / nb_n = -C^* \log n + C_4 \log n = -(C^* - C_4) \log n$$

Therefore, by substituting this into (B.17), we have

$$\mathbb{P}\left(T_{3n}^* > \phi_{2n}^{1/2}\eta_n\right) = \mathbb{P}\left(\max_{1 \le k \le m_n} |U_n(s_k)| > \phi_{2n}^{1/2}\eta_n\right)$$
  
$$\le m_n \sup_{s \in \mathcal{S}} \mathbb{P}\left(|U_n(s)| > \phi_{2n}^{1/2}\eta_n\right) \le 2\frac{m_n}{n^{C^* - C_4}}$$

Now, we can choose  $C^*$  sufficiently large so that  $\sum_{n=1}^{\infty} \mathbb{P}\left(T_{3n}^* > \phi_{2n}^{1/2}\eta_n\right)$  is summable, from which we have

$$T_{3n}^* = O_{a.s.}(\phi_{2n}^{1/2}\eta_n) = O_{a.s.}\left(\left(\phi_{2n}\frac{\log n}{nb_n}\right)^{1/2}\right)$$

by the Borel-Cantelli lemma.

Next, we consider  $T_{1n}^*$ . Note that

$$T_{n}^{\tau}(\gamma;s) - T_{n}^{\tau}(\gamma;s_{k}) = \frac{1}{nb_{n}} \sum_{i=1}^{n} (c_{0}^{\top}x_{i})^{2} \Delta_{i}(\gamma;s) (K_{i}(s) - K_{i}(s_{k})) \mathbf{1}_{\tau_{n}}$$
(B.18)  
 
$$+ \frac{1}{nb_{n}} \sum_{i=1}^{n} (c_{0}^{\top}x_{i})^{2} (\Delta_{i}(\gamma;s) - \Delta_{i}(\gamma;s_{k})) K_{i}(s_{k}) \mathbf{1}_{\tau_{n}}.$$

For the first item in (B.18), using a similar derivation as Lemma A.6 yields that if n is sufficiently large,

$$\mathbb{E}\left[\left|\frac{1}{nb_{n}}\sum_{i=1}^{n}\left(c_{0}^{\top}x_{i}\right)^{2}\Delta_{i}(\gamma;s)\left(K_{i}\left(s\right)-K_{i}\left(s_{k}\right)\right)\mathbf{1}_{\tau_{n}}\right|\right] \\ \leq b_{n}^{-1}\tau_{n}\mathbb{E}\left[\left|\Delta_{i}(\gamma;s)\left(K_{i}\left(s\right)-K_{i}\left(s_{k}\right)\right)\right|\right] \\ \leq C_{5}C_{S}\tau_{n}\phi_{2n}/\left(m_{n}b_{n}\right).$$

for some constant  $C_5 < \infty$ . For the second item in (B.18), without loss of generality, consider that  $\gamma(s) < \gamma(s_k)$  and  $\gamma_0(s) < \gamma_0(s_k)$ . Then by choosing the covering interval length  $C_S/m_n$  smaller than  $\phi_{2n}$ , we have

$$\mathbb{E}\left[\sup_{s\in\mathcal{I}_{k}}\left|\frac{1}{nb_{n}}\sum_{i=1}^{n}\left(c_{0}^{\top}x_{i}\right)^{2}\left(\Delta_{i}(\gamma;s)-\Delta_{i}(\gamma;s_{k})\right)K_{i}\left(s_{k}\right)\mathbf{1}_{\tau_{n}}\right|\right]$$

$$\leq 2C_{6}\tau_{n}\left(\sup_{s\in\mathcal{S}}K(s)\right)\mathbb{E}\left[\sup_{s\in\mathcal{I}_{k}}\left|\frac{1}{n}\sum_{i=1}^{n}\mathbf{1}\left(\gamma_{0}(s)< q_{i}\leq\gamma_{0}(s_{k})\right)\right|\right]$$

$$+\sup_{s\in\mathcal{I}_{k}}\left|\frac{1}{n}\sum_{i=1}^{n}\mathbf{1}\left(\gamma(s)< q_{i}\leq\gamma(s_{k})\right)\right|\right]$$

$$\leq C_{6}\tau_{n}\mathbb{P}\left(\inf_{s\in\mathcal{I}_{k}}\gamma_{0}(s)< q_{i}\leq\sup_{s\in\mathcal{I}_{k}}\gamma_{0}(s)\right)+C_{6}\tau_{n}\mathbb{P}\left(\inf_{s\in\mathcal{I}_{k}}\gamma(s)< q_{i}\leq\sup_{s\in\mathcal{I}_{k}}\gamma(s)\right)$$

$$\leq C_{6}C_{S}\tau_{n}/m_{n},$$

where the last line follows from Taylor expansion and Assumption A-(vi). This bound does not depend on k and hence  $T_{1n}^* = O_p(\tau_n/m_n)$ . Similarly for  $T_{2n}^*$ , Taylor expansion yields that

$$\begin{aligned} |\mathbb{E} \left[ T_n^{\tau} \left( \gamma; s \right) \right] - \mathbb{E} \left[ T_n^{\tau} \left( \gamma; s_k \right) \right] | &\leq b_n^{-1} \tau_n \mathbb{E} \left[ \Delta_i(\gamma; s) K_i\left(s\right) - \Delta_i(\gamma, s_k) K_i\left(s_k\right) \right] \\ &\leq b_n^{-1} \tau_n \mathbb{E} \left[ \Delta_i(\gamma; s) \left( K_i\left(s\right) - K_i\left(s_k\right) \right) \right] \\ &+ b_n^{-1} \tau_n \mathbb{E} \left[ \left( c_0^{\top} x_i \right)^2 \left( \Delta_i(\gamma; s) - \Delta_i(\gamma; s_k) \right) K_i\left(s_k\right) \right] \\ &\leq C_7 \tau_n / m_n \end{aligned}$$

for some  $C_7 < \infty$ , where the last line follows by choosing the covering interval length

 $C_S/m_n$  smaller than  $\phi_{2n}$ . This bound is also uniform in k and hence  $T_{2n}^* = O(\tau_n/m_n)$  as well. Therefore, by choosing  $m_n = [(\phi_{2n}(\log n)/nb_n)^{1/2}/\tau_n]^{-1}$ , we have that  $T_{1n}^*$  and  $T_{2n}^*$ are both the order of  $(\phi_{2n}(\log n)/nb_n)^{1/2}$ . It follows that  $P_{T3n} \leq \eta^{-1}C(\phi_{2n}(\log n)/nb_n)^{1/2}$ for some  $C \in (0, \infty)$  by Markov's inequality.

Finally, if we choose  $\tau_n$  such that  $\tau_n = O(\phi_{2n}^{1/2}((\log n)/nb_n)^{-1/2})$ , we have both  $P_{T1n}$ and  $P_{T2n}$  are also bounded by  $\eta^{-1}C(\phi_{2n}(\log n)/nb_n)^{1/2}$ . A possible choice of  $\tau_n$  is  $n^{\epsilon}$ or larger. This completes the proof.

**Proof of Lemma A.12** Since the proof is similar as that in Lemma A.11, we only highlight the different part. We only present the argument for  $L_n(\gamma; s)$  as the proof for  $\overline{L}_n(\gamma; s)$  is identical. We now define  $\mathbf{1}_{\tau_n} = \mathbf{1}[|c_0^{\top} x_i u_i| < \tau_n]$  for some truncation parameter satisfying  $\tau_n \to \infty$  as  $n \to \infty$ , which can be different from the one chosen in Lemma A.11 above. We let

$$L_n^{\tau}(\gamma; s) = \frac{1}{\sqrt{nb_n}} \sum_{i=1}^n c_0^{\top} x_i u_i \Delta_i(\gamma; s) K_i(s) \mathbf{1}_{\tau_n},$$

and write

$$\mathbb{P}\left(\sup_{s\in\mathcal{S}}|L_{n}\left(\gamma;s\right)|>\eta\right) \\
\leq \mathbb{P}\left(\sup_{s\in\mathcal{S}}|L_{n}^{\tau}\left(\gamma;s\right)-L_{n}(\gamma;s)|>\eta/2\right) + \mathbb{P}\left(\sup_{s\in\mathcal{S}}|L_{n}^{\tau}\left(\gamma;s\right)|>\eta/2\right) \\
\equiv P_{L1n}+P_{L2n},$$

where  $\mathbb{E}[L_n^{\tau}(\gamma, s)] = 0.$ 

To bound  $P_{L1n}$ , similarly as  $P_{T1n}$  in the proof of Lemma A.11, note that

$$\mathbb{E}\left[\sup_{s\in\mathcal{S}}\left|L_{n}^{\tau}\left(\gamma;s\right)-L_{n}(\gamma;s)\right|\right] \\
\leq \mathbb{E}\left[\frac{1}{\sqrt{nb_{n}}}\sum_{i=1}^{n}\left|c_{0}^{\top}x_{i}u_{i}\right|\mathbf{1}\left[\inf_{s\in\mathcal{S}}\gamma_{0}(s)\leq q_{i}\leq \sup_{s\in\mathcal{S}}\gamma_{0}(s)+\overline{r}\phi_{2n}\right]K_{i}\left(s\right)\left(1-\mathbf{1}_{\tau_{n}}\right)\right] \\
\leq (nb_{n})^{1/2}\tau_{n}^{-1}\int_{\inf_{s\in\mathcal{S}}\gamma_{0}(s)}^{\sup_{s\in\mathcal{S}}\gamma_{0}(s)+\overline{r}\phi_{2n}}\mathbb{E}\left[\left(c_{0}^{\top}x_{i}u_{i}\right)^{2}|q,s+tb_{n}\right]f(q,s+tb_{n})K\left(t\right)dqdt \\
\leq C_{1}\phi_{2n}\left(nb_{n}\right)^{1/2}\tau_{n}^{-1}$$

for some  $C_1 \in (0, \infty)$  and hence  $P_{L1n} \leq \eta^{-1} C_1 \phi_{2n} (nb_n)^{1/2} \tau_n^{-1}$  by Markov's inequality. To bound  $P_{L2n}$ , similarly as  $P_{T3n}$  in the proof of Lemma A.11, we write

$$\sup_{s \in \mathcal{S}} \left| L_n^{\tau} \left( \gamma; s \right) \right| \le L_{1n}^* + L_{2n}^*,$$

where

$$L_{1n}^{*} = \max_{1 \le k \le m_{n}} \sup_{s \in \mathcal{I}_{k}} |L_{n}^{\tau}(\gamma; s) - L_{n}^{\tau}(\gamma; s_{k})|$$
$$L_{2n}^{*} = \max_{1 \le k \le m_{n}} |L_{n}^{\tau}(\gamma; s_{k})|$$

and  $\{\mathcal{I}_k\}_{k=1}^{m_n}$  denote  $m_n$  intervals centered at  $s_1, \ldots, s_{m_n}$  with length  $C_S/m_n$  that cover  $\mathcal{S}$  for some  $C_S \in (0, \infty)$ . (The choices of  $m_n$  and  $C_S$  can be different from the ones in Lemma A.11 above.) The bound of  $L_{1n}^*$  can be obtained similarly as  $T_{3n}^*$  above by letting  $Z_{n,i}^{\tau}(s) = (nb_n)^{-1/2} c_0^{\top} x_i u_i \Delta_i(\gamma; s) K_i(s) \mathbf{1}_{\tau_n}$ . In particular, with  $|Z_{n,i}^{\tau}(s)| \leq C_2 \tau_n / (nb_n)^{1/2}$  for all  $i = 1, \ldots, n$  and  $L_n^{\tau}(\gamma; s) = \sum_{i=1}^n Z_{n,i}^{\tau}(s)$ , we have

$$\sup_{s \in \mathcal{S}} \mathbb{P}\left( \left| L_n^{\tau}(\gamma; s) \right| > \phi_{2n}^{1/2} \eta_n \right) \le 2 \exp(-\lambda_n \eta_n + \lambda_n^2 \tau_n^2 C_3) \tag{B.19}$$

for some  $C_3 \in (0,\infty)$ . By choosing  $\lambda_n = (\log n)^{1/2} / \tau_n$  and  $\eta_n = C^* \tau_n (\log n)^{1/2}$  for some finite constant  $C^*$ , we get

$$-\lambda_n\eta_n + \lambda_n^2 \tau_n^2 C_3 = -(C^* - C_3)\log n.$$

Substituting this into (B.19) gives us

$$\sup_{s\in\mathcal{S}}\mathbb{P}\left(\left|L_{n}^{\tau}\left(\gamma;s\right)\right|>\phi_{2n}^{1/2}\eta_{n}\right)\leq2\frac{m_{n}}{n^{C^{*}-C_{3}}},$$

and hence by choosing  $C^*$  sufficiently large

$$L_{2n}^* = O_{a.s.}(\phi_{2n}^{1/2}\eta_n) = O_{a.s.}\left((\phi_{2n}\log n)^{1/2}\right)$$

by the Borel-Cantelli lemma. Regarding  $L_{1n}^*$ , we choose  $m_n = [(\phi_{2n} \log n)^{1/2} / \tau_n]^{-1}$  and use the same argument as bounding  $T_{1n}^*$  above to get

$$\mathbb{E}\left[L_{1n}^*\right] = O\left(\left(\phi_{2n}\log n\right)^{1/2}\right).$$

Therefore, by combining  $L_{1n}^*$  and  $L_{2n}^*$  and using Markov's inequality, we have  $P_{L2n} \leq \eta^{-1}C(\phi_{2n}\log n)^{1/2}$  for some  $C \in (0, \infty)$ .

Finally, if we choose  $\tau_n$  such that  $\tau_n = O(\phi_{2n}^{1/2}((\log n)/(nb_n))^{-1/2})$ , we have  $P_{L1n} \leq \eta^{-1}C(\phi_{2n}\log n)^{1/2}$  as well. A possible choice of  $\tau_n$  is  $n^{\epsilon}$  or larger. This completes the proof.

**Proof of Lemma A.13** We first show (A.33). Consider the case with  $\gamma(s) - \gamma_0(s) \in [r(s)\phi_{2n}, C(s)]$ , where  $0 < \underline{r} = \inf_{s \in S} r(s) \leq \sup_{s \in S} r(s) = \overline{r} < \infty$  and  $\overline{C} = \sup_{s \in S} C(s) < \infty$ ; the other direction can be shown symmetrically. Let

$$\underline{\ell}_D(s) = \inf_{|\gamma(s) - \gamma_0(s)| < \overline{C}(s)} c_0^\top D(\gamma(s), s) c_0 f(\gamma(s), s) > 0 \text{ and } \underline{\ell} = \inf_{s \in \mathcal{S}} \underline{\ell}_D(s) > 0$$

from Assumptions A-(vii) and (viii). Then, from (B.1), we get

$$\sup_{s \in \mathcal{S}} \mathbb{E} \left[ T_n \left( \gamma; s \right) \right] \geq \sup_{s \in \mathcal{S}} \left( \gamma \left( s \right) - \gamma_0 \left( s \right) \right) \left( \underline{\ell} + C_1(s) b_n^2 \right)$$

$$\geq \underline{\ell} \sup_{s \in \mathcal{S}} \left( \gamma \left( s \right) - \gamma_0 \left( s \right) \right) = \underline{\ell} \overline{r} \phi_{2n}$$
(B.20)

because  $0 < C_1(s) < \infty$  for all  $s \in S$  from Assumptions A-(vii) and (viii). Furthermore, Lemma A.11 implies that

$$\mathbb{P}\left(\sup_{s\in\mathcal{S}}\left|T_{n}\left(\gamma;s\right)-\mathbb{E}\left[T_{n}\left(\gamma;s\right)\right]\right|>\eta\right)\leq C_{2}\eta^{-1}\left(\phi_{2n}\frac{\log n}{nb_{n}}\right)^{1/2}$$
(B.21)

for some  $C_2 \in (0, \infty)$ .

We now set  $\gamma_g$  for  $g = 1, \ldots, \overline{g} + 1$  such that, for any  $s \in S$ ,  $\gamma_g(s) = \gamma_0(s) + 2^{g-1}r(s)\phi_{2n}$  where  $\overline{g}$  is the integer satisfying  $\gamma_{\overline{g}}(s) - \gamma_0(s) = 2^{\overline{g}-1}r(s)\phi_{2n} \leq \overline{C}$  and  $\gamma_{\overline{g}+1}(s) - \gamma_0(s) = 2^{\overline{g}}r(s)\phi_{2n} > \overline{C}$ . Then, (B.20) and (B.21) yield that for any fixed  $\eta > 0$ ,

$$\mathbb{P}\left(\max_{1\leq g\leq \overline{g}} \left| \frac{\sup_{s\in\mathcal{S}} T_n\left(\gamma_g; s\right)}{\sup_{s\in\mathcal{S}} \mathbb{E}\left[T_n\left(\gamma_g; s\right)\right]} - 1 \right| > \eta\right) \tag{B.22}$$

$$\leq \mathbb{P}\left(\max_{1\leq g\leq \overline{g}} \frac{\left|\sup_{s\in\mathcal{S}} T_n\left(\gamma_g; s\right) - \sup_{s\in\mathcal{S}} \mathbb{E}\left[T_n\left(\gamma_g; s\right)\right]\right|}{\left|\sup_{s\in\mathcal{S}} \mathbb{E}\left[T_n\left(\gamma_g; s\right)\right]\right|} > \eta\right)$$

$$\leq \mathbb{P}\left(\max_{1\leq g\leq \overline{g}} \frac{\sup_{s\in\mathcal{S}} \left|T_n\left(\gamma_g; s\right) - \mathbb{E}\left[T_n\left(\gamma_g; s\right)\right]\right|}{\left|\sup_{s\in\mathcal{S}} \mathbb{E}\left[T_n\left(\gamma_g; s\right)\right]\right|} > \eta\right)$$

$$\leq \sum_{g=1}^{\overline{g}} \mathbb{P}\left(\sup_{s\in\mathcal{S}} \left|T_n\left(\gamma_g; s\right) - \mathbb{E}\left[T_n\left(\gamma_g; s\right)\right]\right| > \eta \left|\sup_{s\in\mathcal{S}} \mathbb{E}\left[T_n\left(\gamma_g; s\right)\right]\right|\right)$$

$$\leq \sum_{g=1}^{\overline{g}} \frac{C_1\left(\phi_{2n}(\log n)/nb_n\right)^{1/2}}{2^{g-1}\eta \ell \overline{r} \phi_{2n}}$$

$$\leq \frac{C_1}{\eta \ell \overline{r}} \sum_{g=1}^{\infty} \frac{1}{2^{g-1}} \times \frac{1}{n^{\epsilon}}$$

for any  $\varepsilon > 0$ . Then from eq. (33) of Hansen (2000), for any  $\gamma(s)$  such that  $\overline{r}\phi_{2n} \leq \sup_{s \in \mathcal{S}} (\gamma(s) - \gamma_0(s)) \leq \overline{C}$ , there exists some g such that  $\gamma_g(s) - \gamma_0(s) < \gamma(s) - \gamma_0(s) = 0$ .

 $\gamma_{0}\left(s\right) < \gamma_{g+1}\left(s\right) - \gamma_{0}\left(s\right)$ . This implies that

$$\frac{\sup_{s \in \mathcal{S}} T_n(\gamma; s)}{\sup_{s \in \mathcal{S}} |\gamma(s) - \gamma_0(s)|} \\
\geq \frac{\sup_{s \in \mathcal{S}} T_n(\gamma_g; s)}{\sup_{s \in \mathcal{S}} \mathbb{E} \left[ T_n(\gamma_g; s) \right]} \times \frac{\sup_{s \in \mathcal{S}} \mathbb{E} \left[ T_n(\gamma_g; s) \right]}{\sup_{s \in \mathcal{S}} |\gamma_{g+1}(s) - \gamma_0(s)|} \\
= \left( 1 + \left( \frac{\sup_{s \in \mathcal{S}} T_n(\gamma_g; s)}{\sup_{s \in \mathcal{S}} \mathbb{E} \left[ T_n(\gamma_g; s) \right]} - 1 \right) \right) \times \frac{\sup_{s \in \mathcal{S}} \mathbb{E} \left[ T_n(\gamma_g; s) \right]}{\sup_{s \in \mathcal{S}} |\gamma_{g+1}(s) - \gamma_0(s)|},$$

and for any  $\eta > 0$ ,

$$\mathbb{P}\left(\inf_{\bar{r}\phi_{2n}<\sup_{s\in\mathcal{S}}|\gamma(s)-\gamma_{0}(s)|<\overline{C}}\frac{\sup_{s\in\mathcal{S}}T_{n}\left(\gamma;s\right)}{\sup_{s\in\mathcal{S}}|\gamma(s)-\gamma_{0}\left(s\right)|} < C(1-\eta)\right) \\
\leq \mathbb{P}\left(\left(1-\left|\max_{1\leq g\leq \bar{g}}\frac{\sup_{s\in\mathcal{S}}T_{n}\left(\gamma_{g};s\right)}{\sup_{s\in\mathcal{S}}\mathbb{E}\left[T_{n}\left(\gamma_{g};s\right)\right]}-1\right|\right)\frac{\sup_{s\in\mathcal{S}}\mathbb{E}\left[T_{n}\left(\gamma_{g};s\right)\right]}{\sup_{s\in\mathcal{S}}|\gamma_{g+1}\left(s\right)-\gamma_{0}\left(s\right)|} < C(1-\eta)\right) \\
\leq \varepsilon,$$

where the last line follow from (B.20) and (B.22). The proof for (A.34) is similar to that for (A.33) and hence omitted.

For (A.18), Lemma A.12 yields that, for a large enough n,

$$\mathbb{P}\left(\sup_{s\in\mathcal{S}}\left|L_{n}\left(\gamma;s\right)\right|>\eta\right)\leq\eta^{-1}C_{2}\phi_{2n}^{1/2}\left(\log n\right)^{1/2}$$
(B.23)

for some  $C_2 \in (0, \infty)$  similarly as above. Using a similar approach as (B.22), for any fixed  $\eta > 0$ ,

$$\mathbb{P}\left(\max_{1\leq g\leq \overline{g}} \frac{\sup_{s\in\mathcal{S}} \left|L_n\left(\gamma_g;s\right)\right|}{\sqrt{a_n}\sup_{s\in\mathcal{S}}\left(\gamma_g\left(s\right)-\gamma_0\left(s\right)\right)} > \eta\right) \tag{B.24}$$

$$\leq \sum_{g=1}^{\infty} \mathbb{P}\left(\frac{\sup_{s\in\mathcal{S}} \left|L_n\left(\gamma_g;s\right)\right|}{\sqrt{a_n}\sup_{s\in\mathcal{S}}\left(\gamma_g\left(s\right)-\gamma_0\left(s\right)\right)} > \eta\right)$$

$$\leq \sum_{g=1}^{\infty} \frac{C_2\left(\phi_{2n}\log n\right)^{1/2}}{\eta\sqrt{a_n}2^{g-1}\underline{\mu}\overline{r}\phi_{2n}}$$

$$\leq \frac{C_2}{\eta\underline{\mu}\overline{r}}\sum_{g=1}^{\infty} \frac{1}{2^{g-1}}.$$

from (B.20) and (B.23). This probability is arbitrarily close to 0 if  $\bar{r}$  is large enough. Following a similar discussion after (B.5), this result also provides the maximal (or sharp) rate of  $\phi_{2n}$  as  $\log n/a_n$  because we need  $(\log n/a_n)/\phi_{2n} = O(1)$  but  $\phi_{2n} \to 0$  as  $\log n/a_n \to 0$  with  $n \to \infty$ . Finally, for a given g, we define  $\Gamma_g$  as the collection of  $\gamma(s)$  satisfying  $\overline{r}2^{g-1}\phi_{2n} < \gamma(s) - \gamma_0(s) < \overline{r}2^g \phi_{2n}$  for all  $s \in S$ . By a similar argument as (B.24), we have

$$\mathbb{P}\left(\max_{1 \le g \le \overline{g}} \sup_{\gamma \in \Gamma_g} \frac{\sup_{s \in \mathcal{S}} |L_n(\gamma; s)|}{\sqrt{a_n} \sup_{s \in \mathcal{S}} (\gamma(s) - \gamma_0(s))} > \eta\right) \le \frac{C_3}{\eta \overline{r}}$$
(B.25)

for some constant  $C_3 < \infty$ . Combining (B.24) and (B.25), we thus have

$$\mathbb{P}\left(\sup_{\bar{r}\phi_{2n}<\sup_{s\in\mathcal{S}}|\gamma(s)-\gamma_{0}(s)|<\overline{C}}\frac{\sup_{s\in\mathcal{S}}\left|L_{n}\left(\gamma_{g};s\right)\right|}{\sqrt{a_{n}}\sup_{s\in\mathcal{S}}\left(\gamma\left(s\right)-\gamma_{0}\left(s\right)\right)}>\eta\right) \\
\leq 2\mathbb{P}\left(\max_{1\leq g\leq \overline{g}}\frac{\sup_{s\in\mathcal{S}}\left|L_{n}\left(\gamma_{g};s\right)\right|}{\sqrt{a_{n}}\sup_{s\in\mathcal{S}}\left(\gamma_{g}\left(s\right)-\gamma_{0}\left(s\right)\right)}>\eta\right) \\
+2\mathbb{P}\left(\max_{1\leq g\leq \overline{g}}\sup_{\gamma\in\Gamma_{g}}\frac{\sup_{s\in\mathcal{S}}\left|L_{n}\left(\gamma;s\right)\right|}{\sqrt{a_{n}}\sup_{s\in\mathcal{S}}\left(\gamma\left(s\right)-\gamma_{0}\left(s\right)\right)}>\eta\right) \\
\leq \varepsilon$$

for any  $\varepsilon > 0$  if  $\overline{r}$  is sufficiently large. The proof for (A.36) is similar to that for (A.35) and hence omitted.

**Proof of Lemma A.14** For a given  $\gamma$ , since all the convergence results in Lemma A.5 hold uniformly by Lemma A.1, we only need to show  $\sup_{s \in S} |\hat{\gamma}(s) - \gamma_0(s)| \to_p 0$ . To this end, denote  $\overline{\Gamma}$  and  $\underline{\Gamma}$  as the upper and lower bounds of  $\Gamma$ , respectively, and let  $d_{\Gamma} = \overline{\Gamma} - \underline{\Gamma}$ . Since S is compact, it can be covered by the union of a finite number of intervals  $\{\mathcal{I}_k\}_{k=1}^m$  with length  $d_{\Gamma}/m$  and center points  $\{s_k\}_{k=1}^m$ . On the event  $E_n^*$  that  $\hat{\gamma}(s)$  is continuous with probability approaching to one, we can choose a large m such that  $\sup_{s \in \mathcal{I}_k} |\hat{\gamma}(s) - \hat{\gamma}(s_k)| \leq \eta$  for any  $\eta$  and all k. Such a choice is also valid for  $\gamma_0(\cdot)$  since it is also continuous by Assumption A-(vi). Then on the event  $E_n^*$ , using triangular inequality and Lemma A.3, for any  $\eta > 0$  and any  $\varepsilon > 0$ , there is a large enough m such that

$$\begin{split} & \mathbb{P}\left(\sup_{s\in\mathcal{S}}|\widehat{\gamma}(s)-\gamma_{0}(s)|>\eta\right) \\ \leq & \mathbb{P}\left(\max_{1\leq k\leq m}\sup_{s\in\mathcal{I}_{k}}|\widehat{\gamma}(s)-\widehat{\gamma}(s_{k})|>\eta/3\right) + \mathbb{P}\left(\max_{1\leq k\leq m}\sup_{s\in\mathcal{I}_{k}}|\gamma_{0}(s)-\gamma_{0}(s_{k})|>\eta/3\right) \\ & +\mathbb{P}\left(\max_{1\leq k\leq m}|\widehat{\gamma}(s_{k})-\gamma_{0}(s_{k})|>\eta/3\right) \\ \leq & 2\left(1-\mathbb{P}(E_{n}^{*})\right) + \sum_{k=1}^{m}\mathbb{P}\left(|\widehat{\gamma}(s_{k})-\gamma_{0}(s_{k})|>\eta/3\right) \\ \leq & \varepsilon, \end{split}$$

where the last line follows from that  $\mathbb{P}(E_n^*) > 1 - \varepsilon$  for any  $\varepsilon$ . This is because  $\widehat{\gamma}(\cdot)$  is a step function taking values in  $\{q_i\}_{i=1}^n \cap \Gamma$  and hence is piecewise continuous with countable jump points.

**Proof of Lemma A.15** We prove  $\Xi_{n02} = o_p(1)$  and  $\Xi_{n03} = o_p(1)$ . The results for  $\Xi_{n12}$  and  $\Xi_{n13}$  can be shown symmetrically. As in the proof of Theorem 5, we denote the leave-one-out estimator  $\widehat{\gamma}_{-i}(s_i)$  as  $\widehat{\gamma}(s_i)$  in this proof. For expositional simplicity, we only present the case of scalar  $x_i$ .

First, for any continuous function  $\gamma(\cdot) : \mathcal{S} \to \Gamma$ , we define

$$G_n(\gamma) = \frac{1}{\sqrt{n}} \sum_{i=1}^n x_i u_i \mathbf{1} \left[ q_i > \gamma(s_i) + \Delta_n \right] \mathbf{1}_S.$$

For any fixed  $\gamma(\cdot)$ ,  $G_n(\gamma)$  converges to a Gaussian random variable by the random field CLT, where  $\mathbb{E}[x_i u_i \mathbf{1} [q_i > \gamma(s_i) + \Delta_n] \mathbf{1}_S] = 0$  and  $\mathbb{E}[x_i^2 u_i^2 \mathbf{1} [q_i > \gamma(s_i) + \Delta_n] \mathbf{1}_S] < \infty$ from Assumptions ID-(i) and A-(v). Moreover, the convergence holds for any finite collection of  $\gamma(\cdot)$  and the process  $G_n(\gamma)$  is uniformly tight by a similar argument as Lemma A.1. Therefore, we have  $G_n(\gamma) \Rightarrow \mathbb{G}(\gamma)$  as  $n \to \infty$ , where  $\mathbb{G}(\gamma)$  is a Gaussian process with almost surely continuous paths (cf. Lemma A.4 in Hansen (2000)). It follows that, for any  $\gamma(s)$  such that  $\sup_{s \in S} |\gamma(s) - \gamma_0(s)| \leq \overline{r}\phi_{2n}$  for some  $\overline{r} > 0$ , we have

$$G_n(\gamma) - G_n(\gamma_0) \to_p 0$$

as  $G_n(\gamma) - G_n(\gamma_0) \Rightarrow \mathbb{G}(\gamma) - \mathbb{G}(\gamma_0)$ . We now denote  $\overline{\Gamma}_n$  as the set of continuous functions  $\{\gamma(\cdot) : \sup_{s \in \mathcal{S}} |\gamma(s) - \gamma_0(s)| \leq \overline{r}\phi_{2n}\}$ . If we choose  $\overline{r}$  large enough so that  $\mathbb{P}(\widehat{\gamma} \notin \overline{\Gamma}_n) < \varepsilon/2$ , then for any  $\varepsilon > 0$  and  $\eta > 0$ , we have

$$\mathbb{P}\left(|\Xi_{n02}| > \eta\right) \\
= \mathbb{P}\left(|G_n(\widehat{\gamma}) - G_n(\gamma_0)| > \eta\right) \\
= \mathbb{P}\left(|G_n(\widehat{\gamma}) - G_n(\gamma_0)| > \eta \text{ and } \widehat{\gamma} \in \overline{\Gamma}_n\right) + \mathbb{P}\left(|G_n(\widehat{\gamma}) - G_n(\gamma_0)| > \eta \text{ and } \widehat{\gamma} \in \overline{\Gamma}_n^c\right) \\
\leq \mathbb{P}\left(\sup_{\gamma \in E_{n\gamma}} |G_n(\gamma) - G_n(\gamma_0)| > \eta\right) + \mathbb{P}(\widehat{\gamma} \notin \overline{\Gamma}_n) \\
\leq \varepsilon,$$

which gives the desired result.

Second, we consider  $\Delta_n > 0$ . On the event  $E_n^*$  that  $\sup_{s \in \mathcal{S}} |\widehat{\gamma}(s) - \gamma_0(s)| \leq \phi_{2n}$ , we

have

$$\begin{split} \mathbb{E}\left[\left|\Xi_{n03}\right|\right] &= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \mathbb{E}\left[\left|x_{i}^{2} \delta_{0}\right| \mathbf{1}\left[q_{i} \leq \gamma_{0}(s_{i})\right] \mathbf{1}\left[q_{i} > \widehat{\gamma}(s_{i}) + \Delta_{n}\right] \mathbf{1}_{S}\right] \\ &\leq n^{1/2-\epsilon} C \mathbb{E}\left[\mathbf{1}\left[q_{i} \leq \gamma_{0}(s_{i})\right] \mathbf{1}\left[q_{i} > \widehat{\gamma}(s_{i}) + \Delta_{n}\right] \mathbf{1}_{S}\right] \\ &\leq n^{1/2-\epsilon} C \mathbb{E}\left[\mathbf{1}\left[q_{i} \leq \gamma_{0}(s_{i})\right] \mathbf{1}\left[q_{i} > \gamma_{0}(s_{i}) - \phi_{2n} + \Delta_{n}\right] \mathbf{1}_{S}\right] \\ &= n^{1/2-\epsilon} C \int_{\mathcal{S}} \int_{\mathcal{I}(q;s)} f(q,s) dq ds \end{split}$$

for some constant  $0 < C < \infty$ , where  $\mathcal{I}(q; s) = \{q : q \leq \gamma_0(s) \text{ and } q > \gamma_0(s) - \phi_{2n} + \Delta_n\}$ . However, since we set  $\Delta_n > 0$  such that  $\phi_{2n}/\Delta_n \to 0$ , then  $\Delta_n - \phi_{2n} > 0$  holds with a sufficiently large n. Therefore,  $\mathcal{I}(q; s)$  becomes empty for all s when n is sufficiently large. The desired result follows from Markov's inequality and the fact that  $\mathbb{P}(E_n^*) > 1 - \varepsilon$  for any  $\varepsilon > 0$ .

#### References

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