Preference Structures*

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August 8, 2018

Abstract

We suggest the use of two binary relations to describe the preferences of an agent. The first of these, $\succsim$, aims to capture the rankings of the agent that are (subjectively) "obvious/easy." As such, it is transitive, but not necessarily complete. The second one, $\mathbf{R}$, arises from what we observe the agent choose in the context of pairwise choice problems. As such, it is assumed to be complete, but not necessarily transitive. Finally, we posit that $\succsim$ and $\mathbf{R}$ are consistent in the sense that (i) $\mathbf{R}$ is an extension of $\succsim$, and (ii) $\mathbf{R}$ is transitive with respect to $\succsim$. This yields what we call a *preference structure*. It is shown that this model allows for phenomena like rational choice, indeciveness, imperfect ability of discrimination, regret, and advise taking, among others. We show how one may represent preference structures by using (sets of) utility functions, but the bulk of the paper is about choice behavior that arises from preference structures which we model by using the notion of top cycles. It is shown that this leads to a rich theory of choice with a large explanatory power, and still with a surprising amount of predictive power. Under general conditions, we prove that choice correspondences that are induced by preference structures are nonempty-valued, and then identify the largest preference structure that rationalizes a choice correspondence that is known to be rationalizable by some such structure. Thus, this choice theory, while much more general, possess existence and uniqueness properties that parallel those of the classical theory of rational choice.

JEL Classification: D11, D81.

Keywords: Preference structures, Incomplete and nontransitive preferences, multi-utility representation, boundedly rational choice, revealed preference.

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*We thank Alfio Giarlotta and Pietro Ortoleva for their very helpful comments on an earlier version of this paper.

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1 Introduction

The classical approach of describing the preferences of an economic agent on a given set X of choice prospects is to use a binary relation on X. If, according to this relation, an alternative x is ranked (weakly) higher than another alternative y, we understand that the agent (weakly) prefers having x to having y. To get some mileage from the model, one typically imposes some properties on this binary relation, often corresponding to some form of rationality on the part of the agent. In the most standard scenario, of course, we posit that this relation is complete and transitive, and then describe the items that the agent finds choosable from a given feasible menu as those that maximize this relation. Conversely, if the choice behavior of the agent conforms with some standard postulates of rationality, then the choices of the agent over pairwise choice situations *reveal* the preference relation whose maximization rationalizes the choices of the agent over all feasible menus.

This model is not only elegant, but it also possesses an admirable degree of predictive power. However, it is well known that its explanatory power is unduly limited. The experimental demonstrations of nontransitivity of revealed preferences of individuals, for instance, go back to Tversky (1969), which is to cite but one reference from a rather large literature (cf. Loomes and Day (2010)). Besides, numerous explanations and models that accommodate nontransitivity of preferences are offered in the literature, including regret theory (Loomes and Sugden (1982)), nontransitive indifference and similarity (Luce (1956), Fishburn (1970), Beja and Gilboa (1992), and Rubinstein (1988)), and framing effects (Kahneman and Tversky (1979) and Salant and Rubinstein (2008)). Similarly, if we wish to model the occasional indecisiveness of an agent, then completeness hypothesis has to be dropped (as in, say, models of multi-criteria decision making and/or Knightian uncertainty). Moreover, if the economic agent under consideration is, in fact, a group of individuals (such as a board of directors or a family), then positing completeness and transitivity at the outset is not at all warranted. After all the two most standard binary relations that are relevant in this case are the Pareto ordering (which is incomplete) and the majority voting rule (which is nontransitive).

One of the main reasons why the classical model of preferences lacks in explanatory power is that this approach tacitly views that all pairwise choice problems are evaluated by the agent in the same way. However, it is simply unrealistic to presume that every choice problem is equally revealing. Depending on the context, some choices may be "easy," even "trivial," for an agent, while others may be "hard" enough that she may feel justifiably insecure about them. (For instance, most agents would choose the sure lottery that pays them \$10 over that pays them \$5 "easily," while they may find ranking two complicated lotteries with large supports "difficult.") It is only reasonable that such (subjectively) "hard" choice problems may cause the choices of the agent fail the strict requirements of rationality. This, in turn, results in the preference relation revealed through such choices, which is necessarily complete, fail the property of transitivity. By contrast, the choices across "easy" pairwise choice problems (whichever these may be for the agent) would presumably abide by transitivity. But, since some pairwise choice problems may be deemed "difficult" for the agent, the "easy" ones may yield only a part of her overall preferences, thereby failing the property of completeness.

Even though this description is very simple, and seemingly realistic, it cannot be captured by the classical approach of modeling one's preferences by means of a binary relation. In this paper, we instead suggest using an alternative approach that involve two binary relations on X. The first of these, denoted as $\succsim$, aims to capture those rankings of the agent that are (subjectively) "obvious/easy." As it is likely that cyclical choice patterns would not arise from the "easy" pairwise choice problems, we assume that $\succsim$ is reflexive and transitive, but not necessarily complete. Intuitively, and we will formalize this later in the paper, this relation is a dominance relation; when $x \succsim y$, we understand that the agent is entirely confident in choosing x over y. The second binary relation, denoted as $\mathbf{R}$, arises from what we observe the agent choose in the context of all pairwise choice problems. (Thus $\succsim$ is not observable, but $\mathbf{R}$ is.) As it is generated also by "hard" choice problems, this relation may lead to cyclic choice patterns, so it is allowed to be nontransitive, while, naturally, we assume that it is complete. Put precisely, $\mathbf{R}$ is the preference relation revealed through the pairwise choice problems; when $x \mathbf{R} y$, it is understood that an outside observer has seen the agent choose x over y at some observation point.

As $\succsim$ and $\mathbf{R}$ are meant to describe the preferences of the same economic agent, they must be consistent with each other. To ensure this, we assume at the outset that both the weak and strict parts of $\mathbf{R}$ extend those of $\succsim$, respectively. That is, if the agent is certain that x and y are perfect substitutes for her – this is captured by the core relation $\succsim$ declaring x and y indifferent – then the revealed preference $\mathbf{R}$ maintains that x and y are indeed indifferent. Similarly, if the agent thinks x is "obviously" strictly better than y – this is captured by $\succsim$ ranking x strictly above y – then $\mathbf{R}$ reveals indeed that this is the case. In order-theoretic terms, we thus require $\mathbf{R}$ be an *extension* of $\succsim$.

In fact, our interpretation suggests that the connection between $\succsim$ and $\mathbf{R}$ should be even tighter than this. To wit, suppose the agent declares that $x \mathbf{R} y$ and $y \succsim z$ for some alternatives x, y and z. According to our interpretation, this says that the agent declares x superior to y (although she may not be completely confident in this judgement) while she is sure that y is better for her than z. It then seems reasonable that the agent would prefer x over z, albeit, she may be insecure about this decision (that is, $x \mathbf{R} z$ holds, but not necessarily $x \succsim z$). As the analogous reasoning applies also to the case where $x \succsim y$ and $y \mathbf{R} z$, it makes sense to require $\mathbf{R}$ be *transitive with respect to* $\succsim$, which means

$$x \mathbf{R} y \succsim z \quad \text{or} \quad x \succsim y \mathbf{R} z \quad \text{implies} \quad x \mathbf{R} z$$

for every x, y and z in X.

This property completes the model we propose to describe the preferences of an (individual or social) economic agent. We call this model a *preference structure* on X. Put precisely, a preference structure on X is a pair of binary relations $(\succsim, \mathbf{R})$ on X such that (i) $\succsim$ is reflexive and transitive, (ii) $\mathbf{R}$ is complete, (iii) $\mathbf{R}$ is an extension of $\succsim$ (in terms of both indifference and strict preference) and (iv) $\mathbf{R}$ is transitive with respect to $\succsim$. Evidently, this model reduces to the classical one when $\succsim = \mathbf{R}$, but in general, it is much richer than that model. Indeed, the primary objective of the present paper is to explore, if as a first investigation, what the model of preference structures is capable of.[1]

After introducing our basic nomenclature (Section 2), we begin our analysis by demonstrating that quite a number of preference models are captured by preference structures. In addition to the classical rational choice model, among these are the models of incomplete preferences, preferences with imperfect ability of discrimination, regret preferences, and preferences completed by the recommendations of a consultant (Sections 3.2-3). We then prove that for any preference structure $(\succsim, \mathbf{R})$, there is a set of preorders such that $\succsim$ is realized as the intersection of this set and $\mathbf{R}$ as the union of it (Section 3.4). Thus, the first component of any preference structure is a unanimity ordering – this sits well with our interpretation of $\succsim$ as being a dominance relation – while the second component of it is a rationalizable preference in the sense of Cherepanov et al. (2013). In turn, this fact allows us to draw from the very recent literature on the utility representation of nonstandard preferences to obtain a multi-utility representation for (continuous) preference structures (Section 3.5). This provides a convenient method of defining (continuous) preference structures in exactly the same way one often introduces standard preferences in economic models through utility functions.

The main parts of the paper are its Sections 4 and 5. In Section 4, we consider how one may think of an agent making her choices on the basis of her preference structure. That is, we define the set $C(S)$ of all possible choices of an economic agent from a given feasible menu S by using a preference structure. In the classical case, this is done by setting $C(S)$ as the set of all maximum elements of S with respect to the preference relation of the agent. The situation is less clear cut in the context of an arbitrarily given preference structure $(\succsim, \mathbf{R})$. What readily follows from our interpretation of this structure is that the agent would never choose an alternative x from S if there is another alternative in S that strictly dominates x in terms of the "sure" ordering $\succsim$. Thus, $C(S)$ must be contained in $\mathbf{MAX}(S, \succsim)$, the set of all maximal elements in S with respect to $\succsim$. We then posit that the

[1]Describing preferences through two binary relations instead of one is not new. Especially in the literature on decision making under uncertainty, this method is adopted by a number of recent papers. There are also two papers, namely, Giarlotta and Greco (2013) and Giarlotta and Watson (2018), that adopt closely related models in the context of our general (ordinal) setup. The difference between the model of preference structures and those of these two papers are explained at the end of Section 3.1.

"choosable" alternatives in S should "maximize" $\mathbf{R}$ on $\mathbf{MAX}(S, \succsim)$. Unfortunately, as $\mathbf{R}$ need not be transitive, there is no *a priori* reason for the existence of such maxima, even when S contains only three alternatives. This is, of course, a familiar problem to social choice theory, which has thus developed several alternative notions of optima to confront this problem, such as the top-cycle solution, the uncovered set, the Banks set, etc.. We adopt the first of these here, and set $C(S)$ as the top-cycle in $\mathbf{MAX}(S, \succsim)$ with respect to $\mathbf{R}$.[2] This generalizes the rational choice paradigm (because, when $\succsim = \mathbf{R}$, this specification makes $C(S)$ the set of all maxima relative to $\succsim$). Moreover, we find in Sections 4.2-3 that this set is nonempty under some standard (topological) conditions on S and $(\succsim, \mathbf{R})$, and the resulting notion of choice correspondence captures many interesting choice frameworks, among which are the models of rational choice with incomplete preferences, some satisficing models such as choice with constant thresholds, and certain types of sequentially rational choice procedures. Thus, the explanatory power of the alternative choice theory that we examine here is quite superior to the classical theory. And yet, in Section 4.4 we demonstrate that this choice model retains a good part of the predictive power the classical rational choice model, for it satisfies certain rationality postulates such as the Condorcet Criterion and the Aizerman Choice Axiom, along with several monotonicity properties.

We do not attempt to provide a behavioral characterization of our choice model here.[3] Instead, in Section 5, we examine the issue of *identification* of a preference structure from choice behavior, under the hypothesis that that choice behavior arises from a preference structure in the way we have just specified. In the context of rational choice, there is a clear answer to this, for a rational choice correspondence (on a suitable domain) can be rationalized by one, and only one, complete preference relation. Unsurprisingly, the generality of our model does not permit such a strong uniqueness result (as in the case of most boundedly rational choice models). However, under very general conditions, we were able to give a complete characterization of the *largest* preference structure that rationalizes a choice correspondence that is known to be rationalizable by some such structure. Thus, from any choice correspondence rationalizable by a preference structure we can elicit the preference structure (i) which rationalizes that correspondence; and (ii) whose "sure" preferences exhibit the least amount of indecisiveness compatible with that choice correspondence.

As preliminary as many of our results are, they do seem to provide support for entertaining the idea of replacing the classical approach to modeling preferences with that of preference structures. This model does indeed appear amenable to economic analysis – but, needless to say, one needs to see how it would perform in the concrete contexts of consumer theory, equilibrium analysis, game theory, etc. – while capturing

[2] More precisely, $C(S)$ is the smallest subset of $\mathbf{MAX}(S, \succsim)$ such that every element in this set is ranked strictly higher than every $\succsim$-maximal element outside this set with respect to $\mathbf{R}$ (Sections 4.1-2).

[3] This task is undertaken in a separate paper by Evren, Nishimura and Ok (2018).

significantly more behavioral phenomena than the standard model. At any rate, we conclude the paper with some concluding remarks on this issue and an Appendix that contains the proofs of the results that are omitted in the body of the text.

2 Nomenclature

As we deal with somewhat nonstandard preference relations in this paper, which need not be either complete or transitive, we introduce here some terminology that pertains to the general theory of binary relations. We will use this nomenclature throughout the paper.

Binary Relations. Let X be a nonempty set. By a **binary relation** on X, we mean any nonempty subset of $X \times X$. But, for any binary relation $\mathbf{R}$ on X, we often adopt the usual convention of writing $x \mathbf{R} y$ instead of $(x, y) \in \mathbf{R}$. For any nonempty $Y \subseteq X$, by $x \mathbf{R} Y$, we mean $x \mathbf{R} y$ for every $y \in Y$. Moreover, for any binary relations $\mathbf{R}$ and $\mathbf{S}$ on X, we simply write $x \mathbf{R} y \mathbf{S} z$ to mean $x \mathbf{R} y$ and $y \mathbf{S} z$, and so on. For any nonempty subset S of X, the **restriction** of $\mathbf{R}$ to S is defined as the binary relation on S given by

$$\mathbf{R}|_S := \mathbf{R} \cap (S \times S).$$

For any element x of X, the **upper set** of x with respect to $\mathbf{R}$ is defined as $x^{\uparrow,\mathbf{R}} := \{y \in X : y \mathbf{R} x\}$, and the **lower set** of x with respect to $\mathbf{R}$ is $x^{\downarrow,\mathbf{R}} := \{y \in X : x \mathbf{R} y\}$. When either $x \mathbf{R} y$ or $y \mathbf{R} x$, we say that x and y are **R-comparable**, and put

$$\text{Inc}(\mathbf{R}) := \{(x, y) \in X \times X : x \text{ and } y \text{ are not } \mathbf{R}\text{-comparable}\}.$$

This set is symmetric, that is, $(x, y) \in \text{Inc}(\mathbf{R})$ iff $(y, x) \in \text{Inc}(\mathbf{R})$. If $\text{Inc}(\mathbf{R}) = \varnothing$, that is, every x and y in X are $\mathbf{R}$-comparable, we say that $\mathbf{R}$ is **complete** (or **total**).

The **asymmetric** (or **strict**) **part** of a binary relation $\mathbf{R}$ on X is defined as the binary relation $\mathbf{R}^>$ on X with $x \mathbf{R}^> y$ iff $x \mathbf{R} y$ and not $y \mathbf{R} x$, and the **symmetric part** of $\mathbf{R}$ is defined as $\mathbf{R}^= := \mathbf{R} \backslash \mathbf{R}^>$. The **composition** of two binary relations $\mathbf{R}$ and $\mathbf{S}$ on X is defined as $\mathbf{R} \circ \mathbf{S} := \{(x, y) \in X \times X : x \mathbf{R} z \mathbf{S} y \text{ for some } z \in X\}$. We say that $\mathbf{S}$ is a **subrelation** of $\mathbf{R}$, and that $\mathbf{R}$ is a **superrelation** of $\mathbf{S}$, if $\mathbf{S} \subseteq \mathbf{R}$.

We denote the diagonal of $X \times X$ by $\triangle_X$, that is, $\triangle_X = \{(x, x) : x \in X\}$. A binary relation $\mathbf{R}$ on X is said to be **reflexive** if $\triangle_X \subseteq \mathbf{R}$, **antisymmetric** if $\mathbf{R}^= \subseteq \triangle_X$, and **transitive** if $\mathbf{R} \circ \mathbf{R} \subseteq \mathbf{R}$. If $\mathbf{R}$ is reflexive and transitive, we refer to it as a **preorder** on X. (Throughout the paper, generic preorders are denoted as $\succsim$ or $\trianglerighteq$, and the asymmetric parts of $\succsim$ and $\trianglerighteq$ are denoted as $\succ$ and $\triangleright$, respectively.) Finally, an antisymmetric preorder on X is said to be a **partial order** on X. If X is endowed with a prespecified partial order, we may refer to it as a **poset**.

On occasion, we will work with weaker properties than transitititivity. In particular, $\mathbf{R}$ is said to be **quasitransitive** if $\mathbf{R}^>$ is transitive, and **acyclic** if there do not exist a positive integer k and $x_1, ..., x_k \in X$ such that $x_1 \mathbf{R}^> \cdots \mathbf{R}^> x_k \mathbf{R}^> x_1$. (Transitivity implies quasitransitivity, and quasitransitivity implies acyclicity. The converses of these implications are false.)

The **transitive closure** of a binary relation $\mathbf{R}$ on X is the smallest transitive superrelation of $\mathbf{R}$; we denote this relation by $\mathrm{tran}(\mathbf{R})$. This relation always exists; we have $x\ \mathrm{tran}(\mathbf{R})\ y$ iff there exist a $k \in \mathbb{Z}_+$ and $x_0, ..., x_k \in X$ such that $x = x_0\ \mathbf{R}\ x_1\ \mathbf{R} \cdots \mathbf{R}\ x_k = y$. Obviously, $\mathrm{tran}(\mathbf{R})$ is a preorder on X, provided that $\mathbf{R}$ is reflexive.

In general, the asymmetric part of a binary relation $\mathbf{R}$ is not preserved by the asymmetric part of the transitive closure of $\mathbf{R}$, and conversely. But when $\mathbf{R}$ is complete, $x\ \mathrm{tran}(\mathbf{R})^{>}\ y$ implies $x\ \mathbf{R}^{>}\ y$.

Lemma 2.1. *For any complete binary relation* $\mathbf{R}$ *on a nonempty set* X, *we have* $\mathrm{tran}(\mathbf{R})^{>} \subseteq \mathbf{R}^{>}$.

Proof. Take any $x, y \in X$ such that $x\ \mathbf{R}^{>}\ y$ is false. Then, as $\mathbf{R}$ is complete, $y\ \mathbf{R}\ x$, and hence, $y\ \mathrm{tran}(\mathbf{R})\ x$, so $x\ \mathrm{tran}(\mathbf{R})^{>}\ y$ is false. ■

Extension of Binary Relations. Let $\mathbf{R}$ be a binary relation on X. If $\mathbf{S}$ and $\mathbf{S}^{>}$ are subrelations of $\mathbf{R}$ and $\mathbf{R}^{>}$, respectively, we say that $\mathbf{R}$ is an **extension** of $\mathbf{S}$ (or that $\mathbf{R}$ *extends* $\mathbf{S}$). If $\mathbf{R}$ extends $\mathbf{S}$ and it is total, we refer to it as a **completion** of $\mathbf{S}$.[4]

We say that $\mathbf{R}$ is **Suzumura consistent** if $x\ \mathrm{tran}(\mathbf{R})\ y$ implies not $y\ \mathbf{R}^{>}\ x$ for every $x, y \in X$ (which is the same thing as saying that $\mathrm{tran}(\mathbf{R})$ is an extension of $\mathbf{R}$). This property implies acyclicity, and for complete relations, coincides with transitivity. Moreover, it is known that $\mathbf{R}$ can be extended to a total preorder on X iff it is Suzumura consistent (cf. Suzumura (1976)). In particular, every preorder can be extended to a complete preorder; this is known as *Szpilrajn's Theorem.*

Transitivity with Respect to another Binary Relation. Our main focus in this paper is on reflexive, but not necessarily transitive, binary relations. A useful concept in the analysis of such binary relations is the notion of *transitivity with respect to a binary relation.* Put precisely, given any two binary relations $\mathbf{R}$ and $\mathbf{S}$ on a nonempty set X, we say that $\mathbf{R}$ is **$\mathbf{S}$-transitive** if $\mathbf{R} \circ \mathbf{S} \subseteq \mathbf{R}$ and $\mathbf{S} \circ \mathbf{R} \subseteq \mathbf{R}$, which means that either $x\ \mathbf{R}\ y\ \mathbf{S}\ z$ or $x\ \mathbf{S}\ y\ \mathbf{R}\ z$ implies $x\ \mathbf{R}\ z$ for any $x, y, z \in X$. This notion generalizes the classical concept of transitivity, for, obviously, $\mathbf{R}$ is $\mathbf{R}$-transitive iff it is transitive.

The Transitive Core. Let $\mathbf{R}$ be a reflexive binary relation on a nonempty set X. By the **transitive core** of $\mathbf{R}$, we mean the largest subrelation $\mathbf{S}$ of $\mathbf{R}$ such that $\mathbf{R}$ is $\mathbf{S}$-transitive, and denote this subrelation as $\mathsf{T}(\mathbf{R})$. It is plain that $\mathbf{R}$ is transitive iff

[4]Some authors say that $\mathbf{R}$ is an "extension" of $\mathbf{S}$ when $\mathbf{R} \subseteq \mathbf{S}$. While mathematically more convenient, that definition makes the dependence of the extended relation $\mathbf{S}$ on the original relation a very weak one. After all, according to that definition, $X \times X$ is a completion of *any* binary relation on X. (For instance, in the context of choice theory, that definition of extension maintains that the preference relation that says that all alternatives are indifferent extends any preference relation, even those relations according to which all (distinct) alternatives are strictly ranked.) Conceptually speaking, the definition of extension we adopt here is far more useful.

$\mathbf{R} = \mathsf{T}(\mathbf{R})$. A folk theorem of order theory says that $\mathsf{T}(\mathbf{R})$ exists, it is a preorder, and it satisfies: $x \ \mathsf{T}(\mathbf{R}) \ y$ iff $x^{\uparrow,\mathbf{R}} \subseteq y^{\uparrow,\mathbf{R}}$ and $y^{\downarrow,\mathbf{R}} \subseteq x^{\downarrow,\mathbf{R}}$ (cf. Cerreia-Vioglio and Ok (2018).) In order theory, especially in the context of interval orders, $\mathsf{T}(\mathbf{R})$ is sometimes called the *trace* of $\mathbf{R}$ (cf. Doignon et al. (1986)). Here we adopt the terminology of Nishimura (2018) who has provided an axiomatic characterization of the operator T on the class of all complete binary relations on X.

Continuity of a Binary Relation. Let X be a topological space and $\mathbf{R}$ a binary relation on X. There are various ways in which we can think of $\mathbf{R}$ as continuous. In particular, $\mathbf{R}$ is called **closed-continuous** if it is a closed subset of $X \times X$ (relative to the product topology), and it is called **open-continuous** if $\mathbf{R}^{>}$ is an open subset of $X \times X$. When $\mathbf{R}$ is closed-continuous, so is $\mathbf{R}^{=}$, but easy examples show that $\mathbf{R}$ need not be open-continuous. In fact, a famous result of Schmeidler (1971) says that if X is connected and $\mathbf{R}$ is a preorder on X with $\mathbf{R}^{>} \neq \varnothing$, then $\mathbf{R}$ is both closed- and open-continuous only if it is complete.[5] In this paper, we adopt closed-continuity as the primary notion of continuity, and in what follows, refer to a closed-continuous binary relation simply as **continuous**.

3 Preference Structures

3.1 Introduction

Let X be a nonempty set which we take as the collection of all mutually exclusive choice prospects for an economic agent (who may itself be a collection of individuals, such as a board of directors, congress, or a family). This agent is entirely confident in the preferential ranking of *some* of the alternatives in X. We model these rankings by means of a binary relation $\succsim$ on X. So, when $x \succsim y$ for some $x, y \in X$, we understand that the agent is "sure" that x is better than y for her. Of course, $\succsim$ is unobservable (because we do not know when an agent is "sure" about her preferential rankings), but our interpretation mandates $\succsim$ be (reflexive and) transitive: If x is surely better than y, and y is surely better than z, it makes sense that x will be deemed surely better than z. However, and this is where the present theory begins to deviate from the standard theory of rational decision-making, there is no need for $\succsim$ to be complete. The agent may well find the comparison of some alternatives "difficult," an entirely realistic phenomenon.[6]

Suppose the agent is unable to rank two alternatives x and y with respect to $\succsim$. When confronted with the problem of choosing between x and y, we will still observe

[5] Which of these two continuity notions one adopts may have significant consequences in terms of the representation of a given preorder; see, for instance, Evren (2014).

[6] If the agent here is a collection of individuals (each with her own preference relation), then $\succsim$ may, for instance, correspond to the rankings of the alternatives according to the Pareto rule. When the Pareto ranking works, the comparisons are "easy," but of course, there may be many cases in which the Pareto ranking does not apply. (See Example 3.6 below.)

her make a decision.[7] So, in the case of this choice problem, if she chooses x over y, we say that "x is revealed to be preferred to y," and if we have somehow witnessed that she choose x over y at some observation point, and y over x in some other, we say that "x is revealed to be indifferent to y." (We do not adopt a random preference model in this paper.) As such, we model *all* pairwise rankings of the individual, "easy" ones as well as the "hard" ones, by means of a binary relation $\mathbf{R}$ on X (which is observable). The very interpretation of $\mathbf{R}$ mandates it be complete. However, it is only natural that "hard choices" may not act transitively: If the agent has chosen x over y with great difficulty, and was also conflicted about her choice between y over z, but has nonetheless chosen y over z, then it may well be the case she choose z over x (again with difficulty). This not only rings true by daily introspection, but is verified by numerous experimental studies (on the nontransitivity of preferences). Moreover, if our economic agent consists of a set of individuals, then even the most standard methods of aggregating constituent preferences (such as majority voting) may result in the revelation of nontransitive rankings of the alternatives.

These considerations suggest that we model the "preferences" of an economic agent by means of an ordered pair $(\succsim, \mathbf{R})$ of binary relations on X such that $\succsim$ is a preorder and $\mathbf{R}$ is complete. Moreover, it only makes sense that these relations are consistent in the sense that $x \succsim y$ (x is "surely" better than y) implies $x \mathbf{R} y$ (we observe the agent choose x over y), and $x \succ y$ (x is "surely" strictly better than y) implies $x \mathbf{R}^{>} y$ (we observe the agent never choose y over x). Put succinctly, our interpretation of things mandates that $\mathbf{R}$ be a *completion* of $\succsim$.

Before going further, we would like to emphasize that we take $\mathbf{R}$ as an *extension* of $\succsim$ here, not merely as a superrelation of $\succsim$. This is essential for pretty much all of the results that will be reported below; indeed it is this property that distinguishes the present work from the earlier ones. And this is by no means a technical point. The requirement that we have $\succ \subseteq \mathbf{R}^{>}$ (in addition to $\succsim \subseteq \mathbf{R}$) is but only dictated by the very interpretation of $\succsim$ and $\mathbf{R}$. Suppose, for a moment, that we do not insist on this requirement. Then, even though it is "obvious" to the agent that x is strictly better than y (that is $x \succ y$), we would allow for $x \mathbf{R}^{=} y$ which means that the agent would choose y over x at some point, an apparently unacceptable occurrence. For instance, where x is \$1000 and y is \$0, and most agents would "surely" rank the former strictly over y, we would then allow the agent reveal herself to be indifferent between \$1000 and \$0. Similarly, when the economic agent is the coalition of, say, ten individuals, all ten of whom vote for x over y, without the extension requirement, we would allow the coalition to be indifferent between x and y (with respect to the observed preferences $\mathbf{R}$). It seems clear that the present model would be unacceptably coarse without

[7] Of course, the agent may "choose" not to make a choice, but this necessitates that at least some pairwise choice problems (those that do not include the option of not choosing) to be designated as unobservable situations. As in the standard theory, we abstract away from such contingencies here by tacitly allowing all pairwise choice situations within our framework. This will be made formally clear in Section 4.

asking for $\succ \subseteq \mathbf{R}^{>}$ as well as $\succsim \subseteq \mathbf{R}$ (which in turn implies $\sim \subseteq \mathbf{R}^{=}$).

In fact, our interpretation suggests that $\mathbf{R}$ should act in coherence with $\succsim$ in a way that goes beyond the said extension property. Suppose our agent declares that $x \mathbf{R} y$ and $y \succsim z$ for some alternatives x, y and z. We interpret this as saying that the agent likes x better than y, even though she may well be somewhat insecure about this decision, while she prefers y over z in complete confidence. But then it stands to reason that the "obvious" superiority of y over z for this agent would entail that she would like x better than z, but, of course, it is possible that she may not be secure in this judgement either (that is, $x \mathbf{R} z$ holds, but not necessarily $x \succsim z$). Consequently, and since the same reasoning applies when $x \succsim y \mathbf{R} z$ as well, it makes good sense to require $\succsim$ and $\mathbf{R}$ to satisfy the following property:

$$x \mathbf{R} y \succsim z \quad \text{or} \quad x \succsim y \mathbf{R} z \quad \text{implies} \quad x \mathbf{R} z$$

for all $x, y, z \in X$. Put succinctly, we posit $\mathbf{R}$ be $\succsim$*-transitive.* As we shall see, the hypothesis that $\mathbf{R}$ be a $\succsim$-transitive completion of $\succsim$, allows us to learn quite a bit about $\succsim$ (which is unobservable) from $\mathbf{R}$ (which is observable).

These considerations lead us to the main subject of the present paper:

Definition. A **preference structure** on a nonempty set X is an ordered pair $(\succsim, \mathbf{R})$ where $\succsim$ is a preorder on X and $\mathbf{R}$ is a $\succsim$-transitive completion of $\succsim$ on X. In this context, we refer to $\succsim$ as the **core preference relation** of the structure, and to $\mathbf{R}$ as its **revealed preference relation**.

In what follows, we will briefly discuss the relation between preference structures and some similar constructs found in decision theory, and consider several concrete examples of preference structures. We will then provide various characterizations of preference structures, and talk about how we may obtain a "utility representation" for a preference structure. Then, in Section 4, we will develop a theory of choice that is based on preference structures, and in Section 5, we will examine the consistency of this choice theory with the basic interpretation of preference structures as well as the identification of preference structures from choice data.

Relation to the Literature. Modeling individual preferences by means of two binary relations, one incomplete and the other complete, is not new in decision theory. Especially in the literature on decision making under uncertainty, this method is employed by a number of studies. For example, in the Anscombe-Aumann framework, Gilboa et al. (2010) have used two binary relations, the first being a preorder (à la Bewley (1986)) and the second a complete preorder (à la Gilboa and Schmeidler (1989)). However, this model, along with those of all other works that we know in this line of research, not only does not apply in the context of the general (ordinal) framework we consider here, but, in addition, it does not produce a preference structure, because its second (complete) relation is not an extension of its first (complete) preference relation.

There are a very few studies that employ two binary relations to model individual preferences in our general setup. The two papers that are most closely related to the present work are Giarlotta and Greco (2013) and Giarlotta and Watson (2018). The former study focuses on an ordered pair of binary relations $(\succsim, \mathbf{R})$ on X in which $\succsim$ is a preorder and $\mathbf{R}$ is a complete and $\succsim$-transitive superrelation of $\succsim$ such that, for any two alternatives x and y, either $x \succsim y$ or $y \ \mathbf{R} \ x$. (This model is called the *necessary and possible* (NaP) *preference* on X.) While, to our knowledge, Giarlotta and Greco (2013) is the first paper that suggest the use of two binary relations, one transitive and the other complete, in the general context of (ordinal) preferences, the final requirement (which forces any two alternatives that are "hard" to compare to be revealed indifferent) is simply too strong for a general model of individual preferences. Instead, NaP-preferences are found to be of major use in multiple criteria decision analysis in general, and robust ordinal regression in particular. (See Giarlotta (2018) for a comprehensive survey on the theory and applications of NaP-preferences.)

Giarlotta and Watson (2018) drop the requirement that either $x \succsim y$ or $y \ \mathbf{R} \ x$ must hold for all x and y in X, and assume only that $\succsim$ is a preorder and $\mathbf{R}$ is a complete and $\succsim$-transitive superrelation of $\succsim$. This model, which is referred to as a *complete bi-preference* by Giarlotta and Watson, need not be a preference structure because here $\mathbf{R}$ need not be an extension on $\succsim$. To wit, $(\succsim, X \times X)$ is a complete bi-preference – in fact, it is a NaP-preference on X – no matter what $\succsim$ is; in this model the second (revealed) preference relation says that *any* two choice alternatives are indifferent, even though $\succsim$ may indicate that the agent views one of these alternatives "obviously" better than the other. For instance, where X is $\mathbb{R}_+$, whose contents are interpreted as monetary prizes, $(\geq, \mathbb{R}_+ \times \mathbb{R}_+)$ is a NaP-preference that says that the agent inherently prefers to have more money to less, but is observed to be indifferent between any quantities of money, even, say, between 0 dollars and one million dollars. (By contrast, the only preference structure that is compatible with the core preference relation being $\geq$ here is $(\geq, \geq)$.) All in all, it appears that the model of complete bi-preferences is too coarse to serve as a general model of individual preferences.[8]

3.2 Examples

The following examples, in which X stands for an arbitrary nonempty set, aim to illustrate the breadth of the model of preference structures.

[8]Giarlotta and Watson (2018) call a complete bi-preference $(\succsim, \mathbf{R})$ with $\succ \subseteq \mathbf{R}^>$ a *monotonic complete bi-preference*; this is exactly what we refer to as a *preference structure* in the present paper. In fact, they explicitly say that imposing the additional requirement $\succ \subseteq \mathbf{R}^>$ on a complete bi-preference would make the model naturally suited for choice theory, but they do not pursue this direction in their work. Instead, they focus mainly on the theory *comonotonic* bi-preferences $(\succsim, \mathbf{R})$ which, by definition, satisfy $\mathbf{R}^> \subseteq \succ$. As noted by Giarlotta and Watson (2018, Section 4.3), the semantics of such structures are "quite different from that of a monotonic bi-preference, being more related to decision analysis and operations research rather than choice theory."

Example 3.1. Let $\succsim$ be a complete preorder on X. Then, $(\succsim, \succsim)$ is a preference structure on X. (Every complete preference relation may thus be thought of as a preference structure.)

Example 3.2. Let $\mathbf{R}$ be a total binary relation on X. Then, $(\triangle_X, \mathbf{R})$ is a preference structure on X. (Every total binary relation may thus be thought of as a preference structure.)

Example 3.3. Let $\succsim$ be a preorder on X. If $\mathbf{R}$ stands for $\succsim \cup \mathrm{Inc}(\succsim)$, then $(\succsim, \mathbf{R})$ is a preference structure on X.

Example 3.4. Let $\succsim$ and $\succsim^*$ be two preorders on X. If $\succsim^*$ is a completion of $\succsim$, then $(\succsim, \succsim^*)$ is a preference structure on X.

Example 3.5. Fix a positive integer n, and let u_i be a real map on X for each $i = 1, ..., n$. For any $m \in \{1, ..., n\}$, define the preorders $\succsim$ and $\succsim_m$ on X by $x \succsim y$ iff $u_i(x) \geq u_i(y)$ for each $i = 1, ..., n$, and $x \succsim_m y$ iff $u_1(x) + \cdots + u_m(x) \geq u_1(y) + \cdots + u_m(y)$. Then, $(\succsim, \succsim_n)$ is a preference structure on X, but $(\succsim, \succsim_m)$ need not be a preference structure on X if $m < n$.

Example 3.6. (*Aggregation by Majority Voting*) Let $\mathcal{P}$ be a nonempty family of total preorders on X. Then, $\bigcap \mathcal{P}$ is the Pareto ordering induced by this collection. In turn, we define the (*majority voting*) binary relation $\mathcal{P}_{\mathrm{maj}}$ on X as

$$x \; \mathcal{P}_{\mathrm{maj}} \; y \quad \text{iff} \quad |\{\succsim \in \mathcal{P} : x \succ y\}| \geq |\{\succsim \in \mathcal{P} : y \succ x\}|$$

for every $x, y \in X$. Then, $(\bigcap \mathcal{P}, \mathcal{P}_{\mathrm{maj}})$ is a preference structure on X.

Example 3.7. (*Preferences with Imperfect Discrimination*) Let $\mathbf{R}$ be a complete and quasitransitive binary relation on X. Then, $(\triangle_X \sqcup \mathbf{R}^>, \mathbf{R})$ is a preference structure on X. This model allows us to capture the utility model of imperfect discrimination which goes back to Armstrong (1939) and Luce (1956), and studied more recently by Beja and Gilboa (1992), among others. To wit, let $u : X \to \mathbb{R}$ be any function and take any real number $\varepsilon \geq 0$. Define the binary relation $\mathbf{R}$ on X as $x \; \mathbf{R} \; y$ iff $u(x) \geq u(y) - \varepsilon$. This is a complete and quasitransitive binary relation on X with $x \; \mathbf{R}^> \; y$ iff $u(x) > u(y) + \varepsilon$ and $x \; \mathbf{R}^= \; y$ iff $|u(x) - u(y)| \leq \varepsilon$. (The idea is that the agent does not discriminate between alternatives whose utility values are close enough; Luce (1956) thus refers to ε as the *just noticeable difference.*) Then, $(\succsim, \mathbf{R})$ is a preference structure on X where $\succsim$ is the semiorder on X defined by $x \succsim y$ iff either $x = y$ or $u(x) > u(y) + \varepsilon$. The interpretation is that the pairwise ranking of any two alternatives is an "easy" one if the utilities of these alternatives are sufficiently (that is, more than ε) distinct, and "hard" otherwise.

Example 3.8. (*Preferences with Regret*) Let n be any positive integer, and $p := (p_1, ..., p_n)$ a probability vector with $p_i > 0$ for each i. Consider an environment in which there are n many states of the world, and state i obtains with probability p_i. We put $X := \mathbb{R}^n$, and interpret any $x := (x_1, ..., x_n) \in X$ as a state-contingent claim that pays x_i dollars at state i. Let $\succsim$ be the preorder defined by $x \succsim y$ iff $x_i \geq y_i$ for each $i = 1, ..., n$. (Thus, when $x \succsim y$ holds, x is an obviously better prospect than y.) Now let $u : \mathbb{R} \to \mathbb{R}$ and $Q : \mathbb{R} \to (-1, 1)$ be strictly increasing functions with $u(0) = 0$. Furthermore, assume that Q is odd (i.e., $Q(-a) = -Q(a)$ for every $a \in \mathbb{R}$), and convex on $\mathbb{R}_+$. We define the binary relation $\mathbf{R}$ on X as

$$x \mathbf{R} y \quad \text{iff} \quad \sum_{i=1}^{n} p_i Q\left(u(x_i) - u(y_i)\right) \geq 0.$$

This relation, due to Loomes and Sugden (1982), is known as a *regret preference*; it ranks prospects on the basis of their aggregate regret/rejoice due to the (utility) difference between the realized rewards. When $n \geq 3$ and Q is strictly convex on $\mathbb{R}_+$, $\mathbf{R}$ is not transitive, but it is always complete. In fact, it is easy to check that $(\succsim, \mathbf{R})$ is a preference structure on X.

The following example is a generalization of the previous one.

Example 3.9.[9] (*Intra-Dimensional Comparison Heuristics*) Let n be any positive integer, and consider an environment in which every commodity is modeled through n attributes that an agent values. We thus put $X := \mathbb{R}^n$, and interpret any $x := (x_1, ..., x_n) \in X$ as a commodity which possesses x_i units of the attribute i. For each $i \in \{1, ..., n\}$, let us pick any skew-symmetric function $f_i : \mathbb{R}^2 \to (-1, 1)$ that is strictly increasing in the first component, and any strictly increasing and odd $W : (-1, 1)^n \to \mathbb{R}$.[10] We define the binary relation $\mathbf{R}$ on X as

$$x \mathbf{R} y \quad \text{iff} \quad W\left(f_1(x_1, y_1), ..., f_n(x_n, y_n)\right) \geq 0.$$

Here the vector $(f_1(x_1, y_1), ..., f_n(x_n, y_n))$ corresponds to comparisons of the commodities x and y attribute by attribute; we can interpret f_i as measuring either the (dis)similarity of x_i and y_i or the salience of the ith attribute relative to the other attributes. We thus follow Tserenjigmid (2015), who has recently worked out a nice axiomatization for it, by calling $\mathbf{R}$ an *intra-dimensional comparison* (IDC) *relation*. Not only is any regret preference is an IDC relation, but the model of IDC relations contains the additive utility model (Example 3.5), the additive difference model of Tversky (1969) and a version of the salience theory of Bordalo, Gennaioli and Schleifer (2012). The upshot here is that $(\succsim, \mathbf{R})$ is a preference structure on X, where $\succsim$ is the binary relation on X defined by $x \succsim y$ iff $f_i(x_i, y_i) \geq 0$ for each $i = 1, ..., n$.[11]

[9] We thank Pietro Ortoleva for suggesting this example to us.

[10] *Skew-symmetry* of f_i means that $f_i(a, b) = -f_i(b, a)$ for every $a, b \in \mathbb{R}$.

[11] Due to the skew-symmetry and monotonicity of f_is, we actually have $x \succsim y$ iff $x_i \geq y_i$ for each $i = 1, ..., n$ here. This makes the proof of our claim routine.

Example 3.10. Let $\mathbf{R}$ be a complete binary relation on X. Clearly, if $(\succsim, \mathbf{R})$ is a preference structure on X, then $\succsim$ must be a subrelation of $\mathsf{T}(\mathbf{R})$, but $(\mathsf{T}(\mathbf{R}), \mathbf{R})$ need not be a preference structure on X. Indeed, $(\mathsf{T}(\mathbf{R}), \mathbf{R})$ is a preference structure on X iff $\mathsf{T}(\mathbf{R})^{>} \subseteq \mathbf{R}^{>}$ (that is, $\mathbf{R}$ is a completion of $\mathsf{T}(\mathbf{R})$). For example, let u and ε be as in Example 3.7, let $\mathbf{R}$ be the semiorder defined in that example, and assume that $\sup u(X) - \inf u(X) < 2\varepsilon$. Then, as proved by Nishimura (2018), we have $x \ \mathsf{T}(\mathbf{R}) \ y$ iff $u(x) \geq u(y)$. So, $\mathsf{T}(\mathbf{R})^{>} \subseteq \mathbf{R}^{>}$ fails, and hence, $(\mathsf{T}(\mathbf{R}), \mathbf{R})$ is not a preference structure on X.

3.3 Completion by Advise

In this section too we consider an example of a preference structure. As this example is a particularly interesting one from a behavioral viewpoint, we investigate it in some further detail. We will return to this example at various junctures later in the paper.

We wish to model a situation in which an individual, when faced with a "hard" choice problem, seeks the advise of a consultant. The issue of dealing with "easy" choices is again modeled by means of a (core) preference relation $\succsim$ on the given alternative space X. When two alternatives x and y are incomparable with respect to $\succsim$ – this choice is "hard" for the agent – she acts according to the advices of another individual (consultant). We imagine that the consultant is rational in the traditional sense, so her advices arise from a complete preorder $\trianglerighteq$ on X. Moreover, we assume that this preorder is consistent with $\succsim$ in the sense that $\succsim \subseteq \trianglerighteq$. (Otherwise, it would be rather awkward for our agent to trust the recommendations of the advisor, as some of those recommendations would conflict with her core preferences.) This leads us to a particularly interesting preference structure.

An ordered pair $(\succsim, \mathbf{R})$ of two binary relations on a nonempty set X is said to be a **completion by advise** on X if $\succsim$ is a preorder on X, and there exists a complete preorder $\trianglerighteq$ on X such that $\succsim \subseteq \trianglerighteq$ and $x \ \mathbf{R} \ y$ iff

$$\text{either } x \succsim y \text{ or } [x \text{ and } y \text{ are not } \succsim\text{-comparable and } x \trianglerighteq y]. \tag{1}$$

It turns out that this is always a preference structure.

Proposition 3.1. *Every completion by advise on a nonempty set X is a preference structure on X.*

Proof. Let $(\succsim, \mathbf{R})$ be a completion by advise on X, and let $\trianglerighteq$ be a complete preorder on X such that $\succsim \subseteq \trianglerighteq$ and $x \ \mathbf{R} \ y$ iff (1) holds. Then, $\succsim$ is a preorder on X by hypothesis, and $\mathbf{R}$ is obviously a total binary relation on X. Moreover, $\succsim \subseteq \mathbf{R}$ by definition of $\mathbf{R}$, and if $x \succ y$, then $x \ \mathbf{R} \ y$ but $y \ \mathbf{R} \ x$ cannot hold (because neither $y \succsim x$ nor $(y, x) \in \text{Inc}(\succsim)$ holds). Thus, $\mathbf{R}$ is a completion of $\succsim$. It remains to prove that $\mathbf{R}$ is $\succsim$-transitive. To this end, take any x, y and z in X such that $x \ \mathbf{R} \ y \succsim z$. Notice that $z \succ x$ cannot hold, because otherwise $y \succ x$ (by transitivity of $\succsim$), and hence $y \ \mathbf{R}^{>} \ x$ (because $\mathbf{R}$ is an extension of $\succsim$), a contradiction. Thus: Either $x \succsim z$

or $(x,z) \in \mathrm{Inc}(\succsim)$. In the former case, we have $x \mathbf{R} z$ by definition of $\mathbf{R}$, so we are done. Similarly, if $x \succsim y$, then $x \mathbf{R} z$ because $\succsim$ is transitive and $\succsim \subseteq \mathbf{R}$. So, assume that $(x,z) \in \mathrm{Inc}(\succsim)$ and $x \succsim y$ is false. Since $x \mathbf{R} y$, the latter statement and the definition of $\mathbf{R}$ imply that $x \trianglerighteq y$. As $\succsim \subseteq \trianglerighteq$ by hypothesis, therefore, $x \trianglerighteq y \trianglerighteq z$, and hence $x \trianglerighteq z$. It follows that $(x,z) \in \mathrm{Inc}(\succsim)$ and $x \trianglerighteq z$, that is, $x \mathbf{R} z$, as we sought. As we can similarly show that $x \succsim y \mathbf{R} z$ implies $x \mathbf{R} z$, our proof is complete. ∎

Intuitively speaking, if $(\succsim, \mathbf{R})$ is a completion by advise, then $\mathbf{R}$ acts transitively across pairs of alternatives that are not $\succsim$-comparable. The following result provides a simple characterization of completions by advise building on this intuition.

Proposition 3.2. *Let $\succsim$ be a preorder on a nonempty set X, and $\mathbf{R}$ a completion of $\succsim$. Then, $(\succsim, \mathbf{R})$ is a completion by advise on X if, and only if*

$$\mathrm{Inc}(\succsim) \cap \mathbf{R} = \mathrm{Inc}(\succsim) \cap \mathrm{tran}(\mathbf{R}). \tag{2}$$

This observation helps identify some necessary conditions for a preference structure $(\succsim, \mathbf{R})$ to be a completion by advise. For instance, we can use it to verify that $\mathrm{Inc}(\succsim) \cap \mathbf{R}$ must be Suzumura consistent for every completion by advise $(\succsim, \mathbf{R})$.[12] And, in general, if $(\succsim, \mathbf{R})$ is a preference structure such that $\mathrm{Inc}(\succsim) \cap \mathbf{R}$ is Suzumura consistent, then $\mathbf{R}$ is quasitransitive. We thus find that a necessary condition for $(\succsim, \mathbf{R})$ be a completion by advise is that $\mathbf{R}$ be quasitransitive.

3.4 A Characterization of Preference Structures

The following result provides a general representation theorem for preference structures. Its import stems from the fact that it connects the two components of a preference structure by means of a single entity, namely, a collection of preorders.

Theorem 3.3. *Let $\succsim$ and $\mathbf{R}$ be two binary relations on a nonempty set X. Then, $(\succsim, \mathbf{R})$ is a preference structure on X if, and only if, there is a nonempty collection $\mathcal{P}$ of preorders on X such that*

$$(\succsim, \mathbf{R}) = \left(\bigcap \mathcal{P}, \bigcup \mathcal{P}\right)$$

where $\bigcup \mathcal{P}$ is complete and each $\trianglerighteq \in \mathcal{P}$ extends $\succsim$.[13]

[12] To derive a contradiction, suppose $\mathrm{Inc}(\succsim) \cap \mathbf{R}$ is not Suzumura consistent. Then, by Proposition 3.2, $\mathbf{T} := \mathrm{Inc}(\succsim) \cap \mathrm{tran}(\mathbf{R})$ is not Suzumura consistent, so there exists a $k \in \mathbb{N}$ and $x_1, \dots, x_k$ in X such that $x_1 \mathbf{T} \cdots \mathbf{T} x_k \mathbf{T} x_1$ with at least one of these $\mathbf{T}$ holding strictly. By relabeling if necessary, we may assume that $x_k \mathbf{T}^{>} x_1$. Then, we have $x_1 \mathbf{R} \cdots \mathbf{R} x_k \mathbf{T}^{>} x_1$, that is, $x_1 \mathrm{tran}(\mathbf{R}) x_k \mathbf{R}^{>} x_1$. Since x_1 and x_k are not $\succsim$-comparable, it follows that $x_1 \mathbf{T} x_k \mathbf{T}^{>} x_1$, a contradiction.

[13] An analogous result for the so-called NaP preferences was obtained by Giarlotta and Greco (2013).

The "if' part of this result provides a general method of defining preference structures. In turn, its "only if" part provides a "multi-selves" interpretation for any given preference structure $(\succsim, \mathbf{R})$. To wit, let $\mathcal{P}$ stand for a nonempty collection of preorders on X as found in Theorem 3.3. We may think of each element $\trianglerighteq$ in $\mathcal{P}$ as a (potentially incomplete) preference relation of a different "self" of the same individual. (For instance, the agent may not know which of these relations will be the relevant one at the time of consumption, so entertains them all before making her choice.) These "selves" of the agent are perfectly consistent with the core preference relation $\succsim$ of the agent in that every one of them extends $\succsim$. In addition, $\succsim$, being equal to $\bigcap \mathcal{P}$, ranks an alternative x over another alternative y iff every one of her "selves" agree that this is the correct ranking. It is in this sense that $\succsim$ may be thought of as a *dominance* relation. On the other hand, the revealed preference relation $\mathbf{R}$ of the agent, being equal to $\bigcup \mathcal{P}$, ranks x over y iff at least one of her "selves" agree that this is the correct ranking. In this sense, we may think of $\mathbf{R}$ as a *rationalizable* preference on X, borrowing (and slightly abusing) the terminology used by Cherepanov, Feddersen and Sandroni (2013). Importantly, these notions of dominance and rationalizability are compatible, for they are based on the preferences of the same set of "selves" of the agent.

Remark. A natural question is when we can guarantee the completeness of each member of $\mathcal{P}$ in the representation provided in Theorem 3.3. It turns out that this is a very restrictive requirement; we can do this only when $\mathbf{R}$ is obtained from $\succsim$ by rendering every $\succsim$-incomparable pair indifferent. Put more precisely, we have the following theorem: A preference structure $(\succsim, \mathbf{R})$ on a nonempty set X satisfies $\mathbf{R} = \succsim \sqcup \operatorname{Inc}(\succsim)$ iff there is a nonempty collection $\mathcal{P}$ of complete preorders on X such that (i) $(\succsim, \mathbf{R}) = (\bigcap \mathcal{P}, \bigcup \mathcal{P})$ and (ii) each $\trianglerighteq \in \mathcal{P}$ extends $\succsim$.[14] (We omit the proof, which is available upon request.)

3.5 Utility Representation of Preference Structures

In recent years, generalized theories of utility representations are developed for preference relations that lack properties such as completeness and transitivity. Drawing from some of those studies, we introduce in this section a general notion of utility representation for preference structures.

In what follows we simplify our notation by writing

$$\mathcal{U}(x) \geq \mathcal{U}(y)$$

to mean $u(x) \geq u(y)$ for every $u \in \mathcal{U}$, where $\mathcal{U}$ is any nonempty collection of real maps on X and $x, y \in X$. Similarly, we write $\mathcal{U}(x) > \mathcal{U}(y)$ to mean $\mathcal{U}(x) \geq \mathcal{U}(y)$ and $u(x) > u(y)$ for some $u \in \mathcal{U}$.

[14] A classical result of order theory, due to Dushnik and Miller (1941), says that every partial order $\succsim$ on a nonempty set X is the intersection of a nonempty collection $\mathcal{P}$ of linear orders on X. More generally, Donaldson and Weymark (1998) prove that every preorder $\succsim$ on X is the intersection of a nonempty collection $\mathcal{P}$ of total preorders on X. The theorem we just stated, which is based on the Axiom of Choice, not only generalizes the Donaldson-Weymark Theorem, but it also shows that all members of $\mathcal{P}$ in that theorem can be chosen to be extensions of $\succsim$.

The Multi-Utility Representation. Let $\succsim$ be a preorder on a nonempty set X. We say that a nonempty collection $\mathcal{U}_0$ of real maps on X is a **multi-utility representation** for $\succsim$ if

$$x \succsim y \quad \text{iff} \quad \mathcal{U}_0(x) \geq \mathcal{U}_0(y) \tag{3}$$

for each $x, y \in X$. When X is a topological space, and $\mathcal{U}_0 \subseteq \mathbf{C}(X)$, we refer to $\mathcal{U}_0$ as a **continuous multi-utility representation** for $\succsim$.[15] This utility representation notion, introduced to the decision theory literature by Ok (2002), reduces to the classical notion of utility representation of a complete preference relation when $\mathcal{U}_0$ is a singleton. It allows us to think of an incomplete preference relation $\succsim$ as arising from the unanimity of several complete preference relations each of which admits a utility representation in the standard sense. In one form or another, it is adopted quite broadly in the literature on incomplete preferences and multicriteria choice theory. Besides, Evren and Ok (2011) have shown that, for a fairly large class of topological spaces X, every continuous preorder on X admits a continuous multi-utility representation.

The Maxmin Multi-Utility Representation. Any binary relation that admits a multi-utility representation is transitive, so this representation concept has to be generalized for it to apply to nontransitive relations. One such generalization (for reflexive binary relations) is recently introduced by Nishimura and Ok (2016), to which we now turn.

Let $\mathbf{R}$ be a reflexive binary relation on a topological space X. We say that a nonempty collection $\mathbb{U}$ of nonempty sets of continuous real maps on X is a **maxmin multi-utility representation** for $\mathbf{R}$ if

$$x \mathbf{R} y \quad \text{iff} \quad \mathcal{U}(x) \geq \mathcal{U}(y) \text{ for some } \mathcal{U} \in \mathbb{U} \tag{4}$$

for each $x, y \in X$. This notion nests that of multi-utility representation – consider the case where $\mathbb{U}$ is a singleton – and hence the classical concept of utility representation. It allows us to think of the ranking of two alternatives by $\mathbf{R}$ as rationalized by the unanimity of a set of complete preference relations each of which admits a utility representation in the standard sense, but unlike in the case of "multi-utility representation," it permits using different sets of such preference relations for different pairs of alternatives. For various interpretations, and applications, of this utility representation notion, we refer the reader to Nishimura and Ok (2016).

Utility Representation of Preference Structures. We now combine the two utility representation notions considered above to obtain a notion of utility representation for preference structures. Let us first introduce the following terminology: We say that a collection $\mathcal{U}$ of real maps on a nonempty set X is **comonotonic** with respect to another such set $\mathcal{U}_0$ if $\mathcal{U}_0(x) \geq \mathcal{U}_0(y)$ implies $\mathcal{U}(x) \geq \mathcal{U}(y)$ for every $x, y \in X$. If,

[15] By $\mathbf{C}(X)$, we mean the normed linear space of all continuous real maps on X (relative to the sup-norm).

in addition, $\mathcal{U}_0(x) > \mathcal{U}_0(y)$ implies $\mathcal{U}(x) > \mathcal{U}(y)$ for every $x, y \in X$, we say that $\mathcal{U}$ is **strictly comonotonic** with respect to $\mathcal{U}_0$.

Let $\mathbb{U}$ be a maxmin multi-utility representation for a reflexive binary relation $\mathbf{R}$ on X, and suppose that an element $\mathcal{U}_0$ of $\mathbb{U}$ is a multi-utility representation for a preorder $\succsim$ on X. Then, $\succsim$ is a subrelation of $\mathbf{R}$, but $\mathbf{R}$ need not be an extension of $\succsim$, nor it is guaranteed to be $\succsim$-transitive. Both of these difficulties are resolved, however, if $\mathcal{U}$ is strictly comonotonic with respect to $\mathcal{U}_0$. Thus, conversely, given a preference structure $(\succsim, \mathbf{R})$ on X, it is natural to look for a multi-utility for $\succsim$ and a maxmin multi-utility for $\mathbf{R}$ that are connected in this manner. The following theorem shows that under a suitable compactness condition, continuity of $\succsim$ is enough to find such a utility representation for $(\succsim, \mathbf{R})$. In addition, it provides a general method of defining (continuous) preference structures on a compact Hausdorff topological space.

Theorem 3.4. *Let X be a compact Hausdorff space, and $\succsim$ and $\mathbf{R}$ binary relations on X such that $\mathbf{R}$ is complete. Then, $(\succsim, \mathbf{R})$ is a preference structure such that $\succsim$ is continuous if, and only if, there exists a continuous multi-utility representation $\mathcal{U}_0$ for $\succsim$ and a maxmin multi-utility representation $\mathbb{U}$ for $\mathbf{R}$ such that $\mathcal{U}_0 \in \mathbb{U}$ and every $\mathcal{U} \in \mathbb{U}$ is strictly comonotonic with respect to $\mathcal{U}_0$.*[16]

Remark. (*On the Converse of Theorem 3.4*) Let $\mathbb{U}$ be a nonempty collection of nonempty sets of real maps on X, and define the binary relations $\mathbf{R}$ and $\succsim$ on X by (4) and (3), respectively, where $\mathcal{U}_0$ is an element of $\mathbb{U}$ with respect to which all members of $\mathbb{U}$ are strictly comonotonic. Then, $\succsim$ is a preorder and $\mathbf{R}$ is a $\succsim$-transitive extension of $\succsim$. Yet, in general, $(\succsim, \mathbf{R})$ need not be a preference structure, because this representation does not guarantee that $\mathbf{R}$ is complete.

Remark. (*Minmax Representation of Preference Structures*) Let X be a topological space and $\succsim$ and $\mathbf{R}$ two binary relations on X. Suppose that there is a nonempty collection $\mathbb{U}$ of nonempty sets of (continuous) real maps on X such that

$$x \mathbf{R} y \quad \text{iff} \quad \text{for every } \mathcal{U} \in \mathbb{U}, \text{ there is a } u \in \mathcal{U} \text{ with } u(x) \geq u(y),$$

and an element $\mathcal{U}_0$ of $\mathbb{U}$ such that (3) and

$$x \succ y \quad \text{iff} \quad u_0(x) > u_0(x) \text{ for every } u_0 \in \mathcal{U}_0 \tag{5}$$

hold for every $x, y \in X$. In addition, assume that every member of $\mathbb{U}$ is comonotonic with respect to $\mathcal{U}_0$ (or that $\mathcal{U}_0 \subseteq \mathcal{U}$ for every $\mathcal{U} \in \mathbb{U}$). Then, one can show that $(\succsim, \mathbf{R})$ is a preference structure such that $\succsim$ is continuous.[17] This suggests that such a collection $\mathbb{U}$, which we call a (*continuous*) *minmax multi-utility representation*, may be viewed as a useful notion of utility representation for preference structures.[18] Under certain topological conditions, it can be shown that the class of preference structures that admit such a (continuous) representation is the same as those characterized in Theorem 3.4.

[16] This statement remains true so long as X is a locally compact and σ-compact Hausdorff topological space (such as $\mathbb{R}^n$).

[17] If X is a compact metric space, and every member of $\mathbb{U}$ is compact, then $\mathbf{R}$ is continuous here as well. (This is proved in the first paragraph of the proof of Theorem 2b of Nishimura and Ok (2016).)

[18] In the context of decision making under risk, this utility representation notion has recently been investigated in quite some depth by Hara, Ok and Riella (2018).

4 Choice Theory and Preference Structures

In Section 3 we have looked at various examples of preference structures, and their basic properties. In this section, we turn to how "choices" may arise from preference structures. This necessitates that we agree on what it means for an alternative to *maximize* a given complete (but not necessarily transitive) binary relation on a given feasible set, so we start the section with a discussion of this issue.

4.1 Maximization of Complete Binary Relations

Let X be a nonempty set, $\mathbf{R}$ a binary relation on X, and S a nonempty subset of X. An element x of S is called **R-maximal** in S if there is no $y \in S$ with $y \ \mathbf{R}^{>} \ x$, and **R-maximum** in S if $x \ \mathbf{R} \ S$ (that is, $x \ \mathbf{R} \ y$ for every $y \in S$). We denote the set of all $\mathbf{R}$-maximal and $\mathbf{R}$-maximum elements in S by $\mathbf{MAX}(S, \mathbf{R})$ and $\max(S, \mathbf{R})$, respectively. We always have $\max(S, \mathbf{R}) \subseteq \mathbf{MAX}(S, \mathbf{R})$, but this inequality may hold strictly (unless $\mathbf{R}$ is complete).

For binary relations $\mathbf{R}$ that are not transitive, these notions are rarely useful, because in this case $\mathbf{MAX}(S, \mathbf{R})$ may be empty even for a finite set S. For this reason, alternative notions of extrema are developed for binary relations. The most common of these is the notion of *top-cycles* to which we now turn (but in the context of complete binary relations).

Top-Cycles. Let $\mathbf{R}$ be a complete binary relation on X. We say that a nonempty subset A of S is a **highset in S with respect to R**, or more simply, an **R-highset in S**, if

$$x \ \mathbf{R}^{>} \ y \quad \text{for every } x \in A \text{ and } y \in S \backslash A.$$

Notice that the collection of all $\mathbf{R}$-highsets in S is nonempty, because it contains S. Moreover, this collection is linearly ordered by set inclusion $\supseteq$. (*Proof.* Suppose A and B are two $\mathbf{R}$-highsets in S such that $A \subseteq B$ is false. Then, pick any $a \in A \backslash B$, and notice that, for any $b \in B$, we have $b \ \mathbf{R}^{>} \ a$ because $a \in S \backslash B$ and B is an $\mathbf{R}$-highset in S. As A is itself an $\mathbf{R}$-highset in S, and a is in A, this implies that $b \in A$ for each $b \in B$, that is, $B \subseteq A$.) Consequently, if it exists, there is a unique smallest (with respect to $\supseteq$) $\mathbf{R}$-highset in S, and this set equals $\bigcap\{A \in 2^X : A \text{ is an } \mathbf{R}\text{-highset in } S\}$. We thus define the **top-cycle in S with respect to R**, which we denote by $\bigcirc(S, \mathbf{R})$, as

$$\bigcirc(S, \mathbf{R}) := \bigcap\{A \in 2^X : A \text{ is an } \mathbf{R}\text{-highset in } S\}.$$

This set is nonempty iff the smallest $\mathbf{R}$-highset in S exists. In particular, we have $\bigcirc(S, \mathbf{R}) \neq \varnothing$ whenever S is a nonempty finite set. [19]

[19] Mainly in the literature on voting theory, the notion of top cycles are studied extensively in the case where $\mathbf{R}$ is a tournament (that is, an asymmetric total binary relation on a finite set). See, for instance, Laslier (1997). When $\mathbf{R}$ is an arbitrary total binary relation, some authors refer

By "maximization of $\mathbf{R}$ in S," we mean identifying $\bigcirc(S,\mathbf{R})$. This is not only intuitive, but it is also consistent with the standard case (because $\bigcirc(S,\mathbf{R})$ reduces to $\max(S,\mathbf{R})$ when $\mathbf{R}$ is transitive). Yet, it is a bit troublesome that the definition of $\bigcirc(S,\mathbf{R})$ is indirect, and hence not really helpful for computations. Fortunately, there is an operational way of looking at the top-cycle in a set with respect to an arbitrary complete binary relation. Our next result formulates this method, thereby also clarifying that our definition is consistent with how top-cycles are traditionally defined in, say, social choice theory.

Proposition 4.1. *Let S be a nonempty subset of a set X, and $\mathbf{R}$ a complete binary relation on X. Then,*

$$\bigcirc(S,\mathbf{R}) = \max(S, \mathrm{tran}(\mathbf{R}|_S)).$$

Remark. Let us call a nonempty subset A of S an **R-cycle in** S if for any x and y in A, there exist finitely many $a_1, ..., a_k$ in A such that $x \mathbf{R} a_1 \mathbf{R} \cdots \mathbf{R} a_k \mathbf{R} y$. (If S is finite and $\mathbf{R}$ is a complete binary relation on S, then A is an $\mathbf{R}$-cycle in S iff we can enumerate A as $\{x_1, ..., x_n\}$ such that $x_1 \mathbf{R} x_2 \mathbf{R} \cdots \mathbf{R} x_n \mathbf{R} x_1$, so our definition is in concert with how this concept is defined for linear orders.) It follows from Proposition 4.1 that, when nonempty, $\bigcirc(S,\mathbf{R})$ is an $\mathbf{R}$-cycle in S. This justifies the terminology of referring to $\bigcirc(S,\mathbf{R})$ as a "cycle" in S with respect to $\mathbf{R}$. (In fact, when nonempty, $\bigcirc(S,\mathbf{R})$ is the unique $\mathbf{R}$-cycle in S that is also an $\mathbf{R}$-highset in S.)

Existence of Top Cycles. Most works on top-cycles with respect to a complete binary relation take the ground set X of the binary relation as finite. This, in turn, makes the issue of existence of top-cycles a trivial matter. More generally, one can use basic topological hypotheses to guarantee the existence of top-cycles even when X is not finite. The earliest such theorem that we know is given by Kalai and Schmeidler (1977), which allows even for incomplete binary relations. Unfortunately, there is an error in that result, so, we prove here the existence theorem we need from scratch.[20] We note, however, that we do not claim originality with respect to this result, which indeed seems like a folk theorem of social choice theory (cf. Duggan (2007)).

Theorem 4.2. *Let S be a nonempty compact subset of a topological space X, and $\mathbf{R}$ a continuous and complete binary relation on X. Then, $\bigcirc(S,\mathbf{R}) \neq \varnothing$.*

Proof. If $\max(S,\mathbf{R}|_S) \neq \varnothing$, then, obviously, $\max(S,\mathrm{tran}(\mathbf{R}|_S)) \neq \varnothing$, and our assertion follows from Proposition 4.1. Let us then assume that $\max(S,\mathbf{R}|_S) = \varnothing$.

to $\bigcirc(S,\mathbf{R})$ as the *weak top cycle* of $\mathbf{R}$ in S (cf. Ehlers and Sprumont (2008)), and some as the $\mathbf{R}$-*admissible set* in S (cf. Kalai and Schmeidler (1977)).

[20] Theorem 2 of Kalai and Schmeidler (1977) says that if X is a compact topological space and $\mathbf{R}$ an open-continuous binary relation on X, then $\max(X,\mathrm{tran}(\mathbf{R}))$ is nonempty. This is false. (*Example.* Let $X := \{1, \frac{1}{2}, \frac{1}{4}, ..., 0\}$, and note that X is a compact subspace of $\mathbb{R}$. Let $\mathbf{R}$ be the binary relation on X defined by $0 \mathbf{R}^{=} 1$ and $\cdots \mathbf{R}^{>} \frac{1}{4} \mathbf{R}^{>} \frac{1}{2} \mathbf{R}^{>} 1$. Then, $\mathbf{R}$ is open-continuous, and indeed $\mathbf{MAX}(X,\mathbf{R}) \neq \varnothing$, but $\mathbf{MAX}(X, \mathrm{tran}(\mathbf{R})) = \varnothing$.) But if, in the context of the statement of Kalai and Schmeidler's theorem, we assume that $\mathbf{R}$ is antisymmetric, then the result is true. (Indeed, the proof given in Kalai and Schmeidler (1977) settles this case.)

Then, $\{x^{\downarrow\downarrow} : x \in S\}$ is a cover of S, where $x^{\downarrow\downarrow} := \{y \in S : x \ \mathbf{R}^{>} \ y\}$ for each $x \in S$. Since $\mathbf{R}$ is complete and continuous, this is actually an open cover of S (in S), and hence, by compactness of S, there is a finite subset T of S such that $\{x^{\downarrow\downarrow} : x \in T\}$ covers S. Since T is finite, $\max(T,\text{tran}(\mathbf{R}|_S))$ is nonempty. Let x^* be an arbitrarily picked element from $\max(T,\text{tran}(\mathbf{R}|_S))$. Clearly, for any $y \in S$, we have $x^* \ \text{tran}(\mathbf{R}|_S) \ x \ \mathbf{R}^{>} \ y$ for some $x \in T$, and hence, $x^* \ \text{tran}(\mathbf{R}|_S) \ y$. In other words, $x^* \in \max(S,\text{tran}(\mathbf{R}|_S))$. In view of Proposition 4.1, therefore, $\bigcirc(S, \mathbf{R}) \neq \varnothing$. ■

4.2 Rationalization by Preference Structures

We now turn to the fundamental issue of defining how "choices" are made on the basis of a given preference structure. We wish to investigate this matter at a suitably general level (without restricting attention only to finite choice problems). To this end, let X be any nonempty set, and let $\mathfrak{X}$ be any collection of nonempty subsets of X such that (i) $\mathfrak{X}$ contains all singletons, and (ii) $\mathfrak{X}$ is closed under taking finite unions. (In particular, $\mathfrak{X}$ contains all nonempty finite subsets of X). For ease of reference, we will refer to any such ordered pair $(X, \mathfrak{X})$ as a **choice environment**. For example, $(X, 2^X \backslash \{\varnothing\})$ is a choice environment. More generally, where $P(X)$ denotes the collection of all nonempty finite subsets of X, $(X, P(X))$ is a choice environment; this is the environment used by the vast majority of works in the theory of individual choice. Still more generally, $(X, \mathbf{k}(X))$ is a choice environment, where X is a topological space and $\mathbf{k}(X)$ stands for the set of all nonempty compact subsets of X. Our two main theorems below will be stated in the context of this environment.

Given any choice environment $(X, \mathfrak{X})$, by a **choice correspondence on $\mathfrak{X}$**, we mean a set-valued map $C : \mathfrak{X} \rightrightarrows X$ such that $C(S) \subseteq S$ for every $S \in \mathfrak{X}$ and $C(S) \neq \varnothing$ for every finite $S \in \mathfrak{X}$. Such a choice correspondence C is said to be **single-valued** if $C(S)$ is a singleton for every finite $S \in \mathfrak{X}$.

Now take any preference structure $(\succsim, \mathbf{R})$ on X. We say that a choice correspondence C on $\mathfrak{X}$ is **rationalized by** $(\succsim, \mathbf{R})$ if

$$C(S) = \bigcirc(\mathbf{MAX}(S, \succsim), \mathbf{R}) \tag{6}$$

for every $S \in \mathfrak{X}$. Thus, we posit that an agent with a preference structure $(\succsim, \mathbf{R})$ settles on her choice(s) from a given feasible set S by employing a two-step procedure. First, she looks for those alternatives in S that are maximal with respect to her core preference relation $\succsim$. If there is only one such alternative in S, then she chooses that alternative. If there is a multiplicity of such alternatives (which may be due to indifferences and/or incomparabilities instigated by $\succsim$), then she restricts her attention to those alternatives, and evaluates them on the basis of her second (complete) binary relation $\mathbf{R}$. She finalizes her choice(s) by maximizing $\mathbf{R}$ on $\mathbf{MAX}(S, \succsim)$ in the sense of finding the top-cycle in $\mathbf{MAX}(S, \succsim)$ with respect to $\mathbf{R}$. This top-cycle is the set of all alternatives she deems "choosable" in S.

We note that one can certainly obtain alternative notions of "rationalization by a preference structure" by replacing the top-cycle operator by an alternative maximization notion, such as the uncovered set (Lombardi (2008)) or the untrapped set (Duggan (2007)). In what follows, we will derive quite a number of results that will hopefully witness that our way of defining this notion is a useful one, but it goes without saying that one cannot argue that our definition is the "right" one on *a priori* grounds.

For future reference, we note that, in view of Proposition 4.1, C is rationalized by $(\succsim, \mathbf{R})$ iff

$$C(S) = \max(\mathbf{MAX}(S, \succsim), \mathrm{tran}(\mathbf{R}|_{\mathrm{MAX}(S,\succsim)})), \tag{7}$$

for every $S \in \mathfrak{X}$. We will often adopt this alternative formula when studying the properties of C.

The Main (Existence) Theorem. When X is a topological space, we say that a preference structure $(\succsim, \mathbf{R})$ is **continuous** if both $\succsim$ and $\mathbf{R}$ are continuous binary relations on X. The choice correspondence rationalized by such a preference structure is nonempty-valued in most cases of interest.

Theorem 4.3. *For any topological space X, the choice correspondence on $\mathbf{k}(X)$ rationalized by a continuous preference structure $(\succsim, \mathbf{R})$ on X is nonempty-valued.*[21]

Proof. For any $S \in \mathbf{k}(X)$, by a well-known theorem of topological order theory, continuity of $\succsim$ ensures that $\mathbf{MAX}(S, \succsim)$ is a nonempty compact subset of X.[22] Applying Theorem 4.2 then completes the proof. ■

4.3 Examples

The following examples illustrate the breadth of the notion of rationalization by preference structures. Unless otherwise is explicitly stated, $(X, \mathfrak{X})$ stands below for an arbitrarily fixed choice environment.

Example 4.1. Let $\succsim$ be a complete preorder on X. Then, the choice correspondence C on $\mathfrak{X}$ rationalized by the preference structure $(\succsim, \succsim)$ satisfies

$$C(S) = \max(S, \succsim) \quad \text{for every } S \in \mathfrak{X}.$$

We thus see that the choice theory based on preference structures generalizes the standard choice theory that is based on complete preference relations.

Example 4.2. Let $\mathbf{R}$ be a complete binary relation on X. Then, the choice correspondence C on $\mathfrak{X}$ rationalized by the preference structure $(\triangle_X, \mathbf{R})$ satisfies

$$C(S) = \bigcirc(S, \mathbf{R}) \quad \text{for every } S \in \mathfrak{X}.$$

[21] We can relax the continuity assumption to upper semicontinuity here. That is, it is enough to assume in Theorem 4.3 only that both $x^{\uparrow,\succsim}$ and $x^{\uparrow,\mathbf{R}}$ are closed sets in X for every $x \in X$.

[22] The earliest reference for this result seems to be Wallace (1945).

We thus see that the choice theory based on preference structures generalizes the theory of top-cycle choice rules that are commonly used in the theory of social choice and tournaments (cf. Kalai and Schmeidler (1977), Schwartz (1986), Laslier (1997), and Ehlers and Sprumont (2008).)

Example 4.3. Let $\succsim$ be a preorder on X. Then, the choice correspondence C on $\mathfrak{X}$ rationalized by the preference structure $(\succsim, \succsim \sqcup \operatorname{Inc}(\succsim))$ satisfies

$$C(S) = \mathbf{MAX}(S, \succsim) \quad \text{for every } S \in \mathfrak{X}.$$

We thus see that the choice theory based on preference structures generalizes the choice theory that is based on incomplete (but transitive) preference relations. (See, for instance, Eliaz and Ok (2006).)

Example 4.4. Let $\mathcal{P}$ and $\mathcal{P}_{\text{maj}}$ be defined as in Example 3.5. Then, the choice correspondence C on $\mathfrak{X}$ rationalized by the preference structure $(\bigcap \mathcal{P}, \mathcal{P}_{\text{maj}})$ assigns to any feasible set $S \in \mathfrak{X}$ those Pareto optimal outcomes in S that maximizes the transitive closure of the majority voting rule on S. (Here, of course, Pareto optimality and majority voting rule are understood relative to the preference relations in $\mathcal{P}$.)

Example 4.5. (*The Constant Threshold Choice Model*) Let $u : X \to \mathbb{R}$ be any function and take any real number $\varepsilon \geq 0$. Define the binary relation $\mathbf{R}$ on X as x $\mathbf{R}$ y iff $u(x) \geq u(y) - \varepsilon$. Consider first the preorder $\succsim'$ on X defined by $x \succsim' y$ iff either $x = y$ or $u(x) > u(y)$. Then, $(\succsim', \mathbf{R})$ is a preference structure on X, and the choice correspondence C on $\mathfrak{X}$ rationalized by this preference structure is the rational choice model: $C(S) = \arg\max\{u(x) : x \in S\}$ for every $S \in \mathfrak{X}$. Next, consider the preorder $\succsim$ on X defined by $x \succsim y$ iff either $x = y$ or $u(x) > u(y) + \varepsilon$. Then, $(\succsim, \mathbf{R})$ is a preference structure on X – recall Example 3.7 – it is fairly easy to prove that the choice correspondence C on $\mathfrak{X}$ rationalized by this preference structure satisfies

$$C(S) = \{x \in S : \sup u(S) - u(x) \leq \varepsilon\}$$

for every $S \in \mathfrak{X}$. Following Luce (1956), such a correspondence is referred to as a *constant threshold choice model*. Conclusion: Every constant threshold choice model is rationalized by a preference structure.

Example 4.6. (*Choice with Advice*) Let C be the choice correspondence rationalized by a completion by advise $(\succsim, \mathbf{R})$ on X. Take any S in $\mathfrak{X}$ such that $C(S) \neq \varnothing$, and let $\trianglerighteq$ be a complete preorder on X such that $\succsim \subseteq \trianglerighteq$ and x $\mathbf{R}$ y iff (1) holds. Now take any $x, y \in C(S)$. Then, both x and y belong to $\mathbf{MAX}(S, \succsim)$, so either x and y are not $\succsim$-comparable or $x \succsim y \succsim x$. In the former case, (1) entails that x $\mathbf{R}$ y iff $x \trianglerighteq y$. In the latter case, both x $\mathbf{R}$ y and $x \trianglerighteq y$ hold (because $\succsim$ is a subrelation of both $\mathbf{R}$ and $\trianglerighteq$). This proves that $\mathbf{R}|_{\mathbf{MAX}(S,\succsim)} = \trianglerighteq|_{\mathbf{MAX}(S,\succsim)}$. As $\trianglerighteq$ is transitive, therefore, $\operatorname{tran}(\mathbf{R}|_{\mathbf{MAX}(S,\succsim)})$ is none other than the restriction of $\trianglerighteq$ to $\mathbf{MAX}(S,\succsim)$. In view of Proposition 4.1, therefore, we have:

$$C(S) = \max(\mathbf{MAX}(S,\succsim), \trianglerighteq)$$

for every $S \in \mathfrak{X}$ with $C(S) \neq \varnothing$. This accords well with the intuition behind the completion by advise model. When dealing with a choice problem S, an agent whose preferences are captured by this model first identifies the undominated alternatives in S with respect to her inherent (core) preference relation. If there is only one such alternative in S, then her problem is solved by choosing that alternative. If there are more than one such alternative in S, then she is conflicted as to which of these to choose. In that case, she presents her reduced choice problem $\mathbf{MAX}(S,\succsim)$ to her consultant whose preferences are represented by the complete preorder $\trianglerighteq$ on X. This advisor identifies the best alternatives within $\mathbf{MAX}(S,\succsim)$ according to her own preferences, and our principal agent chooses (one of those) alternatives.

Remark. After the important contribution of Manzini and Mariotti (2007) to choice theory, choice correspondences $C : \mathfrak{X} \rightrightarrows X$ of the form $C(S) = \max(\mathbf{MAX}(S,\mathbf{R}_1),\mathbf{R}_2)$, where $\mathbf{R}_1$ and $\mathbf{R}_2$ are binary relations on a finite set X, are often called *sequentially rationalized choice procedures*.[23] Any choice correspondence rationalized by completion by advise is indeed of this form. However, such a choice correspondence is a very particular sequentially rationalizable choice rule because of the tight connection between the two preference relations it uses.

Remark. The spirit of the notion of choice correspondences rationalized by preference structures is certainly in concert with that of sequentially rationalized choice procedures. However, despite the initial appearance of the formula (7), such a choice correspondence is, in general, not a sequentially rationalized choice procedure. This is because, for a feasible set $S \in \mathfrak{X}$, $\operatorname{tran}(\mathbf{R}|_{\mathrm{MAX}(S,\succsim)})$ is not the same relation as $\operatorname{tran}(\mathbf{R})$ in general. Thus, there is no "one" second binary relation used in a choice correspondence rationalized by a preference structure.[24]

The examples above shows that quite a number of choice models are captured by choice correspondences that are rationalized by preference structures. However, not all choice correspondences may be captured in this manner; a choice model induced

[23] Strictly speaking, the model considered by Manzini and Mariotti (2007), the *sequential shortlisting method*, applies only to single-valued choice correspondences. The more general formulation we consider here was studied recently by García-Sanz and Alcantud (2015).

[24] To make this point formally, let X be a finite set and put $\mathfrak{X} := 2^X \backslash \{\varnothing\}$. Proposition 1 of García-Sanz and Alcantud (2015) shows that any sequentially rationalized choice correspondence C on $\mathfrak{X}$ satisfies the so-called *Expansion Property*: $C(S) \cap C(T) \subseteq C(S \cup T)$ for every $S, T \in \mathfrak{X}$. By contrast, a choice correspondence on $\mathfrak{X}$ rationalized by a preference structure need not satisfy this property. For instance, let $X := \{x_1, ..., x_5\}$, $\succsim := \triangle_X \sqcup \{(x_3, x_5), (x_4, x_2)\}$, and consider the complete binary relation $\mathbf{R}$ on X with $\mathbf{R}^> = \{(x_3, x_5), (x_4, x_2), (x_3, x_1), (x_4, x_1)\}$. Then, $(\succsim, \mathbf{R})$ is a preference structure on X. Now, put $S := \{x_1, x_2, x_3\}$ and $T := \{x_1, x_4, x_5\}$. Since each member of S is $\succsim$-maximal, and S is the top cycle in S with respect to $\mathbf{R}$, we have $C(S) = S$, where C is the choice correspondence on $\mathfrak{X}$ rationalized by $(\succsim, \mathbf{R})$. Similarly, $C(T) = T$. Yet, since $\mathbf{MAX}(X,\succsim) = \{x_3, x_4\}$, we have $C(X) = \{x_3, x_4\}$. Thus, x_1 belongs to $C(S) \cap C(T)$, but not to $C(S \cup T)$.

by a preference structure is far from being devoid of predictive power. This issue is explored systematically in the next section, but the following example already provides a simple demonstration.

Example 4.7. Put $X := \{x, y, z\}$, and take any choice correspondence C on $\mathfrak{X} := 2^X \backslash \{\varnothing\}$ such that $x \in C\{x, y\}$ and $\{y\} = C\{x, y, z\}$. To derive a contradiction, suppose C is a choice correspondence rationalized by a preference structure $(\succsim, \mathbf{R})$ on $\mathfrak{X}$. Since $y \in C\{x, y, z\}$, y is $\succsim$-maximal in X, and hence in $\{x, y\}$, so $x \in C\{x, y\}$ implies that $x \mathbf{R} y$ (Proposition 4.1). Since x does not belong to $C\{x, y, z\}$ but y does, therefore, x is not $\succsim$-maximal in X. As $y \succ x$ cannot hold (because $x \in C\{x, y\}$), we thus have $z \succ x$. Then, $y \succ z$ cannot hold, because otherwise $y \succ x$ and hence $\{y\} = C\{x, y\}$, a contradiction. It follows that z is $\succsim$-maximal in X, and hence $y \mathbf{R}^> z$ (because $\{y\} = C\{x, y, z\}$). Thus, $z \succ x \mathbf{R} y$ and $y \mathbf{R}^> z$, but this contradicts the $\succsim$-transitivity of $\mathbf{R}$.

4.4 Properties

Throughout this section, $(X, \mathfrak{X})$ stands for an arbitrarily fixed choice environment, unless otherwise is explicitly specified.

The Weak Axiom of Revealed Preference. We say that a choice correspondence C on $\mathfrak{X}$ satisfies the **Property** α if for every finite $S, T \in \mathfrak{X}$ with $S \subseteq T$,

$$C(T) \cap S \subseteq C(S),$$

and that it satisfies **Property** β if for every finite $S, T \in \mathfrak{X}$ with $S \subseteq T$,

$$x, y \in C(S) \quad \text{and} \quad x \in C(T) \quad \text{imply} \quad y \in C(T).$$

In choice theory, a choice correspondence C on $\mathfrak{X}$ that satisfies both of these properties is said to satisfy the **Weak Axiom of Revealed Preference** (WARP). A fundamental result of this theory says that C satisfies this axiom iff it is rationalizable by a complete prefence relation (in the sense that there exists a total preorder $\succsim$ on X with $C(S) = \max(S, \succsim)$ for every nonempty finite $S \subseteq X$).

Unsurprisingly, a choice correspondence on $\mathfrak{X}$ rationalized by a preference structure $(\succsim, \mathbf{R})$ on X need not satisfy either Property α or Property β.

Example 4.8. Put $X := \{x, y, z\}$, and let $\mathbf{R}$ be the complete binary relation on $\mathfrak{X} := 2^X \backslash \{\varnothing\}$ with $x \mathbf{R}^> y \mathbf{R}^= z \mathbf{R}^= x$. Then, the choice correspondence C on $\mathfrak{X}$ rationalized by the preference structure $(\triangle_X, \mathbf{R})$ fails Property α. (For, $y \in C(X)$ but $\{x\} = C\{x, y\}$.)

Example 4.9. Put $X := \{x, y, z\}$, and let $\succsim$ be the preorder on $2^X \backslash \{\varnothing\}$ with $(x, y), (y, z) \in \text{Inc}(\succsim)$ and $x \succ z$. Then, the choice correspondence C on $2^X \backslash \{\varnothing\}$

rationalized by the preference structure $(\succsim, \succsim \sqcup \mathrm{Inc}(\succsim))$ fails Property β. (For, $\{y,z\} = C\{y,z\}$ but $\{x,y\} = C(X)$.)

Remark. It is an easy exercise to verify that a choice correspondence rationalized by a completion by advise always satisfies Property α. However, such a choice correspondence may fail Property β. (Indeed, any choice correspondence of the form considered in Example 4.3 is a choice correspondence rationalized by a completion by advise.)

Interestingly, a *single-valued* choice correspondence rationalized by a preference structure satisfies the Weak Axiom of Revealed Preference.

Proposition 4.4. *Let C be a choice correspondence on $\mathfrak{X}$ rationalized by a preference structure $(\succsim, \mathbf{R})$ on X. Then, C is single-valued if, and only if, $\mathbf{R}$ is a linear order on X (and hence $C = \max(\cdot, \mathbf{R})$).*

Proof. The "if" part of the assertion is straightforward. Conversely, suppose C is single-valued. If $x \mathbf{R}^{=} y$ for some distinct $x, y \in X$, then neither $x \succ y$ nor $y \succ x$ may hold (because $\mathbf{R}$ extends $\succsim$), and hence, where $S := \{x, y\}$, we have $S = \mathbf{MAX}(S, \succsim)$. Since $x \mathbf{R}^{=} y$, therefore, S is the top-cycle in $\mathbf{MAX}(S, \succsim)$ with respect to $\mathbf{R}$, that is, $S = C(S)$, contradicting C being single-valued. Thus: $\mathbf{R}$ is antisymmetric. Now suppose $\mathbf{R}$ is not transitive. As $\mathbf{R}$ is antisymmetric, this means that $x \mathbf{R}^{>} y \mathbf{R}^{>} z \mathbf{R}^{>} x$ for some $x, y, z \in X$. Notice that $y \succ x$ may not hold (because $\mathbf{R}$ extends $\succsim$). Besides, if $x \succ y$, then $x \mathbf{R} z$ (because $\mathbf{R}$ is $\succsim$-transitive), contradicting $z \mathbf{R}^{>} x$. It follows that x and y are not $\succsim$-comparable. By symmetry, this is true also for y and z, and x and z, so we have $T = \mathbf{MAX}(T, \succsim)$ where $T := \{x, y, z\}$. But then, by the choice of x, y and z, the top-cycle in T with respect to $\mathbf{R}$ is T itself, so $T = C(T)$, contradicting C being single-valued. Conclusion: $\mathbf{R}$ is transitive. (Since $\mathbf{R}$ extends $\succsim$, an immediate application of Proposition 4.1 shows that, when $\mathbf{R}$ is transitive, $\bigcirc(\mathbf{MAX}(S, \succsim), \mathbf{R}) = \max(S, \mathbf{R})$ for every $S \in \mathfrak{X}$. Thus, if $\mathbf{R}$ is a linear order, $C = \max(\cdot, \mathbf{R})$.) ■

Proposition 4.4 shows that the choice theory that is based on preference structures reduces to the standard theory of rational choice in the context of single-valued choice correspondences. (Thus, for instance, the rational shortlisting models of Manzini and Mariotti (2007) and Au and Kawai (2011), as well as the attention filter model of Masatlioglu, Nakajima and Ozbay (2012) and the reason-based choice model of Lombardi (2009), are not captured by this theory.) This is not really surprising. After all, the main goal of the model of preference structures is to capture behavioral traits such as indecisiveness and cyclic choices, and as such, the choice theory that this model induces is primed to make many-valued choice predictions.

Other Behavioral Rationality Properties. We have seen in Example 4.7 that not every choice correspondence is rationalized by a preference structure. This is not an idle example at all. It turns out that we can make quite a few predictions about the choice behavior that is rationalized by a preference structure.

A choice correspondence C on $\mathfrak{X}$ is said to satisfy the **Condorcet Criterion** if for every $S \in \mathfrak{X}$ and $x \in S$,

$$x \in C\{x,y\} \text{ for every } y \in S \quad \text{imply} \quad x \in C(S).$$

On the other hand, if for every finite $S, T \in \mathfrak{X}$ with $S \subseteq T$,

$$x, y \in C(S) \text{ and } x \neq y \quad \text{imply} \quad \{x\} \neq C(T),$$

we say that C satisfies **Property** δ, and if for every finite $S, T \in \mathfrak{X}$ with $S \subseteq T$,

$$C(T) \subseteq S \quad \text{implies} \quad C(S) = C(T),$$

we say that C satisfies the **Aizerman Condition**. The choice behavior that is rationalizable by a preference structure is sure to be consistent with these rationality properties.

Proposition 4.5. *Let C be a choice correspondence on $\mathfrak{X}$ rationalized by a preference structure $(\succsim, \mathbf{R})$ on X. Then, C satisfies the Condorcet Criterion as well as Property δ and the Aizerman Condition.*

Remark. The Aizerman Condition and WARP are identical properties for *single-valued* choice correspondences. Thus, Proposition 4.4 is in fact an immediate corollary of Proposition 4.5.

Proposition 4.5 is useful in distinguishing those choice correspondences rationalized by preference structures from some of the boundedly rational choice correspondences introduced in the recent literature on choice theory. For instance, we see readily that the reference-dependent choice model of Ok, Ortoleva and Riella (2014) is distinct from the present choice model as the former does not satisfy the Condorcet Criterion.

Similarly, the model of choice with limited consideration, introduced recently by Lleras et al. (2017), is distinct from the present choice model. A choice correspondence C on $\mathfrak{X}$ is said to be a *choice correspondence with limited consideration* if $C = \max(\Gamma(\cdot), \trianglerighteq)$ where $\trianglerighteq$ is a complete preorder on X, and Γ is a choice correspondence on $\mathfrak{X}$ that satisfies Property α. (Lleras et al. (2017) refer to Γ as a *competition filter* on $\mathfrak{X}$.) This model appears at first to be more general than the present choice model. But, in general, a choice correspondence with limited consideration does not satisfy the Condorcet Criterion, so it need not be rationalizable by a preference structure.[25]

[25] Neither of the models of Ok, Ortoleva and Riella (2015) and Lleras, et al. (2017) are nested within the present choice model. Indeed, these models do not reduce to the standard rational choice model for single-valued choice correspondences, but the present model does (Proposition 4.4). This is also true for (the original as well as the multi-valued version of) the rationalization model of Cherepanov et al. (2013).

Remark. It is worth noting that $\mathbf{MAX}(\cdot,\succsim)$ is a competition filter on $\mathfrak{X}$. Thus, $\max(\mathbf{MAX}(\cdot,\succsim),\text{tran}(\mathbf{R}))$ is actually a choice correspondence on $\mathfrak{X}$ with limited consideration. However, this correspondence is in general distinct from $\bigcirc(\mathbf{MAX}(S,\succsim),\mathbf{R})$, and indeed, easy examples would show that it need not be rationalizable by a preference structure.

Finally, note that Example 4.5 shows that certain types of *satisficing* rules à la Herbert Simon are captured by the choice theory that is based on preference structures. But not all threshold choice models, let alone all satisficing rules, are rationalizable by preference structures. For example, consider the choice correspondence C on $\mathfrak{X}$ defined by

$$C(S) = \{x \in S : \sup u(S) - u(x) \leq \varepsilon(S)\},$$

where u is any real map on X and $\varepsilon : \mathfrak{X} \to \mathbb{R}_+$ is a function such that $\varepsilon(A) \leq \varepsilon(B)$ for every $A, B \in \mathfrak{X}$ with $A \subseteq B$. (Where X is finite, Frick (2016) has recently provided an axiomatization of such choice correspondences, and called any such correspondence a *monotone threshold choice model*.) It is easily checked that, in general, such a choice correspondence fails Property δ. In view of Proposition 4.5, therefore, we conclude that a monotone threshold model need not be rationalizable by a preference structure.

In passing, we note that a natural question is to determine the behavioral properties that would characterize those choice correspondences rationalized by preference structures. We found this question to be quite challenging. An answer to it is provided in a separate paper by Evren, Nishimura and Ok (2018).

Monotonicity Properties. We have defined the choice correspondence rationalized by a given preference structure $(\succsim,\mathbf{R})$ on X by choosing the top-cycle (with respect to $\mathbf{R}$) among the $\succsim$-maximal elements in any feasible set. This definition seems reasonable and the examples we looked at in Section 4.3 witness its potential. However, one may define this concept in alternative ways (say, by using different solution concepts for tournaments, such as the uncovered set), so it is prudent to examine its consistency properties.

Suppose C is a rationalizable choice correspondence on $\mathfrak{X}$ in the classical sense, that is, $C = \max(\cdot,\succsim)$ for some complete preorder $\succsim$ on X. Let z be a choice from a feasible set S by a decision maker whose choice behavior is modeled by this choice correspondence. If this agent is instead offered the feasible set $S \cup \{x\}$ where x is a new alternative at least as good as z, she would surely deem x choosable from this set: $x \in C(S \cup \{x\})$. It is in this sense that C is *monotonic* with respect to $\succsim$.

Now let C be the choice correspondence rationalized by some preference structure $(\succsim,\mathbf{R})$ on X. We would like to carry out the same query in this case as well, but now monotonicity may be checked either with respect to the core preference relation $\succsim$ of the agent, or her revealed preferences $\mathbf{R}$. Let us first look into the first situation. Suppose $z \in C(S)$ for some $S \in \mathfrak{X}$. Then, if the agent has no doubt in her mind that x is a better alternative than z, that is, $x \succsim z$, one would expect she view x as choosable from $S \cup \{x\}$. The following proposition shows that C possesses this property indeed.

Proposition 4.6. *Let C be a choice correspondence on $\mathfrak{X}$ rationalized by a preference structure $(\succsim, \mathbf{R})$ on X. Then, for any $S \in \mathfrak{X}$,*

$$x \succsim z \in C(S) \quad \textit{implies} \quad x \in C(S \cup \{x\}).$$

Let us now ask the same question with respect to the revealed preference relation $\mathbf{R}$. That is, suppose $z \in C(S)$ for some $S \in \mathfrak{X}$, and that we have observed the agent choose x over z (at least once). Would this agent necessarily deem x choosable from $S \cup \{x\}$? This is less clear than the previous situation. The decision maker may have chosen x over z with serious difficulty, perhaps referring to the preferences of another individual (as in the completion by advise model). Thus, it is possible that some alternatives in S may dominate x, but not z, with respect to the core preferences of the agent, and this may cause x be not chosen from $S \cup \{x\}$ even though z is deemed choosable from S. This may indeed be the case. However, if z is the *only* choice from S, the following proposition shows that C acts still monotonically with respect to $\mathbf{R}$.

Proposition 4.7. *Let C be a choice correspondence on $\mathfrak{X}$ rationalized by a preference structure $(\succsim, \mathbf{R})$ on X. Then, for any finite $S \in \mathfrak{X}$,*

$$x \mathbf{R} z \textit{ and } \{z\} = C(S) \quad \textit{imply} \quad x \in C(S \cup \{x\}).$$

Remark. The requirement "$\{z\} = C(S)$" cannot be relaxed to "$z \in C(S)$" in Proposition 4.7. To see this, let $X := \{x, y, z\}$, and define $\succsim := \triangle_X \sqcup \{(y, x)\}$ and $\mathbf{R} := X^2 \backslash \{(x, y)\}$. Then, $(\succsim, \mathbf{R})$ is a preference structure on X. But for the choice correspondence C on $2^X \backslash \{\varnothing\}$ rationalized by $(\succsim, \mathbf{R})$, we have $C(X) = \{y, z\} = C\{y, z\}$ while $x \mathbf{R} z$.

Propositions 4.6 and 4.7 witness that our definition rationalization by a preference structure $(\succsim, \mathbf{R})$ is duly consistent with how we interpret $(\succsim, \mathbf{R})$.

5 Elicitation of Preference Structures

The classical rational choice theory is built on the hypothesis that the choice correspondence of a rational individual arises from the maximization of a complete preorder (on the set of all choice prospects). Moreover, such a choice correspondence may be rationalized by a *unique* complete preorder. (That is, if $(X, \mathfrak{X})$ is a choice environment and $\max(\cdot, \succsim) = \max(\cdot, \succsim')$ on $\mathfrak{X}$ for some complete preorders $\succsim$ and $\succsim'$ on X, then $\succsim = \succsim'$.) Thus, every complete preorder (interpreted as a preference relation) gives rise to a unique rationalizable choice model, and every rationalizable choice model induces a unique preference relation (that arises from the pairwise choice problems). While trivial, this duality is one of the most useful aspects of rational choice theory.

In this section, we investigate if, and how, such a duality exists for the choice model induced by preference structures. That is, we examine the relation between two preference structures $(\succsim, \mathbf{R})$ and $(\succsim', \mathbf{R}')$ that happen to rationalize the same choice correspondence. The exact analogue of the situation in rational choice theory would be to have $\succsim = \succsim'$ and $\mathbf{R} = \mathbf{R}'$ in this instance. The second of these equations is indeed correct, but mainly because different preorders with the same asymmetric part would declare the same elements as maximal in all feasible sets, the first equation is, in general, false. (For instance, $(\triangle_X, X \times X)$ and $(X \times X, X \times X)$ rationalize the same choice correspondence.) However, in the context of a great variety of choice environments, it is possible to determine "the" largest preference structure – this is the one whose core preference exhibits the least amount of incompleteness – that rationalizes a choice correspondence which is known to be rationalizable by *some* preference structure. Thus, the present choice model too exhibits a useful, and this time entirely nontrivial, duality. Every preference structure gives rise to a unique rationalizable choice model (in the sense of (6) and in the context of a suitably general choice environment), and conversely, every choice model that is rationalizable by a preference structure induces a unique largest preference structure.

Revealed Preferences. Let $(X, \mathfrak{X})$ be any choice environment. Let C be a choice correspondence on $\mathfrak{X}$. There are a number of important preference relations on X that we may define by using C. Perhaps the most obvious is the binary relation $\mathbf{R}_C$ on X defined by

$$x \; \mathbf{R}_C \; y \quad \text{iff} \quad x \in C\{x, y\}.$$

Thus, $x \; \mathbf{R}_C \; y$ means that the agent (with choice correspondence C) would choose x over y when comparing these two alternatives alone. Naturally, we refer to $\mathbf{R}_C$ as the **revealed preference relation** induced by C. This relation is complete (because C is nonempty-valued over finite sets), but without knowing more about C, it is not possible to say anything more about it.

Revealed Core Preferences. The revealed preference relation induced by C arises only through pairwise comparisons of alternatives, and it does not tell us whether or not $x \; \mathbf{R}_C \; y$ entails the superiority of x over y across all feasible sets. To accomodate for this, we define three binary relations on X. First, we define $\succ_C$ on X by

$$x \succ_C y \quad \text{iff} \quad y \notin C(S) \text{ for every } S \in \mathfrak{X} \text{ with } x \in S.$$

In words, $x \succ_C y$ means that the agent would never choose y in any situation in which x is feasible; the presence of x in any menu rules out y being a potential choice. Thus, we refer to $\succ_C$ as the **revealed core dominance** induced by C. This relation is asymmetric, but we do not know much about it otherwise.

Second, we consider the binary relation $\sim_C$ on X defined by $x \sim_C y$ iff

$$x \in C(S \cup \{x\}) \text{ iff } y \in C(S \cup \{y\})$$

and

$$z \in C(S \cup \{x\}) \text{ iff } z \in C(S \cup \{y\})$$

for every $S \in \mathfrak{X}$ and every $z \in S$. In words, $x \sim_C y$ means that x and y are perfect substitutes in that replacing one for the other does not change the choice behavior of the agent in any choice situation. This relation, which we borrow from Riberio and Riella (2017), is called the **revealed core indifference** induced by C. It is a symmetric relation disjoint from $\succ_C$.

Finally, we define $\succsim_C$ as the union of the relations $\succ_C$ and $\sim_C$, and refer to it as the **revealed core preference relation** induced by C. In general, $\succsim_C$ is a subrelation of $\mathbf{R}_C$ distinct from $\mathbf{R}_C$. (In most cases of interest, $\succsim_C$ is not even complete.) But of course, when C satisfies WARP, there is no difference between the revealed core preferences and revealed preferences, that is, $\succsim_C = \mathbf{R}_C$.

The Main (Uniqueness) Theorem. Suppose C is a choice correspondence rationalized by some preference structure $(\succsim, \mathbf{R})$. According to the *interpretation* of preference structures we outlined in Section 3.1, $\mathbf{R}$ is the revealed preference of the involved agent in the sense that if $x \mathbf{R} y$ then we understand that this agent has been observed to choose x over y in a pairwise choice situation. On the other hand, $\mathbf{R}_C$ is the binary relation that is *defined* in exactly this way by using C. Since C is rationalized by $\mathbf{R}_C$, the consistency of our interpretation of $(\succsim, \mathbf{R})$ and the definition of rationalization by a preference structure demands that $\mathbf{R} = \mathbf{R}_C$.

Similarly, according to our interpretation of $(\succsim, \mathbf{R})$, $x \succ y$ means that the agent prefers x over y "obviously," so it stands to reason that she would never choose y in any situation in which x is feasible. But $\succ_C$ is defined in exactly this way by using C. So, the consistency of our models of preference structures and choice correspondences rationalized by them, we must at least have $\succ \subseteq \succ_C$. As precisely the same considerations entail that we have $\sim \subseteq \sim_C$, what we need in fact is that the revealed core preference relation $\succsim_C$ be an extension of the core preference relation $\succsim$.

The following uniqueness theorem, certainly the deepest one among the results of the present paper, establishes that these consistency properties do indeed hold in the context of most choice environments of interest. We state it in the setting of Theorem 4.3.

Theorem 5.1. *Let X be a topological space, and let C be the choice correspondence on $\mathbf{k}(X)$ rationalized by a preference structure $(\succsim, \mathbf{R})$ on X such that $\succsim$ is continuous. Then, $\succsim_C$ is an extension of $\succsim$ and $\mathbf{R} = \mathbf{R}_C$. Moreover, $(\succsim_C, \mathbf{R}_C)$ is a preference structure on X that rationalizes C.*

As we have noted above, the notions of a complete preference relation and a rationalizable choice correspondence are duly (and trivially) consistent with each other. In fact, these two models are equivalent to each other. Theorem 5.1 establishes a similar duality between the notions of a preference structure and a choice correspondence rationalizable by a preference structure. So long as we pick the revealed core preference and revealed preference relations induced by a choice correspondence as "representative," then the models $(\succsim_C, \mathbf{R}_C)$ and C stand dual to each other, provided that C is rationalizable by a preference structure. (That is, $(\succsim_C, \mathbf{R}_C)$ is defined through C, and the choice correspondence that $(\succsim_C, \mathbf{R}_C)$ rationalizes is precisely C.)

In passing, we note that the preference structure $(\succsim_C, \mathbf{R}_C)$ in Theorem 5.1 is quite special. For any two preference structures $(\succsim, \mathbf{R})$ and $(\succsim', \mathbf{R}')$ on X, let us say that $(\succsim, \mathbf{R})$ is **larger than** $(\succsim', \mathbf{R}')$ if $\succsim \supseteq \succsim'$ and $\mathbf{R} \supseteq \mathbf{R}'$. Then, in the context of Theorem 5.1, we see that $(\succsim_C, \mathbf{R}_C)$ is larger than *any* preference structure on X that rationalizes C. Thus:

Corollary 5.2. *In the context of Theorem 5.1, $(\succsim_C, \mathbf{R}_C)$ is the largest preference structure on X that rationalizes C.*

Remark. The topological nature of Theorem 5.1 (and hence of Corollary 5.2) is needed only to guarantee the following property: For every $S \in \mathbf{k}(X)$ and every x in S that is not $\succsim$-maximal, there is a $\succsim$-maximal $y \in S$ with $y \succ x$. Consequently, we can easily formulate Theorem 5.1 in purely order-theoretic terms. In particular, let $(X, \mathfrak{X})$ be any choice environment and $(\succsim, \mathbf{R})$ any preference structure on X. Let $\succcurlyeq$ denote the smallest partial order that extends the asymmetric part of $\succsim$, that is, $\succcurlyeq := \succ \sqcup \triangle_X$. Then, if $(S, \succcurlyeq)$ is a countably chain-complete poset for every $S \in \mathfrak{X}$, the assertions of Theorem 5.1 remain valid.[26]

We conclude with two examples of preference structures on a topological space X. In the first of these, core preference and revealed core preference relations coincide, while in the second, core preference relation is a proper subrelation of its revealed core preference relation.

Example 5.1. (*Revealed Incomplete Preferences*) Following Eliaz and Ok (2006), we say that a preorder $\succsim$ on X is **regular** if for every $(x, y) \in \mathrm{Inc}(\succsim)$, there is a $z \in X$ such that either (i) $(x, z) \in \mathrm{Inc}(\succsim)$ while y and z are $\succ$-comparable; or (ii) $(y, z) \in \mathrm{Inc}(\succsim)$ while x and z are $\succ$-comparable.[27] We let $\succsim$ be a continuous and regular preorder on X, and consider the the preference structure $(\succsim, \succsim \sqcup \mathrm{Inc}(\succsim))$. Then, $\mathbf{MAX}(S, \succsim) \neq \varnothing$ for any $S \in \mathbf{k}(X)$, so, as we have noted in Example 4.3, the choice correspondence C on $\mathfrak{X}$ rationalized by this preference structure satisfies

$$C(S) = \mathbf{MAX}(S, \succsim) \quad \text{for every } S \in \mathfrak{X}. \tag{8}$$

We claim that $\succsim = \succsim_C$. To see this, take any x and y in X. That $\succsim \subseteq \succsim_C$ is obvious. Conversely, take any $x, y \in X$, and suppose $x \succ_C y$. Then, $\{x\} = C\{x, y\}$

[26] Given a poset $(S, \succcurlyeq)$, we say that a nonempty subset T of S is a *$\succcurlyeq$-chain* if $\succcurlyeq|_T$ is complete. In turn, $(S, \succcurlyeq)$ is said to be *countably chain-complete* if every countable $\succcurlyeq$-chain in S has a least upper bound in S with respect to $\succcurlyeq$. Such posets are of great importance for order-theoretic fixed point theory.

[27] For example, where X is an open and convex subset of $\mathbb{R}^n$, and u and v are strictly increasing and continuous real maps on X, the preorder $\succsim$ on X defined by $x \succsim y$ iff $u(x) \geq u(y)$ and $v(x) \geq v(y)$, is regular.

by definition of $\succ_C$, so (8) entails $x \succ y$. Finally, suppose $x \sim_C y$. Then, $\{x, y\} = C\{x, y\}$, so, by (8), either $x \sim y$ or $(x, y) \in \text{Inc}(\succsim)$. To derive a contradiction, suppose the latter case is true. Then, by regularity of $\succsim$, there is a $z \in X$ such that either (i) $(x, z) \in \text{Inc}(\succsim)$, while y and z are $\succ$-comparable; or (ii) $(y, z) \in \text{Inc}(\succsim)$, while x and z are $\succ$-comparable. If (i) holds, then $\{x, z\} = C\{x, z\}$ and $|C\{y, z\}| = 1$ by (8). But, since $x \sim_C y$, $\{x, z\} = C\{x, z\}$ implies $\{y, z\} = C\{y, z\}$, so $|C\{y, z\}| = 2$, a contradiction. Since we can similarly show that (ii) also yields a contradiction, we conclude that $x \sim y$, as we sought.

Example 5.2. (*Revealed Preferences with Imperfect Discrimination*) Assume that X is uncountable, and let $u : X \to \mathbb{R}$ be a continuous surjection. Pick any $\varepsilon > 0$, and consider the preference structure $(\succsim, \mathbf{R})$ on X where $x \mathbf{R} y$ iff $u(x) \geq u(y) - \varepsilon$, and $x \succsim y$ iff either $x = y$ or $u(x) > u(y) + \varepsilon$. We have seen in Example 4.5 that the choice correspondence C on $\mathbf{k}(X)$ rationalized by this preference structure satisfies

$$C(S) = \{x \in S : \max u(S) - u(x) \leq \varepsilon\} \quad \text{for all } S \in \mathfrak{X}. \tag{9}$$

We claim that

$$x \succsim_C y \quad \text{iff} \quad \text{either } u(x) = u(y) \text{ or } u(x) > u(y) + \varepsilon \tag{10}$$

for any $x, y \in X$. (So, we have $\succ = \succ_C$, but $\succsim = \succsim_C$ only if u is injective.) To see this, take any x and y in X. The "if" part of (10) follows readily from the definition of $\succsim_C$ and (9). Conversely, suppose $x \succ_C y$. Then, $\{x\} = C\{x, y\}$ by definition of $\succ_C$, so by (9), $u(x) > u(y) + \varepsilon$. Finally, suppose $x \sim_C y$, and to derive a contradiction, assume $u(x) > u(y)$. Since u is surjective, there exists a $z \in X$ with $u(z) = u(y) - \varepsilon$. Then, $u(x) > u(z) + \varepsilon$, so by (9), we have $z \in C\{y, z\}$ and $\{x\} = C\{x, z\}$, contradicting $x \sim_C y$. By symmetry, $u(y) > u(x)$ cannot hold either, so we conclude that $x \sim_C y$ implies $u(x) = u(y)$, as we sought.

6 Conclusion

In this paper, we proposed a new model to describe the preferences of an economic agent on an arbitrarily given set X of choice prospects. The classical approach is to use a complete preorder, which is typically referred to as a *preference relation*, on X for this purpose. Instead, we suggested the use of two binary relations on X. The first of these, denoted as $\succsim$, aims to capture those rankings of the agent that are (subjectively) "obvious/easy." (This relation is not observable.) As it is hard to imagine that cyclical choice patterns would arise from the "easy" pairwise choice problems, we assume that $\succsim$ is reflexive and transitive, but it need not be complete (because some pairwise choice problems may well be deemed "hard" by the agent). The second binary relation, denoted as $\mathbf{R}$, arises from what we observe the agent choose in the context of all pairwise choice problems. (This relation is observable.) As these include the "hard" ones as well, this relation may exhibit cycles, so it is allowed to be nontransitive, while, naturally, we assume that it is complete. Finally, we posit that $\succsim$ and $\mathbf{R}$ are consistent with each other (as they arise from the preferences of the same agent) in the sense that (i) $\mathbf{R}$ is an extension of $\succsim$, and (ii) $\mathbf{R}$ is transitive with respect to $\succsim$. This way we arrive at what we dubbed here as a *preference structure*.

The entirety of the present paper can be seen as a preliminary investigation of preference structures. First, we have seen that quite a large number of preference models (where the economic agent may be a group of individuals) are captured by preference structures. Among these are the models of incomplete preferences, preferences with imperfect ability of discrimination, regret preferences, and preferences completed by the recommendations of a consultant. Moreover, we have showed that one can represent preference structures by using (sets of) utility functions; this may be useful when working with such structures in economic models. Second, we have outlined how choice behavior that arises from preference structures can be modeled by using the notion of top-cycles. We have seen that this leads to a rich theory of choice which generalizes the classical rational choice theory. The explanatory power of this alternative choice theory is obviously superior to the classical theory. It also has a good deal of predictive power (although, of course, less than the classical theory), for it does satisfy quite a number of rationality properties (such as the Condorcet Criterion, the Aizerman Choice Axiom, etc.). Moreover, this theory has appealing existence and uniqueness properties, paralleling those of the standard rational choice theory. Indeed, the first main result of the present paper established the nonempty-valuedness of choice correspondences that are induced by preference structures, and the second one identified the largest preference structure that rationalizes a choice correspondence that is known to be rationalizable by some such structure. Both of these results hold under very general topological (or order-theoretic) conditions.

We would like to think of the present paper as a beginning of a long research project with numerous avenues to be explored. It would be interesting to revisit the classical consumer theory, this time using preference structures instead of preference relations. Similarly, and even more interestingly, one should investigate how (ordinal) game theory would look like when we model the preferences of the players through preference structures. Then, one should certainly see how the classical theories of decision-making under risk and uncertainty would adapt to preference structures. This would, in turn, open up a whole new set of potential applications. Similarly, it should be interesting to see how one may model time preferences through preference structures, and then revisit the theory of optimal saving. These, and numerous other avenues that remain to be explored, will eventually determine if the notion of preference structures is indeed a useful construct for decision theory at large.

APPENDIX: Proofs

This appendix contains the proofs of the results that were omitted in the body of the text.

Proof of Proposition 3.2

Assume that $(\succsim, \mathbf{R})$ satisfies (2). Put $\trianglerighteq := \operatorname{tran}(\mathbf{R})$, and define the binary relation $\mathbf{S}$ on X by x $\mathbf{S}$ y iff

$$\text{either } x \succsim y \text{ or } [(x,y) \in \operatorname{Inc}(\succsim) \text{ and } x \trianglerighteq y].$$

We wish to show that $\mathbf{R} = \mathbf{S}$. To this end, suppose x $\mathbf{R}$ y. If x and y are $\succsim$-comparable, then we must have $x \succsim y$ (because $y \succ x$ would entail y $\mathbf{R}^{>}$ x, given that $\mathbf{R}$ extends $\succsim$), and hence x $\mathbf{S}$ y. If, on the other hand, $(x,y) \in \operatorname{Inc}(\succsim)$, then x $\mathbf{S}$ y follows readily from the fact that $\mathbf{R} \subseteq \trianglerighteq$. Thus: $\mathbf{R} \subseteq \mathbf{S}$. Conversely, suppose x $\mathbf{S}$ y. If x and y are $\succsim$-comparable, then $x \succsim y$ by definition of $\mathbf{S}$, and hence x $\mathbf{R}$ y. If x and y are not $\succsim$-comparable, then x $\operatorname{Inc}(\succsim) \cap \operatorname{tran}(\mathbf{R})$ y by definition of $\mathbf{S}$, so, by (2), x $\mathbf{R}$ y. Thus: $\mathbf{R} \supseteq \mathbf{S}$.

Now suppose $(\succsim, \mathbf{R})$ is a completion by advise. Then, there is a complete preorder $\trianglerighteq$ on X such that $\succsim \subseteq \trianglerighteq$ and (1) holds. Take any $x, y \in \operatorname{Inc}(\succsim)$ with x $\operatorname{tran}(\mathbf{R})$ y. Then, there exist finitely many $x_1, ..., x_k \in X$ such that x $\mathbf{R}$ x_1 $\mathbf{R}$ $\cdots$ $\mathbf{R}$ x_k $\mathbf{R}$ y. Since $\mathbf{R} \subseteq \succsim \cup \trianglerighteq \subseteq \trianglerighteq$, we then have $x \trianglerighteq x_1 \trianglerighteq \cdots \trianglerighteq x_k \trianglerighteq y$, so given that $\trianglerighteq$ is transitive, we find $x \trianglerighteq y$. Given that x and y are not $\succsim$-comparable, this means x $\mathbf{R}$ y. Conclusion: $\operatorname{Inc}(\succsim) \cap \mathbf{R} \supseteq \operatorname{Inc}(\succsim) \cap \operatorname{tran}(\mathbf{R})$. As the converse containment is obviously true, we are done.

Proof of Theorem 3.3

The proof of the "if" part of the assertion is straightforward, so we focus only on its "only if" part. Let $(\succsim, \mathbf{R})$ be a preference structure on X, and put $\mathbf{T} := \mathbf{R} \backslash \succsim$. We may assume that $\mathbf{T}$ is nonempty, for otherwise there is nothing to prove.

Claim. For every $(x,y) \in \mathbf{T}$, there is a preorder $\succsim_{(x,y)}$ on X such that (i) $\succsim_{(x,y)}$ extends $\succsim$, (ii) $\succsim_{(x,y)} \subseteq \mathbf{R}$, and (iii) $x \succsim_{(x,y)} y$.

Proof of Claim. Fix any $(x,y) \in \mathbf{T}$, and define

$$\succsim_{(x,y)} \quad := \succsim \cup (x^{\uparrow,\succsim} \times y^{\downarrow,\succsim}).$$

That $\succsim_{(x,y)}$ is a preorder with $x \succsim_{(x,y)} y$ is verified routinely. To prove (ii), take any $a, b \in X$ with $a \succsim_{(x,y)} b$. If (a,b) does not belong to $\mathbf{R}$, then it does not belong to $\succsim$ either (because $\mathbf{R}$ is a superrelation of $\succsim$). In that case, then, (a,b) belongs to $x^{\uparrow,\succsim} \times y^{\downarrow,\succsim}$, so we have $a \succsim x$ $\mathbf{R}$ $y \succsim b$, which, by $\succsim$-transitivity of $\mathbf{R}$, implies a $\mathbf{R}$ b, a contradiction. We thus conclude that $\succsim_{(x,y)} \subseteq \mathbf{R}$. It remains to check that $\succsim_{(x,y)}$ extends $\succsim$. Obviously, $\succsim$ is a subrelation of $\succsim_{(x,y)}$. To complete the proof of the claim, then, take any $a, b \in X$ with $a \succ b$. To derive a contradiction, suppose we have $b \succsim_{(x,y)} a$. Then, by definition of $\succsim_{(x,y)}$, (b,a) must belong to $x^{\uparrow,\succsim} \times y^{\downarrow,\succsim}$, and hence, $y \succsim a \succ b \succsim x$. As $\succsim$ is transitive, then, $y \succ x$, and this implies y $\mathbf{R}^{>}$ x (because $\succ$ is a subrelation of $\mathbf{R}^{>}$), but this contradicts the fact that $(x,y) \in \mathbf{R}$. $\|$

For each $(x,y) \in \mathbf{T}$, let $\succsim_{(x,y)}$ be a preorder on X that satisfies the conditions of the claim above, and put

$$\mathcal{P} := \{\succsim_{(x,y)}: (x,y) \in \mathbf{T}\} \cup \{\succsim\}.$$

Then, every element of $\mathcal{P}$ is a preorder on X that extends $\succsim$. As $\succsim$ is a subrelation of $\succsim_{(x,y)}$, and $\succsim_{(x,y)}$ is a subrelation of $\mathbf{R}$, for each $(x,y) \in \mathbf{T}$, it is also plain that $\succsim = \bigcap \mathcal{P}$ and $\bigcup \mathcal{P} \subseteq \mathbf{R}$. On the other hand, if x $\mathbf{R}$ y, then either $x \succsim y$ or $(x,y) \in \mathbf{T}$. In the former case, we obviously have $(x,y) \in \bigcup \mathcal{P}$. In the latter case, $x \succsim_{(x,y)} y$, and we again find $(x,y) \in \bigcup \mathcal{P}$. Conclusion: $\bigcup \mathcal{P} = \mathbf{R}$. Finally, as $\mathbf{R}$ is total, this finding shows that $\bigcup \mathcal{P}$ is total, and our proof is complete.

Proof of Theorem 3.4

Let $(\succsim, \mathbf{R})$ be a preference structure on X such that $\succsim$ is continuous, and define $\mathbf{T}$, and $\succsim_{(x,y)}$ for each $(x,y) \in \mathbf{T}$, as in the proof of Theorem 3.3. Since $\succsim$ is continuous, $x^{\uparrow,\succsim}$ and $y^{\downarrow,\succsim}$ are closed sets in X, so $x^{\uparrow,\succsim} \times y^{\downarrow,\succsim}$ is closed in $X \times X$, for every $(x,y) \in \mathbf{T}$. It follows that $\succsim_{(x,y)}$ is closed in in $X \times X$, for every $(x, y) \in \mathbf{T}$. Now put $\succsim_0 := \succsim$, and define $I := \mathbf{T} \cup \{0\}$ (to serve as an index set). Finally, put $\mathcal{P} := \{\succsim_i : i \in I\}$, and note that every member of $\mathcal{P}$ is a continuous preorder on X. Moreover, as established in the proof of Theorem 3.3, $\succsim_i$ extends $\succsim$ for every $i \in I$, and we have

$$\mathbf{R} = \bigcup_{i \in I} \succsim_i . \tag{11}$$

We now use Theorem 1 of Evren and Ok (2011) for find, for each $i \in I$, a nonempty set $\mathcal{U}_i$ of continuous real maps on X such that

$$x \succsim_i y \quad \text{iff} \quad u(x) \geq u(y) \text{ for every } u \in \mathcal{U}_i,$$

and put $\mathbb{U} := \{\mathcal{U}_i : i \in I\}$. Combining this statement with (11) shows that (3) and (4) hold for every $x, y \in X$. Moreover, if $\mathcal{U}_0(x) \geq \mathcal{U}_0(y)$ for some x and y in X, then, $x \succsim y$, and hence $x \succsim_i y$ (because $\succsim_i$ extends $\succsim$) so $\mathcal{U}_i(x) \geq \mathcal{U}_i(y)$, for each $i \in I$. The analogous argument shows also that $\mathcal{U}_0(x) > \mathcal{U}_0(y)$ implies $\mathcal{U}_i(x) > \mathcal{U}_i(y)$ for each $i \in I$ and $x, y \in X$. We thus conclude that every member of $\mathbb{U}$ is strictly comonotonic with respect to $\mathcal{U}_0$.

The proof of the "if" part of Theorem 3.4 is routine, and hence omitted.

Proof of Proposition 4.1

We will use the following preliminary result to streamline the argument.

Lemma A.1.[28] *Let S be a nonempty set, and $\mathbf{R}$ a complete binary relation on X such that $\bigcirc(S, \mathbf{R}) \neq \varnothing$. Then, x $\mathrm{tran}(\mathbf{R}|_S)$ y for every $x, y \in \bigcirc(S, \mathbf{R})$.*

Proof. Suppose the assertion is false. By completeness of $\mathbf{R}$, then, there is a $y \in \bigcirc(S, \mathbf{R})$ such that $A := \{x \in \bigcirc(S, \mathbf{R}) : x \ \mathrm{tran}(\mathbf{R}|_S)^{>} \ y\}$ is nonempty. Then, A $\mathrm{tran}(\mathbf{R}|_S)^{>}$ z, and hence A $\mathbf{R}|_S^{>}$ z (Lemma 2.1), for every $z \in \bigcirc(S, \mathbf{R}) \backslash A$. But then A is a proper subset of $\bigcirc(S, \mathbf{R})$ which is an $\mathbf{R}$-highset in S, which contradicts $\bigcirc(S, \mathbf{R})$ being the smallest such set. ■

We now turn to the proof of Proposition 4.1. If $\bigcirc(S, \mathbf{R}) \neq \varnothing$, then Lemma A.1, and the fact that $\bigcirc(S, \mathbf{R})$ is an $\mathbf{R}$-highset in S, readily entail that any one element of $\bigcirc(S, \mathbf{R})$ is a maximum element in S with respect to $\mathrm{tran}(\mathbf{R}|_S)$. In other words, nonemptiness of $\bigcirc(S, \mathbf{R})$ entails that $\max(S, \mathrm{tran}(\mathbf{R}|_S))$ is nonempty. Consequently, it is enough to prove the desired equation under the hypothesis that $\max(S, \mathrm{tran}(\mathbf{R}|_S)) \neq \varnothing$. If x belongs to $\max(S, \mathrm{tran}(\mathbf{R}|_S))$ and y is an element of S that does not, then $y \ \mathbf{R} \ x$ cannot hold, because otherwise, $y \ \mathbf{R} \ x \ \mathrm{tran}(\mathbf{R}|_S) \ S$, and hence, $y \ \mathrm{tran}(\mathbf{R}|_S)$ S, which means $y \in \max(S, \mathrm{tran}(\mathbf{R}|_S))$, a contradiction. As $\mathbf{R}$ is complete and $\max(S, \mathrm{tran}(\mathbf{R}|_S))$ is nonempty, therefore, we conclude that $\max(S, \mathrm{tran}(\mathbf{R}|_S))$ is an $\mathbf{R}$-highset in S. To derive a contradiction, suppose there is an $\mathbf{R}$-highset in S, say, B, which is a proper subset of $\max(S, \mathrm{tran}(\mathbf{R}|_S))$. Take any x in $\max(S, \mathrm{tran}(\mathbf{R}|_S))$ which does not belong to B, and fix an arbitrary y in B. As x $\mathrm{tran}(\mathbf{R}|_S)$ y, there exist finitely many $a_1, ..., a_k \in S$ such that $x \ \mathbf{R} \ a_1 \ \mathbf{R} \cdots \mathbf{R} \ a_k \ \mathbf{R} \ y$. Then, since B is an $\mathbf{R}$-highset in S that contains y, it must also contain a_k. Continuing inductively with this argument, we see that each a_i, and in fact, x must belong to B, a contradiction. This completes our proof.

Notational Convention: To simplify our notation, we henceforth adopt the following convention: Where a preorder $\succsim$ on a nonempty set X is given,

$$M(A) := \mathbf{MAX}(A, \succsim) \qquad \text{for any nonempty } A \subseteq X.$$

[28] This result is not new; it was proved by Schwartz (1972) in the case where S is finite.

Notational Convention: For any nonnegative integer k, we put

$$[k] := \{0, ..., k\}.$$

The following result shows that, in the context of a preference structure $(\succsim, \mathbf{R})$, the asymmetric part of $\mathbf{R}$ is transitive relative to the asymmetric part of $\succsim$. We will use this fact at several points below.

Lemma A.2. *Let $(\succsim, \mathbf{R})$ be a preference structure on a nonempty set X. Then,*

$$x \succ y \ \mathbf{R}^{>} \ z \ (\text{or } x \ \mathbf{R}^{>} \ y \succ z) \quad \textit{implies} \quad x \ \mathbf{R}^{>} \ z$$

for every $x, y, z \in X$.

Proof. Take any $x, y, z \in X$ with $x \succ y \ \mathbf{R}^{>} \ z$ but assume that $x \ \mathbf{R}^{>} \ z$ is false. As $\mathbf{R}$ is complete, we then have $z \ \mathbf{R} \ x$. So, $z \ \mathbf{R} \ x \succ y$ and we find $z \ \mathbf{R} \ y$ contradicting $y \ \mathbf{R}^{>} \ z$. The analogous argument shows that $x \ \mathbf{R}^{>} \ y \succ z$ implies $x \ \mathbf{R}^{>} \ z$ as well. ■

Proof of Proposition 4.5

Take any $S \in \mathfrak{X}$, and let x be an element of S such that $x \in C\{x, y\}$ for every $y \in S$. Then, $x \in M(S)$, so, given that $C(S)$ is the top-cycle in $M(S)$ with respect to $\mathbf{R}$, if x did not belong to $C(S)$, we would have $y \ \mathbf{R}^{>} \ x$ for some $y \in M(S)$. But then, for this y, we would have $\{y\} = C\{x, y\}$, a contradiction. Conclusion: C satisfies the Condorcet Criterion.

In the rest of the proof, S and T are arbitrarily fixed finite elements of $\mathfrak{X}$ with $S \subseteq T$. First, suppose that $x \in C(S)$ and $\{x\} = C(T)$. Take any $y \in M(S)\backslash\{x\}$. If $y \in M(T)$, then $x \ \mathbf{R}^{>} \ y$ because $\{x\}$ is the top-cycle in $M(T)$ with respect to $\mathbf{R}$. On the other hand, if y does not belong to $M(T)$, then, $z \succ y$ for some $z \in T$. Since T is finite and $\succsim$ is transitive, it is without loss of generality to assume that $z \in M(T)$. But then, again, $\{x\} = C(T)$ implies that $x \ \mathbf{R}^{>} \ z$. By Lemma A.2, therefore, we find $x \ \mathbf{R}^{>} \ y$ once again. Putting these findings together, we conclude that $x \ \mathbf{R}^{>} \ y$ for every $y \in M(S)\backslash\{x\}$. It follows that $\{x\}$ is the top-cycle in $M(S)$ with respect to $\mathbf{R}$, that is, $\{x\} = C(S)$. Conclusion: C satisfies Property δ.

Next, suppose that $C(T) \subseteq S$. Then, $C(T) \subseteq S \cap M(T) \subseteq M(S)$. Since $x \ \mathrm{tran}(\mathbf{R}|_{C(T)})^{=} \ y$ for every $x, y \in C(T)$ by Proposition 4.1, therefore,

$$x \ \mathrm{tran}(\mathbf{R}|_{M(S)})^{=} \ y \quad \text{for every } x, y \in C(T). \tag{12}$$

Moreover, we have

$$x \ \mathbf{R}^{>} \ y \quad \text{for every } x \in C(T) \text{ and } y \in M(S)\backslash C(T). \tag{13}$$

(Indeed, for any such x and y, suppose $y \ \mathbf{R} \ x$ holds. Then, y cannot be $\succsim$-maximal in T, for otherwise $y \in C(T)$. But if y is not $\succsim$-maximal in T, there is a $z \in T$ with $z \succ y$. As T is finite, we may assume that $z \in M(T)$. Since y is $\succsim$-maximal in S, z cannot belong to S, and hence, it does not belong to $C(T)$. Since x is in the top-cycle in $M(T)$ with respect to R, then, $x \ \mathbf{R}^{>} \ z$, and it follows from Lemma A.2 that $x \ \mathbf{R}^{>} \ y$, a contradiction.) Using (13), we can show that

$$x \ \mathrm{tran}(\mathbf{R}|_{M(S)})^{>} \ y \quad \text{for every } x \in C(T) \text{ and } y \in M(S)\backslash C(T). \tag{14}$$

(Indeed, for any such x and y, suppose $y \ \mathrm{tran}(\mathbf{R}|_{M(S)}) \ x$ holds. Then, there is a positive integer k and $z_0, ..., z_k \in M(S)$ such that $y = z_0 \ \mathbf{R} \ z_1 \ \mathbf{R} \cdots \mathbf{R} \ z_k = x$. Put $\ell := \max\{i \in [k] : z_i \in M(S)\backslash C(T)\}$. (Since $z_0 \in M(S)\backslash C(T)$, this number is well-defined.) Note that $z_k \in C(T)$, so $\ell \in [k-1]$ and $z_{\ell+1} \in C(T)$. But then $z_\ell \ \mathbf{R} \ z_{\ell+1}$ even though $z_{\ell+1} \in C(T)$ and $z_\ell \in M(S)\backslash C(T)$, contradicting (13).)

We are now ready to complete our proof. Clearly, (12) and (14) entail that any x in $C(T)$ maximizes $\text{tran}(\mathbf{R}|_{M(S)})$ over $M(S)$. Thus, by Proposition 4.1, $C(T) \subseteq C(S)$. Conversely, take any $x \in C(S)$. Pick an arbitrary w in $C(T)$, and notice that $w \in C(S)$ (as we now know that $C(T) \subseteq C(S)$). Consequently, $x \ \text{tran}(\mathbf{R}|_{M(S)})^{=} \ w$, and it follows from (14) that $x \in C(T)$. Conclusion: C satisfies the Aizerman Condition.

Proof of Proposition 4.6

Take any $S \in \mathfrak{X}$, and any $x, z \in X$ with $x \succsim z \in C(S)$. We put $T := S \cup \{x\}$; our aim is to show that $x \in C(T)$. Assume first that $x \sim z$ (where $\sim$ is the symmetric part of $\succsim$). In this case, $M(S) = M(T)$, and both x and z belong to $M(T)$. In view of Proposition 4.1, $z \in C(S)$ implies that $z \ \text{tran}(\mathbf{R}|_{M(T)}) \ M(T)$ while $x \ \mathbf{R}^{=} \ z$ (because $\mathbf{R}$ extends $\succsim$). It follows that $x \ \text{tran}(\mathbf{R}|_{M(T)}) \ M(T)$, so, again by Proposition 4.1, $x \in C(T)$.

Assume now that $x \succ z$. In this case x belongs to $M(T)$, but z does not. To derive a contradiction, let us suppose that x does not belong to $C(T)$. By Proposition 4.1, then, there must exist a $y \in M(T)$ such that

$$y \ \text{tran}(\mathbf{R}|_{M(T)})^{>} \ x. \tag{15}$$

Now, since $z \in C(S)$ and $y \in M(T)$, and hence $y \in M(S)$, we have $z \ \text{tran}(\mathbf{R}|_{M(S)}) \ y$, so there is a positive integer k and $w_0, ..., w_k \in M(S)$ such that

$$z = w_0 \ \mathbf{R} \ w_1 \ \mathbf{R} \cdots \mathbf{R} \ w_k = y.$$

Put $\ell := \max\{i \in [k] : x \succ w_i\}$. (This number is well-defined because $x \succ w_0$.) As $y \ \mathbf{R}^{>} \ x$, we cannot have $x \succ w_k$, and hence $\ell \in [k-1]$. But then $w_{\ell+1}, ..., w_k \in M(T)$, and we have

$$x \succ w_\ell \ \mathbf{R} \ w_{\ell+1} \ \mathbf{R} \cdots \mathbf{R} \ w_k = y,$$

so, by $\succsim$-transitivity of $\mathbf{R}$, we find

$$x \ \mathbf{R} \ w_{\ell+1} \ \mathbf{R} \cdots \mathbf{R} \ w_k = y.$$

This means $x \ \text{tran}(\mathbf{R}|_{M(T)}) \ y$, contradicting (15).

Proof of Proposition 4.7

Take any finite $S \in \mathfrak{X}$, and any $x, z \in X$ with $x \ \mathbf{R} \ z$ and $\{z\} = C(S)$. We put $T := S \cup \{x\}$; we wish to show that $x \in C(T)$. Suppose first that x is not $\succsim$-maximal in T. Then $y \succ x$ for some $y \in T$. Since T is finite and $\succsim$ is transitive, it is without loss of generality to assume that $y \in M(T)$. Since $y \succ x \ \mathbf{R} \ z$, we get $y \ \mathbf{R} \ z$ by $\succsim$-transitivity of $\mathbf{R}$. As $\{z\}$ is the top-cycle in $M(S)$, and $y \in M(S)$, we must then have $y = z$. But this means $z \succ x$, and hence $z \ \mathbf{R}^{>} \ x$, contradiction. Conclusion: $x \in M(T)$.

Now, since $x \ \mathbf{R} \ z$, and $\mathbf{R}$ extends $\succsim$, we do not have $z \succ x$. On the other hand, by Proposition 4.6, $x \succsim z$ implies $x \in C(T)$. It remains to consider the case where $(x, z) \in \text{Inc}(\succsim)$. In this case, $z \in M(T)$. Moreover, as $\{z\}$ is the top-cycle in $M(S)$, we have $z \ \mathbf{R}^{>} \ y$ for every $y \in M(S)\backslash\{z\}$. But then, $x \ \mathbf{R} \ z \ \mathbf{R}^{>} \ y$, and hence $x \ \text{tran}(\mathbf{R}|_{M(T)}) \ y$, for every $y \in M(T)\backslash\{x, z\}$. By Proposition 4.1, then, $x \in C(S)$, and we are done.

Proof of Theorem 5.1

We begin with proving two preliminary results that will be needed in the main body of the proof.

Lemma A.3. *Let S be a compact subset of a topological space X, and $\succsim$ a continuous preorder on X. Then, for every $x \in S\backslash M(S)$ there is a $y \in M(S)$ with $y \succ x$.*

Proof. Take any x in $S \backslash M(S)$, and put $T := x^{\uparrow,\succsim} \cap S$. By continuity of $\succsim$, T is a closed subset of S. Since S is compact, therefore, T is a compact set in X. As we have noted in the proof of Theorem 4.3, then, $M(T) \neq \varnothing$. Pick any y in this set. Notice that any $z \in S$ with $z \succsim y$ must belong to T (by transitivity of $\succsim$). It follows that y is $\succsim$-maximal in S as well. And, obviously, $y \succsim x$. Besides, since x is not $\succsim$-maximal in S, we do not have $y \sim x$. ■

The following lemma is stated in the setting of Theorem 5.1.

Lemma A.4. *Let X be a topological space, and let C be the choice correspondence on $\mathbf{k}(X)$ rationalized by a preference structure $(\succsim, \mathbf{R})$ on X such that $\succsim$ is continuous. Take any $x, y, z \in X$ such that $x \ \mathbf{R} \ y \ \mathbf{R}^{>} \ z$ but not $x \succ_C z$. Then, there exist a positive integer k and $w_0, ..., w_k \in X$ such that*

(a) $z = w_0 \ \mathbf{R} \cdots \mathbf{R} \ w_k = x$;
(b) $(w_i, w_j) \in \mathrm{Inc}(\succ)$ *for any distinct* $i, j \in [k]$;
(c) $z \in C\{w_0, ..., w_k\}$.

Proof. Since $x \succ_C z$ is false, there exists a $T \in \mathbf{k}(X)$ such that $x \in T$ and $z \in C(T)$. If $x \in M(T) := \mathbf{MAX}(T, \succsim)$, then Proposition 4.1 says that $z \ \mathrm{tran}(\mathbf{R}|_{M(T)}) \ x$, and thus there is a positive integer k and $w_0, ..., w_k \in M(T)$ such that (a) holds. In addition, (b) holds (because each w_i is $\succsim$-maximal in T) while (c) holds by (a) and Proposition 4.1.

Suppose now that x is not $\succsim$-maximal in T. By Lemma A.3 and Proposition 4.1, there exist a positive integer ℓ and $w'_0, ..., w'_\ell \in M(T)$ such that

$$z = w'_0 \ \mathbf{R} \cdots \ \mathbf{R} \ w'_\ell \succ x.$$

Define $k := \min\{i \in [\ell] : w'_i \succ x\}$. Note that $k > 0$, for, otherwise, $y \ \mathbf{R}^{>} \ z \succ x$, and hence $y \ \mathbf{R}^{>} \ x$ (Lemma A.2), contradicting the hypothesis $x \ \mathbf{R} \ y$. Moreover, $w'_{k-1} \ \mathbf{R} \ w'_k \succ x$ implies $w'_{k-1} \ \mathbf{R} \ x$ (by $\succsim$-transitivity of $\mathbf{R}$). Therefore, setting $w_i := w'_i$ for each $i = 0, ..., k-1$, and $w_k := x$, we obtain (a). Since w_i is $\succsim$-maximal in T for each $i \in [k-1]$, we have $(w_i, w_j) \in \mathrm{Inc}(\succ)$ for any distinct $i, j \in [k-1]$. Moreover, for any $i \in [k-1]$, we may have neither $w_i \succ x$ (by the choice of k) nor $x \succ w_i$ (because w_i is $\succsim$-maximal in T). We cannot have $x \sim_i w_i$ either, because x is not $\succsim$-maximal in T but w_i is. Conclusion: (b) holds. Finally, (c) holds by (a) and Proposition 4.1. ■

We now turn to the proof of Theorem 5.1. As we have noted in Section 5, for any x and y in X, we have $\{x\} = C\{x, y\}$ iff $x \ \mathbf{R}^{>} \ y$, and $\{x, y\} = C\{x, y\}$ iff $x \ \mathbf{R}^{=} \ y$ (because $(\succsim, \mathbf{R})$ rationalizes C). As an immediate consequence, we therefore find that $\mathbf{R}_C = \mathbf{R}$. Moreover, $\succ \subseteq \succ_C \subseteq \mathbf{R}^{>}$ and $\sim_C \subseteq \mathbf{R}^{=}$. We will use these facts below as a matter of routine.

We organize our main argument in terms of several claims.

Claim 1. $\succ_C$ is transitive.

Proof of Claim 1. Take any $x, y, z \in X$ with $x \succ_C y \succ_C z$. To derive a contradiction, suppose that $x \succ_C z$ is false. Then, Lemma A.4 applies (because $\succ_C \subseteq \mathbf{R}^{>}$), so there exist a positive integer k and $w_0, ..., w_k \in X$ such that (a), (b) and (c) of that result hold.

Suppose first that $y \succ w_i$ for some $i \in [k]$. Define $\ell := \max\{i \in [k] : y \succ w_i\}$, and note that $\ell < k$ (because $\ell = k$ means $y \succ x$, and hence $y \succ_C x$, a contradiction). Put $A := \{w_{\ell+1}, ..., w_k, y\}$. By definition of ℓ and (b), $w_{\ell+1}, ..., w_k$ are all $\succsim$-maximal in this set. In fact, y is also $\succsim$-maximal in A, because, since $y \succ w_\ell$, if we had $w_j \succ y$ for some $j \in \{\ell+1, ..., k\}$, we would get $w_j \succ w_\ell$, contradicting (b). Thus: $\mathbf{MAX}(A, \succsim) = A$. Moreover, $y \ \mathbf{R} \ w_{\ell+1}$ for, otherwise, $w_{\ell+1} \ \mathbf{R}^{>} \ y \succ w_\ell$, and by Lemma A.2, we find $w_{\ell+1} \ \mathbf{R}^{>} \ w_\ell$, contradicting (a). By (a) and Proposition 4.1, therefore, $y \in C(A)$, but since $x \in A$, this contradicts $x \succ_C y$.

Suppose now that $y \succ w_i$ is false for every $i \in [k]$. We next consider the possibility that $w_i \succ y$ for some $i \in [k]$. Put $B := \{w_0, ..., w_k, y\}$. By (b), and because $y \succ w_i$ is false for every $i \in [k]$, we see that $w_0, ..., w_k$ are all $\succsim$-maximal in B. So, $\mathbf{MAX}(B, \succsim) = \{w_0, ..., w_k\}$. By (a) and Proposition 4.1, therefore, $z = w_0 \in C(B)$, but since $y \in B$, this contradicts $y \succ_C z$.

In view of the previous two paragraphs, the only remaining case to consider is when neither $y \succ w_i$ nor $w_i \succ y$ holds for any $i \in [k]$. But in this case, $\mathbf{MAX}(S, \succsim) = S$ where $S := \{w_0, ..., w_k, y\}$. Moreover, $x \mathbf{R} y$ (because $x \succ_C y$ and $\succ_C \subseteq \mathbf{R}^>$), so, by (a), we have $z = w_0 \mathbf{R} \cdots \mathbf{R} w_k \mathbf{R} y$, that is, $z \operatorname{tran}(\mathbf{R}|_S) S$. It follows from Proposition 4.1 that $z \in C(S)$, but since $y \in S$, this contradicts $y \succ_C z$. $\|$

Claim 2. $\sim_C$ is an equivalence relation on X.

Proof of Claim 2. This follows from the definition of $\sim_C$ by routine verification. $\|$

Claim 3. $\succ_C$ is $\sim_C$-transitive.

Proof of Claim 3. Take any $x, y, z \in X$ such that $x \succ_C y \sim_C z$. Let S be an arbitrary element of $\mathbf{k}(X)$ with $x \in S$. (We wish to show that $z \notin C(S)$.) If z does not belong to S, there is nothing to prove, so suppose $z \in S$. As $x \succ_C y$ and $x \in S$, we have $y \notin C(S \cup \{y\})$. Thus, since $y \sim_C z$, we have $z \notin C(S \cup \{z\}) = C(S)$. An analogous argument shows that $x \sim_C y \succ_C z$ implies $x \succ_C z$ as well. $\|$

Combining Claims 1, 2 and 3 shows that $\succ_C \sqcup \sim_C$ is transitive. Thus:

Claim 4. $\succsim_C$ is a preorder on X.

Claim 5. $\mathbf{R}$ is $\succsim_C$-transitive.

Proof of Claim 5. Let us first show that $\mathbf{R}$ is $\succ_C$-transitive. Take any $x, y, z \in X$ with $x \mathbf{R} y \succ_C z$. If $x \mathbf{R} z$ is false, then $z \mathbf{R}^> x$ (because $\mathbf{R}$ is complete). One can then check that no two elements of $\{x, y, z\}$ are $\succ$-comparable. Then, by Proposition 4.1, $\{x, y, z\} = C\{x, y, z\}$, but this contradicts $y \succ_C z$. Thus: $\mathbf{R} \circ \succ_C \subseteq \mathbf{R}$. One can similarly prove that $\succ_C \circ \mathbf{R} \subseteq \mathbf{R}$.

We next show that $\mathbf{R}$ is $\sim_C$-transitive. Take any $x, y, z \in X$ with $x \mathbf{R} y \sim_C z$. The second part of this statement entails that $x \in C(\{x\} \cup \{y\})$ iff $x \in C(\{x\} \cup \{z\})$. But $x \in C\{x, y\}$ (because $x \mathbf{R} y$ and $\mathbf{R} = \mathbf{R}_C$), so we find that $x \in C\{x, z\}$, that is, $x \mathbf{R}_C z$. Since $\mathbf{R} = \mathbf{R}_C$, we thus conclude that $\mathbf{R} \circ \sim_C \subseteq \mathbf{R}$. One can similarly prove that $\sim_C \circ \mathbf{R} \subseteq \mathbf{R}$. $\|$

Claim 6. $(\succsim_C, \mathbf{R}_C)$ is a preference structure on X.

Proof of Claim 6. We have noted at the begining of the proof that $\succ_C \subseteq \mathbf{R}^>$ and $\sim_C \subseteq \mathbf{R}^=$, that is, $\mathbf{R}$ is an extension of $\succsim_C$. In view of Claims 4 and 5, and because $\mathbf{R}_C = \mathbf{R}$, therefore, the claim follows. $\|$

Claim 7. $(\succsim_C, \mathbf{R}_C)$ rationalizes C.

Proof of Claim 7. Take any S in $\mathbf{k}(X)$, and put $M(S) := \mathbf{MAX}(S, \succsim_C)$. By definition of $\succ_C$, we have $C(S) \subseteq M(S)$, while, by Proposition 4.1, $\operatorname{tran}(\mathbf{R}|_{C(S)}) = C(S) \times C(S)$. It follows that

$$x \operatorname{tran}(\mathbf{R}|_{M(S)})\ y \quad \text{for every } x, y \in C(S). \tag{16}$$

On the other hand, note that $M(S) \subseteq \mathbf{MAX}(S, \succsim)$ because $\succ \subseteq \succ_C$. Therefore, if $y R x$ holds for some $x \in C(S)$ and $y \in M(S)$, we have $y \in \mathbf{MAX}(S, \succsim)$, and it follows that $y \in C(S)$. Conclusion:

$$x \mathbf{R}^> y \quad \text{for every } x \in C(S) \text{ and } y \in M(S) \backslash C(S). \tag{17}$$

In turn, this implies that

$$x \operatorname{tran}(\mathbf{R}|_{M(S)})^> y \quad \text{for every } x \in C(S) \text{ and } y \in M(S) \backslash C(S). \tag{18}$$

(Indeed, take any $x \in C(S)$ and $y \in M(S) \backslash C(S)$, and suppose $y = w_0 \mathbf{R} \cdots \mathbf{R} w_k = x$ for some $k \in \mathbb{N}$ and $w_0, ..., w_k \in M(S)$. By (17), w_1 cannot belong to $C(S)$, so $w_1 \in M(S) \backslash C(S)$. Then, again by (17), w_2 cannot belong to $C(S)$, and continuing by induction, we see that $w_{k-1}\ \mathbf{R}\ w_k = x$ cannot hold.) Combining (16) and (18), we find that $C(S) = \max(M(S),\text{tran}(\mathbf{R}|_{M(S)}))$. In view of Proposition 4.1, this proves the claim. $\|$

Given Claims 6 and 7, the proof of Theorem 5.1 will be complete if we can show that $\succsim_C$ is an extension of $\succsim$. Since we know that $\succ\ \subseteq\ \succ_C$, it remains to prove that $\sim\ \subseteq\ \sim_C$. To this end, take any $x, y \in X$ with $x \sim y$. Let S be an arbitrarily fixed element of $\mathbf{k}(X)$, and put $T_x := \mathbf{MAX}(S \cup \{x\}, \succsim)$ and $T_y := \mathbf{MAX}(S \cup \{y\}, \succsim)$. Since $\succsim$ is a preorder, it is readily checked that $x \sim y$ implies $T := T_x \cap S = T_y \cap S$.

Now assume $x \in C(S \cup \{x\})$. Then, x is $\succsim$-maximal in $S \cup \{x\}$, and hence, y is $\succsim$-maximal in $S \cup \{y\}$ (because $\succsim$ is a preorder and $x \sim y$), that is, $y \in T_y$. Moreover, by Proposition 4.1, x $\text{tran}(\mathbf{R}|_{T_x})$ T_x. So, if $z \in T \subseteq T_x$, there is a positive integer k such that $x\ \mathbf{R}\ w_0\ \mathbf{R} \cdots \mathbf{R}\ w_k = z$ for some $w_0, ..., w_k \in T_x$. Here, we can in fact assume that $w_0, ..., w_k \in T$ without loss of generality. (For, otherwise, $w_i = x$ for some $i \in [k]$. Then, set $l := \max\{i \in [k] : w_i = x\}$, and we have $x\ \mathbf{R}$ $w_{l+1}\ \mathbf{R} \cdots \mathbf{R}\ w_k = z$ with $w_i \in T$ for all $i = l+1, \ldots, k$.) Since $y \sim x$ and $\mathbf{R}$ is $\succsim$-transitive, this implies $y\ \mathbf{R}\ w_0\ \mathbf{R} \cdots \mathbf{R}\ w_k = z$. Thus, y $\text{tran}(\mathbf{R}|_{T_y})$ T. If $z \in T_y \backslash T$, then $z = y$, and we obviously have y $\text{tran}(\mathbf{R}|_{T_y})$ z. Conclusion: y $\text{tran}(\mathbf{R}|_{T_y})$ T_y. By Proposition 4.1, this yields $y \in C(S \cup \{y\})$, as we sought. By symmetry, therefore, we conclude: $x \in C(S \cup \{x\})$ iff $y \in C(S \cup \{y\})$.

Next, take any z in S with $z \in C(S \cup \{x\})$. Then, z is $\succsim$-maximal in $S \cup \{x\}$, and hence, it is $\succsim$-maximal in $S \cup \{y\}$ (because $\succsim$ is a preorder and $x \sim y$), that is, $z \in T_y$. Moreover, by Proposition 4.1, z $\text{tran}(\mathbf{R}|_{T_x})$ T_x. Now, take any $w \in T_y$. Then, we can show that there is a positive integer k with

$$z\ \mathbf{R}\ w_0\ \mathbf{R}\ w_1\ \mathbf{R}\ \cdots\ \mathbf{R}\ w_k\ \mathbf{R}\ w \qquad \text{for some } w_0, \ldots, w_k \in T_x. \tag{19}$$

(Indeed, if $w \in T \subseteq T_x$, then z $\text{tran}(\mathbf{R}|_{T_x})$ w, and (19) follows at once. If $w \in T_y \setminus T$, then $y = w \in T_y$, which implies $x \in T_x$ and hence z $\text{tran}(\mathbf{R}|_{T_x})$ $x \sim y$. So, there is a positive interger k such that $z\ \mathbf{R}\ w_0\ \mathbf{R}\ w_1\ \mathbf{R}\ \cdots\ \mathbf{R}\ w_k\ \mathbf{R}\ x \sim y$ for some $w_0, \ldots, w_k \in T_x$. This again implies (19) by $\succsim$-transitivity of $\mathbf{R}$.) For the sequence $w_0, ..., w_k$ in (19), define

$$w_i' := \begin{cases} w_i, & \text{if } w_i \neq x \\ y, & \text{if } w_i = x, \end{cases}$$

for each $i \in [k]$, and note that $z\ \mathbf{R}\ w_1'\ \mathbf{R} \cdots \mathbf{R}\ w_k'\ \mathbf{R}\ w$ by $\succsim$-transitivity of $\mathbf{R}$. Since $w_i' \in T_y$ for each $i \in [k]$, this shows that z $\text{tran}(\mathbf{R}|_{T_y})$ w. It then follows from the arbitrary choice of w that z $\text{tran}(\mathbf{R}|_{T_y})$ T_y, that is, $z \in C(S \cup \{y\})$, as we sought. By symmetry, therefore, we conclude: $z \in C(S \cup \{x\})$ iff $z \in C(S \cup \{y\})$ for every $z \in S$. In view of the arbitrariness of S, this establishes that $\sim\ \subseteq\ \sim_C$, and completes the proof of Theorem 5.1.

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