University of Pennsylvania **Economics 705, Fall 2018**

Prelim Examination

Friday August 10, 2018. Time limit: 150 minutes

Instructions:

(i) The number of points for each question is given below.

(ii) The exam is closed book and closed notes.

(iii) To receive full credit for your answers you have to explain your calculations. For example, when you use the weak law of large numbers (WLLN) or the central limit theorem (CLT), state it clearly.

(iv) You may state additional assumptions.

Question 1: Linear Regression Model (30 Points)

Consider the linear regression model

$$y_i = x_i\theta + u_i, \quad u_i|x_i \sim iid(0,1), \quad x_i \geq \epsilon > 0, \quad \mathbb{E}[x_i^2] = Q, \quad i = 1, \ldots, n$$

Moreover, the x_i's are also independent across i. Notice that we assumed that the conditional variance of u given x is known to be one. Moreover, $k = 1$ and both x_i and θ_i are scalar.

(i) (5 Points) Derive the likelihood function under the assumption that the u_i's are in fact normally distributed.

(ii) (5 Points) Derive the maximum likelihood estimator $\hat{\theta}$.

(iii) (2 Points) Show that the MLE $\hat{\theta}$ is consistent even if the observations are not normally distributed.

(iv) (5 Points) Based on the assumption of normality, define the likelihood-ratio test statistic for the null hypothesis that $\theta = \theta_0$.

(v) (10 Points) Derive the large sample distribution of the likelihood ratio statistic (under the null hypothesis, without assuming that the data are normally distributed).

(vi) (3 Points) What are the acceptance and rejection regions for the LR test given a type-I error of $\alpha = 0.05$?

Question 2: Weighted Least Squares Estimator (10 Points)

Let (y_i, x_i), $i = 1, ..., n$ be a i.i.d. sample with $E(y_i|x_i) = x_i'\beta$. Let $X \in R^{n\times k}$ and $Y = R^{n\times 1}$ be the matrix of observations. The weighted least square estimator of β is

$$\widetilde{\beta} = (X'WX)^{-1}X'WY,$$

where $W = diag\{w_1, ..., w_n\}$ and $w_i = x_{ji}^{-2}$, where x_{ji} is the jth element of x_i. Note that when $x_{ji} = 1$, this becomes the OLS estimator.

In what situations do you think $\widetilde{\beta}$ would perform better than OLS?

Question 3: Method of Moments (30 Points)

Let x be a random variable with $\mu = E(x)$ and $\sigma^2 = var(x)$. Define

$$g(x|m, s) = \begin{pmatrix} x - m \\ (x - m)^2 - s \end{pmatrix}.$$

(i) (5 points) Show that $Eg\,(x|m, s) = 0$ if and only if and only if $m = \mu$ and $s = \sigma^2$.

(ii) (5 points) Suppose you have i.i.d. observations $x_1, ..., x_n$. Derive the method of moments estimators for μ and σ^2, call them $\widehat{\mu}$ and $\widehat{\sigma}^2$.

(iii) (5 points) Show $\widehat{\mu}$ and $\widehat{\sigma}^2$ are consistent.

(iv) (10 points) Derive the joint asymptotic distributions of $(\widehat{\mu}, \widehat{\sigma}^2)$.

(v) (5 points) Now suppose we are also interested in $\gamma = E(x - E(x))^3$. How to estimate γ by the method of moments estimator?

Question 4: Minimum Distance Estimation (30 Points)

Take a linear regression model

$$Y = X\beta + U,$$

where X is $n \times k$ and β is $k \times 1$. Assume the data is i.i.d. and $\mathbb{E}[U_i|X_i] = 0$.

Suppose the parameter β is known to satisfy the restrictions

$$\beta = g(\theta),$$

for some unknown parameter $\theta \in \Theta \subset R^m$ and some known function $g(\cdot) : R^m \to R^k$, where $m < k$.

The parameter space Θ is *compact*. The function $g(\theta)$ is *monotonic* and twice continuously differentiable.

Let $\widehat{\beta}$ denote the LS estimator for β. One can also estimate θ from $\widehat{\beta}$ using the minimum distance criterion. Specifically, for some symmetric positive definite $k \times k$ matrix W, define

$$C(\theta) = \left(\widehat{\beta} - g(\theta)\right)' W \left(\widehat{\beta} - g(\theta)\right)$$

and define $\widehat{\theta}$ as the minimizer of $C(\theta)$:

$$\widehat{\theta} = \underset{\theta \in \Theta}{\arg\min} C(\theta).$$

(i) (5 Points) How to identify θ?

(ii) (5 Points) What is the probability limit of $C(\theta)$ as $n \to \infty$?

(iii) (5 Points) How to show $\widehat{\theta}$ is consistent for θ?

(iv) (10 Points) Find the asymptotic distribution for $\widehat{\theta}$.

(v) (5 Points) What is the optimal choice of W for $\widehat{\theta}$?

END OF EXAM