I study a dynamic principal-agent relationship in which the principal invests costly resources in a project of uncertain quality to induce costly effort from an agent. The principal observes the output from the project privately and can be either informed (has learned that project quality is high) or uninformed. The agent learns about project quality through the investments made by the principal. The principal wants to invest less when pessimistic about project quality; however, the agent demands higher investment when pessimistic to exert effort. The principal faces the trade-off between investing optimally and transmitting information about project quality to the agent. The principal’s optimal equilibrium features full information transmission when the uninformed principal has high beliefs (probability that project quality is high) and no information transmission at low beliefs. The informed principal may invest at sub-optimally high levels early in the relationship, but eventually, optimality is restored. That is, the principal’s optimal equilibrium may exhibit distortions in the short run but not in the long run.

Keywords: dynamic agency, experimentation, private learning

JEL codes: C73, M51, D83, D86

1. Introduction

In most organizations, supervisors have a role beyond the passive job of writing contracts with employees and disseminating the terms of the contract. They also play an active role in production, investing resources to augment the quality or quantity of the output produced. When working jointly with an employee on a project of uncertain quality, if a supervisor has private information (such as past output from the project), her investments in the project may transmit her private information to the employee. This
may not be desirable — it may be costly to induce effort from an employee who learns that the project is unlikely to produce output. In such situations, a supervisor faces the trade-off between investing optimally\(^2\) in the project based on her private information and transmitting information about quality of the project to the employee. How should the supervisor manage this trade-off over time? In particular, should the supervisor’s investment be sensitive to her private information? If yes, when and how?

I pursue these questions in a dynamic principal-agent model in which the principal learns privately about a project of uncertain quality (good or bad) over time. The principal invests costly resources into the project. The agent learns about the quality of the project through the investments made by the principal and exerts costly effort. The agent’s motivation (willingness to exert effort) increases in the principal’s investment and the agent’s belief (probability that project quality is good). The principal’s willingness to invest increases with her belief. I solve for the principal’s optimal equilibrium and find three main insights. First, if the relationship is in a optimistic stage (both players have high beliefs), the principal ignores information transmission concerns and invests optimally, i.e. the principal’s investment is sensitive to her private information. Second, if the relationship moves to a pessimistic stage, or starts at a pessimistic stage, the principal ignores her private information when investing — to prevent any further information transmission and thereby preserving the motivation of the agent to exert effort. Third, if the relationship starts at a pessimistic stage, a principal who learns that the project is good, may still be forced to invest sub-optimally. However, such distortions are transient and disappear eventually.

More specifically, I study a continuous time principal-agent relationship where the principal seeks to induce effort from the agent on a project of unknown quality, either good or bad. Initially, both players have a common prior \(\mu_0\) about the quality of the project being good. At each instant, the principal chooses how much to invest \((x_t)\) in the project at a cost that is convex in investment. In the same instant, but after observing the principal’s investment, the agent chooses whether to exert effort at a cost or not. Not exerting effort is cost-less. The agent’s effort is observable, but not contractible. Conditional on the quality of the project being good and the agent exerting effort, lump-sum output worth \(1 + x_t\) arrives at the exponential rate of \(\lambda\). Output is privately observed by the principal. The principal’s investment does not affect the arrival rate of output, however it affects the value of the output if output arrives. That is, the principal can augment the output by investing, but cannot produce an output in the absence of effort from the agent. There is no output if either the project quality is bad or the agent does not exert effort. At the beginning of the relationship, both players agree to a non-negotiable sharing rule of output where the principal receives a share \(\gamma\) and the agent receives a share \(1 - \gamma\) of the output.

\(^2\)Here, optimal investment means the value maximizing investment in the absence of information transmission concerns.
Both players discount future payoffs at the rate $r$. The outside option of both players are normalized to zero. There are no transfers.

At any point in the relationship the principal is one of two types. The “high” type, or type $H$ principal is one who has observed an output and knows that the project is of good quality. The “low” type, or type $L$ principal is one who has not observed an output. Since the investment cost of the principal is convex, in the absence of information transmission concerns, the principal’s optimal investment is increasing in her belief that the project is of good quality. I study Markov Perfect Bayesian Equilibria, henceforth equilibria, of this game. The relevant state variable is the pair of beliefs $(\mu, \mu^a)$ where $\mu$ is the belief of the type $L$ principal that the project is of good quality and $\mu^a$ is the belief of the agent that she faces the type $H$ principal.

The key question I ask is the following: How does the principal manage the trade-off between her current value and future value by controlling the information transmitted to the agent? I find that the answer to this question is critically dependent on the cost of effort of the agent, which captures the severity of misalignment of incentives of the players.

When the cost of effort is low (below some $c^*$), I show that the principal optimal equilibrium is fully separating at every belief of the type $L$ principal (Proposition 2). That is, there is full information transmission to the agent. The type $H$ principal always chooses her optimal investment and the type $L$ principal chooses an investment strictly lower than the type $H$ principal’s optimal investment. At intermediate beliefs type $L$ principal invests above her optimal level to induce effort from the agent and when beliefs are sufficiently low she quits the relationship. The misalignment in the incentives of players results in the type $L$ principal investing above her optimal investment to motivate the agent to exert effort at intermediate beliefs. However, the misalignment is not sufficiently high for it to be optimal to stop the transmission of information about project quality to the agent in equilibrium.

When the cost of effort for the agent is high (above $c^*$), the misalignment in incentives of the players is enough that in the principal optimal equilibrium, in addition to investing above her optimal investment, the type $L$ principal also stops the flow of information to the agent when her beliefs are sufficiently low. In this case, the nature of the principal optimal equilibrium is sensitive to the initial prior $\mu_0$. When $\mu_0$ is high (above some $\bar{\mu}$), the principal optimal equilibrium has three regions of interest in the space of type $L$ principal’s beliefs (Proposition 4). When beliefs are high, there is separation and transmission of information to the agent. However, once beliefs reach a cut-off ($\bar{\mu}$), the pooling phase begins. Both types of principal pool on the type $H$ principal’s optimal investment until beliefs reach another cutoff $\mu$ at which point the type $L$ principal quits the relationship. By stopping the

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$^3$Since the agent does not observe output during the course of the relationship, I interpret the rate of discounting $r$ as the exogenous rate at which the game ends resulting in the agent realizing his share of the accumulated output.
flow of information to the agent at $\bar{\mu}$, the type $L$ principal exploits the agent’s uncertainty in the pooling phase and is able to reduce her investment while inducing effort from the agent.

When $\mu_0$ is intermediate ($\mu_0 \in [\mu_g, \bar{\mu}]$), the separation region does not exist (Proposition 5). Both type of principals start with pooling until beliefs reach $\mu$ at which point type $L$ principal quits. Although qualitatively similar to the case with high initial prior, the nature of investment during pooling is different in this case. When starting with high initial prior, the type $L$ principal optimally chooses when to start pooling ($\bar{\mu}$). However, when initial belief is lower than $\bar{\mu}$, such a luxury does not exist. Since the relationship begins with the agent being more pessimistic ($\mu_0 < \bar{\mu}$) the average investment needed during the pooling phase to induce effort from the agent is strictly higher compared to the case when $\mu_0 > \bar{\mu}$. This implies in this case the average pooling investment is strictly higher than type $H$ principal’s optimal investment. When $\mu_0 < \mu_g$, the average pooling investment needed is high enough that the relationship does not even start.

This leads to an important observation. When the cost of effort is high and the relationship starts at intermediate beliefs, type $H$ principal invests sub-optimally during the pooling phase. This results from the inability of the type $H$ principal to separate herself from type $L$ principal. The sub-optimality of type $H$ principal’s investment is not perpetual. Once the type $L$ principal quits the relationship, type $H$ principal is revealed to the agent and continues to invest optimally thereafter. That is to say, the type $H$ principal’s investment may be sub-optimal in the short run, but in the long run, optimality is restored.

The above insights can be useful in understanding the optimal behavior of managers. In organizations, workers are given feedback through periodic evaluations and appraisals. These represent the cost-less (cheap talk) channel through which managers provide information to workers to motivate them to exert effort. However, how workers learn about the relevant aspects of the production environment (project’s quality/agent’s ability) is not limited to these periodic chunks of information. The day to day actions of a manager, how much interest they show in a workers activities by investing their resources credibly transmits the private information of the manager to the workers affecting their motivation to work. Accounting for this channel of credible information transmission may enhance how managers motivate their workers and better achieve organizational goals.

The paper is organized as follows. I review the connection of this paper with literature in Section 2, followed by a one period example in Section 3. In Section 4, I present the model and then present results in Section 5. Extensions are discussed in Section 6. All proofs are relegated to the appendix.
Motivating agents to exert effort through costly signalling has been previously studied in static settings by Hermalin (1998) and Komai et al. (2007) who analyze models of moral hazard in teams where a leader endowed with private information about a project spends costly resources to signal the quality of the project to her followers in order to induce effort. In contrast, this paper analyzes a dynamic environment where the principal learns about the quality of a project over time and balances between investing optimally and transmitting information to the agent. Dong (2018) studies a similar problem where two players simultaneously, but separately experiment to learn about the state of the world, with one player possessing superior information from the outset, as opposed to my model of joint experimentation where players start the relationship with symmetric information. My results are qualitatively different in that at high beliefs about project quality, our model predicts full information transmission. Halac (2012) studies a game with transfers where a principal with persistent private information induces effort from the agent. She shows that a separating equilibrium does not exist, and in equilibrium, the principal discloses her type gradually. In contrast, my model is without transfers and the principal acquires private information over time which result in very different dynamics.

This paper is related to the literature on private learning. In particular, Bonatti and Hörner (2011) study a model of moral hazards in teams in an exponential bandit framework and show that the incentive to free-ride on other players’ leads to reduction of effort. Augmenting the setting of Bonatti and Hörner (2011), Guo and Roesler (2016) analyze a model of collaboration and private learning with publicly observable exit decisions. However, in their setting players have access to a fully revealing bad signal and communicate their private information by exiting or the lack of it. Bimpikis et al. (2018) study a strategic experimentation model of private learning where information can be credibly disclosed without a cost through commitment. Akcigit and Liu (2016) examine an innovation competition between two firms which decide whether to pursue a risky or a safe project. Since a firm benefits when its competitor works in a less rewarding direction, it never reveals dead-end finding.

More broadly, this paper is related to the literature on experimentation (See, for instance Bolton and Harris (1999), Keller et al. (2005), Keller and Rady (2015)), particularly in principal-agent settings with conflicting interests studied by Guo (2016), Garfagnini (2011) and Kuvalakar and Lipnowski (2018) among others.

3. A ONE PERIOD EXAMPLE

In this section I present a one period example that highlights the main forces I study in this paper. Both the principal (she) and the agent (he) share common prior \( \mu \in [0,1] \) that the quality of a project is good. The principal moves first and invests \( x \in [0, \bar{x}] \) at a cost


of $\frac{1}{2}ax^2$. After observing the principal’s investment, the agent can exert effort ($e = 1$) at a cost of $c$ or not ($e = 0$). Not exerting effort is cost-less. If the project is of good quality and the agent exerts effort then an output $1 + x$ is produced with probability $p$. No output is produced if project quality is bad or the agent does not exert effort. The principal gets a share $\gamma$ and the agent gets a share $1 - \gamma$ of the output.

Supposing that the agent exerts effort when the belief that project quality is good is $\mu$, the principal’s payoff when she invests $x$ is given by:

$$
\gamma \mu p (1 + x) - \frac{1}{2}ax^2.
$$

This gives the principal’s optimal choice of investment as a function of her belief

$$
x^*(\mu) = \frac{\gamma \mu p}{a},
$$

resulting in a value of

$$
\gamma \mu p + \frac{(\gamma \mu p)^2}{2a}.
$$

The agent’s problem is to exert effort or not. Given a belief $\mu$ and the choice of investment $x$ of the principal, the agent exerts effort if

$$
(1 - \gamma)\mu p (1 + x) - c \geq 0.
$$

This gives the minimum the principal needs to invest $x_a(\mu)$ at belief $\mu$ in order to induce effort from the agent,

$$
x_a(\mu) = \frac{c}{(1 - \gamma)\mu p} - 1.
$$

Note that $x_a(\mu)$ is decreasing in $\mu$. As the assessment about the quality of the project decreases, the agent demands higher investment from the principal to be willing to exert effort. That is, the more pessimistic the agent is about the prospect of the project, the more costly it is for the principal to induce effort from the agent.

At any belief $\mu$, there is an upper bound to what the principal is willing to invest to induce effort. This upper bound $x_u(\mu)$ solves

$$
\gamma \mu p (1 + x_u) - \frac{1}{2}ax_u^2 = 0.
$$

Note that $x_u(\mu)$ is increasing in $\mu$. Define $\mu_l$ as the belief at which $x_a(\cdot)$ and $x_u(\cdot)$ intersect. Also, define $\mu_k$ as the belief at which $x_a(\cdot)$ and $x^*(\cdot)$ intersect.

Note from Figure 2 that when beliefs are sufficiently high, i.e. in the region $[\mu_k, 1]$, there is no conflict between the principal and the agent. The principal’s optimal investment is at least as much what the agent requires to exert effort. For intermediate beliefs, i.e. in the region $[\mu_l, \mu_k)$, the agent’s participation constraint binds ($x_a(\cdot) > x^*(\cdot)$). In this region, the principal invests higher than her optimal level in order to appease the agent into exerting effort. For sufficiently low beliefs, i.e. in the region $[0, \mu_l)$, the cost of appeasing the agent
is too high for the principal and the principal prefers to not induce effort from the agent. This observation is summarized in the proposition below.

**Proposition 1.** When belief is $\mu$, the unique Subgame Perfect Nash Equilibrium of the one shot game is given by

$$ x = \begin{cases} \max \left\{ \frac{c}{(1-\gamma)\mu p} - 1, \frac{\gamma \mu p}{\mu} \right\} & \text{if } \mu \geq \mu_l, \\ 0 & \text{if } \mu < \mu_l. \end{cases} $$

$$ e(x) = \begin{cases} 1 & \text{if } x \geq \frac{c}{(1-\gamma)\mu p} - 1, \\ 0 & \text{otherwise}. \end{cases} $$

The one shot game highlights the difficulty in motivating an agent to exert effort when he is pessimistic about the project quality. In a dynamic environment where the principal learns about the project quality privately over time and makes investments into the project, the investment of the principal today not only affects the output today, but also affects how much the agent learns about project quality which in turn affects the continuation value of the principal. In the rest of the paper, I study how the principal optimally manages the trade-off between current value and her continuation value by influencing how the agent learns about project quality.
4. Model

Players: Time $t$ is continuous and runs from 0 to $\infty$. A principal hires an agent to work on a project of unknown quality. The quality of the project is good ($\theta = 1$), or bad ($\theta = 0$). At time 0, both players have common prior $\mu_0$ that project quality is good.

Actions: At each instant, the principal invest an amount $x \in [0, \bar{x}]$ of resource into the project at a cost of $\frac{ax^2}{2}$ where $a$ is a constant. In the same instant, but after having observed the investment of the principal, the agent chooses either to exert effort ($e = 1$) in the project or not ($e = 0$). The agent incurs a cost of $c$ if she exerts ($e = 1$) effort. Not exerting effort is cost-less. The agent’s effort choice is observed by the principal but is not contractible.

Payoffs: The agent’s effort and the quality of the project affects the arrival rate of lump sum output which is exponentially distributed. The arrival rate of output is $\lambda \theta e$ where $\lambda$ is the sensitivity of the production technology. Output is only produced if the project is of good quality and the agent exerts effort. Output is privately observed by the principal. Note that the first arrival of output resolves all the uncertainty for the principal: the realization of the first output results in the posterior of the principal taking the value $\mu = 1$. The output at time $t$, if it arrives is given by $1 + x_t$, where $x_t$ is the principal’s investment at time $t$. The principal’s investment does not affect the arrival rate of output, however it affects the value of the output if output arrives. That is, the principal can only augment the output by investing. I denote by $y_t \in \{\phi, (1 + x_t)\}$ the output at time $t$, where $\phi$ denotes no output.

At the beginning of the relationship, both players commit to a non-negotiable sharing rule of output where the principal receives a share $\gamma$ and the agent receives a share $1 - \gamma$ of the output. Both players discount future payoffs at the rate $r$. Notice that the agent does not observe output during the course of the relationship. I interpret the rate of discounting $r$ as the exogenous rate at which the game ends resulting in the agent realizing his share of the accumulated output. There are no transfers. Letting $T$ denote the set of stochastic times at which an output arrives, the principal’s payoff is given by

$$U(\mu_0) = \mathbb{E}^{\gamma, x} \left[ \gamma \sum_{s \in T} e^{-rs} y_s - \int_{t=0}^{\infty} r e^{-rt} \frac{1}{2} ax_t^2 \, dt \right],$$

and the agent’s payoff is given by

$$V(\mu_0) = \mathbb{E}^{\gamma, x} \left[ (1 - \gamma) \sum_{s \in T} e^{-rs} y_s - \int_{t=0}^{\infty} r e^{-rt} c e \, dt \right].$$

This is an important feature since otherwise there is no information asymmetry between the players and the main tension in the model is absent.
4.1. Strategies and Equilibrium

Since the principal observes the history of outputs privately, she has private information about project quality. Given the fully revealing nature of the production technology, the relevant private information of the principal at time $t$ can be categorized into two classes: either the principal has observed an output or has not observed an output before $t$. Denote the private history of the principal at time $t$ as $h_p^t$ defined as $h_p^t = \{y_s\}_{s=0}^t$. $h_p^t$ contains all information about realized outputs up to time $t$. Denote the set of all feasible private histories for the principal by $H_p^t$. The principal can be of two types at time $t$ depending on her private history $H_p^t$. Type $L$ is a principal who has not observed any output. Type $H$ is a principal who has observed output at some point. I denote the belief (probability that project quality is good) of type $T \in \{L, H\}$ principal by $\mu^T$. Given that the agent exerts effort ($e = 1$) at every instant before $t$, the belief of the type $L$ principal at time $t$ is given by

$$\mu^L = \frac{\mu_0}{\mu_0 + (1 - \mu_0) \exp(\lambda t)}.$$  

The belief of a type $H$ principal at any time is $\mu^H = 1$. At the start of the game, by definition, the principal is type $L$. Also note that type $H$ is an absorbing type.

The agent learns about project quality through the investment of the principal. In particular, the information available to the agent about project quality depends on the strategies of both types of principal. There are two relevant beliefs of the agent that we need to track throughout the game:

1. $\mu^a$: The agent’s belief (probability) that the principal is type $H$.
2. $\bar{\mu}^a$: The agent’s belief (probability) that the project quality is good.

Note that the agent’s beliefs are known to the principal since the agent does not possess any private information. At the start of the game $\mu^a = 0$ and $\bar{\mu}^a = \mu_0$ since at the beginning of the game the principal is type $L$ and there is no information asymmetry. At any time in the game, $\mu^a$ and $\bar{\mu}^a$ are related as follows

$$\bar{\mu}^a = \mu^a \mu^H + (1 - \mu^a) \mu^L$$

$$= \mu^a \left[1 + \left(1 - \mu^a\right) \frac{\mu_0}{\mu_0 + (1 - \mu_0) \exp(\lambda t)}\right].$$

**Solution Concept:** The solution concept used in this paper is Markov Perfect Bayesian Equilibrium. Strategies are Markov if they depend only on payoff-relevant past events that includes a player’s own payoff-relevant private information. The payoff-relevant information for a principal at any point where she makes her investment decision, is her own private type $T \in \{L, H\}$, the belief of the type $L$ principal $\mu^L$ and the public belief of the agent $\mu^a$. The payoff-relevant information for the agent at any point where he makes his effort decision is his belief about the type of principal before investment $\mu^a$, the belief of the
type \( L \) principal \( \mu^L \) and the investment of the principal. The state variable for this problem is the pair \( S = (\mu^L, \mu^a) \in [0,1]^2 \) which is observed by both players. At the beginning of the game, the principal is of type \( L \) and the agent knows he is facing type \( L \) principal. Hence the game starts with the state \( S_0 = (\mu_0, 0) \).

The principal’s strategy depends on her type and the state \( S \). A pure strategy for type \( T \) principal is a function

\[
x_T : [0,1]^2 \rightarrow [0, \bar{x}].
\]

Two histories for the principal that lead to the same posterior for the principal and the agent, result in an identical investment choice by the principal. The agent’s strategy depends on the state \( S \) and the observed investment \( x \in [0, \bar{x}] \). A pure strategy for the agent is a function

\[
e : [0,1]^2 \times [0, \bar{x}] \rightarrow \{0,1\}.
\]

With slight abuse of notation, I denote the strategy of type \( T \) principal as \( x_T(\mu^L, \mu^a) \) and the strategy of the agent as \( e(\mu^L, \mu^a, x) \), where \( \mu^a \) is the updated belief of the agent after having observed investment \( x \).

![Figure 2. Timeline within an instant](image)

**Updating of beliefs off-path:** An important part of the description of a Perfect Bayesian Equilibrium is the updating of beliefs after players observe off-path actions. I maintain consistency in the rule the agent uses to update his belief after observing investments that are off-path, across equilibria presented in this paper. I describe the rule below and refer to it when specifying an equilibrium.

**Definition 1. (Updating of beliefs off-path)** Suppose \( X^\sigma \) is the set of all possible on path investments at state \( S = (\mu, \mu^a) \) under a strategy profile \( \sigma \). Set \( x^1 = \max X^\sigma \) and \( x^2 = \min X^\sigma \).
SUPERVISING TO MOTIVATE

The updated belief of the agent after observing an investment \( x \notin X^\sigma \) is given by

\[
\mu^a_+ (S,x) = \begin{cases} 
\mu^a_+ (S,x^1) & \text{if } x > x^1, \\
\mu^a_+ (S,x^2) & \text{if } x \in (x^2,x^1), \\
0 & \text{if } x < x^2, 
\end{cases}
\]

where \( \mu^a_+ (S,x^1) \) and \( \mu^a_+ (S,x^2) \) are calculated using Bayes’ rule.

Note that this definition applies even if \( \mu^a = 1 \). That is, even if the agent assigns probability 1 that he is facing type \( H \) principal before investment (and thereby \( |X^\sigma| = 1 \)), the agent can update her belief if he observes off-path investment. This updating rule specifies that after observing an off-path investment, the probability the agent assigns to be facing the type \( H \) principal is equal to what he would have assigned after observing the highest on-path investment lower than the investment observed. If the off-path investment is lower than all on-path investments, the agent believes that he is facing type \( L \). Note that if type \( H \) principal invests higher than type \( L \) principal (which will be the case in any equilibrium), this belief updating rule is monotonic in investment. This updating rule specifies the worst beliefs for the agent (\( \mu^a_+ = 0 \)) following any downward deviation of the principal, which results in the strongest incentives for the principal to not deviate from the path of play.

I make an assumption on parameters so that there are no distortions in a relationship where it is common knowledge that the principal is type \( H \). The following suffices.

**Assumption 1.** \((1 - \gamma)\lambda - c > 0\).

Since I am interested in equilibria that maximize the principal’s ex-ante payoff, the following definition is useful.

**Definition 2.** (Principal Optimal Equilibrium) For any initial state \( S_0 = (\mu_0,0) \), denote by \( \Sigma_{S_0} \) the set of all equilibria\(^5\) at \( S_0 \). For any \( \sigma \in \Sigma_{S_0} \) denote the value to the type \( L \) principal at \( S_0 \) as \( U^\sigma_L (S_0) \). I call \( \hat{\sigma} \in \Sigma_{S_0} \) a principal optimal equilibrium if,

\[
U^\hat{\sigma}_L (S_0) = \max_{\sigma' \in \Sigma_{S_0}} U^\sigma_L (S_0).
\]

5. Results

I first describe the equilibrium in the game where it is common knowledge among both players that project is of good quality (\( \theta = 1 \)). Since the principal has no private information, and hence no concern for information transmission, in this case the principal chooses

\(^5\)The set of all equilibria is the set of Markov Perfect Equilibria satisfying the belief updating rule given in Definition 1.
her optimal investment at every instant by maximizing her expected flow payoff given by

$$\gamma \lambda (1 + x) - \frac{ax^2}{2}.$$ 

Her optimal investment is $$\frac{\lambda \gamma}{a}$$. The agent exerts effort at every instant and his payoff from the relationship is denoted as $$z^a$$ given by

\begin{equation}
(1) \quad z^a = (1 - \gamma) \lambda (1 + \frac{\gamma \lambda}{a}) - c,
\end{equation}

which is strictly positive since Assumption 1 guarantees that the agent’s payoff is strictly positive regardless of the investment of the principal. I denote the value to the principal in this relationship as $$z^p$$ given by

\begin{equation}
(2) \quad z^p = \gamma \lambda (1 + \frac{\gamma \lambda}{2a}).
\end{equation}

The above discussion is summarized by the following lemma whose proof I omit.

**Lemma 1.** Suppose both players know that project is of good quality. The unique equilibrium is given by

$$x^* = \frac{\lambda \gamma}{a}; e^* = 1.$$ 

I next describe an autarkic equilibrium which is used as punishment off the path of play in the equilibria I construct.

**Lemma 2.** Suppose $$\mu \leq \frac{c}{\lambda(1 - \gamma)}$$. The strategy profile $$\sigma^u = (x^u_H, x^u_L, e^u)$$ described below is an equilibrium at state $$(\mu, 0)$$.

$$x^u_H(\cdot, \cdot) = x^u_L(\cdot, \cdot) = 0; e^u(\cdot, \cdot, \cdot) = 0.$$ 

Agent updates beliefs after observing off-path investment using the rule described in Definition 1.

The proof can be found in the appendix. Note that the autarkic equilibrium shown above exists at the state when type $$L$$ principal’s belief $$\mu$$ is less than $$\frac{c}{\lambda(1 - \gamma)}$$ and the agent believes he is facing the type $$L$$ principal. When the type $$L$$ principal’s belief is above $$\frac{c}{\lambda(1 - \gamma)}$$, the agent’s value is strictly positive from exerting effort even if the principal does not invest, hence autarky is not an equilibrium when $$\mu > \frac{c}{\lambda(1 - \gamma)}$$. When players revert to autarky, we say that players have quit the relationship.

I begin the characterization of the principal’s optimal equilibria by stating a necessary condition that must be satisfied in any equilibrium that specifies pooling at some point.
**SUPERVISING TO MOTIVATE**

**Lemma 3.** Fix an equilibrium that specifies pooling at some interval \((p_1, p_2)\) of type L principal’s beliefs\(^6\) and the agent exerts effort in this interval. Denote by \(x^p(p)\) the pooling investment function in the interval \((p_1, p_2)\). Let \(D \subseteq (p_1, p_2)\) be a Lebesgue measurable set of beliefs where \(x^p(p) < \frac{\gamma \lambda}{a}\) when \(p \in D\). Then, \(D\) has Lebesgue measure zero.

Note that \(\frac{\gamma \lambda}{a}\) is the type H type principal’s optimal investment. The result above states that in any equilibrium which specifies pooling in an interval \((p_1, p_2)\) of type L principal’s beliefs, the pooling investment must be at least equal to \(\frac{\gamma \lambda}{a}\) except at a measure zero subset of \((p_1, p_2)\). The idea behind the result is as follows. The agent’s belief updating rule given by Definition 1 specifies that during pooling, the agent’s updated belief after observing an investment higher than the equilibrium investment is identical to the updated belief after observing equilibrium investment. If there is a subset of \((p_1, p_2)\) of positive measure where the equilibrium investment function specifies an investment smaller than \(\frac{\gamma \lambda}{a}\), the type H principal can simply deviate to \(\frac{\gamma \lambda}{a}\) for all beliefs in this subset and improve her payoff since she will then invest optimally. Note that the agent’s belief updating will be unchanged by this deviation which implies that continuation equilibrium when type L principal’s belief reaches \(p_1\) will be unchanged following the deviation.

Since the principal’s behavior on a measure zero set of beliefs during pooling has no payoff consequence to either player, going forward, I will assume that in any equilibrium, the investment specified during pooling (if there is pooling) must be at least equal to \(\frac{\gamma \lambda}{a}\).

The above result leads to the following important observation. The amount type L principal is willing to invest in order to induce effort from the agent increases in her belief since her value of the relationship is lower at smaller beliefs. If her beliefs are sufficiently low, she will not invest \(\frac{\gamma \lambda}{a}\) to induce effort. This implies that at sufficiently low beliefs of type L principal, there cannot be pooling in equilibrium. I define \(\mu\) as the cut-off belief at which type L principal is indifferent between quitting and inducing effort from the agent by investing \(\frac{\gamma \lambda}{a}\).

**Definition 3.** Define

\[
\mu = \frac{r \lambda \gamma^2}{2[r \lambda \gamma^2 + a(r \gamma + 2p)]}.
\]

Interpretation of \(\mu\): Suppose type L principal has a choice of investing \(\frac{\gamma \lambda}{a}\) and inducing effort from the agent or quit the relationship. If the belief of the type L principal \(\mu > \mu\), she strictly prefers to induce effort. If \(\mu < \mu\), she strictly prefers to quit. If \(\mu = \mu\), she is indifferent between inducing effort and quitting.

---

\(^6\)The dependence of the principal’s strategy on the agent’s belief has been suppressed for the ease of notation.
In light of the above discussion, a natural question arises: Is the agent willing to exert effort at \( \mu \) following an investment of \( \frac{\gamma \lambda}{a} \) if he knows he is facing the type \( L \) principal? The answer to this condition depends on whether condition \( C_1 \) (defined below) is satisfied or not. If \( C_1 \) is satisfied then the agent does not exert effort, and if \( C_1 \) is violated, then the agent exerts effort.

**Condition 1 (\( C_1 \)):**

\[
 r \left[\mu \lambda (1 - \gamma) \left(1 + \frac{\lambda \gamma}{a}\right) - c\right] + \mu \lambda z^a < 0.
\]

I provide a heuristic intuition behind \( C_1 \). Note that \( r \left[\mu \lambda (1 - \gamma) \left(1 + \frac{\lambda \gamma}{a}\right) - c\right] dt \) is the expected flow value of the agent and \( \mu \lambda z^a dt \) is the expected continuation value which is the product of the probability \( (\mu \lambda dt) \) that an output is produced (implying that the type \( L \) principal transitions to type \( H \)) and the value to the agent from continuing the relationship with the type \( H \) principal going forward \( (z^a) \). The sum of these two values gives the overall value to the agent from exerting effort at \( \mu \). Also note that fixing other parameters of the model, \( C_1 \) is satisfied when the cost of effort of the agent is high and is violated otherwise. In other words, \( C_1 \) can also be stated in terms of a cut-off on the cost of effort of the agent i.e. \( C_1 \) is equivalent to

\[
 c > \mu \lambda (1 - \gamma) \left(1 + \frac{\lambda \gamma}{a}\right) - \frac{\mu \lambda z^a}{r} = c^*.
\]

The nature of principal’s optimal equilibrium critically depends on whether \( C_1 \) is satisfied or not i.e. whether the agent’s cost of effort is high or low. I present the principal optimal equilibrium for both cases in the following sections.

### 5.1. Low Cost of Effort

In this section I present the principal optimal equilibrium when \( C_1 \) is violated i.e. agent’s cost of effort is low in Proposition 2 presented below. Before we go to the proposition, I define some useful objects.

**Definition 4.** Define \( \mu^c \) as the unique value of \( \mu \) that solves

\[
 \frac{\mu \gamma \lambda}{a} = f(\mu),
\]

and \( \mu^s \) as the unique value of \( \mu \) that solves

\[
 r \left[\mu \lambda \gamma (1 + f(\mu)) - a \frac{(f(\mu))^2}{2}\right] + \mu \lambda z^p = 0,
\]

where

\[
 f(\mu) = \frac{c}{\mu \lambda (1 - \gamma)} - \frac{z^a}{r(1 - \gamma)} - 1.
\]
SUPERVISING TO MOTIVATE

Also define type L principal’s optimal investment as a function of her belief as

\[ g(\mu) = \frac{\mu \gamma \lambda}{a}. \]

**Proposition 2.** Suppose C1 does not hold, then, \( \mu^s \leq \mu < \mu^c \), and the principal optimal equilibrium is a fully separating equilibrium \( \sigma^* \) given by,

\[
x^*_H(\mu, \mu^a) = \frac{\gamma \lambda}{a}; \quad x^*_L(\mu, \mu^a) = \begin{cases} \frac{\mu \gamma \lambda}{a} & \text{if } \mu \in [\mu^c, 1], \\ f(\mu) & \text{if } \mu \in [\mu^s, \mu^c), \\ 0 & \text{if } \mu < \mu^s; \end{cases}
\]

\[
e^*(\mu, \mu^a = 1, x) = 1;
\]

\[
e^*(\mu, \mu^a = 0, x) = \begin{cases} 1 & \text{if } \mu < \mu^c \text{ and } x \geq f(\mu), \\ 1 & \text{if } \mu \geq \mu^c, \\ 0 & \text{otherwise}. \end{cases}
\]

Agent updates beliefs using Bayes’ rule on path and using the rule described in Definition 1 after observing off-path investment.

**Figure 3.** Equilibrium when cost of effort is low.
The principal’s optimal equilibrium when $C1$ does not hold is a fully separating equilibrium, i.e. separation is specified for every belief of the type $L$ principal. There is full information transmission, through the investments of the principal, the agent learns perfectly the type of principal she is facing. Type $H$ principal invests optimally at every point in the relationship. Type $L$ principal invests optimally when her beliefs are high ($\mu \in [\mu^e, 1]$), invests higher than her optimal ($g(\mu)$) at intermediate beliefs ($\mu \in [\mu^s, \mu^e]$) and quits the relationship at low beliefs ($\mu < \mu^s$). Note that this implies that if the initial prior $\mu_0$ is less than $\mu^s$, the principal never invests and the agent never exerts effort, i.e. there is no production.

Whenever the type $L$ principal invests higher than her optimal investment, note that the value of the agent facing her must be zero, otherwise she can reduce her investment by a little and still induce effort and consequently be better off. This is indeed the case when $\mu < \mu^e$. Type $L$ principal simply chooses the minimum investment needed to induce effort from the agent using the equation below,

\[
r \left[ \mu \lambda (1 - \gamma) \left( 1 + x_L^T(\mu, \mu^a) \right) - c \right] dt + \mu \lambda z dt = 0.
\]

Note that the expected continuation value to the agent comes only from facing the type $H$ principal in the next instant. This is because, if the agent faces the type $L$ principal at the next instant, as argued above, the value of the agent is zero.

Type $L$ principal’s investment is always weakly less than the optimal investment of type $H$ principal. To see this observe that since since $C1$ is violated,

\[
r \left[ \mu \lambda (1 - \gamma) \left( 1 + \frac{\lambda \gamma}{a} \right) - c \right] dt + \mu \lambda z dt \geq 0,
\]

which implies type $L$ principal can induce effort from the agent at $\mu$ by investing weakly lower than $\frac{\lambda \gamma}{a}$, and strictly so if the above inequality is strict. Also note that type $L$ principal’s investment can never be greater than $\frac{\lambda \gamma}{a}$ when $\mu < \mu$ by the definition of $\mu$. Since type $L$ principal’s equilibrium investment is strictly decreasing in $\mu$ when $\mu \in [\mu^s, \mu^e]$, her investment is always weakly less than $\frac{\lambda \gamma}{a}$.

When the agent’s cost of effort is low, the misalignment in the incentives of the players results in the type $L$ principal investing above her optimal investment to motivate the agent to exert effort at intermediate beliefs. However, the misalignment is not sufficiently high to stop the flow of information about the project quality to the agent. As we will see in the next section, if the cost of effort for the agent is high enough, in the principal optimal equilibrium, in addition to investing above her optimal investment, the type $L$ principal behavior results in the stoppage of the flow of information to the agent when her beliefs are sufficiently low.
5.2. High cost of effort

When the agent’s cost of effort is high, i.e. condition C1 holds, the principal’s optimal equilibrium is sensitive to the initial prior $\mu_0$ about the project quality. Before we proceed, I define a belief $\bar{\mu}$ which will be useful in our analysis.

**Definition 5.** Suppose starting at state $(\mu,0)$ with $\mu > \bar{\mu}$ type H principal invests $\frac{\lambda \gamma}{a} \mu$ in perpetuity and type L principal invests $\frac{\lambda \gamma}{a}$ until beliefs drift down to $\bar{\mu}$ and then quits. Denote by $V(\mu)$ the value of the agent if he exerts effort following investment. Define $\bar{\mu}$ as the unique belief at which $V(\bar{\mu}) = 0$.

As in the case with low cost of effort, if the initial prior is sufficiently low, the only equilibrium is autarky, i.e. the principal invests zero and the agent never exerts effort. I state this in the following proposition.

**Proposition 3.** Suppose C1 holds. There exists $\mu_g \in (\bar{\mu}, \bar{\mu})$ such that if $\mu_0 < \mu_g$, the only equilibrium is autarky, i.e. players quit right away. If $\mu_0 \geq \mu_g$ an equilibrium with production exists.

The proof is delivered by Lemma 13 which can be found in the appendix. The principal’s optimal equilibrium is qualitatively different depending on whether the initial prior $\mu_0$ is greater than $\bar{\mu}$ or not. I first consider the case when $\mu_0 > \bar{\mu}$. Recalling the definition of $\mu^c$ and $f(\mu)$ given in Definition 4 before Proposition 2, the principal’s optimal equilibrium is given in the proposition below.

**Proposition 4.** Suppose C1 holds. If $\mu_0 > \bar{\mu}$, the principal’s optimal equilibrium $\sigma^*$ is a three phase equilibrium given by,

- **Separating Phase:** $\mu > \bar{\mu}$:

  \[ x^*_H(\mu, \mu^a) = \frac{\gamma \lambda}{a} \mu \]

  \[ x^*_L(\mu, \mu^a) = \begin{cases} \frac{\mu \gamma \lambda}{a} & \text{if } \mu \in (\max\{\bar{\mu}, \mu^c\}, 1], \\ f(\mu) & \text{if } \mu \in (\bar{\mu}, \max\{\bar{\mu}, \mu^c\}]. \end{cases} \]

  \[ e^*(\mu, \mu_+^a = 1, x) = 1; \]

  \[ e^*(\mu, \mu_+^a = 0, x) = \begin{cases} 1 & \text{if } \mu > \max\{\bar{\mu}, \mu^c\}, \\ 1 & \text{if } \mu \in (\bar{\mu}, \max\{\bar{\mu}, \mu^c\}) \text{ and } x \geq f(\mu), \\ 0 & \text{if } \mu \in (\bar{\mu}, \max\{\bar{\mu}, \mu^c\}) \text{ and } x < f(\mu); \end{cases} \]

---

7Intervals of the form $(a,a]$, $(a,a)$ and $[a,a)$ are considered as empty sets.

17
• **Pooling Phase**: $\mu \in [\overline{\mu}, \bar{\mu}]$:

\[
x^*_H(\mu, \mu^a) = x^*_L(\mu, \mu^a) = \frac{\gamma \lambda}{a};
\]

\[e^*(\mu, \mu^a, x) = 1 \text{ if } \mu^a \neq 0;\]

Players revert to autarky if at any point in this phase the agent assigns $\mu^a = 0$.

• **Quitting Phase**: $\mu < \overline{\mu}$:

\[
x^*_H(\mu, \mu^a) = \frac{\gamma \lambda}{a}; \quad x^*_L(\mu, \mu^a) = 0;
\]

\[e^*(\mu, \mu^a, x = 1) = 1;
\]

\[e^*(\mu, \mu^a, x = 0) = 0;\]

Agent updates beliefs using Bayes’ rule on path and using the rule described in Definition 1 after observing off-path investment.

![Figure 4](image-url)

**Figure 4.** Equilibrium when cost of effort is high and prior is high. This figure is for the case when $\mu^c > \bar{\mu}$.

The principal optimal equilibrium exhibits three phases when initial prior $\mu_0 > \bar{\mu}$. When beliefs of the type $L$ principal are high, separation occurs. Type $L$ principal invests less than type $H$ principal who invests optimally. In this region, the behavior of the principal is qualitatively similar to the fully separating equilibrium presented in Proposition 2. Within the separating region the type $L$ principal invests optimally when $\mu > \max\{\bar{\mu}, \mu^c\}$. If
\( \mu^c > \bar{\mu} \), there can be a region \((\bar{\mu}, \mu^c)\) within the separating phase where type L principal invests above her optimal level since the agent’s participation constraint begins to bind at beliefs lower than \( \mu^c \). Once beliefs reach \( \bar{\mu} \), the pooling phase begins where both players pool on the type H principal’s optimal investment until belief reach \( \underline{\mu} \), at which point type L principal quits the relationship.

Even when cost of effort is high, if the relationship begins with a high belief that the project is of good quality, it is optimal for both types of principals to invest optimally and allow full information transmission to the agent, at least until beliefs drift down enough where the conflict between the type L principal and the agent begins. Anticipating the high costs (investment) of motivating an agent who is pessimistic (low belief that project is of good quality) to exert effort in future, type L principal has incentives to stop the flow of information to the agent, which leads to pooling at intermediate beliefs in equilibrium. A natural question arises then — When does pooling begin? By Definition 5, if pooling begins at \( \bar{\mu} \) and both types of principal invest type H principal’s optimal investment \( \frac{\gamma \lambda}{\bar{a}} \) then the agent gets a value of zero at \( \bar{\mu} \). If pooling is delayed, i.e. pooling begins at \( \mu' < \bar{\mu} \), the average investment in the pooling phase must be strictly higher than \( \frac{\gamma \lambda}{\bar{a}} \) to induce effort from the agent since the agent is pessimistic at \( \mu' \) compared to \( \bar{\mu} \). However note that in this case type L principal can deviate at belief \( \mu' + \epsilon < \bar{\mu} \). By choosing the type H principal’s separating investment at \( \mu' + \epsilon \), type L principal can guarantee she doesn’t have to invest any more than \( \frac{\gamma \lambda}{\bar{a}} \) in future to induce effort. If pooling begins earlier at \( \mu' > \bar{\mu} \), note that the agent’s value at \( \mu' \) is strictly positive since the minimum possible investment during pooling is \( \frac{\gamma \lambda}{\bar{a}} \). By decreasing the pooling cutoff, the type L principal can separate and invest less than \( \frac{\gamma \lambda}{\bar{a}} \) for a little longer and hence is better off. Therefore in an ex-ante principal optimal equilibrium pooling must begin at \( \bar{\mu} \). Note that during the pooling phase, after observing investment, the agent’s belief that he faces the type H principal given by \( \mu' + \epsilon \) is strictly positive (and increases as \( \mu \) goes down over time) since there is a positive probability that the principal has observed output since the end of the separating phase. Any downward deviation by the principal in this phase leads to the agent updating \( \mu' + \epsilon \) to 0 and players continue with autarky (quitting the relationship).

By stopping the flow of information to the agent at \( \bar{\mu} \), the type L principal exploits the agent’s uncertainty \( \mu_+ \) is increasing as \( \mu \) goes down) in the pooling phase and is able to reduce her investment while inducing effort from the agent. In particular, she is able to induce effort from the agent when her belief \( \mu \in [\underline{\mu}, \mu_g) \) which would not have been possible if the agent was aware that he is facing her.

Next, we turn to the principal’s optimal equilibrium when the initial prior \( \mu_0 \in [\mu_g, \bar{\mu}] \). Before I present it, the following definition will be useful.
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**DEFINITION 6.** Denote by \( \mathcal{X}(\mu_0) \) as set of pooling investment function defined from \([\mu, \mu_0]\) to \([\frac{\lambda \gamma}{a}, x]\) that give both the type L principal and the agent, a non negative value in the pooling region \([\mu, \mu_0]\) when the relationship begins at the state \((\mu_0, 0)\). Define \( x^p(\cdot; \mu_0) \in \mathcal{X}(\mu_0) \) as the function that maximizes the ex-ante value of the type L principal at the state \((\mu_0, 0)\).

I state the proposition characterizing the principal’s optimal equilibrium when \( \mu_0 \in [\mu_g, \bar{\mu}] \) below.

**PROPOSITION 5.** Suppose C1 holds. If \( \mu_0 \in [\mu_g, \bar{\mu}] \), the principal’s optimal equilibrium \( \sigma^* \) is a two phase equilibrium given by,

- **Pooling Phase:** \( \mu \in [\mu, \bar{\mu}] \):
  
  \[ x_H^*(\mu, \mu^a) = x_L^*(\mu, \mu^a) = x^p(\mu; \mu_0); \]
  
  \[ e^*(\mu, \mu^a, x) = 1 \text{ if } \mu^a \neq 0, \text{ or } \mu = \mu_0; \]

  Players revert to autarky if at any point in this phase the agent assigns \( \mu^a = 0 \), except when \( \mu = \mu_0 \).

- **Quitting Phase:** \( \mu < \mu \):
  
  \[ x_H^*(\mu, \mu^a) = \frac{\gamma \lambda}{a}; \quad x_L^*(\mu, \mu^a) = 0; \]
  
  \[ e^*(\mu, \mu^a, 1, x) = 1; \]
  
  \[ e^*(\mu, \mu^a, 0, x) = 0; \]

Agent updates beliefs using Bayes’ rule on path and using the rule described in Definition 1 after observing off-path investment.

The principal’s optimal equilibrium in this case is qualitatively similar to when \( \mu_0 > \bar{\mu} \) except that the separation region does not exist and pooling starts at the outset. When \( \mu_0 \in [\mu_g, \bar{\mu}] \) the agent is already pessimistic and demands to be appeased to exert effort. For the reasons argued in the preceding discussion, it is optimal to pool from the start of the relationship. However, the nature of pooling investment is different now. When pooling begins at a belief \( \mu_0 < \bar{\mu} \), for the reasons pointed out in the preceding discussion, the average\(^8\) investment in the pooling phase must be higher than \( \frac{\gamma \lambda}{a} \) in order to induce effort from the agent. In particular, as \( \mu_0 \) decreases, the average investment needed during pooling starting at \( \mu_0 \) increases. Figure 5 shows that for lower initial prior \( \mu'_0 < \mu_0 \), the average investment is higher in the pooling phase. Eventually at beliefs lower than \( \mu_g > \mu \), the investment required to induce effort is too high and the type L principal prefers not induce effort if \( \mu_0 < \mu_g \). The principal’s optimal pooling equilibrium investment function

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\(^8\)For any pooling investment function staring at the state \((\mu_0, 0)\), the average investment is the constant pooling investment that delivers the same value to the agent at state \((\mu_0, 0)\).
SUPERVISING TO MOTIVATE

\[ x^P(\mu; \mu_0) \] is the pooling investment function that satisfies the participation constraint of both the type L principal and agent in the pooling region and maximizes the value of the type L principal at \( \mu_0 \), the initial prior. As in the case with high \( \mu_0 \), in the pooling phase, equilibrium is supported using the threat of autarky following a downward deviation of the principal.

5.3. Equilibrium distortions

Since the agent’s cost of effort is strictly positive in this model, the conflict between the agent and the type L principal is inevitable. Once the belief of type L principal is low enough, the agent will demand investment higher than the optimal investment of type L in order to exert effort. As one would expect, this results in distortion i.e. sub-optimally high levels of investment of type L principal to appease the agent at low beliefs.

Perhaps interestingly, the type H principal’s investment may also be distorted in equilibrium. As Proposition 5 shows, if the cost of effort is high enough and initial beliefs are in the intermediate range, in the pooling region, the type H principal invests strictly above her optimal investment on average. This results from the inability of the type H principal to separate herself from type L principal in any equilibrium. To see why, suppose that type H could separate herself at some belief in the pooling region, then, post separation she would invest at her optimal level and induce effort from the agent who knows that he is facing type H principal. But then, type L principal could simply mimic type H’s
investment, convince the agent that she is type \( H \) and induce effort by investing type \( H \) principal’s optimal investment which on average is lower than the equilibrium pooling investment.

The sub-optimality of type \( H \) principal’s investment during pooling is not perpetual. Once the type \( L \) principal’s beliefs reach \( \mu \) and she quits the relationship, type \( H \) principal is revealed to the agent and continues to invest optimally thereafter. That is to say, the type \( H \) principal’s investment may be sub-optimal in the short run, but in the long run, optimality is restored. This discussion is summarized in the proposition below.

**Proposition 6.** If \( \mu_0 \in [\mu_L, \bar{\mu}] \) and \( c > \bar{c} \), then the type \( L \) principal invests sub-optimally in the pooling phase. However, once type \( L \) principal quits, type \( L \) principal invests optimally in perpetuity.

Note that when cost of effort \( c \leq \bar{c} \) or initial prior \( \mu_0 > \bar{\mu} \), there is no distortion in type \( H \) principal’s investment. When \( c \leq \bar{c} \) the type \( L \) principal can induce effort by investing lower than type \( H \) principal’s optimal investment whenever she wants which is why there is no distortion. When \( c > \bar{c} \), and \( \mu_0 > \bar{\mu} \), the intuition is as follows. Note that during the relationship, type \( L \) principal transitions to become type \( H \) if she observes output. That is, distortion in type \( H \) principal’s investment also hurts the type \( L \) principal who anticipates that she could transition to become type \( H \) in future. In particular, this is indeed a concern for the type \( L \) principal at the initial belief \( \mu_0 \). This concern for the type \( L \) principal at \( \mu_0 \) results in no distortions for type \( H \) principal in the principal optimal equilibrium.

### 5.4. Discussion

In this model, the conflict between the principal and the agent has two sources: the cost of effort for the agent and the beliefs of the players about the quality of the project being good. Propositions 2, 4 and 5 highlight how these sources of conflict interact and shape the principal’s optimal behavior in equilibrium. In particular, note that if the cost of effort for the agent is zero, there is no conflict: the agent does not need to be appeased to exert effort, and hence there is no distortion in the investment of both types of principal. When the cost of effort is low \( (c < \bar{c}) \), there is conflict when players are sufficiently pessimistic which results in type \( L \) principal investing sub-optimally and appeasing the agent to exert effort. However, the conflict is not enough to warrant stopping information transmission to the agent, and hence, the equilibrium is fully separating at every belief. When the cost of effort is high \( (c > \bar{c}) \), the conflict between players is high enough that equilibrium exhibits stoppage of information transmission at low beliefs. In this case the equilibrium exhibits information transmission at high beliefs if players are optimistic to begin with, but eventually, as the beliefs of players fall, the flow of information to the agent is stopped to preserve his motivation, anticipating the high investment costs of motivating the agent to
exert effort if he becomes more pessimistic. Proposition 6 shows that if the cost of effort is high and relationship starts with low priors, even the type $H$ principal’s investment is distorted owing to her inability to credibly convince the agent of her type. However, such distortions are transient and go away in the long run.

In the model presented, the uncertainty in the environment is about the project quality. One can also interpret the uncertainty to be about the agent’s ability, that is, it is the agent’s ability that affects whether output is produced or not and, the principal, who is experienced and better skilled, learns about the agent’s ability privately.

In organizations, workers are given feedback through periodic evaluations and appraisals. These represent a cost-less (cheap talk) channel through which managers provide information to workers to motivate them to exert effort. However, how workers learn about the relevant aspects of the production environment (project’s quality / agent’s ability) is not limited to these periodic chunks of information. The day to day actions of a manager, how much interest they show in a workers activities by investing their resources credibly transmits the private information of the manager to the workers affecting their motivation to work. This novel channel of credible information transmission, which, if accounted for, may enhance how managers motivate their workers and better achieve organizational goals.

6. Extensions

6.1. Continuous effort choice

In this paper I assumed that the agent’s effort choice is binary, either she exerts effort at a cost $c$ or does not exert effort. I now show that if we allow the agent chose effort with a constant marginal cost, the results of this paper are unaffected. Suppose the agent can choose $e \in [0,1]$ at a cost of $ce$. As before $e = 0$ is cost-less and $e = 1$ costs $c$. First note that effort has two effects, first it affects probability of arrival of output, and second it affects the speed of learning. In particular, with continuous choice of $e$, the rate of arrival of output $\lambda e \theta$ can be controlled more precisely by the agent. However, note that since marginal cost of effort is constant, and the decision problem of the agent at any instant is linear (due to exponential arrival rates), the optimal effort choice must be bang-bang. In other words, allowing for continuous effort with constant marginal cost has no effect on our results. However, if the marginal cost of effort is not constant, i.e. optimal effort allocation may be strictly interior, the results in this paper may not hold.

6.2. Breakdowns

In this paper, the observation of an output reveals to the principal that project is of good quality. That is, this is a model of fully revealing good news. Suppose instead of observing
a fully revealing output, the principal privately observes breakdowns, i.e. fully revealing signals that confirm that project quality is bad. Output is hidden from both the principal and the agent. In this case note that the type $L$ principal (the one who knows project is bad quality for sure) has no incentive to mimic the optimal investment of type $H$ principal (the one who is optimistic since the start of the game on account of not seen a breakdown yet). Hence any equilibrium where the principal invests and the agent exerts effort must be fully separating at all beliefs of type $H$ principal and both types invest their optimal investment. That is, there is no distortion in equilibrium.

7. Conclusion

This paper studies a simple principal-agent relationship with private learning where the principal faces a trade-off between optimally choosing her actions and transmitting information to the agent. There are three main takeaways. First, if the relationship is in an optimistic stage, i.e. both parties believe the relationship is likely to bear fruits in future, the principal should ignore information transmission concerns and invest optimally, i.e. the principal’s actions should be sensitive to her private information. Second, if relationship moves to a pessimistic stage, or starts at a pessimistic stage, the principal should ignore her private information in choosing her actions, to prevent any information transmission and hence preserving the motivation of the agent to work. Third, if the relationship starts at a pessimistic stage, a principal who learns that the project is good, may still be forced to choose her actions sub-optimally in the short run, however, in the long run, she chooses her optimal actions.

The insights of this paper can be useful in understanding the optimal behavior of managers in organizations. In particular, feedback on project quality/agent’s ability that affects the motivation of a worker is not limited to appraisals (periodic and cost-less information transmission), but also regularly through how much a manager actively invests in the employee (or the project he is working on) as the relationship progresses. This novel channel of credible information transmission, if accounted for, may enhance how managers motivate their workers and better achieve the objectives of their organization.

A. Appendix

Definition 7. I define two classes of strategy profiles as follows.

1. Three phase strategy profile: A three phase strategy profile $\sigma = (x^H_{\mu}, x^L_{\mu}, e^\sigma)$ is a strategy profile that satisfies the following on path. For some cutoff $\bar{\mu}$,

- **Separating Phase:** When $\mu^L > \bar{\mu}$: $x^H_{\mu^L, \mu^a} \neq x^L_{\mu^L, \mu^a}$.
- **Pooling Phase:** When $\mu^L \in [\mu, \bar{\mu}]$: $x^H_{\mu^L, \mu^a} = x^L_{\mu^L, \mu^a} = x^p(\mu^L)$. 

24
SUPERVISING TO MOTIVATE

- **Quitting Phase:** When $\mu_l < \mu$: $x_H^r(\mu_l, \mu^a) = \frac{\gamma \lambda}{a}$; $x_L^r(\mu_l, \mu^a) = 0$.

\[(2) \text{ Two phase strategy profile:} \text{ A two phase strategy profile } \sigma = (x_H^e, x_L^e, e^e) \text{ is a strategy profile that satisfies the following on path.} \]

- **Pooling Phase:** When $\mu_l \geq \mu$: $x_H^r(\mu_l, \mu^a) = x_L^r(\mu_l, \mu^a) = x_P^e(\mu_l)$.
- **Quitting Phase:** When $\mu_l < \mu$: $x_H^r(\mu_l, \mu^a) = \frac{\gamma \lambda}{a}$; $x_L^r(\mu_l, \mu^a) = 0$.

**PROOF OF LEMMA 2:** Note that starting at a state $(\mu, 0)$, the strategy profile prescribes that the agent never exerts effort and the principal invests 0. Suppose the agent deviates and exerts effort for a small time interval $\Delta$, then the agent’s payoff is given by

$$\mu \lambda \Delta r (1 - \gamma) - rc \Delta,$$

$$= \Delta [\mu \lambda r (1 - \gamma) - rc].$$

For any positive $\Delta$, the payoff to the agent from this deviation is negative if $\lambda r (1 - \gamma) - rc < 0$ or $\mu \leq \frac{c}{\lambda (1 - \gamma)}$. Hence the agent has no incentive to deviate. Given the agent’s behavior, it is easy to see that the behavior of both types of principal are optimal. 

**PROOF OF LEMMA 3:** Suppose $L(D) > 0$. Consider the deviation of type $H$ principal to an investment level of $\frac{\gamma \lambda}{a}$ for all $p \in D$. By Definition 1, everywhere in the interval $(p_1, p_2)$ the agent’s belief is equal to what it would have been in the absence of deviation. Hence, the continuation equilibrium at $p_1$ is unchanged given the agent exerts effort everywhere in $(p_1, p_2)$. The agent’s flow utility is higher when the principal invests $\frac{\gamma \lambda}{a}$ compared to the on-path investment level $x_P^e(p)$ when $p \in D$, hence, given that continuation play is identical at $p_1$, the agent continues to exert effort in-spite of the deviation. Since continuation play does not change at $p_1$, and $L(D) > 0$, the utility of types $H$ is strictly higher by deviating to $\frac{\gamma \lambda}{a}$ for $p \in D$, since type $H$ is now choosing her optimal level of investment.

**A.1. LOW COST OF EFFORT**

In this section, I provide the proofs of the case when $C_1$ is violated, i.e. cost of effort is low.

**PROOF OF PROPOSITION 2:** We first show that the strategy profile is an equilibrium, we then argue that it is indeed the principal’s optimal equilibrium.

First we observe that since the strategy profile is fully separating, on equilibrium path the agent knows perfectly the type of principal he is facing when making his effort decision. I now show that no player has any incentive to deviate from the candidate equilibrium strategy profile.
- Type $H$ principal: Under our candidate equilibrium strategy profile, type $H$ principal chooses her flow optimal investment $\frac{\lambda \gamma}{a}$ after any history while inducing effort from the agent. Clearly, type $H$ principal cannot achieve a higher payoff by deviating to any other strategy.

- Type $L$ principal: First we observe that the optimal level of investment of type $L$ with belief $\mu$ is equal to $\frac{\mu \lambda \gamma}{a}$ since it maximizes the flow value of the type $L$ principal given by

$$\mu \lambda \gamma (1 + x) - a \frac{x^2}{2}.$$ 

Note that $\frac{\mu \gamma \lambda}{a}$ is linear and increasing in $\mu$. Also, $\frac{c}{\mu \lambda (1 - \gamma)} - \frac{z^a}{r(1 - \gamma)} - 1$ is decreasing in $\mu$, taking a value of $\infty$ as $\mu$ goes to 0 and is less than $\frac{\gamma \lambda}{a}$ when $\mu = 1$ since $z^a = \lambda (1 - \gamma) (1 + \frac{\gamma \lambda}{a}) - c > 0$ by definition. Hence, there must exist a unique $\mu^c$ such that

$$\frac{\mu^c \gamma \lambda}{a} = \frac{c}{\mu^c \lambda (1 - \gamma)} - \frac{z^a}{r(1 - \gamma)} - 1.$$ 

Next, define $\mu_1$ such that

$$f(\mu_1) = \frac{c}{\mu_1 \lambda (1 - \gamma)} - \frac{z^a}{r(1 - \gamma)} - 1 = \frac{\lambda \gamma}{a}.$$ 

Note that since $C1$ does not hold, $\mu_1 \leq \mu$. Also $f(\mu) > \frac{\lambda \gamma}{a}$ for all $\mu < \mu_1$. Recalling the definition of $\mu$, we can say that there exists a $\mu^s \in [\mu_1, \mu]$ such that if $\mu < \mu^s$ the type $L$ principal strictly prefers to quit the relationship, if $\mu > \mu^s$ the type $L$ principal strictly prefers to induce effort by investing $f(\mu)$, and is indifferent when $\mu = \mu^s$. Also, it is easy to see that if $\mu_1 < \mu$ then $f(\mu^s) < \frac{\lambda \gamma}{a}$. Since the principal is indifferent between hiring and firing at $\mu^s$, the value of the type $L$ principal at $\mu^s$ is given by

$$U^*_L(\mu^s, \mu^s) = 0,$$ 

which implies that

$$r \left( \mu^s \lambda \gamma (1 + x^*_L(\mu^s)) - a \frac{(x^*_L(\mu^s))^2}{2} \right) + \mu^s \lambda z^p = 0.$$ 

Now observe that $r \left( \mu \lambda \gamma (1 + x^*_L(\mu)) - a \frac{(x^*_L(\mu))^2}{2} \right) + \mu \lambda z^p$ is increasing in $\mu$, which implies that the value of the principal is strictly positive when $\mu \in [\mu^s, \mu^c]$. Therefore, the type $L$ has no incentive to deviate when $\mu^L \in [\mu^s, \mu^c]$ since by deviating to a lower investment than $x^*_L(\mu^L)$ type $L$ principal only pauses the relationship
(since the agent does not exert effort) and by deviating to a higher investment the principal does no better because \( x_H^*(\mu) < x_H^0(\mu) \) for all \( \mu \in \mu^c, \mu^c \). Note that when \( \mu \in \mu^c, 1 \), type \( L \) principal is investing at her myopic optimal level and cannot do better by deviating to any other investment level. Hence I have shown that type \( L \) principal has no incentive to deviate from her candidate equilibrium strategy.

- Agent: I first show that if \( \mu_L \leq \mu^c \), then the value to the agent facing a type \( L \) principal is 0 in the candidate equilibrium. The value of an agent facing type \( L \) principal can be written as

\[
V^*(\mu_L, \mu_L^a = 0, x_L^*) = rdt[\mu_L^1(1-\gamma)(1+x_L^*) - c] + (1-rdt)[\mu_L^1 \lambda dtz^a + (1-\mu_L^1 \lambda dt)V^*(\mu_L + d\mu_L^1, \mu_L^a = 0, x_L^*)],
\]

where

\[
d\mu_L = -\lambda \mu_L^1(1-\mu_L^1)dt.
\]

The above expression can be rearranged to give

\[
V^*(\mu_L, \mu_L^a = 0, x_L^*) = \underbrace{dt[r(\mu_L^1(1-\gamma)(1+x_L^*) - c) + \mu_L^1 \lambda z^a]}_{A(\mu^L)} + (1-rdt)(1-\mu_L^1 \lambda dt)V^*(\mu_L + d\mu_L^1, \mu_L^a = 0, x_L^*).
\]

The value of the agent facing a type \( L \) principal can be decomposed into an pseudo flow term \( A \) and a continuation value \( B \) adjusted for discounting and the probability that agent will face a type \( L \) principal after \( dt \). Plugging in for \( x_L^* \), it is straightforward to see that \( A(\mu^L) = 0 \). Also note that we can decompose the continuation value after \( B \) in a similar manner. Note that for any belief of type \( L \) principal \( \mu \leq \mu^c \), we have \( A(\mu) = 0 \). \( V^*(\mu_L, \mu_L^a = 0, x_L^*) \) is just a weighted aggregate of \( A(\mu) \) for \( \mu \in \mu^c, \mu^c \) and hence must be equal to 0. Therefore, the agent has no incentive to deviate and terminate the relationship when \( \mu_L \in \mu^c, \mu^c \).

Next, I show that the value of the agent is strictly positive when \( \mu_L \in (\mu^c, 1) \). To see this, simply note that \( r(\mu_L^1(1-\gamma)(1+x_L^*) - c) + \mu_L^1 \lambda z^a > 0 \) when \( \mu_L \in (\mu^c, 1) \).

Therefore, the agent has no incentive to deviate and terminate the relationship when \( \mu_L \in (\mu^c, 1) \).

This establishes that the strategy profile is indeed an equilibrium. I now show that this is also the principal optimal equilibrium. First we observe that the type \( L \) principal always invests strictly lower than type \( H \) principal’s investment who invests optimally. Hence, from Lemma 3 we know that any equilibrium with pooling can be improved upon. This implies that the principal optimal equilibrium must be a fully separating equilibrium. Note that in our candidate equilibrium, the type \( L \) principal invests above her optimal investment only when the value of the agent is 0, i.e. the agent’s participation constraint
binds. Hence, the type $L$ principal cannot decrease her investment when the agent’s IR binds and be better off. This establishes that the equilibrium is indeed optimal for the principal.

**Lemma 4.** Consider any equilibrium $\sigma$ with separation at $(\mu^L, \mu^a)$ and the agent facing type $L$ principal exerts effort following investment $x_L(\mu^L, \mu^a) > \frac{\mu^L \gamma \lambda}{a}$. Define $V^\sigma(\mu^L, \mu^a_+ = 0, x_L(\mu^L, \mu^a))$ as the value of the agent facing type $L$ principal under $\sigma$. Then $V^\sigma(\mu^L, \mu^a_+ = 0, x_L(\mu^L, \mu^a)) = 0$

**Proof.** We start by observing that $\frac{\mu^L \gamma \lambda}{a}$ is the flow optimal level of investment of type $L$ principal at $(\mu^L, \mu^a)$. Suppose $V^\sigma(\mu^L, \mu^a_+ = 0, x_L(\mu^L, \mu^a)) > 0$, then consider the deviation of type $L$ at $(\mu^L, \mu^a)$ to $x' = \frac{\mu^L \gamma \lambda}{a}$. We observe that any downward deviation of the type $L$ principal does not change the continuation play as long as the agent continues to exert effort at $(\mu^L, \mu^a)$. This is because after a downward deviation the agent continues to believe that he is facing the type $L$ principal and the principal’s deviation at $(\mu^L, \mu^a)$ does not affect her belief going forward given that the agent continues to exert effort. There are two possible cases

1. The value of the agent after the deviation, $V^\sigma(\mu^L, \mu^a_+ = 0, x') > 0$. In this case the agent continues to exert effort and the type $L$ principal is better off because she improves her instantaneous payoff while keeping the continuation unchanged.

2. The value of the agent after the deviation, $V^\sigma(\mu^L, \mu^a_+ = 0, x') \leq 0$. In this case consider an alternate deviation from type $L$ principal $x'' = x_L(\mu^L, \mu^a) - \epsilon > \frac{\mu^L \gamma \lambda}{a}$ where $\epsilon$ is chosen such that $V^\sigma(\mu^L, \mu^a_+ = 0, x'') > 0$. Under this alternate deviation the agent continues to exert effort and the type $L$ principal is better off because she improves her instantaneous payoff while keeping the continuation unchanged.

Hence $V^\sigma(\mu^L, \mu^a_+ = 0, x_L(\mu^L, \mu^a)) = 0$.

**A.2. High cost of effort**

In this section, I provide the proofs of the case when $C1$ is satisfied, i.e. cost of effort is high.

**Definition 8.** Suppose $\mu_0 \leq \bar{\mu}$. Consider a two phase strategy profile $\bar{\sigma}$ such that

- $\bar{x}_H(\mu^L, \mu^a) = \bar{x}_L(\mu^L, \mu^a) = \bar{x}(\mu^L)$, such that $\bar{x}(\mu_L) \in [\frac{\lambda \gamma}{a}, \bar{x}]$ if $\mu^L \in [\mu, \mu_0]$;

- $\bar{x}_H(\mu^L, \mu^a) = \frac{\gamma \lambda}{a}$ if $\mu^L < \mu$;

- $\bar{x}_L(\mu^L, \mu^a) = 0$ if $\mu^L < \mu$;
We call \( \tilde{\sigma} \) feasible if it satisfies the individual rationality (IR) constraints of type L principal (see lemma 5) and the agent (see lemma 6) for all \( \mu^L \in [\underline{\mu}, \mu_0] \). Note that if \( \tilde{\sigma} \) is IR for the type L principal then it is IR for type H principal.

**Lemma 5.** The IR constraint of type L principal under \( \tilde{\sigma} \) when \( \mu^L = \mu \in [\underline{\mu}, \mu_0] \) is given by

\[
\left( IC^L_\mu \right)
\]

\[
\int^{\mu}_{\underline{\mu}} \left[ \frac{1 - \mu}{\mu} \frac{\phi}{1 - \phi} \right]^\frac{1}{2} r[\mu \lambda \gamma(1 + \bar{x}(\phi)) - \frac{1}{2} \lambda \bar{x}^2(\phi)] \frac{d\phi}{\lambda \phi(1 - \phi)} + \left[ \frac{\mu}{1 - \mu} \right] r \cdot \frac{\mu - \mu^L}{1 - \mu^L} \geq 0.
\]

**Proof.** Starting at \( \mu^L = \mu \in [\underline{\mu}, \mu_0] \), the probability that a type L principal observes a output and transitions to become a type H principal by the time \( \mu^L = \mu \) is given by \( \mu(1 - e^{-\lambda(t(\mu) - t(\mu))}) \) where \( t(\mu) - t(\mu) \) is the time it takes for \( \mu^L \) to drift down from \( \mu \) to \( \underline{\mu} \). Recall that in the absence of observing a output

\[
\frac{d\mu^L}{dt} = -\mu^L(1 - \mu^L)\lambda.
\]

Rearranging and integrating both sides I get

\[
\int^{\mu}_{\underline{\mu}} \frac{d\phi}{\phi(1 - \phi)} = -\lambda \int_{t(\mu)}^{\mu} dt,
\]

which gives us

\[
e^{-\lambda(t(\mu) - t(\mu))} = \frac{\mu}{1 - \mu} \frac{1 - \mu}{\mu},
\]

leading to

\[
\mu(1 - e^{-\lambda(t(\mu) - t(\mu))}) = \frac{\mu - \mu^L}{1 - \mu^L}.
\]

Note that if the principal is type H at \( \mu^L = \mu \), then under \( \sigma \) her value is \( z^p \). The value of type L principal under \( \tilde{\sigma} \) when \( \mu^L = \mu \) is then given by

\[
U_\tilde{L}^\sigma(\mu) = \int_{t(\mu)}^{\mu} e^{-r(s - t(\mu))} [\mu(s) \lambda \gamma(1 + \bar{x}(\mu(s))) - \frac{1}{2} \lambda \bar{x}^2(\mu(s))] ds + e^{-r(t(\mu) - t(\mu))} \frac{\mu - \mu^L}{1 - \mu^L} z^p.
\]

Changing the variable of integration from time to type L principal’s belief we get

\[
U_\tilde{L}^\sigma(\mu) = \int^{\mu}_{\underline{\mu}} \left[ \frac{1 - \mu}{\mu} \frac{\phi}{1 - \phi} \right]^\frac{1}{2} r[\mu \lambda \gamma(1 + \bar{x}(\phi)) - \frac{1}{2} \lambda \bar{x}^2(\phi)] \frac{d\phi}{\lambda \phi(1 - \phi)} + \left[ \frac{\mu}{1 - \mu} \right] r \cdot \frac{\mu - \mu^L}{1 - \mu^L} z^p.
\]
\textbf{Lemma 6.} The IR constraint of the agent under $\bar{\sigma}$ when $\mu^L = \mu \in [\underline{\mu}, \mu_0]$ is given by

$$\text{(IC}_L^\mu)$$

\[ \int_{\underline{\mu}}^\mu \left[ \frac{1 - \mu}{\mu} \right] \frac{1}{1 - \phi} r [\mu_0 \lambda (1 - \gamma) (1 + \hat{x}(\phi)) - c] \frac{d\phi}{\lambda \phi (1 - \phi)} + \left[ \frac{\mu}{1 - \mu} - \frac{1 - \mu}{1 - \mu} \right] \frac{\hat{\mu}_0 - \mu}{1 - \mu} \sigma^a \geq 0. \]

\textbf{Proof.} Starting at $\mu^L = \mu_0$, the probability that the agent will face type $H$ principal is given by $\mu_0 (1 - e^{-\lambda (t(\mu) - t(\mu_0))})$ where $t(\mu) - t(\mu_0)$ is the time it takes for $\mu^L$ to drift down from $\mu_0$ to $\underline{\mu}$. Following the steps in the proof of lemma 5, we get

$$\mu_0 (1 - e^{-\lambda (t(\mu) - t(\mu_0))}) = \frac{\mu_0 - \mu}{1 - \mu}. \]

Note that if the principal is type $H$ at $\mu^L = \mu$, then under $\bar{\sigma}$ the value to the agent at $\mu^L = \mu$ is $\sigma^a$. The value of the agent under $\bar{\sigma}$ when $\mu^L = \mu$ is then given by

$$V^{\sigma}(\mu) = \int_{t(\mu)}^{t(\mu)} r e^{-r(s-t(\mu))} [\mu_0 \lambda (1 - \gamma) (1 + \hat{x}(\mu)) - c] ds + e^{-r(t(\mu) - t(\mu))} \frac{\mu_0 - \mu}{1 - \mu} \sigma^a. \]

Changing the variable of integration from time to type $L$ principal’s belief we get

$$V^{\sigma}(\mu) = \int_{\mu}^{\mu} \left[ \frac{1 - \mu}{\mu} \right] \frac{1}{1 - \phi} r [\mu_0 \lambda (1 - \gamma) (1 + \hat{x}(\phi)) - c] \frac{d\phi}{\lambda \phi (1 - \phi)} + \left[ \frac{\mu}{1 - \mu} \right] \frac{\mu_0 - \mu}{1 - \mu} \sigma^a. \]

\textbf{Definition 9.} We call the two phase strategy profile $\bar{\sigma}$ defined in 8 just feasible if under $\bar{\sigma}$ the IR constraint for type $L$ principal binds for all $\mu \in [\underline{\mu}, \mu_0]$. I denote a just feasible strategy profile by $\bar{\sigma}$ and the corresponding pooling investment function by $\hat{x} : [\underline{\mu}, \mu_0] \rightarrow \left[ \frac{\lambda x}{a}, \hat{x} \right]$. 

\textbf{Lemma 7.} Suppose $\bar{\sigma}$ is a just feasible strategy profile. The just feasible pooling investment function $\hat{x}$ is the unique solution to the first order non linear differential equation given by:

$$\hat{x}'(\mu) = \frac{\mu \lambda \gamma (1 + \hat{x}(\mu)) (1 + \frac{\hat{x}}{2})(\frac{\hat{x}}{2} + 2\mu - 1)}{\mu (1 - \mu)[a \hat{x}(\mu) - \mu \lambda \gamma]].$$

\textbf{Proof.} Denote by $U^\sigma_T(\mu)$, the value of a type $T \in \{L, H\}$ principal under $\bar{\sigma}$ at $\mu^L = \mu \leq \mu_0$. Then by definition of $\bar{\sigma}$, we must have $U^\sigma_L(\mu) = 0$ for all $\mu \in [\underline{\mu}, \mu_0]$. This means

$$U^\sigma_L(\mu) = r d t \left[ \mu \lambda \gamma (1 + \hat{x}(\mu) - \frac{\mu \hat{x}(\mu)}{2} \right] + (1 - r d t)[\mu \lambda d t U^\sigma_H(\mu + d \mu) + (1 - \mu \lambda d t) U^\sigma_L(\mu + d \mu)] = 0.$$
Since by definition of \( \hat{\sigma} \), \( U^\hat{\sigma}_L(\mu + d\mu) = 0 \) we get
\[
U^\hat{\sigma}_L(\mu) = r dt \left[ \mu \lambda \gamma (1 + \hat{x}(\mu)) - \frac{a \hat{x}^2(\mu)}{2} \right] + (1 - r dt) \mu \lambda dt U^\hat{\sigma}_L(\mu + d\mu) = 0.
\]
Since \( \hat{x}(\mu) \in [\frac{\lambda}{a}, \bar{x}] \) and \( 0 \leq U^\hat{\sigma}_L(\mu) \leq z^p \) for all \( \mu \in [\mu, \mu_0] \), \( U^\hat{\sigma}_H(\mu) \) is continuous on \([\mu, \mu_0] \) (Perhaps a lemma for this). This implies that \( \lim_{\Delta \mu \to 0} U^\hat{\sigma}_L(\mu + \Delta \mu) = U^\hat{\sigma}_L(\mu) \). Which implies that we can write
\[
U^\hat{\sigma}_L(\mu) = dt \left[ \mu \lambda \gamma (1 + \hat{x}(\mu)) - \frac{a \hat{x}^2(\mu)}{2} \right] + \mu \lambda U^\hat{\sigma}_L(\mu) = 0.
\]
Since by definition of \( \hat{\sigma} \), \( U^\hat{\sigma}_L(\mu) = 0 \), which implies that
\[
(4) \quad r \left[ \mu \lambda \gamma (1 + \hat{x}(\mu)) - \frac{a \hat{x}^2(\mu)}{2} \right] + \mu \lambda U^\hat{\sigma}_L(\mu) = 0.
\]
for all \( \mu \in [\mu, \mu_0] \) except for any set \( D \subset [\mu, \mu_0] \) of measure zero. Further simplification of equation 4 yields
\[
\hat{x}(\mu) = \frac{\mu \lambda \gamma + \sqrt{r^2 \mu^2 \lambda^2 \gamma^2 + 2ar \mu \lambda (r \gamma + U^\hat{\sigma}_L(\mu))}}{ar}.
\]
Observe that \( U^\hat{\sigma}_H(\mu) = z^p \) which implies that
\[
(5) \quad \hat{x}(\mu) = \frac{\mu \lambda \gamma + \sqrt{r^2 \mu^2 \lambda^2 \gamma^2 + 2ar \mu \lambda (r \gamma + z^p)}}{ar} = \frac{\lambda \gamma}{a}.
\]
Recall from lemma 5 that \( U^\hat{\sigma}_L(\mu) \) is given by
\[
U^\hat{\sigma}_L(\mu) = \int_{\mu}^{\bar{\mu}} \left[ \frac{1 - \mu}{\mu} \Phi \right] \hat{\Phi} \frac{ \lambda \gamma (1 + \hat{x}(\phi)) - \frac{1}{2} a \hat{x}^2(\phi) }{\frac{1}{\phi} - \phi} \frac{d\phi}{\phi} + \left[ \frac{\mu}{1 - \mu} \frac{1 - \mu}{\mu} \right] \frac{\mu - \mu \hat{\Phi}}{1 - \mu} z^p.
\]
Similarly \( U^\hat{\sigma}_H(\mu) \) is given by
\[
U^\hat{\sigma}_H(\mu) = \int_{\mu}^{\bar{\mu}} \left[ \frac{1 - \mu}{\mu} \Phi \right] \hat{\Phi} \frac{ \lambda \gamma (1 + \hat{x}(\phi)) - \frac{1}{2} a \hat{x}^2(\phi) }{\frac{1}{\phi} - \phi} \frac{d\phi}{\phi} + \left[ \frac{\mu}{1 - \mu} \frac{1 - \mu}{\mu} \right] \frac{\mu - \mu \hat{\Phi}}{1 - \mu} z^p.
\]
Evaluating \( U^\hat{\sigma}_H(\mu) - U^\hat{\sigma}_L(\mu) \) we get:
\[
U^\hat{\sigma}_H(\mu) - U^\hat{\sigma}_L(\mu) = (1 - \mu) \int_{\mu}^{\bar{\mu}} \left[ \frac{1 - \mu}{\mu} \Phi \right] \hat{\Phi} \frac{ \lambda \gamma (1 + \hat{x}(\phi)) }{\frac{1}{\phi} - \phi} \frac{d\phi}{\phi} + \left[ \frac{\mu}{1 - \mu} \frac{1 - \mu}{\mu} \right] \frac{\mu - \mu \hat{\Phi}}{1 - \mu} z^p.
\]
Under \( \hat{\sigma} \), \( U^\hat{\sigma}_L(\mu) = 0 \), hence we get
\[
U^\hat{\sigma}_H(\mu) = (1 - \mu) \int_{\mu}^{\bar{\mu}} \left[ \frac{1 - \mu}{\mu} \Phi \right] \hat{\Phi} \frac{ \lambda \gamma (1 + \hat{x}(\phi)) }{\frac{1}{\phi} - \phi} \frac{d\phi}{\phi} + \left[ \frac{\mu}{1 - \mu} \frac{1 - \mu}{\mu} \right] \frac{\mu - \mu \hat{\Phi}}{1 - \mu} z^p.
\]
\[ \frac{dLHS}{d\mu} = r \left[ \hat{x}'(\mu)(a\hat{x}(\mu) - \mu \lambda \gamma - \lambda \gamma(1 + \hat{x}(\mu))) \frac{1}{\mu \lambda (1 - \mu)} \right]^\frac{\hat{x}}{x} \]
\[ + r \frac{1}{2} a \hat{x}(\mu)^2 - \mu \lambda \gamma(1 + \hat{x}(\mu))) \frac{1}{\lambda \mu^2 (1 - \mu)^2} \left[ \frac{\mu}{1 - \mu} \right]^\frac{\hat{x}}{x}. \]

Differentiating the RHS above equation with respect to \( \mu \) we get

\[ \frac{dRHS}{d\mu} = \left[ \frac{\mu}{1 - \mu} \right]^\frac{\hat{x}}{x} \frac{r[\gamma(1 + \hat{x}(\mu))]}{\mu(1 - \mu)}. \]

On equating \( \frac{dLHS}{d\mu} \) to \( \frac{dRHS}{d\mu} \) and and simplifying we get

\[ \hat{x}'(\mu) = \frac{\mu \lambda \gamma(1 + \hat{x}(\mu))(1 + \frac{r}{x}) - a \hat{x}(\mu)^2}{\mu(1 - \mu)[a \hat{x}(\mu) - \mu \lambda \gamma]} \cdot \frac{r + 2\mu - 1}{\lambda \mu^2 (1 - \mu)^2}. \]

Note that the above expression is a first order non linear differential equation and standard results in the theory of differential equation guarantee existence and uniqueness of the solution. The precise solution \( \hat{x}(\mu) \) is pinned down by using the boundary condition given by equation 5.
SUPERVISING TO MOTIVATE

**Definition 10.** For any pooling investment function \( x(\cdot) \) defined for \( \mu \in [\mu_0, \mu] \), define the principal average investment under \( x(\cdot) \) at \( \mu_1 \), \( A(x, \mu_1) \) as the unique solution of

\[
\int_{t(\mu_1)}^{t(\mu)} e^{-r(s-t(\mu_1))} x(\mu(s)) \, ds = \int_{t(\mu_1)}^{t(\mu)} e^{-r(s-t(\mu_1))} A(x, \mu_1) \, ds.
\]

Note that \( A(x, \mu_1) \) is a constant for a given pooling investment function \( x \) and belief \( \mu_1 \).

**Lemma 8.** \( A(\hat{x}, \mu) \) is increasing in \( \mu \).

**Proof.** Recall from lemma 7 that

\[
\hat{x}'(\mu) = \frac{\mu \lambda \gamma (1 + \hat{x}(\mu))(1 + \frac{\xi}{\lambda}) - \frac{a \hat{x}(\mu)^2}{2}(\frac{r}{\lambda} + 2\mu - 1)}{\mu(1 - \mu)[a \hat{x}(\mu) - \mu \lambda \gamma]}.
\]

It is easy to see that the denominator \( \mu(1 - \mu)[a \hat{x}(\mu) - \mu \lambda \gamma] \) is always positive. We focus on the numerator. In particular I define \( \check{x}(\mu) \) such that the numerator is equal to 0. That is

\[
\mu \lambda \gamma (1 + \check{x}(\mu))(1 + \frac{r}{\lambda}) - \frac{a \check{x}(\mu)^2}{2}(\frac{r}{\lambda} + 2\mu - 1) = 0.
\]

On solving the quadratic equation we have

\[
\check{x}(\mu) = \beta(\mu) + \sqrt{\beta(\mu)^2 + 2\beta(\mu)},
\]

where

\[
\beta(\mu) = \frac{\mu \lambda \gamma (1 + \frac{\xi}{\lambda})}{a(\frac{r}{\lambda} + 2\mu - 1)}.
\]

It can be shown that

\[
\beta'(\mu) = \begin{cases} 
\geq 0 & \text{if } \frac{\xi}{\lambda} \geq 1, \\
< 0 & \text{if } \frac{\xi}{\lambda} < 1.
\end{cases}
\]

This implies that

\[
\check{x}'(\mu) = \begin{cases} 
\geq 0 & \text{if } \frac{\xi}{\lambda} \geq 1, \\
< 0 & \text{if } \frac{\xi}{\lambda} < 1.
\end{cases}
\]

Note that

\[
\hat{x}'(\mu) > 0 \text{ if } \hat{x}(\mu) < \check{x}(\mu), \\
\hat{x}'(\mu) = 0 \text{ if } \hat{x}(\mu) = \check{x}(\mu), \\
\hat{x}'(\mu) < 0 \text{ if } \hat{x}(\mu) > \check{x}(\mu).
\]

It can be shown that

\[
\lim_{\mu \uparrow 1} \hat{x}(\mu) = \lim_{\mu \uparrow 1} \check{x}(\mu) = \frac{\lambda \gamma + \sqrt{\lambda^2 \gamma^2 + 2a \lambda \gamma}}{a}.
\]

and \( \check{x}'(\mu) > 0 \), implying that \( \hat{x}(\mu) < \check{x}(\mu) \).
CLAIM 1. If $\frac{r}{x} \geq 1$ then $\hat{x}(\mu)$ is increasing in $[\mu, 1]$.

**Proof.** We have $\dot{x}(\mu) < \ddot{x}(\mu)$ and $\lim_{x \to 1} \ddot{x}(\mu) = \lim_{x \to 1} \dot{x}(\mu)$. Suppose $\dot{x}(\mu)$ and $\ddot{x}(\mu)$ intersect in $[\mu, 1]$. In particular, suppose that the smallest belief at which they intersect is $\mu_1 \in [\mu, 1]$. That is $\dot{x}(\mu_1) = \ddot{x}(\mu_1)$. Then we know that $\dot{x}'(\mu_1) = 0$. We also know that $\ddot{x}'(\mu_1) \geq 0$ since $\frac{r}{x} \geq 1$. This implies that $\dot{x}(\mu_1 - \epsilon) \geq \ddot{x}(\mu_1 - \epsilon)$ for some $\epsilon > 0$, which contradicts that $\mu_1$ is the smallest belief at which $\dot{x}(\mu) = \ddot{x}(\mu)$. Therefore I have shown that $\dot{x}(\mu)$ and $\ddot{x}(\mu)$ never intersect when $\mu < 1$. This implies that $\dot{x}(\mu)$ is increasing in $[\mu, 1]$.

CLAIM 2. If $\frac{r}{x} < 1$ then $\hat{x}(\mu)$ is either increasing or single peaked in $[\mu, 1]$.

**Proof.** I first show that if $\dot{x}(\mu)$ and $\ddot{x}(\mu)$ intersect, then they intersect exactly once. To that end suppose that $\dot{x}(\mu_2) = \ddot{x}(\mu_2)$. Note that since $\frac{r}{x} < 1$, $\dot{x}'(\mu_2) < 0$. Also $\ddot{x}'(\mu_2) = 0$, which implies that $\dot{x}(\mu) > \ddot{x}(\mu)$ when $\mu \in (\mu_2, \mu_2 + \epsilon)$ for some $\epsilon > 0$. Using an argument identical to claim 1 we can say that $\dot{x}(\mu)$ and $\ddot{x}(\mu)$ never intersect in $(\mu_2, 1)$. This implies that if $\dot{x}(\mu)$ and $\ddot{x}(\mu)$ intersect, then they intersect exactly once. Note that when $\dot{x}(\mu)$ and $\ddot{x}(\mu)$ intersect, $\dot{x}(\mu)$ is single peaked (increasing before intersection and decreasing after). If $\dot{x}(\mu)$ and $\ddot{x}(\mu)$ never intersect then $\dot{x}(\mu)$ is increasing in $[\mu, 1]$ because $\dot{x}(\mu) < \ddot{x}(\mu)$ when $\mu \in [\mu, 1]$.

From Claim 1 and Claim 2 there are two possible cases.

1. $\dot{x}(\mu)$ is increasing in $[\mu, 1]$: In this case it is easy to see that $A(\dot{x}, \mu)$ is increasing in $\mu$.

2. $\dot{x}(\mu)$ is single peaked in $[\mu, 1]$: First we observe that $A(\dot{x}, 1) = \frac{\lambda \gamma + \sqrt{\lambda^2 \gamma^2 + 2 a \lambda \gamma}}{a}$ because when $\mu = 1$, the the principal is essentially of type $H$ and the investment of a type $L$ principal such that her value is equal to 0 is given by $\frac{\lambda \gamma + \sqrt{\lambda^2 \gamma^2 + 2 a \lambda \gamma}}{a}$. Next, $A(\dot{x}, \mu) < A(\dot{x}, 1)$ when $\mu < 1$ since the type $L$ principal’s average investment under $\dot{x}$ is always less than that of the type $H$ principal. Suppose $\dot{x}(\mu)$ is increasing in $[\mu, 1)$ and decreasing in $(\mu, 1)$. As shown in case(1), $A(\dot{x}, \mu)$ is increasing when $\mu \in [\mu, 1)$. Since $\dot{x}(\mu)$ is decreasing in $(\mu, 1)$, it must be the case that $\dot{x}(\mu) > \lim_{\mu \to 1} \dot{x}(\mu) = \frac{\lambda \gamma + \sqrt{\lambda^2 \gamma^2 + 2 a \lambda \gamma}}{a}$ when $\mu \in [\mu, 1)$. In particular note that $\dot{x}(\mu) > \frac{\lambda \gamma + \sqrt{\lambda^2 \gamma^2 + 2 a \lambda \gamma}}{a} > A(\dot{x}, \mu)$ when $\mu \in [\mu, 1)$. This implies that $A(\dot{x}, \mu)$ is increasing when $\mu \in [\mu, 1)$. Hence $A(\dot{x}, \mu)$ is increasing in $\mu$.

**Definition 11.** Define the agent minimum average investment at $\mu_1 > \mu$, $B(\mu_1)$ as the unique solution of

$$\int_{t(\mu_1)}^{t(\mu_1)} r e^{-r(s-t(\mu_1))} \left[ \mu_1 \gamma (1 - \gamma) (1 + B(\mu_1)) - c \right] ds + e^{-r(t(\mu_1)-t(\mu_1))} \frac{\mu_1 - \mu}{1 - \mu} z a = 0.$$
**SUPERVISING TO MOTIVATE**

**Lemma 9.** $B(\mu^\dagger)$ is decreasing in $\mu^\dagger$.

**Proof.** I integrate the expression

$$
\int_{t(\mu^\dagger)}^{t(\mu)} r e^{-r(s-t(\mu^\dagger))} \left[ \mu^\dagger \lambda (1 - \gamma) (1 + B(\mu^\dagger)) - c \right] ds + e^{-r(t(\mu)-t(\mu^\dagger))} \frac{\mu^\dagger - \mu}{1 - \mu} z^a = 0.
$$

to get

$$
1 - \left[ \frac{\mu}{1 - \mu} \frac{1 - \mu^\dagger}{\mu^\dagger} \right] \frac{\mu^\dagger}{1 - \mu} \lambda (1 - \gamma) (1 + \frac{\lambda \gamma a}{1 - \mu}) - c + \left[ \frac{\mu}{1 - \mu} \frac{1 - \mu^\dagger}{\mu^\dagger} \right] \frac{\mu^\dagger - \mu}{1 - \mu} z^a = 0.
$$

Which can be rewritten as

$$
0 = \left[ 1 - \left[ \frac{\mu}{1 - \mu} \frac{1 - \mu^\dagger}{\mu^\dagger} \right] \frac{\mu^\dagger}{1 - \mu} \lambda (1 - \gamma) (1 + \frac{\lambda \gamma a}{1 - \mu}) - c \right]
+ \left[ \frac{\mu}{1 - \mu} \frac{1 - \mu^\dagger}{\mu^\dagger} \right] \frac{\mu^\dagger - \mu}{1 - \mu} z^a
+ \left[ 1 - \left[ \frac{\mu}{1 - \mu} \frac{1 - \mu^\dagger}{\mu^\dagger} \right] \frac{\mu^\dagger}{1 - \mu} \right] \left[ \mu^\dagger \lambda (1 - \gamma) \left( B(\mu^\dagger) - \frac{\lambda \gamma a}{1 - \mu} \right) \right].
$$

I define $\mathcal{V}(\mu; B)$ as

$$
\mathcal{V}(\mu; B) = \left[ 1 - \left[ \frac{\mu}{1 - \mu} \frac{1 - \mu^\dagger}{\mu^\dagger} \right] \frac{\mu^\dagger}{1 - \mu} \lambda (1 - \gamma) (1 + \frac{\lambda \gamma a}{1 - \mu}) - c \right]
+ \left[ \frac{\mu}{1 - \mu} \frac{1 - \mu^\dagger}{\mu^\dagger} \right] \frac{\mu^\dagger - \mu}{1 - \mu} z^a
+ \left[ 1 - \left[ \frac{\mu}{1 - \mu} \frac{1 - \mu^\dagger}{\mu^\dagger} \right] \frac{\mu^\dagger}{1 - \mu} \right] \left[ \mu^\dagger \lambda (1 - \gamma) \left( \bar{B} - \frac{\lambda \gamma a}{1 - \mu} \right) \right].
$$

It is easy to see that if $\bar{B}_1 > \bar{B}_2$ then $\mathcal{V}(\mu; \bar{B}_1) > \mathcal{V}(\mu; \bar{B}_2)$ for all $\mu$. Therefore, if $\mathcal{V}(\mu; \bar{B}) = 0$, then $\bar{B}$ must be unique, which in turn implies that as $\bar{B}$ goes up the belief $\mu$ that solves $\mathcal{V}(\mu; \bar{B}) = 0$, decreases.

**Definition 12.** Define $\bar{\mu}$ as the belief at which

$$
B(\bar{\mu}) = \frac{\lambda \gamma a}{1 - \mu}.
$$

**Lemma 10.** Autarkic equilibrium exists at state $(\mu, 0)$ when $\mu \in [\mu, \bar{\mu}]$.

**Proof.** Consider a strategy profile $\sigma^m$, where

$$
\chi^m_{\text{H}}(\mu, \mu^a) = \frac{\lambda \gamma a}{1 - \mu} \chi^m_{\text{L}}(\mu, \mu^a) = \begin{cases} \frac{\lambda \gamma a}{1 - \mu} & \text{if } \mu \geq \mu \smallskip \\
0 & \text{if } \mu < \mu \end{cases}
$$

35
RAVI

\[ e^m(\mu, \mu_+, x) = \begin{cases} 
0 & \text{if } \mu_+ = 0, \\
1 & \text{if } \mu_+ = 1.
\end{cases} \]

At the state \((\bar{\mu}, 0)\), note that by the definition of \(\bar{\mu}\), the value of the agent \(V^m(\bar{\mu})\) under \(\sigma^m\) is equal to 0. Note that the autarkic strategy profile (defined in the proof of Lemma 2) starting at state \((\bar{\mu}, 0)\) specifies a strictly lower level of investment for type \(L\) principal in when \(\mu^L \in [\mu, \bar{\mu}]\) compared to \(\sigma^m\). Hence, the value of the agent under the autarkic strategy profile is strictly negative if the agent chooses to deviate and exert effort at any point. Hence, an autarkic equilibrium exists at state \((\bar{\mu}, 0)\). It is easy to see that the above argument holds for all states \((\mu, 0)\) such that \(\mu \in [\mu, \bar{\mu}]\) and hence, an autarkic equilibrium exists at state \((\mu, 0)\) such that \(\mu \in [\mu, \bar{\mu}]\).

**Lemma 11.** Suppose there exists a feasible two phase strategy profile \(\sigma\) at state \(S = (\mu_1, 0)\) with \(\mu_1 < \bar{\mu}\) and \(V^\sigma(S) > 0\). Then, there exists another feasible two phase strategy profile \(\sigma_0\) at \(S\) such that \(V^{\sigma_0}(S) = 0\).

**Proof.** Suppose \(x^\sigma\) is the pooling investment function in the pooling region \([\mu, \mu_1]\) under \(\sigma\). First observe that the agent’s value at any \(\mu_L = \mu \in [\mu, \mu_1]\) under \(\sigma\) is given by

\[ V^\sigma(\mu) = \int_{\mu}^{\mu_1} \left[ \frac{1 - \mu}{\mu} \frac{\phi}{1 - \phi} \right] \frac{\xi}{r} \phi \left( 1 + c \right) \frac{d\phi}{\mu(1 - \phi)} + \left[ \frac{\mu}{\mu_L} \right] \frac{\xi}{1 - \mu} \mu_1 - \mu \lambda n. \]

Note that \(V^\sigma(\mu)\) is continuous in \([\mu, \mu_1]\). Since \(V^\sigma(\mu_1) > 0\), we have two cases.

**Case 1:** \(V^\sigma(\mu) > 0\) for all \(\mu \in [\mu, \mu_1]\): Consider an alternate two phase strategy profile \(\sigma'\) with the following investment function in the pooling phase:

\[ x^{\sigma'}(\mu) = \begin{cases} 
x^\sigma(\mu) & \text{if } \mu \in [\mu, \bar{\mu}], \\
\lambda & \text{if } \mu \in (\bar{\mu}, \mu_1].
\end{cases} \]

where \(\bar{\mu} \in [\mu, \mu_1]\). The value of the agent at \(\mu_L = \mu_1\) under \(\sigma'\) is given by

\[ V^{\sigma'}(\mu_1; \bar{\mu}) = \int_{\mu}^{\mu_1} \left[ \frac{1 - \mu}{\mu} \frac{\phi}{1 - \phi} \right] \frac{\xi}{r} \phi \left( 1 + x^\sigma(\phi) \right) \frac{d\phi}{\mu(1 - \phi)} + \int_{\bar{\mu}}^{\mu_1} \left[ \frac{1 - \mu}{\mu} \frac{\phi}{1 - \phi} \right] \frac{\xi}{r} \phi \left( 1 + \frac{\lambda n}{a} \right) \frac{d\phi}{\mu(1 - \phi)} + \left[ \frac{\mu}{\mu_1} \right] \frac{\xi}{1 - \mu} \mu_1 - \mu \lambda n. \]
Next, note that $V^{\sigma'}(\mu_1; \hat{\mu} = \mu) < 0$. To see this note that from lemma 9 we know that $B(\mu_1) > \frac{\lambda \gamma}{a}$ since $B(\hat{\mu}) = \frac{\lambda \gamma}{a}$ and $\mu_1 < \hat{\mu}$. Since $B(\mu_1) > \frac{\lambda \gamma}{a}$, we must have

$$\int_{\mu}^{\mu_1} \left[ \frac{1 - \mu_1}{\mu} \frac{\phi}{1 - \phi} \right] \frac{\partial}{\partial \phi} \left[ \mu_1 \lambda (1 - \gamma)(1 + \frac{\lambda \gamma}{a}) - c \frac{\partial \phi}{\lambda \phi (1 - \phi)} + \left( \frac{\mu}{1 - \mu} \frac{1 - \mu_1}{\mu} \right) \right] \frac{\partial \mu}{\partial \mu_1} < 0.$$ 

Also note that $V^{\sigma'}(\mu_1; \hat{\mu} = \mu) = V^{\sigma}(\mu_1) > 0$ by assumption. It is easy to see that $V^{\sigma'}(\mu_1; \hat{\mu})$ is continuous in $\hat{\mu}$ which implies that there must exist a $\hat{\mu}$ such that $V^{\sigma'}(\mu_1; \hat{\mu} = \hat{\mu}) = 0$. I now show that the strategy profile $\sigma'$ with $\hat{\mu} = \hat{\mu}$ is feasible. I first show that $\sigma'$ satisfies the IR constraints of the type $L$ principal for all $\mu \in [\mu, \mu_1]$. Since $\sigma$ is feasible, the IR constraints of the the type $L$ principal for all $\mu \in [\mu, \bar{\mu}]$ are satisfied. Note that $x^\sigma(\mu) = x^{\sigma'}(\mu) = \frac{\lambda \gamma}{a}$ for all $\mu \in (\hat{\mu}, \bar{\mu}]$ which implies that the IR constraints of the the type $L$ principal for all $\mu \in (\hat{\mu}, \bar{\mu}]$ are satisfied.

Next I show that $\sigma'$ satisfies the IR constraints of the agent for all $\mu \in [\mu, \mu_1]$. On the lines argued above, it is easy to see that the IR constraints of the the agent for all $\mu \in [\mu, \bar{\mu}]$ are satisfied. Next note that

$$\mu_1 \lambda (1 - \gamma)(1 + \frac{\lambda \gamma}{a}) - c < 0,$$

which implies that the value of the agent under $\sigma'$ at $\mu_1 = \mu$, $V^{\sigma'}(\mu)$ is increasing in $\mu$ when $\mu \in (\hat{\mu}, \mu_1]$. Since $V^{\sigma'}(\mu_1) = 0$, it must be the case that $V^{\sigma'}(\mu) > 0$ for all $\mu \in (\hat{\mu}, \mu_1]$, which establishes that which implies that the IR constraints of the the agent for all $\mu \in (\hat{\mu}, \bar{\mu}]$ are satisfied.

Since $\sigma'$ with $\hat{\mu} = \hat{\mu}$ satisfies the IR constraints of both type $L$ principal and the agent when $\mu \in [\mu, \mu_1]$, $\sigma'$ with $\hat{\mu} = \hat{\mu}$ is feasible and $V^{\sigma'}(\mu_1) = 0$ and hence proof is complete.

Case 2: There exists a set $\mathcal{M}^\sigma \subset [\mu, \mu_1]$ such that $V^{\sigma}(\mu) = 0$ for $\mu \in \mathcal{M}^\sigma$: In this case, define $\bar{\mu} = \sup \mathcal{M}^\sigma$. Consider an alternate two phase strategy profile $\sigma'$ with the following investment function in the pooling phase:

$$x^{\sigma'}(\mu) = \begin{cases} x^{\sigma}(\mu) & \text{if } \mu \in [\mu, \bar{\mu}], \\ \frac{c}{\mu_1 \lambda (1 - \gamma)} - 1 & \text{if } \mu \in (\bar{\mu}, \mu_1]. \end{cases}$$

Note that $\mu_1 \lambda (1 - \gamma)(1 + x^{\sigma'}(\mu)) - c = 0$ when $\mu \in (\bar{\mu}, \mu_1]$, this implies that $V^{\sigma'}(\mu) = 0$ when $\mu \in (\bar{\mu}, \mu_1]$. This implies that the agent’s IR constraint’s are satisfied for $\mu \in [\mu, \mu_1]$ under $\sigma'$. Next observe that $V^{\sigma}(\mu) > V^{\sigma'}(\mu) = 0$ when $\mu \in (\bar{\mu}, \mu_1]$. This implies that if $\sigma'$ is IR for type $L$ principal, since $\sigma'$ specifies a weakly lower level of investment compared to $\sigma$ when $\mu \in (\bar{\mu}, \mu_1]$. Since $\sigma'$ satisfies the IR constraints of both type $L$ principal and the
agent when $\mu \in [\underline{\mu}, \overline{\mu}]$, $\sigma'$ is feasible and $V^\sigma'(\mu_1) = 0$ and hence proof is complete.

**Lemma 12.** Suppose there exists a feasible two phase strategy profile at state $S = (\mu_1, 0)$ where $\mu_1 < \bar{\mu}$. Then, there exists a feasible two phase strategy profile at state $S' = (\mu'_1, 0)$ where $\mu_1 < \mu'_1 \leq \bar{\mu}$.

**Proof.** Recall from Lemma 9 that the agent’s minimum average investment at $\mu'_1$ is smaller than the agent’s minimum average investment at $\mu_1$. Also note from lemma 8 we know that the type L principal’s average maximum investment is increasing in $\mu$. Note that since a feasible two phase strategy profile exists at state $(\mu_1, 0)$ and the minimum average investment demanded by the agent is smaller at $(\mu'_1, 0)$ and the maximum average investment that type L is willing to invest is higher at $(\mu'_1, 0)$, it must be the case that a feasible two phase exists at $(\mu_1, 0)$.

**Lemma 13.** There exists a belief $\underline{\mu} \in (\underline{\mu}, \bar{\mu})$ such that,

1. if $\mu \geq \underline{\mu}$, then there exist a two feasible two phase strategy profile at state $(\mu, 0)$;
2. if $\mu < \underline{\mu}$, then there does not exist a feasible two phase strategy profile at state $(\mu, 0)$.

**Proof.** By definition of $\underline{\mu}$, the type L principal does not invest any higher than $\frac{\gamma a}{\lambda}$ at $\underline{\mu}$. From lemma 9 we know that $B(\mu)$ is decreasing in $\mu$. We also know that $B(\bar{\mu}) = \frac{\gamma a}{\lambda}$. Since $\bar{\mu} > \underline{\mu}$, we know that $B(\underline{\mu}) > \frac{\gamma a}{\lambda}$. Hence there does not exist a two feasible two phase strategy profile at state $(\underline{\mu}, 0)$. By continuity, there exists an $\epsilon > 0$ such that there does not exist a two feasible two phase strategy profile at state $(\mu, 0)$ when $\mu \in [\underline{\mu}, \underline{\mu} + \epsilon)$. Suppose $\underline{\mu}$ is the smallest belief such that a feasible two phase strategy profile exists at state $(\mu, 0)$ when $\mu \in [\underline{\mu}, \mu_1]$, then we know from lemma 12 that a feasible two phase strategy profile exists at all states $(\mu, 0)$ such that $\mu \geq \underline{\mu}$.

**Lemma 14.** Suppose $\mu_1 \in [\underline{\mu}, \bar{\mu}]$. Then, there exists a two phase equilibrium at state $(\mu_1, 0)$.

**Proof.** We know from Lemma 13 that there exists a feasible two phase strategy profile at $(\mu_1, 0)$. Let us denote a feasible two phase strategy profile by $\hat{\sigma} = (\hat{x}_H, \hat{x}_L, \hat{e})$ given by

- $\hat{x}_H(\mu, \mu^a) = \hat{x}_L(\mu, \mu^a) = \hat{x}(\mu)$ such that $\hat{x}(\mu) \in [\frac{\lambda \gamma}{a}, \bar{x}]$ for $\mu \in [\underline{\mu}, \mu_1]$;
- $\hat{x}_H(\mu, \mu^a) = \frac{\gamma a}{\lambda}$; $\hat{x}(\mu^L, \mu^a) = 0$ for $\mu < \underline{\mu}$;
- $\hat{e}(\mu, \mu^a, x) = \begin{cases} 1 & \text{if } \mu^a_+ = 0 \text{ and } \mu = \mu_1; \\
0 & \text{if } \mu^a_+ = 0 \text{ and } \mu \neq \mu_1, \\
1 & \text{otherwise} \end{cases}$.
I now show that the above two phase strategy profile can be supported as an equilibrium using the threat of autarky following any downward deviation by the principal when \( \mu^L \in [\underline{\mu}, \mu_1] \). Lemma 10 tells us that an autarkic equilibrium exists at state \((\mu, 0)\) when \( \mu \in [\underline{\mu}, \bar{\mu}] \) which implies that an autarkic equilibrium exists at state \((\mu, 0)\) when \( \mu \in [\underline{\mu}, \mu_1] \).

I first show that both types of principal have no incentive to deviate in the pooling region \((\mu^L \in [\underline{\mu}, \bar{\mu}])\). Note that the value of both types of principal is non negative at any point of the relationship starting at state \((\mu_1, 0)\) since \( \sigma^L \) is feasible. Any upward deviation principal during the pooling phase is unprofitable because it only reduces the principal’s value without affecting the agent’s behavior. Any downward deviation by the principal during the pooling phase leads to autarky that yields 0 to the principal and hence is unprofitable. In the firing phase, by deviating to atleast \( \gamma a \), the type \( L \) principal can induce effort from the agent, but this deviation leaves the type \( L \) principal strictly worse off. Type \( H \) principal has no incentive to deviate in the firing phase since she is investing her optimal level of investment and agent is exerting effort.

The agent has no incentive to deviate in the pooling phase since \( \sigma^L \) is feasible and any deviation leads to autarky which cannot improve the agent’s payoff. Hence \( \sigma \) can be supported as an equilibrium by using the threat of autarky.

**Lemma 15.** Suppose \( \mu \in (\underline{\mu}, \bar{\mu}] \). There does not exist an equilibrium at state \((\mu, 0)\) that specifies separation at \( \mu^L \in (\underline{\mu}, \bar{\mu}] \).

**Proof.** Suppose there exists an equilibrium \( \sigma \) at the state \((\mu, 0)\) that specifies separation at \( \mu^L = \mu_1 \in (\underline{\mu}, \bar{\mu}] \). Note that post separation, type \( H \) principal invests \( \frac{\lambda \gamma}{a} \) going forward since the agent now knows that he is facing the type \( H \) principal and hence willing to exert effort. Note that for \( \sigma \) to be feasible, it must specify average investments are higher than \( \frac{\lambda \gamma}{a} \) starting at state \((\mu_1, 0)\). This is because from lemma 9 we know that the agent minimum average investment at state \((\mu, 0)\) given by \( B(\mu) \) is decreasing in \( \mu \) and \( B(\bar{\mu}) = \frac{\lambda \gamma}{a} \). Note that this means that the type \( L \) principal has a deviation. She can mimic type \( H \) principal when \( \mu^L = \mu_1 \) and induce effort from the agent by investing \( \frac{\lambda \gamma}{a} \) until beliefs reach \( \mu \) at which point she quits. Note that by mimicking type \( H \), she can guarantee herself an average investment that is strictly lower. Also note that her investment does not vary with time post her deviation. Since the principal’s investment cost is convex, this guarantees that type \( L \) principal improves her payoff by deviating. Hence there cannot be separation before \( \mu^L = \mu \) in any equilibrium starting at state \((\mu, 0)\) if \( \mu \in (\underline{\mu}, \bar{\mu}] \).

**Proof of Proposition 5.** We know from lemma 14 that a two phase equilibrium exists at state \((\mu_0, 0)\) where \( \mu \in [\underline{\mu}, \bar{\mu}] \). Moreover, Lemma 15 tells us that any equilibrium where the agent exerts effort at state \((\mu_0, 0)\) with \( \mu_0 \in [\underline{\mu}, \bar{\mu}] \) cannot exhibit separation at any point.
This implies that a principal optimal equilibrium, if it exists must be a two phase equilibrium. I now argue that a principal optimal equilibrium, does in fact exist. To that end, note that every two phase equilibrium at \((\mu_0,0)\) is characterized by the corresponding pooling investment function that maps \([\mu,\mu_0]\) to \([\frac{\lambda\gamma}{\alpha}, \bar{x}]\). Denote by by \(\mathcal{X}(\mu_0)\) the set of all equilibrium pooling investment functions at state \((\mu_0,0)\). Note that each of these equilibrium pooling functions is bounded and satisfies the type L principal’s IR \((IC_L)\) and agent’s IR \((IC_A)\) at every \(\mu \in [\mu,\mu_0]\). I show that the set \(\mathcal{X}(\mu_0)\) is compact which implies that there exists a pooling equilibrium function that maximizes the principal’s value over \(\mathcal{X}(\mu_0)\).

Proof of Proposition 4. I first show that the strategy profile is an equilibrium. I then show that it is indeed the principal’s optimal equilibrium.

We first observe that the type H principal has no incentive to deviate at any stage since she is investing at her optimal level \((\frac{\lambda\gamma}{\alpha})\) and inducing effort from the agent. Next I consider type L principal’s behavior. We start with the quitting phase. Note that in this phase, the agent will exert effort if the type L principal invests \((\frac{\lambda\gamma}{\alpha})\) from the beginning of the phase. But note that by the definition of \(\mu\), the type L principal strictly prefers to not induce effort by investing \((\frac{\lambda\gamma}{\alpha})\) when her belief about project quality is strictly smaller than \(\mu\), which implies that the type L principal is behaving optimally by not investing.

In the pooling phase, I first show that under this strategy profile, the value of the type L principal is strictly positive when \(\mu \in (\mu, \bar{\mu}]\). To see this consider a specific pooling investment function \(\hat{x}\) defined when \(\mu \in [\mu, \bar{\mu}]\) with the property that the value of type L principal is 0 in the pooling phase, i.e. the individual rationality constraint of type L principal binds at all beliefs in the pooling phase. Lemma 7 and lemma 8 together show that \(\hat{x}(\mu) > \frac{\lambda\gamma}{\alpha}\) for all \(\mu \in (\mu, \bar{\mu}]\). Note that in our candidate equilibrium, the investment during pooling phase is \(\frac{\lambda\gamma}{\alpha}\), i.e. strictly lower than \(\hat{x}\). This implies that the value of type L principal is strictly positive in the pooling phase when \(\mu \in (\mu, \bar{\mu}]\). Note that any upward deviation by type L principal in the pooling phase does not alter the agent’s behavior but decreases the payoff of the type L principal. Hence there is no incentive to deviate and invest higher. However, any downward deviation leads to the commencement of autarky which yields 0 to type L principal. Hence there is no incentive to deviate downward in the pooling region. This establishes the optimality of type L principal’s behavior in the pooling region.

Now we turn to the separating region. First note that when \(\mu \geq \max\{\bar{\mu}, \mu_c\}\), the type L principal invests her optimal investment \(\frac{\mu\gamma}{\alpha}\) and induces effort from the agent and hence has no incentive to deviate. When \(\mu \in (\bar{\mu}, \max\{\bar{\mu}, \mu_c\})\), the type L’s investment \(c\frac{\mu}{\mu(1-\gamma)} - \frac{z^2}{r(1-\gamma)} - 1\) is strictly smaller than \(\frac{\lambda\gamma}{\alpha}\). Since the agent exerts effort following this investment, the type L principal has no incentive to deviate and invest higher. If type L principal invests lower, then the continuation equilibrium is autarky which yields 0 to type
SUPERVISING TO MOTIVATE

L principal and hence there is no inventive to deviate and invest lower than the specified investment. This completes the argument to show that type L principal has no incentive to deviate.

Lastly, I show that the agent’s behavior is a best response. We start with the firing region. In the firing region, it is optimal for the agent to exert effort if he knows if she is facing a type L principal only if the investment of type L principal is at least $\frac{c}{\mu\lambda(1-\gamma)} - \frac{x^2}{\gamma(1-\gamma)} - 1$. This is because the agent’s flow value is equal to 0 when the type L principal invests $\frac{c}{\mu\lambda(1-\gamma)} - \frac{x^2}{\gamma(1-\gamma)} - 1$. When the agent knows he is facing type H principal, he is willing to exert effort regardless of the investment since his value is strictly positive even if the type H principal does not invest.

In the pooling region, by Definition 12 we know that the agent’s value at the beginning of pooling (when type L principal’s belief is $\bar{\mu}$), the agent’s value is 0. Note that the agent’s value in the pooling region is strictly positive, except at $\bar{\mu}$. This is because the agent’s value at $\bar{\mu}$ in the the pooling region is given by

$$\left(1 - \left[\frac{\mu}{1 - \mu} \frac{1 - \bar{\mu}}{\bar{\mu}}\right]^{\frac{x}{a^2}}\right) \left[\bar{\mu}\lambda(1-\gamma) \left(1 + \frac{\lambda a}{a}\right) - c\right] + \left[\frac{\mu}{1 - \mu} \frac{1 - \bar{\mu}}{\bar{\mu}}\right]^{\frac{x}{a^2}} \frac{\bar{\mu} - \mu}{1 - \mu} z^a = 0.$$

This implies that

$$\bar{\mu}\lambda(1-\gamma) \left(1 + \frac{\lambda a}{a}\right) - c < 0.$$  

Note that the agent’s value at $\mu < \bar{\mu}$ in the the pooling region is given by

$$\left(1 - \left[\frac{\mu}{1 - \mu} \frac{1 - \bar{\mu}}{\bar{\mu}}\right]^{\frac{x}{a^2}}\right) \left[\bar{\mu}\lambda(1-\gamma) \left(1 + \frac{\lambda a}{a}\right) - c\right] + \left[\frac{\mu}{1 - \mu} \frac{1 - \bar{\mu}}{\bar{\mu}}\right]^{\frac{x}{a^2}} \frac{\bar{\mu} - \mu}{1 - \mu} z^a.$$  

The agent value at any belief $\mu$ in the pooling region is a linear combination of $\bar{\mu}\lambda(1-\gamma) \left(1 + \frac{\lambda a}{a}\right) - c$ and $\frac{\bar{\mu} - \mu}{1 - \mu} z^a$. Note that the weight on $\bar{\mu}\lambda(1-\gamma) \left(1 + \frac{\lambda a}{a}\right) - c$ is decreasing in $\mu$. Since $\bar{\mu}\lambda(1-\gamma) \left(1 + \frac{\lambda a}{a}\right) - c < 0$, the agent’s value must be strictly decreasing in $\mu$ in the pooling region. Since value at $\bar{\mu}$ is 0, the value of the agent must be strictly positive when $\mu \in [\underline{\mu}, \bar{\mu})$. This implies that the agent has no incentive to deviate and not exert effort in the pooling region since not exerting effort leads to autarky which we know exists when $\mu \leq \bar{\mu}$.

In the separating region, when $\mu \in (\bar{\mu}, \max\{\mu^c, \bar{\mu}\})$, the agent’s value is 0. To see this note that the type L principal’s investment in this region is $\frac{c}{\mu\lambda(1-\gamma)} - \frac{x^2}{\gamma(1-\gamma)} - 1$ which implies that the flow value of the agent is 0 in this region. Note that eventually beliefs reach $\bar{\mu}$ at which point pooling begins. We know from the previous paragraph that the agent’s value is equal to 0 at $\bar{\mu}$, hence the value of the agent is equal to zero when $\mu \in$
(\bar{\mu}, \max\{\mu^c, \bar{\mu}\})$. Hence the agent cannot do any better by deviating since any deviation leads to autarky and a value of 0 for the agent.

Lastly, in the region where $\mu > \max\{\mu^c, \bar{\mu}\}$, the agent’s flow is strictly positive since the type $L$ principal invests $\frac{\mu \lambda \gamma a}{\rho (1-\gamma)}$ which is strictly above $\frac{\mu^c \lambda \gamma a}{\rho (1-\gamma)} - \frac{\epsilon^2}{\rho (1-\gamma)} - 1$ in this region. Hence the agent gets a strictly positive value in this region. Note that by deviating the agent only delays the value that he will get and hence it is suboptimal for the agent to deviate. This completes the proof that the specified strategy profile is indeed an equilibrium.

Now I show that $\sigma^*$ is indeed the principal optimal equilibrium. We know from Lemma 3 that in the pooling phase the pooling investment level must be at least $\frac{\lambda \gamma a}{\rho}$. We also know from Lemma 15 that the continuation equilibrium at $\bar{\mu}$ must specify pooling when $\mu \in [\bar{\mu}, \bar{\mu}]$. Note that in the pooling phase under $\sigma^*$, the investment is the least admissible for pooling. Hence no other pooling continuation equilibrium at $\bar{\mu}$ can improve the principal’s payoff. Next, given that pooling begins at $\bar{\mu}$, I show that the behavior in the separating phase is optimal. I will show this through two cases.

Case 1: $\mu^c \leq \bar{\mu}$. In this case both types of principal invest their optimal investment in the separating phase. Hence no other separating behavior in this phase can improve the ex ante payoff of the principal.

Case 1: $\mu^c > \bar{\mu}$. In this case the the type $L$ principal invests optimally when $\mu \geq \mu^c$ and above her optimal investment when $\mu \in (\bar{\mu}, \mu^c)$. Note that in any principal optimal equilibrium it must be the case that type $L$ principal invests optimally when $\mu \in (\bar{\mu}, \mu^c)$, otherwise the principal’s payoff can be improved. Note that when $\mu \in (\bar{\mu}, \mu^c)$, type $L$ principal’s investment is above her optimal and the agent’s value is equal to 0, which implies that type $L$ principal cannot reduce her investment and improve her payoff since that will result to the violation of the agent’s IR constraint and the agent will stop exerting effort.

The only question remaining to be answered is that can the cutoff at which pooling begins $\bar{\mu}$ be increased while improving the ex ante payoff of the principal? I show that it is not the case. Suppose the pooling phase begins at $\mu_1 > \bar{\mu}$. From Lemma 9 we know that $B(\mu_1) < B(\bar{\mu}) = \frac{\lambda \gamma a}{\rho}$. We know that in the pooling phase the optimal investment must be $\frac{\lambda \gamma a}{\rho}$, this implies that the value of the agent at the beginning of pooling ($\bar{\mu}$) must be strictly positive. Consider a belief $\mu_1 - \epsilon > \bar{\mu}$ for some $\epsilon > 0$. Suppose the pooling phase begins at $\mu_1 - \epsilon$ with the same pooling investment level. Note that now the type $L$ can invest strictly lower than $\frac{\lambda \gamma a}{\rho}$ and induce effort from the agent. Note that ex ante, the principal is better off in this case because the type $L$ principal invests strictly lower in the region $(\mu_1 - \epsilon, \mu_1)$ and still induces effort from the agent. This establishes that the optimal belief at which pooling phase begins must be $\bar{\mu}$. Therefore $\sigma^*$ is the principal optimal equilibrium. ■
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