Wage Dynamics with Developing Asymmetric Information

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Abstract

A worker's ability to switch jobs is important in understanding individual wage growth and wage offered in the labor market, as shown by Burdett and Mortensen (1998). This paper reconciles the tension between the theory of wage growth by on-the-job search and the negative correlation between job mobility and wage. Workers are heterogeneous in productivity, and when a poaching firm contacts an employed worker, it is possible that the incumbent firm knows the worker's type, while the poaching firm does not. This introduces asymmetry of information when a poaching firm and the incumbent firm plays first-price auction game as in Postel-Vinay and Robin (2002). When the incumbent is better informed than the poaching firm, low-type workers change jobs more frequently and job-to-job transitions convey negative information about worker type. The model implies that the policy which bans employers from inquiring about applicants' wage histories decreases wage dispersion between types, but might have an unintended consequence of increased adverse selection.

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1 Introduction

In this paper, I propose a theory of workers' lifetime wage dynamics that incorporates asymmetry of information –the current employer (the incumbent) knows more about the worker than an outsider (the poacher)– in the Employment-to-Employment transition. By doing so, I can identify the channel through which on-the-job search affects both the individual wage profile and equilibrium inference on worker's quality. This is important because on-the-job worker search alone cannot account for qualitative differences in job transition –a transition may occur as a result of poor performance or upon receipt of a better outside offer regardless of job performance. Indeed, data show that a worker's wage is negatively correlated with his transition frequency. This contradicts the theoretical result which indicates that worker search causes wage growth. [1]

Job-to-job transition is an important element in understanding a worker's wage growth. Many researchers have modeled it as a worker's ability to engage in on-the-job search and analyzed its implications to wages, most notably, Burdett and Mortensen (1998) and Postel-Vinay and Robin (2002). Indeed, in many industries, bringing in a competitor and negotiating wage is widely accepted as a norm for driving up a worker's wage. An anecdote from the consulting industry suggests that, on average, a consultant triggers wage renegotiation following on-the-job search once every three years.[2] This observation is more generic, as Moscarini and Postel-Vinay (2017) shows, the macro-level Employment-to-Employment (henceforth EE) transition rate co-moves strongly with wage growth. [3]

In the model, information about worker quality is revealed gradually over time, and the learning is shared only among the matched party, the worker and his/her current employer. The information is modeled with a Poisson news process that generates arrival over time with rate α. Outside firms other than the matched party do not observe this news process. However, I assume that each worker's job separation and hiring are public information, as these events are recorded in a credible and verifiable CV that the worker carries. This is the public information that is available to outside firms.

An employee working for this firm (incumbent) brings about a meeting with a new firm (poacher) at Poisson rate λ, which denotes the worker's on-the-job search intensity.[4] When

[1]Light and McGarry (1998) consider two different models of job transition, the "search goods" model and the "experience goods" model. They show that workers who are more mobile have lower wage paths, which is consistent with the mechanics of 'experience goods' model, in which low current match expectations cause turnover.

[2]I thank Jay Park, who is working in the industry, for this comment.

[3]The authors conclude that this possibly reflects workers' ability to extract rents following an outside offer. In another paper, Moscarini and Postel-Vinay (2016) consider an extension of the Burdett and Mortensen (1998) model. The model incorporates business cycle fluctuations, and its results fit well with the dynamics of the EE transition rate and wage growth.

[4]I follow the notations in Burdett and Mortensen (1998), Postel-Vinay and Robin (2002). I do not distinguish between search by workers who are employed and workers who are unemployed.

an outside firm (poacher) contacts this worker, the firm competes against the current firm (incumbent) for the future service of the worker by playing a first-price auction game like that in Postel-Vinay and Robin (2002). This is a two-player auction where the bidders are asymmetrically informed.

In this setting, I first focus on the informational content of the worker's employment history. When the current employer knows more about the quality of the worker, the new firm faces the problem of adverse selection – low-quality workers that current employers have low willingness to pay are the workers most likely to be attracted by the poaching firm. Therefore, low-quality workers shift jobs more often, while high-quality workers selectively stay longer in the match. In turn, the average quality of workers increases with tenure.

Next, starting from initial uncertainty about a worker's type, I analyze how the worker's history affects his wage determination. A high-quality worker makes fewer transitions, and the worker's wage gradually increases over time. The employer of the worker earns information rent until the poachers' estimate about the worker's quality is accurate. Due to the asymmetry of information, a low-quality worker can exploit new employers by taking advantage of their inaccurate belief of his type. However, in the long run, their wage decreases with job transition, which is consistent with data.

The model has implications for legislation recently introduced in some states, which bars employers from asking job applicants about wage history. In auctions with asymmetrically-informed bidders, it is known that the incumbent firm's information rent decreases as additional dimensions of worker information (e.g., wage history) become public. In my setting, this implies that, without the wage information, the wage dispersion between high- and low-quality workers increases. The low-quality workers gain from the policy, while high-quality workers' wage growth is hurt by adverse selection.

The paper is organized as follows. In the next subsection, I review the literature that is related to this paper. In Section 2, I present the model, and in Section 3, I define the equilibrium of the model. In Section , I present two cases that yield closed-form solutions and analyze their properties. In Section 5, I discuss the implications of the model and the role of underlying assumptions. Section 6 concludes.

1.1 Literature Review

This paper contributes to the literature on the labor markets with search friction by adding in the elements of adverse selection and asymmetric information.

Burdett and Mortensen (1998) first showed that a worker's ability to engage in on-the-job search can support continuum of posted wages in equilibrium. For their results, it was crucial that firms commit to posted wage contracts, that they could not respond to outside offers. Postel-Vinay and Robin (2002) and Moscarini (2005) relaxed the assumption and solved for the wage-bargaining problem after the arrival of a poacher. In this paper I adopt their premise that wage is determined by an auction. [5]

Alongside the "search goods" model of the job mobility, many researchers focus on "experience goods" nature of the jobs, incorporating gradual learning of match quality (Jovanovic (1979) and Jovanovic (1984).) Moscarini (2005) tried to synthesize both approaches by nesting the "experience goods" job search model into a general equilibrium. My model focuses on learning about worker quality, rather than idiosyncratic match quality, but replicates the turnover pattern whereby low-quality matches dissolve quickly. This effect seems to be prevalent in the data, as shown by Light and McGarry (1998).

My innovation is in incorporating the component of asymmetric-information into these models. I use the first-price auction model with asymmetrically informed bidders studied extensively by Engelbrecht-Wiggans, Milgrom, and Weber (1983), and Milgrom and Weber (1982). In a static setting, the auction game has a well-defined solution, while my technical contribution is solution of the game in a dynamic environment. [6] I solve for the dynamic equilibrium where the value of an object sold in the market is endogenous, as it takes into account the future outcome of the auction. This is also true in Postel-Vinay and Robin (2002) and Moscarini (2005), but in my model, the values are non-stationary because the information asymmetry also evolves over time.

The information asymmetry in my model is generated by the arrival of private information to an incumbent firm. In that regard, this paper is also closely related to the literature on dynamic adverse selection, notably, Kim (2017), Hwang (2018) and Camargo and Lester (2014). These papers analyze trading dynamics in markets for goods and financial assets when a buyer's inference on the quality is influenced both by the public information (calendar time), and the correct anticipation of the seller's equilibrium behavior. It is reasonable to believe that these effects are also present in labor markets where current employers know worker characteristics better than others.

Surprisingly little attention has been paid to information frictions in wage determination, with the exception of Carrillo-Tudela and Kaas (2015), which introduces adverse selection and screening considerations to the framework used in Burdett and Mortensen (1998). We both consider an environment in which a worker's quality is initially unknown to firms. However,

[5] There are some papers that explicitly solves for the alternating-offers bargaining, such as Cahuc, Postel-Vinay, and Robin (2006). However, their bargaining equilibrium is qualitatively similar to to that of the Bertrand auction equilibrium. Modeling the wage-negotiation process as an auction is a commonly accepted practice in the macro-labor context, although some authors attempt to endogenize the choice of bargaining protocol as in Doniger (2015) and Flinn, Mabli, and Mullins (2017). Contrary to the "experience goods" models described in the next paragraph, these papers focus on aggregate equilibrium.

[6] Wolinsky (1988) solved for seller valuation in a sequential auction game in which a random number of bidders are attracted over time. In my setting, because I am interested in wage negotiation initiated by on-the-job search, the auction always has two bidders. Furthermore, Wolinsky (1988) focuses on stationary equilibrium, while my model exhibits non-stationarity due to the evolution of an observable component of a worker's history.

in their setting, workers know their quality, and in return, firms offer screening contracts that separate good types from bad types. They also assume a particular class of contract that promotes/demotes based on the realization of a perfectly revealing signal.

In this paper, I focus instead on information asymmetry between firms. I think this is a natural assumption for many occupations, unless all past worker performance is public. Although I focus on interaction between firms in job transitions, I generate a similar dynamic to that of Carrillo-Tudela and Kaas (2015), because the good worker and the bad worker are treated differently by the incumbent in a wage auction. We both show that unobserved quality (worker type) can account for the correlation between high job mobility and low wages.

My work also contributes to the understanding of wage-tenure profile as in Burdett and Coles (2003) and Stevens (2004), which show that firms optimally choose to backload wages in order to retain workers. Contrary to their findings, my model generates reverse causality, in which tenure in a firm increases a worker's bargaining power. I also provide a learning channel through which workers drive up their wages, contrary to other explanations such as human capital accumulation.

2 Model

2.1 Basic Setting

Model Setup

- *Time*: Time is continuous. Calendar time is indexed by $t \in [0, \infty)$.
- *Firms*: The economy is populated with measure 1 of identical, risk neutral firms. A firm can hire multiple workers.
- *Workers*: Workers are either of type H and L (with notation: $\{H, L\}$) standing for *High* and *Low* productivity. Both risk neutral. Assume that worker types are sole input into the production.
- *Types (Productivity)*: Type L workers produce observable flow output normalized to 0. In a small interval of time, type H workers might generate a lump-sum output Y (*breakthrough*), at Poisson rate α. Otherwise, they produce 0.
- *Payoffs*: For a firm with discount rate $r > 0$, the expected continuation value of a type H worker's output is $\frac{\alpha Y}{r}$. The continuation value is 0 for a type L worker's output. Accordingly, a firm hiring a worker who is H with probability p, at flow wage w, accrues expected flow profit of

  $$p\alpha Y - w.$$

  Flow payoff of the worker is wage w.

- *Learning*: Firms learn from the output their workers generate. If an employee first produces a positive output (Y), then the employer immediately knows that the employee is an H. The employer is uncertain about an employee that has produced nothing to date. Nevertheless, the longer an employee of a firm produces nothing, the more likely his/her employer thinks the worker is unproductive.
- *On-the-job Search*: Outside employment opportunities for a worker arrive at Poisson rate λ, in ther form of a competing wage offer from an outside *poaching* firms.
- *Asymmetry of Information*: I assume that the output is observed only by the current employer (*incumbent*). Outsiders, or a *poacher* only observes a worker's employment history.

Discussion Since the focus of this paper is on the adverse selection, I assume that worker types are the sole input into the production and abstract away from idiosyncratic match productivity. Also, I will adopt the setting of Burdett and Mortensen (1998), in which a firm can hire multiple workers, which is not true in a matching model. I also analyze in the level of an individual worker following the worker's employment history. A worker's history effectively starts with the first job and is not affected by unemployment, which I do not include in my model.

According to the payoff structure, a worker's type is learned through a *good news* process that generates news with rate α for H workers but generates no news for L workers. It is reasonable to think of this arrival of this process as a private '*breakthrough*'. There are several ways to think about the private breakthrough process. First of all, it may be the output of a worker which cannot be transferred out of the firm because it is the firm's property, or because it is confidential. Examples include research output, coded program, or sales performance. On the other hand, we can think of the information as a subjective performance measure, which is accurate and correlated across firms. Examples would be a senior professor's evaluation of assistant professor, beyond the public output, such as publication. Lastly, the model might apply to a worker's teamwork ability, and leadership which has the feature of an experience good the poachers have less accurate knowledge over.

Even though the perfect good news is assumed for tractability, the mechanics of the model goes through as long as the incumbent is better informed than a poacher.

2.2 Histories and Beliefs

In this section, I define two equilibrium objects that evolve as a function of a worker's observable history. I assume that the worker's employment history is a public information, as carried around in the form of a resume, or a CV.

Definition 1 (history)**.** A worker's (employment) history, *$h(t)$, is a chronological list of all firms and tenures up until age t:*

$$h(t) = (\tau_1, \tau_2, \ldots, \tau_n), \quad t = \sum_{i=1}^{n} \tau_i$$

where τ_i is the tenure at the i-th firm this worker was employed at.

The firms $1, 2, \ldots, n$ are chronologically ordered so that the n-th firm is the last firm to employ the worker: the current employer, which we call an *incumbent* firm. Note that τ_n is defined to be $t - \sum_{i=1}^{t-1} \tau_i$. In particular, in contrast to the previous tenures (τ_i, with $i \neq n$) that terminated by a transition, the worker need not be ending his tenure at the current firm (n) at time t. For later reference, it is useful to distinguish between histories continuing with tenure τ_n and histories with switch at time t.

Definition 2. *Define by $h(t)$ the history continuing with tenure τ_n at time t, and $\tilde{h}(t)$ the history with tenure τ_n ending at t.*

I define the two equilibrium objects.

Definition 3 (beliefs)**.** *$p(h(t))$ is the* incumbent's belief *that the worker is H, if the incumbent did not observe any good news output (breakthrough) after hiring him/her (for τ_n duration).*

$x(h(t))$ is the poacher's belief *about the incumbent's knowledge that the worker is H.*

Discussion on Histories The space of histories consists of partitions of age t, (work history of length t) into a vector of past tenures in the firms that the worker was employed at. Formally, we denote the full set of public employment histories of a worker of age t by $\mathcal{H}(t)$, where

$$\mathcal{H}(t) = \{(\tau_1, \tau_2, \ldots, \tau_n) \mid \sum_{i=1}^{n} \tau_i = t\},$$

while a particular employment history is an element $h(t) \in \mathcal{H}(t)$. A worker is never unemployed in my model, and the set of tenures add up to the age t. Firm n is the incumbent firm that currently hires the worker. Note that τ_n is the continuation of current tenure. Unlike other tenures τ_i, ($i < n$), τ_n might or might not terminate at t.

Since I assume a continuum of firms being drawn randomly in a meeting process, any two firms in a worker's employment history are distinct firms. Furthermore, concerning the meeting process (happening at a Poisson rate λ) outlined above, note that the history reflects only the meetings that resulted in transition. It may be that a worker generated a meeting, but did not transit, as these events are not reflected in the history.

Potentially, there may be other elements of a worker's history that are also observable and informative, such as wages, or the identity of the firm. I abstract away from the possibilities for now, but later discuss what happens when the wage is also public.

Discussion on Beliefs Poachers (outside firms) do not observe the output process and form belief about the information in the current match, conditioning only on the public information, the employment history.

To elaborate on the belief p, note first that the incumbent's belief that the worker's quality is high, P, is a mapping

$$P:\ \mathcal{H}(t) \times \mathbf{1}(\tau_n) \rightarrow [0,1],$$

where $\mathbf{1}(\tau_n)$ is a indicator random variable of whether the incumbent firm has observed the H output for the duration of τ_n.

Given an employment history $h(t)$, the mapping is given by

$$\begin{cases} P(h(t),1) = 1 \\ P(h(t),0) = p(h(t)) := \frac{p_0 e^{-\alpha\tau_n}}{p_0 e^{-\alpha\tau_n} + (1-p_0)} \end{cases}$$

where p_0 is the initial expectation of the worker's quality at the point of hire. I relegate to the later sections the details on p_0. In essence, it is pinned down by the history of the worker up until $t-\tau_n$, $(\tilde{h}(t-\tau_n))$, and the bid the incumbent firm made at the point of attracting the worker.

Since the arrival of a good output follows a Poisson news process of arrival rate α, the probability that a H type worker generates no arrival for the duration of τ_n $(\tau_n > 0)$ is $e^{-\alpha\tau_n}$. The belief $p(h(t))$ is obtained using Bayes' rule, or by solving the ODE for the Poisson good-news drift starting from an initial belief p_0:[7]

$$p'(\tau_n) = -\alpha p(\tau_n)(1-p(\tau_n)), \quad p(0) = p_0.$$

The drift equation reflects the fact that the incumbent firm becomes more pessimistic about the worker quality, as time elapses without observing a good output. It is sufficient to track only the pessimistic belief $p(h(t))$, since the belief jumps up to 1 with the arrival of a good output.

Now, we focus on the belief of the *poacher* (outside firm), $x(h(t))$. Define $x(h(t))$ as the poacher's Bayes rational belief over the incumbent's observation of output:

$$x(h(t)) := Pr(\{P(h(t), \mathbf{1}(\tau_n)) = 1\}).$$

The belief is affected by several elements. First, it is affected by the incumbent's additional

[7] Derivation is contained in Appendix A.1 for the readers who are not familiar with continuous time belief process with Poisson news.

	Incumbent	Poacher
Tenure in the current firm	O	O
Past jobs and transitions	O	O
Good output (Y)	O	X
Failed poaching attempts	O	X
Belief	$p(h(t))$ or 1	$x(h(t))$

Table 1: Summary of Information Structure

information about the worker, belief about the initial expectation (p_0) and the good output realization within τ_n. In order to avoid complication, assume the incumbent and the poacher agrees on the initial expectation p_0.

Assumption 1. *Assume that the incumbent and the poacher agrees on the initial expectation about the worker's quality p_0, given the public employment history $h(t-\tau_n)$.*

A particular example of an implementation of this assumption is presented in Section 4.1, in the form of Assumption 5.

If both firms agree on the initial quality, p_0, then the only divergence in information is whether the incumbent has observed a good ouput for the last τ_n duration. From this fact, it might be tempting to say that $x(h(t))$ is given by

$$x(h(t)) = p_0\Big(1 - e^{-\alpha\tau_n}\Big).$$

However, this is not the whole story because the poacher does not know if there was any other failed poaching attempts that are not reflected in the public employment history. The Bayes-rational outsider knows the following: the poaching attempts arrive at the Poisson rate λ, and among them, only those that resulted in a job transition are shown in the employment history. The outsider's estimate is biased without taking into account the failed poaching attempts that are not observed. I summarize the elements of information structure in Table 1.

2.3 Wage Auction

While the worker is hired, he/she generates meeting with a new firm (poacher) at Poisson rate λ. This is an exogenous process at which the worker meets another firm, and is the only opportunity for a worker to shift to a new job. If the worker does not shift to a new job, the worker continues in the current job.

Assume that, once the poacher contacts the worker, the two firms (incumbent and poacher) compete for the future service of the worker through a *common value first-price sealed-bid* *auction.* Furthermore, we assume that, by initiating this auction, the worker fully commits to accept the result of the auction, by shifting to whichever firm that offers higher bid.

From the assumption, strategic players of an auction are the informed incumbent and the uninformed poacher. Engelbrecht-Wiggans, Milgrom, and Weber (1983) have already studied the first price auction game between informed and uninformed bidders, in a static and symmetric payoff setting. For reference, we summarize the main results of the paper in the Appendix A.2 and proceed to think about how the result modifies in our setting.

In our setting, the informed bidder receives a fully revealing signal about the value of an object (the employee): the firm's value takes on one of two possible numbers, depending on whether the bidder received information (arrival of α) or not. The Bayes rational belief about the firms' information x, corresponds to the distribution of the firm's signal. Formally:

Definition 4. *The symmetric-payoff, common-value first-price auction game with asymmetric information at time t, when the beliefs are $x = x(h(t))$ and $p = p(h(t))$, consists of*

- *Two bidders: Informed I (Incumbent), and Uninformed P (Poacher)*
- *Informed I can be of two types: $\{Ih, I0\}$ for observing/not observing the news.*

  Uninformed P is of only one type: $P\emptyset$ since he does not know about the news arrival.
- *Two ex-post valuations: $0 \leq \Pi_{I0} < \Pi_{Ih}$, for I, while the expected value of the worker*

$$\Pi_{P\emptyset} = x\Pi_{I0} + (1-x)\Pi_{Ih}$$

- *Signals: I observes binary signal that informs*

$$\begin{cases} \Pi_{Ih} & \textit{with probability } x \\ \Pi_{I0} & \textit{with probability } 1-x \end{cases}$$

Applying the result from Engelbrecht-Wiggans, Milgrom, and Weber (1983):

Proposition 1. *The game has a unique Bayesian Nash equilibrium where:*

1. *The support of the bids is $[\Pi_{I0}, \tilde{\Pi}]$, where $\tilde{\Pi} = x\Pi_{Ih} + (1-x)\Pi_{I0}$.*
2. *Both players submit mixed bids according to the distribution*

$$G(b) = \frac{(1-x)(\Pi_{Ih} - \Pi_{I0})}{\Pi_{Ih} - b}, \quad b > \Pi_{I0}$$

where, mixed strategy for player $Ih, I0$ and $P\emptyset$ satisfy:

$$G_{P\emptyset}(b) = G(b),$$

$$xG_{Ih}(b) + (1-x)G_{I0}(b) = G(b),$$

and $G_{I0}(b) = 1$ *for all* $b \geq \Pi_{I0}$ *and* 0 *otherwise.*

Figure 1 depicts a particular pair of distributions. This result is derived using indifference conditions as in Appendix A.3. We note that the equilibrium strategies imply that with positive probability there are ex-post instances where bidder $P\emptyset$ regrets winning. This is the well-known "winner's curse" in auctions with common values and asymmetric information. However, in order for the bidder $P\emptyset$ to participate in the auction, and to bid non-trivial bids, there has to be some instances in which bidder $P\emptyset$ wins positive profit, which happens when Ih loses the auction.

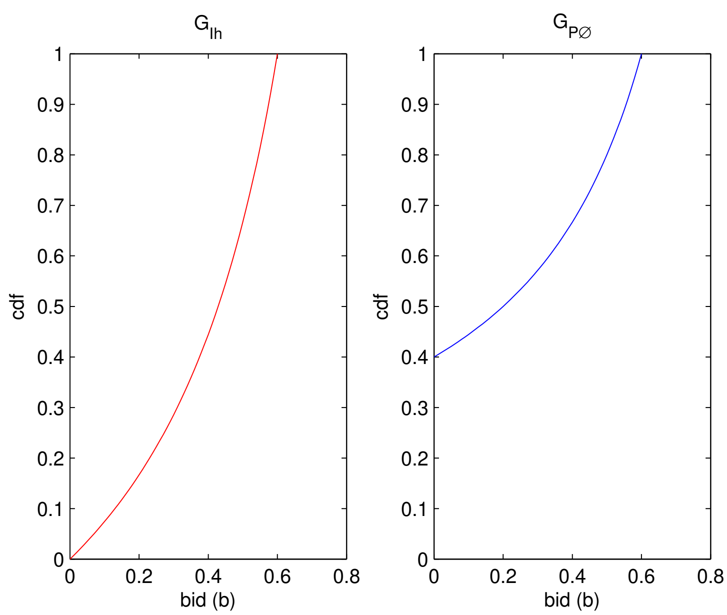

Figure 1: Distribution of Bids G_{Ih} and $G_{P\emptyset}$, with $x = 0.6$, $\Pi_{Ih} = 1$, $\Pi_{I0} = 0$

Note that, in the equilibrium of the auction game, there is non-zero probability of tie at Π_{I0}. For later sections, in order to make probability calculations easy, I assume that in case of a tie, $P\emptyset$ wins. Later, I show that this corresponds to an efficient tie breaking rule.

Assumption 2. *Tie is resolved in favor of the poacher* $P\emptyset$*.*

Discussion I am effectively ruling out the worker's decision in the wage determination. It is restrictive, because as we later show in the equilibrium section, that a worker might have profitable deviation of reneging on his/her commitment to the auction outcome, and wait for the next wage auction where the poachers have more favorable belief about the worker. However, although continual switching is not optimal in the early phase of the career, the result will go through for later phase of the worker's career, because the worker has to ultimately cash-in the benefits by initiating an auction. Furthermore, a worker's incentive to renege on the auction outcome will be less if the worker anticipates that by reneging, the incumbent will exploit the worker until the next auction. Hence, we highlight our main assumption as:

Assumption 3. *The players in the wage auction game are two firms (incumbent and poacher). The worker abides by the auction rule and chooses a firm that offers better wage.*

3 Equilibrium

I solve for equilibrium the in a dynamic environment. In essence, the value of the firm from winning the auction is the annuity value of the worker's output net the future bids the firm has to make in order to retain the worker. The auction equilibrium with the given values exhibits differential rate at which H and L workers are leaving the current match. Hence, the Bayes rational beliefs take into account the win probabilities of the auction. The path of beliefs in turn affects the evolution of values, which affects the present discounted value of the future profits. The equilibrium is a fixed point in which the objects are consistent with each other.

Formally, the equilibrium is objects are given by

Proposition 2 (Equilibrium)**.** *The equilibrium consists of paths of beliefs $x, p : \mathcal{H}(t) \to [0, 1]$, paths of values $\Pi_{Ih}, \Pi_{I0} : \mathcal{H}(t) \to \mathbb{R}$, and the mixed bidding strategies $G_{Ih}, G_{I0}, G_{P\emptyset}$ that maps from $\mathcal{H}(t)$ such that:*

- *During the continuation of tenure, $x(t) = x(h(t))$ evolves according to the ODE:*

$$x'(t) = \Big(\int G_{P\emptyset}(v|h(t))\, dG_{Ih}(v|h(t))\Big)\lambda x(t)(1 - x(t)) + \alpha(1 - x(t))p(h(t))$$

- *During the continuation of tenure, $p(t) = p(h(t))$ evolves according to the ODE:*

$$p'(t) = -\alpha p(t)(1 - p(t)),$$

- *During the continuation of tenure, $\Pi_{Ih}(t) = \Pi_{Ih}(h(t))$ and $\Pi_{I0}(t) = \Pi_{I0}(h(t))$ solves the ODE's:*

$$-\Pi_{Ih}'(t) + (r + \lambda)\Pi_{Ih}(t) = \alpha Y + \lambda \int (\Pi_{Ih}(t) - v) G_{P\emptyset}(v|h(t))\, dG_{Ih}(v|h(t)) \tag{Ih}$$

$$-\Pi_{I0}'(t) + (r + \lambda + \alpha p(t))\Pi_{I0}(t) = \alpha p(t)\Big(Y + \Pi_{Ih}(t)\Big) \tag{I0}$$

- $\{G_{Ih}, G_{I0}, G_{P\emptyset}\}$ *are equilibrium mixed strategies of an auction with values* $\Pi_{Ih}(h(t))$, $\Pi_{I0}(h(t))$ *for an incumbent where* $h(t)$ *is a history with continuation of tenure at time* t*, and values* $\Pi_{Ih}(\tilde{h}(t))$, $\Pi_{I0}(\tilde{h}(t))$ *for a poacher, where* $\tilde{h}(t)$ *is a history with job shift at time* t.

The equilibrium solves for the fixed points of the beliefs, values and the auction equilibrium. The readers who are interested in the characterizations of the equilibrium can skip to the next Section 4. For detailed derivation of the conditions, readers can refer to the Appendix A.4 where we invoke a limit argument from a discrete model as $dt \to 0$.

Discussion on Beliefs The drift of belief p is coming from the Bayes rule, taking into account the incumbent's growing pessimism with no arrival of good output. The initial condition is what the incumbent firm, and the outside firms, believe about the quality of the worker when the incumbent poached the worker from the previous firm. Note that, with maximum bid on the support, the firm expects to win for sure, and the expected quality of the worker is at most $x + (1-x)p < 1$ because of the adverse selection.

Lastly, the drift equation for the belief x is the addition of two components:

$$x'(t) = \underbrace{\Big(\int G_{P\emptyset}(v|t)\, dG_{Ih}(v|t)\Big)\lambda x(t)(1-x(t))}_{(1)} + \underbrace{\alpha(1-x(t))p(h(t))}_{(2)}$$

The first component (1) is the bad news drift coming from the differential rate at which the good and bad workers are leaving the firm. Due to the tie-breaking rule assumption, if any other poacher was to arrive (at rate λ) before a poacher's arrival at τ, the earlier poacher must have taken the no news worker for sure, while the good workers have non-zero probability of being retained by the incumbent. Therefore, job-to-job transition in a worker's history is an imperfect bad news about the worker's type. (1) shows that there is growing optimism about the worker's type as the worker's tenure in the firm increases. Part (2) is additional flow of learning (αp) from the pool of workers with no news so far.

Discussion on (Ih)**,** $(I0)$ These are all local conditions, or annuity equations for the expected future value of the firms. Although most of the results follow directly from the definition of the auction equilibrium and annuity equations, it is worth mentioning a few intuitions from the expressions. Note that the last condition implies, from no knowledge about the worker's type except for the initial expectation p, that the poacher's value is Π_{I0} with the starting beliefs. Integrating the annuity equation $(I0)$ yields the following expression for the value:

$$\Pi_{I0}(\tilde{h}(t)) = \alpha p(\tilde{h}(t)) \int_0^\infty e^{-(\alpha+\lambda+r)s}\Big(Y + \Pi_{Ih}(\tilde{h}(t+s))\Big)\, ds \tag{I0'}$$

where $\tilde{h}(t+s)$ is the history with continuation in this firm for duration of s. The derivation of the result is relegated to Appendix A.5.

The equation is intuitive because it is the discounted value of transition to the learned state Ih at any future point $t+s$, with discount rate is the sum of three components, (1) the rate of transition to state Ih, α, (2) the rate of losing the worker before any transition arrival, λ, (3) and the discount rate r. Since the poacher does not observe any output, $x(\tilde{h}(t)) = 0$. The match starts from initial belief $p(\tilde{h}(t))$. Note, however, the Π_{I0} is not stationary because Π_{Ih} is changing over time.

Now we move on to the value of learning the good type at the history of continuation at time t, $\Pi_{Ih}(h(t))$. The discounted value of the future auctions matters both in terms of continuation probability and the expected bid payment. However, since all the bids on the support of the auction equilibrium induces same expected payoff, we can use the indifference condition to write down the value equation as if the incumbent firm wins all future auctions with probability 1:

$$-\Pi'_{Ih}(t) + (r+\lambda)\Pi_{Ih}(t) = \alpha Y + \lambda\Big(\Pi_{Ih}(t) - \bar{V}(t)\Big) \tag{Ih'}$$

where $\bar{V}(t)$ is the maximum bid over the support of $G_{P\emptyset}(v|h(t))$. The flow profit is given by

$$\alpha Y - \lambda \bar{V}(t)$$

flow expected produce minus the expected cost of retainment. Integrating over time yields:

$$\Pi_{Ih}(t) = \frac{\alpha Y}{r} - \frac{\lambda}{r}\int_t^\infty e^{-r(s-t)}\tilde{V}(s)\,ds$$

which has a clear interpretation of expected future value of the worker net the expected future (maximum) bids, in order to retain the worker.

In particular, when the cost is assumed to be held constant forever at $\bar{V}$,

$$\begin{aligned}\Pi_{Ih}(t) &= \frac{\alpha Y}{r} - \lambda\int_t^\infty e^{-r(s-t)}\bar{V}\,ds \\ &= \frac{\alpha Y}{r} - \frac{\lambda}{r}\bar{V}.\end{aligned}$$

A natural reason for $\bar{V}$ to be fixed is because the worker quality is known to be H and the belief does not evolve. In case, $\bar{V}$ is the value of starting a new employment from belief 1, which is given by:

$$\Pi_{I0} = \alpha\int_0^\infty e^{-(\alpha+\lambda+r)s}[Y + \Pi_{Ih}]\,ds.$$

Substituting in the Π_{Ih} allows us to solve for Π_{I0}:

$$\Pi_{I0} = \frac{\alpha}{\alpha+\lambda+r}\Big(Y + \frac{\alpha Y}{r} - \frac{\lambda}{r}\Pi_{I0}\Big)$$

Algebra yields:[8]

$$\Pi_{I0} = \Pi_{Ih} = \frac{\alpha Y}{r+\lambda},$$

a stationary expression for both Ih and $I0$. This is intuitive because when the worker's type is known to be *High*, everytime the worker meets another firm, Bertrand auction drives down the value of the firm to 0. Furthermore, both Ih and $I0$ know the worker's type and is anticipated to receive the same expected profit.

Discussion on Auction The auction equilibrium characterized in Section 2.3 mostly applies to the equilibrium. However, the auction is equilibrium is more general because the values of winning the auction is different for the two bidders. The discrepancy comes from the history of the worker. If the incumbent firm wins, the worker's history is a continuation history $h(t)$, but if the poacher wins, the transition is made public in the new history $\tilde{h}(t)$. In general, the firm's values as calculated by $(I0)$ and (Ih) need not coincide for the two histories. In the next section, I focus on the cases in which I can characterize the equilibrium despite the complications.

4 Characterization of the Equilibrium

In order to solve for the equilibrium, it is necessary to keep track of the evolution of beliefs and the values at the same time. Below, I provide two examples where I can solve for the equilibrium objects in the closed form. For the first case, I assume short-lived firms whose values are stationary. From the first case, I derive an implication for the inference on worker quality from the observed employment history. For the second case, I impose an additional assumption on the auction rule. By doing so, I characterize the evolution of a firm's value, and its implication to the lifetime wage profile.

[8] Expanding,

$$\Big(1 + \frac{\alpha}{\alpha+\lambda+r}\frac{\lambda}{r}\Big)\Pi_{10} = \frac{\alpha}{\alpha+\lambda+r}\frac{\alpha+r}{r}Y$$
$$\frac{(r+\alpha)(r+\lambda)}{(\alpha+\lambda+r)r}\Pi_{10} = \frac{\alpha}{\alpha+\lambda+r}\frac{\alpha+r}{r}Y$$

and

$$\Pi_{1h} = \frac{\alpha Y}{r} - \frac{\lambda}{r}\frac{\alpha Y}{r+\lambda} = \frac{\alpha Y}{r+\lambda}.$$

4.1 Solvable Case 1: Stationarity by Replacement of Firms

In this section I impose an assumption that makes firm values stationary. From there, I can solve for the Markov equilibrium in which the aforementioned two beliefs are state variables.

Note that the complications arise because the expected future profit of the firm has to take into account the future auction outcomes. In order to get around the issue, I assume that the firms are 'replaced' by a new firm every time a poacher arrives at rate λ. This can have interpretation of a new department within the same firm competing for the worker, or that there is a new manager introduced every new round of an auction. Another way to justify it is to say that the lifetime of a manager is short enough compared to the poaching attempt, which arrives only occassionally. The auction game is still played between the informed bidder (successor firm/manager of the 'incumbent') and an uninformed bidder ('poacher').

Assumption 4. *The manager is replaced with the arrival of a poacher (at rate λ), and his/her expected lifetime is $\frac{1}{\lambda}$.*

Values Assuming so, the future auction outcomes are effectively ruled out from the value calculation, and I take into account only the use value of this worker until the next auction. The values are stationary due to the memoryless property of a Poisson process. The expected duration of the match is always $\frac{1}{\lambda}$.

Proposition 3 (Values $(I0)$, (Ih))**.** *Given a history $h(t)$ and beliefs $x = x(h(t))$ and $p = p(h(t))$, the values of the incumbent are $\Pi_{Ih}(h(t)) = \frac{\alpha Y}{r+\lambda}$, and $\Pi_{I0}(h(t)) = p\frac{\alpha Y}{r+\lambda}$.*

The value of a poacher is $\Pi_{I0}(\tilde{h}(t)) = \tilde{p}\frac{\alpha Y}{r+\lambda}$, where $\tilde{p}$ is the initial belief on worker quality, having won the auction. $\tilde{p}$ satisfies $p \leq \tilde{p} \leq x + (1-x)p$.

In this environment, the value of a worker is linear in its belief about quality p. With the observation of a good output, the value jumps to $\frac{\alpha Y}{r+\lambda}$.

Auction Equilibrium Since the values are stationary, it suffices to know only the point beliefs $x = x(h(t))$ and $p = p(h(t))$ to characterize the values at the time of an auction.

Although information variables evolve over time, the auction game itself is a repeated static auction: after a transition shock (λ), there is a new auction game between two bidders; one informed and the other uninformed.

The informed bidder's value for the worker takes two points: $\frac{\alpha Y}{r+\lambda}$ with probability x, and $\alpha p\frac{Y}{r+\lambda}$ ($p = p(h(t))$) with probability $1-x$. Since the winning poacher does not know whether the worker is H at the time of poaching, the game is different from the static auction, in which the value of the object is revealed instantly after winning. However, the two auctions are very similar in the sense that, in the static auction, the uncertainty is taken into account at the bidding stage, while in this environment, the uncertainty matters after winning the auction. Indeed, the auction equilibrium is very similar to the static case as shown below:

Proposition 4. *The equilibrium bids of the auction game with belief $p = p(h(t))$ and $x = x(h(t))$ come from the support*

$$\big[p\tfrac{\alpha Y}{r+\lambda},\ (x+(1-x)p)\tfrac{\alpha Y}{r+\lambda}\big].$$

Bidder $I0$ bids $p\frac{\alpha Y}{r+\lambda}$, and the equilibrium expected profit of the uninformed poacher is 0.

The distribution of bidder Ih bids, G_{Ih}, solves the indifference condition for bidder $P\emptyset$:

$$\Big(\frac{xG_{Ih}(v)+(1-x)p}{xG_{Ih}(v)+(1-x)}\Big)\frac{\alpha Y}{r+\lambda}-v=0,$$

for all $v \in \big[p\frac{\alpha Y}{r+\lambda},\ (x+(1-x)p)\frac{\alpha Y}{r+\lambda}\big]$.

The distribution of bidder $P\emptyset$ bids, $G_{P\emptyset}$, solves the indifference condition for bidder Ih:

$$G_{P\emptyset}(v)\Big(\frac{\alpha Y}{r+\lambda}-v\Big)=\frac{\alpha Y}{r+\lambda}-(x+(1-x)p)\frac{\alpha Y}{r+\lambda},$$

for all $v \in \big[p\frac{\alpha Y}{r+\lambda},\ (x+(1-x)p)\frac{\alpha Y}{r+\lambda}\big]$.

From the assumption, in case of a tie, which happens with positive probability at $p\frac{\alpha Y}{r+\lambda}$, the poacher wins the worker.

Beliefs Given the auction equilibrium, I can calculate the winning probabilities. The incumbent $I0$ loses the worker for sure, while the incumbent Ih wins with probability $1-\frac{1}{2}x$. The probability can be calculated explicitly from the bidding strategies, as in Appendix A.6. Using this information, the drift of beliefs with the continuation in tenure forms an autonomous system:

Proposition 5. *The pair of beliefs (x, p) as function of tenure τ in a firm solves the following ODE's:*

$$x'(\tau)=\lambda\Big(1-\frac{1}{2}x(\tau)\Big)x(\tau)(1-x(\tau))+\alpha(1-x(\tau))p(\tau)$$
$$p'(\tau)=-\alpha p(\tau)(1-p(\tau))$$

from starting belief $p(0)$ and $x(0)=0$.

The evolution of beliefs reflect that the workers with good output is retained in the firm with probability $1-\frac{1}{2}x$.

Earlier, in Section 2.2, I made an assumption (Assumption 1) that the incumbent and the poacher agree on the starting belief p_0. In the current setting, this can be done by making a technical assumption that the poaching wage is publicly observed.

Assumption 5. *A worker's employment history shows the winning bid a poacher has made at the time of the worker's transition.*

The assumption is made purely for a technical reason and does not have a substantive element. Under this assumption, the poacher's expected profit from winning an auction is 0, and the starting belief from a winning bid v is given by:

$$\frac{xG_{Ih}(v) + (1-x)p}{xG_{Ih}(v) + (1-x)},$$

where $x = x(h(t))$, $p = p(h(t))$, and the bid distribution, G_{Ih}, is from the auction equilibrium calculated above, with beliefs x and p. In essence, given the equilibrium bidding strategies, it is the expected quality of a worker derived from Bayes rule.

To summarize, the divergence of histories $h(t)$ and $\tilde{h}(t)$ at time t is reflected in the beliefs (x, p) given by $\big(x(h(t)), p(h(t))\big)$, and

$$x(\tilde{h}(t)) = 0, \quad p(\tilde{h}(t)) = \frac{xG_{Ih}(v) + (1-x)p}{xG_{Ih}(v) + (1-x)}.$$

Simulated Wage Paths Since the system of beliefs is autonomous, an individual's career path can be readily simulated given a prior belief, transition shocks, and the outcomes of auctions, the bids from which governing the starting belief in a new tenure.

Figures 2, 3 depict a simulated path of beliefs and wages for two types of workers, using the identical transition shocks that arrived at times $(\ 0.61 \quad 1.22 \quad 2.73 \quad 3.84 \quad 6.42 \quad 7.06 \quad 11.63\ )$. The arrival of transition shocks are denoted with vertical dotted red lines.

Note that the L type worker shifted every time there was a transition shock because incumbent did not want to bid for the worker. These episodes are shown by the vertical dashed red lines exhibiting sharp drop in beliefs. The last episode at 11.63 pushed the belief down close to 0 because the poacher could win the worker with a very small bid.

Notice also that the belief path of H type workers is more smooth because some of the transition shocks did not result in transition. For this example, poaching attempts at time 0.61, 1.22, 3.84 and 11.63 were deterred by the incumbent; note the wage jumps at these points despite the smooth evolution of beliefs. These are instances of the incumbent matching the worker's outside offer. Even when the worker transitted to a new firm, (points 2.73, 6.42, and 7.06), the drop in beliefs were milder compared to the L workers because the poacher had to bid high enough to win against the incumbent. But also note that two close-by transition shocks at 6.42 and 7.06 resulted in sharp drop in the belief at point 7.06. Probably what happened is that the worker could not generate news during the short duration of tenure, and the poacher at 7.06 could win the worker with a relatively small bid.

In the end, the beliefs converge to the correct level. The histories diverge for the two workers because high type workers on average has longer tenure in a firm; with the arrival of a good output, the H workers are bid by two firms, while the L workers are always bid only by the poacher. In the limit, for a long enough history, public information would be enough to distinguish the two types. However, note that the belief about H worker's type jumps down whenever there is a shift to a new firm, since L workers are more likely to leave the match (this is an imperfect bad news); while the L workers also exhibit upward belief drift between any two transition shocks.

4.2 Solvable Case 2: Restriction on Auction Rule

For this case, I make the following assumption which is convenient, while retaining the main mechanism of the auction game:

Assumption 6. *Recall option: The incumbent can buy the worker back from the poacher by bribing the poacher in case the poacher won the H worker. The incumbent pays the future value of the worker to the poaching firm.*

The incumbent is indifferent between paying out the future value of the worker and losing the worker. The poacher still has incentive to bid in the auction because there is positive probability that he might win over the high type worker and is bribed, or reimbursed in cash by the incumbent. This assumption has the benefit of fixing the transition rule ex-ante for different types of workers (H worker always stays, while L worker always shifts), and making the auction game that of *symmetric* values.

Beliefs Under this new auction rule, when the poacher wins the auction and realizes that the incumbent does not bribe, the firm immediately knows that the worker has not generated a news in the previous firm. On top of that, since the worker transfers if and only if the worker did not generate any news in any of the previous employers, the belief p is effectively the function of starting belief p_0 and elapsed time in the market t only:

$$p(h(t)) = p(t),$$

where

$$p(t) = \frac{p_0 e^{-\alpha t}}{p_0 e^{-\alpha t} + (1 - p_0)}$$

from intial belief p_0.

The belief x reverts back to 0 everytime a worker makes transition, and the drift of belief at tenure τ, calendar time t is:

$$x'(t) = \lambda x(t)(1 - x(t)) + \alpha(1 - x(t))p(t)$$

starting from $x(t - \tau) = 0$. This equation can be solved for x in closed form, as attached in Appendix A.7.

Figure 2: Simulated Belief and Wage Paths for Case 1, L type worker

Figure 3: Simulated Belief and Wage Paths for Case 1, H type worker

Auction Equilibrium The time variable t is calendar time, with knowledge of the point of the last job transition, $\sum_{i=1}^{n-1} \tau_i$. $\Pi_{Ih}(h(t))$ and $\Pi_{I0}(h(t))$ are the values of the incumbent. It is clear that the candidate lower bound of the bids is $\Pi_{I0}(h(t))$ since $I0$ bidder never bids above it.

The poacher's value at time t is evaluated over a different history, $\tilde{h}(t)$. However, from the bribing assumption, the poacher only obtains a no news worker. That is, when the poacher indeed wins a worker, the expected value of the worker $\Pi_{I0}(\tilde{h}(t))$ starts from initial belief $p(t)$ and $x(t) = 0$:

$$\Pi_{I0}(\tilde{h}(t)) = \alpha p(t) \int_0^\infty e^{-(\alpha+\lambda+r)s} \Pi_{Ih}(\tilde{h}(t+s))\, ds$$

In case the poacher is bribed, the poacher receives transfer

$$\Pi_{Ih}(h(t))$$

from the incumbent.

Overall, the poacher expects at most

$$x(t)\Pi_{Ih}(h(t)) + (1 - x(t))\Pi_{I0}(\tilde{h}(t)),$$

from bidding the upper bound of the support, which is

$$x(t)\Pi_{Ih}(h(t)) + (1 - x(t))\Pi_{I0}(h(t)).$$

Therefore, the poacher's expected profit from the auction is:

$$(1 - x(t))(\Pi_{I0}(\tilde{h}(t)) - \Pi_{I0}(h(t)))$$

where $\tilde{h}(t)$ is continuation from starting belief $p(t)$ and $x(t) = 0$, in comparison to $h(t)$ from $p(t)$ and $x(t) \geq 0$. I later verify that the expression is positive in equilibrium. Intuitively, the two paths agree on the quality of the worker $p(t)$, but differs only in what others believe about the worker, $x(t)$. $h(t)$ is a history that is more costly to fight off a poacher.

In this case, given $x = x(t)$, the indifference condition for the poacher that defines distribution G_{Ih} is given by:

$$xG_{Ih}(v|t)(\Pi_{Ih}(h(t)) - v) + (1 - x)(\Pi_{I0}(\tilde{h}(t)) - v) = (1 - x)(\Pi_{I0}(\tilde{h}(t)) - \Pi_{I0}(h(t))).$$

In turn, the support of the bids is

$$\Big[\Pi_{I0}(h(t)),\ x(t)\Pi_{Ih}(h(t)) + (1 - x(t))\Pi_{I0}(h(t))\Big].$$

Since the upper bound of the support contains information about how much the incumbent has to pay in order to keep the match with probability 1, this feature allows us to write the incumbent's value autonomously in terms of $\Pi_{Ih}(h(t))$ and $\Pi_{I0}(h(t))$ only, without knowing about the worker's value in a new match.

Values Denote by $\Pi_{Ih}(t)$ and $\Pi_{I0}(t)$ the continuation values for the incumbent, following history $h(t)$. From the set of beliefs, the values solve the system of equations:

- $$-\Pi'_{I0}(t) + (r + \lambda + \alpha p(t))\Pi_{I0}(t) = \alpha p(t)[Y + \Pi_{Ih}(t)] \tag{I0}$$

- $$-\Pi'_{Ih}(t) + (r + \lambda)\Pi_{Ih}(t) = \alpha Y + \lambda \int (\Pi_{Ih}(t) - v) G_{P\emptyset}(v|t)\, dG_{Ih}(v|t) \tag{Ih}$$

Proposition 6. *Using the fact that* $\bar{v} = x(t)\Pi_{Ih}(t) + (1 - x(t))\Pi_{I0}(t)$*, the second ODE can be rewritten:*

$$-\Pi'_{Ih}(t) + (r + \lambda)\Pi_{Ih}(t) = \alpha Y + \lambda(1 - x(t))(\Pi_{Ih}(t) - \Pi_{I0}(t)). \tag{Ih'}$$

Note that this is a non-homogeneous system of ODE's ($x(t)$ and $p(t)$ are changing over time) with two variables Π_{Ih} and Π_{I0}. Still, it is an autonomous system of two variables, in which I can characterize some properties of the solution:

Proposition 7. *Fix time* t*, and the beliefs* $x(t)$*, and* $p(t)$*. The value at time* t *that the incumbent firm has to pay in order to retain the worker for sure (i.e., upper bound of the bidding support) is given by*

$$\alpha Y \int_t^{\infty} e^{-(r+\lambda)(z-t)} \Big(x(z) + (1 - x(z))p(z) \Big)\, dz,$$

which is the integral over future path of the quality of the workers in the firm $x(z) + (1 - x(z))p(z)$*, for* $z \in [t, \infty)$*.*

The expression is intuitive in the sense that it integrates over the path of future beliefs regarding the quality of the worker staying in the firm. The firm's profit is shown to be decreasing over time, as the cost term increases with time. The proof involves solving the system of equations using substitution, and the readers can refer to Appendix A.8 for full proof. The full path of firm profits Π_{Ih}, Π_{I0} can be also found accordingly, although the expressions are much messier. Note that in the limit, $x(z) + (1 - x(z))p(z)$ approaches 1, and the cost term goes to

$$\frac{\alpha Y}{r + \lambda}.$$

This verifies that a firm's information rent decreases with time.

A few basic properties of the value function is outlined here:

Proposition 8. *For any $h(t)$, $\Pi_{Ih}(h(t))$ is decreasing in α and decreasing in λ. A poacher's value $\Pi_{I0}(\tilde{h}(t))$ is increasing in α and increasing in $p_0 = p(\tilde{h}(t))$.*

Proof. Since the closed form of x and p are known, the closed form for the $\tilde{x}(t) = x(t) + (1 - x(t))p(t)$ is:

$$\tilde{x}(t) = \frac{p_0 \frac{\alpha}{\alpha+\lambda}\left(1 - e^{-(\alpha+\lambda)t}\right) + p_0 e^{-(\alpha+\lambda)t}}{p_0 \frac{\alpha}{\alpha+\lambda}\left(1 - e^{-(\alpha+\lambda)t}\right) + e^{-\lambda t}\left(p_0 e^{-\alpha t} + (1 - p_0)\right)}.$$

The result is just a substitution of this expression into the Proposition 7. Details of the procedure is contained in Appendix A.9 □

Simulated Path of Wages Below is a simulated path of beliefs x and p for a particular realization of Poisson arrivals, when both λ and α are set at $\frac{1}{3}$. The blue line stands for the poacher's belief x about the news arrival in the incumbent; while the green line stands for the incumbent's belief that the worker's type is H, p. The green line is gradually decreasing from p_0 according to a good news drift. Note that with the arrival of output at point 4.2, the incumbent's belief (green line, on the right) jumps up to 1 and stays there after.

Note also that in the left graph of Figure 4, the slope of the poacher's belief gets smaller as the time goes on. This is because the growth of x is affected by learning, which happens at rate $\alpha p(t)$. Everytime the worker transits, the belief starts to grow again from 0, and the initial growth rate is affected by the initial level of optimism about the worker's type. Therefore, even for the same H type of workers, if one happens to draw the news at a later point in career, the wage growth rate thereafter will be slower than when the worker happened to draw the news at an early point in career.

For these two paths simulated for the length of 30, we see that they start to diverge after the arrival of the good news. Before then, the transition shocks (rate λ) affects the two workers in the same way by driving the outside firm's belief back into 0. For L type workers, everytime the worker meets another firm, the worker transits to a new firm, which drives down the poacher's belief, x, back to 0. H type workers are not explicitly affected since there is no transition. However, the competitors infer postive news about the worker's type as the worker's tenure increases (smooth increasing part in the right figure). Ultimately, the belief reaches the correct one, $x = 1$ for sufficiently long tenure.

I now investigate how the belief path translates to wages paid to the worker. I use the proxy, the retaining cost term, which we wrote as

$$\alpha Y \int_t^\infty e^{-(r+\lambda)(s-t)}\left(x(s) + (1 - x(s))p(s)\right) ds$$

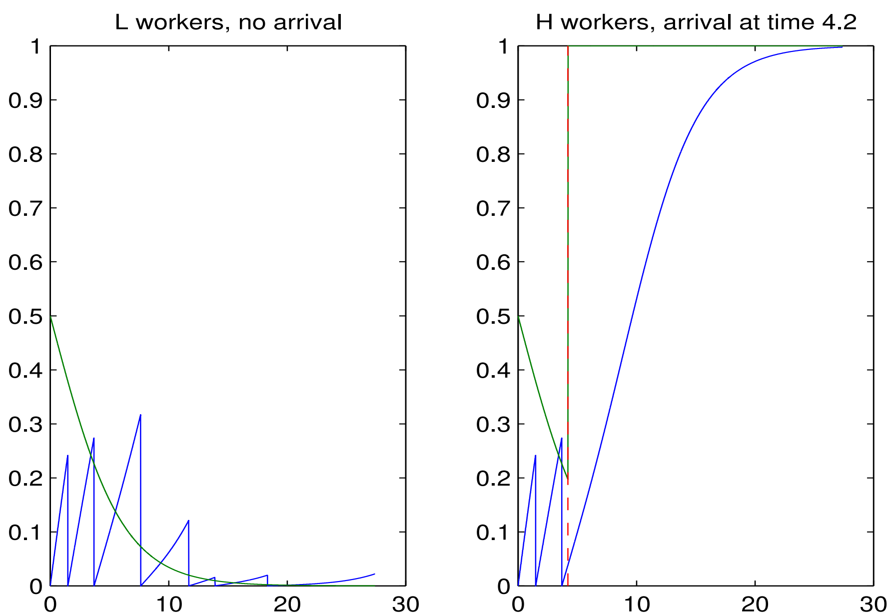

Figure 4: Simulated path of beliefs, x in blue and p in green. Initial belief set at $p_0 = 0.5$. Arrival of transition shocks, which arrives at rate λ, shown by the kinks at the left Figure. For H workers, learning shock occured at time 4.2, indicated by red dotted line.

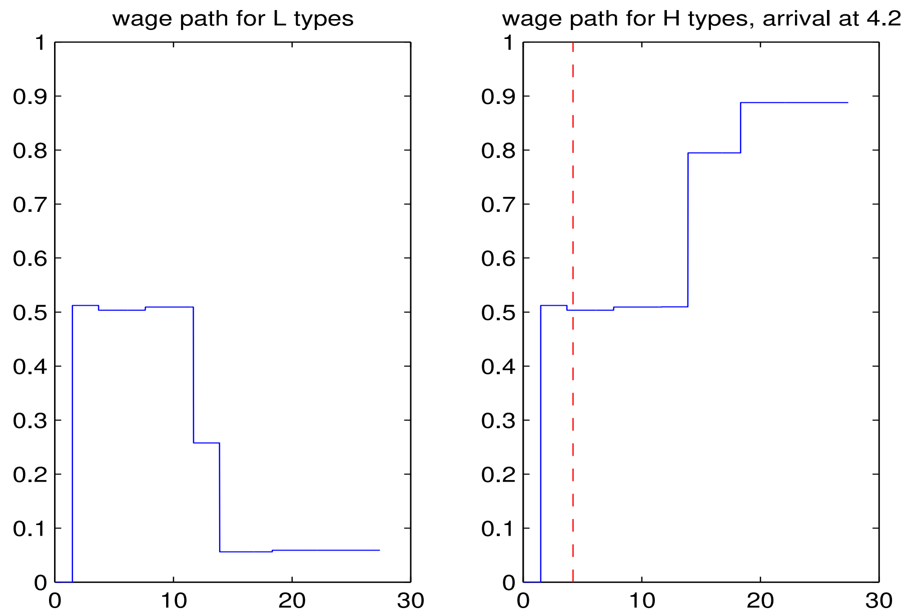

Figure 5: Sample Wage Paths for the Two Types of Workers

These are calculated from the last observed transition, and is a function of tenure. We solve for this integral and normalize for the termination rate of the match $r + \lambda$ to be the proxy for the constant wage paid to the worker. Note that this is an upper bound of the actual wage paid since actual wage is determined by the outcome of the auction. We graph a corresponding sample wage paths below:

The wage paths start to diverge after the arrival of news: for the H type workers, longer duration of stay in the match signals outside firms that the worker is of high quality, and the incumbent firm has to pay more in order to fight off the poachers who are bidding more aggressively for this worker. In the end, the wage is driven up to the productivity of the match, αY. For the L types, although they might initially draw high wage by luck, the wage converges to 0 as the work history drives belief down close to 0.

I expect the wage path of L workers to exhibit a hump pattern. Although the wage growth in the middle age is driven by the fact that they start from wage 0, it is also magnified by firm's information rent.

5 Discussions

5.1 Role of Job Transition in Wage Growth and Long-Run Convergence

For both the case 1 and case 2, the long-run convergence is obvious. For case 2, a worker with a good news is kept with probability 1, and the belief about the worker's quality gradually converges to 1. In case 1, whenever a transition shock arrives, L workers shift with probability 1, while H workers shift with probability less than 1. After observing employment history for long enough, the firms can distinguish the two types.

Proposition 9. *For case 1, the average tenure of a L worker is $\frac{1}{\lambda}$, while the average tenure of a H worker is greater than $\frac{1}{\lambda}$. For long enough job history, the firms correctly learn the worker's type.*

Since both workers start from a pool with initial belief p_0, the proposition implies that H workers, in the long run, raise their wage to 1, while for L workers, wage goes down to 0.

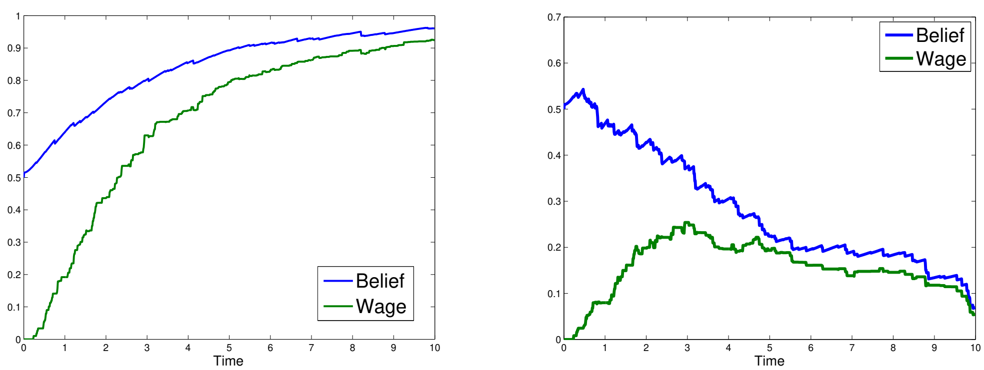

Figure 6: Simulated Average Belief and Wage Path for H Types (Left) and L Types (Right)

Figure 6 shows an average path of belief and wage for the two types respectively, an average path of 50 different simulated wage paths, following case 1 environment. Note that the path still exhibits many kinks due to a large heterogeneity of path realizations. However, there is noticeable trend of the belief and the wage converging to the correct level.

The model implies that, on average, workers with frequent job transitions are likely to be low-quality workers, and the number of transitions is negatively correlated with wage. Light and McGarry (1998) shows that unobserved heterogeneity can account for negative correlation between number of job transitions and low wages. This is in contrast to 'search goods' explanation of job transition, in which job transition indicates that a worker moving to a better paying job, and hence, more transitions increase a worker's wage. The authors conclude that the data is more consistent with 'experience goods' model, in which the transition is initiated by dissolution of a match that is doing poorly.

Carrillo-Tudela and Kaas (2015) has also pointed out that perfect information model of job transition cannot account for the unobserved heterogeneity. Their modelling method differs from this paper, in that they focus on firms' screening and worker sorting. Both the theory and data shows that it is important to take into account unobservable heterogeneity of the workers, and a natural modelling choice is to incorporate asymmetry of information. This paper presents one way of modelling asymmetry of information, in a natrual environment in which some employers know more about the worker than the others.

In a different line, there are papers showing that lifetime earning paths diverge significantly for workers in different percentiles of income. A report by Guvenen, Karahan, Ozkan, and Song (2015) analyses the lifetime earnings of more than 4 million workers using their W2 record. What they show is a large heterogeneity in lifetime earnings growth. Bottom 20 percentile of workers exhibit lifetime earnings decrease, while a median worker experiences 38% increase in lifetime earnings. Even after controlling for extreme outliers, top percentile workers' gain in lifetime earnings is about 1500%.

There are many ways to account for the heterogeneity, and learning is one possible explanation. Farber (2005) used survey data of displaced workers (due to slack work, plant closing, or position abolishment) and showed that about 10% of the displaced workers permanently leave the labor force after displacement. Even for those who return, 13% are hired at part-time jobs after losing a full-time job, and the re-employment income is on average 13% less than the previous job. There are many factors underlying this phenomenon, including the age effect (the authors focus on life-cycle earnings), and firm specific human capital that is lost at displacement. However, it is also documented that there is noticeable 'stigma' effect to workers who lose jobs, and in order to account for the effect, it is necessary to incorporate learning in the model. This paper suggests one step into the direction, and future work will involve identifying the effect of learning within a comparable sample of small cohort with similar characteristics (same college, MBA program, etc.) and checking if the job transition affect these workers differently.

5.2 Implications to the Information Policy

It is worthwhile comparing with the benchmark case where the learning is a public news. The benchmark case is a very simple example of Postel-Vinay and Robin (2002) model where there is no firm heterogeneity and workers can take at most two types. Since our first price auction limits to the Bertrand competition when there is no asymmetry of information, every time there is an auction, the wage gets driven up to the belief about the worker at the point of auction.

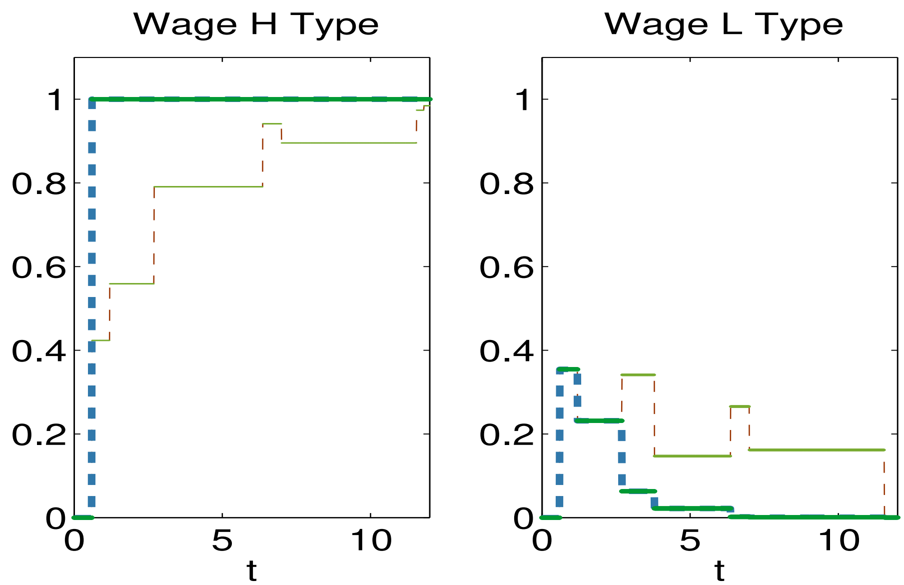

Figure 7: Simulated Wage Profile for Full-Information, as in Postel-Vinay and Robin (2002). The thicker lines (full-information wage profile) allow for comparison with the case with the thin lines (wage profile under asymmetric information, from the previous section). Note that H type workers boost up the wage to 1 in early career, while the wage of L type workers gradually decline along the belief path $p(t)$.

Due to the Bertrand competition, the value of a H worker to a firm is discounted by rate λ, arrival of a poacher. The profit the firm can obtain by keeping the worker is

$$\frac{\alpha Y}{r+\lambda}.$$

Therefore, starting from the initial belief p about the worker's type, the value of the worker is

$$\alpha p \int_0^\infty e^{-(\alpha+\lambda+r)t}[Y + \frac{\alpha Y}{r+\lambda}]\, dt$$

which can be decomposed into the use value of the worker when the worker actually generates a breakthrough before a poacher arrives (y term) and the continuation value starting from the breakthrough, $\frac{\alpha Y}{r+\lambda}$, appropriately discounted taking into account transition and poaching rate. The value can be extended to be

$$\alpha p\Big(\frac{Y}{\alpha+\lambda+r} + \frac{1}{\alpha+\lambda+r}\frac{\alpha Y}{r+\lambda}\Big) = \frac{\alpha p Y}{r+\lambda}$$

Therefore, we expect that, everytime a worker with no breakthrough shifts a job, to receive the future value $p\frac{\alpha Y}{r+\lambda}$ at that time, and the wage to drive up to the marginal productivity $\frac{\alpha Y}{r+\lambda}$ with the arrival of a breakthrough.

From this example, it is worth noting that the worker who has generated a news does benefit from making the news public. While it has negative consequences for bad workers because their wage deterministically drifts down along with the belief. In our model example of two worker types, the wage is almost perfectly informative about the worker's type because any wage increase by retention is a perfect signal that the worker is a High type.

This sheds light onto the recent California legislature which bans firms from inquiring about the worker's past wage history. The legislature was introduced on the grounds that the worker quickly loses bargaining power once the wage information is revealed, especially for the workers who were receiving wage that are below average/expectation to the new employer. According to our model, the high type workers who have generated the good news would still want to credibly convey the wage information to the new firm, while the low type workers would want to hide the information. Along with consideration for pooling/what the firms can infer from declined report, the model tells us what the effect would be if the ban is to be strictly enforced to preclude all wage information. Furthermore, we expect our main mechanism to go through if there is small chance that high type workers would also like to hide their information about their wage history, because if the current wage is not a favorable signal to the worker's type, the worker benefits by hiding it rather than disclosing it.

It should be noted, however, that this exercise is only a benchmark. A limitation is that I assumed short-lived firms. If firms internalize the informational content of the wage in future auctions, the element would alter the firms' bidding strategy. However, restricting to a perfect news benchmark is not a large deviation from the policy exercise. In the perfect good news output, as I describe here, a poacher would immediately tell that a worker is H type if the worker received a promotion in the current firm. If the poacher sees that a worker did not receive a promotion, then the poacher is more pessimistic about the worker's quality.

5.3 Role of the Firm's Commitment

In this section, I explore the role of firm's commitment in driving the main mechanism of the model. This can serve both as a robustness check for the main results, or serve as the gauge for the genericity of the model in terms of capturing different real world dynamics. For instance, if we consider an object that is open for appraisal, such as real-estate property, it is more natural to think that the good assets trade faster, while the less attractive ones stay in the market. In the relevant paper, Kaya and Kim (2018) shows how the trading dynamics would flip if we take into account these 'appraisal' possibility.

We construct a realistic example with drastically different results. Assume that the firms instead learn by 'bad news'. That is, instead of assuming that α accompanies the lump-sum produce of Y, think of it as accompanying a lump-sum deficit $-Y$, which makes it inefficient for the firm to retain the worker afterwards. Assume that a worker with no news generates flow profit of $b > 0$ to the firm. Expected flow profit of the H worker is $b > 0$, while for L worker, it is $b - \alpha Y \leq 0$.

The match starts with prior belief p_0 and drifts according to the equation

$$p'(t) = \alpha p(t)(1 - p(t)), \quad p(t) = \frac{p_0 e^{\alpha t}}{p_0 e^{\alpha t} + (1 - p_0)}.$$

Since the bad news worker is not kept in this firm, both the poacher and the incumbent agree that a worker with tenure t is likely to be H type with probability $p(t)$. Opportunity for the poacher to 'appraise' the good, or to 'interview' a worker will make the poacher better informed than the incumbent.

Definition 5. *With an 'interview', the poacher privately observes the news generated by the worker for next T duration of time.*

We assume that the news process the interviewer observes is the same as the bad news breakthrough process that they see as an employer. The result still goes through for other types of news process, such as perfect good news process, or any other imperfect news. Assuming bad news process, from the incumbent's point of view, the poacher's belief about the worker takes at most two points:

$$\begin{cases} \tilde{p} = \frac{p(t)}{p(t)+(1-p(t))e^{-\alpha T}} > p(t) & \text{if no bad news, probability } x(t) \\ 0 & \text{if bad news, probability } 1 - x(t) \end{cases}$$

where, according to the definition of interview: $x(t) = p(t) + (1 - p(t))e^{-\alpha T}$ and $1 - x(t) = (1 - p(t))(1 - e^{-\alpha T})$. This is exactly the mirror case of the main model, where now we let $x(t)$ be the probability that the *incumbent* attaches to the event that there were no bad news observed in the interview. Effectively, the incumbent now has to worry about overpaying for the worker who the poacher identified as unproductive.

In the replacement case (Example 2), the support of the bids is the interval $[0, p(t)]$, where the poacher with no bad news ($P0$) wins with probability $1-\frac{1}{2}x(t) = 1-\frac{1}{2}\left(p(t)+(1-p(t))e^{-\alpha T}\right)$ and bids to make incumbent firm ($I\emptyset$) satisfy zero-profit, indifference condition: (assume that the incumbent firm's bid is disclosed)

$$0 = \frac{x\tilde{p}G_{P0}(v) + (1 - x) \cdot 0}{xG_{P0}(v) + (1 - x)} \frac{\alpha Y}{r + \lambda} - v$$

Since the probability of winning is 1 at the upper bound of the support, $\bar{v}$, we pin down the upper bound:

$$\bar{v} = (x\tilde{p} + (1 - x) \cdot 0) \frac{\alpha Y}{r + \lambda} = p(t)\frac{\alpha Y}{r + \lambda} = \Pi(p(t)).$$

Poacher with bad news (PL) wins with zero probability, while the incumbent firm ($I\emptyset$) bids in order to make $P0$ bidder indifferent:

$$G_{I\emptyset}(v)=\frac{\Pi(\tilde{p})-\Pi(p(t))}{\Pi(\tilde{p})-v}=\frac{(\tilde{p}-p(t))\frac{\alpha Y}{r+\lambda}}{\tilde{p}\frac{\alpha Y}{r+\lambda}-v}\quad \text{over } v\in\left[0,p(t)\frac{\alpha Y}{r+\lambda}\right].$$

Since only a no bad news poacher, $P0$, makes non-trivial bid, successful poaching drives up belief to $\tilde{p}$ from $p(t)$. If the incumbent succeeded in retaining the worker, due to the interview, the poacher might have been type PL who saw bad news about the worker. Every time the worker is retained with bid v, belief jumps down taking into account the adverse selection:

$$p(h(t))=\frac{x\tilde{p}G_{P0}(v)+(1-x)\cdot 0}{xG_{P0}(v)+(1-x)}<p(t).$$

6 Conclusion

This paper focuses on the relationship between the wage profile and the job-to-job transition of a worker, by exploring the interaction between the micro-auction game and the macro-labor model. I incorporate learning and information friction into the traditional model of on-the-job search, and show that asymmetry of information can generate high job transition rate for low type workers, and low job transition for high type workers. This finding is consistent with data which shows that escape from a bad match is a more likely cause of job transition than worker search alone. Using the results from the literature on auctions, I characterize the auction equilibrium in the dynamic setting, and analyse the effect of information policy. I show that hiding a worker's wage history helps low type workers, while it might have a negative consequence of hurting high type workers. At the same time, a worker may on average be worse off when the law switches off learning from wage history. The model shows that the interaction between a well-informed incumbent and less-informed poacher results in positive signal on worker quality as a worker's tenure grows. Depending on the industry, this might not be a case. If a worker's performance is public, or if the firm lacks commitment to keep a bad worker, it is possible that a longer tenure convey negative signal about quality. I hope this discussion can shed light to different connotations to job transitions attached in various industries.

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A Appendix

A.1 Good News Drift

Starting from initial probability $p(t) \in [0, 1]$, the probability that no news arrives in dt interval is given by:

$$P(H)Pr(\text{no news}|H) + P(L)Pr(\text{no news}|L) = p(t)(1 - \alpha dt) + (1 - p(t)).$$

Applying Bayes rule, $p(t + dt)$ is given by:

$$p(t + dt) = \frac{p(t)(1 - \alpha dt)}{p(t)(1 - \alpha dt) + (1 - p(t))}.$$

Subtracting $p(t)$ from both sides:

$$\begin{aligned} p(t + dt) - p(t) &= \frac{p(t)(1 - p(t))(1 - \alpha dt) - p(t)(1 - p(t))}{p(t)(1 - \alpha dt) + (1 - p(t))} \\ &= \frac{p(t)(1 - p(t))(-\alpha dt)}{p(t)(1 - \alpha dt) + (1 - p(t))}. \end{aligned}$$

Dividing both sides by dt and taking limit $dt \to 0$,

$$p'(t) = -\alpha p(t)(1 - p(t)).$$

In general, this is true even if match dissolves with some positive rate, as long as the rate for H and L types are identical. Starting from any $p_0 = p(t)$, and using the Bayes' rule, the belief after dt is given by

$$p(t + dt) = \frac{p(t)(1 - rdt - \alpha dt - \lambda dt)}{p(t)(1 - rdt - \alpha dt - \lambda dt) + (1 - p(t))(1 - rdt - \lambda dt)}$$

$$p(t + dt) - p(t) = \frac{p(t)(1 - p(t))(1 - rdt - \alpha dt - \lambda dt) - p(t)(1 - p(t))(1 - \lambda dt)}{p(t)(1 - rdt - \alpha dt - \lambda dt) + (1 - p(t))(1 - rdt - \lambda dt)}$$

$$p'(t) = -\alpha p(t)(1 - p(t)), \quad p(0) = p_0$$

This is true as long as both the high and low types leave at the same rate λ.

A.2 Results from Engelbrecht-Wiggans, Milgrom, and Weber (1983)

Definition 6. *Z is the value of the object, X is the private signal of an informed bidder, U is an independent uniform random variable on $[0,1]$. Let $H = E[Z|X]$.*

Definition 7. *Define the informed bidder's distributional type be $T = T(H,U)$, uniformly distributed on $[0,1]$, where*

$$T(h,u) = Pr(\{H < h, \text{ or } H = h \text{ and } U < u\}).$$

Let

$$H(t) = \inf\{h|P(H \leq h) > t\}$$

to have $H = H(T)$ almost surely.

We solve for the equilibrium strategies $\beta : [0,1] \to \mathbb{R}_+$ for the informed bidder, and the bid distribution G for uninformed bidder.

Proposition 10 (Engelbrecht-Wiggans, Milgrom, and Weber (1983))**.** *The equilibrium bid distribution β of the informed bidder is,*

$$\beta(t) = E[H(T)|T \leq t] = \frac{1}{t}\int_0^t H(s)ds$$

with $\beta(0) = H(0)$, and $\beta(1) = E[H]$.

The equilibrium bid distribution G of the uninformed bidder is

$$G(b) = Pr(\beta(T) \leq b).$$

Proof. Suppose the informed bidder type is $T = t$. If he bids $\beta(\tau)$, then he wins with probability τ, yielding an expected payoff

$$[H(t) - \beta(\tau)]\tau = \int_0^\tau (H(t) - H(s))\, ds$$

which is maximized at $\tau = t$.

For any uninformed bidder, any bid below $H(0)$ yields zero payoff, while bid greater than $E[H]$ generates negative payoff. Consider a bid $b = \beta(t)$. Its expected payoff is

$$E[Z - \beta(t)|T \leq t]t$$

However, $E[Z - \beta(t)|T \leq t] = E[H(T)|T \leq t] - \beta(t) = 0$. □

Applying the results to our environment, we get the distribution

$$H(t) = \begin{cases} \Pi_{I0} & t \le 1-x \\ \Pi_{Ih} & t > 1-x \end{cases}$$

which implies

$$\beta(t) = \frac{1}{t}\int_0^t H(s)\,ds = \begin{cases} \Pi_{I0}, & t \le 1-x \\ \frac{1}{t}\Big((1-x)\Pi_{I0} + (t-(1-x))\Pi_{Ih}\Big), & t > 1-x \end{cases}$$

A.3 Common Value Auction: A Constructive Proof

In this Appendix, we show that in equilibrium:

- $I0$ submits degenerate bid Π_{I0}. Ih and $P\emptyset$ submit randomized bids over support $[\Pi_{I0}, \tilde{\Pi}]$, where $\tilde{\Pi} = x\Pi_{Ih} + (1-x)\Pi_{I0}$.
- Bidding distributions for the three players are

$$G_{Ih}(b) = \frac{1-x}{x}\frac{b-\Pi_{I0}}{\tilde{V}-b}$$

$$G_{P\emptyset}(b) = \frac{(1-x)(\Pi_{Ih}-\Pi_{I0})}{(\Pi_{Ih}-b)}$$

  G_{I0} is degenerate at Π_{I0}.

Claim 1. *There does not exist a pure strategy NE of this auction game.*

Proof. The value of the object at sale is at least Π_{I0}.

Any bid $b < \Pi_{I0}$ is not played in equilibrium because any opponent player can make profitable deviation to b' with $b < b' < \Pi_{I0}$.

Suppose bidder $P\emptyset$ always bids $b = \Pi_{I0}$. Then bidder $1H$ has profitable deviation for any bid $b' > \Pi_{I0}$, from which he can bid slightly less, $b' > b'' > \Pi_{I0}$.

Suppose bidder $P\emptyset$ bids $\Pi_{I0} < b \le x\Pi_{Ih} + (1-x)\Pi_{I0}$. Bidder Ih's best response is to bid slightly above b and win for sure. In which case, the bidder 2 expects to gain negative profit $\Pi_{I0} - b < 0$, and would rather get 0 payoff by bidding Π_{I0}.

The bidder $P\emptyset$ never bids above $x\Pi_{Ih} + (1-x)\Pi_{I0}$ in equilbrium. Since bidder $I0$'s best response to $b > \Pi_{I0}$ of bidder $P\emptyset$ is to bid below b, bidder $P\emptyset$ always wins over $I0$ in equilibrium. Hence his maximum willingness to pay for the object is $x\Pi_{Ih} + (1-x)\Pi_{I0}$. □

Therefore, if there is an equilibrium, it has to be in mixed strategies, the support is given by the following claim.

Claim 2. *In equilibrium, bidder $P\emptyset$'s bid distribution $G_{P\emptyset}$ is continuous and strictly increasing over* $(\Pi_{I0}, x\Pi_{Ih} + (1-x)\Pi_{I0}]$.

Proof. First, note that given the candidate bidder $P\emptyset$'s strategy, bidder $I0$ never puts positive probability on bids above Π_{I0}. Therefore, the strategy is best response to bidder Ih's. Furthermore, Ih and $P\emptyset$ shares the same support. We can also rule out any atom in the interior of the support, otherwise there is profitable deviation. We can also rule out $G_{P\emptyset}$ having atom at $\tilde{\Pi}$ because Ih would deviate. □

In equilibrium, Ih and $P\emptyset$ mixes over bids to make the opponent indifferent over all bids on the support. That is, G_{Ih} solves:

$$xG_{Ih}(b)(\Pi_{Ih} - b) + (1-x)(\Pi_{I0} - b) = 0$$

and $G_{P\emptyset}$ solves:

$$G_{P\emptyset}(b)(\Pi_{Ih} - b) = (\Pi_{Ih} - \tilde{\Pi}) = (1-x)(\Pi_{Ih} - \Pi_{I0}).$$

When ties are resolved with the toss of a fair coin, the winning probabilities are:

$$\begin{cases} \frac{1}{2}x + (1-x) & \text{for } Ih \\ \frac{1}{2}(1-x) & \text{for } I0 \\ \frac{1}{2} & \text{for } P\emptyset \end{cases}$$

Conditional on type realizations, Ih or $I0$, probability that bidder $P\emptyset$ wins is

$$Pr(P\emptyset \text{ wins}|Ih) = \frac{1}{2}x, \quad Pr(P\emptyset \text{ wins}|I0) = x + \frac{1}{2}(1-x)$$

That is, bidder $P\emptyset$'s expected payment is

$$\frac{1}{2}x^2\Pi_{Ih} + (x(1-x) + \frac{1}{2}(1-x)^2)\Pi_{I0}$$

Bidder Ih expects to pay

$$\frac{1}{2}x\Pi_{Ih} + (1-x)\Pi_{I0}$$

Bidder $I0$: $\frac{1}{2}(1-x)\Pi_{I0}$.

A.4 Proposition 2: Derivation of Equilibrium Conditions

Beliefs Time t stands for tenure in the firm. The differential equation is obtained using Bayes rule:

$$x(t+dt) = \frac{x(t)\big(1-\lambda dt + \lambda dt Pr(Ih \text{ wins}|t)\big) + (1-x(t))p(t)\alpha dt}{x(t)\big(1-\lambda dt + \lambda dt Pr(Ih \text{ wins}|t)\big) + (1-x(t))(1-\lambda dt + \lambda dt Pr(I0 \text{ wins}|t))}$$

where, given the bid distributions at time t,

$$Pr(Ih \text{ wins}|t) = \int G_{P\emptyset}(v|t)\, dG_{Ih}(v|t).$$

We assume that in case of a tie, the worker shifts to $P\emptyset$. In this case, $Pr(I0 \text{ wins}|t)$ is 0, and

$$x(t+dt)-x(t) = \frac{x(t)(1-x(t))\big(1-\lambda dt + \lambda dt Pr(Ih \text{ wins}|t)\big) + (1-x(t))\big(p(t)\alpha dt - x(t)(1-\lambda dt)\big)}{x(t)\big(1-\lambda dt + \lambda dt Pr(Ih \text{ wins}|t)\big) + (1-x(t))(1-\lambda dt)}$$

Dividing by dt and taking $dt \to 0$:

$$x'(t) = \underbrace{\lambda Pr(Ih \text{ wins}|t) x(t)(1-x(t))}_{(1)} + \underbrace{\alpha(1-x(t))p(t)}_{(2)}$$

Note that the differential rate at which I wins affects the drift of the belief x. Intuitively, when the $I0$ workers are leaving with probability 1, staying at this firm is a partial good news about the worker's type, which is reflected in the drift component in (1). The second component, (2), is additional breakthrough flowing from $1-x(t)$ to $x(t)$ pool.

Values It is informative to look at the recursion of the value equations in the discrete time:

$$\Pi_{Ih}(t) = Y\,\alpha dt + (1-rdt-\lambda dt)\Pi_{Ih}(t+dt) + \lambda dt \int (\Pi_{Ih}(t+dt)-v) G_{P\emptyset}(v|t+dt) dG_{Ih}(v|t+dt)$$

where the recursive expression $\Pi_{Ih}(t+dt)$ takes into account the the location of the future beliefs, which is taken as exogenous by the firm at the time of auction.

Since the highest bid $\bar{V}(t)$ wins the auction game for sure, substituting the indifference condition:

$$\Pi_{Ih}(t) = Y\,\alpha dt + (1-rdt-\lambda dt)\Pi_{Ih}(t+dt) + \lambda dt(\Pi_{Ih}(t+dt) - \bar{V}(t))$$

Subtracting both sides by $\Pi_{Ih}(t)$ and dividing by dt:

$$0 = \alpha Y + \Pi'_{Ih}(t) - (r+\lambda)\Pi_{Ih}(t) + \lambda(\Pi_{Ih}(t) - \bar{V}(t)). \tag{1h'}$$

Since this holds for any t, integrating it for $[s, \infty)$ after mutiplying by $e^{-(r+\lambda)t}$ yields

$$0 = \frac{\alpha Y}{r+\lambda} e^{-(r+\lambda)s} - e^{-(r+\lambda)t}\Pi_{Ih}(t) + \int_t^\infty e^{-(r+\lambda)s}(\Pi_{Ih}(s) - \bar{V}(s))\, ds.$$

Similarly, for Π_{I0}, the value it derives are from expected value of transition to state Π_{Ih},

$$\Pi_{I0}(t) = (1 - r\,dt - \lambda dt - \alpha p(t)dt)\Pi_{I0}(t+dt) + \alpha p(t)\,dt\Pi_{Ih}(t+dt),$$

$$0 = \Pi'_{I0}(t) - (r + \lambda + \alpha p(t))\Pi_{I0}(t) + \alpha p(t)\Pi_{Ih}(t) \tag{I0}$$

A.5 Section 3: Expected Flow Value $I0$

Normalize all the expressions so that the starting period t is 0. First note that Π_{I0} starting from initial belief $\tilde{p}(0)$, by a direct integration of $I0$, is given by:

$$\Pi_{I0}(0) = \int_0^\infty e^{-\int_0^s (r+\lambda+\alpha\tilde{p}(z))dz} \alpha\tilde{p}(s)\Big(Y + \Pi_{Ih}(s)\Big)\, ds,$$

where $\Pi_{Ih}(s)$ is the value after learning the type of the worker, evaluated over the Bayes rational belief path starting from $\tilde{p}(0)$. We note that Π_{I0} is the expected future value from transition to the learned state Π_{Ih} and the lump-sum payoff Y that arrives at rate α in case the worker is indeed $High$ quality.

Due to our good news learning assumption, the rate of transition at tenure τ, $\alpha\tilde{p}(\tau)$, and the evolution of $\tilde{p}(\tau)$ exactly cancels out in the integral:

$$\begin{aligned}
\alpha\tilde{p}(s)e^{-\int_0^s \alpha\tilde{p}(z)\,dz} &= \alpha\tilde{p}(s)e^{-\alpha s + \int_0^s \alpha(1-\tilde{p}(z))\,dz} \\
&= \alpha\tilde{p}(s)e^{-\alpha s - \int_0^s \frac{\tilde{p}'(z)}{\tilde{p}(z)}\,dz} \\
&= \alpha\tilde{p}(s)e^{-\alpha s}\frac{\tilde{p}(0)}{\tilde{p}(s)} = \alpha\tilde{p}(0)e^{-\alpha s}
\end{aligned}$$

Therefore, the expression can be simplified to:

$$\Pi_{I0}(0) = \alpha\tilde{p}(0)\int_0^\infty e^{-(r+\lambda+\alpha)s}\Big(Y + \Pi_{Ih}(s)\Big)\, ds.$$

Intuitively, Π_0 is the expected value of transition to Π_1 at rate α, with effective discount rate $\lambda + r$, coming from the dissolution of the match at rate λ. The good news drift allows us to replace the effect from change in transition rate over time, $\alpha\tilde{p}(s)$, with the initial probability $\tilde{p}(0)$ and constant rate of transition α.

A.6 Section 4.1: Case 1 Equilibrium Auction Outcomes

Using the indifference condition, $G_{P\emptyset}$ is given by:

$$G_{P\emptyset}(v) = \frac{\Pi_1 - \Pi_{I0}(x + (1-x)p)}{\Pi_1 - v} = \frac{(1-x)(1-p)\frac{\alpha Y}{r+\lambda}}{\frac{\alpha Y}{r+\lambda} - v}$$

with atom $1 - x$ at the lower bound, $p\frac{\alpha Y}{r+\lambda}$.

Again, using the indifference condition, G_{Ih} is given by:

$$G_{Ih}(v) = \frac{1-x}{x}\Big(\frac{1-p}{1 - v\frac{r+\lambda}{\alpha Y}} - 1\Big)$$

Since the value is a linear transformation of a belief, it is convenient to convert the v variables into corresponding beliefs $\tilde{p} = \frac{r+\lambda}{\alpha Y}v$. Define the bid distributions $\tilde{G}_{Ih}$ and $\tilde{G}_{P\emptyset}$ as

$$\tilde{G}_{Ih}(\tilde{p}) = G_{Ih}(\tilde{p}\frac{\alpha Y}{r+\lambda}) = \tilde{G}_{Ih}(\tilde{p}) = \frac{1-x}{x}\Big(\frac{1-p}{1-\tilde{p}} - 1\Big),$$

$$\tilde{G}_{P\emptyset}(\tilde{p}) = G_{P\emptyset}(\tilde{p}\frac{\alpha Y}{r+\lambda}) = \frac{(1-x)(1-p)}{(1-\tilde{p})}.$$

for $\tilde{p} \in [p, x + (1-x)p]$.

The probability that the Ih type wins over $P\emptyset$ is given by:

$$\int_p^{x+(1-x)p} \tilde{G}_{P\emptyset}(\tilde{p})\, d\tilde{G}_{Ih}(\tilde{p}) = \int_p^{x+(1-x)p} \frac{(1-x)(1-p)}{1-\tilde{p}} \frac{1-x}{x} \frac{(1-p)}{(1-\tilde{p})^2}\, d\tilde{p}$$

The antiderivative of $\frac{1}{(1-\tilde{p})^3}$ being $\frac{1}{2(1-\tilde{p})^2}$, the definite integral that is to be multiplied by $\frac{(1-x)^2(1-p)^2}{x}$ is

$$\frac{1}{2}\frac{1}{(1-\tilde{p})^2}\Big|_p^{x+(1-x)p} = \frac{1}{2}\Big(\frac{1}{(1-x)^2(1-p)^2} - \frac{1}{(1-p)^2}\Big) = \frac{1}{2}\Big(\frac{1-(1-x)^2}{(1-x)^2(1-p)^2}\Big) = \frac{1}{2}\frac{x(2-x)}{(1-x)^2(1-p)^2}.$$

In the end, the probability is $\frac{1}{2}(2-x) = 1 - \frac{1}{2}x$.

A.7 Solution for x in Case 2

Using

$$\begin{aligned}\int_0^t e^{-\lambda s}\alpha e^{-\alpha s}\, ds &= \frac{\alpha}{\alpha+\lambda}\int_0^t (\alpha+\lambda)e^{-(\alpha+\lambda)s}\, ds \\ &= \frac{\alpha}{\alpha+\lambda}\Big(1 - e^{-(\alpha+\lambda)t}\Big),\end{aligned}$$

Starting from initial belief p_0 and $x(0) = 0$, x is given by

$$\begin{aligned} x(t) &= \frac{p_0 \int_0^t e^{-(r+\lambda)s} d(1-e^{-\alpha s})}{p_0 \int_0^t e^{-(r+\lambda)s} \, d(1-e^{-\alpha s}) + p_0 e^{-(r+\alpha+\lambda)t} + (1-p_0)e^{-(r+\lambda)t}} \\ &= \frac{p_0 \frac{\alpha}{\alpha+\lambda}(1-e^{-(\alpha+\lambda)t})}{p_0 \frac{\alpha}{\alpha+\lambda}(1-e^{-(\alpha+\lambda)t}) + e^{-\lambda t}(p_0 e^{-\alpha t} + (1-p_0))} \end{aligned}$$

Intuitively, only the subset of good workers who received good news (with probability $1-e^{-\alpha s}$ within duration s) before the first poaching attempt, are the revealed good workers in this firm. It is easily seen in the expression $\frac{\alpha}{\alpha+\lambda}$ that the order of arrival of two independent Poisson processes, rates α and λ, matter for the numerator. The denominator represents the probability of survival in this firm for the duration of t, either by the arrival of news, or no poaching attempt ($e^{-\lambda t}$).

To verify that the solution to the ODE is indeed as above, we check our algebra by plugging in the expression for x into the differential equation:

$$x'(t) = \frac{p_0 \alpha e^{-(\alpha+\lambda)t}}{A(t)} - \frac{p_0 \frac{\alpha}{\alpha+\lambda}(1-e^{-(\alpha+\lambda)t})A'(t)}{A(t)^2},$$

where

$$A(t) = p_0 \frac{\alpha}{\alpha+\lambda}(1-e^{-(\alpha+\lambda)t}) + e^{-\lambda t}(p_0 e^{-\alpha t} + (1-p_0)),$$

and

$$\begin{aligned} A'(t) &= p_0 \alpha e^{-(\alpha+\lambda)t} - p_0(\alpha+\lambda)e^{-(\alpha+\lambda)t} - \lambda(1-p_0)e^{-\lambda t} \\ &= -\lambda p_0 e^{-(\alpha+\lambda)t} - \lambda(1-p_0)e^{-\lambda t}. \end{aligned}$$

Meanwhile,

$$1 - x(t) = \frac{p_0 e^{-(\alpha+\lambda)t} + (1-p_0)e^{-\lambda t}}{A(t)},$$

hence,

$$\begin{aligned} \frac{x'(t)}{1-x(t)} &= \frac{p_0 \alpha e^{-(\alpha+\lambda)t}}{p_0 e^{-(\alpha+\lambda)t} + (1-p_0)e^{-\lambda t}} - \frac{p_0 \frac{\alpha}{\alpha+\lambda}(1-e^{-(\alpha+\lambda)t})(-\lambda)}{A(t)} \\ &= \alpha p(t) + \lambda x(t) \end{aligned}$$

A.8 Proof of Proposition 7

Define

$$D(t) = \Pi_{Ih}(t) - \Pi_{I0}(t).$$

Subtract the equations in the system to yield the following ODE in terms of D only:

$$-D'(t) + (r+\lambda)D(t) = \alpha(1-p(t))Y + (\lambda(1-x(t)) - \alpha p(t))D(t)$$

$$-D'(t) + (r + \lambda x(t) + \alpha p(t))D(t) = \alpha(1-p(t))Y$$

Use the drift equation to substitute $\lambda x(t) + \alpha p(t)$ with $\frac{x'(t)}{1-x(t)}$:

$$-D'(t) + (r + \frac{x'(t)}{1-x(t)})D(t) = \alpha(1-p(t))Y$$

Since $\frac{x'(t)}{1-x(t)} = \frac{d}{dt}(-\log(1-x(t)))$, we have

$$\exp(-rt + \log(1-x(t))) = \exp(-rt)(1-x(t))$$

multiplying by this number,

$$-e^{-rt}(1-x(t))D'(t) + (re^{-rt}(1-x(t)) + e^{-rt}x'(t))D(t) = \alpha e^{-rt}(1-x(t))(1-p(t))Y$$

which can be rearranged by

$$-\frac{d}{dt}e^{-rt}(1-x(t))D(t) = e^{-rt}(1-x(t))(1-p(t))\alpha Y$$

Integrating from t to infinity, noting that D and x are bounded:

$$e^{-rt}(1-x(t))D(t) = \int_t^\infty e^{-rs}(1-x(s))(1-p(s))\alpha Y\, ds$$

Plugging this information into (Ih'):

$$-\Pi'_{Ih}(t) + (r+\lambda)\Pi_{Ih}(t) = \alpha Y + \lambda(1-x(t))(\Pi_{Ih}(t) - \Pi_{I0}(t)) \tag{Ih'}$$

$$-e^{-(r+\lambda)t}\Pi'_{Ih}(t) + (r+\lambda)e^{-(r+\lambda)t}\Pi_{Ih}(t) = \alpha Y e^{-(r+\lambda)t} + \lambda e^{-(r+\lambda)t}(1-x(t))(\Pi_{Ih}(t) - \Pi_{I0}(t))$$

$$\begin{aligned}\frac{d}{dt} - e^{-(r+\lambda)t}\Pi_{Ih}(t) &= \alpha Y e^{-(r+\lambda)t} + \lambda e^{-\lambda t}e^{-rt}(1-x(t))D(t) \\ &= \alpha Y e^{-(r+\lambda)t} + \lambda e^{-\lambda t}\int_t^\infty e^{-rs}(1-x(s))(1-p(s))\alpha Y\, ds\end{aligned}$$

Integrating over $[t, \infty)$:

$$e^{-(r+\lambda)t}\Pi_{Ih}(t) = \frac{\alpha Y}{r+\lambda}e^{-(r+\lambda)t} + \lambda\int_t^\infty e^{-\lambda s}\int_s^\infty e^{-rz}(1-x(z))(1-p(z))\alpha Y\, dz\, ds$$

Changing the order of integration

$$e^{-(r+\lambda)t}\Pi_{Ih}(t) = \frac{\alpha Y}{r+\lambda}e^{-(r+\lambda)t} + \int_t^\infty \big(e^{-\lambda t} - e^{-\lambda z}\big)e^{-rz}(1-x(z))(1-p(z))\alpha Y\, dz$$

Multiplying by $e^{(r+\lambda)t}$

$$\Pi_{Ih}(t) = \frac{\alpha Y}{r+\lambda} + \int_t^\infty \big(e^{-r(z-t)} - e^{-(r+\lambda)(z-t)}\big)(1-x(z))(1-p(z))\alpha Y\, dz$$

Use this to back out the $x(t)\Pi_{Ih}(t) + (1-x(t))\Pi_{I0}(t) = \Pi_{Ih}(t) - (1-x(t))D(t)$:

$$\begin{aligned}
&\Pi_{Ih}(t) - (1-x(t))D(t)\\
&= \frac{\alpha Y}{r+\lambda} + \int_t^\infty \big(1-e^{-\lambda(z-t)}\big)e^{-r(z-t)}(1-x(z))(1-p(z))\alpha Y\, dz - \int_t^\infty e^{-r(s-t)}(1-x(s))(1-p(s))\alpha Y\, ds\\
&= \frac{\alpha Y}{r+\lambda} - \int_t^\infty e^{-(r+\lambda)(z-t)}(1-x(z))(1-p(z))\alpha Y\, dz\\
&= \alpha Y \int_t^\infty e^{-(r+\lambda)(z-t)}\Big(x(z) + (1-x(z))p(z)\Big)\, dz
\end{aligned}$$

A.9 Proposition 8: Properties of Value Functions

Expression for $\tilde{x}$ Denote by $C(t)$, the maximum cost in order for the incumbent to retain the worker:

$$C(t) = \alpha Y \int_t^\infty e^{-(r+\lambda)(s-t)}\big(x(s) + (1-x(s))p(s)\big)\, ds$$

Closed form for the $x(s) + (1-x(s))p(s) := \tilde{x}(s)$:

$$\tilde{x}(t) = \frac{p_0\frac{\alpha}{\alpha+\lambda}\Big(1-e^{-(\alpha+\lambda)t}\Big) + p_0 e^{-(\alpha+\lambda)t}}{p_0\frac{\alpha}{\alpha+\lambda}\Big(1-e^{-(\alpha+\lambda)t}\Big) + e^{-\lambda t}\Big(p_0e^{-\alpha t} + (1-p_0)\Big)}$$

normalizing by p_0:

$$\begin{aligned}
\tilde{x}(t) &= \frac{\frac{\alpha}{\alpha+\lambda}\Big(1-e^{-(\alpha+\lambda)t}\Big) + e^{-(\alpha+\lambda)t}}{\frac{\alpha}{\alpha+\lambda}\Big(1-e^{-(\alpha+\lambda)t}\Big) + e^{-\lambda t}\Big(e^{-\alpha t} + \frac{1-p_0}{p_0}\Big)}\\
&= \frac{\frac{\alpha}{\alpha+\lambda} + \frac{\lambda}{\alpha+\lambda}e^{-(\alpha+\lambda)t}}{\frac{\alpha}{\alpha+\lambda} + \frac{\lambda}{\alpha+\lambda}e^{-(\alpha+\lambda)t} + \frac{1-p_0}{p_0}e^{-\lambda t}}
\end{aligned}$$

Lemma 1.

$$\frac{\partial \tilde{x}}{\partial \lambda} > 0.$$

Proof. The result follows from the property of x. Suppose there are two paths of x's, which I denote as x_1 and x_2 for parameters $\lambda_1 < \lambda_2$, starting from the same p_0. For t close to 0,

$x_2(t) > x_1(t)$ and the inequality should not flip because if so, there is an intersection and at the intersection, $x = x_2 = x_1$, we have $x_2' > x_1'$.

Therefore, an increase in λ increases x and $\tilde{x} = x + (1-x)p$. $\square$

Lemma 2.

$$\frac{\partial \tilde{x}}{\partial \alpha} > 0.$$

Proof. From the last expression for $\tilde{x}$, focus on the term

$$\frac{\alpha}{\alpha+\lambda} + \frac{\lambda}{\alpha+\lambda} e^{-(\alpha+\lambda)t} := A$$

which I denote A. Its derivative with respect to α is given by:

$$(-1)\frac{\lambda e^{-(\alpha+\lambda)t}\big((\alpha+\lambda)t - e^{(\alpha+\lambda)t} + 1\big)}{(\alpha+\lambda)^2}$$

It is immediately shown that $1 + (\alpha+\lambda)t - e^{-(\alpha+\lambda)t} < 0$ and the result follows. $\square$

Expression for Π_{Ih} From

$$\Pi_{Ih}(t) = \frac{\alpha Y}{r} - \lambda \int_t^\infty e^{-r(z-t)} C(z)\, dz,$$

we have

$$\begin{aligned}
\Pi_{Ih}(t) &= \frac{\alpha Y}{r} - \lambda(\alpha Y) \int_t^\infty e^{-r(z-t)} \int_z^\infty e^{-r(s-z)} \tilde{x}(s)\, ds\, dz \\
&= \frac{\alpha Y}{r} - \lambda(\alpha Y) \int_t^\infty \int_t^s e^{-r(z-t)} e^{-r(s-z)} e^{-\lambda(s-z)} \tilde{x}(s)\, dz\, ds \\
&= \frac{\alpha Y}{r} - \lambda(\alpha Y) \int_t^\infty e^{-r(s-t)} \tilde{x}(s) \int_t^s e^{-\lambda(s-z)}\, dz\, ds \\
&= \frac{\alpha Y}{r} - \lambda(\alpha Y) \int_t^\infty e^{-r(s-t)} \tilde{x}(s) \Big(\frac{1}{\lambda}(1 - e^{-\lambda(s-t)})\Big)\, ds \\
&= \alpha Y \int_t^\infty e^{-r(s-t)} \Big(1 - \tilde{x}(s)(1 - e^{-\lambda(s-t)})\Big)\, ds
\end{aligned}$$

Last line by changing the order of integration. From the expression, it is immediate that

Proposition 11. *Suppose αY is a constant. Then, an increase in α decreases $\Pi_{Ih}(t)$.*

Proposition 12. *An increase in λ increases $\tilde{x}$ and $(1 - e^{-\lambda s})$. Overall, decreases the term.*

Expression for Π_{I0} Let's turn to Π_{I0}. Starting from p_0, the value is

$$\alpha p_0 \int_0^\infty e^{-(\alpha+\lambda+r)t} \Big[Y + \Pi_{Ih}(t)\Big]\, dt$$

Expanding

$$\underbrace{\alpha p_0 \frac{Y}{\alpha+\lambda+r}}_{(1)} + \underbrace{\alpha p_0 \int_0^\infty e^{-(\alpha+\lambda+r)t} \int_t^\infty e^{-r(s-t)} \Big(1 - \tilde{x}(s)(1 - e^{-\lambda(s-t)})\Big)\, ds\, dt\, (\alpha Y)}_{(2)}$$

Let $r = 0$ and $\alpha Y = 1$ for simplicity, and focus on the second term, (2):

$$\alpha p_0 \int_0^\infty e^{-(\alpha+\lambda)t} \int_t^\infty \Big(1 - \tilde{x}(s)(1 - e^{-\lambda(s-t)})\Big)\, ds\, dt$$

Changing the order of integration:

$$\begin{aligned}
&\alpha p_0 \int_0^\infty \int_0^s e^{-(\alpha+\lambda)t} \Big(1 - \tilde{x}(s)(1 - e^{-\lambda(s-t)})\Big)\, dt\, ds \\
&= \alpha p_0 \int_0^\infty \int_0^s e^{-(\alpha+\lambda)t}(1 - \tilde{x}(s))\, dt\, dt + \alpha p_0 \int_0^\infty \int_0^s e^{-\alpha t} \tilde{x}(s) e^{-\lambda s}\, dt\, ds
\end{aligned}$$

The first part:

$$\alpha p_0 \int_0^\infty \frac{1}{\alpha+\lambda} \Big(1 - e^{-(\alpha+\lambda)s}\Big)(1 - \tilde{x}(s))\, ds$$

The second part:

$$\alpha p_0 \int_0^\infty \int_0^s e^{-\alpha t} \tilde{x}(s) e^{-\lambda s}\, dt\, ds = \alpha p_0 \int_0^\infty \frac{1}{\alpha}(1 - e^{-\alpha s}) \tilde{x}(s) e^{-\lambda s}\, ds$$

Overall,

$$\underbrace{\frac{p_0}{\alpha+\lambda}}_{(1)} + \underbrace{p_0 \int_0^\infty \frac{\alpha}{\alpha+\lambda} \Big(1 - e^{-(\alpha+\lambda)s}\Big)(1 - \tilde{x}(s))\, ds + p_0 \int_0^\infty (1 - e^{-\alpha s}) \tilde{x}(s) e^{-\lambda s}\, ds}_{(2)}$$

Expanding (2):

$$p_0 \int_0^\infty \frac{\alpha}{\alpha+\lambda}(1 - e^{-(\alpha+\lambda)s}) - \tilde{x}(s) \frac{\alpha}{\alpha+\lambda}(1 - e^{-(\alpha+\lambda)s}) - \tilde{x}(s) e^{-(\alpha+\lambda)s} + \tilde{x}(s) e^{-\lambda s}\, ds$$

$$p_0 \int_0^\infty \frac{\alpha}{\alpha+\lambda}(1 - e^{-(\alpha+\lambda)s}) - \tilde{x}(s) \frac{\alpha}{\alpha+\lambda} - \frac{\lambda}{\alpha+\lambda} \tilde{x}(s) e^{-(\alpha+\lambda)s} + \tilde{x}(s) e^{-\lambda s}\, ds$$

$$p_0 \int_0^\infty \frac{\alpha}{\alpha+\lambda}(1 - \tilde{x}(s)) + \frac{\lambda}{\alpha+\lambda} e^{-(\alpha+\lambda)s}(1 - \tilde{x}(s)) - \underbrace{e^{-(\alpha+\lambda)s}}_{(*)} + \tilde{x}(s) e^{-\lambda s}\, ds$$

The underbraced term $(*)$ cancels out with (1). Therefore, the expression is:

$$p_0 \int_0^\infty \Big(\frac{\alpha}{\alpha+\lambda} + \frac{\lambda}{\alpha+\lambda} e^{-(\alpha+\lambda)s}\Big)(1 - \tilde{x}(s)) + \tilde{x}(s) e^{-\lambda s}\, ds$$

To see how this increases with α, let

$$A(\alpha) = \frac{\alpha}{\alpha+\lambda} + \frac{\lambda}{\alpha+\lambda} e^{-(\alpha+\lambda)s}$$

and let

$$B = \frac{1-p_0}{p_0} e^{-\lambda s}.$$

It follows immediately that

$$\tilde{x}(s) = \frac{A}{A+B}.$$

Rewrite the above expression in terms of A and B:

$$p_0 \int_0^\infty A\Big(\frac{B}{A+B}\Big) + \frac{A}{A+B}\frac{p_0}{1-p_0} B\, ds$$

$$p_0 \int_0^\infty \frac{A}{A+B}\Big(\frac{1}{1-p_0}\Big) B\, ds$$

Overall,

$$\int_0^\infty \frac{A}{A+B} e^{-\lambda s}\, ds$$

Note that p_0 affects the expression only through B. Increase in p_0 decreases B, and increases the expression. Therefore, I show increment in α and p_0.