Sovereign Debt, Default Risk, and the Liquidity of Government Bonds

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Abstract

The secondary market for sovereign bonds is illiquid and the liquidity is endogenous. Such endogenous liquidity has important effects on the credit spread and the probability of default. To study equilibrium implications of such liquidity, I integrate directed search in the secondary market into a macro model of sovereign default. The model generates liquidity endogenously because investors in the secondary market face a trade-off between the transaction costs and the trading probability. This trade-off varies with the aggregate state of the economy, creating a time-varying liquidity premium over the business cycle. I show that trade flows in the secondary market significantly affect the price of sovereign bonds and amplify the effect of default risk on credit spreads. The importance of liquidity in the secondary market increases when the economic conditions of the issuing country worsen. Illiquidity increases with default risk and accounts for a sizable fraction of credit spreads, ranging from 10% to 50%.

Keywords: Sovereign Debt; Default; Liquidity; OTC Markets; Directed Search.
JEL classification: D83, E32; E43; F34; G12.

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1 Introduction

A country's government often issues long-term bonds to sell in the international market. Before such sovereign bonds mature, they are traded in over-the-counter (OTC) markets. Trading in this secondary market is decentralized, costly and time consuming.[1] The liquidity of the secondary market affects not only the price of outstanding bonds, but also the price of new issuances and, hence, the government's decision on whether to default on the bonds. In turn, the liquidity of sovereign bonds *endogenously* depends on the state of the economy in addition to trading frictions in the secondary market. In this paper, I study sovereign default incorporating the role of the liquidity of the secondary market, which has been largely ignored in the sovereign default literature originated in the seminal work of Eaton and Gersovitz (1981).[2] To endogenize the liquidity of bonds, I integrate search frictions in the secondary market into a general equilibrium model of sovereign debt with default risk. I use the model to qualitatively and quantitatively study the role of liquidity of bonds on interest rate spreads and the government's decision on debt issuance and default. In addition, the model provides insights on the effects of policy interventions in the secondary markets.

Recent empirical studies have emphasized the role of liquidity on sovereign bond markets. For example, Beber, Brandt, and Kavajecz (2009) document the effects of liquidity on sovereign credit spreads over safe bonds, especially during times of heightened market uncertainty. For the Euro area, Nguyen (2014) reports that, during the 2010-2012 European debt crisis, even countries with very liquid bonds faced illiquidity periods. She documents that the relative bid-ask spread - a standard measure of liquidity - of Italian bonds reached 667 basis points,[3] an unprecedented level for Italian bonds whose bid-ask spreads are usually below 50 basis points.[4] Large bid-ask spreads were also observed in Ireland and Portugal.

[1] See, for example, Duffie (2012) for details on OTC markets for bonds and World-Bank and IMF (2001) for details structure of sovereign debt markets.

[2] The only exception in the literature is Passadore and Xu (2018), who impose exogenous trading frictions on individuals selling sovereign bonds in the secondary market.

[3] The relative bid-ask spread, a standard measure of liquidity, is defined as

$$S^{B-A} \equiv \frac{p^A - p^B}{\frac{1}{2}\left(p^A + p^B\right)} \times 10,000,$$

where p^A is the ask price (the price at which bond dealers sell bonds), p^B is the bid price (the price at which bond dealers buy bonds), and the ratio is multiplied by $10,000$ to measure it in basis points.

[4] For more details on the importance of liquidity shock in the Eurozone debt crisis, see, for example, Calice,

However, the most extreme example is Greece. Bloomberg data shows that bid-ask spreads for 10 year Greek bonds were about $2,000$ basis points, on average, during the fourth quarter of 2011, a couple of months before the debt restructuring of March, 2012.

The size of bid-ask spreads during the European debt crisis makes endogenous liquidity in the secondary market interesting on its own. The policy responses of the European Central Bank (ECB) and the International Monetary Fund (IMF) make it even more so. In May, 2010, the ECB launched the Securities Market Programme (SMP) that involved direct sovereign bond purchases in secondary markets to improve liquidity conditions and help stabilize distressed sovereign bond yields. As stated by the ECB Press Release of May 10th, 2010, one of the goals of the SMP interventions was "to ensure depth and liquidity in those market segments which are dysfunctional."[5] Bond purchases in secondary markets do not reduce the debt-to-GDP ratio of the issuing country nor directly improve its fiscal deficit or aggregate production. How does this type of intervention help overcome a debt crisis? When are such interventions effective and better than others? The literature does not provide answers to these questions because it has largely abstracted from endogenous liquidity.

To model endogenous liquidity, I incorporate frictions in bond markets in the same spirit as Shi (1995) and Trejos and Wright (1995), for fiat money, and Duffie, Garleanu, and Pedersen (2005), for corporate bonds. More specifically, the model assumes that the sovereign government sells its debt in a centralized primary market to dealers that act as intermediaries between the government and foreign investors.[6] Those dealers then trade bonds with investors in a decentralized secondary market. In the model, dealers do not have reasons to hold bonds other than to re-sell them to investors. For their intermediation service, dealers charge investors a transaction fee. On the other side of the market, investors demand sovereign bonds to maximize expected returns and optimize their portfolio composition. In

Chen, and Williams (2013) and Nguyen (2014). For emerging market bonds, see Hund and Lesmond (2008), who estimate that liquidity frictions are the main source of differences across credit spreads for countries within the same credit rating category.

[5] See the press release at https://www.ecb.europa.eu/press/pr/date/2010/html/pr100510.en.html. For more details on the SMP, see Trebesch and Zettelmeyer (2018).

[6] In reality, only a few large banks can trade in primary bond markets. All other investors, such as individual investors, institutional investors, and investment funds, need to buy bonds in OTC markets. In the model, dealers represent agents of those banks. For the case of Greece, the list of primary dealers can be found at https://www.bankofgreece.gr/Pages/en/Markets/HDAT/members.aspx.

order to be able to trade a bond, investors need to meet dealers in OTC markets, which are subject to search frictions. An investor's valuation for a bond incorporates the cost of intermediation fees, the expected time to trade, and the default risk.

In the secondary market, search is competitive (or directed).[7,8] In every period, dealers and investors choose to visit one specific submarket in order to search for a trading counterpart. Each submarket is characterized by a transaction fee that the investor needs to pay to the dealer if they trade. A matching technology determines the number of trades in each submarket given the numbers of dealers and investors. In equilibrium, investors and dealers face a trade-off between the intermediation fee and the trading probability. For an investor, the higher the intermediation fee an investor pays to a dealer, the higher the investor's probability of trading. For a dealer, the higher the intermediation fee the dealer charges, the lower the probability of trading. Facing this trade-off, the entry decision of investors and dealers into submarkets endogenously determine the liquidity of bonds, as well as transaction fees, the trading volume, and the frequency of trades.

On the qualitative side, the model generates two main new insights. First, trade flows between investors and dealers in the secondary market affect the price of newly issued government bonds in the primary market. For example, if the investors' flow of orders in the secondary market to buy bonds from dealers increases, the demand for bonds by dealers in the centralized primary market increases. In this case, the bond price must increase to clear the market and restore equilibrium. Even if the government does not change debt issuances.

Second, there is a positive correlation between default risk and illiquidity in the secondary market that amplifies the bonds' interest rates. In equilibrium, the higher the default probability, the lower the incentives for investors to purchase bonds and the smaller the mass of investors that show up in the secondary market as buyers. Moreover, as the probability of default increases, investors holding bonds have higher incentives to find a counterpart to

[7] This modeling is consistent with evidence on the structure of secondary markets. Li and Schurhoff (2018) analyze dealer networks for US municipal bonds and document that: (*i*) there is a systematic price dispersion across dealers, with 20-40% dealer-specific variation in markups, (*ii*) trading costs increase with centrality of the dealer in the network and central dealers charge up to 80% larger spreads, (*iii*) central dealers place bonds more readily with investors than other dealers, consistent with the notion that they face smaller search frictions, and (*iv*) central dealers provide more liquidity immediacy to investors than peripheral dealers. This evidence suggests that investors who desire to trade bonds have a wide range of dealers to visit and that they face a trade-off between the markup charged by dealers and the trading probability.

[8] Well-known examples of directed search models are Peters (1991), Montgomery (1991), Moen (1997), Julien, Kennes, and King (2000), Burdett, Shi, and Wright (2001), and Shi (2001).

sell the bonds. Thus, if the sovereign government wants to issue a certain amount of debt when default probability is high, it will need to induce more dealers to sell bonds in the secondary market. Because dealers' revenues are transaction fees, these fees must increase in order to induce more dealers to sell bonds in the secondary market. A higher transaction fee discourages even more investors from purchasing bonds. As a result, the price of bonds in the primary market must fall sufficiently to induce investors to purchase the desired amount of newly issued debt. Therefore, the price for bonds in the primary market falls by more than the amount required to compensate investors for the increased default risk.

I calibrate the model to quantify the effects of liquidity frictions on interest rates. Liquidity frictions significantly contribute to credit spreads. Pricing the pure default probability and comparing it with total spreads, I find that in normal times the model generates credit spreads that are between $1.5 - 2$ times larger than credit spreads needed to compensate only for default risk. Additionally, credit spreads are more sensitive to negative output shocks. After a negative output shock, the response in credit spreads at impact is more than 4 times larger than the response in default risk alone. Thus, liquidity frictions and their interactions with default risk create a quantitatively important amplification mechanism. Finally, I use the model to decompose credit spreads in Greece before the debt restructure of 2012.[9] I find that between $2006Q1 - 2011Q4$ the contribution of liquidity frictions to total spreads varies between $10\% - 50\%$, reaching 50% right before the Greek debt restructuring.

Related Literature

Sovereign Default. The large body of research on sovereign debt with strategic default originates in Eaton and Gersovitz (1981), with a strong quantitative focus after the work by Neumeyer and Perri (2005), Aguiar and Gopinath (2007), and Arellano (2008). Because debt has long term, my work is closer to Hatchondo and Martinez (2009) and Chatterjee and Eyigungor (2012)[10]. I contribute to this literature by endogenizing liquidity in the sec-

[9] I focus on Greece because it is one of the few countries with a default episode since sovereign bonds trade in OTC markets. Among those countries, Greece is the only country with available data on liquidity of the secondary market.

[10] I abstract from maturity choice decisions. For papers that allow for maturity choice in models with perfectly liquid sovereign bonds see for example Arellano and Ramanarayanan (2012), Bocola and Dovis (2016), and Sanchez, Sapriza, and Yurdagul (2018). See Kozlowski (2018) for a model of maturity choice in the context of illiquid corporate bonds.

ondary market to study the equilibrium implications for the interest rate spreads and default probability of sovereign bonds. The framework provides a tool to understand how liquidity and risk premia interact in equilibrium and how quantitatively important the interaction is for sovereign bonds' credit spreads, government revenues from new debt issuance, and incentives to default. Furthermore, the framework uses bid-ask spreads and volumes traded in the secondary market to quantify the importance of liquidity frictions, without compromising on the tractability of the model. Most papers in the literature on sovereign default have abstracted from the secondary market completely. Exceptions Broner, Martin, and Ventura (2010) and Bai and Zhang (2012), incorporate the secondary market into their model, but they assume the secondary market to be frictionless.[11]

The most closely related paper is Passadore and Xu (2018), who build on Chen, Cui, He, and Milbradt (2018) to incorporate trading frictions in the secondary market into a standard model of sovereign default. Passadore and Xu (2018) impose two assumptions that differ from mine. First, they assume that investors purchase bonds in the primary market, although investors re-sell bonds in the secondary market. Second, they exogenously fix the probability of being able to re-sell bonds in the secondary market. Because of these assumptions, their model is unable to capture how liquidity in the secondary market endogenously responds to changes in the issuing country's economic conditions. To capture endogenous liquidity, I assume that investors both buy and sell bonds in the secondary market and use directed search to endogenize the trading probabilities in this market.

Endogenous liquidity is important to assess the effects of policy interventions in the sovereign bond market, such as the securities market programme (SMP) implemented by the ECB in 2010-2011.[12] In my model, interventions of this kind directly affect the price, the bid-ask spreads, and the liquidity premium of newly issued bonds, by changing the net demand for bonds in the secondary market. In contrast, if trading probabilities are exogenous, such interventions may not affect the price of newly issued bonds nor the terms

[11] This paper also relates to recent papers on the European debt crisis. For example, Bocola and Dovis (2016) study the effects of rollover risk on Italian credit spreads during the crisis, Dovis and Kirpalani (2018) focus on the effects of bailout expectations on interest rate spreads dynamics, and Gutkowski (2018) studies the role of sovereign bonds as collateral and the consequences of secondary markets disruptions on output, employment, and investment. I contribute to those papers by incorporating time-varying liquidity frictions and quantifying their effect on the dynamics of interest rate spreads in Greece.

[12] In section 4.4 I discuss two alternative ways of modeling such interventions and their implications.

of trade in bilateral meetings, because the interventions do not affect the outstanding level of debt or default probabilities.

Asset Liquidity in OTC Markets. This paper is also related to the literature following Duffie et al. (2005) where assets are traded in OTC markets and liquidity is modeled using search and matching frictions.[13] In this literature, there are two type of investors, high and low valuation investors. In equilibrium, high valuation investors want to purchase debt, while low valuation investors want to sell because they suffer from a cost for holding the asset. In order to trade, investors need to visit a market with search frictions.

In my model, the trading structure in the secondary market is close to Lagos and Rocheteau (2009) and Lester, Rocheteau, and Weill (2015), where (*i*) trades occur only between a dealer and an investor but not directly between investors; and, (*ii*) dealers do not need to hold inventories as they have permanent access to the centralized primary market that serves as a clearing system.[14] The main contribution of my paper to this literature is to build a tractable equilibrium model with endogenous liquidity over the business cycle and strategic default decisions, while this literature usually focuses on steady states and no default. Another contribution is to endogenize the supply of assets in the secondary market. The response of this supply to market conditions is critical for understanding endogenous liquidity, but it is absent in this literature.[15]

Previous studies also consider endogenous default decisions on corporate bonds. One example is He and Milbradt (2014). However, the model assumes a stationary environment where both the characteristics and the supply of assets are fixed over time. Chen et al. (2018) extend the analysis of He and Milbradt (2014) to allow the aggregate state of the economy to take two values and show how exogenous reductions in trading probabilities increase credit spreads. As a result, their model misses two critical features of my model: an endogenous supply of bonds and endogenous trading probabilities in the secondary market. They assume that the supply of bonds is fixed over time and that the selling probability is

[13] This literature derives for monetary theory models of fiat money in Shi (1995) and Trejos and Wright (1995).

[14] For a model with market makers (dealers) that accumulate inventories, see Weill (2007).

[15] Exceptions are Geromichalos and Herrenbrueck (2018) where two bond issuers determine the endogenous supply of competing bonds in a steady state equilibrium and Kozlowski (2018) where companies choose the optimal size of investment projects and larger projects require larger asset issuance to raise enough funds. However, in both papers the asset supplied is chosen once at $t = 0$ and kept fixed afterwards.

exogenously fixed. As explained above, both features are necessary for understanding how liquidity of sovereign bonds responds to changes in the state of the economy and affects the price of new issuance.

Layout. The remainder of the paper is organized as follows. Section 2 describes the environment of the model economy. Section 3 defines an equilibrium of this economy and characterizes its main theoretical implications. Section 4 illustrates the new channels in the model. Section 5 calibrates the model and provides quantitative results. Section 6 studies the Greek debt crisis. Finally, section 7 concludes. The Appendix contains proofs, the solution algorithm, the description of the data used for the calibration of the model, and some additional details on the calibration.

2 Environment

Time is discrete and infinite. There are three types of agents: (i) a sovereign government; (ii) dealers; and (iii) foreign investors.

The country of interest faces a random endowment process $y_t \in \bar{Y} \equiv [y_{\min}, y_{\max}]$, which is Markovian. The government maximizes the lifetime value of the representative household in the country, given by

$$\sum_{t=0}^{\infty} \beta^t U(c_t),$$

where $U(\cdot)$ is strictly increasing and concave, c_t is household's consumption, and $\beta \in (0,1)$ is the discount factor. The government can save or borrow from international credit markets, described later in this section.

At the beginning of each period the outstanding debt is B_t. The government chooses whether to default or not on its outstanding debt obligations. If the government defaults on the debt, there are two costs. The first cost is a temporary exclusion from financial markets that prevents the government from borrowing or saving while in financial autarky. While the government is in default, every period it can re-gain access to financial markets with exogenous probability $\phi \in (0,1)$.[16] Upon re-gaining access to credit markets, the government

[16] For models with endogenous market re-access see Yue (2010) and Benjamin and Wright (2013).

starts with no outstanding debt. The second cost is an output cost, i.e., under default the endowment is given by $h(y_t) = y_t(1 - \omega(y_t))$, where $\omega(y_t)$ is a (weakly) increasing function of y_t.

If the country does not default on debt, the country is in good credit conditions. In this case the government can save or issue debt, $B_{t+1} \in \bar{B} \equiv [B_{\min}, B_{\max}]$, in the primary bond market ($B > 0$ means debt while $B < 0$ means credit). When the government issues debt, the bonds are sold to dealers in the primary market, which is Walrasian.[17] The sovereign government takes as given the price schedule of bonds $p(y_t, B_t, \Omega_t, B_{t+1})$, which is determined in equilibrium. The arguments of this pricing schedule are the current level of endowment, which allows investors to forecast next period's endowment, the current level of debt, B_t, the current distribution of investors with respect to their types and asset holdings, Ω_t, and B_{t+1}.

The maturity of bonds is determined by a parameter λ. As it is usually assumed in the literature, each unit of debt matures with probability $\lambda \in [0, 1]$ in every period, independently of when that unit of debt was issued (e.g., Hatchondo and Martinez (2009)). Thus, the average time to maturity of each bond is $\frac{1}{\lambda}$ periods. Each unit of unmatured bond pays a coupon $z \geq 0$ every period. Therefore, the period t budget constraint of the sovereign government that is in good credit conditions is given by

$$c_t + [\lambda + (1 - \lambda) z] B_t \leq y_t + p(y_t, B_t, \Omega_t, B_{t+1}) [B_{t+1} - (1 - \lambda) B_t].$$

The left hand side represent expenditures on consumption, coupon payments, and repayment of matured bonds. The right hand side represents incomes from endowment and new debt issuances.

Dealers are risk-neutral. They can access the primary market without cost and purchase bonds issued by the government at the competitive price, $p(y_t, B_t, \Omega_t, B_{t+1})$. That is, there are no frictions in the primary market. Also, it is assumed that dealers have permanent access to this primary market so that they only purchase bonds when they want to sell it to an investor. This simplification avoids working with dealers that hold inventories of the bond.

[17] The government could organize an auction to sell the bonds. Since in this model there is complete and perfect information about valuations of the bonds, the auction could be designed to extract all the surplus from dealers. That is, dealers would be acting as if there is perfect competition among them.

In addition, dealers have access to a frictional secondary market where they can trade with foreign investors. This secondary market is characterized by directed search. Specifically, there is a continuum of submarkets that are characterized by the transaction fee that dealers charge to investors in case a trade occurs. Entry into submarkets is competitive. To enter any submarket, a dealer needs to pay a per-period flow cost, γ.[18] A dealer compensates the entry cost by the bid-ask spread between submarkets. As the only intermediators between the primary and the secondary market, dealers collect the orders from the secondary market and clear the net demand (or supply) in the primary market at the end of each period.

Investors in the secondary market are from foreign countries with a fixed measure $\bar{I} \geq B_{\max}$. They can trade bonds only by meeting a dealer. In order to trade, investors choose the submarket to enter. That is, investor's search for dealers is also directed. For simplicity, I assume that investors can only hold either zero or one unit of the bond. I denote an investor's bond holdings $a \in \{0, 1\}$. There are two types of investors, denoted ℓ and h. Type $i \in \{\ell, h\}$ investors have preferences u_i over the bond, with $u_h > u_\ell$, in addition to other consumption c. These different preferences will generate gains from trade for investors in equilibrium. Investors enter the economy as type h and without bonds. In equilibrium these investors will be the ones willing to purchase sovereign debt. Once a type h investor acquires a unit of the bond, the investor starts to face a preference shock with probability $\zeta \in (0, 1)$, which changes the investor into type ℓ. In the equilibrium, type ℓ investors are the ones willing to sell the bonds in the secondary market. For simplicity, I assume that once an investor that gets rid of the bond, the investor leaves the economy and is replaced by a new type h investor who does not have the bond. There are two ways that an investor can get rid the bond: (i) by selling it to a dealer in the secondary market, or (ii) by waiting for the bond to mature, which occurs every period with probability λ. Finally, investors have access to a risk-free, perfectly liquid, one period zero-coupon bond that pays an exogenous return $r > 0$.

In each submarket in the secondary market, there is a constant returns to scale order processing technology denoted $\mathcal{M}(d, n)$, where d is the number of dealers in a submarket and n is the number of investors. Each order is equally likely to be executed at any time. The probability of an order being executed is given by $\alpha(\theta) \equiv \frac{\mathcal{M}(d,n)}{n} = \mathcal{M}(\theta, 1)$, with

[18] This cost can be interpreted as a constant marginal cost of allocating a dealer into a submarket, for a bank that participates in the primary market.

$\theta \equiv \frac{d}{n}$. The amount of orders executed by a dealer in a period of time is then $\rho(\theta) \equiv \frac{\mathcal{M}(d,n)}{d}$. I assume that $\mathcal{M}(\cdot,\cdot)$ is such that $\alpha(0) = 0$, $\alpha(\infty) = 1$, $\rho(\infty) = 0$, and $\alpha(\cdot)$ is strictly increasing and concave.

Within each period, the timing of actions within a period is as follows:

1. Shock y is observed. The government decides whether or not to default. If the government defaults the bond is not available as a possible investment choice for investors anymore, and investor's and dealer's problems are irrelevant.

2. If the government repays, then the government chooses issuances, B', optimally.

3. A fraction λ of B matures. Their owners are replaced by h investors without bonds. Principal of matured bonds is paid to current bond owners. Unmatured bonds pay coupon, z, and yield utility u_i to investors of type $i \in \{\ell, h\}$.

4. Investors' preference shock is realized and a fraction ζ of type h investors with a unit of a bond become type ℓ investors.

5. The centralized primary market and the decentralized secondary markets open. In the centralized primary market, the government and dealers trade at a competitive price. Investors and dealers decide optimally which submarket to visit and, those who meet a counterpart, trade in the secondary market.

Notice that by the time investors make their submarket choice decisions in step 5, government's debt issuance B' is already known. Thus, if the government is under good credit conditions, the relevant state of the economy for investors is given by $s_t = (y_t, B_t, \Omega_t, B_{t+1}) \in \overline{S}$, where $\overline{S}$ represents the space of all possible values for s_t. That is, the current aggregate state of the economy consists of the current level of output, y_t, the outstanding level of debt at the beginning of period t, B_t, the distribution of investors' types and bond holdings at the beginning of period t, Ω_t, and government's choice of next period's level of debt, B_{t+1}.

In the following subsections I formulate the government's, investors', and dealers' problems.

2.1 Government

At the beginning of each period, the government chooses whether to default, $\delta = 1$, or repay, $\delta = 0$, and the optimal debt issuance in case of repayment. The government's value function is:

$$V(y, B, \Omega) = \max_{\delta \in \{0,1\}} \left\{ (1-\delta) V^R(y, B, \Omega) + \delta V^D(y) \right\}, \tag{1}$$

where $V^R(\cdot)$ is the value of repaying debt obligations and $V^D(\cdot)$ is the value of default. The values come from domestic households' intertemporal utility. The value of default is given by

$$V^D(y) = U(h(y)) + \beta \mathbb{E}_{y'|y} \left[\phi V(y', 0, \Omega_0) + (1-\phi) V^D(y') \right]. \tag{2}$$

That is, if the government decides to default, it does not need to repay outstanding debt so it will consume the total output of the current period. However, there is an output cost associated to the default decision so today's consumption is given by the function $h(y) = y_t(1-\omega(y_t))$. In addition, the continuation value is a weighted sum of the value of re-gaining credit access, which happens with probability ϕ, and starting next period in default, which happens with probability $(1-\phi)$. Notice that if the government re-gains access to credit, it starts with zero outstanding debt and investors distribution Ω_0, with all investors being high type with no bond holdings. In the case of not defaulting the debt, the government chooses consumption of the domestic household, c, and the new stock of debt, B'. The value of repaying debt is given by

$$\begin{aligned} V^R(y, B, \Omega) &= \max_{c, B'} \left\{ U(c) + \beta \mathbb{E}_{y'|y} V(y', B', \Omega') \right\}, \ s.t.: \\ [BC_G] &: \ c + [\lambda + (1-\lambda) z] B = y + p(y, B, \Omega, B') [B' - (1-\lambda) B]. \end{aligned}$$

z is the coupon paid to the bond. The price schedule $p(y, B, \Omega, B')$ is determined endogenously in the primary debt market and depends on the amount of new debt issued by the government. The government internalizes how the price of debt issuances changes as it changes the level of debt issued for next period but it takes as given this pricing function.

Using the constraint to substitute c, the objective function becomes:

$$
\begin{aligned}
V^R(y,B,\Omega) &= \max_{B'}\{U(y+p(y,B,\Omega,B')[B'-(1-\lambda)B]-[\lambda+(1-\lambda)z]B) \\
&\quad +\beta\mathbb{E}_{y'|y}V(y',B',\Omega')\}.
\end{aligned} \tag{3}
$$

2.2 Foreign Investors

Foreign investors trade the bond in the secondary market. To describe an investor's decision, I denote the set of submarkets as $\bar{F}=[f_{\min},f_{\max}]$ which is a compact space. To define this space, let $f_{\min}\equiv\gamma$ since no dealer will be willing to enter a submarket that does not pay enough to cover the entry cost. In addition, the price for a bond that matures with probability λ, with a flow return $(z+u_h)$ every period until maturity, and that never defaults would be $\frac{\lambda+(1-\lambda)(z+u_h)}{\lambda+r}$. Thus, we can define $f_{\max}\equiv\frac{\lambda+(1-\lambda)(z+u_h)}{\lambda+r}$ since no investor would be willing to pay a higher intermediation fee, even if the trading probability is equal to one.

Denote the value functions of a type i investor holding a unit of the bond as I_i^a, where $i\in\{\ell,h\}$ and $a\in\{0,1\}$. Note that the combination $i=\ell$ and $a=0$ never occurs because, after getting rid of the bond, the investor leaves the economy and is replaced by a new type h investor with $a=0$. I formulate the decision problem for an investor below.

2.2.1 High Type Investors without a Bond

For each state $s=(y,B,\Omega,B')\in\overline{S}$, the value for a type h investor with $a=0$ is given by

$$
\begin{aligned}
I_h^0(s) &= \max_f\{\alpha(\theta(f))\left[-p(s)-f+\frac{1}{1+r}\mathbb{E}_{y'|y}[1-\delta(y',B',L_1')]I_h^1(s')\right] \\
&\quad +\frac{1-\alpha(\theta(f))}{1+r}\mathbb{E}_{y'|y}[1-\delta(y',B',L_1')]I_h^0(s')\}.
\end{aligned} \tag{4}
$$

The investor chooses optimally which submarket to visit (how much transaction fee, f, he wants to pay) in order to purchase a unit of the sovereign bond. In submarket f, the investor will be able to trade with a dealer with probability $\alpha(\theta(f))$. Once matched, the investor purchases a unit of the bond after paying its price in the primary market, $p(s)$, plus the transaction fee, f. In addition, holding a bond derives a continuation value of being a type h investor in the next period, $I_h^1(s')$, provided that the government does not default on the bond when next period starts. The continuation value is discounted at the rate r, which is the rate of return on the perfectly liquid, risk-free bond. If the investor is not matched with a dealer, which happens with probability $1-\alpha(\theta(f))$, the investor receives the discounted continuation value of being type h with $a=0$ at the beginning of the next period, conditional on the government not defaulting on the bond. If the government defaults on the bond, the continuation value of holding the bond is zero.

2.2.2 High Type Investors with a Bond

The value function for a type h investor with $a=1$ is as follows,

$$\begin{aligned} I_h^1(s) &= \lambda+(1-\lambda)(u_h+z)+\zeta\left[I_\ell^1(s)-\lambda-(1-\lambda)(u_\ell+z)\right] \\ &\quad +(1-\lambda)(1-\zeta)\max_f\{\alpha(\theta(f))[p(s)-f] \\ &\quad +\frac{[1-\alpha(\theta(f))]}{1+r}\mathbb{E}_{y'|y}[1-\delta(y',B',L_1')]\,I_h^1(s')\}. \end{aligned} \tag{5}$$

The investor obtains the face value, 1, if the bond matures, which occurs with probability λ. If the bond does not mature, the investor enjoys utility u_h from holding it and the coupon payment z. In addition, the investor is hit by the preference shock with probability ζ, which changes the investor to a type ℓ investor with $a=1$. The payoff in this case is described below. The other terms multiplied by ζ inside the squared bracket are there to avoid double counting flows of payments and utility of the bond (see equation 7 below). If the investor is not hit by the preference shock and the bond does not mature, the investor might be matched to a dealer in submarket f or not. In the case of a match the investor sells the bond and receives the price of the bond $p(s)$ minus the transaction fee, f, paid to the dealer. If the investor is not matched with a dealer there is no trade and the investor receives the discounted continuation value of being type h with $a=1$ at the beginning of next period, conditional on the government not defaulting on the bond. If the government defaults on the bond, the continuation value is zero.

For a type h investor with $a=1$ to chose to enter a submarket $f>0$, the price $p(s)$ should be very attractive as these investors like holding the bond. The following condition is necessary and sufficient for this to happen:

$$f > 0 \iff p(s) > \frac{1}{1+r}\mathbb{E}_{y'|y}\left[1-\delta(y', B', L_1')\right] I_h^1(s'). \tag{6}$$

Notice that, since $I_h^0(s') \geq 0$ and the benefit of purchasing the bond for the type h investors with $a = 0$ is given by

$$\frac{1}{1+r}\mathbb{E}_{y'|y}\left[1-\delta(y', B', L_1')\right]\left[I_h^1(s') - I_h^0(s')\right],$$

then, whenever (6) holds no investor is willing to purchase a bond. Thus, only investors trying to sell will show up in secondary markets, which can only occur in the case that the sovereign government decides to retire a large amount of bonds from the market. Therefore, re-purchasing bonds is expensive for the government because it needs to pay a price higher than the valuation of type h investors.

2.2.3 Low Type Investors with a Bond

The value function for a type ℓ investor with $a = 1$ is given by

$$\begin{aligned} I_\ell^1(s) &= \lambda + (1-\lambda)(u_\ell + z) \\ &\quad + (1-\lambda)\max_f\{\alpha(\theta(f))\left[p(s) - f\right] \\ &\quad + \frac{\left[1-\alpha(\theta(f))\right]}{1+r}\mathbb{E}_{y'|y}\left[1-\delta(y', B', L_1')\right] I_\ell^1(s')\}. \end{aligned} \tag{7}$$

If the bond matures, with probability λ, the investor receives the principal of the bond, 1, and exits the market. If the bond does not mature, the investor receives the coupon payment z and the flow utility u_ℓ. In addition, the investor chooses optimally a submarket, f, to sell the bond. If there is a match, the investor receives the price of the bond, $p(s)$, minus the transaction fee, f. If the investor is not matched with a dealer, the investor receives the discounted continuation value of being type ℓ with $a = 1$ at the beginning of next period, conditional on the government not defaulting on the bond. If the government defaults on the bond, the continuation value is zero.

2.3 Dealers

Dealers participate competitively in debt markets. Each dealer chooses an intermediation fee to be charged to investors for intermediation services. To enter any given submarket a dealer needs to pay a flow cost $\gamma > 0$. A dealer posts the intermediation fee to maximize expected profits:

$$\Pi = \max_{f \in F} \{\rho\left(\theta\left(f\right)\right) f - \gamma\}, \tag{8}$$

where $\rho\left(\cdot\right)$ represents the probability of being able to execute an order, derived from the order execution technology $\mathcal{M}$ described earlier. Competitive entry of dealers implies that the following condition of complementary slackness

$$\Pi\left(f\right) \leq 0 \text{ and } \theta\left(f\right) \geq 0. \tag{9}$$

Whenever expected profits for dealers are negative in a submarket f, the market tightness $\theta\left(f\right)$ in this submarket is zero since no dealers have incentives to participate in this submarket. On the other hand, whenever the market tightness is positive, a positive mass of dealers participate of the submarket, in which case expected profits for each dealer should equal zero.

Condition (9) provides a mapping from each submarket intermediation fee, f, to the tightness in that submarket. This mapping is given by

$$\theta\left(f\right) = \left\{ \begin{array}{cc} \rho^{-1}\left(\frac{\gamma}{f}\right) & \text{if } \Pi\left(f\right) = 0, \\ 0 & \text{otherwise.} \end{array} \right. . \tag{10}$$

2.4 Market Clearing

The primary market is Walrasian and only government and dealers can access it. In each state $s \in \overline{S}$ and conditional on the government being in good credit standards, the price $p\left(s\right)$ must clear the bonds primary market.

Recall that $\bar{I}$ is the total mass of investors. Let H_0 be the mass of type h investors with $a = 0$, H_1 the mass of type h investors with $a = 1$, and L_1 the mass of type ℓ investors with $a = 1$. All of these are measured at the beginning of a period. At the beginning of any given period, $\bar{I} = H_0 + H_1 + L_1$. In addition, all outstanding bonds must be held by some investors. That is, $B = H_1 + L_1$. Using this notation, the total supply of bonds in the primary market is given by

$$\underbrace{\max\{B',0\} - (1-\lambda)B}_{\text{Government's supply}} + \underbrace{\alpha\left(\theta_\ell^1\right)\zeta(1-\lambda)H_1}_{\text{New sellers' supply}}$$
$$+\underbrace{\alpha\left(\theta_\ell^1\right)(1-\lambda)L_1}_{\text{Old sellers' supply}} + \underbrace{\alpha\left(\theta_h^1\right)(1-\lambda)(1-\zeta)H_1}_{\text{Potential type } h \text{ sellers}}.$$

The first term is the government's new bond issuances. The operator $\max\{B',0\}$ captures the possibility that the government chooses $B' < 0$, in which case the government can at most demand the $(1-\lambda)B$ outstanding bonds in the primary market. The second term is the supply of bonds from "new sellers", i.e., type h investors who are hit by the preference shock (ζ of them) to become type ℓ investors. They sell their bond holdings. Since a fraction λ of them will see their bond mature, only $1-\lambda$ will be trying to sell the bond. Among the sellers, only a fraction $\alpha\left(\theta_\ell^1\right)$ will get matched with a dealer. θ_ℓ^1 is the tightness in the submarket optimally chosen by type ℓ investors. The third term above is the supply of bonds by "old sellers." Similarly to the second term, it consists of those type ℓ investors that are holding a bond who were not able to sell it in the past, for which the bond did not mature, and who got matched with a dealer. Finally, the fourth term is the potential supply of bonds by type h investors with $a = 1$, which is positive if and only if condition (6) holds.

On the other side of the primary market, the demand for bonds, are the buying orders received by dealers from the fraction of type h investors, $\alpha\left(\theta_h^0\right)$:

$$\underbrace{\alpha\left(\theta_h^0\right)H_0}_{\text{Old Buyers' demand}} + \underbrace{\alpha\left(\theta_h^0\right)\lambda B}_{\text{New Buyers' demand}}.$$

The first term represents "old buyers," i.e. type h investors with $a = 0$ who are in the market from the last period. The second term, represents the "new buyers," i.e. those type h investors who entered the economy in the current period to replace the investors that left the economy after their unit of the bond matured. The tightness θ_h^0 corresponds to the submarket optimally chosen by type h investors with $a = 0$.

The distribution of investors types and bond holdings is Ω, which consist of the three measures $\{H_0, H_1, L_1\}$. However, since $H_1 = B - L_1$, and since all type ℓ investors visit the same submarket independently of their type in the previous period, the total supply from investors trying to sell their unit of the bond can be written as

$$\alpha\left(\theta_{\ell}^{1}\right)(1-\lambda)\left[\zeta B+(1-\zeta) L_{1}\right].$$

In addition, $\bar{I}=H_0+H_1+L_1=H_0+B$, and so $H_0=\bar{I}-B$. Hence, I define the excess demand function for each state $s \in \overline{S}$ as

$$\begin{aligned} ED(s) \equiv & \underbrace{\alpha\left(\theta_{h}^{0}(s)\right)\left[\bar{I}-(1-\lambda) B\right]}_{\text{Buyers' demand}}-\underbrace{\left[\max\{B',0\}-(1-\lambda) B\right]}_{\text{Government's supply}} \\ & -\underbrace{\alpha\left(\theta_{\ell}^{1}(s)\right)(1-\lambda)\left[\zeta B+(1-\zeta) L_{1}\right]}_{\text{Sellers' supply}}-\underbrace{\alpha\left(\theta_{h}^{1}(s)\right)(1-\lambda)(1-\zeta)\left(B-L_{1}\right)}_{\text{Potential type } h \text{ sellers}}. \end{aligned}$$

Now, the only element of Ω in this excess demand function is L_1. Therefore, we only need to keep track of L_1 instead of Ω as the aggregate state variable describing the distribution of investor types and bond holdings. As I will show later, this excess demand function is consistent with only one price $p(s)$ clearing the market, for each state s.

Finally, the law of motion for the aggregate state variable L_1 is given by

$$\begin{aligned} L_{1}' &= (1-\lambda)\left[1-\alpha\left(\theta_{\ell}^{1}\right)\right] L_{1}+\zeta(1-\lambda)\left[1-\alpha\left(\theta_{\ell}^{1}\right)\right] H_{1} \\ &= (1-\lambda)\left[1-\alpha\left(\theta_{\ell}^{1}\right)\right]\left[(1-\zeta) L_{1}+\zeta B\right]. \end{aligned} \tag{11}$$

There is no uncertainty about the future value L_1' since it is completely determined by the current state and the tightness, θ_{ℓ}^{1}, at the optimally chosen submarket f_{ℓ}^{1}.

3 Equilibrium

I first define an equilibrium for this economy in section 3.1 and then proceed to characterize the properties of the equilibrium in section 3.2.

3.1 Equilibrium Definition

The equilibrium concept used here is recursive competitive equilibrium.

Definition 1 *A Recursive Competitive Equilibrium (RCE) in this economy consists of a set of value functions* $\left\{V, V^R, V^D, I_h^0, I_h^1, I_\ell^1, \Pi\right\}$*, a set of policy functions* $\{\delta, B', f_h^0, f_h^1, f_\ell^1\}$*, a tightness function* θ*, and a pricing function* p*, such that for all* $s = (y, B, L_1, B') \in \overline{S}$*:*

1. *Given functions* $p(s)$*,* $f_\ell^1(s)$*,* $\theta(s)$*, the functions* $V(y, B, L_1)$*,* $V^R(y, B, L_1)$*,* $V^D(y)$*,* $\delta(y, B, L_1)$*,* $B'(y, B, L_1)$*, solve the sovereign government's problem in (1)-(3);*

2. *Given* $p(s)$*,* $\delta(y, B, L_1)$*,* $B'(y, B, L_1)$*,* $\theta(s)$*, the functions* $I_h^0(s)$*,* $I_h^1(s)$*,* $I_\ell^1(s)$*,* $f_h^0(s)$*,* $f_h^1(s)$*,* $f_\ell^1(s)$ *solve the investor's problem in (4), (5), and (7);*

3. *The tightness function* $\theta(s)$ *is consistent with free entry of dealers and determined by (10);*

4. *The function* $p(s)$ *clears the primary market of bonds; and*

5. *The expected law of motion for the aggregate state* L_1 *is consistent with policy functions and given by (11).*

3.2 Equilibrium Characterization

3.2.1 Government's Problem

Government's problem defined by equations $(1)-(3)$, looks exactly like a standard sovereign default model. The new channels arising due to liquidity frictions in secondary markets only enter into the problem through the budget constraint in the value of repaying debt, $V^R(y, B, L_1)$, in (3). In particular, liquidity friction affect the price schedule for new bond issuances, $p(y, B, L_1, B')$, through the state variable L_1, which measure the mass of low type investors at the beginning of current period. However, for any given price schedule the government problem satisfies standard properties in the literature. In Appendix A.1, I repeat some of the standard arguments to characterize the solution of government's problem for a given price schedule, $p(s)$. Later, I will examine the existence of the price schedule in equilibrium and how it is affected by liquidity frictions.

3.2.2 Investor's Problem

In this section I characterize investors' demand and supply for sovereign bonds in the secondary market and how trades in the secondary market affect demand and supply of bonds in the centralized primary market. In particular, I analyze how net demand for bonds that dealers bring to the primary market responds to changes in bond prices, leaving everything else constant. The main result of this section is in lemma 3. The lemma show that, for each state $s \in \overline{S}$, the net demand of bonds carried by dealers into the primary market (see equation (12)) is strictly decreasing in price. This partial equilibrium result is then used in subsection 3.2.3 to characterize equilibrium price schedule for given government policies and in subsection 3.2.4 to establish existence of an equilibrium.

Definition 2 *For a given bond price p, I say that an* <u>*investor participates in the secondary market*</u> *for bonds if there exists an intermediation fee $f \geq \gamma$ at which he would be willing to trade.*

The definition focuses on transaction fees $f \geq \gamma$ because, on the other side of the market, no intermediary is willing to trade for a fee $f < \gamma$ and, in such a case, an investor's probability of trading is zero.

To save some space, for each state $s \in \overline{S}$, I define the following default adjusted expected values

$$\begin{aligned} E_h^0(s) &\equiv \mathbb{E}_{y'|y}\left[1-\delta\left(y', B', L_1'\right)\right]\left[I_h^1(s') - I_h^0(s')\right], \\ E_\ell^1(s) &\equiv \mathbb{E}_{y'|y}\left[1-\delta\left(y', B', L_1'\right)\right] I_\ell^1(s'). \end{aligned}$$

Also, for each $s \in \overline{S}$, define investors' aggregate net demand for bonds as

$$\begin{aligned} ND(s) \equiv\ & \underbrace{\alpha\left(\theta_h^0(s)\right)\left[\overline{I} - (1-\lambda)B\right]}_{\text{Buyers' demand}} - \underbrace{\alpha\left(\theta_\ell^1(s)\right)(1-\lambda)\left[\zeta B + (1-\zeta)L_1\right]}_{\text{Sellers' supply}} \\ & \underbrace{-\alpha\left(\theta_h^1(s)\right)(1-\lambda)(1-\zeta)(B-L_1)}_{\text{Potential type } h \text{ sellers}}. \end{aligned} \tag{12}$$

I now show that it is decreasing and continuous in $p(s)$, for all $s \in \overline{S}$.

Lemma 3 *For any $s \in \overline{S}$, and given a government's default policy function $\delta(y, B, L_1)$, investors' aggregate net demand defined in (12) is continuous and decreasing in $p(s)$. Moreover, if*

$$\frac{1}{1+r} E_\ell^1(s) + \gamma < \frac{1}{1+r} E_h^0(s) - \gamma,$$

it is strictly decreasing for all $p(s) \in \mathbb{R}_+$, and if

$$\tilde{p}_2(s) \equiv \frac{1}{1+r} E_\ell^1(s) + \gamma \geq \frac{1}{1+r} E_h^0(s) - \gamma \equiv \tilde{p}_1(s),$$

it is constant for all $p(s) \in [\tilde{p}_1(s), \tilde{p}_2(s)]$ and strictly decreasing for all $p(s)$ in the complement of this set in $\mathbb{R}_+$.

Proof. See Appendix A.2. ■

I provide the details of the proof as well as necessary intermediate results in Appendix A.2. The result in lemma 3 is intuitive. It states that given a government's default policy function the dealers' net demand for bonds in the primary market is decreasing in the price of the bond, and strictly decreasing in most of the cases. This follows because the mass of type h investors with $a = 0$ buying from dealers is decreasing in the price of the bond and the mass of investors holding a bond that sell bonds to dealers is increasing in the price of the bond. The mass of type h investors with $a = 0$ that meet dealers is decreasing in p because the dealer charges the investor $p + f_h^0(s)$, and so the investor pays more for the same expected return. Therefore, the investor responds by optimally reducing the intermediation fee $f_h^0(s)$. As a result, dealers earn lower expected profits and there is less entry into the secondary market, which in turn reduces the matching probability of investors trying to purchase bonds. As the each investor's trading probability decreases, a smaller mass of them trade. An opposite argument explains why the mass of investors selling their bonds to dealers increases with p.

3.2.3 Bond Market Clearing Prices

I now use the results in section 3.2.2 to show that for each state $s \in \overline{S}$ and given a government's default policy function $\delta(y, B, L_1)$, there is a unique price that is consistent with market clearing. Then, I characterize the pricing schedule faced by the government. Throughout this subsection I denote $B(s)$, $L_1(s)$, and $B'(s)$ the second, third, and fourth component of $s = (y, B, L_1, B')$, respectively.

For each $s \in \overline{S}$ and any given price p, define the excess demand function for bonds in the primary market as

$$ED(s;p) \equiv ND(s;p) - [B'(s) - (1-\lambda) B(s)], \tag{13}$$

with $ND(s;p)$ defined as in (12).

Proposition 4 *If*

$$\frac{1}{1+r} E_\ell^1(s) + \gamma < \frac{1}{1+r} E_h^0(s) - \gamma,$$

for any policy function $\delta(y, B, L_1)$ and any $s \in \overline{S}$ such that $B'(s) > 0$, there is a unique price $p(s) \in \mathbb{R}_+$ consistent with

$$p(s)\, ED(s; p(s)) = 0. \tag{14}$$

Moreover, either $p(s) > 0$ and $ED(s; p(s)) = 0$, or $p(s) = 0$ and $ED(s; p(s)) \leq 0$. In addition, when

$$\tilde{p}_2(s) \equiv \frac{1}{1+r} E_\ell^1(s) + \gamma \geq \frac{1}{1+r} E_h^0(s) - \gamma \equiv \tilde{p}_1(s),$$

the result still holds except when $B'(s) = (1-\lambda) B(s)$, in which case any price within $[\tilde{p}_1(s), \tilde{p}_2(s)]$, is consistent with $p(s)\, ED(s; p(s)) = 0$.

Proof. See Appendix A.3. ■

Given this result, we can then characterize the price schedule faced by a government conditional on a given policy function $\delta(y, B, L_1)$.

Corollary 5 *The price schedule faced by a government conditional on a given policy function $\delta(y, B, L_1)$ is given by*

$$p(s) = \begin{cases} \{x \in \mathbb{R}_+ : ED(s; x) = 0\} & \text{if } ND(s;0) > B'(s) - (1-\lambda) B(s) \\ 0 & \text{if } ND(s;0) \leq B'(s) - (1-\lambda) B(s) \end{cases}. \tag{15}$$

Proof. Directly follows from the previous results. ■

The price schedule defined in (15) replaces the standard no-arbitrage condition usually found in the literature of sovereign default. Proposition 4 states that the price that clears the primary market for sovereign bonds is unique, except for a particular cases in which neither the government nor the dealers participate in the primary market. To be more precise, this uniqueness statement is conditional on given a government's default policy function, investors value functions, and future expected prices. However, the result highlights the parallelism of the pricing schedule to the standard no-arbitrage condition that maps future prices and a default policy function into current prices. In this sense, solving this model is not harder than a standard model of sovereign default. Instead of having a closed form expression for the price as in standard no arbitrage condition, I just need to find the price consistent with (14).

3.2.4 Equilibrium Existence

In this section I show existence of an equilibrium for the economy outlined in section 2.

Proposition 6 *An equilibrium exists.*

Proof. See Appendix A.4. ■

The arguments of the existence proof follow standard arguments in the literature making use of the Kakutani-Fan-Glicksberg fixed point theorem (see Aliprantis and Border (2006) theorem 17.55). The new important part of the argument is replacing the standard no arbitrage condition by the market clearing price in the centralized primary market characterized by the mapping in (15). Using the results in section 3.2.3 I can show that the new mapping for the pricing schedule satisfies all required conditions and standard arguments can be applied to the model described in section 2. The details of the proof are provided in Appendix A.4.

4 Main Mechanisms and the Role of Liquidity

4.1 An Illustrative Example

To build intuition, I consider a particular case in which I can write a closed form expression for the pricing schedule defined in (15).[19] In particular, I assume that the order processing technology (matching function) is Cobb-Douglas and given by

$$\mathcal{M}(d,n) = \theta_0 d^{1/2} n^{1/2}.$$

In addition, I assume that high type investors never become low type, $\zeta = 0$, and $u_h = u_\ell = 0$. If no investor becomes low type, the measures of investors at the beginning of each period are

$$H_0 = \left[\overline{I} - (1-\lambda)B\right],\ H_1 \equiv (1-\lambda)B,\ L_1 \equiv 0.$$

Under these assumptions, it is easy to solve for the price schedule. Whenever the government is issuing new debt, I can write the price in the primary market as

$$p(s) = \underbrace{\frac{1}{1+r}\mathbb{E}_{y'|y}\{\left[I_h^1(s') - I_h^0(s')\right]}_{\text{Value of holding bond}}\underbrace{\left[1-\delta(y',B',0)\right]}_{\text{Default Risk}}\} - \underbrace{2\frac{\gamma}{\theta_0^2}\frac{B'-(1-\lambda)B}{\overline{I}-(1-\lambda)B}}_{\text{Liquidity Component}}.$$

Some remarks are in order. The price is divided into three component: (i) investor's expected discounted value of acquiring a bond, (ii) an adjustment for default risk, and (iii) a liquidity component.[20] The first term in the right hand side corresponds to components (i) and (ii) and is very similar to the standard no arbitrage condition of sovereign default models. The only difference is that $p(s')$ is replaced by the value of becoming bond holder, $[I_h^1(s') - I_h^0(s')]$. The second term in the right hand side is the liquidity component, which contains the following ingredients. First, there terms $2\gamma/\theta_0^2$ represent the importance of intermediation frictions. The more dealers have to pay to participate in the secondary market (higher γ) or the less efficient is the order processing technology

[19] In the general cases a closed form expression is not available. In particular, the assumptions of section 5 do allow me a to find a closed form expression for the pricing schedule.

[20] Notice that some part of the effects of liquidity are hidden inside the term $\left[I_h^1(s') - I_h^0(s')\right]$ which takes into account future liquidity conditions and their effects on the value for holding the bond. The purpose of this example is to build some intuition.

(lower θ_0), the larger is the price discount from the liquidity component. Second, the ratio $[B' - (1-\lambda)B] / [\overline{I} - (1-\lambda)B]$, represents the size of new issuances relative to potential investors' demand for bonds. The larger is the amount of new debt issued the larger is the price discount from the liquidity component. This is because the larger is the debt issuance, the more investors need to be matched with dealers in the secondary market, which is only possible if more entry of dealers to the secondary market. This occurs if investors visit a submarket where they pay a larger intermediation fee. Therefore, since the total amount paid by investors is $p + f$, to induce investors to pay a higher intermediation fee and attract more dealers, the price in the primary market has to fall to compensate them. This term highlights how the flows of bonds traded impact the price for bonds in the primary market. Finally, the last ingredient of the liquidity component is the maturity probability λ. Since, $\bar{I} > B'$ it is always true that the longer the maturity of the bond (smaller λ), the lower is the price discount due to the liquidity component. The reason is that in order to achieve certain new stock of debt B' a smaller flow of debt issuance is needed when a smaller fraction of bonds mature every period.

In this example only one type of investor trade bonds. The example does not consider the effect of bonds sold in the secondary market by type ℓ investors, which compete for buyers with government's newly issued bonds. In section 4.3 I describe the behavior of the model in the general case.

4.2 Interest Rate Spreads and Liquidity Measures

The model proposed in section 2 produces trading probabilities for dealers and investors as well as intermediation fees. In this section I show how trading probabilities and intermediation fees in the model can be mapped to the measures of liquidity observed in the data such as the bid-ask spread, volume traded, and the turnover rate of bonds.

I first consider the bid-ask spread. The bid-ask spread is defined as the difference of the ask price that an investor pays to buy a bond and the bid price that an investor gets for selling a bond. This spread is measured as a proportion of some mid price for the bond. I denote p^A the ask price, p^B the bid price, and p^M the mid-price of the bond. The bid-ask spread of a bond, measured in basis points, is

$$S^{B-A} = \frac{p^A - p^B}{p^M} \times 10,000.$$

For each $s \in \overline{S}$, in the model I define

$$\begin{aligned} p^A(s) &\equiv p(s) + f_h^0(s), \\ p^B(s) &\equiv p(s) - f_\ell^1(s), \text{ and} \\ p^M(s) &\equiv \frac{p^A(s) + p^B(s)}{2}. \end{aligned}$$

So the model counterpart of the bid-ask spread is given by the sum of the intermediation fees divided by the mid-price, i.e.,

$$S^{B-A}(s) \equiv \frac{f_h^0(s) + f_\ell^1(s)}{p^M(s)} \times 10,000. \tag{16}$$

Using the model, I also construct the traded volume in secondary markets in each state $s \in \overline{S}$. In equilibrium it is given by

$$\begin{aligned} Vol(s) \equiv\ & \alpha\left(\theta\left(f_h^0(s)\right)\right)\left[\overline{I} - (1-\lambda) B(s)\right] \\ & + \alpha\left(\theta\left(f_\ell^1(s)\right)\right)(1-\lambda)\left[\zeta B(s) + (1-\zeta) L_1(s)\right]. \end{aligned}$$

Finally, I define the turnover rate for bonds in each state $s \in \overline{S}$ as

$$Turnover(s) \equiv \frac{Vol(s)}{B(s)}. \tag{17}$$

In addition, I compute the interest rate spread of the risky sovereign bond over a perfectly liquid risk free bond that pays an interest rate r every period. To compute the total spread of a government bond over the risk free bond, denoted $S^R(s)$, I calculate the return rate $r_g(s)$ which makes the present discounted value of the promised sequence of future payments on a bond equal to the price. That is, $p(s) = \frac{\lambda + (1-\lambda) z}{\lambda + r_g(s)}$. Then, the total interest rate spread is given by

$$S^R(s) \equiv (1+r(s))^4 - (1+r)^4 \tag{18}$$
$$= \left[1 + \frac{\lambda + (1-\lambda) z}{p(s)} - \lambda\right]^4 - (1+r)^4 .$$

I measure annualized interest rate spreads. Since in section 5 I calibrated the model at the quarterly frequency, the power 4 in 18 it to calculate annualized spreads. In section 5 I use available information on data counterparts for $S^{B-A}(s)$, $S^R(s)$, and $Turnover(s)$ together with standard variables used in the sovereign default literature to calibrate the parameters of the model.

4.3 Impulse Response Functions

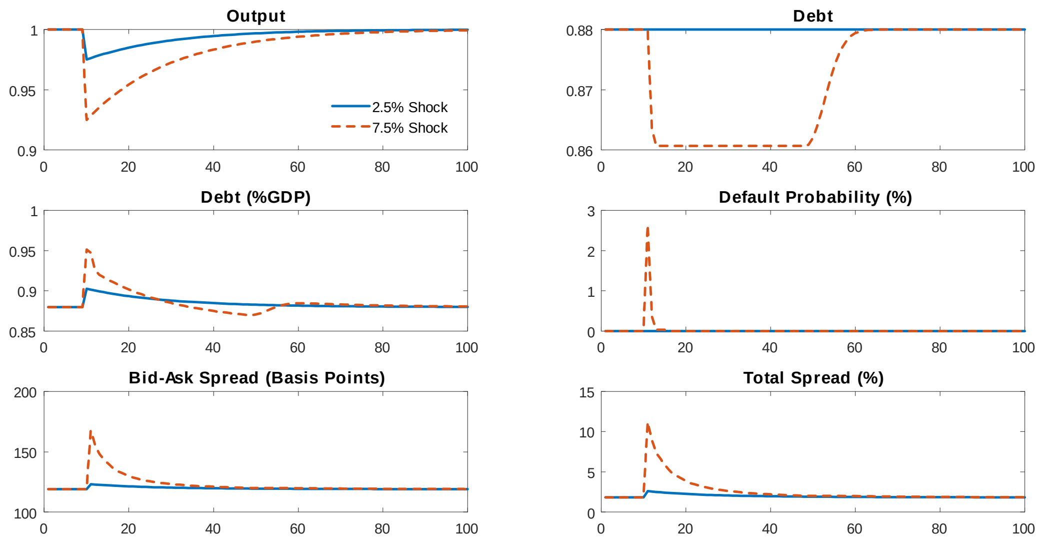
To better understand the mechanisms of the model, in this section I analyze the effects of output shocks on endogenous variables in the model.[21] Since the model is nonlinear, the state at the moment of the shock matters. To pick the starting point, I simulate the model for $1,010$ periods feeding the model with an endowment level constant at the mean of the endowment distribution, $\bar{y} = 1$. I drop the first $1,000$ periods and keep the remaining 10 periods. I then use period 10 as the starting point and shock the model in period 11. After the shock, I let the model run assuming that innovations to output process $\varepsilon_t = 0$ for all $t > 11$.

Figure 1 shows the sequence of endowment (top-left panel) considered in this exercise and the endogenous evolution of government debt (top-right), debt as a percentage of endowment (center-left), and the corresponding probability of default implied by the endowment process and government choices (center-right). Figure 1 also shows the response of the market in terms of bid-ask spread as defined by (16) (bottom-left), and the response of the credit spread as defined by (18) (bottom-right). For each of these variables, the figure plots two lines, which correspond to two endowment shocks of different sizes. The blue solid line shows the dynamics after a 2.5% decrease in endowment while the red

[21] The functional forms and parameters used in the sections are those described in section 5.

dashed line depicts the case in which the endowment decreases 7.5%.

Figure 1. Output shocks and equilibrium responses.



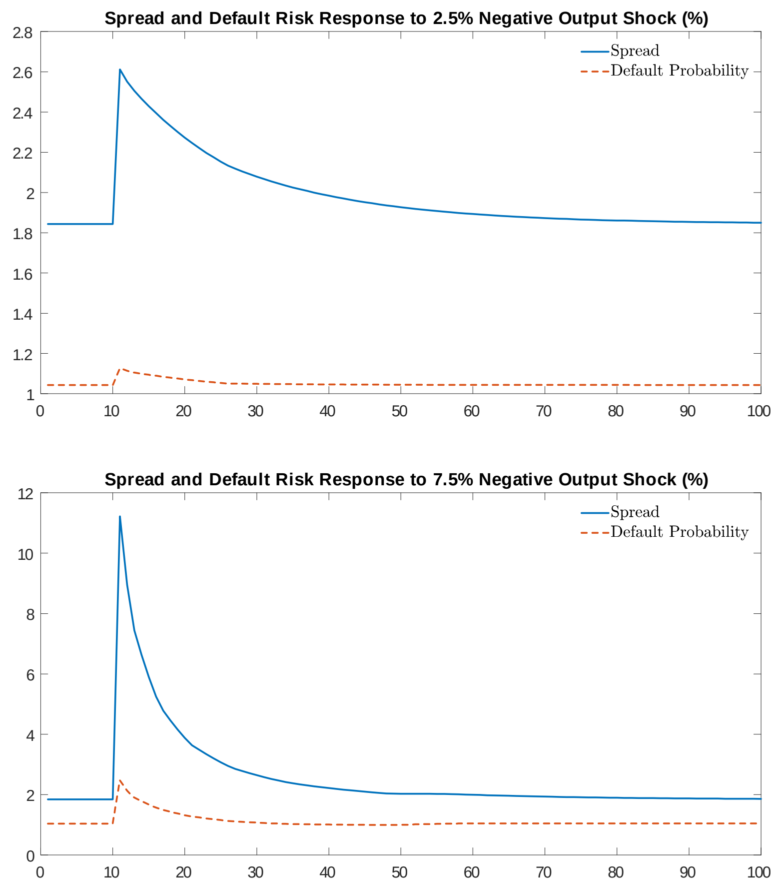
Note: The blue solid line shows the dynamics after a 2.5% decrease in endowment while the red dotted line depicts the case where the endowment decreases 7.5%.

Consider first a small endowment shock that reduces output by 2.5% (blue solid lines). In this case, the government decides to keep the level of debt constant and just issue enough debt to repay maturing bonds. Then, debt to output increases only due to the reduction in output. As output decreases, the probability of default increases since it is more likely that further negative shocks push the government towards the default region. The bid-ask spread and the total interest rate spread increase as a response in the reduction in output and the increased probability of default.

The case of a large endowment shock with a reduction in output of 7.5%, (red dashed lines), has similar implications. The main difference with the small shock is that now, as the output fall is larger, the probability of default and total interest rate spread increase considerably more. Here, the increase in interest rates is so large that the government responds by reducing the outstanding level of debt. At impact, the debt to output ratio increases but it converges back to initial levels as the endowment recovers.

Figure 2 shows the response in the credit spread (blue solid line) together with the response in of the pure default risk component of credit spreads (red dashed line).[22] The top panel shows the case of a 2.5% reduction in endowment while the bottom one shows the case of a 7.5% shock.

Figure 2. Effecto of output shocks on credit spread and default risk.



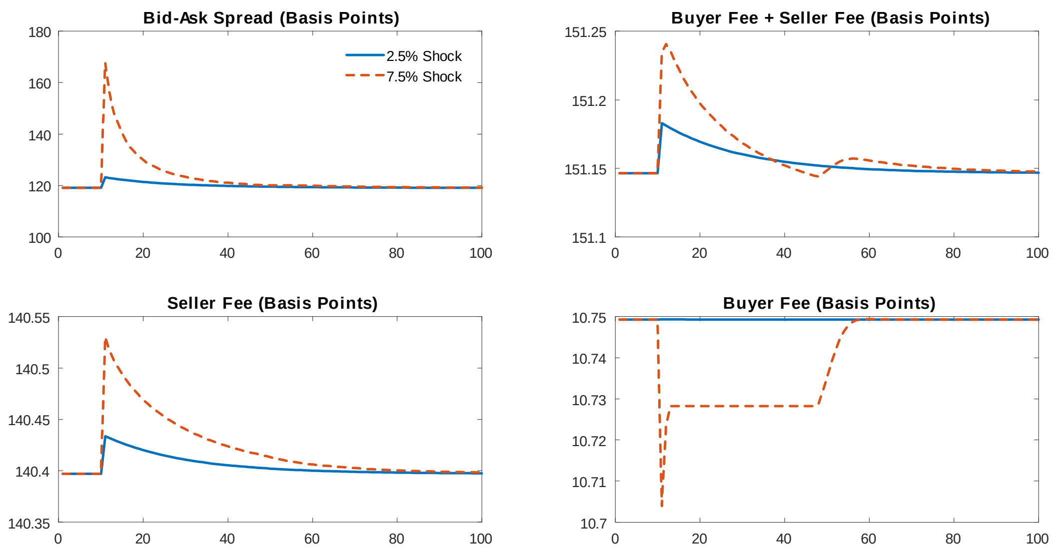
Note: Response of credit spreads and default risk component of credit spread after a 2.5% negative endowment shock (top panel) and a 7.5% negative endowment shock (bottom panel). The blue solid line is the response of total interest rate spread to output shock while the red dashed line is the response in spread needed to compensate investors for the higher default risk.

[22] The pure default risk component is obtained by calculating the price of a perfectly liquid bond under the same government's issuance and default policy functions. The distance to total interest rate spreads corresponds to the effect of liquidity frictions and their interactions with default risk. For more details on this calculation see section 6.3 and equation (19).

Figure 2 highlights the importance of liquidity frictions on interest rate spreads. In both cases, the response in total spreads is larger than the pure default risk component. This is because at the same time that output decreases, liquidity conditions deteriorate. That is, the additional increase in credit spreads reflects a larger compensation for worse liquidity conditions in secondary markets.

Finally, Figure 3 analyzes the changes in bid-ask spreads. The top-left panel shows the bid-ask spread defined as in (16), measured in basis points. The top-right panel shows the sum of the intermediation fees paid both by type h investors with $a = 0$, f_h^0, and type ℓ investors with $a = 1$, f_ℓ^1. Then, the bottom-left and bottom-right figures show the response on f_ℓ^1 and f_h^0, respectively. All panels show the effects of a 2.5% endowment shock (blue solid line) and the effect of a 7.5% endowment shock (red dashed line).

Figure 3. Intermediation Fees and the Bid-Ask Spread.



Note: The blue solid line shows the dynamics after a 2.5% decrease in endowment while the red dotted line depicts the case where the endowment decreases 7.5%.

Recall that after the 2.5% endowment shock the government does not change the outstanding level of debt. Hence, the full adjustment in bid-ask spreads arises from responses in investors search decisions and their effects on the market clearing price at the centralized primary market. In this case, there is an increase in both f_ℓ^1 and f_h^0 that result in a higher bid-ask spread. As outstanding stock of debt is fixed and equal to the pre-shock level, the increase in default probability increases type ℓ investors' bond supply, as they are now willing to pay larger intermediation fees to sell their bonds faster. As f_ℓ^1 increases, the mass of type ℓ investors that trade increases and the supply of bonds in primary markets is larger. To clear the market, the mass of type h investors trading has to increase until it matches the supply. This can only happen if investors are willing to pay higher intermediation fees f_h^0 to meet a dealer with higher probability. Thus, the price of the bond in the primary market has to fall enough so that type h investors with $a = 0$ are willing to pay such larger fees. In other words, the fall in the price of the bond in the centralized primary market has to more than compensate type h investors for the larger probability of default and induce them to pay larger intermediation fees, f_h^0. This effect amplifies the response in total interest rate spread.

The dynamics are richer when the endowment shock reduces output by 7.5%. In this case, the increase in the interest rate spread is larger and the government responds by reducing the outstanding level of debt. At the same time, type ℓ investors holding the bonds, are willing to pay higher intermediation fees, f_ℓ^1. This is because a larger intermediation fee allows these investors to trade with higher probability and avoid default risk. Hence, the mass of type ℓ investor selling bonds increases. In net, at impact, the reduction in government's supply of bonds is larger than the increase in supply by type ℓ investors selling their bonds. This is consistent with market clearing only if less type h investors are purchasing bonds. Thus, market clearing is consistent with lower f_h^0. As it can be seen in the bottom panels f_ℓ^1 increases while f_h^0 decreases, at impact. However, the sum of the two intermediation fees increases, as show in the top-right panel. Therefore, the bid-ask spread increases after the shock. Since the bid-ask spread increases, the interest rate spread increases more than compensating for default risk.

4.4 Discussion: The Role of Key Assumptions

Before moving on to the quantitative implications of the model, I discuss the importance of specific assumptions that differentiate my models from existing ones in the literature. These different assumptions lead to new important insights and different implications for variables defined in section 4.2. The closest paper in the literature is Passadore and Xu (2018) who build on He and Milbradt (2014) and Chen et al. (2018) to incorporate liquidity frictions in a standard model of sovereign default.

The key differences in assumptions are the following. First, they assume that high valuation investors purchase bonds in the primary market, which is a centralized competitive markets. In contrast, I assume that both types of investors have to trade in secondary markets. Second, they assume that in secondary markets the matching probability of low valuation investors is given by a fixed exogenous parameter. Instead, I assume that search is competitive and investors that want to meet a dealer face a trade-off between the transaction fee that they pay to dealers and the probability of trading. In my model, the optimal balance of this trade-off changes with the state of the economy, thus, trading probabilities and intermediation fees change as the state of the economy fluctuates. Third, in their setup the pool of buyers is infinite and any amount of debt can be transferred to them immediately at any point in time. In contrast, I assume that the pool of investors is a large but finite mass.[23]

The mechanisms driving the bid ask spreads are very under these two sets of assumptions. The positive correlation between default risk and the absolute difference of the ask price and the bid price (dollar bid-ask spread) does not arise naturally when the trading probability is exogenous. To generate this prediction, Passadore and Xu (2018) assume that after default the recovery value of a bond is positive and that low type investors exogenously lose their bargaining power when the bond is in default. Thus, the outside option for a low type investor is lower as the probability of default increases due to a higher risk of losing all bargaining power in future meetings if the government defaults.

In contrast, my model can capture the positive correlation between dollar bid-ask spreads without assuming exogenous changes in technological parameters, even in cases where the recovery rate of a defaulted bond is equal to zero.[24] The mechanism is as follows. When the

[23] By large I mean that the mass of the investors pool is larger than the upper bound for government debt issuances. Thus, if markets were frictionless, the size of the pool of investors would be irrelevant and equivalent to assuming that there are infinite investors.

[24] With zero recovery rate the bid-ask spread is zero when the bond is in default (more precisely not well defined) but it is strictly positive whenever the bond is not in default. The jump to zero comes as a discontinuity of the bid-ask spread on the probability of default as the probability of default becomes one. For the baseline calibration used in section 5 the dollar bid-ask spread is strictly increasing in default probability for all probabilities strictly lower than one. In addition, It is straightforward to incorporate an exogenous positive recovery rates.

probability of default increases, low valuation investors are willing to pay higher intermediation fees in order to sell their bonds faster. Therefore, more investors sells their bonds. But, since high valuation investors also trade bonds in secondary markets, more buyers have to meet dealers to acquire the larger amount of bonds sold by low type valuations. That can only happen if more dealers enter submarkets with high type investors, which in turns happens only if high type investors pay higher intermediation fees to attract them. Thus, to induce high type investors to pay higher intermediation fees, the price of the bond in the centralized primary market, where dealers purchase bonds, has to fall more than what would compensate investors for higher default probability. The mechanism highlights the importance of allowing for endogenous trading probabilities as a determinant of secondary market liquidity in equilibrium.

In addition, changes in the supply of bonds have very different implications for the liquidity of bonds. If high type investors can access the frictionless primary market, changes in the supply of bonds have no direct impact on the bid-ask spread and the liquidity premium, as type h investors absorb all new issuances.[25] Moreover, if the matching probability is exogenously fixed, new issuances do not have any effect on future trading probabilities. In my model, endogenous matching probabilities, together with the assumption that type h investors purchase bonds in the secondary market, make it increasingly more difficult to find a buyer for an extra unit of newly issued bonds. Finding buyers for more new bonds requires more dealers in the secondary market and dealers can only be attracted to enter with larger intermediation fees. Thus, a larger supply affects the price of bonds through a larger liquidity premium.

Importantly, and related to the last point, different effects of bond supply on liquidity conditions, result in different assessments of policy interventions. For example, during the European debt crisis the ECB directly purchased sovereign bonds in the secondary market. Interventions of this kind would affect liquidity very differently in the two models. One potential way of modeling such an intervention is by assuming that the ECB sends agents to buy bonds, who face the same frictions than type h investors. If type h investors purchase bonds in the frictionless primary market, adding ECB agents purchasing bonds in the primary

[25] In Passadore and Xu (2018), bond issuances indirectly affect liquidity premium through the effect on default probability. Changes in the supply of bonds that do not change default risk have no direct or indirect effect on liquidity, if the trading probability is exogenous.

market does not affect the price of bonds paid by type h investors. In addition, if the trading probability in secondary markets is exogenously fixed, extra demand for bonds would not affect the liquidity for low valuation investors nor their negotiation outcomes when meeting intermediaries or high valuation investors. In my framework instead, investors purchase bonds in the secondary market. So, independently of ECB agents' submarket choice, as long as it is profitable for dealers to visit those submarkets, there would be dealers transferring bonds from the centralized primary market to the ECB through its agents. That is, ECB agents' buying orders add to investors' net demand, and thus, the results in sections 3.2.2 and 3.2.3 imply that the price in the primary market increases.

An alternative is to assume that the ECB is a big player and can open a trading window where all investors can sell their bonds at a given price, maybe with some limit in the amount of the intervention. To simplify the argument, suppose it is a one time and unexpected intervention. If sellers trading probability is fixed and the ECB does not purchase all bonds held by type ℓ investors, this intervention does not increase the probability of trading for those that do not sell their bonds to the ECB. It also does not change trading conditions in favor of type ℓ investors when meeting a type h investor. Hence, bid-ask spreads would remain unchanged. In addition, transfers from investors to the ECB do not change the amount of outstanding debt and there are no indirect effect through changes in default risk. In my model instead, if some sellers directly sell their bonds to the ECB, the remaining number of investors selling bonds to dealers will decrease. This will result in a higher investors' net demand and a higher price in the primary market.

5 Quantitative Analysis

This section takes the model to the data. I first describe how to extend the solution method proposed by Chatterjee and Eyigungor (2012) to solve models of long term debt when the price schedule is determined by a market clearing condition instead of the standard no-arbitrage condition. I leave details to Appendix B. Then, I describe the functional forms used in the quantitative exercises. Next, I calibrate the model to match some time series moments for the Greek economy around the sovereign debt crisis of 2010-2012. I conclude the section showing how the model fits non-targeted moments in the data usually analyzed in the literature.

5.1 Model Solution

5.1.1 Solution Method

I make use of the results obtained in section 3.2 to solve the model. It is well know that long-term debt models have convergence issues. To get convergence, I adapt the method proposed by Chatterjee and Eyigungor (2012) to my model. The main difference is that I cannot price bonds using the standard operator defined by a no-arbitrage condition. Instead, I have to solve the price that clears the centralized primary market for bonds which is unique, except for region in the state space with measure zero where the government does not issue debt, as shown in section 3.2.3. Updating the pricing schedule requires solving for investors' optimal choices and the net excess demand from investors for each solution of the government problem. Then, given the solution of the government problem and investors' net demand for bonds, I can solve for the market clearing price, $p(s)$, for each $s \in \overline{S}$. Although the solution requires a few additional steps than solving standard models, it does not significantly increase the computational burden. For more details please refer to Appendix B.

5.1.2 Model Simulation

I simulate the model over $T = 500,000$ periods. Then, I burn the $T_1 = 1,000$ initial periods. Find $N_1 = 300$ episodes of length $T = 69$ periods where the 69th period is a default episode and none of the previous 99 periods are default periods. I discard the first $T_0 = 30$ periods and keep 69 periods before default. Length T is chosen to be 69 because I use data on Greek GDP from 1995Q1 and default happens 68 quarters later in $2012Q1$. Since after re-gaining access to international financial markets the government re-enters with $B = L_1 = 0$, I choose to discard T_0 periods before the beginning of each replica of the economy so that I let the model reach the ergodic set. Then, I keep the first $N_2 = 100$ of those episodes and I compute the summary statistics for each of them. Finally, I average over these episodes and report these averages of the moments.

5.2 Functional Forms

The utility function of the government and the output cost of default have the following functional form

$$\begin{aligned} u(c) &= \frac{c^{1-\sigma}}{1-\sigma}, \\ \omega(y) &= \max\left\{0, d_0 y + d_1 y^2\right\} \\ h(y) &= y - \omega(y). \end{aligned}$$

Utility function is a standard CRRA while output cost of default has the same functional form as in Chatterjee and Eyigungor (2012). The persistent stochastic process for output is given by

$$\begin{aligned} \log(y_t) &= \rho_y \log(y_{t-1}) + \varepsilon_t, \text{ with } \varepsilon_t \sim N\left(0, \eta_y\right), \\ y_t &= \tilde{y}_t + m_t, \end{aligned}$$

where output shocks m_t are transitory *i.i.d.* shocks drawn from a normal distribution, $N(0, \eta_m)$, truncated between $\{-\bar{m}, \bar{m}\}$. I assume that trading probabilities for investors and dealers are given by

$$\alpha(\theta) = \frac{\theta}{1+\theta} \text{ and } \rho(\theta) = \frac{\alpha(\theta)}{\theta},$$

where market tightness $\theta \equiv \frac{d}{n}$. So, if a submarket has measure n of investors orders and tightness θ, then the measure of dealers is θn and the measure of matches is

$$\mathcal{M}(n, \theta n) = \alpha(\theta) n = \frac{n \times (\theta n)}{n + (\theta n)}.$$

This function is a Dagum (1975) function, also known as the telephone line matching function. Some convenient properties of this matching function are that $\alpha(\theta) \in [0,1]$ and that $\alpha(\cdot)$ is twice continuously differentiable for all $\theta \in \mathbb{R}_+$.[26]

[26] These properties are not satisfied by Cobb-Douglas matching functions, for example.

5.3 Parameters

I calibrate the model at a quarterly frequency and it has in total 17 parameters. From those parameters, 2 of them are set exogenously, 7 of them are calibrated directly from the data and the remaining 8 are estimated to target some moments in the data.

Exogenously Set The following parameters are taken exogenously from previous literature.

Table 1: Exogenously set parameters

Parameter	Value	Source
σ	2.000	standard in macro
ϕ	0.050	Cruces and Trebesch (2013)

The risk aversion parameter $\sigma = 2$ is a standard value used in macroeconomic models. The probability of regaining access to international credit markets after a default is chosen to be $\phi = 0.05$, which implies an average exclusion length of 5 years, in line with evidence in Cruces and Trebesch (2013).[27]

Calibrated Table 2 summarizes the values and targets for the group of parameters directly calibrated from the data.

Table 2: Parameters directly calibrated from the data.

Parameter	Value	Target
ρ_y	0.987	persistence of AR(1) process for quarterly GDP
η_y	0.015	variance of the residual of quarterly GDP process
$\bar{m}$	0.024	to get convergence.
σ_m	0.048	to get convergence.
λ	0.039	pre-crisis average maturity
γ	0.001	minimum $\left(p^a - p^b\right)/2$
r	0.010	pre-crisis average German 3 months rate

[27] This number is also within the 2 year median duration calculated by Gelos, Sahay, and Sandleris (2011) since 1990 and the more than 7 year median duration computed by Benjamin and Wright (2013).

The output process is discretized using Tauchen-Hussey method. After discretization, income follows a Markov process with transition probabilities given by $\Pr(y_{t+1} = y_m | y_t = y_n) = \pi^y_{n,m}$, for all $n, m \in \{1, 2, ..., N\}$, we choose $N = 51$. Then, I pick the values for $\bar{m}$ and σ_m to achieve convergence within $3,000$ iterations starting from all initial guesses equal to zero in all states. I then calibrate ρ_y and η_y to match the persistence for the AR(1) process for the GDP Cycle in Greece for the period $1995Q1 - 2017Q4$, and η_y to match the standard deviation of the residuals. The maturity parameter is set to $\lambda = 0.039$, which represents an average expected time to maturity of 6.5 years. This corresponds to the average time to maturity of outstanding bonds for the pre-crisis period (until $2007Q4$). Dealers' entry cost, γ, determines the minimum transaction fee that can arise in the model. Since I do not have direct estimations of the intermediation cost I set $\gamma = 0.0010$, which is 10 basis points of the average price, and about 10 times smaller than the average bid-ask spread. That is, the size of the bid-ask spread is not driven by setting a minimum transaction cost close to the average bid-ask spread. This is consistent with the very low bid-ask spreads observed around 2006 after Greece joined the Euro and before US subprime crisis. Finally, $r = 0.01$ corresponds to the average interest rates for 3-month German bonds. In the model the risk free rate is assumed to correspond to perfectly liquid bonds. German bonds are almost perfectly liquid with bid-ask spreads below 5 basis points in most of trading dates, including the period of the European debt crisis.

Estimated The remaining 8 parameters are estimated using the model. Table 3 summarizes the targets for these parameters.

Table 3: Parameters estimated to match moments in the data.

Parameter	Value	Target
β	0.977	default probability of 0.675% per year
d_0	-0.323	average $r(s) - r$
d_1	0.425	standard deviation of $r(s) - r$
z	0.030	to get an average price of 1.00
u_h	0.009	Expected utility flows equal zero.
u_ℓ	-0.190	average S^{B-A}
ζ	0.325	match pre-crisis average turnover rate at HDAT
$\bar{I}$	5.000	$\bar{I} > \bar{B}$.

Although all moments respond to all parameters, I discuss how each of the parameters is targeted towards one particular moment. The discount factor β is used to calibrate the probability of default to the observed frequency in data. That is, the number of events (1) divided by number of quarters in good credit standings since $1975Q1$ (148 until $2012Q1$).[28] All else equal, the more impatient the government (lower β) the closer it decides to go the default region and thus the larger the probability of default. The estimated discount factor is $\beta = 0.977$. This is a high discount factor compared to the standard values used in the literature. To generate sizeable spreads it is standard to assume $\beta \approx 0.95$ for quarterly models.[29] An important difference is that here the credit spread includes a liquidity component and the interest rate spread is higher than the implied probability of default. Parameters of the output cost function d_0 and d_1 are estimated to match the average credit spread between Greek and German 10-year bonds and its standard deviation. The higher the cost of default the lower the probability of default and the more willing the government is to take amounts of debt that are closer to the default region. The higher the d_1 (curvature) of the default cost, the larger the volatility of credit spreads as it is very costly for the government to default in

[28] This is the same calculation that Chatterjee and Eyigungor (2012) do for Argentina to find an annual probability of default of 12%.

[29] For example, Arellano (2008) estimates $\beta = 0.953$, Chatterjee and Eyigungor (2012) estimates $\beta = 0.954$, and Hatchondo and Martinez (2009) $\beta = 0.95$. Some exceptions are provided by Bocola and Dovis (2016) and Benjamin and Wright (2013) who estimate $\beta = 0.97$.

high endowment states relative to low endowment states. As a result debt issuance decisions tend to be more volatile. Thus, spreads tend to be more volatile.[30] As it is standard, the coupon rate is chosen so that on average the bond price is 1, which means that bonds are traded at par.

The new parameters estimated in my model are: u_h, u_ℓ, ζ, and $\bar{I}$. The difference between u_h and u_ℓ is a key determinant of the bid-ask spread. Those two parameters are calibrated to match the average bid ask spread in the data. In addition, the parameters u_h, u_ℓ, are set such that the ex-ante expected flow utility of holding the bond is zero so that the price for a buyer is not artificially distorted by flow utilities. High type investors holding one unit of a bond become low type with probability ζ. In equilibrium, low type investors have incentives to sell their bonds. Thus, the fraction of high type investors that become low type is a key determinant of the turnover rate for bonds in secondary markets (defined in equation (17)). Therefore, the parameter $\zeta = 0.350$ is the one consistent with the turnover rate in secondary markets, which is 78% per-quarter according with data from the Greek Central Bank. Finally, the measure of investors $\bar{I}$ is set to be larger than the upper bound on debt issuance so that it is never the case that the government would want to choose a debt level in the grid where there are not enough investors in the economy to purchase such amount. As the upper bound on debt issuance is $\overline{B} = 4$, I set the measure of investors to be $\bar{I} = 5$. Appendix D provides additional details on the computation of the turnover rate and the constraint on u_h and u_ℓ, while appendix C describes data sources.

5.4 Model Implications for Business Cycle Moments

Before focusing on the analysis of Greek debt crisis, I look at the implications of the model for business cycle moments often analyzed in the sovereign default literature. Table 4 shows the model implications for standard business cycle moments that are not targeted in the calibration and the corresponding values in Greek data. Symbols σ_x represent standard deviation of x while $\rho_{x,y}$ correlation between x and y. Variables y, c, tb/y, and S^R, represent

[30] This cost function is introduced by Chatterjee and Eyigungor (2012). A more detailed description can be found in their paper.

output, consumption, trade balance to GDP ratio, and credit spreads.

Table 4: Moments not targeted

Moments	Model	Data
σ_c/σ_y	1.30	1.06
$\rho_{S^R,tb/y}$	0.57	0.63
$\rho_{S^R,c}$	-0.57	-0.40
ρ_{y,S^R}	-0.48	-0.48
$\rho_{y,tb/y}$	-0.29	-0.39
$\rho_{y,c}$	0.97	0.86

The model is able to capture the fact that consumption is more volatile than output and the sign of correlations. Model predictions for correlations are also very close in magnitudes. However, it exaggerates the volatility of consumption relative to output and the correlation of these two variables.

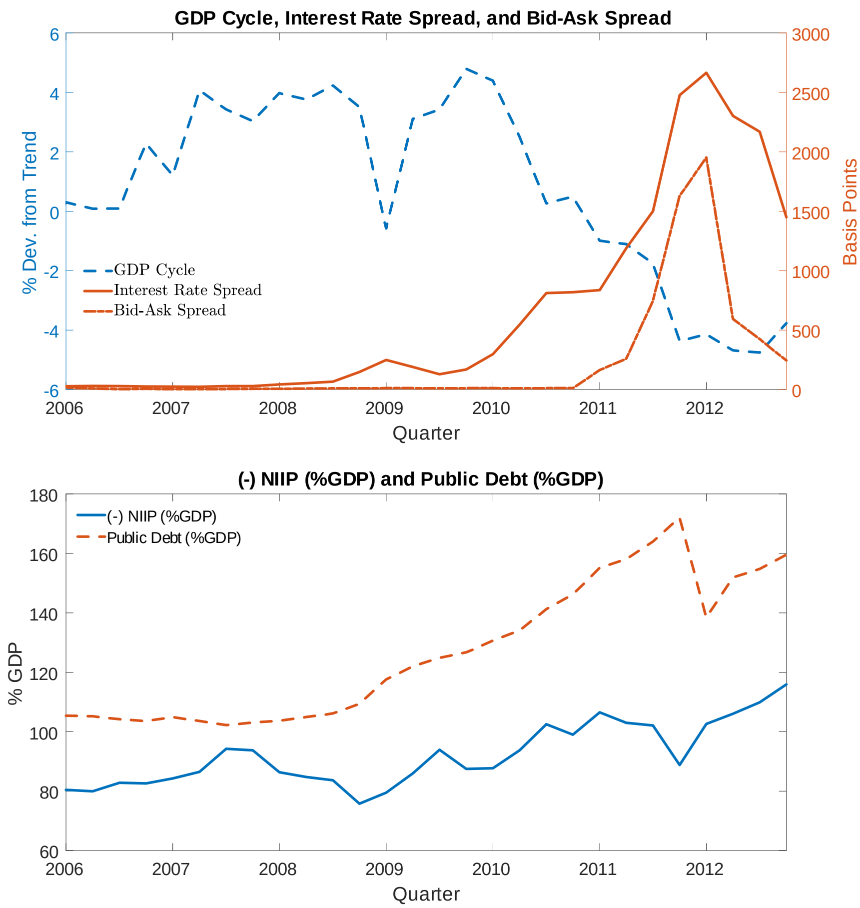
6 Case Study: Greek's Debt Crisis

6.1 Greek Time Series

I now focus on Greek economy around the sovereign debt crisis of 2010-2012 that resulted in a debt restructuring (or default episode) in the first quarter of 2012. Figure 4 shows the time series for macroeconomic variables from 2006$Q1$ until 2012$Q4$. The top panel shows Greek GDP cycle (blue dashed line with scale in left axis), the interest rate spread of Greek long-term bonds compared to same maturity German bonds as defined in equation (18) (red solid line with scale in right axis), and the bid-ask for Greek bonds in secondary markets as defined in equation (16) (red dashed-dotted line with scale in right axis). The bottom panel shows the negative of net international investment position (NIIP) as percentage of GDP[31] (blue solid line) together with total public and publicly guaranteed debt as percentage of GDP (red dashed line). As the model focus on external debt of a country, the counterpart NIIP is the appropriate counterpart in the data to debt in the model.

[31] A positive number means a negative NIIP with the rest of the world.

Figure 4: Greek time series 2006$Q1$ − 2012$Q4$.



Note: The figure shows quarterly data time series of Greek economy from 2006Q1 to 2012Q4. The top panel shows the evolution of the GDP cycle (blue dashed line) together with the interest rate spread (red solid line) and the bid-ask spread (red dashed-dotted line). The bottom panel plots the negative of the net international investment position (NIIP) as percentage of GDP (blue solid line) together with total public debt as percentage of GDP. Source: Bloomberg and Eurostats.

Between 2006$Q1$ and 2010$Q4$ Greece experienced sustained economic growth above the trend, only temporarily interrupted in 2009$Q1$ during the sub-prime crisis. However, during 2010 the GDP gap decreased sharply from growing around 4.5% above trend in 2010$Q1$ to around 0.5% above trend in 2010$Q4$. During 2011 the GDP gap kept falling and was always bellow trend until the end of the sample.

Before the subprime crisis, Greek government was able to take debt at very low interest rates that were only around $20-50$ basis points above German rates, as it can be observed in the bottom left panel of Figure 4. Cheap interest rates allowed Greece to accumulate debt over the years until 2008, when U.S. crisis hit international financial markets and Greek interest rate spreads began to increase, reaching a first spike at 250 basis points in the first quarter of 2009. As interest rate started increasing, Greece began reducing its stock of debt during 2008 and the beginning of 2009. In 2009 output recovered and interest rate spreads went down to stay around 150 basis points during the whole year. By the end of 2009 interest rate spreads of Greek bonds started increasing at fast rate, going from 170 basis points in $2009Q4$ to around 900 basis points during $2010Q2$, amid a political crisis and the revelation that Greece had been understating its debt and deficit figures for years. Attempts to stop the crisis during 2010 were not successful and by the end of 2011 interest rate spreads were above 2500 basis points. In $2012Q1$ Greek debt was restructured involving bond swaps and a 65% haircut on investor's net present value of Greek bond holdings. On March of 2012 the International Swaps and Derivatives Association declared a triggering credit event. In other words, a default. During 2012 interest rate spreads decreased due to the debt relief on Greek bond in $2012Q1$ following the restructure but remained high for the rest of the year. In $2012Q4$ Greece bought back a large fraction of the newly issued bonds over the debt restructuring, which increase the market price of bonds on 20%, significantly reducing the interest rate spreads. See Zettelmeyer, Trebesch, and Gulati (2013) and Trebesch and Zettelmeyer (2018) for detailed and clear exposition of events in during Greek debt crisis.

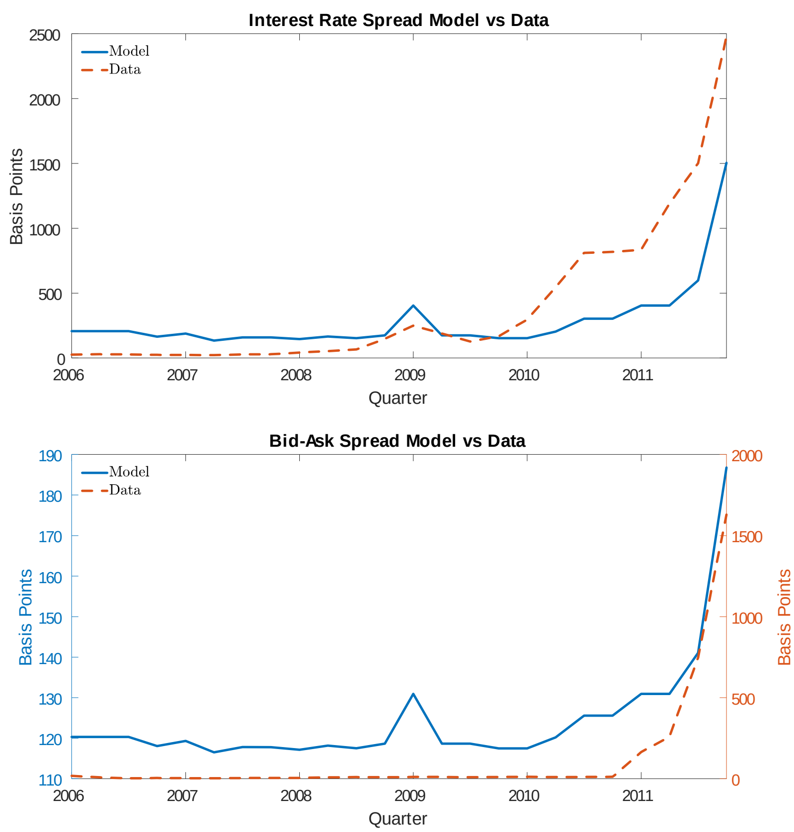
The bid-ask spreads, defined as in 16, remained below 50 basis points during the period $2006Q1-2010Q4$. However, bid-ask spreads sharply increase from 165 basis points during $2011Q1$ to about 2000 basis points before the debt restructure of March in 2012, when total interest rate spreads where between $2600-3000$ basis points. The model presented in section 2 can be used to assess how the endowment process affects the rest of variables depicted in Figure 4 and determined how much of interest rate spreads reflects probability of default and how much of it is due to liquidity frictions.

6.2 Model Time Series

Here, I do the following exercises. I feed the GDP cycle plotted in the top-left panel of Figure 4 into the model as the realizations of the endowment process for $t \in \{1995Q1, ..., 2011Q4\}$.

I stop the time series in $2011Q4$ because Greece default in $2012Q1$ and the model shuts down secondary markets after default since recovery value is zero. I let the government, investors, and dealers react optimally by choosing debt issuances, defaulting or not, and optimal submarket to visit. Using optimal choices I compute the model implied interest rate spreads, S_t^R, and the bid-ask spreads, S_t^{B-A}, as defined in 18 and 16, respectively. Figure 5 shows the models predictions compared to the data. The blue solid lines represent model's predictions while red dashed lines correspond to data.

Figure 5. Interest Rate Spreads and Bid-Ask Spreads.



Note: The blue solid lines are model predictions for interest rate spreads (top panel) and bid-ask spread (bottom panel) while the red dashed lines are the analogous time series in the data.

The top panel shows the evolution of credit spreads as defined in equation (18). The model does a good job to capture the dynamics of spreads and it is able to explain a large fraction of the changes in magnitude in the data. However, it is not able to get the quantities exactly right. This is not surprising given that the model is calibrated to match the average spreads in Greece in a time period with very large changes. Before the subprime crisis in the US, Greece was able to take debt at German rates while after the subprime interest rate spreads behaved as those ones in a very risky emerging market. In addition, the model abstracts from bailout expectations and time varying changes in investors discount factors, which have been shown to be important determinants of interest rate spreads in European countries in the period of analysis.[32]

The bottom panel shows the evolution of bid-ask spreads as defined in equation (16). The figure shows that the model is able to capture the fact that bid-ask spreads increase when the economy gets closer to a default episode and the qualitative dynamics over the business cycle. However, the model (blue solid line with scale in left axis) is not able to get the extremely large increase in bid-ask spreads in the data (red dashed line with scale in right axis). In the model, the bid-ask spread at the peak of the crisis are around 60% higher than pre-crisis levels while in the data bid-ask spreads at the peak of the crisis are 5000% larger. Again, the calibration of the model targets average bid-ask spreads in the sample. Before the crisis Greek bonds were almost perfect substitutes to German bonds and very liquid and when the crisis hit Greek bonds became extremely illiquid. Data from the Bank of Greece shows that the monthly turnover rate for Greek bonds in HDAT went from above 25% before the crisis to less than 0.5% after the debt restructure.[33]

6.3 Spread Decomposition

How large is liquidity premium? To answer this question, I use the structure of the model to decompose the predicted interest rate spreads, S_t^R, into a default risk component and a liquidity component. To decompose interest rate spreads I do the following exercise. I take optimal government policies from the model and calculate the bond prices that would reflect the probability of default in an alternative model in which bonds are perfectly liquid. That,

[32]See Bocola and Dovis (2016) for a decomposition of interest spreads that accounts time varying investors discount factors and Dovis and Kirpalani (2018) for the effects of bailout expectations on interest rate spreads dynamics.

[33]HDAT is one of the most important electronic secondary securities market. For more information visit https://www.bankofgreece.gr/Pages/en/Markets/HDAT/default.aspx.

is I solve the following Bellman equation for prices

$$\tilde{p}(y, B, L_1 = 0) = \frac{1}{1+r}\mathbb{E}_{y',m'|y}\left\{\left[1 - \delta^*(y', m', B'^*, L_1' = 0)\right]\left[\lambda + (1-\lambda)\left(z + \tilde{p}(y', B'^*, L_1' = 0)\right)\right]\right\}, \tag{19}$$

where $B'^* \equiv B'(y, B, L_1 = 0)$ is the optimal policy of the government in the model with liquidity frictions. Using the counterfactual price for bonds we can calculate the interest rate consistent with default risk. For each pair (y, B), this interest rate is given by

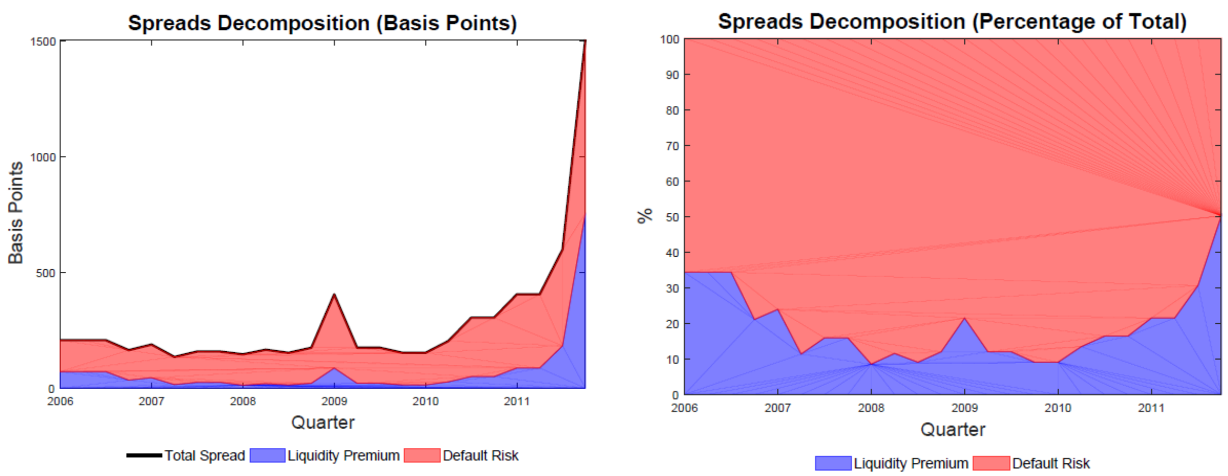
$$r^d(y, B) = \frac{\lambda + (1-\lambda) z}{\tilde{p}(y, B)} - \lambda.$$

Then, for each state (y, B, L_1) I decompose interest rate spreads, $S^R(y, B, L_1)$, into a default risk component, $S^d(y, B, L_1)$, and a liquidity component, $S^\ell(y, B, L_1)$, which are given by

$$\begin{aligned} S^R(y, B, L_1) &= S^d(y, B, L_1) + S^\ell(y, B, L_1), \\ S^d(y, B, L_1) &\equiv \left(1 + r^d(y, B)\right)^4 - (1+r)^4, \\ S^\ell(y, B, L_1) &= S^R(y, B, L_1) - S^d(y, B, L_1). \end{aligned}$$

The liquidity component captures both pure liquidity frictions plus the feedback interactions between liquidity risk and default risk. Figure 6 shows the results of this decomposition. In the left panel, the black solid line is the total interest rate spread, S_t^R, the red area represents the amount of the spread representing default risk while the blue area is the liquidity component. The right panel show the shares of total spread that are due to default risk and liquidity in red and blue, respectively.

Figure 6. Interest Rate Spreads Decomposition.



Note: The left panel shows the interest rate spread predicted by the model (black line) decomposed into a default risk component (red area) and a liquidity component (blue area). The right panel shows the same two components as a fraction of the total interst rate spread.

The model interpretation of the data is that liquidity frictions can significantly contribute to interest rate spreads. In addition, the contribution of liquidity premium to total interest rate spreads is about 50% in the last quarter of 2011, just before the debt restructuring. That is, as the economy's fundamental deteriorated, interest rate spreads were fueled by worsened liquidity conditions that induced even higher cost of credit.

7 Conclusions

In this paper, I incorporate endogenous liquidity frictions into a standard quantitative model of sovereign default. Liquidity is modeled by introducing directed search into the secondary market for sovereign bonds, where investors need to meet dealers in order to trade. Since search is directed, investors and dealers face a trade-off between the intermediation fee and the trading probability. For an investor, the higher the intermediation fee paid the higher the probability of trading. For a dealer, the higher the intermediation fee charged the lower the probability of trading. In addition, the optimal balance of this trade-off varies with the state of the economy. Thus, as trading probabilities and intermediation fees are endogenous and time varying, the liquidity of the secondary market for bonds is also endogenous and time varying over the business cycle.

This paper contributes to two different literatures. It provides a framework to study liquidity as an endogenous outcome in a model of sovereign debt with equilibrium default a la Eaton and Gersovitz (1981). The model provides a micro-foundation for transactions of bonds in the secondary market that highlights the importance of taking into account the size of trade flows to determine the price of both outstanding and newly issued bonds. In addition, this paper contributes to the literature OTC market that follows the work by Duffie et al. (2005). The proposed framework allows me to study the endogenous evolution of the liquidity of assets over the business cycle and its interactions with the equilibrium default risk. In addition, it takes into account that the issuer of the asset (the government in my model) responds to markets conditions by changing the supply of assets in the market. Taking into account the optimal response of the issuer is also important to understand the equilibrium default risk and liquidity premium as drivers of interest rate spreads.

The model presented in this paper is also useful to understand the effects of policy interventions in the secondary market. In section 4, I described two potential ways to implement direct bond purchases in the secondary market, which could resemble ECB interventions like the SMP. The mechanisms of the model rationalize how such interventions reduce the liquidity premium, default risk, and interest rate spreads. Analyzing the effects of secondary market interventions is not possible in models where trade flows do not affect equilibrium prices.

From a quantitative perspective, the model based decomposition of interest rates shows that liquidity premium significantly contributed to explain total spreads in the last debt crisis in Greece. Between $2006Q1 - 2011Q4$, the liquidity premium accounted for $10 - 50\%$ of the interest rate spreads predicted by the model. In addition, the liquidity component of interest rate spread is increasing in the probability of default and reaches its highest share on total spreads right before the default episode of $2012Q1$.

The analysis can be extended in several ways. One important dimension that is not considered in this framework is allowing for maturity choice. On the one hand, longer maturities allow government to rollover smaller amount of debt, making it easier to find enough buyers. This reduces the effect of liquidity risk on interest rate spreads. However, bonds with longer maturities are traded more times in secondary markets before maturing. This second effect increases liquidity premium. Which of the two effects dominates is an open research question. Another interesting aspect related to maturity choice is that there seems to be a trade-off between offering a wide set of maturities that satisfy the needs of different types of investors and the liquidity of each of the alternative bonds issued. In the data, governments tend to offer a wide range of maturities but usually a couple of them are much more liquid than the others. More research efforts are still needed to understand the optimal balance of this trade-off.

One more important issue omitted in this paper is the effects of changes in the risk free rate on liquidity conditions of risky bonds due to changes in investors' discount factor. Such changes would affect interest rate spreads by the usual channels studied in the literature[34] but could also generate interesting amplification dynamics in liquidity premium. In addition, taking into account investors' risk aversion opens the door to understand the interactions between bond and CDS markets. Both types of assets are traded in OTC markets and their liquidity conditions interact with each other. In models of risk neutral investors the amount of CDS contracts traded is indeterminate.

Finally, the model abstracts from government bonds held by domestic households. As pointed out by Broner et al. (2010) bonds transactions in secondary markets may rule out default episodes in equilibrium as foreign investors have incentives to sell bonds to domestic households and the government may not want to default on domestic households. However, my model predicts that secondary markets endogenously become more illiquid in bad times, exactly when foreign investors have incentives to transfer bonds to domestic ones. Moreover, the size of the flows matter in my model. Thus, the composition of debt holders across foreign and domestic investors may significantly affect the cost of bonds re-allocation.

[34] See Lizarazo (2013), Bocola and Dovis (2016), and Tourre (2017).

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A Proofs

A.1 Government's Problem

I begin by analyzing government's problem and showing that the value functions of the government have a unique solution for a given pricing function $p(s)$ and a given policy function $f_\ell^1(s)$.

Lemma 7 *Given any continuous pricing function $p(s)$, there exists a unique solution for the government's problem defined in equations (1) − (3).*

Proof. First, we show that given functions $\{V^D, p, \theta_\ell^1\}$, where function $\theta_\ell^1 \equiv \rho^{-1}\left(\frac{\gamma}{f_\ell^1}\right)$ and f_ℓ^1 is the policy function for type ℓ investor with bond holdings $a = 1$, there is a unique solution to the country's borrowing problem. Let $\bar{L} \equiv [0, B_{\max}]$ and $\mathcal{V}\left(\bar{Y} \times \bar{B} \times \bar{L}\right)$ be the space of all continuous real functions on $\bar{Y} \times \bar{B} \times \bar{L}$, bounded below and above by

$$\begin{aligned}
\underline{V} &\equiv V^D\left(\underline{y}\right), \\
\overline{V} &\equiv \frac{U\left(c_{\max}\right)}{1-\beta}, \\
c_{\max} &\equiv \bar{y} + \frac{1}{1+r}\left[B_{\max} - (1-\lambda) B_{\min}\right] - \left[\lambda + (1-\lambda) z\right] B_{\min}.
\end{aligned}$$

Now, for any function $f \in \mathcal{V}\left(\bar{Y} \times \bar{B} \times \bar{L}\right)$ define the mapping T^V by

$$\left(T^V f\right)\left(y, B, L_1; V^D, p, \theta_\ell^1\right) \equiv \max\left\{\max_{B'} U\left(c\left(y, B, L_1, B'\right)\right) + \beta \mathbb{E}_{y'|y} f\left(y', B', L_1'\right), V^D\left(y; p, \theta_\ell^1\right)\right\}.$$

with

$$c\left(y, B, L_1, B'\right) \equiv y + p\left(y, B, L_1, B'\right)\left[B' - (1-\lambda) B\right] - \left[\lambda + (1-\lambda) z\right] B.$$

Notice also that

$$L_1' = (1-\lambda)\left[1 - \alpha\left(\theta_\ell^1\left(y, B, L_1, B'\right)\right)\right]\left[L_1(1-\zeta) + \zeta B\right].$$

We now suppress the dependence of the mapping on $\left(V^D, p, \theta_\ell^1\right)$ for notational purposes. Now, notice that T^V is a monotone operator. That is, for any two function $f_1, f_2 \in$

$\mathcal{V}\left(\bar{Y}\times\bar{B}\times\bar{L}\right)$ such that $f_1 \geq f_2$, we have that

$$\begin{aligned}
\left(T^V f_2\right)(y,B,L_1) &= \max\left\{\max_{B'}\left\{U\left(c\left(y,B,L_1,B'\right)\right)+\beta\mathbb{E}_{y'|y}f_2\left(y',B',L_1'\right)\right\},V^D(y)\right\} \\
&= \max\left\{U\left(c\left(y,B,L_1,B_2'^{*}\right)\right)+\beta\mathbb{E}_{y'|y}f_2\left(y',B_2'^{*},L_1'\right),V^D(y)\right\} \\
&\leq \max\left\{U\left(c\left(y,B,L_1,B_2'^{*}\right)\right)+\beta\mathbb{E}_{y'|y}f_1\left(y',B_2'^{*},L_1'\right),V^D(y)\right\} \\
&\leq \max\left\{\max_{B'}\left\{U\left(c\left(y,B,L_1,B'\right)\right)+\beta\mathbb{E}_{y'|y}f_1\left(y',B',L_1'\right)\right\},V^D(y)\right\} \\
&= \left(T^V f_1\right)(y,B,L_1).
\end{aligned}$$

So,

$$Tf_1 \geq Tf_2.$$

Next we show that operator T^V satisfies discounting. That is, consider the mapping $(Tf+a)$ given by

$$\begin{aligned}
\left(T^V f+a\right)(y,B,L_1) &= \max\left\{\max_{B'}\left\{U\left(c\left(y,B,L_1,B'\right)\right)+\beta\mathbb{E}_{y'|y}\left[f\left(y',B',L_1'\right)+a\right]\right\},V^D(y)\right\} \\
&= \max\left\{\max_{B'}\left\{U\left(c\left(y,B,L_1,B'\right)\right)+\beta\mathbb{E}_{y'|y}\left[f\left(y',B',L_1'\right)\right]\right\}+\beta a,V^D(y)\right\} \\
&\leq \max\left\{\max_{B'}\left\{U\left(c\left(y,B,L_1,B'\right)\right)+\beta\mathbb{E}_{y'|y}\left[f\left(y',B',L_1'\right)\right]\right\}+\beta a,V^D(y)+\beta a\right\} \\
&= \max\left\{\max_{B'}\left\{U\left(c\left(y,B,L_1,B'\right)\right)+\beta\mathbb{E}_{y'|y}\left[f\left(y',B',L_1'\right)\right]\right\},V^D(y)\right\}+\beta a \\
&= (Tf)(y,B,L_1)+\beta a.
\end{aligned}$$

Therefore, operator T^V is a contraction mapping with modulus β, and hence it has a unique fixed point for any given functions $\left\{V^D,p,\theta_\ell^1\right\}$. In addition, since $U(\cdot)$ is continuous and using the theorem of the maximum we can show that T^V maps continuous functions into continuous functions. Now, given the fixed point of T^V, $V^*\left(y,B,L_1;V^D,p,\theta_\ell^1\right)$, we can just express the value of repayment as

$$V^R(y,B)=\max_{B'}\left\{U\left(c\left(y,B,L_1,B'\right)\right)+\beta\mathbb{E}_{y'|y}V^*\left(y',B',L_1'\right)\right\}.$$

In addition, we can solve for $V^D\left(y;p,\theta_\ell^1\right)$ using the mapping $\left(T^D f\right)\left(y;p,\theta_\ell^1\right)$ defined as

$$\left(T^D f\right)(y;p)=U\left(y^{def}\right)+\beta\mathbb{E}_{y'|y}\left[\phi V^*\left(y',0,0;f,p,\theta_\ell^1\right)+(1-\phi)f\left(y';p,\theta_\ell^1\right)\right].$$

Notice that $V^*\left(y, B, L_1; V^D, p, \theta_\ell^1\right)$ is non-decreasing in V^D. Therefore, it is straightforward to show that T^{VD} is a monotone mapping. Also, notice that for any $a \in \mathbb{R}$

$$\begin{aligned}
\left(T^V f\right)\left(y, B, L_1; V^D + a, p\right) &= \max\left\{\max_{B'}\left\{U\left(c\left(y, B, L_1, B'\right)\right) + \beta \mathbb{E}_{y'|y}\left[f\left(y', B', L_1'\right)\right]\right\}, V^D(y) + a\right\} \\
&\leq \max\left\{\max_{B'}\left\{U\left(c\left(y, B, L_1, B'\right)\right) + \beta \mathbb{E}_{y'|y}\left[f\left(y', B', L_1'\right)\right]\right\}, V^D(y)\right\} + a \\
&= \left(T^V f\right)\left(y, B, L_1; V^D, p, \theta_\ell^1\right) + a.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left[T^{VD}(f + a)\right]\left(y; p, \theta_\ell^1\right) &= U\left(y^{def}\right) + \beta \mathbb{E}_{y'|y}\left\{\phi V^*\left(y', 0, 0; f + a, p, \theta_\ell^1\right) + (1 - \phi)\left[f\left(y'; p, \theta_\ell^1\right) + a\right]\right\} \\
&\leq U\left(y^{def}\right) + \beta \mathbb{E}_{y'|y}\left\{\phi\left[V^*\left(y', 0, 0; f, p, \theta_\ell^1\right) + a\right] + (1 - \phi)\left[f\left(y'; p, \theta_\ell^1\right) + a\right]\right\} \\
&= U\left(y^{def}\right) + \beta \mathbb{E}_{y'|y}\left\{\phi V^*\left(y', 0, 0; f, p, \theta_\ell^1\right) + (1 - \phi) f\left(y'; p, \theta_\ell^1\right)\right\} + \beta a.
\end{aligned}$$

Thus, T^{VD} satisfies discounting. Therefore, there exists a fixed point for the mapping $\left(T^{VD} f\right)(y)$, $V^{D*}(y)$. Again, by continuity of $U(\cdot)$ and since $V^*(\cdot)$ is continuous we can show that T^{VD} maps continuous functions into continuous functions.

So, we can conclude that there is a unique pair of functions $V, V^D \in \mathcal{V}\left(\bar{Y} \times \bar{B} \times \bar{L}\right)$ that solve government's problem for any given continuous pricing function. ■

A.2 Investors Problems

Definition 8 *I say that for a given bond price p, an investor participates in the secondary market for bonds if there exists an intermediation fee $f \geq \gamma$ at which he would be willing to trade. I denote $\mathcal{P}_i^a(s)$ the set of prices at which investors with holdings $a \in \{0, 1\}$ and type $i \in \{\ell, h\}$ participates in secondary markets when state of the economy is $s \in \overline{S}$.*

Throughout this section, for each state $s \in \overline{S}$ we define we define the expected following expected continuation values

$$\begin{aligned}
E_h^0(s) &\equiv \mathbb{E}_{y'|y}\left[1 - \delta\left(y', B', L_1'\right)\right]\left[I_h^1(s') - I_h^0(s')\right], \\
E_h^1(s) &\equiv \mathbb{E}_{y'|y}\left[1 - \delta\left(y', B', L_1'\right)\right] I_h^1(s'), \\
E_\ell^1(s) &\equiv \mathbb{E}_{y'|y}\left[1 - \delta\left(y', B', L_1'\right)\right] I_\ell^1(s').
\end{aligned}$$

and the gains from trade, before including transaction costs, for each type of investor as

$$\begin{aligned} R_h^0(s) &\equiv -p(s) + \frac{1}{1+r} E_h^0(s), \\ R_h^1(s) &\equiv p(s) - \frac{1}{1+r} E_h^1(s), \\ R_\ell^1(s) &\equiv p(s) - \frac{1}{1+r} E_\ell^1(s). \end{aligned}$$

Before proving lemma 3 it is handy to show some intermediate results. First, I show that, for each state $s \in \overline{S}$, we can find price thresholds that determine the set of investors that participate in the secondary market. In addition, I show that there might be a price region in which the secondary market completely shuts down.

Lemma 9 *For any state $s \in \overline{S}$, If*

$$\frac{1}{1+r} E_\ell^1(s) + \gamma < \frac{1}{1+r} E_h^0(s) - \gamma, \tag{20}$$

there exist three price thresholds $\{\tilde{p}_1(s), \tilde{p}_2(s), \tilde{p}_3(s)\}$ such that

Proposition 10 *(i) If $p(s) < \tilde{p}_1(s)$, $p(s) \in \mathcal{P}_h^0(s)$ and $p(s) \notin \mathcal{P}_\ell^1(s), \mathcal{P}_h^1(s)$. Only type h investors with $a = 0$ participate in secondary markets;*

(ii) If $p(s) \in (\tilde{p}_1(s), \tilde{p}_2(s))$, $p(s) \in \mathcal{P}_h^0(s), \mathcal{P}_\ell^1(s)$ and $p(s) \notin \mathcal{P}_h^1(s)$. Both type h investors with $a = 0$ and type ℓ investors with $a = 1$ participate in secondary markets;

(iii) If $p(s) \in (\tilde{p}_2(s), \tilde{p}_3(s))$, $p(s) \in \mathcal{P}_\ell^1(s)$ and $p(s) \notin \mathcal{P}_h^0(s), \mathcal{P}_h^1(s)$. Only type ℓ investors with $a = 1$ participate in secondary markets; and

(iv) If $p(s) > \tilde{p}_3(s)$, $p(s) \in \mathcal{P}_\ell^1(s), \mathcal{P}_h^1(s)$ and $p(s) \notin \mathcal{P}_h^0(s)$. Both type ℓ investors with $a = 1$ and type h investors with $a = 1$ participate in secondary markets.

If the inequality in (20) is reversed, there exist three price thresholds $\{\tilde{p}_1(s), \tilde{p}_2(s), \tilde{p}_3(s)\}$ such that

(i) If $p(s) < \tilde{p}_1(s)$, $p(s) \in \mathcal{P}_h^0(s)$ and $p(s) \notin \mathcal{P}_\ell^1(s), \mathcal{P}_h^1(s)$. Only type h investors with $a = 0$ participate in secondary markets;

(ii) If $p(s) \in (\tilde{p}_1(s), \tilde{p}_2(s))$, $p(s) \notin \mathcal{P}_h^0(s), \mathcal{P}_\ell^1(s), \mathcal{P}_h^1(s)$. No investor has incentives to participate and secondary markets shut down;

(iii) If $p(s) \in (\tilde{p}_2(s), \tilde{p}_3(s))$, $p(s) \in \mathcal{P}_\ell^1(s)$ and $p(s) \notin \mathcal{P}_h^0(s), \mathcal{P}_h^1(s)$. Only type ℓ investors with $a = 1$ participate in secondary markets; and

(iv) If $p(s) > \tilde{p}_3(s)$, $p(s) \in \mathcal{P}_\ell^1(s), \mathcal{P}_h^1(s)$ and $p(s) \notin \mathcal{P}_h^0(s)$. Both type ℓ investors with $a = 1$ and type h investors with $a = 1$ participate in secondary markets.

Proof. Notice that investors with $a = 0$ can only participate in trades in which they purchase a bond and investors with $a = 1$ can only participate in trades where they sell a bond. We start by analyzing the case in which

$$\frac{1}{1+r}E_\ell^1(s) + \gamma < \frac{1}{1+r}E_h^0(s) - \gamma.$$

Here we define

$$\begin{aligned}
\tilde{p}_1(s) &\equiv \frac{1}{1+r}E_\ell^1(s) + \gamma, \\
\tilde{p}_2(s) &\equiv \frac{1}{1+r}E_h^0(s) - \gamma, \\
\tilde{p}_3(s) &\equiv \frac{1}{1+r}E_h^1(s) + \gamma.
\end{aligned}$$

Notice that if

$$p(s) < \tilde{p}_1(s) < \tilde{p}_2(s),$$

then no type ℓ investor is willing to participate in secondary markets since it will imply a negative gain from trade for all $f \geq \gamma$. In addition, assuming that $I_\ell^1(s') < I_h^1(s')$,[35] then we have that

$$R_h^1(s) < R_\ell^1(s),$$

so also type h investors with $a = 1$ have no incentives to sell their bonds. Finally, if

$$\tilde{p}_2(s) > p(s),$$

[35] This is true in equilibrium since it is more valuable to hold a bond when individuals are type h than when they are type ℓ.

we have that $R_h^0(s) > \gamma$. Therefore, type h investors with $a = 0$ participate on secondary markets. Next, if the price is such that

$$\tilde{p}_1(s) < p(s) < \tilde{p}_2(s),$$

we get that $R_\ell^1(s), R_h^0(s) > \gamma$. So, both of these types of investors participate on secondary markets. Also, since $I_h^0(s') \geq 0$, we have that

$$p(s) < \tilde{p}_2(s) \leq \tilde{p}_3(s),$$

which implies that $R_h^1(s) < \gamma$, and hence type h investors with $a = 1$ do not participate in secondary markets. Now, lets consider the case in which the price is such that

$$\tilde{p}_2(s) < p(s) < \tilde{p}_3(s).$$

Here we get that $R_\ell^1(s) > \gamma > R_h^1(s), R_h^0(s)$. Thus, type ℓ investors with $a = 1$ are the only ones that participate in secondary markets. Finally, if $\tilde{p}_3(s) < p(s)$, we get that $R_\ell^1(s) > R_h^1(s) > 0 > R_h^0(s)$, and all investors with $a = 1$ participate in secondary markets.

It remains to analyze the case in which

$$\frac{1}{1+r}E_\ell^1(s) + \gamma \geq \frac{1}{1+r}E_h^0(s) - \gamma.$$

Here we define

$$\begin{aligned}
\tilde{p}_1(s) &\equiv \frac{1}{1+r}E_h^0(s) - \gamma, \\
\tilde{p}_2(s) &\equiv \frac{1}{1+r}E_\ell^1(s) + \gamma, \\
\tilde{p}_3(s) &\equiv \frac{1}{1+r}E_h^1(s) + \gamma.
\end{aligned}$$

So, if $p(s) < \tilde{p}_1(s)$, we get that $R_h^0(s) > \gamma > R_\ell^1(s), R_h^1(s)$, and only type h investors with $a = 0$ participate in secondary markets. If $p(s) \in (\tilde{p}_1(s), \tilde{p}_2(s))$, then we have that $R_h^0(s), R_h^1(s), R_\ell^1(s) < \gamma$. So, secondary markets shut down since no investor has incentives to participate. In a similar way as before, it can be checked that if $p(s) \in (\tilde{p}_2(s), \tilde{p}_3(s))$

we get that $R_\ell^1(s) > \gamma > R_h^0(s), R_h^1(s)$, so only type ℓ investors with $a = 1$ participate in secondary markets. Finally, since the definition of $\tilde{p}_3(s)$ has not change, it is straightforward to see that $p(s) > \tilde{p}_3(s)$ implies that all investors holding a bond participate in secondary markets. This completes the proof. ■

Next, I show that whenever investors participate in the secondary market, optimal transaction fee $f_h^0(s)$ is decreasing and $f_\ell^1(s), f_h^1(s)$ are increasing in $p(s)$.

Lemma 11 *Given an aggregate state $s = (y, B, L_1, B')$ and taking government policy functions $\delta(y, B, L_1)$, $B'(y, B, L_1)$ as given:*

Proposition 12

(i) *The optimal submarket choice $f_h^0(s)$ is unique, continuous, and strictly decreasing in $p(s)$, for all $p(s) \in int(\mathcal{P}_h^0(s))$.*

(ii) *The optimal submarket choice $f_\ell^1(s)$ is unique, continuous, and strictly increasing in $p(s)$, for all $p(s) \in int(\mathcal{P}_\ell^1(s))$.*

(iii) *The optimal submarket choice $f_h^1(s)$ is unique, continuous, and strictly increasing in $p(s)$, for all $p(s) \in int(\mathcal{P}_h^1(s))$.*

(iv) *$f_i^a(s) = 0$ is optimal for all $p(s) \notin int(\mathcal{P}_i^a(s))$, all $i \in \{\ell, h\}$, and all $a \in \{0, 1\}$.*

Proof. Let the aggregate state of the economy be an arbitrary $s = (y, B, L_1, B') \in \overline{S}$. In all cases we focus on the price region in which investors are willing to participate in secondary markets, characterized in proposition 9.

(i) Using the free entry condition (10) we have that in any active submarket $\rho(\theta) f = \gamma$ and, by properties of the matching function we have that $\frac{\alpha(\theta)}{\rho(\theta)} = \theta$. So, we can re-write the problem of a type h with $a = 0$ as if the investor chooses θ instead of f. That is,

$$\begin{aligned} I_h^0(s) &= \max_\theta \alpha(\theta)\left[-p(s) + \frac{1}{1+r}\mathbb{E}_{y'|y}\left[1 - \delta(y', B', L_1')\right] I_h^1(s')\right] - \gamma\theta \\ &\quad + \frac{1 - \alpha(\theta)}{1+r}\mathbb{E}_{y'|y}\left[1 - \delta(y', B', L_1')\right] I_h^0(s'). \end{aligned}$$

Now, since $\alpha(\cdot)$ is differentiable, we can take first order condition with respect to θ to get

$$[\theta] : \alpha'(\theta)\left\{-p(s) + \frac{1}{1+r}\mathbb{E}_{y'|y}\left[1-\delta(y', B', L'_1)\right]\left[I_h^1(s') - I_h^0(s')\right]\right\} = \gamma.$$

We define

$$R_h^0(s) \equiv -p(s) + \frac{1}{1+r}\mathbb{E}_{y'|y}\left[1-\delta(y', B', L'_1)\right]\left[I_h^1(s') - I_h^0(s')\right],$$

which is independent of θ and decreasing in $p(s)$. Remember we focus in the region of prices in which $R_h^0(s)$ is positive, else investors would prefer not to purchase the asset and optimal tightness with be zero. Thus, we have that the optimal choice of θ in state s is given by

$$\theta_h^0(s) = \alpha'^{-1}\left(\frac{\gamma}{R_h^0(s)}\right).$$

Next, notice that since $\alpha(\cdot)$ is strictly concave, $\alpha'(\cdot)$ is strictly decreasing in its argument. Therefore, its inverse is also strictly decreasing. So, since $R_h^0(s)$ is decreasing in $p(s)$, $\gamma/R_0^0(s) > 0$, and $\alpha'^{-1}(\cdot)$ is strictly decreasing, we have that $\theta_h^0(s)$ is decreasing in $p(s)$. Finally, since in any open submarket $\rho(\theta) f = \gamma$ and $\rho(\cdot)$ is strictly decreasing, we have that $f_h^0(s)$ is strictly decreasing in $p(s)$. Continuity follows from continuity of $R_h^0(s)$ on $p(s)$ and by continuity of $\alpha'(\cdot)$.

(ii) Similarly, we can write the problem of type ℓ investors with $a = 1$ as

$$\begin{aligned} I_\ell^1(s) &= \lambda + (1-\lambda)(u_\ell + z) \\ &\quad + (1-\lambda)\max_\theta\{\alpha(\theta)p(s) - \gamma\theta \\ &\quad + (1-\lambda)\frac{[1-\alpha(\theta)]}{1+r}\mathbb{E}_{y'|y}\left[1-\delta(y', B', L'_1)\right] I_\ell^1(s')\}. \end{aligned}$$

Taking first order conditions we get that

$$[\theta] : \alpha'(\theta)\left\{p(s) - \frac{1}{1+r}\mathbb{E}_{y'|y}\left[1-\delta(y', B', L'_1)\right] I_\ell^1(s')\right\} = \gamma.$$

So, defining

$$R_\ell^1(s) \equiv p(s) - \frac{1}{1+r}\mathbb{E}_{y'|y}\left[1-\delta(y',B',L_1')\right] I_\ell^1(s'),$$

which is positive and increasing in $p(s)$, we can find that the optimal submarket choice is given by

$$\theta_\ell^1(s) = \alpha'^{-1}\left(\frac{\gamma}{R_\ell^1(s)}\right).$$

Using similar arguments than in (i) since $R_\ell^1(s)$ is strictly increasing in $p(s)$ we get that $\theta_\ell^1(s)$ is strictly increasing in $p(s)$ and so is $f_\ell^1(s)$. Continuity follows from continuity of $R_\ell^1(s)$ on $p(s)$ and by continuity of $\alpha'(\cdot)$.

(iii) Finally, we can write the problem of type h investors with $a=1$ as

$$\begin{aligned} I_h^1(s) &= \lambda + (1-\lambda)(u_h+z) + \zeta\left[I_\ell^1(s) - \lambda - (1-\lambda)(u_\ell+z)\right] \\ &\quad + (1-\lambda)(1-\zeta)\max_\theta \alpha(\theta)\{p(s)-\gamma\theta \\ &\quad + (1-\lambda)(1-\zeta)\frac{[1-\alpha(\theta)]}{1+r}\mathbb{E}_{y'|y}\left[1-\delta(y',B',L_1')\right] I_h^1(s')\}. \end{aligned}$$

Now, using the definition

$$R_h^1(s) \equiv p(s) - \frac{1}{1+r}\mathbb{E}_{y'|y}\left[1-\delta(y',B',L_1')\right] I_h^1(s'),$$

and taking first order condition, we get

$$[\theta] : \alpha'(\theta)\, R_h^1(s) = \gamma.$$

So, we have that

$$\theta_h^1(s) = \alpha'^{-1}\left(\frac{\gamma}{R_h^1(s)}\right),$$

and noticing that $R_h^1(s)$ is strictly increasing in $p(s)$ and similar arguments as before we get the proposed result.

■

Proof for Lemma 3

Now we are ready to prove the result state in lemma 3.

Proof. First, notice that for any given $s = (y, B, L_1, B')$, $ND(s)$ is continuous since $\alpha(\cdot)$ is continuous by assumption and by proposition 11 we know that $\theta_h^0(s)$ and $\theta_\ell^1(s)$ are also continuous in $p(s)$. In addition, notice in the case in which

$$\frac{1}{1+r} E_\ell^1(s) + \gamma < \frac{1}{1+r} E_h^0(s) - \gamma,$$

from proposition 9 there is always at least one type of investors participating in secondary markets, so from proposition 11 it follows that $ND(s)$ is strictly decreasing in $p(s)$. This is because $\alpha(\cdot)$ is a strictly increasing function, θ_h^0 is strictly decreasing in $p(s)$ so buyers' demand is strictly decreasing in $p(s)$, and because $\theta_\ell^1(s)$ and $\theta_h^1(s)$ are strictly increasing in $p(s)$ so then the negative of sellers' supply is strictly decreasing. In the case in which

$$\frac{1}{1+r} E_\ell^1(s) + \gamma \geq \frac{1}{1+r} E_h^0(s) - \gamma,$$

there is always at least one type of investor participating in secondary markets as long as $p(s) \notin [\tilde{p}_1(s), \tilde{p}_2(s)]$, so from proposition 11 it follows that $ND(s)$ is strictly decreasing in $p(s) \notin [\tilde{p}_1(s), \tilde{p}_2(s)]$. When, $p(s) \in [\tilde{p}_1(s), \tilde{p}_2(s)]$, from proposition 9 we know that there are no investors participating in the secondary market. Therefore, $ND(s) = 0$ and constant for the whole interval. ∎

A.3 Market Clearing Price

Proof to Proposition 4

Proof. Consider first the case in which s is such that, $ND(s;0) \leq B'(s) - (1-\lambda) B(s)$. Then, since $p \geq 0$ and from proposition 3 $ND(s;p)$ is decreasing in p, either $ND(s;0) = \max\{B'(s), 0\} - (1-\lambda) B(s)$ or there is no price such that $ND(s,p) = B'(s) - (1-\lambda) B(s)$. Therefore, for any $p \in \overline{P}$ there is an excess supply of bonds in the primary market (i.e. $ED(s;p) \leq 0$). Thus, the unique price consistent with (14) is $p(s) = 0$.

Next, consider the case in which s is such that $ND(s;0) > B'(s) - (1-\lambda) B(s)$. Here we have two cases. First, if

$$\frac{1}{1+r} E_\ell^1(s) + \gamma < \frac{1}{1+r} E_h^0(s) - \gamma,$$

from proposition 3 we know that $ND(s;p)$ is strictly decreasing in p, for all $p \in \mathbb{R}_+$. Then we just need to increase the price until $ND(s;p) = B'(s) - (1-\lambda)B(s)$. The second case, is the case in which

$$\tilde{p}_2(s) \equiv \frac{1}{1+r}E_\ell^1(s) + \gamma \geq \frac{1}{1+r}E_h^0(s) - \gamma \equiv \tilde{p}_1(s).$$

Now, if $ND(s,0) > B'(s) - (1-\lambda)B(s) > 0$, from proposition 9 we know that for any $p \in [0, \tilde{p}_1(s)]$ $ND(s)$ is strictly decreasing and continuous, and also we know that $ND(s, \tilde{p}_1(s)) = 0$, since above that price no type h investor with $a = 0$ is participating in secondary markets. Thus, by the intermediate value theorem, there must exist a price between 0 and $\tilde{p}_1(s)$ such that $ED(s, p(s)) = 0$. A similar argument applies if $B'(s) - (1-\lambda)B(s) < 0$. In this case we will find a price above $\tilde{p}_2(s)$ such that $ED(s, p(s)) = 0$. The only case that is a little more subtle is the case in which $B'(s) = (1-\lambda)B(s)$. Here, we have that government supply is zero. In addition, we know that for any $p(s) \in [\tilde{p}_1(s), \tilde{p}_2(s)]$, secondary markets shut down, so $ND(s, p(s)) = 0$. In this case, $ED(s, p(s)) = 0$ for any $p(s) \in [\tilde{p}_1(s), \tilde{p}_2(s)]$. This multiplicity arises because the government is not trying to sell or buy bonds, and investors have no incentives to participate in secondary markets. ■

A.4 Equilibrium Existence

In this section, I provide an equilibrium existence proof for the case in which the sets $\bar{Y}$ and $\bar{B}$ have a finite number of elements. Since to compute the model I have to discretize these sets this assumption does not imply a further loss of generality for the quantitative results.

Proposition 13 *The mapping $(T^p p)(y, B, L_1, B')$ defined as*

$$(T^p p)(s) = \begin{cases} \{x \in \mathbb{R}_+ : ED(s;x) = 0\} & \text{if } ND(s;0) > B'(s) - (1-\lambda)B(s) \\ 0 & \text{if } ND(s;0) \leq B'(s) - (1-\lambda)B(s) \end{cases}.$$

is a closed interval in $\mathbb{R}$ and the correspondence $(T^p p)(s)$ has a closed graph.[36]

[36] *Disclaimer: The following version of the proof corresponds to the case with $\lambda = 1$ and divisible assets. Check for updated versions of the paper for the proof for the model presented in section 2.*

Proof. By proposition 4, if

$$\frac{1}{1+r}E_{\ell}^{1}(s)+\gamma<\frac{1}{1+r}E_{h}^{0}(s)-\gamma,$$

the corrrespondence defined by $(T^p p)(s)$ is single-valued for each s, therefore $(T^p p)(s)$ is a function and is a closed interval in $\mathbb{R}_+$. If the signed of this inequality is reversed and $B'(s)=(1-\lambda)B(s)$, then we have that $p(s)\in[\tilde{p}_1(s),\tilde{p}_2(s)]$ which is also a closed interval in $\mathbb{R}_+$. has closed graph trivially. Thus, $(T^p p)(s)$ has closed graph trivially. ■

Proof of Proposition 6

Proof. As seen in proposition 7 I can define a composite mapping $\left[T^G\left(V,V^D\right)\right]\equiv\left[T^{VD}\left(V^D;T^V\left(V;V^D\right)\right)\right]$ $\mathcal{V}\left(\bar{Y}\times\bar{B}\times\bar{L}\right)\times\mathcal{V}\left(\bar{Y}\times\bar{B}\times\bar{L}\right)\to\mathcal{V}\left(\bar{Y}\times\bar{B}\times\bar{L}\right)\times\mathcal{V}\left(\bar{Y}\times\bar{B}\times\bar{L}\right)$, and it returns a pair of continuous functions on the compact space $\mathcal{V}\left(\bar{Y}\times\bar{B}\times\bar{L}\right)$ for any given continuous pricing function. Since these functions are single valued then $\left[T^G\left(V,V^D\right)\right](y,B,L_1)$ is trivially a closed set in $\mathbb{R}^3$, convex-valued, and has closed graph for each (y,B,L_1). Then, the mapping $\left[T^G\left(V,V^D\right)\right]=\prod_{(y,B,L_1)\in\bar{Y}\times\bar{B}\times\bar{L}}\left[T^G\left(V,V^D\right)\right](y,B,L_1)$ inherits all these properties (See Aliprantis and Border (2006) Theorem 17.23 and Theorem 17.28).

I first define an upper bound for the space of prices. This upper bound is given by

$$p_{\max}\equiv\left\{x\in\mathbb{R}_+:U\left(y_{\max}-x\left(1-\lambda\right)B_1-\left[\lambda+\left(1-\lambda\right)z\right]B_1\right)+\beta\mathbb{E}_{y'|y}V\left(y',0,0\right)=V^D\left(y_{\min}\right)\right\},$$

where B_1 is the smaller element in $\bar{B}$ that is greater than zero. That is, the maximum price that the government would be willing to pay to go from $B=B_1$ to $B=0$ and staying in good credit standings rather than going from $B=B_1$ to $B=0$ by defaulting. As in proposition 13, I define the price correspondence $(T^p p)(s)\in\bar{P}\equiv[0,p_{\max}]$ as

$$(T^p p)(s)==\begin{cases}\{x\in\mathbb{R}_+:ED(s;x)=0\} & \text{if } ND(s;0)>B'(s)-(1-\lambda)B(s)\\ 0 & \text{if } ND(s;0)\leq B'(s)-(1-\lambda)B(s)\end{cases}.$$

Thus, $(T^p p)(s)$ is the set of prices for $s=(y,B,L_1,B')$ that are consistent with bonds market clearing given the price function $p(\cdot)$. Now, by proposition 13, $(T^p p)(s)$ is a closed interval in $\mathbb{R}$ and $(T^p p)(s)$ has a closed graph. Therefore, $(T^p p)(s)$ is an upper hemicontinuous correspondence. For any $p\in\bar{P}$, let $(T^p p)\in\bar{P}$ be the product correspondences $\prod_{s\in\bar{S}}(T^p p)(s)$.

Since $(T^p p)(s)$ is convex-valued for each s, the product correspondence is convex-valued as well. Furthermore, since $(T^p p)(s)$ is upper hemicontinuous with compact values for each s, the product correspondence $(T^p p)$ is also upper hemicontinuous with compact values (See Aliprantis and Border (2006) Theorem 17.28). Therefore, $(T^p p)$ is a closed convex-valued correspondence that takes elements of the compact, convex set $\bar{P}$ and returns sets in $\bar{P}$.

In addition, we have can define the correspondence $\left(T^D D\right)(B, L_1) \in [0,1]$.

$$\left(T^D D\right)(B, L_1) \equiv \left\{y \in Y : V^R(y, B, L_1) \leq V^D(y)\right\}$$

which determines the set of states (y, B, L_1) such that default is weakly optimal and consistent with government's optimal default set $D(B, L_1)$. Since$\left(T^D D\right)(B, L_1)$ is a closed interval in $\mathbb{R}$ and $\left(T^D D\right)(B, L_1)$ has a closed graph. Therefore, $\left(T^D D\right)(B, L_1)$ is an upper hemicontinuous correspondence. So, for any $D(B) \in \bar{Y}$, let $\left(T^D D\right) \in \bar{Y}$ be the product correspondences $\prod_{(B,L_1)\in \bar{B}\times\bar{L}} \left(T^D D\right)(B, L_1)$. Since $\left(T^D D\right)(B, L_1)$ is convex-valued for each B, the product correspondence is convex-valued as well. Furthermore, since $\left(T^D D\right)(B, L_1)$ is upper hemicontinuous with compact values for each (B, L_1), the product correspondence $\left(T^D D\right)$ is also upper hemicontinuous with compact values. Therefore $\left(T^D D\right)$ is a closed convex-valued correspondence that takes elements of the compact, convex set $\bar{Y}$ and returns sets in $\bar{Y}$.

Finally, define $T^E(D, p) \equiv \left[T^G\left(V, V^D\right)\right] \times \left(T^D D\right) \times (T^p p)$. This is a closed convex-valued correspondence that takes convex sets $\mathcal{V}\left(\bar{Y} \times \bar{B} \times \bar{L}\right) \times \mathcal{V}\left(\bar{Y} \times \bar{B} \times \bar{L}\right) \times \bar{Y} \times \bar{P}$ and returns sets in $\mathcal{V}\left(\bar{Y} \times \bar{B} \times \bar{L}\right) \times \mathcal{V}\left(\bar{Y} \times \bar{B} \times \bar{L}\right) \times \bar{Y} \times \bar{P}$, as the product of $\left[T^G\left(V, V^D\right)\right]$, $\left(T^D D\right)$, and $(T^p p)$ (again using Aliprantis and Border (2006) Theorem 17.28). Therefore, by Kakutani-Fan-Glicksberg fixed point theorem (See Aliprantis and Border (2006) Theorem 17.55) there is a $\left(V^*, V^{D*}, D^*, p^*\right) \in \mathcal{V}\left(\bar{Y} \times \bar{B} \times \bar{L}\right) \times \mathcal{V}\left(\bar{Y} \times \bar{B} \times \bar{L}\right) \times \bar{Y} \times \bar{P}$ such that $\left(V^*, V^{D*}, D^*, p^*\right) \in T^E\left(V^*, V^{D*}, D^*, p^*\right)$. Hence, there is an equilibrium pair of government value function $\left(V^*, V^{D*}\right)$, an equilibrium default set $D^*(B, L_1)$, and an equilibrium bond price function $p^*(s)$. $\blacksquare$

B Solution Algorithm

B.1 The Approximated Model

I first show the equations of the approximated model that I solve. As in Chatterjee and Eyigungor (2012), I solve an approximation of my model in which I add an *i.i.d.* output shock $m_t \sim trunc\,\mathcal{N}\left(0, \sigma_m^2\right)$ with $m_t \in [-\bar{m}, \bar{m}]$. The shock m is realized at the same time as the shock y. So, the problem of the government is now given by

$$\begin{aligned}
V\left(y,m,B,L_1\right) &= \max_{\delta\in\{0,1\}} \left(1-\delta\right) V^R\left(y,m,B,L_1\right) + \delta V^D\left(y\right), \\
V^D\left(y\right) &= U\left(h\left(y\right)\right) + \beta \mathbb{E}_{y',m'|y}\left[\phi V\left(y',m',0,0\right) + \left(1-\phi\right) V^D\left(y'\right)\right] \\
V^R\left(y,m,B,L_1\right) &= \max_{B'}\{U\left(y + p\left(y,B,L_1,B'\right)\left[B' - \left(1-\lambda\right)B\right] - \left(\lambda+z\right)B\right) \\
&\quad + \beta\mathbb{E}_{y',m'|y} V\left(y',m',B',L_1'\right)\}.
\end{aligned}$$

The value functions of the investors, for all $s = \left(y, B, L_1, B'\right)$, are given by

$$\begin{aligned}
I_h^0\left(s\right) &= \max_\theta \left\{\alpha\left(\theta\right)\left(-p\left(s\right) + \frac{1}{1+r^f}\mathbb{E}_{y',m'|y}\left[1-\delta\left(y',m',B',L_1'\right)\right]\left[I_h^1\left(s'\right) - I_h^0\left(s'\right)\right]\right) - \gamma\theta\right\} \\
&\quad + \frac{1}{1+r^f}\mathbb{E}_{y',m'|y}\left[1-\delta\left(y',m',B',L_1'\right)\right] I_h^0\left(s'\right)
\end{aligned}$$

$$\begin{aligned}
I_h^1\left(s\right) &= \left(1-\lambda\right)\left(1-\zeta\right)\max_\theta\left\{\alpha\left(\theta\right)p\left(s\right) - \gamma\theta + \frac{1-\alpha\left(\theta\right)}{1+r^f}\mathbb{E}_{y',m'|y}\left[1-\delta\left(y',m',B',L_1'\right)\right] I_h^1\left(s'\right)\right\} \\
&\quad + \zeta\left[I_\ell^1\left(s\right) - \lambda - \left(1-\lambda\right)\left(z+u_\ell\right)\right] + \lambda + \left(1-\lambda\right)\left(u_h + z\right)
\end{aligned}$$

$$I_\ell^1\left(s\right) = \lambda + \left(1-\lambda\right)\left(u_\ell + z\right) + \left(1-\lambda\right)\max_\theta\left\{\alpha\left(\theta\right)p\left(s\right) - \theta\gamma + \frac{1-\alpha\left(\theta\right)}{1+r^f}\mathbb{E}_{y',m'|y}\left[1-\delta\left(y',m',B',L_1'\right)\right] I_\ell^1\left(s'\right)\right\}$$

where $s' = \left(y', B', L_1', B''\left(y',m',B',L_1'\right)\right)$.So, the optimal tightness are given by

$$\begin{aligned}
\theta_h^0\left(s\right) &= \alpha'^{-1}\left(\frac{\gamma}{-p\left(s\right) + \frac{1}{1+r^f}\mathbb{E}_{y',m'|y}\left[1-\delta\left(y',m',B',L_1'\right)\right]\left[I_h^1\left(s'\right) - I_h^0\left(s'\right)\right]}\right) \times \mathbb{I}_{\left\{R_h^0(s)>\gamma\right\}}, \\
\theta_h^1\left(s\right) &= \alpha'^{-1}\left(\frac{\gamma}{p\left(s\right) - \frac{1}{1+r^f}\mathbb{E}_{y',m'|y}\left[1-\delta\left(y',m',B',L_1'\right)\right] I_h^1\left(s'\right)}\right) \times \mathbb{I}_{\left\{R_h^1(s)>\gamma\right\}} \\
\theta_\ell^1\left(s\right) &= \alpha'^{-1}\left(\frac{\gamma}{p\left(s\right) - \frac{1}{1+r^f}\mathbb{E}_{y',m'|y}\left[1-\delta\left(y',m',B',L_1'\right)\right] I_\ell^1\left(s'\right)}\right) \times \mathbb{I}_{\left\{R_\ell^1(s)>\gamma\right\}},
\end{aligned}$$

where $\mathbb{I}_{\{x\}}$ is an indicator function that takes value equal one when x is true and zero otherwise, and $R_i^a(s)$ are defined as in section 3.2.2. Next, I will define

$$\begin{aligned}
\hat{Z}(y, B', L_1') &\equiv \mathbb{E}_{y',m'|y} V(y', m', B', L_1') \\
\hat{E}_h^0(y, B', L_1') &\equiv \mathbb{E}_{y',m'|y} [1 - \delta(y', m', B', L_1')] \left[I_h^1(s') - I_h^0(s')\right] \\
\hat{E}_h^1(y, B', L_1') &\equiv \mathbb{E}_{y',m'|y} [1 - \delta(y', m', B', L_1')] I_h^1(s') \\
\hat{E}_\ell(y, B', L_1') &\equiv \mathbb{E}_{y',m'|y} [1 - \delta(y', m', B', L_1')] I_\ell^1(s').
\end{aligned}$$

Notice that $L_1'(s) = (1-\lambda)\left[1 - \alpha\left(\theta_\ell^1(s)\right)\right][(1-\zeta)L_1 + \zeta B]$, so I can also write

$$\begin{aligned}
\hat{Z}(s) &\equiv \mathbb{E}_{y',m'|y} V(y', m', B', L_1'(s)) \\
\hat{E}_h^0(s) &\equiv \mathbb{E}_{y',m'|y} [1 - \delta(y', m', B', L_1'(s))] \left[I_h^1(s') - I_h^0(s')\right] \\
\hat{E}_h^1(s) &\equiv \mathbb{E}_{y',m'|y} [1 - \delta(y', m', B', L_1'(s))] I_h^1(s') \\
\hat{E}_\ell^1(s) &\equiv \mathbb{E}_{y',m'|y} [1 - \delta(y', m', B', L_1'(s))] I_\ell^1(s').
\end{aligned}$$

Then, knowing the law of motion for L_1, and the policy functions δ and $B'(y, m, B, L_1)$, I can pin down

$$\begin{aligned}
\theta_h^0(s) &= \alpha'^{-1}\left(\frac{\gamma}{-p(s) + \frac{1}{1+r^f}\hat{E}_h^0(s)}\right) \times \mathbb{I}_{\left\{R_h^0(s) > \gamma\right\}} \\
\theta_h^1(s) &= \alpha'^{-1}\left(\frac{\gamma}{p(s) - \frac{1}{1+r^f}\hat{E}_h^1(s)}\right) \times \mathbb{I}_{\left\{R_h^1(s) > \gamma\right\}} \\
\theta_\ell^1(s) &= \alpha'^{-1}\left(\frac{\gamma}{p(s) - \frac{1}{1+r^f}\hat{E}_\ell^1(s)}\right) \times \mathbb{I}_{\left\{R_\ell^1(s) > \gamma\right\}}.
\end{aligned}$$

Using this results in the result in the excess demand function

$$\begin{aligned}
ED(s) \equiv\ & \alpha\left(\theta_h^0(s)\right)\left[\overline{I} - (1-\lambda)B\right] - [B' - (1-\lambda)B] \\
& -\alpha\left(\theta_\ell^1(s)\right)(1-\lambda)[\zeta B + (1-\zeta)L_1] - \alpha\left(\theta_h^1(s)\right)(1-\lambda)(1-\zeta)(B - L_1),
\end{aligned}$$

I can pin down the unique price that is consistent with $p(s)\,ED(s) = 0$. These steps provide me an algorithm to solve the model. This algorithm begins with a guess of the price schedule, updates the policy functions of the government, solves for investors optimal submarket choice as a function of price $p(s)$ in each state $s \in \overline{S}$, and finally finds the price $p(s)$ consistent with zero excess demand in primary markets, for each $s \in \overline{S}$.

C Data Description

C.1 Data Sources

I collect the following time series for Greek economy.

National Accounts. Quarterly time series from 1995Q1 - 2017Q4 for consumption, exports, imports, and GDP are obtained from Eurostats. I use seasonal adjusted and calendar adjusted chain-linked (2010) in million euros.

Debt and Investment Position. Data on net international investment position (as % of GDP) for the period 2003Q4 - 2017Q4 is obtained from Eurostat. Public and Publicly Granted debt (as % of GDP) for the period 2000Q1 - 2017Q3 is obtained from the World Development Indicators database from the World Bank.

Interest Rates and Spreads. Interest rates data is collected from Eurostats and Bloomberg. Interest rate spreads is calculated as the difference in the annual interest rates between Greek and German long term government yields in Eurostats. Long term debt yields are composed from central government bonds with residual maturity of around 10 years. Computing interest rate spreads using generic central government bonds from Greece and Germany collected from bloomberg results in almost identical time series. Data on daily bid and ask prices is collected from Bloomberg. I compute quarterly time series using average of active days in each quarter. I use time series for bid and ask prices for 10 year bonds but using bid and ask prices for 5 year bonds results in almost identical bid-ask spreads time series. The problem with 5 year bonds is that the time series has some missing values.

Secondary Market Volumes and Turnover Rates. Information on secondary market trade volumes is obtained from the electronic secondary securities market (HDAT) available at the Bank of Greece website.[37] Monthly traded volumes is available from January 2001 to December 2017. Quarterly time series are calculated as the sum of monthly traded volumes.

D Details on the Calibration

D.1 Calculating utility from holding bonds

I calibrate preferences bonds u_h and u_ℓ such that they satisfy to conditions: (i) that ex-ante expected utility from holding the asset is zero for a given ζ, and (ii) such that $u_h - u_\ell$ targets the average bid-ask spread in the data. Condition (i) imposes the following restriction

$$\begin{aligned}
0 &= u_h + \sum_{j=1}^{\infty}\left\{\left[\frac{\Pr\{\delta_j = 0\}}{1+r}(1-\lambda)(1-\zeta)\right]^j u_h + \left[\frac{\Pr\{\delta_j = 0\}}{1+r}\zeta\left(1-\alpha\left(\theta_{\ell,j}\right)\right)\right]^j u_\ell\right\} \\
&= \sum_{j=0}^{\infty}\left[\frac{\Pr\{\delta_j = 0\}}{1+r}(1-\lambda)(1-\zeta)\right]^j u_h + \sum_{j=1}^{\infty}\left[\frac{\Pr\{\delta_j = 0\}}{1+r}\zeta\left(1-\alpha\left(\theta_{\ell,j}\right)\right)\right]^j u_\ell \\
&\approx \sum_{j=0}^{\infty}\left[\frac{\bar{\delta}}{1+r}(1-\lambda)(1-\zeta)\right]^j u_h + \sum_{j=1}^{\infty}\left[\frac{\bar{\delta}}{1+r}\zeta(1-\bar{\alpha})\right]^j u_\ell \\
&= \frac{(1+r)\,u_h}{1+r-\bar{\delta}(1-\lambda)(1-\zeta)} + \frac{(1+r)\,u_\ell}{1+r-\bar{\delta}\zeta(1-\bar{\alpha})}
\end{aligned}$$

$$\implies u_h = -u_\ell \frac{1+r-\bar{\delta}(1-\lambda)(1-\zeta)}{1+r-\bar{\delta}\zeta(1-\bar{\alpha})}.$$

Notice that $\bar{\alpha} \equiv \sum_{j=1}^{\infty}\alpha\left(\theta_{\ell,j}\right)$ does not depend on u_h.

D.2 Computing Turnover Rates

The turnover rate is 78% in HDAT in Greece is per quarter. This includes transactions between dealers and investors as well as interdealer transactions. To calculate the turnover rate in the model we have to compute all the transactions that happen in secondary markets between dealers and investors and also interdealers transactions in primary markets. This is not exactly what happens in reality as some dealers hold inventories and do not need to trade with other dealers and some other trades occur through a long chain of dealer to dealer

[37] https://www.bankofgreece.gr/Pages/en/Markets/HDAT/statistics.aspx

transactions. We will assume that the number of transactions in the model approximates the amount of transactions in the data. In the model, we can calculate the amounts of transactions in both the primary and secondary market as follows. In the primary market, whenever the government issues debt it is purchased by a dealer. Then, the number of transactions in the primary market is given by

$$\max\{B', 0\} - (1-\lambda) B.$$

Now, in the secondary markets, the following trades occur between a dealer and an investor:

$$\begin{aligned} \text{New seller to dealer} &: \alpha\left(\theta_\ell^1\right)\zeta(1-\lambda) H_1 \\ \text{Old seller to dealer} &: \alpha\left(\theta_\ell^1\right)(1-\lambda) L_1 \\ \text{Potential } h \text{ seller to dealer} &: \alpha\left(\theta_h^1\right)(1-\lambda)(1-\zeta) H_1 \\ \text{Dealer to old buyer} &: \alpha\left(\theta_h^0\right) H_0 \\ \text{Dealer to new buyer} &: \alpha\left(\theta_h^0\right)\lambda B. \end{aligned}$$

Since we do not model the amounts of trades in the interdealers market this is the minimum amount of transactions in secondary markets. The maximum amount of transactions adds to these transactions the maximum amount of possible trades between dealers in the interdealers market, so we have to clean the turnover rate in the data to remove those trades. This is the sum of the following trades.

$$\begin{aligned} \text{Primary buyers to selling dealers} &: \max\{B', 0\} - (1-\lambda) B \\ \text{Secondary buyers to selling dealers} &: \alpha\left(\theta_\ell^1\right)\zeta(1-\lambda) H_1 + \alpha\left(\theta_\ell^1\right)(1-\lambda) L_1 \alpha\left(\theta_h^1\right)(1-\lambda)(1-\zeta) H_1. \end{aligned}$$

Since primary market clears we can simplify this calculation using total purchases by investors,

$$\text{Total investors' purchases} : \alpha\left(\theta_h^0\right) H_0 + \alpha\left(\theta_h^0\right)\lambda B = \alpha\left(\theta_h^0\right)\left[\overline{I} - (1-\lambda) B\right].$$

Then, the maximum amount of transactions in secondary markets, given that interdealers market is competitive, is given by

$$X^{\max} = 2\left[\alpha\left(\theta_h^0\right)H_0 + \alpha\left(\theta_h^0\right)\lambda B\right] + \alpha\left(\theta_\ell^1\right)\zeta\left(1-\lambda\right)H_1 + \alpha\left(\theta_\ell^1\right)\left(1-\lambda\right)L_1 + \alpha\left(\theta_h^1\right)\left(1-\lambda\right)\left(1-\zeta\right)H_1.$$

In reality the minimum amount of trades in secondary markets is given by

$$X^{\min} = \alpha\left(\theta_h^0\right)H_0 + \alpha\left(\theta_h^0\right)\lambda B + \alpha\left(\theta_\ell^1\right)\zeta\left(1-\lambda\right)H_1 + \alpha\left(\theta_\ell^1\right)\left(1-\lambda\right)L_1 + \alpha\left(\theta_h^1\right)\left(1-\lambda\right)\left(1-\zeta\right)H_1.$$

This happens when the same dealer is connecting both the investor selling and the investor buying and just acting as a bridge. As mentioned before, the number of transactions in reality could be lower or higher than X^{SM} as long chains of dealers would be require to transfer one bond from an investor to another one. Li and Schurhoff (2018) find that for municipal bonds in United States, the average chain involves 1.5 dealers. So, we can compute an intermediate amount of trades in secondary markets as

$$X^{mean} = 1.5\left[\alpha\left(\theta_h^0\right)H_0 + \alpha\left(\theta_h^0\right)\lambda B\right] + \alpha\left(\theta_\ell^1\right)\zeta\left(1-\lambda\right)H_1 + \alpha\left(\theta_\ell^1\right)\left(1-\lambda\right)L_1 + \alpha\left(\theta_h^1\right)\left(1-\lambda\right)\left(1-\zeta\right)H_1.$$

We will use X^{mean} to compute the model's implied turnover rate as

$$\text{Turnover rate} = \frac{X^{mean}}{B}.$$