Market Design and Walrasian Equilibrium[†]

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Abstract

We establish the existence of Walrasian equilibrium for economies with many discrete goods and possibly one divisible good. Our goal is not only to study Walrasian equilibria in new settings but also to facilitate the use of market mechanisms in resource allocation problems such as school choice or course selection. We consider all economies with quasilinear gross substitutes preferences. We allow agents to have limited quantities of the divisible good (limited transfers economies). We also consider economies without a divisible good (nontransferable utility economies). We show the existence and efficiency of Walrasian equilibrium in limited transfers economies and the existence and efficiency of strong (Walrasian) equilibrium in nontransferable utility economies. Finally, we show that various constraints on minimum and maximum levels of consumption and aggregate constraints of the kind that are relevant for school choice or course selection problems can be accommodated by either incorporating these constraints into individual preferences or by incorporating a suitable production technology into nontransferable utility economies.

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1. Introduction

In this paper, we establish the existence of Walrasian equilibrium in economies with many discrete goods and either with a limited quantity of one divisible good or without any divisible goods. Our goal is not only to study Walrasian equilibria in new settings but also to facilitate the use of market mechanisms in resource allocation problems such as school choice or course selection. To this end, we develop techniques for analyzing allocation problems in economies with or without transfers and for incorporating additional constraints into allocation rules. In particular, we show that individual distributional requirements (for example, a rule stating that every BA student must take at least 2 science courses, 2 humanities courses and one 1 social science course for credit) can be incorporated into preferences in a manner consistent with the resulting economy having a Walrasian equilibrium. Such requirements are common in US universities. We also show that aggregate constraints that restrict the total number of seats in a set of classes can be rendered consistent with the existence of Walrasian equilibrium by incorporating a suitable production technology into the economy.

In Kelso and Crawford (1982)'s formulation of the competitive economy, there is a finite number of goods, and a finite number of consumers with quasilinear utility functions that satisfy the *substitutes property.* Kelso and Crawford also assume that each consumer is endowed with enough divisible good to ensure that she can purchase any bundle of discrete goods at the equilibrium prices. This last condition would be satisfied, for example, if each consumer had more of the divisible good than the value she assigns to the aggregate endowment of indivisible goods in the economy. We call the Kelso-Crawford setting the *transferable utility economy.* Kelso and Crawford's ingenious formulation of the substitutes property facilitates their existence theorem as well as a tatonnement process/dynamic auction for computing Walrasian equilibrium. Subsequent research has identified various important properties of Walrasian equilibrium in transferable utility economies.

Our first goal is to do away with the assumption that each consumer has enough of the divisible good to purchase whatever she may wish at the equilibrium prices. In particular, we allow for arbitrary positive endowments of the divisible good. We call this the *limited transfers economy.* We also consider the *nontransferable utility economy*; that

is, we consider economies in which there is no divisible good. This setting is particularly well-suited for the analysis of many allocation problems such as school choice or course selection where transfers are not allowed.

Substitutes preferences have been used to analyze a variety of market design problems[1]. Then, results suggesting that the Walrasian equilibrium correspondence is nearly incentive compatible when there are sufficiently many agents (Roberts and Postlewaite, 1976) have been invoked to argue that Walrasian methods can play a role in market design. In most of these applications it is unreasonable to assume that each agent has enough divisible good to acquire whatever she wishes. In many applications, transfers (i.e., the divisible good or equivalently, money) are ruled out altogether and the problem is one of assigning efficiently and fairly a fixed number of objects to individuals. Hence, both the limited transfers economy and the nontransferable utility economy are of interest.

Theorem 1 establishes the existence of a Walrasian equilibrium (henceforth, equilibrium) in random allocations for limited transfers economies. In the transferable utility case, randomization is not necessary since in such economies, a random equilibrium allocation at prices p is simply a probability distribution over deterministic equilibrium allocations at prices p. However, in both limited transfers and nontransferable utility economies, randomization is necessary for the existence of equilibrium. The following simple example establishes the necessity of randomization.

Example 1: There are two agents and a single good. Both agents' utility for the indivisible good is 2 but both have only one unit of the divisible good (in the limited transfers economy) or fiat money (in the nontransferable utility economy). Without randomization, if the price is less than or equal to 1, both agents will demand the good; if the price is greater than 1 neither will demand the good. Since there is exactly one unit of the good, there can be no deterministic equilibrium for this economy. If randomization is allowed, the equilibrium price of the indivisible good is 2 and each agent will get the indivisible good with probability $1/2$.

[1] See, for example, the literature on matching with contracts starting with Hatfield and Milgrom (2005).

Example 1, above, features utilities that satisfy the substitutes property and, as a result, there exists a Walrasian equilibrium with a random allocation. Example 2, below, illustrates how equilibrium existence fails if utilities do not satisfy the substitutes property.

Example 2: The economy has three agents and three indivisible goods. Initially, agents 1 and 2 each have 1 unit of the divisible good and no divisible goods. Agent 3's initial endowment consists of the three goods and zero units of the divisible good. For agents 1 and 2,

$$u_i(A) = \begin{cases} 0 & \text{if } |A| < 2 \\ 2 & |A| \geq 2 \end{cases}$$

while $u_3(A) = 0$ for all A. Since the three goods are perfect substitutes, in equilibrium, all three must have the same price. Let r be this common price. Clearly, $r = 0$ is impossible in any equilibrium since both agents 1 and 2 would demand at least 2 goods with probability 1 and market clearing would fail. Then, in any equilibrium, $r > 0$ and agents 1 and 2 must consume the three goods with probability 1. This implies $r \cdot 3 \leq 2$ and hence $r \leq 2/3$. At $r \leq 2/3$ both agent 1 and 2 will want to consume any 2 of goods with probability $1/(2r)$ and 0 goods with probability $1 - /(2r)$. Market clearing requires, at a minimum, that the expected total consumption of the these two agents is 3. Hence, $2 \cdot 2 \cdot 1/(2r) = 3$ and therefore $r = 2/3$. This means that the unique optimal random consumption bundle for agent $i = 1, 2$ at these prices is the distribution that assigns her 2 goods with probability $3/4$ and zero goods with probability $1/4$. This pair of random consumptions is feasible in expectation but is not *implementable.* That is, there is no random allocation that yields this random consumption to both consumers at prices $p_1 = p_2 = p_3 = 2/3$. To see why, note that in any state of the world in which player 1 is allocated 2 goods, player 2 must be allocated either 1 good, which is never optimal for him, or 0 goods. However, consuming 0 goods with probability $3/4$ is not optimal for player 2.

The utility function of agents 1 and 2 in Example 2 does not satisfy the substitutes property: consider $p^1 = p^2 = 0.5, p^3 = 3$, then the optimal bundle is $\{1, 2\}$; however, when we increase p^2 to $q^2 = 3$ and keep other prices unchanged, there is no optimal bundle that includes good 1. Our main results show that examples such as the one above cannot be constructed with utility functions that satisfy the substitutes property.

The assignment of courses to students typically requires a mechanism without transfers, i.e., without a divisible good. To address this and related applications, Theorem 2 demonstrates the existence of a competitive equilibrium for the nontransferable utility economy. Hylland and Zeckhauser (1979) first proposed Walrasian equilibria as an allocation mechanism for the unit demand nontransferable utility economy. They showed that some equilibria may be Pareto inefficient because local non-satiation need not hold in this setting. Nonetheless, Hylland and Zeckhauser (1979) showed that efficient equilibria always exist. Mas-Colell (1992) coins the term *strong equilibrium* for a competitive equilibrium in which every consumer chooses the cheapest utility maximizing consumption and shows that strong equilibria are efficient. Our Theorem 2 establishes the existence of a strong and, therefore, Pareto efficient equilibrium.

Allocation problems often feature constraints on individual or group consumption. In course assignment problems, university rules may constrain students' course selection either by imposing distributional requirements of the kind described above or by limiting the number of courses that a student can take for credit from a specified list of courses (as in Princeton's rule of 12 discussed below). In a school choice problem, administrators may restrict parents' choices based on the location of their residence; and, finally, in office allocation problems, choices may be constrained by employee seniority. We analyze such constrained allocation problems in Theorem 3. There, we consider a broad range of constraints on individual consumption and show that our model can incorporate them. In some applications, groups of individuals may face constraints on their joint consumption or there may be aggregate constraints. For example, a university may reserve a certain number of seats in a class for those students who must take this class as a requirement of their majors. In addition, there may be aggregate constraints on lab space that limit the total number of seats available in a collection of related classes. We analyze such constraints in section 3.

1.1 Related Literature

Kelso and Crawford (1982) establish the existence of a Walrasian equilibrium in the transferable utility economy using an ascending tatonnement process. They show that this process converges to a Walrasian equilibrium price vector. Gul and Stacchetti (1999) argue

that, in a sense, Kelso and Crawford's substitutes property is necessary for the existence of equilibrium: given any utility function that violates the substitutes property, it is possible to construct an N-person economy consisting of an agent with this utility function and $N-1$ agents with utility functions that satisfy the substitutes property, where there is no Walrasian equilibrium.[2] Hence, their result shows that it is impossible to extend Kelso and Crawford's existence result to a larger class of utility functions than those that satisfy the substitutes property.

Sun and Yang (2006) provide a generalization of the Kelso-Crawford existence result that allows for some complementarities in a transferable utility economy. They circumvent Gul and Stacchetti's impossibility result by imposing joint restrictions on agent's preferences. In particular, they assume that the indivisible goods can be partitioned into two sets such that all agents consider goods within each element of the partition as substitutes and goods in different partition elements as complements.

As noted above, a special class of substitutes preferences are the unit demand preferences. These describe situations in which agents can consume at most one unit of the divisible good. Leonard (1983) studies transferable utility unit demand economies and identifies an allocation rule that generalizes the second-price auction and has strong incentive and efficiency properties. His allocation rule is the Walrasian rule together with the lowest equilibrium prices. Hylland and Zeckhauser (1979) are the first to study what we have called a nontransferable utility unit demand economy. They establish the existence of an efficient Walrasian equilibrium in such economies. Hylland and Zeckhauser's work has led to a literature on competitive equilibrium solutions to market design problems: Ashlagi and Shi (2016) study competitive equilibrium with equal incomes in a market with continuum of agents. Le (2017), He, Miralles, Pycia and Yan (2017) and Echenique, Miralles and Zhang (2018) maintain the assumption of unit demand preferences, but allow for general endowments, non-EU preferences[3] or priority-based allocations. Mas-Colell (1992) and McLennan (2018) study more general convex economies with production.

There are two major differences between these papers and ours. First, most of these papers introduce some notion of "slackness" into the definition of Walrasian equilibrium

[2] Yang (2017) finds an error in Gul and Stacchetti's proof and supplies an alternative proof.
[3] That is, in the convexified economy, they allow utility functions that are nonlinear in probabilities.

to guarantee its existence, while the equilibrium notion in our paper is standard. Second, they focus on convex (or convexified) economies and thus there is no implementability problem. In our setup implementability is the key issue. We provide a discussion of the second point after the statement of Theorem 1.

Budish, Che, Kojima and Milgrom (2013) study a variety of probabilistic assignment mechanisms. Our work relates to section 4 of their paper, where they define and show the existence of what they call pseudo-Walrasian equilibrium.[4] In that section, they consider fully separable preferences and establish the existence of efficient pseudo-Walrasian equilibrium. They also describe how individual constraints can be incorporated into pseudo-Walrasian equilibrium.

In Appendix B of their paper, they consider a richer class of preferences adopted from Milgrom (2009). These preferences amount to the closure of unit demand preferences under satiation and convolution.[5] Ostrovsky and Paes Leme (2015) prove that the closure of unit demand preferences under endowment and convolution yields a strict subset of substitutes preferences. They identify a rich class of preferences that belong to the latter but not the former. It is easy to check that this class of preferences is also excluded from the class described in Appendix B of Budish, Che, Kojima and Milgrom (2013). Thus, compared to Budish, Che, Kojima and Milgrom (2013), we consider a richer class of preferences and a richer class of constraints. In particular, their analysis of pseudo-Walrasian equilibrium does not include group constraints or aggregate constraints, other than bounds on the aggregate supply of each good.

Kojima, Sun and Yu (2018) study constraints in a transferable utility economy. They show that imposing upper and lower bounds of quantities consumed (i.e., interval constraints) on gross substitutes preferences preserves the gross substitutes property. They also show that a slight generalization of interval constraints are the only ones that preserve the gross substitutes property for every gross substitutes utility function. Lemma 1 below and the discussion prior to it is related to their first result regarding interval constraints.

[4] Presumably, the qualifier "pseudo" is to indicate the interjection of fiat money and also to acknowledge various additional constraints that are typically not a part of the definition of a competitive economy. By incorporating these constraints into preferences and technology and by assuming that the mechanism designer/seller values the fiat money, we are able to interpret our equilibria as proper Walrasian equilibria.

[5] See section 2.1 for a discussion of closures under substitutes preserving operations.

Our main results focus on limited transfers and nontransferable utility economies and allow for joint restrictions on the utility function and the constraints (and hence permit a larger set of constraints than Kojima, Sun and Yu (2018)).

2. The Substitutes Property and the Limited Transfers Economy

Let $H = \{1, \ldots, L\}$ be the set of goods. Subsets of H are consumption bundles.[6] We identify each $A \subset H$ with $x \in X := \{0,1\}^L$ such that $x^j = 1$ if and only if $j \in A$. Hence, $o = (0, \ldots, 0) \in X$ is identified with the empty set.

For any $x \in X$, let $\text{supp}\,(x) = \{k \in H | x^k = 1\}$ and $\sigma(x) = \sum_j x^j$. A utility on X is a function $u : X \to \mathbb{R} \cup \{-\infty\}$. The effective domain of u, denoted as $\text{dom}\, u$, is the set $\text{dom}\, u = \{x \in X : u(x) > -\infty\}$. Without loss of generality, we normalize u so that $u(x) \geq 0$ for all $x \in \text{dom}\, u$. Throughout, we adopt the following convention: $-\infty + (-\infty) = -\infty \geq -\infty$. We assume that every agent's overall utility function is quasilinear in the divisible good. Given any price vector $p \in \mathbb{R}^L_+$, we let $U_i(x,p) = u(x) - p \cdot x$ denote the agent's objective function.[7]

For $x, y \in X$, we write $x \leq y$ to mean that each coordinate of x is no greater than the corresponding coordinate of y and let $x \wedge y$ denote $z \in X$ such that $z^j = \min\{x^j, y^j\}$ for all j. Similarly, let $x \vee y$ denote $z \in X$ such that $z^j = \max\{x^j, y^j\}$ for all j. Without risk of confusion, we sometimes refer to u as the utility function (instead of saying the utility index associated with the utility function U). We let $\chi^j \in X$ denote the good j; that is, $\chi^j(k) = 1$ if $k = j$; otherwise, $\chi^j(k) = 0$. Similarly, for any set of indivisible goods $A \subset H$, define $\chi^A \in X$ as follows:

$$\chi^A(k) = \begin{cases} 1 & \text{if } k \in A \\ 0 & \text{otherwise} \end{cases}$$

Throughout, we will assume that $\text{dom}\, u \neq \emptyset$ and u is monotone; that is, $x \leq y$ implies $u(x) \leq u(y)$.

[6] We are assuming that there is a single unit of each good. This assumption makes the analysis of the implementability problem easier and is without loss of generality, since we can label each of the multiple units of a good as a distinct good. Equilibrium will ensure that each of these units has the same price.

[7] If the agent has an endowment of indivisible goods, the objective function is unchanged since the value of the endowment enters the utility function as a constant.

2.1 The Substitutes Property and the Transferable Utility Economy

Define the transferable utility demand correspondence for u as follows:

$$D_u(p) := \{x \in X \mid u(x) - p \cdot z \geq u(y) - p \cdot y \text{ for all } y \in X\}$$

Since $\operatorname{dom} u \neq \emptyset$ and $q \in \mathbb{R}^L_+$, $D_u(p)$ will always lie in the effective domain of u. The substitutes property states the following: let x be an optimal consumption bundle at prices p and assume that prices increase (weakly) to some $\hat{p}$. Then, the agent must have an optimal bundle at $\hat{p}$ which has her consuming at least as much of every good that did not incur a price increase. The formal definition is as follows:

Definition: *The function u has the substitutes property if $x \in D_u(p)$, $p \leq \hat{p}$, $\hat{p}^j = p^j$ for all $j \in A$ implies there exists $y \in D_u(\hat{p})$ such that $y^j \geq x^j$ for all $j \in A$.*

Kelso and Crawford (1982) introduced the substitutes property. Since then, numerous alternative characterizations have been identified. For example, the substitutes property is equivalent to $M^\natural$-concavity: the function u is *$M^\natural$-concave* if for all $x, y \in \operatorname{dom} u$, $x^j > y^j$ implies $[u(x - \chi^j) + u(y + \chi^j) \geq u(x) + u(y)$ or there is k such that $y^k > x^k$, $u(x - \chi^j + \chi^k) + u(y + \chi^j - \chi^k) \geq u(x) + u(y)]$.[8] Gul and Stacchetti (1999) show that that if u satisfies the substitutes property, then it must be submodular:[9]

$$u(x) + u(y) \geq u(x \vee y) + u(x \wedge y)$$

If the inequality above always holds with equality, then u is additive.

Perhaps the best-known subclass of substitutes utility functions are unit demand utilities. These utility functions are appropriate for situations in which each agent can consume at most one unit of the indivisible goods: u is a *unit demand utility* if

$$u(x) = \max\{u(\chi^j) \mid \chi^j \leq x\}$$

Below, we describe operations on gross substitutes utility functions that enable us to derive new gross substitutes utility functions. In section 3, we use these operations

[8] See, for example, Fujishige and Yang (2003) or Theorem 4.1 in Shioura and Tamura (2015).

[9] Gul and Stacchetti (1999) show this result for u such that $\operatorname{dom} u = H$. Their result extends immediately to utilities functions u such that $\operatorname{dom} u \neq X$.

to incorporate additional restrictions in school choice and course selection problems. Let $k > 0$ be an integer, $z \in X$ and u, v be two substitutes utility functions. Define,

$$u^z(x) = u(x \wedge z)$$

$$u_z(x) = u(x \vee z) - u(z)$$

$$(u \odot v)(x) = \max_{y \leq x} \{u(y) + v(x - y)\}$$

$$[u]^k(x) = \max_{\substack{y \leq x \\ \sigma(y) \leq k}} u(y)$$

$$[u]_k(x) = \begin{cases} \max_{\substack{y \leq x \\ \sigma(y) \geq k}} u(y) & \text{if } \sigma(x) \geq k \\ -\infty & \text{otherwise.} \end{cases}$$

Call u^z the z-constrained u, u_z the z-endowed u, $u \odot v$ the convolution (or aggregation) of u, v, $[u]^k$ the k-satiation of u and $[u]_k$ the k-lower bound u. It is easy to verify that a z-endowed u satisfies the substitutes property whenever u does and that the convolution of u and v satisfies the substitutes property whenever u and v both satisfy the substitutes property. Similarly, verifying that a z-constrained utility satisfies the gross substitutes property whenever $\operatorname{dom} u \cap \{x \,|\, x \leq z\} \neq \emptyset$ is straightforward.[10]

Bing, Lehmann and Milgrom (2004) prove that the k-satiation of the substitutes utility u is a substitutes utility (provided there is at least one set in $\operatorname{dom} u$ with k or fewer elements). The following lemma establishes the substitutes property for k-lower bound u. All proofs are in the Appendix.

Lemma 1: *If u satisfies substitutes property and $\operatorname{dom} u \cap \{x \in X | \sigma(x) \geq k\} \neq \emptyset$, then $[u]_k$ satisfies the substitutes property.*

For any given class of utility functions, $\mathcal{U}$, and set of substitutes preserving operations, τ, let $\tau(\mathcal{U})$ denote the set of all utility functions that can be derived from the elements of $\mathcal{U}$ by repeatedly applying various operations in τ. We will call $\tau(\mathcal{U})$ the τ-closure of $\mathcal{U}$. In other words, $\tau(\mathcal{U})$ is the smallest family of utility functions that includes $\mathcal{U}$ and is closed under operations in τ. Clearly, if each element of $\mathcal{U}$ satisfies the substitutes property, then so does the τ-closure of $\mathcal{U}$.

[10] In some cases, it is easier to verify that the new utility function satisfies $M^\natural$-concavity which, as we noted above, is equivalent to the substitutes property.

Ostrovsky and Paes Leme (2015) show that the endowment and convolution closure of the set of unit demand preferences is a strict subset of the set of all substitutes preferences.[11] They also provide a rich class of examples that satisfy the substitutes property but are not in the endowment and convolution closure of the set of unit demand preferences. Let τ be the five substitutes preserving operations discussed above and let $\mathcal{U}$ be the set of all unit demand preferences. Then, using Ostrovsky and Paes Leme's arguments, it is easy to verify that $\tau(\mathcal{U})$ is a strict subset of substitutes preferences and excludes the same rich class of preferences that these authors have identified.

Next, we define a new class of utility functions that are useful for describing student preferences over class schedules. We call these utility functions *academic preferences* since they incorporate curricular requirements into the agent's utility function. The following example provides an illustration of academic preferences.

Example 3: Students are required to take at least three courses for credit and satisfy a distributional requirement by taking courses from at least four out of five categories a, b, c, d, e. Each class that a student takes for credit can potentially meet two of these four requirements. There are ten classes, $H = \{ab, ac, ad, ae, bc, bd, be, cd, ce, de\}$, each identified by the requirements that it might satisfy. Students can use at most one class to meet two different requirements simultaneously. The remaining two requirements must be met with distinct classes.[12] A student may get credit for an additional course if the four courses together meet all five requirements. Hence, a student can meet the distributional requirement by taking either three or four courses for credit, but students who take four courses for credit must cover all of the five categories.

For example, the schedules $\{ab, ac, ad\}$ and $\{ab, bc, cd, de\}$ are both feasible; the former yields three course credits, while the latter yields four. The schedule $\{ab, bc, ac\}$ is not feasible since it only covers three distinct categories; the schedule $C = \{ab, bc, cd, da\}$ is

[11] Ostrovsky and Paes Leme (2015) conjecture that the endowment and convolution closure of the set of all weighted matroids is the set of substitutes preferences. They note that results from Murota (1996), Murota and Shioura (1999), and Fujishige and Yang (2003) ensure that every *weighted matroid* and hence every *rank function* satisfies the substitutes property. For the definitions of weighted matroid, rank function and other relevant terms and results from matroid theory, see Appendix A. There we also provide a short proof that weighted matroids satisfy the substitutes property based on Fujishige and Yang (2003)'s result that the substitutes property is equivalent to $M^\natural$-concavity.

[12] Undergraduates need to satisfy distributional requirements in most universities. Northwestern's version allows multiple requirements to be met with a single course in some but not all situations.

feasible but only yields three course credits since it only covers four categories. Let Y be the set of all feasible schedules with all courses for credit. That is, each $z \in Y$ either has three courses and meets four distributional requirements or has four courses and meets five distributional requirements. Let v be an additive utility function on X. When the student takes course $i \in H$ for credit, she enjoys utility $v(\xi^i)$. Her objective is to choose a feasible course schedule that maximizes the total utility of the courses she takes for credit.[13] That is,

$$u(x) = \begin{cases} \max_{\substack{y \in Y \\ y \leq x}} v(y) & \text{if there is } y \in Y \text{ such that } y \leq x \\ -\infty & \text{otherwise} \end{cases}$$

In Appendix A, we offer a general definition of academic preferences, show that they include the example above and that they satisfy the gross substitutes property.

We conclude this section by discussing the existence of equilibrium in a transferable utility economy. Let N be the number of agents in the economy, $\xi \in X^N$ be an allocation and let ξ_i denote agent i's consumption in the allocation ξ. Then (ξ, p) is a (deterministic) Walrasian equilibrium in the transferable utility economy if $\sum_{i=1}^N \xi_i \leq \chi^H$, $u_i(\xi_i) - p \cdot \xi_i \geq u_i(x) - p \cdot x$ for all $x \in X$, i and $\sum_{i=1}^N \xi_i^a = 1$ if $p^a > 0$ for all $a \in H$. Kelso and Crawford (1982) showed that if preferences satisfy monotonicity and the substitutes property, and the effective domain is X for all agents, then there exists an equilibrium with (deterministic) allocation in the transferable utility economy. To see how the result can be extended to general effective domains, note that since $\operatorname{dom} u_i \neq \emptyset$, the demand $D_{u_i}(p) \subseteq \operatorname{dom} u_i$ for all i and p. Hence, for any candidate equilibrium allocation ξ, we have $\xi_i \in \operatorname{dom} u_i$ for each i. The existence of such an allocation is a necessary condition for the existence of an equilibrium and we will incorporate it into the definition of a transferable utility economy.

Definition: *$\mathcal{E} = \{(u_i)_{i=1}^N\}$ is a transferable utility economy if, u_i satisfies the substitutes property for all i and there exists an allocation ξ such that $\sum_{i=1}^N \xi_i \leq \chi^H$ and $\xi_i \in \operatorname{dom} u_i$ for all i.*

The following lemma states that the additional condition above is also sufficient for the existence of an equilibrium when preferences satisfy the substitutes property.

[13] The example is meant to show that a rich set of curricular restrictions can be accommodated with academic preferences. Simpler versions can easily be constructed. For example, replacing the condition "a student may get credit for an additional course if the four courses together meet all five distributional requirements" with "a student who fulfills the requirements for 3 course credits can get an additional credit by taking a fourth course" would also yield an academic preference.

Lemma 2: *The transferable utility economy has a (deterministic) equilibrium.*[14]

Notice that efficient allocations of indivisible goods, optimal demands and Walrasian equilibria are independent of the initial endowments in the transferable utility economy. Therefore, the definition of the transferable utility economy omits them. However, endowments will matter in the limited transfers economy, defined in the next section.

2.2 The Limited Transfers Economy

Example 1 shows that when agents have limited budgets, a deterministic equilibrium may not exist even in the simplest limited transfers economies. Thus, we need to extend our definition of an allocation to allow for randomness: a random consumption (of indivisible goods) $\theta : X \to [0,1]$ is a probability distribution on X; that is, $\sum_{x \in X} \theta(x) = 1$. Let Θ denote the set of all random consumptions. For $\theta \in \Theta$, let $\bar{\theta} \in \mathbb{R}^L_+$ denote the coordinate-by-coordinate mean of θ; that is $\bar{\theta}^j = \sum_{x \in X} \theta(x) \cdot x^j$. We assume that u is also the agent's von Neumann-Morgenstern utility function over indivisible goods. Hence,

$$u(\theta) = \sum_{z \in X} u(z)\theta(z)$$

The effective domain of u on Θ consists of all the random consumptions such that $\theta(x) > 0$ implies $x \in \operatorname{dom} u$.[15]

Quasilinearity ensures that we do not have to worry about randomness in the consumption of the divisible good; we can identify every such random consumption with its expectation. Hence, we define the von Neumann-Morgenstern utility function U as follows:

$$U(\theta, p) = u(\theta) - p \cdot \bar{\theta}$$

Let $w_i \in X$ denote agent i's endowment of indivisible goods and let b_i denote her endowment of the divisible good. For some applications, it is useful to have an additional agent, the seller or market designer, who holds some or all of the aggregate endowment of

[14] Variations of Lemma 2 are also proved in Fujishige and Yang (2002), Murota and Tamura (2003) and Kojima, Sun and Yu (2018). See also Murota (2003) Chapter 11 and Shioura and Tamura (2015).

[15] The function $u : \Theta \to [0,1]$ is continuous on the effective domain but not on the whole domain. For example, suppose that $\operatorname{dom} u = X \backslash \{o\}$, $x \in \operatorname{dom} u$ and take a sequence of random consumptions θ^n such that $\theta^n(o) = 1/n$ and $\theta^n(x) = 1 - 1/n$. Clearly, $u(\theta^n) = -\infty$ for all n, but $\lim \theta^n = \theta$ such that $\theta(x) = 1$ and therefore, $u(\theta) \neq -\infty$.

the indivisible goods. We will sometimes refer to this agent as agent 0 and assume that she derives no utility from the indivisible goods; she only values the divisible good. The aggregate endowment of indivisible goods in the economy is $\chi^H := (1, \ldots, 1) \in X$ and, therefore, the seller's endowment of the indivisible goods is $w_0 = \chi^H - \sum_{i=1}^N w_i$. We will assume that $w_i \in \operatorname{dom} u_i$ for each i to guarantee that agent i can afford at least one bundle in the effective domain.

Definition: *$\mathcal{E} = \{(u_i, w_i, b_i)_{i=1}^N\}$ is a limited transfers economy if, for all i, u_i satisfies the substitutes property, $b_i > 0$ and $w_i \in dom\, u_i$.*

A random allocation (of indivisible goods) for this economy is a probability distribution $\alpha : X^N \to [0, 1]$. For any such α, let α_i denote the i'th marginal distribution of α; that is, $\alpha_i \in \Theta$ is the random consumption of agent i, where

$$\alpha_i(x) = \sum_{\{\xi : \xi_i = x\}} \alpha(\xi)$$

A random allocation α is feasible for the economy $\mathcal{E}$ if, for all ξ such that $\alpha(\xi) > 0$, $\xi_i \in \operatorname{dom} u_i$ for all i and $\sum_{i=1}^N \xi_i \leq \chi^H$.

The budget (set) of an agent with endowment w, b at prices p is

$$B(p, w, b) = \left\{ \theta \in \Theta \,\middle|\, p \cdot \bar{\theta} \leq p \cdot w + b \right\}$$

Then, $\theta \in B(p, w, b)$ is optimal for agent i given budget $B(p, w, b)$ if

$$U_i(\theta, p) \geq U_i(\theta_o, p)$$

for all $\theta_o \in B(p, w, b)$.

Definition: *A price vector $p \in \mathbb{R}_+^L$ and a random allocation α is an equilibrium for the limited transfers economy $\mathcal{E}$ if*

(1) α is feasible for the economy $\mathcal{E}$;

(2) for all $i \geq 1$, α_i is optimal for agent i given budget $B(p, w_i, b_i)$;

(3) $p^j > 0$ and $\alpha(\xi) > 0$ imply $\sum_{i=1}^N \xi_i^j = 1$.

Theorem 1: *The limited transfers economy $\mathcal{E} = \{(u_i, w_i, b_i)_{i=1}^N\}$ has an equilibrium.*

One possible way to prove Theorem 1 would be to prove the existence of equilibrium for the *convexified economy*; that is, an economy in which agents have convex consumption sets. To see how this can be done, let $Z = [0,1]^L$ and define, $\hat{u} : Z \to \mathbb{R}$ the *convexified version* of u as follows:

$$\hat{u}(z) = \max\{u(\theta) \,|\, \bar{\theta} \leq z\}$$

and define $\hat{U}(z,p) = \hat{u}(z) - p \cdot z$. Hence, $\hat{u}(z)$ is the maximum utility that the agent with von Neumann-Morgenstern utility index u could get by choosing a lottery θ over indivisible goods such that the expected consumption of indivisible good j is no greater than z^j. Note that $\hat{U}$ is defined on the convex set $Z \times \mathbb{R}^L_+$. An allocation $(z_1, \ldots, z_N)$ is feasible in the convexified economy if $\sum z_i \leq \chi^H$.

Establishing the existence of equilibrium in the convexified economy using standard techniques is straightforward.[16] The last step of this proof approach would be to implement the equilibrium of the convexified economy with a random allocation. This last step is not difficult for a unit demand economy since, for that case, an appeal to the Birkhoff-von Neumann Theorem ensures existence of the desired random allocation. This line of argument does not work generally when agents can consume multiple units of indivisible goods and preferences are not separable. Example 2 in the introduction shows that, without the substitutes property, the implementability problem is, in general, insurmountable.

Our proof relies on the existence of equilibrium in the transferable utility economy. We seek a $\lambda_i \in (0,1]$ for each agent i and a Walrasian equilibrium (p, α) for the modified *transferable utility* economy with random consumption in which each u_i is replaced by

$$\hat{u}_i = \lambda_i u_i$$

such that each agent i spends, in expectation, (1) no more that b_i on indivisible goods and (2) exactly b_i on indivisible goods if $\lambda_i < 1$. It is possible to decrease an agent's equilibrium spending as much as needed by decreasing that agent's λ_i. Hence, we can satisfy condition (1). A fixed-point argument ensures that we can also satisfy condition

[16] For example, it can be shown that the convexified version of the utility function satisfies the conditions in McLennan (2018) and thus existence of equilibrium in the convexified economy can be guaranteed for all utility index $\{u_i\}_{i=1}^N$. However, as is shown in the Example 2, without substitutes property, there might be no equilibrium in the economy even though there is one in its convexified version.

(2). The Walrasian equilibria for this modified economy are then shown to be equilibria of the original economy.

3. Nontransferable Utility Economies and Constraints

In this section, we will consider allocation problems in settings without a divisible good. We call this type of an economy a *nontransferable utility economy*. In many applications, nontransferable utility economies impose constraints on individual consumption, or on the consumption of groups. To address some of these applications, we show how our model can incorporate a variety of individual, group, and aggregate constraints. An individual constraint restricts the number of goods that a single agent can consume from a specified set of goods. A group constraint restricts the total number of goods from a specified set of perfect substitutes that can be consumed by a particular group of agents. Finally, aggregate constraints restrict the various combinations of goods available for the entire population.

An example of an individual constraint is Princeton University's *rule of 12*. According to this rule, no more than 12 courses in a student's major may be counted towards the 31 courses needed to obtain the A.B. degree. Distributional requirements are a second type of individual constraints. For example, Art and Archaeology students at Princeton University must take at least one course in each of the following three areas: group 1 (ancient), group 2 (medieval/early modern), and group 3 (modern/contemporary). An example of a group constraint is the requirement that at least 50 percent of the slots in each school should go to students who live in the school's district. Similarly, the so-called "controlled choice" constraints in school assignment that require schools to balance the gender, ethnicity, income, and test score distributions among their students, are group constraints.[17] Aggregate constraints define the feasible allocations for the entire economy. For example, suppose two versions of introductory physics are being offered: Phy 101, the version that does not require calculus and Phy 103, the version that does require calculus. Suppose each of these classes can accommodate 120 students, but because both courses have lab requirements and lab facilities are limited, the total enrollment in the two courses can be no greater than 200 students.

[17] See Abdulkadiroğlu, Pathak and Roth (2005) for examples of such constraints in practice.

In the next subsection, we describe the nontransferable utility economy, define a strong (Walrasian) equilibrium and establish its existence and efficiency. Section 3.2 deals with individual, group and aggregate constraints.

3.1 Nontransferable Utility Economies

In a nontransferable utility economy, each agent i has a substitutes utility function u_i and a quantity b_i of fiat (or artificial) money. Initially, the entire aggregate endowment of indivisible goods belongs to the market designer. Each agent's utility depends only on her consumption of indivisible goods. That is, agents solve the following utility maximization problem:

$$U_i(p, b_i) = \max u_i(\theta) \text{ subject to } p \cdot \bar{\theta} \leq b_i$$

Hence, U_i is the indirect utility function of agent i.

Definition: *$\mathcal{E}^* = \{(u_i, b_i)_{i=1}^N\}$ is a nontransferable utility economy if, for all i, u_i satisfies the substitutes property, $o \in dom\, u_i$ and $b_i > 0$.*

In the nontransferable utility setting, Walrasian mechanisms provide a rich menu of allocation rules with desirable properties. The designer may accommodate fairness concerns by choosing the agents' endowments of fiat money (the b_i's) appropriately. In particular, choosing the same b_i for every agent ensures that the resulting allocations are envy-free. This is the setting for many allocation problems such as school choice, course selection or office selection (for example, when a business or a department moves into a new building). In such markets, the Walras correspondence can serve both as real allocation mechanism and as a benchmark for evaluating other mechanisms.

Hylland and Zeckhauser (1979) note that in a nontransferable utility economy with unit demand preferences, some Walrasian equilibria are inefficient. Specifically, nontransferable utility economies may have equilibria in which some agents do not purchase the least expensive optimal option in their budget sets and equilibria with this property may be inefficient. To address this problem, Mas-Colell (1992) introduces the concept of a *strong* equilibrium; that is, a Walrasian equilibrium in which every consumer chooses the least expensive optimal bundle and proves that strong equilibria are Pareto efficient.

Definition: *A price $p \in \mathbb{R}^L_+$ and a random allocation α is a strong equilibrium for the nontransferable utility economy $\mathcal{E}^*$ if*

(1) α is feasible for the economy $\mathcal{E}^*$;

(2) for all $i \geq 1$, α_i is optimal for agent i given budget $B(p, b_i)$ and costs no more than any other optimal random consumption;

(3) $p^j > 0$ and $\alpha(\xi) > 0$ imply $\sum_{i=1}^N \xi_i^j = 1$.

The theorem below establishes the existence of a strong and, therefore, Pareto efficient equilibrium for the nontransferable utility economy.

Theorem 2: *The nontransferable utility economy has a strong equilibrium.*

Our proof of Theorem 2 relies on Theorem 1: we consider a sequence of limited transfers economies $\mathcal{E}_n = \{(nu_i, w_i, b_i)_{i=1}^k\}$ for $n = 1, 2 \ldots$ where $w_i^j = 0$ for all j and i. Hence, $\mathcal{E}_n$ is a limited transfers economy in which agent i's endowment of indivisible goods is equal to her endowment of indivisible goods in $\mathcal{E}^*$ (i.e., zero), her endowment of the divisible good is the same as her endowment of fiat money in $\mathcal{E}^*$ and her utility function is n-times her utility function in $\mathcal{E}^*$. Then, we appeal to Theorem 1 to conclude that each $\mathcal{E}_n$ has an equilibrium (p^n, α^n). Since this sequence lies in a compact set, it has a limit point. Then, we show that this limit point must be an equilibrium of $\mathcal{E}^*$. Since this equilibrium is a limit-point of a sequence of equilibria for limited transfers economies; that is, equilibria in which money has intrinsic value, it must be a strong equilibrium.

3.2 Group Constraints

In many applications, one group is given priority over another. For example, suppose that the maximal enrollment in a particular physics class is n and there are $m < n$ physics majors who are required to take that class. Thus, at most $n - m$ non-majors can enroll in the class. More generally, a group constraint (A, n) for the group $I \subset \{1, \ldots, N\}$ with $n \leq |A|$ states that the agents in I can collectively consume at most n units from the set A, where A is a collection of *perfect substitutes* (for all agents).

To accommodate this constraint, pick any subset B of A with $|A| - n$ elements. Then, replace each u_i for $i \in I$ with u_i' such that

$$u_i'(x) = u_i\left(x \wedge \chi^{B^c}\right)$$

Thus, the new utility for members of group I is the their original utility restricted to the complement of B. As we noted above, restrictions of utilities to a subset of choices satisfy the substitutes property if the original utility satisfies the substitutes property. Moreover, since elements of A are perfect substitutes, restricting the consumption of members of group I to B^c is equivalent to restricting their aggregate consumption of the goods in A. Thus, a group constraint can be accommodated by modifying some agents' utility functions.

3.3 Individual Constraints

The simplest individual constraints are bounds on the number of goods an agent may consume from a given set of goods. For example, a student may be required to take 4 classes each semester, but may be barred from enrolling in more than 6. We can incorporate this constraint by modifying the student's unconstrained utility function u as follows:

$$u^m(x) = \max_{\substack{y \leq x \\ \sigma(y) \leq 6}} \hat{u}(y)$$

where

$$\hat{u}(x) = \begin{cases} u(x) & \text{if } \sigma(x) \geq 4 \\ -\infty & \text{if } \sigma(x) < 4 \end{cases}$$

The modified utility u^m incorporates the lower bound constraint by restricting the effective domain of u to those bundles that satisfy the constraint. It incorporates the upper bound by imposing satiation above the constraint. Next, we generalize these constraints and impose bounds on overlapping subsets of goods. To preserve the gross substitutes property, we require the utility function to be separable across subsets of goods that must satisfy a constraint. A collection of goods, $A \subset H$, is a *module* of the utility u if for all $x \in X$,

$$u(x) = u(x \wedge \chi^A) + u(x \wedge \chi^{A^c})$$

Note that this condition is symmetric: if A is a module of u, then so is A^c. For example, suppose that A is the set of all humanities courses and A^c is the set of all other courses. If a student's utility over various combinations of humanities courses is independent of her utility over various combinations of the other courses, then A is a module. A collection of

sets, $\mathcal{H}$, is a *hierarchy* if $A, B \in \mathcal{H}$ and $A \cap B \neq \emptyset$ implies $A \subset B$ or $B \subset A$. Given any u, we say that the hierarchy $\mathcal{H}$ is *modular* if each element of $\mathcal{H}$ is a module of u.

A modular constraint places bounds on the agent's consumption for subsets of items that form a modular hierarchy. The collection $c = \{(A(k), l(k), h(k))_{k=1}^{K}\}$ is a *constraint* if $l(k), h(k)$ are integers, $A(k) \subset H$, and $X_c \cap X^c \neq \emptyset$, where

$$X_c := \{x \in X | \, \forall k, \sigma(x \wedge \chi^{A(k)}) \geq l(k)\}$$

are consumptions that satisfy the lower bound and

$$X^c = \{x \in X | \, \forall k, \sigma(x \wedge \chi^{A(k)}) \leq h(k)\}$$

are consumptions that satisfy the upper bound. The constraint $c = \{(A(k), l(k), h(k))_{k=1}^{K}\}$ is a *modular constraint* for u if $\mathcal{H} = \{A(1), \ldots, A(K)\}$ is a modular hierarchy for u and $(X^c \cap X_c) \cap \operatorname{dom} u \neq \emptyset$.

As an example of a modular constraint, suppose that students must take at least 3 humanities classes and at least 4 social science classes; moreover, each student is required to take at least 8 but no more than 12 classes overall. In this case, the constraint is modular if the student's utility over combinations of science courses is independent of her utility over combinations of humanities courses.

Given a utility function u, let $\hat{u}_c$ be the utility function with effective domain $X_c \cap \operatorname{dom} u$; that is,

$$\hat{u}_c(y) = \begin{cases} u(y) & \text{if } y \in X_c \\ -\infty & \text{otherwise.} \end{cases}$$

Finally, define $u(c, \cdot)$ as follows:

$$u(c, x) = \max \left\{ \hat{u}_c(y) \,\middle|\, y \in X^c, y \leq x \right\}$$

Then, the effective domain of $u(c, \cdot)$ is $\operatorname{dom} u(c, \cdot) = X_c \cap \operatorname{dom} u \neq \emptyset$.

Lemma 3: *If u satisfies the substitutes property and c is a modular constraint for u, then $u(c, \cdot)$ satisfies the substitutes property.*

To see how the substitutes property may fail if the constraint is not modular, consider the utility function described in equation (1) below. Let $H = \{0, 1, 2, 3\}$. Then,

$$u(x) = \begin{cases} 2 & \text{if } x^j \cdot x^{j\oplus 1} > 0 \text{ for some } j \in H \\ 0 & \text{if } x = o \\ 1 & \text{otherwise} \end{cases} \tag{1}$$

where $\oplus$ denotes addition modulo 4. Note that u satisfies the substitutes property.[18] Let $A = \{0, 1\}$ and suppose that the agent is restricted to consuming at most one unit from A. Clearly, A is not a module of u. To see that the resulting constrained utility function does not satisfy the substitutes property, set $p^0 = p^1 = p^2 = p^3 = 1/2$. Then, $\{0, 3\}$ is an optimal consumption set at prices p. The substitutes property fails since at prices q such that $q^3 = 2$ and $q^j = p^j$ for $j \neq 3$, there is no optimal bundle that contains item 0.

To see how the substitutes property may fail if the sets in the constraint are modules but fail to form a hierarchy, consider the following utility: $u(x) = \sigma(x)$. Let $H = \{0, 1, 2, 3\}$, then any subset of H is a module of u. Suppose the constraints are $(\{1, 2\}, 0, 1)$, $(\{0, 1\}, 0, 1)$ and $(\{0, 1, 2, 3\}, 0, 2)$. Then, at $p^j = 1/2$ for all $j \in H$, $\{1, 3\}$ is an optimal consumption set at p. Again, the substitutes property fails since at prices q such that $q^3 = 2$ and $q^j = p^j$ for $j \neq 3$, there is no optimal consumption set that contains 1.

To establish existence of equilibrium we must guarantee that the "interior" of each agent's budget set contains some consumption in the effective domain of her modified utility. To address this issue, we add two assumptions to our earlier model. First, we assume that there is a division of the aggregate resources into $N + 1$ consumption bundles that meet every consumer's lower bound constraint. Second, we assume that all agents have equal endowments of fiat money, which we normalize to 1.

Definition: $\mathcal{E}_c^* = \{(u_i, 1)_{i=1}^N, \{c_i\}_{i=1}^N\}$ *is a nontransferable utility economy with modular constraints if,*

(i) for all i, u_i satisfies the substitutes property;

(ii) for all i, c_i is a modular constraint for u_i;

(iii) there exist $x_1, \cdots, x_{N+1}$ such that $\sum_{k=1}^{N+1} x_k \leq \chi^H$ and $x_k \in \operatorname{dom} u_i(c_i, \cdot)$ for all k, i.

[18] It is the convolution of the two unit demand preferences v and $\hat{v}$ where v takes the value 1 at any x such that $x^0 > 0$ or $x^2 > 0$ and is equal to zero otherwise and $\hat{v}$ takes the value 1 at any x such that $x^1 > 0$ or $x^3 > 0$ and is equal to zero otherwise.

Clearly, a strong equilibrium (p, α) of $\mathcal{E}_c^*$ will satisfy all of the lower bound constraints. If $\alpha(\xi) > 0$ and ξ_i violates some upper bound constraint, then there must be some $x \leq \xi_i$ that satisfies all of the constraints such that $u_i(c_i, x) = u_i(c_i, \xi_i)$. Then, $p^j = 0$ for all j such that $\xi_i^j > x^j$; otherwise, ξ could not be a *strong* equilibrium. Hence, we can replace α with a random allocation β that satisfies all of the upper bound (and lower bound) constraints such that (p, β) is also a strong equilibrium of $\mathcal{E}_c^*$. Hence, the existence of a strong equilibrium in $\mathcal{E}_c^*$ ensures the existence of a strong equilibrium in $\mathcal{E}_c^*$ that satisfies all of the constraints in c.

Theorem 3: *The nontransferable utility economy with modular constraints has a strong equilibrium.*

The additional two assumptions in the definition of a nontransferable utility economy with modular constraints are introduced to ensure that every agent can afford a consumption in the effective domain of her modified utility function. Alternatively, we could assume that there is a subset of goods in abundant supply (i.e., goods that have zero prices in equilibrium) and every agent can choose a consumption bundle in the effective domain from that subset. In the course selection problem, it may be the case that a subset of classes is never oversubscribed and students can choose courses from that subset to meet the requirements. In this case, the additional two assumptions can be dispensed with.

3.4 Aggregate Constraints

The example at the beginning of this section with a bound on the total enrollment in two physics is an illustration of an aggregate constraint. Alternatively, suppose an economics department schedules classes in labor economics, intermediate microeconomics and in corporate finance. There are two types of TAs: those that can cover labor economics and intermediate microeconomics, and those that can cover intermediate microeconomics and corporate finance. There are 60 TAs of each type and, for simplicity, we assume that each TA can be responsible for one student. TA time is fungible across different classes so that at most 60 students can enroll in labor economics, at most 60 students enroll in corporate finance and at most 120 students can enroll in any of the three types of classes.

In these examples, we can describe the aggregate constraint as a hierarchy $\mathcal{H}$ that limits the supply of available items. That is, the aggregate constraint has the form

$\{(A(k), n(k))_{k=1}^K\}$ such that the $A(k) \subseteq H$ for all k, $\{A(k)\}_{k=1}^K$ is a hierarchy and each $n(k)$ is a natural number describing the maximal quantity of indivisible goods that can be supplied from set $A(k)$.

Hierarchies are a special case of a class of constraints that can be described as matroids. A collection of sets $\mathcal{I} \subset X$ is a *matroid* if (i) $\emptyset \in \mathcal{I}$, (ii) $y \in \mathcal{I}$, $x \leq y$ implies $x \in \mathcal{I}$ and (iii) $x, y \in \mathcal{I}$, $\sigma(x) < \sigma(y)$ implies there is j such that $x^j < y^j$ and $x + \chi^j \in \mathcal{I}$.[19] We will re-interpret the aggregate constraint as a production technology; that is, we will define a production possibility set for the economy that only includes the output combinations consistent with the desired constraints.[20]

A nontransferable utility production economy is a collection $\tilde{\mathcal{E}} = \{(u_i, b_i)_{i=1}^N, \mathcal{I}\}$ such that $\mathcal{I}$ is a matroid. To see how we can embed a collection of aggregate restrictions into a matroid production set, let (A, n) denote a single aggregate restriction. Hence, the set of feasible production plans given any X and the restriction (A, n) is:

$$X(A, n) = \{x \in X \mid \sigma(x \wedge \chi^A) \leq n\}$$

Given any hierarchy of aggregate restrictions $d = \{(A(k), n(k))_{k=1}^K\}$, let $\mathcal{I}_d$ denote the set of all production plans consistent with d; that is,

$$\mathcal{I}_d = \bigcap_{a \in d} X(a)$$

Lemma 4: *If d is a hierarchy of aggregate restrictions, then $\mathcal{I}_d$ is a matroid.*

In the economy with production, a random allocation α is a probability distribution over $X^N \times \mathcal{I}$. For any such α, the marginal α_i is the random consumption for agent $i = 1, \ldots, N$ and the marginal α_{N+1} is the production plan for the producer or seller.

Definition: *$\tilde{\mathcal{E}} = \{(u_i, b_i)_{i=1}^N, \mathcal{I}\}$ is a production economy with nontransferable utility if, for all i, u_i satisfies the substitutes property, $o \in \operatorname{dom} u_i$ and if $\mathcal{I}$ is a matroid.*

[19] A matroid can be defined in a variety of equivalent ways. In Appendix A, we offer two alternative definitions and a few other relevant notions and results from matroid theory.

[20] Note that (i) corresponds to the standard requirement that inaction is possible, while (ii) is the usual comprehensiveness property of production possibility sets. The final condition, (iii) is a type of discrete convexity.

A random allocation α is *feasible* for the economy $\tilde{\mathcal{E}} = \{(u_i, b_i)_{i=1}^N, \mathcal{I}\}$ if, for all (ξ, z) such that $\alpha(\xi, z) > 0$, $\sum_{i=1}^N \xi_i \leq z$, $z \in \mathcal{I}$ and $\xi_i \in \operatorname{dom} u_i$ for all i. The definitions of budget sets and consumer optimality remain unchanged. The random allocation α is *producer optimal* if $\alpha(\xi, z) > 0$ implies $p \cdot z \geq p \cdot z'$ for all $z' \in \mathcal{I}$.

Definition: *A price $p \in I\!R_+^L$ and a random allocation α is a strong equilibrium for the production economy with nontransferable utility $\tilde{\mathcal{E}}$ if*

(1) α is feasible for $\tilde{\mathcal{E}}$;

(2) for all i, α_i is optimal given budget $B(p, b_i)$ and costs no more than any other optimal random consumption;

(3) α is producer optimal;

(4) $p^j > 0$ and $\alpha(\xi, z) > 0$ imply $\sum_{i=1}^N \xi_i^j = z^j$.

Hence, with production, a Walrasian equilibrium specifies prices and a random allocation over consumptions and production plans. The implied random consumption and production plans must be feasible and optimal for both the consumers and the producer. The definition of a strong equilibrium is the same as in the previous section: the Walrasian equilibrium (p, α) is a strong equilibrium if for each agent i, α_i is the cheapest optimal random consumption for i given the budget constraint.

Theorem 4: *The production economy with nontransferable utility has a strong equilibrium. Strong equilibrium allocations are Pareto efficient.*

Let $\mathcal{B}$ be the production possibility frontier of the technology $\mathcal{I}$; that is, $\mathcal{B} = \{x \in \mathcal{I} \,|\, y \geq x, y \in \mathcal{I} \text{ implies } y = x\}$. Each element in $\mathcal{B}$ is called an efficient production plan. Part (3) of the definition below requires that every efficient production plan can be divided into $N+1$ consumptions that satisfy every consumer's lower bound constraint. This mirrors a similar assumption in Theorem 3. The corollary below facilitates individual, group and aggregate constraints by combining Theorems 3 and 4.

Definition: $\tilde{\mathcal{E}}_c = \{(u_i, 1)_{i=1}^N, \{c_i\}_{i=1}^N, \mathcal{I}\}$ *is a production economy with nontransferable utility and modular constraints if*

(1) u_i satisfies the substitutes property for all i;

(2) c_i is a modular constraint for u_i for all i;

(3) $\mathcal{I}$ is a matroid;

(4) for all $z \in \mathcal{B}$, there exist $x_1, \cdots, x_{N+1}$ such that $\sum_{k=1}^{N+1} x_k \leq z$ and $x_k \in dom\, u_i(c_i, \cdot)$ for all k, i.

Corollary: *The production economy with nontransferable utility and modular constraints has a strong equilibrium.*

4. Conclusion

Our results suggest that Walrasian methods can be employed in a variety of market design problems whenever preferences satisfy the substitutes property. Gul and Stacchetti (1999) show that given any utility function that does not satisfy the substitutes property, we can construct a transferable utility economy with N agents, one with the preference in question and $N-1$ with substitutes preferences such that no equilibrium exists. Hence, it seems unlikely that a general existence result for the nontransferable utility economy that permits a larger set of preferences than the substitutes class can be proved.

However, Sun and Yang (2006) provide a generalization of the Kelso-Crawford existence result that allows for some complementarities in consumption. In particular, they show that if the goods can be partitioned into two sets such that all agents consider goods within each element of the partition substitutes and consider goods in different elements complements, then a Walrasian equilibrium exists in the corresponding transferable utility economy. A generalization of this result is offered in Shioura and Yang (2015). One possible extension of the current work would be the see if equilibrium also exists with Sun-Yang preferences in the limited transfers and nontransferable utility economies.

5. Appendix A

Unless indicated otherwise, the definitions and results below can be found in Oxley (2011):

Recall that a *matroid* $\mathcal{I} \subset X$ is a collection of sets such that (I1) $\emptyset \in \mathcal{I}$, (I2) $y \in \mathcal{I}$, $x \leq y$ implies $x \in \mathcal{I}$ and (I3) $x, y \in \mathcal{I}$, $\sigma(x) < \sigma(y)$ implies there is j such that $x^j < y^j$ and $x + \chi^j \in \mathcal{I}$.

There are various alternative ways to describe a matroid. One way is by characterizing it maximal elements. For any matroid $\mathcal{I}$, let $\mathcal{B}(\mathcal{I}) = \{x \in \mathcal{I} \,|\, y \geq x \text{ and } y \in \mathcal{I} \text{ implies } y = x\}$ be the set of all maximal elements of $\mathcal{I}$. Then, $\mathcal{B}(\mathcal{I})$ is a *basis system*; that is, (B1) $\mathcal{B}(\mathcal{I})$ is nonempty and (B2) $x, y \in \mathcal{B}(\mathcal{I})$ and $x^j > y^j$ implies there is k such that $y^k > x^k$ and $x - \chi^j + \chi^k \in \mathcal{B}(\mathcal{I})$.

If $\mathcal{B} \subset X$ satisfies (B1) and (B2), then $\mathcal{I} = \{x \in X \,|\, x \leq y \text{ for some } y \in \mathcal{B}\}$ is a matroid and $\mathcal{B} = \mathcal{B}(\mathcal{I})$. Every basis system $\mathcal{B}$ satisfies the following stronger version of (B2): (B2*) $x, y \in \mathcal{B}(\mathcal{I})$ and $x^j > y^j$ implies there is k such that $y^k > x^k$ and $x - \chi^j + \chi^k, y - \chi^k + \chi^j \in \mathcal{B}(\mathcal{I})$. Also, all elements of a basis system have the same cardinality; that is, if $x, y \in \mathcal{B}$ and $\mathcal{B}$ is a basis system, then $\sigma(x) = \sigma(y)$. Hence, for any matroid $\mathcal{I}$, $\mathcal{B}(I)$ is the set of elements of $\mathcal{I}$ with the maximal cardinality; $\mathcal{B}(\mathcal{I}) = \{x \in \mathcal{I} \,|\, y \in \mathcal{I} \text{ implies } \sigma(x) \geq \sigma(y)\}$.

Gul and Stacchetti (2000) show that if u satisfies the substitutes property, then the set of elements of $D_u(p)$ with the smallest cardinality is a basis system for every p.

For any $\mathcal{B}$, let $\mathcal{B}^\perp = \{\chi^H - x \,|\, x \in \mathcal{B}\}$. If $\mathcal{B}$ is a basis system, then $\mathcal{B}^\perp$ is also a basis system and is called the *dual* of $\mathcal{B}$.

A function $r : X \to \mathbb{N}$ is a *rank function* if (R1) $0 \leq r(x) \leq \sigma(x)$, (R2) $x \leq y$ implies $r(x) \leq r(y)$ and (R3) $r(x \vee y) + r(x \wedge y) \leq r(x) + r(y)$. For any rank function, r, the set of all minimal (in the natural order on $\mathbb{R}^L$) maximizers of r is a basis system. Also, given any matroid $\mathcal{I}$, the function r defined by $r(x) = \max\{\sigma(y) \,|\, y \leq x, y \in \mathcal{I}\}$ is a rank function.

A *weighted matroid* is a function ρ defined as follows: for any additive and monotone utility function v and any matroid $\mathcal{I}$, let $\rho(x) = \max_{\substack{y \leq x \\ y \in \mathcal{I}}} v(y)$. A rank function is a special case of a weighted matroid where $v(x) = \sigma(x)$.

To define academic preferences, we adopt the following concept from Yokote (2017): $Y \subset X$ is an *$M^\natural$-convex set* if $x, y \in Y$ and $x^j > y^j$ implies either $x - \chi^j, y + \chi^j \in Y$ or there is k such that $y^k > x^k$ and $x - \chi^j + \chi^k, y - \chi^k + \chi^j \in Y$. It is easy to see that a set Y is $M^\natural$-convex if and only if the function I^*_Y such that $I^*_Y(x) = 0$ if $x \in Y$ and $I^*_Y(x) = -\infty$ if $x \notin Y$ is $M^\natural$-concave.

The utility function u is an *academic preference* if there exists an additive and monotone utility function v and an $M^\natural$-convex set Y such that

$$u(x) = \begin{cases} \max_{\substack{y \in Y \\ y \leq x}} v(y) & \text{if there is } y \leq x,\ y \in Y \\ -\infty & \text{otherwise} \end{cases}$$

Fact: Every academic preference satisfies the substitutes property.

Proof: Shioura (2009) shows that a weighted matroid is $M^\natural$-concave. The same argument establishes that an academic preference is $M^\natural$ concave. Since $M^\natural$-concavity is equivalent to the substitute property, the fact follows. □

Next, we will show that the utility function in Example 3 is an academic preference. Identify H with the edges of an undirected graph with vertices $\{a, b, c, d, e\}$. Then, the set of feasible schedules with all courses for credit, Y, is a collection of 3 or 4 edges that contain no cycles. Let Z be the set of all subsets of H that contain no cycles. It is well-known that Z is a matroid and therefore, Y^*, the set of maximal elements of Z is a basis system. Clearly, each element of Y^* has four edges and $Y^* \subset Y$. To prove that Y is $M^\natural$ convex, first assume that $x, y \in Y^*$ and $x^j > y^j$. Then, by the strong exchange property B2*, there is k such that $y^k > x^k$ and $x - \chi^j + \chi^k, y - \chi^k + \chi^j \in Y^*$. If $x \in Y^*$, $y \notin Y^*$ and $x^j > y^j$ (hence, x contains four edges and y contains three edges), we can always find $\hat{y} \in Y^*$ such that $y \leq \hat{y}$ and $\hat{y}^j < x^j$. Then, by the strong exchange property B2*, there is k such that $x - \chi^j + \chi^k, \hat{y} - \chi^k + \chi^j \in Y^*$. If $y^k = 1$, then clearly $x - \chi^j + \chi^k, y - \chi^k + \chi^j \in Y$. If $y^k = 0$, then $\hat{y} - \chi^k + \chi^j = y + \chi^j \in Y$ and $x - \chi^j \in Y$. Now we consider the case where $x \notin Y^*$, $y \in Y^*$ and $x^j > y^j$. Choose $\hat{x} \in Y^*$ such that $x \leq \hat{x}$ and $\hat{x}^j > y^j$. Then, by B2*, there is k such that $\hat{x} - \chi^j + \chi^k, y - \chi^k + \chi^j \in Y^*$ and hence $x - \chi^j + \chi^k, y - \chi^k + \chi^j \in Y$. Finally, if $x, y \notin Y^*$ and $x^j > y^j$, choose $\hat{x} \in Y^*$ such that $x \leq \hat{x}$ and clearly $\hat{x}^j > y^j$. Then we are back to the previous case for $\hat{x} \in Y^*$ and $y \notin Y^*$ and we know that either there exists k such that $\hat{x} - \chi^j + \chi^k, y - \chi^k + \chi^j \in Y$ or $\hat{x} - \chi^j, y + \chi^j \in Y$. In the former case, $\hat{x} - \chi^j + \chi^k \in Y^*$ and hence $x - \chi^j + \chi^k, y - \chi^k + \chi^j \in Y$. In the latter case, note $x - \chi^j$ has only two edges and there is at most one other edge j' such that $x - \chi^j + \chi^{j'}$ forms a cycle. Find $k' \neq j'$ and $y^{k'} > 0$, then $x - \chi^j + \chi^{k'}, y + \chi^j - \chi^{k'} \in Y$. Thus, Y is $M^\natural$ and the utility function in Example 3 is an academic preference.

6. Appendix B

6.1 Proof of Lemma 1

First, we will modify the definition of the single improvement property (SI) (Gul and Stacchetti, 1999) as follows:

Definition: *The function u has the single improvement property (SI) if for all p such that $D_u(p) \subset dom\, u$ and all $x \in dom\, u - D_u(p)$, there is y such that $U(x,p) < U(y,p)$, $|supp\,(x) - supp\,(y)| \leq 1$ and $|supp\,(y) - supp\,(x)| \leq 1$.*

Theorem 4.1 and Theorem 5.1 in Shioura and Tamura (2015) establish that the substitutes property (SI) and $M^\natural$-concavity are equivalent. Also, a utility function u is submodular if it satisfies the substitutes property. Gul and Stacchetti (1999) show that (SI) is equivalent to the substitutes property if the effective domain is X. Their proof reveals that the above modified definition of (SI) is equivalent to the substitutes property for a general effective domain.

The following proof is similar to the proof that k-satiation preserves substitutes property in Bing, Lehmann and Milgrom (2004). We first prove two auxiliary lemmas. Lemma B1 provides an alternative characterization of $M^\natural$-concavity.

Lemma B1: *Let u be a utility function that satisfies the substitutes property. If $x, y \in dom\, u$ with $x \not\geq y$ and $y \not\geq x$, then there is k, j such that $x^j > y^j$, $y^k > x^k$ and $u(x - \chi^j + \chi^k) + u(y + \chi^j - \chi^k) \geq u(x) + u(y)$.*

Proof: Since u satisfies the substitutes property, u is $M^\natural$-concave. Since $y \not\geq x$, there exists j with $x^j > y^j$. Since $x, y \in \mathrm{dom}\, u$, $M^\natural$-concavity implies that either there is k such that $y^k > x^k$ such that $u(x - \chi^j + \chi^k) + u(y + \chi^j - \chi^k) \geq u(x) + u(y)$ and we are done, or that $u(x - \chi^j) + u(y + \chi^j) \geq u(x) + u(y)$. That is,

$$u(y + \chi^j) - u(y) \geq u(x) - u(x - \chi^j)$$

Similarly, since $x \not\geq y$, there exists l with $y^l > x^l$. It follows from $M^\natural$-concavity that either there is k with $x^k > y^k$ such that $u(x - \chi^k + \chi^l) + u(y + \chi^k - \chi^l) \geq u(x) + u(y)$ and we are done, or

$$u(x + \chi^l) - u(x) \geq u(y) - u(y - \chi^l)$$

The above two inequalities together with the submodularity of u imply

$$u(x-\chi^j+\chi^l)-u(x-\chi^j)\geq u(x+\chi^l)-u(x)\geq u(y)-u(y-\chi^l)$$
$$u(y-\chi^l+\chi^j)-u(y-\chi^l)\geq u(y+\chi^j)-u(y)\geq u(x)-u(x-\chi^j)$$

Hence,

$$u(x-\chi^j+\chi^l)+u(y-\chi^l+\chi^j)\geq u(x)+u(y)$$

as desired. □

The following lemma states that if a bundle with n elements does not maximize utility among all bundles with at least n elements, then we can increase its utility by either adding an element to it or replacing one of its elements with a different one.

Lemma B2: *Let u be a utility function that satisfies the substitutes property. Let A, B be such that $|B|\geq n=|A|$, $\chi^A\in \operatorname{dom} u$ and $U(\chi^B,p)>U(\chi^A,p)$. Then, either there exists $l\notin A$ such that $U(\chi^A+\chi^l,p)>U(\chi^A,p)$ or there exists $k\in A$, $l\notin A$ such that $U(\chi^A+\chi^l-\chi^k,p)>U(\chi^A,p)$.*

Proof: Let $\mathcal{D}$ denote the utility maximizing bundles at price p among all bundles with at least n elements. Denote the Hausdorff distance between any two sets $\hat{A},\hat{B}$ as $d(\hat{A},\hat{B})=|\hat{A}-\hat{B}|+|\hat{B}-\hat{A}|$. Let B^* minimize the Hausdorff distance from A among the elements of $\mathcal{D}$. By assumption, $U(\chi^{B^*},p)>U(\chi^A,p)$. Clearly $B^*-A\neq\emptyset$, otherwise, since $|A|=n, |B^*|\geq n$ and $B^*\subseteq A$, we have $A=B^*$, a contradiction. Since $\chi^A\in\operatorname{dom} u$, we have $\chi^{B^*}\in\operatorname{dom} u$.

First, assume $A-B^*\neq\emptyset$. By Lemma B1, there exists k,l with $k\in A-B^*$, $l\in B^*-A$ such that

$$u(\chi^A)+u(\chi^{B^*})\leq u(\chi^A-\chi^k+\chi^l)+u(\chi^{B^*}-\chi^l+\chi^k)$$

Since the total cost of bundles on either side of the above inequality is the same, we have

$$U(\chi^A,p)+U(\chi^{B^*},p)\leq U(\chi^A-\chi^k+\chi^l,p)+U(\chi^{B^*}-\chi^l+\chi^k,p)$$

By assumption, $U(\chi^{B^*},p)\geq U(\chi^{B^*}-\chi^l+\chi^k,p)$. If $U(\chi^{B^*},p)=U(\chi^{B^*}-\chi^l+\chi^k,p)$, then $\chi^{B^*}-\chi^l+\chi^k$ is also optimal at price p among all bundles with at least n elements and

$d(A, B^* \cup \{k\} - \{l\}) < d(A, B^*)$, which contradicts the definition of B^*. Thus, $U(\chi^{B^*}, p) > U(\chi^{B^*} - \chi^l + \chi^k, p)$ and by the inequality above, $U(\chi^A, p) < U(\chi^A - \chi^k + \chi^l, p)$.

Second, assume $A - B^* = \emptyset$. Then, since $A \neq B^*$, A is a strict subset of B^* and $|B^*| \geq n+1$. For any $j \in B^* - A$, $d(B^* - \{j\}, A) < d(B^*, A)$ and $|B^* - \{j\}| \geq n$. Then, $U(\chi^{B^*}, p) > U(\chi^{B^*} - \chi^j, p)$ and therefore,

$$p^j < u(\chi^{B^*}) - u(\chi^{B^*} - \chi^j)$$

Since u is submodular, u has decreasing marginal returns. Recall that $\chi^A + \chi^j \in \operatorname{dom} u$, $\chi^A \leq \chi^{B^*}$ and $j \notin A$. Hence,

$$p^j < u(\chi^{B^*}) - u(\chi^{B^*} - \chi^j) \leq u(\chi^A + \chi^j) - u(\chi^A)$$

That is, $U(\chi^A, p) < U(\chi^A + \chi^l, p)$. □

Proof of Lemma 1: Since $M^\natural$-concavity is equivalent to (SI), it suffices to show that $[u]_k$ satisfies the latter. Since the effective domain of $[u]_k$ is nonempty, there is $s \in \operatorname{dom} [u]_k$ such that $\sigma(s) \geq k$, $u(s) = [u]_k(s)$ and s is optimal at price p for the utility function $[u]_k$. Let $[U]_k$ be consumer s utility function over both indivisible goods and transfers; that is, $[U]_k(x, p) = [u]_k(x) - p \cdot x$.

Suppose that $x \in \operatorname{dom} [u]_k - D_{[u]_k}(p)$, then $\sigma(x) \geq k$. We need to show there exists y such that $[U]_k(x, p) < [U]_k(y, p)$, $|\operatorname{supp}(x) - \operatorname{supp}(y)| \leq 1, |\operatorname{supp}(y) - \operatorname{supp}(x)| \leq 1$.

If $\sigma(x) = k$, then since $\sigma(s) \geq k$ and $U(s, p) > U(x, p)$, Lemma B2 yields the desired conclusion.

If $\sigma(x) > k$, then $\sigma(z) \geq k$ for any z such that $|\operatorname{supp}(x) - \operatorname{supp}(z))| \leq 1$ and $|\operatorname{supp}(z) - \operatorname{supp}(x)| \leq 1$. Recall that u satisfies (SI) and $x \notin D_u(p)$, $x \in \operatorname{dom} [u]_k \subseteq \operatorname{dom} u$. Then, there exists y such that $U(x, p) < U(y, p)$, $|\operatorname{supp}(x) - \operatorname{supp}(y)| \leq 1, |\operatorname{supp}(y) - \operatorname{supp}(x)| \leq 1$. Since $\sigma(y), \sigma(x) \geq k$, $[U]_k(x, p) = U(x, p)$, $[U]_k(y, p) = U(y, p)$ and therefore, $[U]_k(x, p) < [U]_k(y, p)$. □

6.2 Proof of Lemma 2

This is a corollary of Theorem 8.2 in Shioura and Tamura (2015). Although they assume $\{o, \chi^H\} \in \operatorname{dom} u_i$, their proof can be applied under our assumption that there exists

a feasible allocation ξ such that $\sum_{i=1}^{N} \xi_i \leq \chi^H$ and $\xi_i \in \operatorname{dom} u_i$ for all i. By monotonicity, our assumption implies that there exists ξ^* such that $\xi_i^* \in \operatorname{dom} u_i$, $\sum_{i=1}^{N} \xi_i^* = \chi^H$ and

$$\sum_{i=1}^{N} u_i(\xi_i^*) = \max \left\{ \sum_{i=1}^{N} u_i(x_i) | x_i \in \operatorname{dom} u_i, \forall i, \sum_{i=1}^{N} x_i = \chi^H \right\}$$

Then, the remainder of the proof follows Theorem 8.2 in Shioura and Tamura (2015). □

6.3 Proof of Theorem 1

Suppose $\mathcal{E} = \{(u_i, w_i, b_i)_{i=1}^N\}$ is a limited transfers economy. The proof for existence of an equilibrium relies on Lemma 2. We first introduce some properties of equilibria in a transferable utility economy $\mathcal{E}_o = \{(u_i)_{i=1}^N\}$. In $\mathcal{E}_o$, as each agent has enough divisible goods to buy whatever bundle of indivisible goods she desires, the set of equilibrium prices and the set of equilibrium allocations of indivisible goods are independent of the initial endowments of both divisible and indivisible goods. Hence we can state the consumers' problem as maximizing the following function over x for given prices p:

$$U_i(x, p) = u_i(x) - p \cdot x$$

By assumption, $w_i \in \operatorname{dom} u_i$ and u_i satisfies the substitutes property for each i. Then, Lemma 2 establishes the existence of an equilibrium with deterministic allocations for the transferable utility economy. It is easy to see that for any given allocation α, the set of prices that support α (that is, p such that (p, α) is an equilibrium) is defined by a finite set of linear weak inequalities and therefore is compact and convex. Since we are in a transferable utility setting, any Pareto efficient allocation must maximize social surplus. It is also easy to verify that if (p, ξ) is a (deterministic) equilibrium, and $\hat{\xi}$ is a social surplus maximizing allocation, then $(p, \hat{\xi})$ is also an equilibrium. The following *exchangeability property* is a consequence of the last two observations: if (p, α) and $(\hat{p}, \hat{\alpha})$ are both equilibria, then $(p, \hat{\alpha})$ is also an equilibrium. Then, it also follows that the set of random equilibrium allocations is simply the convex hull of the set of deterministic equilibrium allocations and hence, the set of equilibrium prices for random allocations is the same as the set of equilibrium prices for deterministic allocations. It follows that for any transferable utility economy, there is

a set of prices P^* and a set of random allocations $\mathcal{A}^*$ such that the set of equilibria is $P^* \times \mathcal{A}^*$.

Since every price vector in P^* supports the same set of allocations, P^* is a nonempty, convex and compact set as it is defined by a finite set of linear weak inequalities. Since $\mathcal{A}^*$ is the set of social surplus maximizing allocations, it is also a nonempty, convex and compact set. We summarize these observations in Lemma B3 below.

Lemma B3: *For any transferable utility economy $\mathcal{E}_o$, the set of equilibria is $P^* \times \mathcal{A}^*$ for some nonempty compact convex set of prices P^* and the nonempty compact convex set of social surplus maximizing random allocations $\mathcal{A}^*$.*

Given the transferable utility economy $\mathcal{E}_o$, let $\mathcal{A}_o$ be the set of all feasible random allocations contained in the product of the convex hulls of agents' effective domains. Restricting attention to $\mathcal{A}_o$ is without loss of generality; for any random allocation $\alpha \notin \mathcal{A}_o$, there must be some agent with utility $-\infty$ so that α cannot be efficient and thus $\alpha \notin \mathcal{A}^*$. Since the set of deterministic feasible allocations is finite and $w_i \in \operatorname{dom} u_i$ for each i, $\mathcal{A}_o$ is a nonempty, compact and convex subset of a Euclidian space. For any $\lambda = (\lambda_1, \ldots, \lambda_N) \in [0,1]^N$, we define the folllowing maximization problem:

$$M(\lambda) = \max_{\alpha \in \mathcal{A}_o} \sum_i \lambda_i u_i(\alpha_i)$$

We set $-\infty \times 0 = -\infty$; that is, when $\lambda_i = 0$, $\lambda_i u_i(\cdot)$ has the same effective domain as u_i and remains 0 on the effective domain. Note that $M(\lambda)$ is a linear programming problem. Let $\Delta(\lambda) \subset \mathcal{A}_o$ denote the set of solutions to this problem. For $\lambda \in [0,1]^N$, define $\mathcal{E}_o(\lambda) = \{(\lambda_i u_i)_{i=1}^N\}$. As for each i, $\operatorname{dom} u_i = \operatorname{dom} \lambda_i u_i$ and λu_i satisfies the substitutes property, $\mathcal{E}_o(\lambda)$ is also a transferable utility economy.

By Lemma B3, $\Delta(\lambda)$ is the set of equilibrium allocations for the economy $\mathcal{E}_o(\lambda)$. Let $a = \sum_i u_i(\chi^H)$ and $\mathcal{P} = [0,a]^N$. Then any equilibrium price p of the transferable utility economy $\mathcal{E}_o(\lambda)$ must lie in $\mathcal{P}$. Let $P^*(\lambda)$ be the set of all equilibrium prices for $\mathcal{E}_o(\lambda)$.

Lemma B4: *For any limited transfers economy $\mathcal{E}$, there exists $\lambda \in [0,1]^N$ and an equilibrium (p, α) of the corresponding transferable utility economy $\mathcal{E}_o(\lambda)$ such that for each i, if $\lambda_i < 1$, then $p \cdot \bar{\alpha}_i = b_i$ and if $\lambda_i = 1$, then $p \cdot \bar{\alpha}_i \leq b_i$.*

Proof: Let U_i^λ be consumer i's utility function in the transferable utility economy $\mathcal{E}_o(\lambda)$; that is, $U_i^\lambda(\theta, p) = \lambda_i u_i(\theta) - p \cdot \bar{\theta}$.

By Lemma B3, the correspondences Δ and P^* are nonempty, compact and convex valued. Since $\Delta(\lambda)$ is also the solution of the maximization problem defined above, Berge's Theorem ensures that Δ is upper hemi-continuous (uhc). Next, we will show that P^* is uhc as well. Since P^* compact-valued, it is enough to show that $\lambda(t) \in [0,1]^N$, $p(t) \in P^*(\lambda(t))$ for all $t = 1, 2, \ldots$, $\lim_{t\to\infty} \lambda(t) = \lambda$ and $\lim_{t\to\infty} p(t) = p$ imply that $p \in P^*(\lambda)$.

Choose $\alpha(t) \in \Delta(\lambda(t))$ for all t. Since $\mathcal{A}_o$ is compact, we can assume, by passing to a subsequence if necessary, that $\alpha(t)$ converges. Let $\alpha = \lim_{t\to\infty} \alpha(t)$. Since Δ is uhc, $\alpha \in \Delta(\lambda)$. Let β be any feasible allocation. Then, the efficiency of α and the feasibility of β imply

$$U_i^{\lambda(t)}(\alpha_i(t), p(t)) = \lambda_i(t) u_i(\alpha_i(t)) - p(t) \cdot \bar{\alpha}_i(t) \geq \lambda_i(t) u_i(\beta_i) - p(t) \cdot \bar{\beta}_i = U_i^{\lambda_i(t)}(\beta_i, p(t))$$

Then, the continuity of u_i ensures that $U_i^\lambda(\alpha_i, p) \geq U_i^\lambda(\beta_i, p)$ for all β and for all i. This implies that $p \in P^*(\lambda)$ and establishes the upper hemi-continuity of P^*.

Next, for each i define correspondence Γ_i from $\mathcal{P} \times [0,1]^L$ to $[0,1]$ as follows:

$$\Gamma_i(p, z) = \begin{cases} [0,1] & \text{if } p \cdot (z - w_i) = b_i \\ 0 & \text{if } p \cdot (z - w_i) > b_i \\ 1 & \text{if } p \cdot (z - w_i) < b_i \end{cases}$$

Clearly, for each i, Γ_i is nonempty, convex, compact valued and uhc.

Let $S = \mathcal{P} \times \mathcal{A}_o \times [0,1]^N$ and let f be a correspondence from S to itself defined as

$$f(p, \alpha, \lambda) = P^*(\lambda) \times \Delta(\lambda) \times \Gamma_1(p, \bar{\alpha}_1) \times \cdots \times \Gamma_N(p, \bar{\alpha}_N)$$

Since P^*, Δ and Γ_i's are nonempty, convex, compact valued and uhc, and the mapping $\alpha \to \bar{\alpha}_i$ is continuous, f is also nonempty, convex, compact valued and uhc. Then, by Kakutani's Fixed-Point Theorem, there is an $s^* = (p^*, \alpha^*, \lambda^*)$ such that $f(s^*) = s^*$. Thus, (p^*, α^*) is an equilibrium of the transferable utility economy $\mathcal{E}_o(\lambda^*)$.

We claim that $\lambda_i^* > 0$ for all i. Suppose $\lambda_i^* = 0$ for some i. Then agent i's utility is identically 0 on the effective domain of u_i and since s^* is a fixed point of f, $p \cdot (\bar{\alpha}_i^* - w_i) \geq$

$b_i > 0$. But this contradicts the fact that α_i solves the consumer i's utility maximization problem.

To complete the proof of the lemma, we will show that $p^* \cdot (\bar{\alpha}_i^* - w_i) \leq b_i$ for all i and that the inequality is an equality if $\lambda_i^* < 1$. Since s^* is a fixed point of f and $\lambda_i^* > 0$, by definition of Γ_i, we must have $p^* \cdot (\bar{\alpha}_i^* - w_i) \leq b_i$. Moreover, if $\lambda_i^* < 1$, again by definition of Γ_i, we must have $p^* \cdot (\bar{\alpha}_i^* - w_i) = b_i$. □

To conclude the proof of Theorem 1, we will show that (p^*, α^*) is an equilibrium of the limited transfers economy $\mathcal{E}$. Consider any θ that i can afford in $\mathcal{E}$; that is, $p^* \cdot (\bar{\theta} - w_i) \leq b_i$. The optimality of α_i^* for i in the transferable utility economy implies

$$\lambda_i^*(u_i(\theta) - u_i(\alpha_i^*)) \leq p^* \cdot \bar{\theta} - p^* \cdot \bar{\alpha}_i^* \tag{2}$$

If $\lambda_i^* = 1$, then equation (2) implies that α_i solves the utility maximization problem of agent i in the limited transfers economy $\mathcal{E}$. If $\lambda_i^* < 1$, since s^* is a fixed-point of f, the right-hand side of equation (2) must be less than or equal to zero as $p^* \cdot (\bar{\alpha}_i^* - w_i) = b_i$. Then, we have $\lambda_i^*(u_i(\theta) - u_i(\alpha_i^*)) \leq 0$ and hence $u_i(\theta) - u_i(\alpha_i^*) \leq \lambda_i^*(u_i(\theta) - u_i(\alpha_i^*)) \leq p^* \cdot \bar{\theta} - p^* \cdot \bar{\alpha}_i^*$. Again, this shows that α_i solves the utility maximization problem of agent i in $\mathcal{E}$. □

6.4 Proof of Theorem 2

Fix the nontransferable utility economy $\mathcal{E}^* = \{(u_i, b_i)_{i=1}^N\}$ and define the sequence of limited transfers economies $\mathcal{E}_n = \{(nu_i, w_i, b_i)_{i=1}^N\}$ for $n = 1, 2 \ldots$ where $w_i^j = 0$ for all j, i. Since $o \in \operatorname{dom} u_i$ for all i, by Theorem 1, each $\mathcal{E}_n$ has an equilibrium (p^n, α^n).[21] By monotonicity, $o \in \operatorname{dom} u_i$ implies that $\operatorname{dom} u_i = X$ and hence, we need not worry about the issue of effective domains. Let $P = [0, \sum_i b_i]^L$. Note that p^n must lie in P. Hence, the sequence (p^n, α^n) lies in a compact set and therefore has a limit point (p, α). By passing to a subsequence if necessary, we assume that (p^n, α^n) converges to (p, α). To conclude the proof, we show that this limit point must be a strong equilibrium of $\mathcal{E}^*$.

Clearly, α is feasible for $\mathcal{E}^*$. Since $p^n \cdot \bar{\alpha}_i^n \leq b_i$ for all n, we know $p \cdot \bar{\alpha}_i \leq b_i$ and hence α_i is affordable for i in $\mathcal{E}^*$. Take any other affordable random allocation θ for i in $\mathcal{E}^*$; that

[21] When there is no risk of confusion, we use superscripts to specify the particular indivisible good (with generic element j) and the particular element in a sequence of prices or allocations (with generic element n, m). Otherwise, we use double superscripts.

is, $p \cdot \bar{\theta} \leq b_i$. We need to show that $u_i(\theta) \leq u_i(\alpha_i)$. First, assume $p \cdot \bar{\theta} < b_i$. Then, there exists $\epsilon > 0$ small enough such that for any $p' \in B_\epsilon(p) \cap I\!R^L_+$, $p' \cdot \bar{\theta} < b_i$, where $B_\epsilon(p)$ is the ϵ-ball centered at p. Since $\lim_{n\to\infty} p^n = p$, we can find $M > 0$ such that for all $n \geq M$, $p^n \cdot \bar{\theta} < b_i$. Hence, θ is affordable for i in $\mathcal{E}_n$ for $n \geq M$. Since α_i^n is an optimal consumption at p^n, $nu_i(\theta) - p^n \cdot \bar{\theta} \leq nu_i(\alpha_i^n) - p \cdot \bar{\alpha}_i^n$; that is, $u_i(\theta) - u_i(\alpha_i^n) \leq (p^n \cdot \bar{\theta} - p^n \cdot \bar{\alpha}_i^n)/n \leq b_i/n$ for all $n \geq M$. Then, the continuity of u_i ensures $u_i(\theta) \leq u_i(\alpha_i)$ as desired.

Next, assume $p \cdot \bar{\theta} = b_i$. Then, there exists a sequence θ^n converging to θ such that $p \cdot \bar{\theta}^n < b_i$ for all n. In the previous paragraph, we showed that $u_i(\theta^n) \leq u_i(\alpha_i)$. Taking a limit and appealing to the continuity of u_i yields $u_i(\theta) \leq u_i(\alpha_i)$. Thus, α_i is optimal for i at prices p in $\mathcal{E}^*$.

To prove that all goods with strictly positive prices are allocated to the agents, it is enough to show that $p^j \cdot (1 - \sum_{i=1}^N \bar{\alpha}_i^j) = 0$ for all j. This follows since $p^{nj} \cdot (1 - \sum_{i=1}^N \bar{\alpha}_i^{nj}) = 0$ for all j, n and hence, the same equality holds in the limit as n goes to infinity. Thus, (p, α) is an equilibrium of $\mathcal{E}^*$.

To conclude, we will show that (p, α) is a strong equilibrium; that is, for all i, $u_i(\theta) = u_i(\alpha_i)$ implies $p \cdot \bar{\theta} \geq p \cdot \bar{\alpha}_i$. If not, assume that $p \cdot \bar{\theta} < p \cdot \bar{\alpha}_i$ for some θ such that $u_i(\theta) = u_i(\alpha_i)$ and consider two cases: (1) agent i is satiated at θ; that is, $u_i(\theta) = u_i(\alpha_i) = u_i(\chi^H)$ or (2) she is not satiated at θ.

If (1) holds, then for sufficiently large n, purchasing θ instead of α_i is affordable for i at p^n and $nu_i(\theta) - p^n \cdot \bar{\theta} > nu_i(\alpha_i) - p^n \cdot \bar{\alpha}_i^n$, contradicting the optimality of α_i^n for agent i in $\mathcal{E}_n$. If (2) holds, then choose $0 < r < 1$ such that $p(r\chi^H + (1-r)\bar{\theta}) < p\bar{\alpha}_i$. Again, for n sufficiently large, the random consumption $r\delta_{\chi^H} + (1-r)\theta$, where δ_{χ^H} is the degenerate lottery that yields χ^H for sure, is affordable at p^n and yields a higher utility than α_i^n, contradicting its optimality for agent i in $\mathcal{E}_n$. □

6.5 Proof of Lemma 3 and Theorem 3

We call $\mathcal{E}^* = \{(u_i, 1)_{i=1}^N\}$ a nontransferable utility economy with general consumption sets if u_i satisfies the substitutes property for all i and there is $x_1, \cdots, x_{N+1}$ such that $\sum_{k=1}^{N+1} x_k \leq \chi^H$ and $x_k \in \operatorname{dom} u_i$ for all k, i. Unlike a nontransferable utility economy, we do not require that $o \in \operatorname{dom} u_i$. Hence, the definition of a nontransferable utility economy with general consumption sets is similar to that of a nontransferable utility economy

with modular constraints; instead of modular constraints, there are more general utility functions for which $\operatorname{dom} u_i = X$ need not hold.

Lemma B5: *A nontransferable utility economy with general consumption sets has a strong equilibrium.*

Proof: Fix $\mathcal{E}^* = \{(u_i, 1)_{i=1}^N\}$ as a nontransferable utility economy with general consumption sets. By definition, there exists $x_1, \cdots, x_{N+1}$ such that $x_i \in \operatorname{dom} u_j$ for all $i = 1, ..., N+1$, $j = 1, ..., N$. Since u_i is monotone, we can assume $\sum_{i=1}^{N+1} x_i = \chi^H$. Define the random consumptions θ^0, θ^1 as follows: $\theta^0(x_i) = 1/N$ for all $i = 1, \ldots, N-1$ and $\theta^0(x_N + x_{N+1}) = 1/N$; $\theta^1(x_i) = 1/(N+1)$ for all $i = 1, \ldots, N+1$. Note that $\bar{\theta}^{0j} = 1/N$ and $\bar{\theta}^{1j} = 1/(N+1)$ for all $j \in H$. By monotonicity again, every realization of θ^0 and θ^1 lies in the effective domain of every agent's utility function. Let α^0 be a random allocation such that $\alpha_i^0 = \theta^0$ for all $i = 1, \ldots, N$. Easy to see that such a random allocation exists.

Consider the following limited transfers economy with random endowments of indivisible goods. Each consumer i has the utility u_i, the random endowment of indivisible goods θ^0 and the endowment of the divisible good $b_i = 1$. Consumer i's budget set is

$$B(p, \theta^0, 1) = \{\theta \in \Theta |\, p \cdot (\bar{\theta} - \bar{\theta}^0) \leq 1\}$$

Otherwise, this economy is identical to the limited transfers economy. Extending the existence proof in Theorem 1 to this economy is straightforward since every realization of the random endowment is in the effective domain of u_i for each agent i.

Next, define the sequence of limited transfers economies $\mathcal{E}_n = \{(nu_i, \theta^0, 1)_{i=1}^N\}$ for $n = 1, 2 \ldots$. Then, each $\mathcal{E}_n$ has an equilibrium (p^n, α^n). The monotonicity of the utility functions implies that we can assume $\sum_{i=1}^N \bar{\alpha}_i^n = \chi^H$ for each n without loss of generality. We distinguish two cases:

Case 1: $\{p^n\}_{n \geq 1}$ has a bounded subsequence.

As $\{\alpha^n\}_{n \geq 1}$ is bounded, by passing to a subsequence if necessary, we assume that (p^n, α^n) is bounded and converges to (p^*, α). Let $p = \gamma p^*$ for $\gamma = 1/(1 + p^* \cdot \bar{\theta}^0)$. We will show that (p, α) is an equilibrium of the economy $\mathcal{E}^* = \{(u_i, 1)_{i=1}^N\}$.

Clearly, α is feasible for $\mathcal{E}^*$ and $p^n \cdot (\bar{\alpha}_i^n - \bar{\theta}^0) \leq 1$ for all n, i. Therefore, the definition of γ ensures $p \cdot \bar{\alpha}_i \leq 1$ for all i. Hence, α_i is affordable for i in $\mathcal{E}^*$. Take any other

affordable random allocation θ for i in $\mathcal{E}^*$; that is, $p \cdot \bar{\theta} \leq 1$. We need to show that $u_i(\theta) \leq u_i(\alpha_i)$. First, assume $p \cdot \bar{\theta} < 1$. Then, there exists $\epsilon > 0$ small enough so that for any $p' \in B_\epsilon(p) \cap I\!R^L_+$, $p' \cdot \bar{\theta} < 1$. Since $\lim p^n = p^*$, we can find $M > 0$ such that for all $n \geq M$, $p^n \cdot (\bar{\theta} - \bar{\theta}^0) < 1$. That is, θ is affordable for i in $\mathcal{E}_n$ for $n \geq M$. Since α_i^n is optimal, $nu_i(\theta) - p^n \cdot \bar{\theta} \leq nu_i(\alpha_i^n) - p^n \cdot \bar{\alpha}_i^n$. Hence, $u_i(\theta) - u_i(\alpha_i^n) \leq p^n \cdot (\bar{\theta} - \bar{\alpha}_i^n)/n$ for all $n \geq M$. Since $\{p^n\}$ is bounded, the right-hand side goes to 0 as n goes to infinity. Passing to the limit and invoking the continuity of u_i yields $u_i(\theta) \leq u_i(\alpha_i)$ for all $p \cdot \bar{\theta} < 1$.

Next, assume $p \cdot \bar{\theta} = 1$. If $u_i(\theta) = -\infty$, then we are done. Now suppose $u_i(\theta) \in I\!R$; that is, θ's support is contained in $\operatorname{dom} u_i$. Recall that $u_i(\theta^0) \in I\!R$ and by definition of p, $p \cdot \bar{\theta}^0 < 1$. Then, let $\theta^n = (1/n)\theta^0 + (1 - 1/n)\theta$ and note that $p \cdot \bar{\theta}^n < 1$ and $u_i(\theta_n) \in I\!R$. Then, the argument in the previous paragraph yields $u_i(\theta^n) \leq u_i(\alpha_i)$. Taking yet another limit and appealing to the continuity of u_i establishes $u_i(\theta) \leq u_i(\alpha_i)$; that is, α_i is optimal for agent i at prices p in $\mathcal{E}^*$.

Case 2: $\sum_{j=1}^L p^{nj} \to \infty$.

Let $\hat{p}^n = N \cdot p^n / (\sum_{l=1}^L p^{nl})$ and note that $\hat{p}^n \in [0, N]^L$. Let (p, α) be a limit point of $(\hat{p}^n, \alpha^n)$ and assume, without loss of generality, that $(\hat{p}^n, \alpha^n)$ converges to (p, α). We will show that this (p, α) must be an equilibrium of $\mathcal{E}^* = \{(u_i, 1)_{i=1}^N\}$. Clearly, α is feasible for $\mathcal{E}^*$. For any n, as α_i^n is affordable for agent i at p^n in economy $\mathcal{E}_n$, we have

$$p^n \cdot \bar{\alpha}_i^n / p^n \cdot \bar{\theta}^0 \leq (1 + p^n \cdot \bar{\theta}^0)/p^n \cdot \bar{\theta}^0$$

Recall that $\bar{\theta}^{0j} = 1/N$ for all j and $\sum_{j=1}^L p^{nj} \to \infty$, we know $p^n \cdot \bar{\theta}^0 \to \infty$ and hence

$$\lim_{n \to \infty} (1 + p^n \cdot \bar{\theta}^0)/p^n \cdot \bar{\theta}^0 = 1$$

Also by the definition of $\hat{p}^n$, $p^n \cdot \bar{\alpha}_i^n / p^n \cdot \bar{\theta}^0 = p^n \cdot \bar{\alpha}_i^n$. Then passing the limit in the above inequality implies that $p \cdot \bar{\alpha}_i \leq 1$ and thus α_i is affordable at p as desired.

Since (p^n, α^n) is a Walrasian equilibrium and $\sum_{i=1}^N \bar{\alpha}_i^n = \chi^H$, for each n,

$$\sum_{i=1}^N \hat{p}^n \cdot \bar{\alpha}_i^n = \hat{p}^n \cdot \chi^H = N$$

The last equation comes from the definition of $\hat{p}^n$. Passing to the limit yields

$$\sum_{i=1}^{N} p \cdot \bar{\alpha}_i = N$$

Since $p \cdot \bar{\alpha}_i \leq 1$ for each i, we have $p \cdot \bar{\alpha}_i = 1$ for all i.

Consider any affordable random allocation θ in $\mathcal{E}^*$; that is, $p \cdot \bar{\theta} \leq 1$. We will show that $u_i(\theta) \leq u_i(\alpha_i)$ for all i. First, assume $p \cdot \bar{\theta} < 1 = p \cdot \bar{\alpha}_i$. Then, there exists $\epsilon > 0$ such that for any $p' \in B_\epsilon(p) \cap I\!R^L_+$ and random consumption $\theta' \in B_\epsilon(\alpha_i)$, we have $p' \cdot \bar{\theta} < p' \cdot \bar{\theta}'$. Since $p = \lim \hat{p}^n$ and $\alpha_i = \lim \alpha_i^n$, we can find $M > 0$ such that $\hat{p}^n \cdot \bar{\theta} < \hat{p}^n \cdot \bar{\alpha}_i^n$ for all $n \geq M$; that is, $p^n \cdot \bar{\theta} < p^n \cdot \bar{\alpha}_i^n$. Then, since α_i^n is affordable for i in $\mathcal{E}_n$ for $n \geq M$, so is θ. Since α_i^n is optimal, we can derive $u_i(c_i, \theta) - u_i(c_i, \alpha_i^n) \leq p^n \cdot (\bar{\theta} - \bar{\alpha}_i^n)/n < 0$ for all $n \geq M$. Passing to the limit, the continuity of u_i implies $u_i(\theta) \leq u_i(\alpha_i)$. Next, assume $p \cdot \bar{\theta} = 1$ and recall that $u_i(\theta^1) \in I\!R$ for any agents i and $p \cdot \bar{\theta}^1 < p \cdot \bar{\theta}^0 = 1$. Then, the result follows from the corresponding argument in Case 1.

To prove that all goods with strictly positive prices are allocated to agents (for both case 1 and case 2), it is sufficient to show that $p^j \cdot (1 - \sum_{i=1}^N \bar{\alpha}_i^j) = 0$ for all such j. This is true since $p^{nj} \cdot (1 - \sum_{i=1}^N \bar{\alpha}_i^{nj}) = 0$ for all $n = 1, 2 \ldots$ and for all $j \in H$ such that $p^j > 0$ and, therefore, $p^j \cdot (1 - \sum_{i=1}^N \bar{\alpha}_i^j) = 0$. Thus, (p, α) is an equilibrium of $\mathcal{E}^*$.

Finally, we will show that (p, α) is a strong equilibrium of $\mathcal{E}^*$; that is, for all i, $u_i(\theta) = u_i(\alpha_i)$ implies $p \cdot \bar{\theta} \geq p \cdot \bar{\alpha}_i$. If not, assume that $p \cdot \bar{\theta} < p \cdot \bar{\alpha}_i$ for some θ such that $u_i(\theta) = u_i(\alpha_i)$. Then, for sufficiently large n, purchasing θ instead of α_i is affordable for i at p^n in $\mathcal{E}_n$ and $-p \cdot \bar{\theta} > -p \cdot \bar{\alpha}_i^n$, contradicting the optimality of α_i^n for i in $\mathcal{E}_n$. □

Proof of Lemma 3: By assumption, the effective domain of $u(c, \cdot)$ is nonempty. Recall that the operations that transfrom u to u^z (the z-constrained u), $[u]^k$ (the k-satiation of u) and $[u]_k$ (the k-lower bound u) all preserve the substitutes property. Similarly, the binary operation $u \odot v$ (the convolution of u, v) also preserves the substitutes property. Then, to complete the proof of the lemma, we note that given any modular constraint $c = \{(A(k), l(k), h(k))_{k=1}^K\}$ for u, we can express $u(c, \cdot)$ as a finite composition of these operations applied to u. This is straightforward; for example, let $c = \{\{(A(k), l(k), h(k))_{k=1}^4\}$

where $A_1, A_2, A_3 \subset H$ are disjoint sets and $A_4 = A_1 \cup A_2$. Then, define

$$\hat{u} = \left([[v \odot w]^{h_4}]_{l_4} \odot [[u^{z_3}]^{h_3}]_{l_3}\right) \odot u^{z_5} \qquad \text{where}$$
$$v = [[u^{z_1}]^{h_1}]_{l_1} \quad \text{and} \quad w = [[u^{z_2}]^{h_2}]_{l_2}$$

$z_i = \chi^{A_i}$ for all $i = 1, 2, 3, 4$ and $z_5 = \chi^{H-A}$ for $A = \bigcup_{i=1}^{4} A_i$. Since each utility function on the right-hand side of the equation above satisfies the substitutes property and all of the operations applied to them preserve the substitutes property, $\hat{u}$ satisfies the substitutes property as well, and since c is a modular constraint for u, $\hat{u} = u(c, \cdot)$. □

Proof of Theorem 3: Fix a nontransferable utility economy with modular constraints $\mathcal{E}_c^* = \{(u_i, 1)_{i=1}^N, \{c_i\}_{i=1}^N\}$. Since u_i is monotone, so is $u(c_i, \cdot)$. By Lemma 3, $u_i(c_i, \cdot)$ satisfies the substitutes property. Hence, $\mathcal{E}^* = \{(u_i(c_i, \cdot), 1)_{i=1}^N\}$ is a nontransferable utility economy with general consumption sets and by Lemma B5, $\mathcal{E}_c^*$ has a strong equilibrium.□

6.6 Proof of Lemma 4, Theorem 4 and the Corollary

Proof of Lemma 4: Clearly, $o \in \mathcal{I}_d$ and $x, y \in \mathcal{I}_d$ and $x \leq y$ implies $x \in \mathcal{I}_d$. Hence, we need only prove that $x, y \in \mathcal{I}_d$ and $\sigma(x) < \sigma(y)$ implies there is j such that $x^j < y^j$ and $x^j + \chi^j \in \mathcal{I}_d$.

We order d, the hierarchy of constraints, in the obvious way: $(A, k) \succ (B, n)$ if $A \neq B$ and $B \subset A$. Call j a *free* element in d if j is not an element of any A such that $(A, n) \in d$ for some n. Otherwise, call j a constraint element. Let $F \subset H$ be the set of free elements in d and let $F^c = H \backslash F$ be the set of constraint elements. Suppose there is $j \in F$ such that $y^j > x^j$. Then, clearly $x + \chi^j \in \mathcal{I}_d$ and we are done. Otherwise, $x^j \geq y^j$ for all $j \in F$ and hence there must be some $\succ$-maximal constraint, (A, n), such that $\sigma(y \wedge \chi^A) > \sigma(x \wedge \chi^A)$.

Let $A_1 = A, n_1 = n$. Then, either there is $j \in A_1$ such that $y^j > x^j$, $j \notin B$ for any $(B, n') \in d$ such that $(A_1, n_1) \succ (B, n')$ in which case we have $x + \chi^j \in \mathcal{I}_d$ and we are done, or there is no such j. In the latter case, there must be a maximal element of the set $\{(B, n') \in d \,|\, (A_1, n_1) \succ (B, n')\}$ such that $\sigma(y \wedge \chi^B) > \sigma(x \wedge \chi^B)$. Let $A_2 = B, n_2 = n'$ and continue in this fashion until we end up with (A_l, n_l) where $\sigma(y \wedge \chi^{A_l}) > \sigma(x \wedge \chi^{A_l})$ and there is no $(C, n'') \in d$ such that $(A_l, n_l) \succ (C, n'')$. Hence, $x + \chi^j \in \mathcal{I}_d$. □

Proof of Theorem 4: Fix the substitutes economy with matroid technology $\tilde{\mathcal{E}} = \{(u_i, b_i)_{i=1}^N, \mathcal{I}\}$. Let $\mathcal{B}$ be the production possibility frontier of $\mathcal{I}$. Hence, $\mathcal{B}$ is a basis system (Appendix A). We will assume, without loss of generality, that there are at least two elements in $\mathcal{B}$. Otherwise, we are back to the standard case without production covered in Theorem 3. Define

$$H = \{1, \ldots, L\} := \{i \,|\, x_i > 0 \text{ for some } x \in \mathcal{B}\} \tag{3}$$

Let $\mathcal{B}^\perp = \{\chi^H - x \,|\, x \in \mathcal{B}\}$. Then $\mathcal{B}^\perp$ is a basis system since $\mathcal{B}$ is a basis system (Appendix A). Let r be the rank function associated with $\mathcal{B}^\perp$; that is,

$$r(x) = \max\{\sigma(x \wedge y) \,|\, y \in \mathcal{B}^\perp\}$$

Since every weighted matroid satisfies the substitutes property, so does r (Appendix A). Now, consider the sequence of $N+1$-agent nontransferable utility economies without production $\mathcal{E}_n = \{(u_i, b_i)_{i=1}^{N+1}\}$ with aggregate endowment of indivisible goods χ^H, $u_{N+1} = r$ and $b_{N+1} = n$. By Theorem 2, let (p^n, α^n) be a strong equilibrium for economy $\mathcal{E}_n$. Without loss of generality, we assume that α^n converges and each p^{nj} converges either to a real number or to infinity.

Let $n^* = \sigma(x)$ for all $x \in \mathcal{B}^\perp$ and let $\hat{n}$ be the cardinality of $\mathcal{B}^\perp$. Since $\mathcal{B}^\perp$ is a basis system, n^* is well defined and $\hat{n} \geq 2$ since $|\mathcal{B}| \geq 2$. We claim that there exists m such that for all $n > m$, the equilibrium utility of agent $N+1$ is maximal, i.e., equal to n^*. To see why this is true, first assume that $\lim p^{nj} < \infty$ for all j. Then, it is immediate that agent $N+1$'s utility is maximal since she can afford to buy the entire endowment in $\hat{\mathcal{E}}_n$ when n is sufficiently large.

Next, assume that $\lim p^{nj} = \infty$ for all j. Then, agent $N+1$ must be consuming each j with arbitrarily high probability as n approaches infinity. In particular, there must be some m such that for all $n > m$, the probability with which agent $N+1$ consumes j, $\bar{\alpha}^j_{N+1}$, must be greater than $(\hat{n}-1)/\hat{n}$. Equation (3) above ensures that for all $j \in H$, there is $y \in \mathcal{B}$ such that $y^j > 0$ and therefore, there is $y \in \mathcal{B}^\perp$ such that $y^j = 0$. These observations imply that agent $N+1$ can afford the random consumption θ such that $\theta(x) = 1/\hat{n}$ for all $x \in \mathcal{B}^\perp$. Clearly, this θ gives agent $N+1$ her maximal utility.

Finally, if there exists at least one j such that $\lim p^{nj} < \infty$ and some j's such that $\lim p^{nj} = \infty$, then choose $\epsilon \in (0, 1-(\hat{n}-1)/\hat{n})$. The argument of the preceding paragraph establishes that for large enough n, the probability with which agent $N+1$ consumes any good whose price goes to infinity must be greater than $\epsilon + (\hat{n}-1)/\hat{n}$ for n large enough. By consuming $\epsilon/2$ less of one such good, this agent will have enough money to consume all of the goods in the first category; that is, the goods such that $\lim p^{nj} < \infty$. Hence, again, agent $N+1$ can afford the random consumption θ described above and her equilibrium utility must be n^*.

Let θ^n be the equilibrium consumption of agent $N+1$ in the economy $\mathcal{E}_n$ for some n large enough to ensure that her payoff is maximal; that is, $u_{N+1}(\theta^n) = n^*$. Since (p^n, α^n) is a strong equilibrium, we can assume, without loss of generality, that agent $N+1$ only consumes bundles in $\mathcal{B}^\perp$. That is, $\theta^n(x) > 0$ implies $x \in \mathcal{B}^\perp$. This means that $\alpha^n(\xi) > 0$ implies $\sum_{i=1}^N \xi_i \leq z$ for some $z \in \mathcal{B}$. Since u_i is monotone for each i, it is without loss of generality to assume $y := \sum_{i=1}^N \xi_i = z$. If instead $y^j < z^j$, then the fact that α^n is a strong equilibrium ensures $p^{nj} = 0$ in which case we can give good j to any agent $i = 1, \ldots, N$ and still end up with optimal bundles for these agents.

Then, to construct a strong equilibrium (p^n, β) of $\tilde{\mathcal{E}} = \{(u_i, b_i)_{i=1}^N, \mathcal{I}\}$ and let

$$\beta(x_1, \ldots, x_N, z) = \alpha^n(x_1, \ldots, x_N, \chi^H - z)$$

for every deterministic allocation $(x_1, \ldots, x_n)$ and $z \in \mathcal{B}$. Since α^n is a strong equilibrium for the $(N+1)$-agent exchange economy $\mathcal{E}_n$, $\alpha^n_{N+1}(\chi^H - z) > 0$ implies $(\chi^H - z) \in \mathcal{B}^\perp$ is the cheapest optimal consumption for agent $N+1$; that is, $p^n(\chi^H - z) \leq p^n(\chi^H - y)$ for all $y \in \mathcal{B}$. This means that z maximizes profit among all $y \in \mathcal{B}$. Other requirements of a strong equilibrium follow immediately from the fact that (p^n, α) is a strong equilibrium of the exchange economy $\mathcal{E}_n$. Hence, (p^n, β) is a strong equilibrium of the production economy with nontransferable utility $\tilde{\mathcal{E}} = \{(u_i, b_i)_{i=1}^N, \mathcal{I}\}$.

The Pareto-efficiency of strong equilibria follows from standard arguments. □

Proof of the Corollary: Fix $\tilde{\mathcal{E}}_c = \{(u_i, 1)_{i=1}^N, \{c_i\}_{i=1}^N, \mathcal{I}\}$, a production economy with nontransferable utility and modular constraints. Let $\tilde{\mathcal{E}} = \{(u_i(c_i, \cdot), 1)_{i=1}^N, \mathcal{I}\}$. Notice that

$\tilde{\mathcal{E}}$ is not a production economy with nontransferable utility as $\operatorname{dom} u_i(c_i, \cdot) \neq X$ so that Theorem 4 does not directly apply. However, similar to the proof for Theorem 3, we can define $\tilde{\mathcal{E}}^* = \{(u_i, 1)_{i=1}^N\}$ as a production economy with nontransferable utility and general consumption sets if (1) u_i satisfies the substitutes property for all i; (2) $\mathcal{I}$ is a matroid and (3) for all $z \in \mathcal{B}$, there exist $x_1, \cdots, x_{N+1}$ such that $\sum_{k=1}^{N+1} x_k \leq z$ and $x_k \in \operatorname{dom} u_i(c_i, \cdot)$ for all k, i. Clearly, $\tilde{\mathcal{E}}$ satisfies all those conditions. Then a combination of ideas in the proofs of Theorem 3 and Theorem 4 can guarantee the existence of a strong equilibrium(p, α) in $\tilde{\mathcal{E}}$. It is easy to see (p, α) is also a strong equilibrium of $\tilde{\mathcal{E}}_c$. □

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