Level-k DSGE and Monetary Policy

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Abstract

This paper develops a new framework of level-k DSGE for monetary policy analysis. Incomplete markets are introduced to guarantee the eductive stability of the equilibrium. k=1.265 is identified using growth and inflation expectations from the Michigan Survey of Consumers, capturing the missing indirect channels and weakened direct channels in households' forecast rules, as well as the wedge between expectations and reality. With more empirical support, the model is applied in four different issues related to monetary policy. First, the real effects of monetary shocks are accumulative when reasoning levels are low and households' planning horizons are short. Second, inflation targeting confuses households with the dynamics of GDP, and hence weakens demand stabilization. Third, in liquidity traps, recovery can be fast, slow or even impossible, depending on how deep the recession is. Forth, the initial effect of monetary shocks in far future is dampened by level-k, but the cumulative effects across time can be large. When $k \to +\infty$, the level-k DSGE reduces to a basic New Keynesian model as in Galí (2015).

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1 Introduction

The prevalent DSGE models for monetary policy analysis[1] usually impose two assumptions on expectations. First, agents know the aggregate states. Second, agents know the aggregate law of motion. Although these assumptions are often rejected by data, these is less consensus on what alternatives[2] we should make, and in what circumstances it is necessary. In this paper, I relax the second assumption, and develop a new framework of level-k DSGE, based on the idea proposed by Farhi and Werning (2017). The essence of level-k is to turn off the subtle general equilibrium effects in expectations that arise from more than $k-1$ layers of feedbacks. This framework is appealing because it has straight forward setup, transparent mechanisms, sharp empirical support, and multiple important applications.

My first contribution is to lay the foundation for level-k DSGE models. The standard setup of level-k in games (Crawford, Costa-Gomes, and Iriberri, 2013) that level-k players best reply to level-(k-1) is no long sufficient in a DSGE environment for two reasons. First, the ex post budget balance requires agents to observe the prices when making decisions, so that a temporal equilibrium (Grandmont, 1977) structure needs to be imposed as in Farhi and Werning (2017). Second, there can potentially be endogenous state variables[3]. As a result, states determine expectations, expectations drive decisions and decisions affect states. This loop needs to be addressed using a recursive structure. Perceiving all others as one level below is formalized as taking the actual equilibrium objects one level below as the perceived equilibrium objects for decision making. All forecasts are made based on rules as functions of the aggregate states. In addition, the model also allows for non-integer levels by assuming a level-1.3 agent perceiving 30% of the others as level-1 and the rest as level-0[4]. Therefore, the level-k DSGE I propose is more general than those in García-Schmidt and Woodford (2016); Farhi and Werning (2017);

[1] Galí (2015) provides the benchmark for small-scale DSGE, while Christiano, Eichenbaum, and Evans (2005) and Smets and Wouters (2007) provie that for medium-scale DSGE.

[2] Limited attention relaxes the first assumption (Mankiw and Reis, 2002; Woodford, 2003; Angeletos and La'O, 2013; Gabaix, 2014; Afrouzi, 2017), while least-squared learning relaxes the second (Marcet and Sargent, 1989; Evans and Honkapohja, 1998; Milani, 2007; Eusepi and Preston, 2011).

[3] For instance, the interest rate responses to inflation and output gaps is inertial, and level-0 is specified to anchor their decisions to the last period.

[4] 30% percentage of the others are level-1 is different from for 30% probability that all others are level-1, due to recursive structure of the model.

Iovino and Sergeyev (2017).

Another issue is "Eductive Stability". The recursive level-k equilibrium is well defined only if it converges to the recursive competitive equilibrium when $k \to +\infty$. This property does not hold in real business cycles (Evans, Guesnerie, and McGough, 2017), because households respond too aggressively to interest rate expectations. As a result, the interest rate implied by households' decisions would exceed the initial range of interest rate expectations. In my model, I introduce incomplete markets to make the planning horizons of households shorter so that they would respond less aggressively. The extent of market incompleteness is disciplined by the average marginal propensity of consume (MPC hereafter) of the unconstrained (non hand-to-mouth) households in Kaplan and Violante (2014).

The equilibrium conditions can no longer be formulated as intertemporal conditions between current period and next period as in recursive competitive equilibrium. The recursive form is more general, and allows for a state space representation. Multiple steps ahead forecasts can be obtained only by iterating on aggregate states using the perceived aggregate law of motion. This feature resembles "Infinite-horizion Learning" as in Eusepi and Preston (2011). However, due to the incomplete markets, the model has a simple purely forward looking representation only in a special case when the steady state real interest rate is zero. All theoretical results in this paper are obtained under this assumption. In the appendix, a more general algorithm is provided for the computation of the full model.

The specification of level-0 and timing is also worth mentioning. Level-0 households and firms fully anchor their spending the pricing decisions to the last period. This is consistent with the principle in level-k games that level-0 agents should be as "dumb" as possible. In order to circumvent multiple equilibria generated by the simultaneity between decisions making and expectation formation. I specify the timing such that agents do not update their expectations until they finish making decisions at the end each period, even if they have previously observed some new information before. This timing arrangement is neutral under rational expectations, because no observations are informative to the agents if they have already anticipated them using the correct aggregate law of motion. This setup implicitly assumes away other forms of learning, and isolates the eductive learning.

My second contribution is to unravel the essence of level-k DSGE. Level-k DSGE is not the only model to allow for a wedge between the perceived aggregate law of motion and the actual aggregate law of motion. Gabaix (2017) also explicitly specifies a pair of perceived and actual objects. Level-k DSGE is sharper in the sense that it predicts in which dimension the wedge would be larger. The model implied that if $k \in [1, 2]$, then in all one-step ahead forecasts, indirect channels are missing and direct channels are weaker. In multiple-step ahead forecasts, this result will still hold approximately if k is close to 1.

Let's use simple notations to demonstrate these channels. Denote the real GDP as Y, nominal interest rate as R, and inflation as Π. Then, $R \to Y$ and $Y \to \Pi$ are direct channels that are at least partially understandable by agents with $k > 1$, while $Y \to \Pi \to Y$ and $R \to Y \to \Pi$ are indirect ones, and partially understandable only by agents with $k > 2$.

These results only exist in one-step ahead forecasts, because in multiple-step ahead forecasts, the perceived law of motion starts to play a role. Take $Y \to \Pi \to Y$ for example. Households do not understand the effect of inflation expectation of the others, but as their own inflation expectations go up, they anticipate interest rate movement, and hence understand $Y \to \Pi \to Y$ through $R \to Y$. Quantitatively, this channel is much weaker than the previous one.

Another interesting feature of level-k is that the forecasts of the forecasts of others are not identical to the direct forecasts on the same objects. As the forecast horizon becomes longer, this difference becomes larger. It implies that the infinite horizon assumption is not innocuous. This feature is not unique in level-k. Expectations modeled as forecast rules anchored to the past is likely to be incompatible to the common knowledge of individual rationality which can be used to achieve the irrelevance of planning horizons. Angeletos and Lian (2016) and Farhi and Werning (2017) have also mentioned the importance of planning horizons for firms and households respectively, while my paper uncovers the essence of it more generally.

My third contribution is to provide sharp disciplines for level-k DSGE. Despite various empirical works in level-k games to identify the parameter k (Camerer, Ho, and Chong, 2004), there are no empirical counterparts in macroeconomics. There may be two reasons for this. First, it seems that models in which households keep on learning from the past is more plausible for business cycle related issues in normal times. Second, it not clear how to identify k.

I argue that level-k is still relevant given historical data for learning, as expectations data show that households' forecast rules are systematically biased. There are two possible reasons why learning does not make households more rational as players are in dynamic experiments. First, the payoffs of decisions are much less clear along business cycles. Second, recalling and analyzing historical data for business cycles is much more costly.

I show that k can be identified by exploiting the co-movements between macroeconomic data and forecast data. Hence, the DSGE structure actually helps identify level-k by providing dynamics of multiple macroeconomic variables. The missing co-movements for indirect channels identify $k \in [1, 2]$, while the weakened direct channels help identify the exact value of k.

In the estimation, I adopt Bayesian approach, and use five time series including quarterly GDP growth rate, quarterly inflation, quarterly federal fund rate, one-year ahead growth forecasts, and one-ahead inflation forecasts. k is jointly estimated with other parameters, and also fits the macroeconomics data. The prior of k is set to be very dispersed and cover the interval of $[1, +\infty)$. Yet, the posterior has a tight 95% confidence interval $[1.197, 1.327]$.

I also discuss why level-k is not observationally equivalent to limited attention. This argument can be justified by showing that households' backcasts are highly correlated with the actual growth rate with no delays. In addition, interest rates and output levels both have asymmetric predictive power for growth and inflation, so that they are unlikely to be fully ignored by the households. Coibion, Gorodnichenko, and Kamdar (2017) summarizes that many models of expectation formation share the same patterns in expectations. Level-k model does not seem to suffer from this problem.

My forth contribution is to explore the potential of level-k DSGE for monetary policy analysis. A natural concern for the necessity of level-k is why not simply using exogenous forecast rules as in Cole and Milani (2017). I argue that the level-k approach is useful because (1) it provides a way of inspecting mechanisms, and (2) it induces endogenous changes of forecast rules and output dynamics that are consistent with data under different monetary regimes. In another word, level-k has both internal interpretation and external validation. These two good features are reflected in the following four applications.

This first application is to study the transmission of monetary shocks. SVAR evidence shows that lower interest rates are associated with faster output growth, instead of higher output levels. In level-k model, this is a natural feature of expectations anchoring under which the real effects of monetary shocks are accumulative across time. Both a low value of k and short planning horizon are required to get this result. With either high k or long planning horizon, the output responses to monetary shocks become much weaker and less persistent.

The second application is to study the trade-off between inflation targeting and GDP target. In the data, growth forecasts before the Volker regime are mean reverting while after that they are uncorrelated with output levels. This is consistent with the switch of monetary regimes from more GDP targeting to more inflation targeting. Households are more confused about the output dynamics under inflation targeting because it is more difficult to understand. As a result, inflation target is less effective for demand stabilization.

The third application is to study the protracted liquidity trap. This phenomenon is a natural implication of level-k. During the liquidity trap when the natural rate of interest is permanently low, the expected recovery will become very slow. A certain amount of lower inflation expectations will induce the households to consume less in reality then their expectations. As a result, the actual recovery can be very slow. Deeper downturns imply more time to recover in expectations, and hence can induce slower or even no recovery. This model implication is supported by the evidence that households do not expect the economy to bounce back during the Great Recession.

The forth application is to compare the effects of forward guidance with the literature. Unlike the theoretical work of McKay, Nakamura, and Steinsson (2016); Angeletos and Lian (2016); Farhi and Werning (2017) which show how forward guidance is dampened, it is accumulative in empirically relevant level-k models, and can ultimately have a large aggregate effect, if the monetary authority has full commitment power in forward guidance.

The rest of the paper is organized as the following. Section 2 presents some motivating facts. Section 3 develops the level-k DSGE model. Section 4 characterizes it. Section 5 estimates it. Section 6 applies the model to monetary policy analysis. Section 7 concludes.

2 Motivation

A key implication of the full information (know the states) rational expectations (know the law of motion) assumption is that agents are able to use all available information to make forecasts. This is in sharp contrast to the expectation data in Michigan Survey of Consumers (MSC hereafter) as part of the co-movements between macroeconomic variables are missing in expectations. In this section, I will take growth expectations as an example, and show that higher output does not induce lower growth rate as in macroeconomic data, but higher interest rate does. Inflation expectations have opposite patterns and will be discussed in Section 5.

2.1 Growth Expectations

MSC does not provide direct measures for households' growth rate expectations. Instead, it ask two related questions. The first one is on the current business conditions compared with a year ago. The second on in the expectation change in business conditions in a year. The answers are qualitative and contains three options: better, worse and do not know. An index is constructed for each based on the distribution of the answers. A brief review of the results are shown in Figure 1.

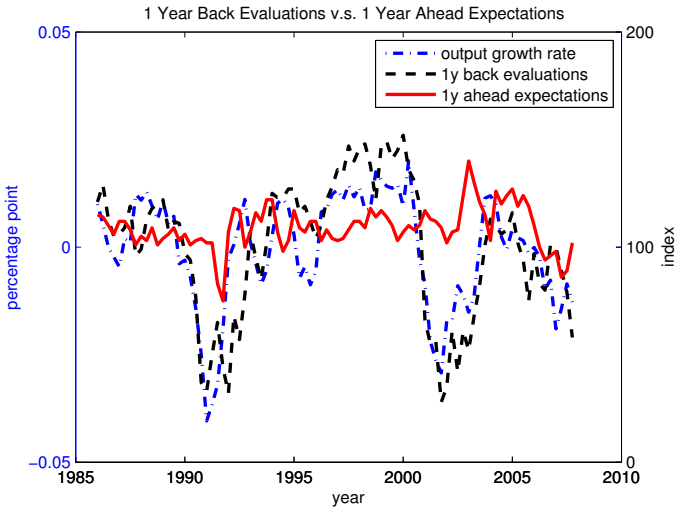

Figure 1: Growth Expectations in MSC 1985-2007

Figure 1 shows that the one year back evaluations on business conditions are highly correlated to with output growth rate, while the one year ahead expectations are have much smaller variations. The co-movements between one year back evaluations and growth rate are very strong, but the co-movements between one year ahead expectations and it are very weak.

In order to impute growth rate expectations from these indexes, I assume that households can observe what happens during the last year perfectly, and use this to rescale the index so that it has a unit of output growth rate. This approach will only amplify the co-movement between expectations and growth rate, and hence obtain the upper bound of rationality in growth expectations. The following equation helps construct the rescaling.

$$\frac{\Delta\text{expected growth rate}}{\Delta\text{forward index}} = \frac{\Delta\text{ex post growth rate}}{\Delta\text{backward index}}.$$

I also compare the expectations with realizations in output levels. Figure 2 shows the results. Even though the volatility if growth expectations has been maximized, the correlation between one year ahead expectations and the output level is still very weak.

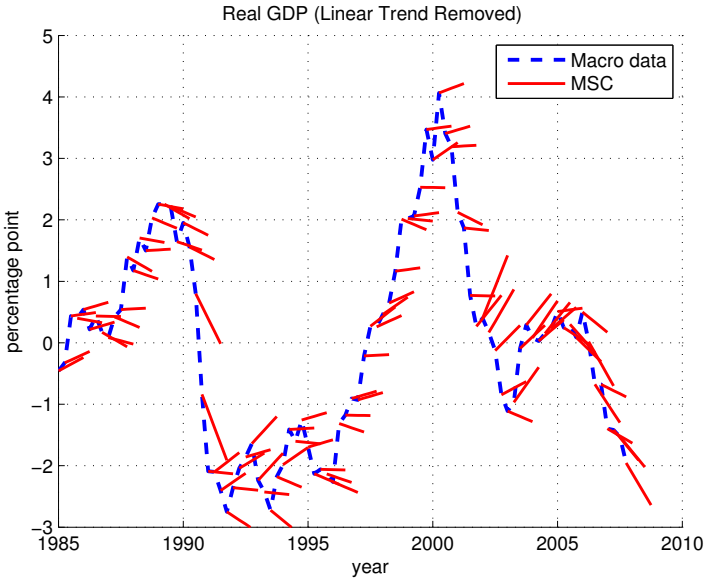

Figure 2: Output Expectations in MSC

2.2 Asymmetric Reasoning

This subsection examines what predicts growth expectations in more details. The results is summarized by multiple regression in the Table 1. All data are linearly or log-linear detrended. The first 6 columns of this table compare the predictability of the expected one year output growth with its realized counterpart, while the last column shows the predictability of forecast errors.

Table 1: Asymmetric Reasoning

	$\hat{y}^e_{t+4} - \hat{y}_t$	$\hat{y}_{t+4} - \hat{y}_t$	$\hat{y}^e_{t+4} - \hat{y}_t$	$\hat{y}_{t+4} - \hat{y}_t$	$\hat{y}^e_{t+4} - \hat{y}_t$	$\hat{y}_{t+4} - \hat{y}_t$	$\hat{y}^e_{t+4} - \hat{y}_{t+4}$
$\hat{y}_t$	-0.018	-0.429^{***}	0.053^{**}	-0.269^{***}	0.073^{***}	-0.187^{*}	0.305^{***}
$\hat{r}_t$					-0.148^{***}	-0.194^{**}	-0.105
$\hat{\pi}_t$							0.742^{***}
$\hat{y}_t - \hat{y}_{t-4}$	0.031	0.454^{***}					
$\hat{r}_t - \hat{\pi}^{e,year}_{t+4}$			-0.121^{***}				
$\hat{r}_t - \hat{\pi}^{year}_{t+4}$				-0.063			
Obs.	84	84	88	88	88	88	88
R^2	0.014	0.303	0.190	0.143	0.358	0.176	0.287

* Standard errors are in the parenthesis. *,**,*** denote significance at the 10%, 5% and 1% levels.

"Asymmetric Reasoning" is summarized by the following patterns. Column 1-2 show that the one year ahead realized growth rate is strongly correlated with the current output level and output growth, while the expected growth rate has almost no correlation. Column 3-6 show that both the ex ante real interest rate and the actual nominal interest rate can predict the expected output growth, and the predictability is no weaker than that for the realized output growth. Column 7 shows the predictability of forecast errors. It indicates that households are over-confident on the expected output growth when the current output level and inflation is high, while conditional on interest rate, this over-confidence is insignificant.

Discussion. These results indicate that households use the interest rate related information but not output related information to forecast output growth. This pattern does not suffer from the concern that households are less aware of the aggregate output because otherwise, the way how I construct the expected output growth will induce no predictability of growth expectations at all. Now, two questions naturally arises. How should this pattern be modeled? What implications it has for monetary policy?

3 The Level-k DSGE Framework

I consider a basic New Keynesian DSGE model as in Galí (2015). The competitive equilibrium is replaced by a recursive level-k equilibrium. Two additional features are introduced so that the model can work well under recursive level-k equilibrium, but is not observationally different under competitive equilibrium. First, expectations are updated only at the end of each period when agents observe all of the economic conditions and aggregate shocks. Second, households are subject to idiosyncratic preference shocks and borrowing constraints in a way that do not induce wealth heterogeneity.

3.1 Households

There are a measure one of infinitely-lived households with a constant discount factor β, and an aggregate stochastic demand wedge η^d multiplied to it. They choose consumption c, labor supply ℓ, and real bond b each period. The instantaneous utility function is $u(c, \ell)$.

Timing. Within each period, events happen in the following order:
(1) Households inherit observations and expectations from the end of last period.
(2) Households observe the current gross inflation rate Π. The real net wealth a is determined by the last period real bond position b_-, the last period nominal gross interest rate R_-, and the current gross inflation rate Π through $a = b_- R_- / \Pi$.
(3) Households are hit by idiosyncratic preference shocks $\zeta \in \{1, \overline{\zeta}\}$, with $\overline{\zeta} \geq 1$ and transition probability $\Pr(\zeta|\zeta_-) = \lambda_{\zeta|\zeta_-}$. This yields an unconditional probability $\Pr(\zeta) = \lambda_\zeta$.
(4) Households observe the real wage rate W and receive a lump sum transfer of real dividend D. They consume c and supply labor ℓ to get utility $\zeta[u(c) - v(\ell)]$. The rest of the budget is saved in real bond b, with borrowing constraint $b \geq 0$ only for $\zeta = \overline{\zeta}$.
(5) Households observe the aggregate output Y as well as all aggregate shocks $\epsilon = (\epsilon^r, \epsilon^d, \epsilon^z)$ that drives the current gross nominal interest rate R, as well as the next period demand wedge and technology level $(\eta^{d\prime}, \eta^{z\prime})$.
(6) Households form new expectations based on these observations[5].

[5] I assume that households do not update expectations until the end of each period using the newly arrived information of $\{\Pi, W, D\}$ to aviod a twoway feedback between current equilibrium outcomes and expectations

States and Equilibrium Objects. Denote the vector of aggregate states as S. Individual states include the real net wealth a and idiosyncratic preference shocks ζ. Level-k households take as given the following equilibrium objects:
(1) perceived and actual real wage rate $\{W^{e,(k)}(S), W^{(k)}(S)\}$,
(2) perceived and actual real dividend $\{D^{e,(k)}(S), D^{(k)}(S)\}$,
(3) perceived gross nominal interest rate $R^{e,(k)}(S, \epsilon^r)$,
(4) perceived gross inflation rate $\Pi^{e,(k)}(S)$,
(5) perceived aggregate law of motion $H^{e,(k)}(S, \epsilon)$.

Households' Problems. Households have perceptions on their equilibrium policy and value functions $\{c^{e,(k)}, \ell^{e,(k)}, b^{e,(k)}, V^{h,e,(k)}\}$ on (ζ, a, S) for the future. These functions solve

$$\begin{aligned} V^{h,e,(k)}(\zeta, a, S) = \max_{\{c,\ell,b\}} &\{\zeta[u(c) - v(\ell)] + \beta \exp(\eta^d) \cdot \mathbb{E}[V^{h,e,(k)}(\zeta', a', S')|(\zeta, a, S)]\} \\ s.t.\ b =& - c + W^{e,(k)}(S)\ell + D^{e,(k)}(S) + a, \\ b \geqslant& 0 \text{ when } \zeta = \overline{\zeta}, \\ a' =& b \cdot R^{e,(k)}(S, \epsilon^r)/\Pi^{e,(k)}(S'), \\ S' =& H^{e,(k)}(S, \epsilon). \end{aligned}$$

In the current period, households observe the actual real wage and dividend $\{W^{(k)}(S), D^{(k)}(S)\}$, have continuation values given by the perceived value function $V^{h,e,(k)}$. The actual equilibrium policy and value functions $\{c^{(k)}, \ell^{(k)}, b^{(k)}, V^{h,(k)}\}$ on (ζ, a, S) solve

$$\begin{aligned} V^{h,(k)}(\zeta, a, S) = \max_{\{c,\ell,b\}} &\{\zeta[u(c) - v(\ell)] + \beta \exp(\eta^d) \cdot \mathbb{E}[V^{h,e,(k)}(\zeta', a', S')|(\zeta, a, S)]\} \\ s.t.\ b =& - c + W^{(k)}(S)\ell + D^{(k)}(S) + a, \\ b \geqslant& 0 \text{ when } \zeta = \overline{\zeta}, \\ a' =& b \cdot R^{e,(k)}(S, \epsilon^r)/\Pi^{e,(k)}(S'), \\ S' =& H^{e,(k)}(S, \epsilon). \end{aligned}$$

on future. As a result, expectations are inferred exclusively from the aggregate states. Under full information rational expectations, expectations inferred from the aggregate states are consistent with equilibrium outcomes, hence $\{\Pi, W, D\}$ provide no additional information. In my specification of level-k reasoning, there are ex post forecast errors in $\{\Pi, W, D\}$ but households do not learn from them.

This can be viewed as a recursive representation of temporary equilibrium (Grandmont, 1977). Conditioning on the aggregate states, households may end up with decisions and values different from what they expect, because the actual equilibrium wage and dividend may be different from the expected ones. By specification, neither actual nor expected budget is violated.

Aggregation. With the assumption that bond b is in zero net supply, and $b \geq 0$ binds only when $\zeta = \overline{\zeta}$, an initially degenerate wealth distribution will always induce equilibrium with degenerate wealth distribution. Start with a degenerate wealth distribution with $b = 0$ for all households. Those with $\zeta = \overline{\zeta}$ would like to consume more and work less, but are borrowing constrained. For others with $\zeta = 1$, the equilibrium wage and dividend[6] will clear the goods market such that they would like to choose $b = 0$. This can be formalized in the following.

Assumption 1. *$u(\cdot)$ and $-v(\cdot)$ are twice continuously differentiable, strictly increasing and strictly concave.*

Assumption 2. $B^{(k)}(S) = \lambda_1 b^{(k)}(1, 0, S) + (1 - \lambda_1) b^{(k)}(\overline{\zeta}, 0, S) = 0$.

Proposition 1. *The aggregate equilibrium objects $\{B^{(k)}, L^{(k)}, C^{(k)}\}$ satisfy*

$$\begin{aligned} B^{(k)}(S) =& b^{(k)}(1, 0, S) = b^{(k)}(\overline{\zeta}, 0, S) = 0, \\ L^{(k)}(S) =& \ell^{(k)}(1, 0, S) = \ell^{(k)}(\overline{\zeta}, 0, S), \\ C^{(k)}(S) =& c^{(k)}(1, 0, S) = c^{(k)}(\overline{\zeta}, 0, S) = W^{(k)}(S)\ell^{(k)}(1, 0, S) + D^{(k)}(S). \end{aligned}$$

Variety Demand. Each household's consumption c is made of a measure one of varieties $\{c_j\}$ with $j \in [0, 1]$. Assume Dixit-Stiglitz aggregator[7] for both varieties and their prices

$$c = \left(\int_0^1 c_j^{\frac{\varepsilon - 1}{\varepsilon}} dj \right)^{\frac{\varepsilon}{\varepsilon - 1}}, \qquad P = \left(\int_0^1 p_j^{1-\varepsilon} dj \right)^{\frac{1}{1-\varepsilon}},$$

where $\varepsilon > 1$. This yields the individual variety demand and its aggregate counterpart

$$c_j = \left(\frac{p_j}{P} \right)^{-\varepsilon} c, \qquad C_j = \left(\frac{p_j}{P} \right)^{-\varepsilon} C.$$

[6] Interest rate will not help clear any market because by the timing specification, it is known only ex post.
[7] A market structure of final good producers and intermediate good producers is not necessary here.

3.2 Firms

There are a measure one of infinitely-lived firms. Firm $j \in [0,1]$ produces variety j exclusively, using labor input n_j via a linear technology $Y_j = \exp(\eta^z) n_j$. They set prices to attract variety demand C_j and produce $Y_j = C_j$. Profits are discounted by real interest rates.

Timing. Within each period, events happen in the following order:
(1) A random fraction θ of the firms are drawn to keep their previous prices no changed.
(2) Each firm j from the other fraction $1-\theta$ sets a price before observing $\{\Pi, W, C_j\}$.
(3) Each firm $j \in [0,1]$ observes (W, C_j), produces C_j, and pays W to each unit of labor input.
(4) Profits are paid as dividend to households. The aggregate dividend is D.
(5) Firms observe aggregate output Y and all aggregate shocks ϵ.
(6) Firms form new expectations based on these observations.

States and Equilibrium Objects. The firms that reset prices choose $p^a = p/P_-$ as the new price over the previous aggregate price. The firm not reseting prices have individual state $p_-^n = p_-/P_-$. Level-k firms take as given the following equilibrium objects:
(1) perceived gross inflation rate $\Pi^{e,(k)}(S)$,
(2) perceived real wage rate $W^{e,(k)}(S)$,
(3) perceived aggregate output $Y^{e,(k)}(S)$,
(4) perceived nominal gross interest rate $R^{e,(k)}(S, \epsilon^r)$,
(5) perceived aggregate law of motion $H^{e,(k)}(S, \epsilon)$.

Firms' Problems. The equilibrium actual policy function and perceived value functions $\{p^{a,(k)}, V^{a,e,(k)}, V^{n,e,(k)}(p_-^n, \cdot)\}$ on S solve the problem of the firms that reset prices

$$
\begin{aligned}
V^{a,e,(k)}(S) = \max_{p^a} & \underbrace{\left(\frac{p^a}{\Pi^{e,(k)}(S)} - \frac{W^{e,(k)}(S)}{\exp(\eta^z)}\right)}_{\text{relative price} - \text{marginal cost}} \underbrace{\left(\frac{p^a}{\Pi^{e,(k)}(S)}\right)^{-\varepsilon} Y^{e,(k)}(S)}_{\text{variety demand}} \\
& + \mathbb{E}\left[\frac{\Pi^{e,(k)}(S')}{R^{e,(k)}(S,\epsilon^r)}\left(\theta V^{n,e,(k)}\left(\frac{p^a}{\Pi^{e,(k)}(S)}, S'\right) + (1-\theta) V^{a,e,(k)}(S')\right) \middle| S\right], \\
s.t.\ S' = & H^{e,(k)}(S, \epsilon),
\end{aligned}
$$

as well as the bellman equation of the firms that do not reset prices

$$
\begin{aligned}
V^{n,e,(k)}(p_-^n, S) = &\left(\frac{p_-^n}{\Pi^{e,(k)}(S)} - \frac{W^{e,(k)}(S)}{\exp(\eta^z)}\right)\left(\frac{p_-^n}{\Pi^{e,(k)}(S)}\right)^{-\varepsilon} Y^{e,(k)}(S) \\
&+ \mathbb{E}\left[\frac{\Pi^{e,(k)}(S')}{R^{e,(k)}(S,\epsilon^r)}\left(\theta V^{n,e,(k)}\left(\frac{p_-^n}{\Pi^{e,(k)}(S)}, S'\right) + (1-\theta)V^{a,e,(k)}(S')\right)\middle| S\right], \\
s.t.\ S' = &H^{e,(k)}(S,\epsilon).
\end{aligned}
$$

Aggregation. The aggregate inflation and dividend are determined by

$$
\begin{aligned}
\Pi^{(k)}(S) &= \left(\theta + (1-\theta)p^{a,(k)}(S)^{1-\varepsilon}\right)^{\frac{1}{1-\varepsilon}}, \\
D^{(k)}(S) &= Y^{(k)}(S) - \frac{W^{(k)}(S)Y^{(k)}(S)}{\exp(\eta^z)}.
\end{aligned}
$$

3.3 Monetary Policy, Market Clearning and Aggregate Shocks

Monetary Policy. Denote steady state nominal gross interest rate, gross inflation rate and output as $\{R_{SS}, \Pi_{SS}, Y_{SS}\}$. Assume these are common knowledge to households. The monetary authority chooses a nominal gross interest rate based on these objects.

In normal times, the monetary authority follows a conventional Taylor Rule

$$
\frac{R^{(k)}(S,\epsilon^r)}{R_{SS}} = \frac{R^{Taylor,(k)}(S,\epsilon^r)}{R_{SS}} = \left(\frac{R_-}{R_{SS}}\right)^{\rho_r}\left(\frac{\Pi^{(k)}(S)}{\Pi_{SS}}\right)^{(1-\rho_r)\phi_\pi}\left(\frac{Y^{(k)}(S)}{Y_{SS}}\right)^{(1-\rho_r)\phi_y}\exp(\sigma_r\epsilon^r).
$$

where ρ_r denotes the level of interest rate smooth, (ϕ_π, ϕ_y) denotes the response coefficients of nominal interest rate to inflation and output, ϵ^r denotes the i.i.d. federal fund rate shock, and σ_r denotes its standard deviation. If not specified, monetary policy follows Taylor Rule.

When discussing liquidity trap related issues, monetary policy rule must satisfy $R^{(k)}(S,\epsilon^r) \geq 0$. In addition, there can be two monetary regimes. Before the economy recovers, $R^{(k)}(S,\epsilon^r) = 0$ regardless of $R^{Taylor,(k)}(S,\epsilon^r) \geq 0$. After the recovery, $R^{(k)}(S,\epsilon^r) = R^{Taylor,(k)}(S,\epsilon^r)$ again.

Market Clearing. In each period when agents are making decisions, supply must be equal to demand in goods market, labor market and bond market.

$$\begin{aligned} Y^{(k)}(S) =& C^{(k)}(S), \\ L^{(k)}(S) =& Y^{(k)}(S)/\exp(\eta^z), \\ 0 =& B^{(k)}(S). \end{aligned}$$

Aggregate Shocks. The aggregate shocks $\epsilon = (\epsilon^r, \epsilon^d, \epsilon^z)$ and processes (η^d, η^z) satisfy

$$\begin{aligned} \epsilon^r \sim& \mathcal{N}(0,1), \\ \eta^{d\prime} =& \rho_d \eta^d + \sigma_d \epsilon^d, \quad \epsilon^d \sim \mathcal{N}(0,1), \\ \eta^{z\prime} =& \rho_z \eta^z + \sigma_z \epsilon^z, \quad \epsilon^z \sim \mathcal{N}(0,1). \end{aligned}$$

Use $(\eta^{d\prime}, \eta^{z\prime}) = H^{\eta}(\eta^d, \eta^z, \epsilon)$ for short notation.

3.4 Recursive Level-k Equilibrium

This subsection establishes a Recursive Level-k Equilibrium. Both households and firms are level-k. Perceiving others as one level below is equivalent to using the actual equilibrium objects of this level as perceived equilibrium objects. Therefore, the recursive level-k equilibrium can be established by iterating on the equilibrium objects. The Recursive Level-k Equilibrium nests the definition of level-k equilibrium in Farhi and Werning (2017) as a special case, and allows for a state space representation of the model.

Level-0 Initialization. Level-0 needs to be specify to initialize the iteration on equilibrium objects. In order to capture the non-rational expectations on output growth and inflation, it is natural to assume that level-0 agents' expenditure and level-0 firms' price choice are fully anchored to the last period[8]. In order to avoid too much complexity and flexibility, no other assumptions are made to distort the level-0 equilibrium.

[8] Fehr and Tyran (2008) and Gill and Prowse (2016) have provided experimental evidence showing that in a dynamic setting, players' decisions are indeed anchored to the past.

Assumption 3. *The level-1 expectations are given by the following statements.*
(1) Level-0 households do not change consumption expenditure, and the level-0 firms do not reset prices. Aggregate labor supply $L^{e,(1)}$ satisfies the production technology.

$$(Y^{e,(1)}(S), \Pi^{e,(1)}(S), L^{e,(1)}(S)) = (\underbrace{Y_-}_{\textit{level-0}}, 1, \underbrace{Y_-/exp(\eta^z)}_{\textit{technology}}).$$

(2) $W^{e,(1)}$ ensures the optimality of $L^{e,(1)}$, and $D^{e,(1)}$ satisfies the aggregate resource constraint.

$$(W^{e,(1)}(S), D^{e,(1)}(S)) = (\underbrace{v_\ell(L^{e,(1)}(S))/u_c(Y_-)}_{\textit{optimal labor supply}}, \underbrace{Y_- - W^{e,(1)}(S)L^{e,(1)}(S)}_{\textit{aggregate resource constraint}}).$$

(3) Given $\{\Pi^{e,(1)}, Y^{e,(1)}\}$, $R^{e,(1)}$ satisfies Taylor Rule

$$\frac{R^{e,(k)}(S,\epsilon^r)}{R_{SS}} = \left(\frac{R_-}{R_{SS}}\right)^{\rho_r} \left(\frac{\Pi^{e,(k)}(S)}{\Pi_{SS}}\right)^{(1-\rho_r)\phi_\pi} \left(\frac{Y^{e,(k)}(S)}{Y_{SS}}\right)^{(1-\rho_r)\phi_y} \exp(\sigma_r \epsilon^r),$$

in normal times, and is consistent with the actual monetary policy rules in other regimes.
(4) The perceived law of motion $H^{e,(1)}$ is consistent with the perceived equilibrium objects and H^η in stationary environments, and also with time index in non-stationary environments.

Level-k Updating. For $\forall j \geq 1$ and $j \in \mathbb{N}_+$, expectations are updated according to

$$(Y^{e,(j+1)}, \Pi^{e,(j+1)}, W^{e,(j+1)}, D^{e,(j+1)}, R^{e,(j+1)}, H^{e,(j+1)}) = (Y^{(j)}, \Pi^{(j)}, W^{(j)}, D^{(j)}, R^{(j)}, H^{(j)}).$$

For $\forall k \geq 1$, and $j \leq k \leq j+1$, expectations are defined[9] as

$$(Y^{e,(k)}, \cdots) = (j+1-k)(Y^{e,(j)}, \cdots) + (k-j)(Y^{e,(j+1)}, \cdots).$$

The solution to the temporary equilibrium[10] with given expectations yields the mapping

$$T : (Y^{e,(k)}, \cdots) \longrightarrow (Y^{(k)}, \cdots).$$

[9] It extends integer k in most level-k models in a way different from García-Schmidt and Woodford (2016), but has more transparent implications in more complex models and when $k \in [1, 2]$.
[10] The equilibrium for each level-k is a temporary equilibrium (Grandmont, 1977).

Aggregate States. In the Taylor Rule monetary regimes, the aggregate state can be summarized by $S = (Y_-, R_-, \eta^d, \eta^z)$[11], while in other regimes, it may also include the time index.

Recursive Level-k Equilibrium.

Definition 1. *The Recursive Level-k Equilibrium consists of a set of*
(1) actual policy and value functions $\{c^{(k)}, \ell^{(k)}, b^{(k)}, V^{h,(k)}\}$ *on* (ζ, a, S) *for households,*
(2) perceived policy and value functions $\{c^{e,(k)}, \ell^{e,(k)}, b^{e,(k)}, V^{h,e,(k)}\}$ *on* (ζ, a, S) *for households,*
(3) actual policy function and value functions $\{p^{a,(k)}, V^{a,e,(k)}, V^{n,e,(k)}(p^n_-, \cdot)\}$ *on* S *for firms,*
(4) actual aggregate objects $\{Y^{(k)}, \Pi^{(k)}, W^{(k)}, D^{(k)}, C^{(k)}, L^{(k)}, B^{(k)}, R^{(k)}(\cdot, \varepsilon^r)\}$ *on* S*, and*
(5) perceived aggregate ones $\{Y^{e,(k)}, \Pi^{e,(k)}, W^{e,(k)}, D^{e,(k)}, C^{e,(k)}, L^{e,(k)}, B^{e,(k)}, R^{e,(k)}(\cdot, \varepsilon^r)\}$ *on* S*, such that*
1. *individual policy and value functions solve the corresponding problems,*
2. *individual policy functions are consistent with the law of motion of individual states,*
3. *actual individual policy functions are consistent with actual aggregate objects,*
4. *monetary policy follows a given policy rules,*
5. *market clearing conditions hold, and*
6. *perceived aggregate objects are determined by level-k updating.*

Definition 2. *Replacing the level-k updating with consistency between the actual and perceived objects yields the Recursive Competitive Equilibrium.*

By definition, when $k \to +\infty$, if the Recursive Level-k Equilibrium converges, it must converge to the Recursive Competitive Equilibrium.

State Space Representation. Definition 1 naturally allows for a transition equation

$$S_{t+1} = H^{(k)}(S_t, \epsilon_t).$$

The actual output, as an example of the actual aggregate variables, is given by

$$Y_{t+s} = Y^{(k)}(S_{t+s}),$$

[11] The distribution of variety prices does not have aggregate effects as firms are assumed to produce regardless of their profit margins.

while the output forecast, as an example of the perceived aggregate variables, is given by

$$Y^{e,(k)}_{t+1+s|t} = Y^{e,(k)}(S^{e,(k)}_{t+s|t}),$$

where $S^{e,(k)}_{t+s|t} = S_t$ for $s = 0$. For $s \geq 1$,

$$S^{e,(k)}_{t+s|t} = H^{e,(k)}(S^{e,(k)}_{t-1+s|t}, \epsilon_{t-1+s}).$$

Here $S^{e,(k)}_{t+s|t}$ is a random variable depending on the sequence of shocks $\{\epsilon_{t+s}\}$.

Infinite Horizon Representation. When all aggregate shocks are turned off, the s period ahead forecast of the aggregate states becomes[12]

$$S^{e,(k)}_{t+s|t} = (H^{e,(k)})^s(S_t).$$

As a result, all aggregate variables become perfect foresighted in forecasts, although the forecasts are biased. The whole sequence of forecasts at different horizons uniquely pins down the optimal decisions of households and firms, and hence the equilibrium. Similar results can be obtained if shocks are not turned off and the model is certainty equivalent. One example is the linear approximation of the equilibrium.

Iterated Expectations. When there is no aggregate uncertainty, agents would still revise their forecasts after they have new observations that are inconsistent with what they expected ex ante. When they try to forecast the forecasts of the others, the law of motion that is used will be downgraded by one level. This will not be equivalent to the direct forecasts, and the deviation will be amplified by the time horizon.

Example 1. *Consider* $k \in \mathbb{N} \cap [2, +\infty)$*, then we have*

$$\begin{aligned} S^{e,(k)}_{t+1+s|t} =& (H^{(k-1)})^{s+1}(S_t), \\ S^{e,(k)}_{(t+1+s|t+1)|t} =& S^{e,(k-1)}_{t+1+s|t+1} = (H^{(k-2)})^s H^{(k-1)}(S_t). \end{aligned}$$

[12] Eusepi and Preston (2011) also iterate foreward on the perceived aggregate law of motion to obtain long horizon expectations.

4 Theoretical Analysis

This section characterizes the model in the following aspects. First, it transforms the linearized equilibrium condition into a pair of interdependent dynamic beauty contests. Second, it derives a set of testable forecast rules for agents with $k \in [1, 2]$. Third, it discusses the role of planning horizons. Forth, it addresses the "eductive stability" issue. These results constitute the first theory of level-k DSGE model.

4.1 Dynamic Beauty Contests

Demand Side. Impose two assumptions to make the results more transparent.

Assumption 4. *Consumption and labor supply are additively separable in households' utility function. The intertemporal elasticity of substitution ω and the frisch elasticity of labor supply ξ are both constant. The utility function is parameterized as*

$$u(c, \ell) = \frac{c^{1-\frac{1}{\omega}}}{1-\frac{1}{\omega}} - \frac{\ell^{1+\frac{1}{\xi}}}{1+\frac{1}{\xi}}.$$

Assumption 5. *The steady state net real interest rate is zero, i.e. $R_{SS} = \Pi_{SS}$.*

Assumption 6. *$\lambda_{\bar{\zeta}|1}\bar{\zeta}$ is large enough, so that assuming $b^{e,(k)}(\bar{\zeta}, a, S) = 0$ is innocuous.*

Use $(\hat{y}, \hat{\pi}, \hat{r}, \hat{w}, \hat{\tau}, \hat{c}, \hat{\ell})$ to denote the real aggregate output, inflation rate, federal fund rate, real wage rate, price markup, real aggregate consumption, and aggregate labor supply, respectively. In particular, price markup $\hat{\tau} = (D - D_{SS})/(W_{SS}L_{SS})$ is used to make notation simpler. Use time index such that the period $t-1$ information set is $\hat{s}_t = (\hat{y}_{t-1}, \hat{r}_{t-1}, \eta^d_t, \eta^z_t)$.

Lemma 1. *In the recursive level-k equilibrium, output satisfies*

$$\hat{y}^{(k)}_t = (1-\gamma)\sum_{s=0}^{+\infty}\gamma^s \hat{y}^{e,(k)}_{t+1+s|t-1} - \omega\sum_{s=0}^{+\infty}\gamma^s(\hat{r}^{e,(k)}_{t+s|t-1} - \hat{\pi}^{e,(k)}_{t+1+s|t-1} + \mathbb{E}_{t-1}\eta^d_{t+s}), \tag{1}$$

where $\gamma = 1 - \sqrt{\frac{\lambda_{\bar{\zeta}|1}\bar{\zeta}}{\lambda_{\bar{\zeta}|1}\bar{\zeta}+1-\lambda_{\bar{\zeta}|1}}}$.

Equation (1) is simple and resembles its counterpart in Angeletos and Lian (2016). Yet, it is not straight forward in at least two aspects. First, the parameter γ, which is affected by both the frequency of binding borrowing constraints and the size of idiosyncratic shocks, plays a similar role as the discount factor β. Second, the expectations on labor market variables can be summarized by the expectations on output, so that wage, dividend and labor supply does not show up in the equation for equilibrium output.

The determinant of the parameter γ indicates that when borrowing constraints are more likely to bind, households have shorter planning horizons. Farhi and Werning (2017) uses positive mortality rate to capture this effect in their theoretical framework. In their model, households react less to long horizon events because higher return rate makes the present value of future incomes smaller. This assumption is no longer satisfactory in quantitative work because an unrealistically high mortality rate is needed, which has no clear interpretation. In my model, a 1% quarterly binding rate is sufficient to generate 10% quarterly discount rate on responses to future events, due to the incentive of the households to keep a certain level of precautionary saving, while the present value effect is much less important.

The nice property that output expectations summarize expectations on real wage, dividend and the labor supply of others does not hold for arbitrary subjective expectations, because the households may not expect themselves to supply the same amount of labor as others, and hence their expected individual incomes are also different from their expected national incomes even if the representative agents assumption has been imposed. Level-k equilibrium can get around this issue by carefully specifying level-0, such that the intratemporal optimality condition and market clearing condition still hold in expectations for all agents at all levels. The whole effect of inconsistent expectations becomes zero in present values. This nice property automatically holds in Angeletos and Lian (2016) due to rational expectations.

The simplicity of equation (1) allows for transparent mechanisms. First, equilibrium output is driven by the weighted average of output expectations at all horizons. When γ gets smaller, households will care more about the near future than the far future. Second, the effects of expected interest rate are cumulative. A permanent shift in interest rate expectations shifts the equilibrium output to a large extent if γ is close to one.

Supply Side. After aggregation, the optimality condition of the firms' pricing problem can be summarized in the following.

Lemma 2. *In the recursive level-k equilibrium, inflation satisfies*

$$\hat{\pi}_t^{(k)} = (1-\theta)\sum_{s=0}^{+\infty}\theta^s\hat{\pi}_{t+s|t-1}^{e,(k)} + (1-\theta)^2\sum_{s=0}^{+\infty}\theta^s[(\omega^{-1}+\xi^{-1})\hat{y}_{t+s|t-1}^{e,(k)} - (1+\xi^{-1})\mathbb{E}_{t-1}\eta_{t+s}^z] \quad (2)$$

The price rigidity parameter θ plays two roles. The first one is similar to γ, which determines firms' planning horizon. The second one is the slop of Phillips Curve. A similar expression can be found in Coibion et al. (2017). Unlike equation (1), (2) holds for arbitrary expectations.

Interest Rate. The expected effects of interest rates can be summarized in the following.

Lemma 3. *The expected cumulative effect of interest rate is,*

$$\sum_{s=0}^{+\infty}\gamma^s\hat{r}_{t+s|t-1}^{e,(k)} = \frac{1}{1-\gamma\rho_r}\left[\rho_r\hat{r}_{t-1} + (1-\rho_r)\sum_{s=0}^{+\infty}\gamma^s(\phi_\pi\hat{\pi}_{t+s|t-1}^{e,(k)} + \phi_y\hat{y}_{t+s|t-1}^{e,(k)})\right]. \quad (3)$$

This derivation of equation (3) relies on the assumption of level-0. I assume that Taylor Rule is common knowledge to agents at all levels, and hence can be used to compute the perceived law of motion. Long horizon expectations can be obtained by iterating on that.

Beauty Contests. Taking stock, we can obtain the dynamic beauty contest representation of the equilibrium.

Proposition 2. *The recursive level-k equilibrium must satisfy*

$$\hat{y}_t^{(k)} = (1-\gamma)\sum_{s=0}^{+\infty}\gamma^s(\varphi_y\hat{y}_{t+1+s|t-1}^{e,(k)} + \varphi_\pi\hat{\pi}_{t+1+s|t-1}^{e,(k)}) - \frac{\omega}{1-\gamma\rho_r}\hat{r}_{t|t-1}^{e,(k)} - \frac{\omega}{1-\gamma\rho_d}\eta_t^d, \quad (4)$$

$$\hat{\pi}_t^{(k)} = (1-\theta)\sum_{s=0}^{+\infty}\theta^s(\kappa\hat{y}_{t+s|t-1}^{e,(k)} + \hat{\pi}_{t+s|t-1}^{e,(k)}) - \kappa\omega\frac{1-\theta}{1-\theta\rho_z}\eta_t^z, \quad (5)$$

where $\varphi_y = 1 - \frac{\omega\gamma(1-\rho_r)}{(1-\gamma)(1-\gamma\rho_r)}\phi_y$, $\varphi_\pi = \frac{\omega}{1-\gamma} - \frac{\omega\gamma(1-\rho_r)}{(1-\gamma)(1-\gamma\rho_r)}\phi_\pi$ *and* $\kappa = (1-\theta)(\omega^{-1}+\xi^{-1})$.

From this representation, it is obvious that higher expectations on future output and inflation always raise the current inflation rate, but not necessarily the current output. The reason is that the interest rate response to higher output level and inflation rate would possibly reverse the incentive of spending, and hence result in even lower current output level.

4.2 Testable Forecast Rules

Asymmetric Reasoning. When $k \in (1, 2]$, macroeconomic co-movements that arise from feedback effects are not understandable by the agents, so that part of the state variables are missing in their forecast rules.

Proposition 3. *Level-k expectations for* $k \in [1, 2]$ *are given by*

$$\begin{aligned}\hat{y}^{e,(k)}_{t|t-1} - \hat{y}_{t-1} =& (k-1)\omega\left(-\frac{1-\rho_r}{1-\gamma\rho_r}\frac{\phi_y}{1-\gamma}\hat{y}_{t-1} - \frac{\rho_r}{1-\gamma\rho_r}\hat{r}_{t-1} - \frac{1}{1-\gamma\rho_d}\eta^d_t\right),\\ \hat{\pi}^{e,(k)}_{t|t-1} =& (k-1)\kappa\left(\hat{y}_{t-1} - \omega\frac{1-\theta}{1-\theta\rho_z}\eta^z_t\right).\end{aligned}$$

This proposition has the following implications. First, the monetary policy rule is only partly understood, in the sense that only the interest rate response to output fluctuations is incorporated into the one-quarter ahead forecast rules. Second, if interest rate does not respond to output, then the forecast rules exhibits "asymmetric reasoning", in the sense that interest rate is only used to forecast output growth, while output level is only used to forecast inflation in the next quarter. Third, agents' reasoning is asymmetric in a way that only the direct effects shows up in forecast rules, while the feedback effects are absent. Forth, the coefficients on all state variables in forecast rules are proportional to $k-1$, which looks as if $k-1$ represents households' awareness of direct effects.

Testable Missing Connections. In Michigan Survey of Consumers, we only have one-year ahead forecasts data. Although these forecast rules are less transparent, it is still possible to obtain some intuitions when the monetary policy rule is inflation targeting and only monetary shocks drive aggregate fluctuations.

Corollary 1. *When* $\phi_y = \eta_t^d = \eta_t^z = 0$ *and* $k \in [1, 2]$*, we have*

$$\frac{\partial \hat{y}^{e,(k)}_{t+3|t-1}}{\partial \hat{y}_{t-1}} - 1 = -\psi\phi_\pi(1 + \rho_r^2 - \psi\phi_\pi),$$
$$\frac{\partial \hat{\pi}^{e,(k)}_{t+3|t-1}}{\partial \hat{r}_{t-1}} = -\frac{\psi(1 + \rho_r + \rho_r^2 - \psi\phi_\pi)}{1 - \rho_r},$$

where $\psi = (k-1)^2(1-\theta)(1+\omega\xi^{-1})\frac{\rho_r(1-\rho_r)}{1-\gamma\rho_r}$,

The missing channels in Proposition 3 are no longer missing here. Yet, their sizes are in the same order of magnitude as the small ψ. Agents are aware of the self-stabilizing forces because they have non-trivial inflation expectations, and understand that interest rate responds to it. These two channels will be used to compute the perceived law of motion. However, both of them are much weaker than reality due to the dampening effects of level-k when k is close to 1. In addition, growth and inflation expectations are formed without using the expectations of the others, which also makes expectations less responsive than reality.

As a result, as long as k is close to 1, the "asymmetric reasoning" is still a distinct feature of the one-year ahead forecast rules. Therefore, we can still use the data implied forecast rules to test whether expectations are correctly specified in the level-k model.

4.3 Planning Horizons

Long Horizon Expectations. The following corollary derived from Lemma 1 can be used to demonstrate the role of planning horizons under non-rational expectations.

Corollary 2. *The level-k IS curve can be reformulated in the following*

$$\hat{y}_t^{(k)} = \hat{y}^{e,(k)}_{t+1|t-1} - \omega(\hat{r}^{e,(k)}_{t|t-1} - \hat{\pi}^{e,(k)}_{t+1|t-1} + \eta_t^d) + (1-\gamma)\sum_{s=1}^{+\infty}\gamma^s(\hat{y}^{e,(k)}_{t+1+s|t-1} - \hat{y}^{e,(k)}_{(t+1+s|t)|t-1})$$
$$- \omega\sum_{s=1}^{+\infty}\gamma^s[(\hat{r}^{e,(k)}_{t+s|t-1} - \hat{r}^{e,(k)}_{(t+s|t)|t-1}) - (\hat{\pi}^{e,(k)}_{t+1+s|t-1} - \hat{\pi}^{e,(k)}_{(t+1+s|t)|t-1})]. \quad (6)$$

According to Corollary 2, planning horizon matters because the "Law of Iterated Expectation" no longer holds across agents. For example, a level-2 agent form expectations as if others are level-1, while they forecast the forecasts of others as if they arise from level-0.

This issue not only exists in level-k models. The derivation of Corollary 2 implicitly assumes that agents understand "individual rationality". If we relax the level-k assumptions expectations but instead impose "Law of Iterated Expectation" on Corollary 2, then the admissible set of subjective expectations will be very restricted. See the following example for illustration.

Example 2. *Consider an example in which*
(1) level-k is relaxed,
(2) $1-\theta=\epsilon^r=\eta^d=0$, $\phi_y\geq 0$, *and*
(3) the perceived Taylor Rule is $\hat{r}^e_{t+s|t-1}=\phi_y\hat{y}^e_{t+s|t-1}$, *then*

$$\hat{y}^e_{t+1|t-1}=(1-\gamma)\sum_{s=0}^{+\infty}\gamma^s\left(\hat{y}^e_{t+2+s|t-1}-\frac{\omega\phi_y}{1-\gamma}\hat{y}^e_{t+1+s|t-1}\right).$$

Now, consider a perceived law of motion $\hat{y}^e_{t|t-1}=\rho^e_y\hat{y}^e_{t-1}$. *Assume that the "Law of Iterated Expectation" also holds across time, which is a natural assumption when individual rationality is not violated, we can iterate on the perceived law of motion to obtain long horizon expectations. As a result, we must have either* $\rho^e_y=0$ *or* $\rho^e_y=1+\omega\phi_y\geq 1$. *In another word, anchored expectations are not admissible in this example.*

There are two cases in which planning horizon does not play a role. The first case is $k\to+\infty$, which corresponds to rational expectations as is standard in most DSGE models. In this case, the "Law of Iterated Expectations" holds across both time and agents, but expectations will no long be anchored by historical data. The second case is $\gamma\to 0$, which corresponds to the "Euler Equation Learning" approach as in Milani (2007) and Milani (2011). In this case, long run expectations are assumed not to play a role. As in Example 2, anchored expectations are not likely to be compatible with the common knowledge of individual rationality. In summary, planning horizon is likely to play a key role in determining equilibrium output, if we would like to have expectations anchored to the past. This issue exists even if the anchoring does not arise from non-rational expectations (Angeletos and Lian, 2016).

Real Effects. The role of planning horizons is more clear when the equilibrium output is expressed as a function of the weighted average of output expectations in all horizons.

Proposition 4. *When $\eta_t^d = 0$ and $k \in [1, 2]$, the recursive level-k equilibrium must satisfy*

$$\hat{y}_t^{(k)} = -\frac{\omega}{1-\gamma\rho_r}\hat{r}_{t|t-1}^{e,(k)} + (k-1)(1-\gamma)\kappa\varphi_\pi\hat{y}_{t|t-1}^{e,(k)} + \delta(1-\gamma)\sum_{s=0}^{+\infty}\gamma^s\hat{y}_{t+1+s|t-1}^{e,(k)}, \tag{7}$$

where $\delta = \varphi_y + (k-1)\gamma\kappa\varphi_\pi = 1 - \frac{\omega\gamma}{1-\gamma}\left[(k-1)(1-\theta)(\omega^{-1}+\xi^{-1})\left(\frac{1-\rho_r}{1-\gamma\rho_r}\gamma\phi_\pi - 1\right) + \frac{1-\rho_r}{1-\gamma\rho_r}\phi_y\right]$.

This proposition indicates that $(1-\gamma)\sum_{s=0}^{+\infty}\gamma^s\hat{y}_{t+1+s|t-1}^{e,(k)}$ is crucial in determining the current equilibrium output, and δ captures the size of this effect. Here, γ plays two roles. First, smaller γ implies that households care more about the near future than the far future. Second, smaller γ leads to larger δ and hence makes households more responsive to expectations. Taking stock, when level-k output expectations anchors the past, smaller γ leads stronger anchoring.

In addition to γ, $\{k, \theta, \phi_\pi, \phi_y\}$ all affect δ. $\{k, \theta\}$ affect δ in the same way, as lower level reasoning and price flexibility both dampen the self-stabilizing channel in expectations through making inflation expectations less responsive. $\{\phi_\pi, \phi_y\}$ both dampens the effects of expectations, but the relative role of ϕ_π is affected by $k-1$ because it has to operate through inflation expectations, and affect the equilibrium output only indirectly.

4.4 Eductive Stability

Analytical Results. The Convergence to rational expectations when $k \to +\infty$ is difficult to characterized in the full model. Still some transparent results can be obtained when $\rho_r = \phi_\pi = 0$. The goal is to show why γ plays a crucial role in "Eductive Stability". The role of γ in the full model is similar.

Proposition 5. *When $\rho_r = \phi_\pi = \eta^d = \eta^z = 0$ and $k \in [1, +\infty) \cap \mathbb{N}$, the law of motion for output satisfies $\hat{y}_t^{(k)} = \rho_y^{(k)}\hat{y}_{t-1}$, and $\rho_y^{(k)}$ satisfies*

$$\frac{\rho_y^{(k)}}{\rho_y^{(k-1)}} = -\omega\phi_y + \left(1 - \frac{\omega\gamma}{1-\gamma}\phi_y\right)\frac{(1-\gamma)\rho_y^{(k-1)}}{1-\gamma\rho_y^{(k-1)}}. \tag{8}$$

Definition 3. *Under the same environment as in Proposition 5, "Eductive Stability" is defined as the following:* $\exists M \in [0,1)$*, such that for* $\forall \rho_y^{(k)}$ *with* $|\rho_y^{(k-1)}| \leq 1$*,* $|\frac{\rho_y^{(k)}}{\rho_y^{(k-1)}}| \leq M$*.*

If we allow $\rho_y^{(0)}$ to be specified in an arbitrary way, then "Eductive Stability" will become a sufficient and necessary condition to guarantee that the convergence to rational expectations when $\kappa \to +\infty$ is monotonic in absolute values. Due to the monotonicity of $\frac{\rho_y^{(k)}}{\rho_y^{(k-1)}}$ in $\rho_y^{(k-1)}$ for $|\rho_y^{(k-1)}| \leq 1$, it is easy to prove the following corollary.

Corollary 3. *When* $\rho_r = \phi_\pi = \eta^d = \eta^z = 0$ *and* $k \in [1, +\infty) \cap \mathbb{N}$*, the sufficient and necesary condition for "Eductive Stability" is* $\gamma \in (\frac{1}{2}\omega\phi_y, 1 - \frac{1}{2}\omega\phi_y)$*.*

In standard parameterization, the $\gamma > \frac{1}{2}\omega\phi_y$ is always satisfied, while $\gamma < 1 - \frac{1}{2}\omega\phi_y$ is usually not. For instance, when $(\omega, \phi_y) = (0.5, 0.2)$, "Eductive Stability" requires $\gamma \in (0.05, 0.95)$. Complete market models are observationally equivalent to $\gamma = \beta > 0.95$, while incomplete market models can have $\gamma < 0.95$. This result resembles the findings in Evans et al. (2017) that the long planning horizon destroys the "Eductive Stability".

Full Model. The proof of "Eductive Stability" is complex and less illuminating in the full model. Yet, the main mechanism is similar. Too strong self-stabilizing feedback from Taylor Rule and consumption response would make the equilibrium output responding negatively to output expectations, which is strongly contradictory to common sense.

There is a crucial parameter δ we could take a closer look. This parameter describes how the equilibrium output reacts to its expectations along all horizons. "Eductive Stability" requires this parameter to be large enough. From the expression of δ as in Proposition 4, we can see that (1) γ plays a similar role as in Proposition 5; (2) smaller k and $1 - \theta$ both increase the likelihood of "Eductive Stability" directly, and in addition makes inflation targeting less likely to destroy "Eductive Stability" indirectly; (3) ρ_r does not play a large role. Therefore, we can conclude that level-k model is more likely to be a useful tool for business cycle questions in the presence of incomplete markets and nominal rigidities.

5 Empirical Analysis

5.1 Estimation

This section presents the estimation results, model fit and the identification of level-k.

State Space Representation. The state space representation has a transition equation

$$\hat{s}_{t+1} = \begin{bmatrix} \hat{y}_t \\ \hat{r}_t \\ \eta^d_{t+1} \\ \eta^z_{t+1} \end{bmatrix} = \Gamma^{ss,(k)} \begin{bmatrix} \hat{y}_{t-1} \\ \hat{r}_{t-1} \\ \eta^d_t \\ \eta^z_t \end{bmatrix} + \begin{bmatrix} 0 \\ \sigma_r \epsilon^r_t \\ \sigma_d \epsilon^d_t \\ \sigma_z \epsilon^z_t \end{bmatrix},$$

and measurement equation

$$\begin{bmatrix} \hat{y}_{t+1} - \hat{y}_t \\ \hat{p}_{t+1} - \hat{p}_t \\ \hat{r}_t \\ \hat{y}^e_{t+4} - \hat{y}_t \\ \hat{p}^e_{t+4} - \hat{p}_t \end{bmatrix} = \begin{bmatrix} \Gamma^{ys,(k)} - \Gamma^y \\ \Gamma^{\pi s,(k)} \\ \Gamma^r \\ \Gamma^{ys,e,(k)}\left(\Gamma^{ss,e,(k)}\right)^3 - \Gamma^y \\ \Gamma^{\pi s,e,(k)} \sum_{\tau=0}^{3} \left(\Gamma^{ss,e,(k)}\right)^\tau \end{bmatrix} \hat{s}_{t+1} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \sigma_y \epsilon^y_t \\ \sigma_p \epsilon^p_t \end{bmatrix},$$

where $\Gamma^{ss,(k)}$ and $\Gamma^{ss,e,(k)}$ denote the actual and perceived law of motion for the recursive level-k equilibrium, $\Gamma^{ys,(k)}$ and $\Gamma^{ps,(k)}$ denote the actual function of output and inflation, $\Gamma^{ys,e,(k)}$ and $\Gamma^{ps,e,(k)}$ denote the perceived functions or the forecast rules of output and inflation, Γ^y and Γ^r extract output and interest rate from the vector of aggregate states. $\{\epsilon^r_t, \epsilon^d_t, \epsilon^z_t, \epsilon^y_t, \epsilon^p_t\}$ are i.i.d. standard normally distributed, in which $\{\epsilon^r_t, \epsilon^d_t, \epsilon^z_t\}$ stand for exogenous shocks driving aggregate states, while $\{\epsilon^y_t, \epsilon^p_t\}$ are measurement errors. $\{\sigma^r_t, \sigma^d_t, \sigma^z_t, \sigma^y_t, \sigma^p_t\}$ denote the standard deviation of all these shocks.

The five time series used for estimation includes the quarterly GDP growth rate $\hat{y}_{t+1} - \hat{y}_t$, quarterly GDP deflator based inflation rate $\hat{p}_{t+1} - \hat{p}_t$, federal fund rate $\hat{r}_t$, one year ahead GDP growth rate forecasts (imputed from the aggregate index of business condition change) $\hat{y}^e_{t+4} - \hat{y}_t$, and the one year ahead inflation expectation $\hat{p}^e_{t+4} - \hat{p}_t$.

Bayesian Estimation. All parameters but $\{\xi, \gamma\}$ estimated using Bayesian approach. Without data from labor market, ξ cannot be identified, so I simply choose $\xi = +\infty$ to induce a linear disutility from labor. γ should be estimated from micro level data. However, estimating planning horizon is very difficult. I make use of the connection between planning horizon and marginal propensity to consume (MPC) of the unconstrained households to infer γ. $\gamma = 0.93$ such that the MPC of unconstrained households is 7% as in Kaplan and Violante (2014). The estimation results of other parameters are summarized in Table 2.

Table 2: Prior and Posterior Distribution of Parameterss

Parameters	Prior shape	Prior Mean	Prior S.D.	Post. Mean	95% Band
k	Uniform 1/k			1.265	[1.197,1.327]
ω	Normal	1.000	0.500	0.074	[0.054,0.101]
θ	Uniform			0.983	[0.976,0.989]
ϕ_π	Normal	1.500	0.500	1.518	[1.079,1.894]
ϕ_y	Normal	0.200	0.100	0.003	[-0.003,0.014]
ρ_r	Uniform			0.870	[0.830,0.907]
ρ_d	Uniform			0.279	[0.103,0.441]
ρ_z	Uniform			0.998	[0.996,1.000]
$400\sigma_r$	InvGamma	0.500	4.000	0.452	[0.395,0.518]
$100\sigma_d$	InvGamma	1.000	4.000	3.776	[2.429,5.227]
$100\sigma_z$	InvGamma	5.000	4.000	14.518	[9.493,22.437]
$100\sigma_y$	InvGamma	0.300	4.000	0.309	[0.267,0.356]
$100\sigma_p$	InvGamma	0.500	4.000	0.360	[0.309,0.425]

$k = 1.265$ is close to the results in experimental games as in Camerer et al. (2004). Although ω is much smaller than 1, it is consistent with macro level estimates as summarized by Havranek (2015). θ is very close to 1 due to the absence of wage rigidity. $\phi_\pi = 1.5$ is quite standard. ϕ_y is close to 0 because higher values will lead to very strong self-stabilizing force in expectations, which is absent in the data. $\rho_r = 0.870$ is standard in the literature. ρ_d much smaller than 1 indicates that the model has strong internal propagation mechanism. ρ_z very close to 1 indicates that the technology process is very persistent. σ_y and σ_p are both smaller than the standard deviations of growth and inflation forecast, which are very difficult to get if the expectations formation process is more properly specified.

5.2 Model Fit

Forecast Wedge. A distinct feature of level-k DSGE model is the wedge between forecasts and reality. Table 3 summarizes the wedge by comparing the corresponding law of motion for output growth and inflation.

Table 3: Forecast Wedge

	$\hat{y}^e_{t+4} - \hat{y}_t$		$\hat{y}_{t+4} - \hat{y}_t$	
	Model	Data	Model	Data
$\hat{y}_t$	-0.008	-0.017 [-0.078,0.044]	-0.456	-0.262 [-0.494,-0.030]
$\hat{r}_t$	-0.294	-0.530 [-0.681,-0.379]	-2.089	-1.481 [-2.054,-0.908]
η^d_{t+1}	-0.037	-0.010 [-0.022,0.001]	-0.129	-0.148 [-0.191,-0.105]
η^z_{t+1}	0.000	0.004 [0.000,0.008]	0.028	0.015 [-0.001,0.031]
Obs.	92	92	92	92
R^2		0.397		0.526
	$\hat{p}^e_{t+4} - \hat{p}_t$		$\hat{p}_{t+4} - \hat{p}_t$	
	Model	Data	Model	Data
$\hat{y}_t$	0.241	0.269 [0.194,0.344]	0.636	0.663 [0.574,0.753]
$\hat{r}_t$	-0.030	0.458 [0.273,0.642]	-1.158	-0.307 [-0.528,-0.086]
η^d_{t+1}	-0.006	0.010 [-0.004,0.024]	-0.094	-0.014 [-0.031,0.003]
η^z_{t+1}	-0.016	-0.020 [-0.025,-0.015]	-0.047	-0.039 [-0.045,-0.033]
Obs.	92	92	92	92
R^2		0.550		0.714

The wedge exists in both direct and indirect channels. First, indirect channels are missing in forecasts. Higher output levels are followed by lower output growth rates in reality, but not in forecasts. Higher interest rates are followed by lower inflation rates in reality, but not in forecasts. This is highlighted by the shaded areas in the table. Second, direct channels are weaker in forecasts. Higher interest rates are followed by lower growth rates in forecasts, but not as much as the reality. Higher output levels are followed by higher inflation rates, but not as much as the reality. The model is consistent with data in terms of the wedge.

Forecast Path. The forecast wedge can also be illustrated by comparing the path of forecasts in the model, forecasts in the data, and reality in the data in Figure 3.

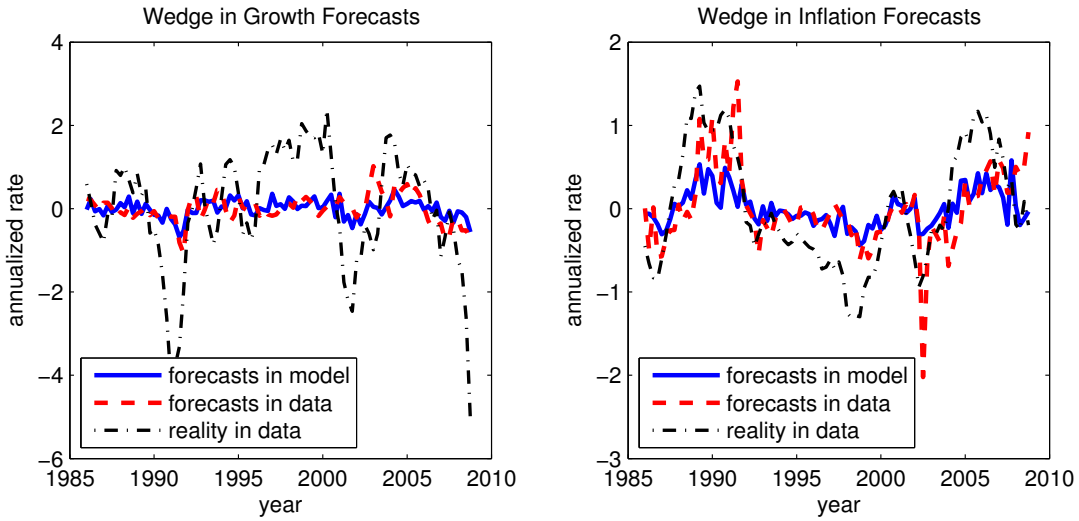

Figure 3: Forecast Wedge

The wedge in growth forecasts is depicted in the left panel. The volatility of forecasts in both model and data is much lower than the reality. The forecast errors are large, persistent and countercyclical, which indicates that households expectations' are driven by endogenous waves of optimism and pessimism. The model implied forecasts generally fits the data counterpart well. The rise of growth forecasts during the early 2000s, and the fall of them during the late 2000s in the data lead the reality. This indicates that there may be other forces, such as news in Barsky and Sims (2012), driving households' growth expectations.

The wedge in inflation forecasts is depicted in the right panel. Inflation forecasts are also less volatile than the reality. The model generally captures the overpredict of inflation during the IT boom in the late 1990s, and the underpredict of inflation during the housing booms in the middle of 2000s. Yet, it does not fully capture the rise of inflation expectations during the late 1980s. This discrepancy is very difficult to clean up because k is exogenous in the model. Once I raise k to capture the inflation expectations during the late 1980s, the fit of inflation expectations during other periods will become much worse.

5.3 Identification

Forecast Rules. The main identification of level-k comes from the forecast rules. Figure 4 plots the coefficient differences between the model implied forecast rules and the data implied forecast rules, normalized by the standard errors of the later. Closer to zero implies better fit.

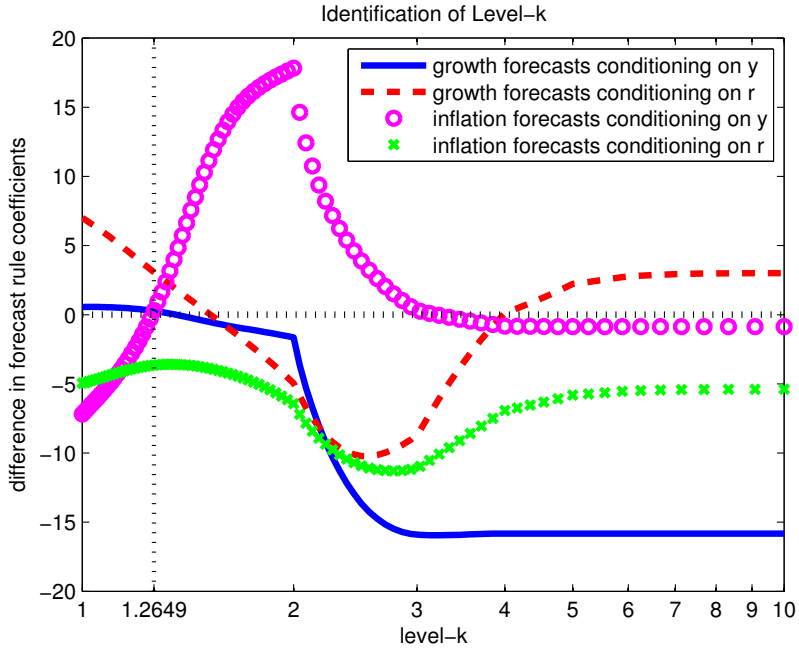

Figure 4: Identification of Level-k

A few things can be learned from Figure 4. First, the model has good convergence property, and has forecast rules close to the fixed points beyond level-5. Second, the convergence is not monotonic, and has some overshooting feature at level-2 and level-3, which has also been documented in Angeletos and Lian (2017). Third, the two indirect channels, growth forecasts conditioning on output and inflation forecast conditioning on interesting rate, can identify $k \in [1, 2]$, because they obtain best fit points only in this range; while the two direct channels, growth forecasts conditioning on interest rate and inflation forecast conditioning on output, only help identify the exact value of k within $[1, 2]$, because they have additional best fit points outside $[1, 2]$ but are more sensitive to k within $[1, 2]$.

Amount the indirect channels, growth forecasts conditioning on output seems to be more important than inflation forecasts conditioning on interest rate. One possible reason is purely forward looking nature of output under rational expectations. If other frictions such as habit are also introduced, then the wedge in this channel will be smaller. Another reason is that forecast data indicate that households tend to expect higher inflation rate following higher interest rate, which is difficult to capture in the simple model of this paper.

Observational Equivalence. A natural concern of using level-k is that maybe the missing connection between macroeconomic data and forecast data is due to the limited attention on macroeconomic data. Two empirical facts actually show that it is not the case.

The first fact comes from a backcast question in Michigan Survey of Consumers asking households' subjective evaluation of the business condition change during the last year. The answers are aggregated in an index. The index is highly correlated with the actually GDP growth rate with R^2 as high as 75%, and no time delays. This indicate that limited attention may not be the major concerns here.

The second fact comes from the data implied forecast rules. If households do not pay attention to output and interest rate, then we should also observe missing connections between current output and inflation forecasts, as well as between current interest rate and growth forecasts. Table 3 shows that these two connections are indeed not missing.

A better interpretation of level-k is that households do observe the macroeconomic variables, but do not use some of them in their forecast rules. In another word, households understand the aggregate states of the world, but not the aggregate law of motion. In this sense, level-k is actually orthogonal to models with inflation rigidity, dispersed information and limited attention, which all assume that households understand the aggregate law of motion, but not the aggregate states of the world.

These two orthogonal ways of modeling imperfect knowledge has very different policy implications. For example, the level-k approach implies that the effectiveness of policy depends on whether it relies on simple direct channels or subtle indirect channels, while the information approach implies that it relies on whether the policy can be precisely communicated.

6 Monetary Policy Analysis

This section applies the level-k DSGE model in four topics related to monetary policy, including the transmission of monetary shocks, the design of monetary policy rules, the consequence of liquidity traps, and the effectiveness of forward guidance.

6.1 Transmission of Monetary Shocks

Transmission Through Expectations. Figure 5 illustrates how expectations affect the transmission of -1 standard deviation of federal fund rate shocks. The model generated interest rate and output responses are compared with their counterparts in a simple structural VAR with 4 lags and three variables including output, inflation and federal fund rate.

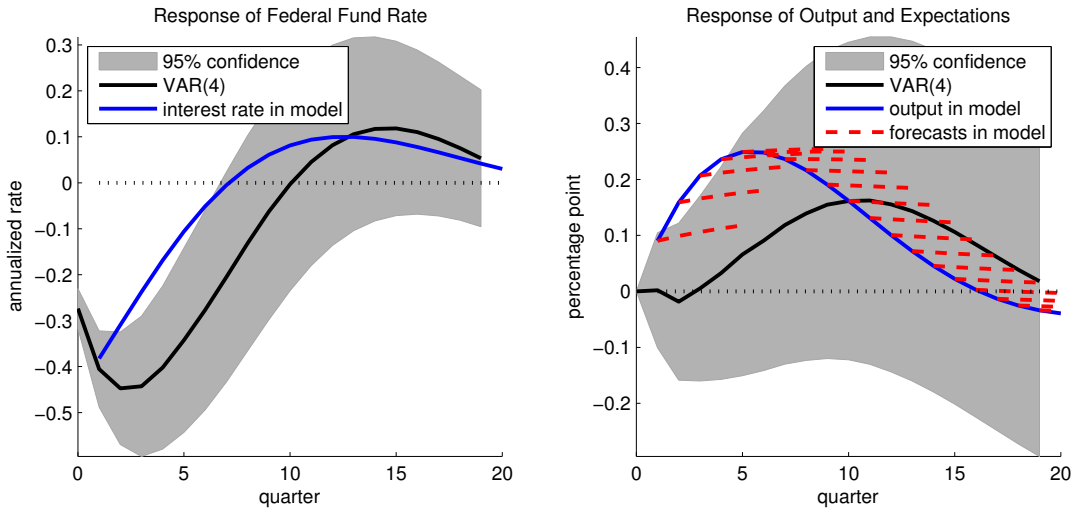

Figure 5: Transmission Through Expectations

An interesting common feature of both model and data is that output rises when federal fund rate is below zero, and declines then it is above. The standard way of capture this pattern as in Christiano et al. (2005) and Smets and Wouters (2007) is to introduce habit, such that households' expenditure is anchored to last period by assumption. The right panel of Figure 5 shows that in the level-k DSGE model, expectations are anchored, so that the output responses also arise from the cumulative effects of interest rates as with habit.

This transmission mechanism can be justified by Corollary 1 and Proposition 4. According to the estimation results, we have $\psi = 0.0007$, which implies that output expectations are almost fully anchored; as well as $\delta = 0.9962$, which implies that anchored expectations induce anchored equilibrium output.

This result is in sharp contrast to Gabaix (2016), which emphasizes that bounded rationality in perceived law of motion will dampen the effect of monetary policy. This difference comes from the assumption on the most ignorant agents. Gabaix (2016) assumes that they perceive output movement to be purely transitory so that a simple discounted Euler Equation can be derived, while I assume that they perceive output movement to be permanent so that the model can match the data of households' forecasts.

Propagation and Amplification. Figure 6 shows that compared with rational expectations and complete markets, level-k and incomplete markets can propagate and amplify the effects of monetary shocks, if combined. The impulse responses of output with different k and γ, and identical other parameters are plotted to compare with the benchmark result. The comparison between $k = 1.2649$ and $k = 10$ and the comparison between $\gamma = 0.93$ and $\gamma = 0.99$ show that level-k and incomplete markets are both necessary for the strong propagation and amplification.

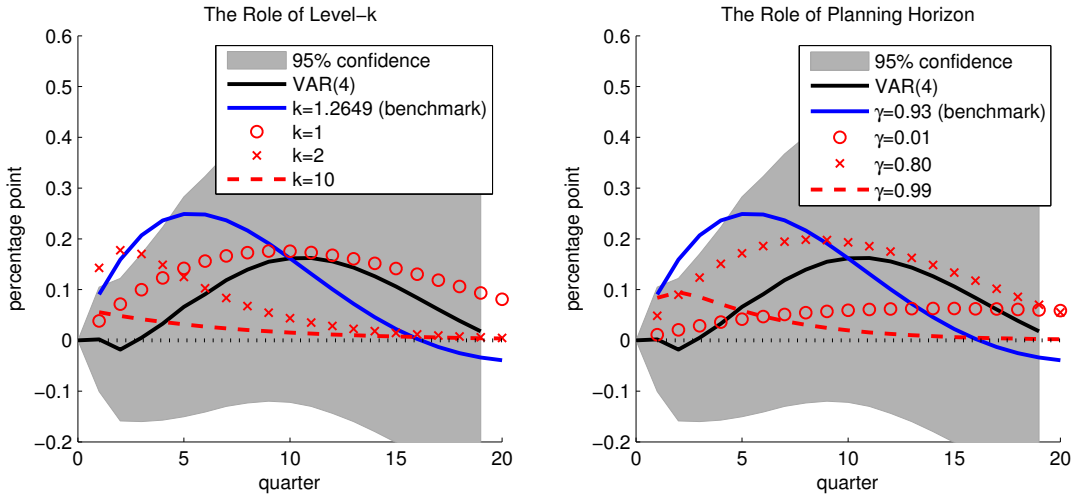

Figure 6: Propogation and Amplification

The propagation effects are monotonically decreasing in k and γ. As k get smaller, output expectations are more anchored to the past, and hence are more persistent[13]. As γ gets smaller, households care more about the near future, and hence the decline of output expectations in far future plays a smaller role, which reinforces the over-persistence effects of output expectations.

In contrast, the amplification effects are not monotonically decreasing in either k or γ. Smaller k dampens not only the indirect self-stabilizing mechanisms, but also the direct reactions of spending, and the total effects are ambiguous. Smaller γ makes households care more about the near future, but according to the right panel of Figure 5, output expectations in the periods closely following the current period will not be very different from the anchored output levels.

It is also interesting to compares these results with two strands of literature. The first one is "Euler Equation Learning" represented by Milani (2007). Their approach is to first derive the equilibrium conditions under rational expectations, and then impose subjective expectations. This approach corresponds to $\gamma \to 0$, which can generate very strong propagation without much amplification as in Figure 6. The limitation is the lack of microfoundation when expectations are adaptive, as is illustrated in Example 2. The second one is "Heterogeneous Agent New Keynesian Model", or "HANK" for short, represented by Kaplan, Moll, and Violante (2017). Their model emphasizes the constrained households' strong reaction to income changes when γ is not close to 1. Although the transmission mechanism becomes very different from models with complete markets, the aggregate effect is less clear. In level-k DSGE, it is the unconstrained households' precautionary saving motive, that really matters for both the transmission and the aggregate effect, while the constrained households play no role. In this sense, our findings are orthogonal.

This role of γ is level-k model is first formalized by Farhi and Werning (2017). They show that level-k and incomplete markets are complementary in dampening the effects of forward guidance in a highly stylized model. Compared to this paper, part of the contribution of my paper is to make their idea more transparent, more general and more empirically relevant by introducing a full-fledged level-k DSGE model. The multiple variable structure of DSGE helps identify level-k through forecast wedge, and also allows for broader applications.

[13] Rozsypal and Schlafmann (2017) also finds similar patterns in micro level data from the Michigan Survey of Consumers.

6.2 Design of Monetary Policy Rules

Inflation v.s. GDP Targeting. The trade off between inflation targeting and GDP targeting in with level-k is very different from that with rational expectations. Compared with ϕ_y, ϕ_π is much less powerful in stabilizing household expectations and hence the real allocations.

More specifically, when $k \in [1, 2]$, positive ϕ_y induces mean reversion in output expectations, while ϕ_π has very limited such effect. Proposition 3 and Corollary 1 have explicitly demonstrated the main intuition, that inflation targeting is more difficult to understand compared with GDP targeting. When interest rate targets on output gaps, higher output level induces higher interest rate if $\phi_y > 0$, and households with $k > 1$ understand that it will further slow down output growth. Higher output also induces higher inflation rate, which raises real interest rate if $\phi_\pi > 1$ and then slows down output growth. This channel involves one layer of feedback effect from the firms setting prices, which can only be understood by households with $k > 2$. Households with $k \in (1, 2]$ can still expect interest rate rise, use this to obtain the perceived law of motion, and ultimately expect some growth decline after multiple periods. Yet, this indirect decline is almost negligible.

In addition, given output expectations, ϕ_π still has smaller effect on equilibrium output compared with ϕ_y, because it has to operate through inflation expectations, which is once again dampened by $k-1$ if $k-1$ is close to zero. In contrast, the anchoring of output expectations to the past will not be dampened by level-k and hence can have larger effect on real allocations.

The Volker Regime. It is widely believed that the U.S. monetary policy regime switched from more GDP targeting to more inflation targeting during Paul Volker' term of office. For example, Boivin (2006) has shown that before 1979q3, we roughly have $(\phi_\pi, \phi_y) = (1.1, 0.12)$ ($\phi_y = 0.48$ in annualized rate), and after that, it becomes $(\phi_\pi, \phi_y) = (1.5, 0.00)$.

This switch of monetary policy regime provides a case for external validation. Table 4 compares the model implied (forecast rules) mean reversion in growth expectations for $(\phi_\pi, \phi_y) = (1.5, 0.00)$ (1985q1-2007q4) and $(\phi_\pi, \phi_y) = (1.1, 0.12)$ (1960q1-1979q3), with their counterparts in the data (OLS regression).

Table 4: Monetary Policy Regimes and Growth Expectations

	$\hat{y}^e_{t+4} - \hat{y}_t$			
	$(\phi_\pi, \phi_y) = (1.5, 0.00)$	1985q1-2007q4	$(\phi_\pi, \phi_y) = (1.1, 0.12)$	1960q1-1979q3
$\hat{y}_t$	-0.008	0.005 [-0.038,0.048]	-0.100	-0.097 [-0.150,-0.044]
Obs.		92		92
R^2		0.001		0.147

*The 1979q4-1984q4 periods during which Paul Volker was in office but the inflation was not yet stabilized is not included.

This comparison result supports the model implication that GDP targeting induces mean reversion in output expectations. In the post-Volker regime, there is no connection between current output level and growth expectations, while in the pre-Volker regime, the connection is significant, and accounts for 14.7% of the variation in growth expectations.

It is also interesting to compare the transmission of monetary shocks and the impact of demand shocks under these two monetary regimes. Figure 7 plots the impulse response functions to -1 standard deviation of monetary shocks and demand shocks under two monetary regimes.

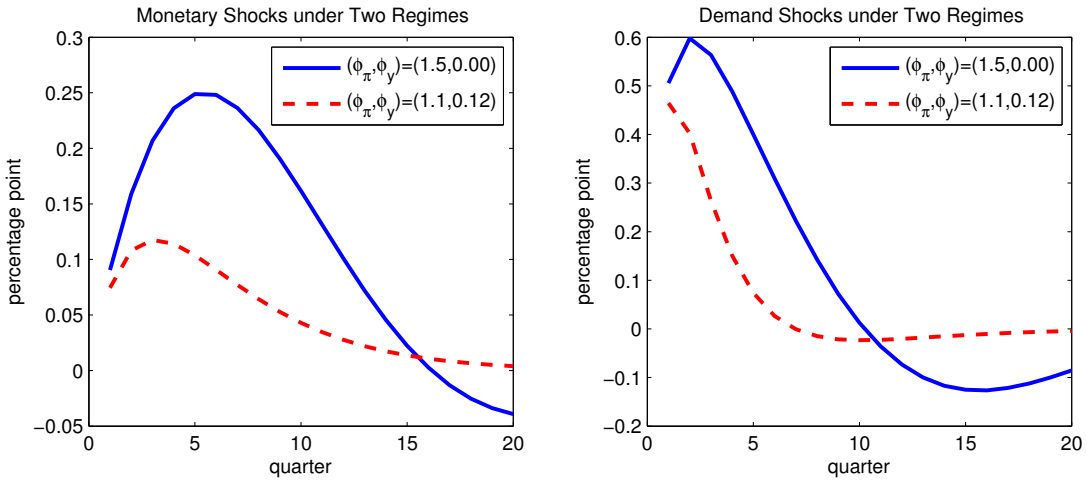

Figure 7: Monetary and Demand Shocks under Two Regimes

The results indicate that GDP targeting, once precisely implemented, is more much powerful than inflation targeting. It justifies the use of GDP targeting during deep recessions in which we observe the negative output gaps with less noise.

6.3 Consequence of Liquidity Traps

Chronic Demand Deficiency. In standard New Keynesian DSGE models, demand driven recessions can not be permanent. Otherwise, deflation will explode. This intuition is clear in Corollary 4, which is derived from Lemma 2.

Corollary 4. *Consider* $k \to +\infty$*, when* $\eta^z = 0$*,*

$$\hat{\pi}_t^{(+\infty)} = \frac{\kappa(1-\theta)}{\theta} \sum_{s=0}^{+\infty} \hat{y}_{t+s}^{e,(+\infty)}.$$

In contrast, in level-k DSGE models, permanent output gap does not induce deflation explosion. The equilibrium inflation rate in level-k DSGE is demonstrated in Corollary 5, which is derived from Lemma 2, Proposition 3, and the definition of recursive level-k equilibrium.

Corollary 5. *Consider* $k \in [1, 2]$*, when* $\eta^z = 0$*,*

$$\hat{\pi}_t^{(k)} = \kappa \left\{ (k-1)(1-\theta)\hat{y}_{t-1} + [1+\theta(k-1)](1-\theta) \sum_{s=0}^{+\infty} \theta^s \hat{y}_{t+s|t-1}^{e,(k)} \right\}.$$

The roles of price stickiness θ in these two models are also different. Under rational expectations, θ does not affect how current inflation depends on output gaps expectations far in the future. In contrast, under level-k expectations, when θ gets smaller, output gaps expectations far in the future becomes less important. It is also consistent with our intuition that news for far future should not have large impact on the current inflation. As a result, the cumulative effects of permanent demand deficiency can only have finite impact on current inflation.

Protracted Liquidity Trap. One limitation of the New Keynesian models under rational expectations is that deeper recessions in the liquidity trap triggered by insufficient aggregate demand must be associated with fast recoveries (Cochrane, 2017). In the same time, during the Great Recession, professional forecasts continuously revised their prospects of the recovery downward, suggesting that the models they used were not able to predict a protracted liquidity trap. The level-k New Keynesian DSGE does not suffer from this limitation.

Under level-k reasoning with $k \in [1, 2]$, the expected output recovery speed is proportional to the difference between inflation target and the drop in natural rate of interest, and it takes longer time to recover when the downturn is deeper. In the expectations of level-k households, other households do not react to growth or inflation expectations, and make spending decisions purely based on nominal interest rate. Proposition 6 formalizes this intuition.

Proposition 6. *Denote $\hat{\pi}_{SS}$ as the inflation target, and $\hat{r}^n$ as the natural rate of interest, and there are no exogenous shocks. For $k \in [1, 2]$, during the liquidity trap, we have*

$$\begin{aligned}\hat{y}^{e,(k)}_{t+s|t-1} =& \hat{y}_{t-1} + (k-1)(s+1)\frac{\omega}{1-\gamma}(\hat{\pi}_{SS} + \hat{r}^n),\\ \hat{\pi}^{e,(k)}_{t+s|t-1} =& (k-1)\kappa\hat{y}^{e,(k)}_{t-1+s|t-1}.\end{aligned}$$

Consider an economy initially with a negative output gap, and no other shocks. The natural rate of interest has declined permanently, and the federal fund rate is binded as zero until the output gap becomes non-negative. Once recovered, the whole economy is forced to stay in the steady state. The output dynamics is computed and plotted in Figure 8.

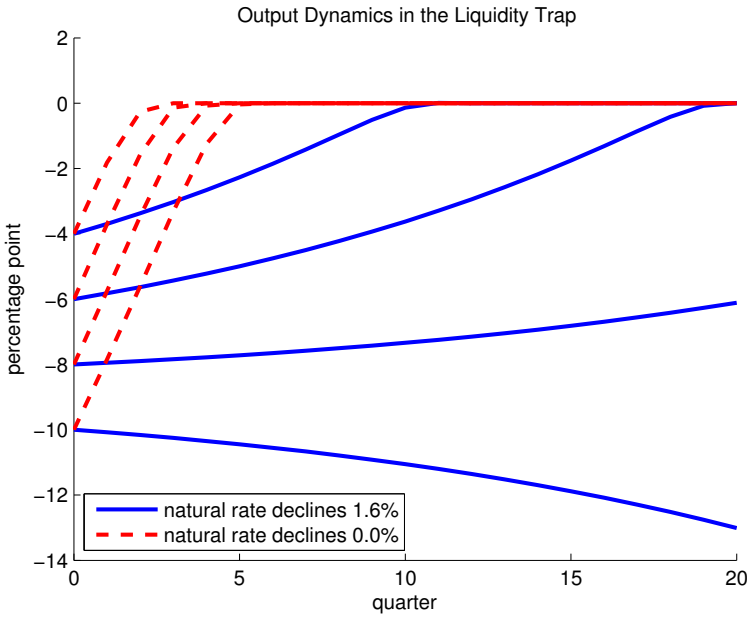

Figure 8: Protracted Liquidity Trap

Figure 8 plots the recovery dynamics for different initial levels of output, and different natural rate of interest declines. When natural rate of interest rate does not decline, recovery is always very fast. When it declines but no more than the inflation target, lower initial levels outputs induce slower recovery. When the initial output level is too low, recovery become impossible.

The main intuition comes from Proposition 6. When the natural rate of interest declines less than the inflation target, households always expect recovery. The expected recovery speed is driven by this gap. When the gap is small, the expected recovery can be slow. The households expecting recovery may not choose spending to justify it because their inflation expectations drive down their expected real interest rate. Lower initial output levels implies lower inflation rates, and hence weaker recoveries. In addition, lower output level also implies that complete recovery takes longer time, hence households' consumption incentive is even weaker.

Figure 9 plots the expected business condition change in the next year from MSC. The average expectations during the recovery periods starting from 2009 is not above the average level before 2009. This indicates that households' recovery expectations are actually stagnant after the deep downturns, which is consistent with the level-k model.

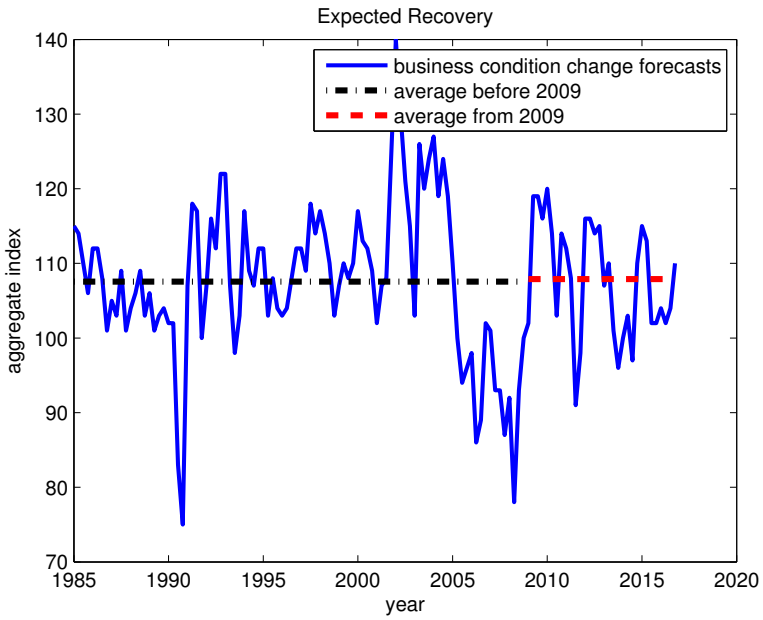

Figure 9: Stagnant Recovery Expectations

6.4 Effectiveness of Forward Guidance

Amplifying or Dampening. Similar to Farhi and Werning (2017), the initial response of output to an interest rate shock in future will be dampened by level-k. However, due to the different specification of level-0, the effects of monetary shocks are accumulated across time. The ultimate effect of forward guidance may not be small. It just takes a while to fully realize.

Policy Experiment. Now consider a thought experiment that the $t+\tau$ period interest rate expectation is shocked by -1 percentage point. All other shocks are turned off. Interest rates are pegged before the shock, and follow the Taylor Rule otherwise.

Proposition 7. *Under this environment, for* $k \in [1,2]$ *and* $s \in [0, \tau-1] \cap \mathbb{N}$,

$$\begin{aligned}\hat{y}^{e,(k)}_{t+1+s|t-1} =& \hat{y}_{t-1} - (k-1)\omega \frac{\gamma^{\tau-s-1} - \gamma^{\tau+1}}{1-\gamma} \hat{r}^{e,(k)}_{t+\tau|t-1},\\ \hat{\pi}^{e,(k)}_{t+1+s|t-1} =& (k-1)\kappa \hat{y}^{e,(k)}_{t+s|t-1}.\end{aligned}$$

Figure 10 plots the dynamic effects of forward guidance with a -1% shock in interest rate at 1-5 years horizons. The results indicate that forward guidance at shorter horizons have larger initial responses, but those at longer horizons can have larger cumulative effects.

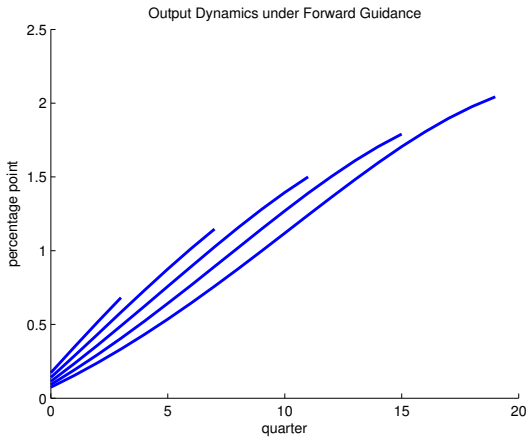

Figure 10: Horizon Effect of Forward Guidance

7 Conclusion

This paper establishes a level-k DSGE framework for monetary policy analysis. A recursive level-k equilibrium is established to handle endogenous state variables, and incomplete markets are introduce to discipline households' planning horizons, as well as guarantee eductive stability. The framework is easy to use and has transparent mechanisms.

The model structure and expectation data help identify parameter k. The interaction between households and firms allows us to use output growth and inflation expectations to separate direct and indirect transmission channels. The expectation data support the model implication that indirect channels are missing, while direct channels are weak in households' forecast rules. The formal evidence identifies $k \in [1, 2]$, while the later evidence identifies the exact value of k. Level-k also has data supported implications different from limited attention.

The model is applied in four issues related to monetary policy: (1) the transmission of monetary shocks, (2) the trade-off between inflation targeting and GDP targeting, (3) the protracted liquidity traps, and (4) the effect of forward guidance. The consistency between the model implied expectations and allocation and data in different monetary regimes provides external validation for the level-k model.

There are still a few related research questions worth further exploring. First, planning horizon has larger effects on equilibrium dynamics under non-rational expectations, but it is still not clear under what general conditions, this part should be explicitly specified. Second, the dual dynamic beauty contests in this paper only provides an example to separate direct and indirect effects in expectation data. Wage stickiness will add one more beauty contest, and provides sharper views for inflation. Third, reasoning in terms of real variables actually involves some level of rationality. The level-k model can be used to deal with this issue by initializing level-0 with nominal anchors. Forth, the level-k framework could be applied to finance related topics in which long horizon expectations play a role, such as bubbles and private money. I leave all these valuable questions for future research.

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Appendix A: Proofs

Proof of Proposition 1

Suppose $b^{(k)}(\overline{\zeta}, 0, S) > 0$, Assumption 2 implies that $b^{(k)}(1, 0, S) < 0$.
Consider households' budget $c^{(k)}(\zeta, 0, S) = W^{(k)}(S)\ell^{(k)}(1, 0, S) + D^{(k)}(S) - b^{(k)}(\zeta, 0, S)$.
Hence, $W^{(k)}(S)\ell^{(k)}(\overline{\zeta}, 0, S) - c^{(k)}(\overline{\zeta}, 0, S) > W^{(k)}(S)\ell^{(k)}(1, 0, S) - c^{(k)}(1, 0, S)$.
Consider the optimality condition for labor supply $v_\ell(\ell^{(k)}(\zeta, 0, S)) = W^{(k)}(S)u_c(c^{(k)}(\zeta, 0, S))$.
According to Assumption 1, we must have $c^{(k)}(\overline{\zeta}, 0, S) < c^{(k)}(1, 0, S)$.
This is contradictory to the concavity of perceived and actual value functions.

Proof of Lemma 1

Consider perfect foresighted $\{\hat{w}, \hat{\tau}, \hat{\pi}, \hat{y}, \hat{r}, \eta^d, \eta^z\}$. Denote wy_{SS} as the steady state labor share. The linearized consumption of a just constrained household i in period $t+1$ satisfies

$$\begin{aligned}\hat{c}^{con}_{i,t+1} =& \hat{a}^{unc}_{i,t+1} + wy_{SS}(\hat{w}_{t+1} + \hat{\tau}_{t+1} + \hat{\ell}^{con}_{i,t+1}),\\ \hat{\ell}^{con}_{i,t+1} =& \xi\hat{w}_{t+1} - \xi\omega^{-1}\hat{c}^{con}_{i,t+1}.\end{aligned}$$

Denote $\kappa_\ell = 1 + wy_{SS}\xi\omega^{-1}$. The solution for $\hat{c}^{con}_{i,t+1}$ is

$$\hat{c}^{con}_{i,t+1} = \kappa_\ell^{-1}\{\hat{a}^{unc}_{i,t+1} + wy_{SS}[(1+\xi)\hat{w}_{t+1} + \hat{\tau}_{t+1}]\}. \tag{9}$$

Denote $\lambda = \lambda_{\overline{\zeta}|1}\overline{\zeta}$. The consumption of an unconstrained household i in period t satisfies

$$\hat{c}^{unc}_{i,t} = (1-\lambda)\hat{c}^{unc}_{i,t+1} + \lambda\hat{c}^{con}_{i,t+1} - \omega(\eta^d_t + \hat{r}_t - \hat{\pi}_{t+1}). \tag{10}$$

$\hat{a}^{unc}_{i,t+1}$ for unconstrained household i satisfies

$$\hat{a}^{unc}_{i,t+1} = \hat{a}_{i,t} - \kappa_\ell\hat{c}^{unc}_{i,t} + wy_{SS}[(1+\xi)\hat{w}_t + \hat{\tau}_t]. \tag{11}$$

The aggregate output $\hat{y}$ satisfies[14]

$$\hat{y}_t = wy_{SS}(\hat{w}_t + \hat{\tau}_t + \mathbb{E}\hat{\ell}_{-i,t}) = wy_{SS}[(1+\xi)\hat{w}_t + \hat{\tau}_t - \xi\omega^{-1}\hat{y}_t]. \tag{12}$$

Combining equation (9)(10)(11)(12) yields

$$(1+\lambda)\hat{c}^{unc}_{i,t} = (1-\lambda)\hat{c}^{unc}_{i,t+1} + \lambda(\kappa_\ell^{-1}\hat{a}_{i,t} + \hat{y}_t + \hat{y}_{t+1}) - \omega(\eta^d_t + \hat{r}_t - \hat{\pi}_{t+1}). \tag{13}$$

Use guess and verify approach to find the expression of $\hat{c}^{unc}_{i,t}$

$$\hat{c}^{unc}_{i,t} = \kappa_a\hat{a}_{i,t} + \kappa_0\hat{y}_t + \sum_{s=0}^{+\infty}\gamma^s[\kappa_y\hat{y}_{t+s} - \kappa_r\omega(\eta^d_{t+s} + \hat{r}_{t+s} - \hat{\pi}_{t+1+s})].$$

$\hat{c}^{unc}_{i,t+1}$ for households unconstrained in period t satisfies

$$\hat{c}^{unc}_{i,t+1} = \kappa_a[\hat{a}_{i,t} + \kappa_\ell(\hat{y}_t - \hat{c}^{unc}_{i,t})] + \kappa_0\hat{y}_{t+1} + \sum_{s=0}^{+\infty}\gamma^s[\kappa_y\hat{y}_{t+1+s} - \kappa_r\omega(\eta^d_{t+1+s} + \hat{r}_{t+1+s} - \hat{\pi}_{t+2+s})].$$

Compare the coefficient of $\hat{a}_t$ in equation (13) to get κ_a

$$(1+\lambda)\kappa_a = (1-\lambda)\kappa_a(1-\kappa_\ell\kappa_a) + \lambda\kappa_\ell^{-1} \implies \kappa_a = \frac{1}{\kappa_\ell}\frac{\sqrt{\lambda}}{1+\sqrt{\lambda}}.$$

Compare the coefficient of $\{\hat{y}_t, \hat{y}_{t+2}, \hat{y}_{t+1}\}$ in equation (13) to get $\kappa_0 + \kappa_y$, γ and κ_y

$$(1+\lambda)(\kappa_0+\kappa_y) = (1-\lambda)\kappa_a\kappa_\ell(1-\kappa_0-\kappa_y) + \lambda \implies \kappa_0 + \kappa_y = \kappa_\ell\kappa_a.$$
$$(1+\lambda)\gamma^2\kappa_y = (1-\lambda)(-\kappa_a\kappa_\ell\gamma^2\kappa_y + \gamma\kappa_y) \implies \gamma = 1 - \sqrt{\lambda}.$$
$$(1+\lambda)\gamma\kappa_y = (1-\lambda)(-\kappa_a\kappa_\ell\gamma\kappa_y + \kappa_0 + \kappa_y) + \lambda \implies \kappa_y = \gamma^{-1}(\kappa_0+\kappa_y).$$

[14]This equation still holds when level-k expectations are imposed on all variables.

Compare coefficients on $\eta_t^d + \hat{r}_t - \hat{\pi}_{t+1}$ in equation (13) to get κ_r

$$-(1+\lambda)\kappa_r\omega = (1-\lambda)\kappa_a\kappa_\ell\kappa_r\omega - \omega \implies \kappa_r = 1-(\kappa_0+\kappa_y).$$

In the equilibrium, we have $\hat{y}_t = \hat{c}_{i,t}^{unc}$ and $\hat{a}_{i,t} = 0$ and then

$$\hat{y}_t = (1-\gamma)\sum_{s=0}^{+\infty}\gamma^s\hat{y}_{t+1+s} - \omega\sum_{s=0}^{+\infty}\gamma^s(\eta_{t+s}^d + \hat{r}_{t+s} - \hat{\pi}_{t+1+s}).$$

Now, impose level-k on it and obtain

$$\hat{y}_t^{(k)} = (1-\gamma)\sum_{s=0}^{+\infty}\gamma^s\hat{y}_{t+1+s|t-1}^{e,(k)} - \omega\sum_{s=0}^{+\infty}\gamma^s(\hat{r}_{t+s|t-1}^{e,(k)} - \hat{\pi}_{t+1+s|t-1}^{e,(k)} + \mathbb{E}_{t-1}\eta_{t+s}^d).$$

Proof of Lemma 2

The optimal price satisfies

$$\begin{aligned}
\hat{p}_t^a =& (1-\theta)\sum_{s=0}^{+\infty}\theta^s\sum_{\tau=0}^{s}\hat{\pi}_{t+\tau|t-1}^{e,(k)} + (1-\theta)\sum_{s=0}^{+\infty}\theta^s(\hat{w}_{t+s}^{e,(k)} - \mathbb{E}_{t-1}\eta_{t+s}^z)\\
=& (1-\theta)\sum_{\tau=0}^{+\infty}\sum_{s=\tau}^{+\infty}\theta^s\hat{\pi}_{t+\tau|t-1}^{e,(k)} + (1-\theta)\sum_{s=0}^{+\infty}\theta^s(\hat{w}_{t+s}^{e,(k)} - \mathbb{E}_{t-1}\eta_{t+s}^z)\\
=& \sum_{s=0}^{+\infty}\theta^s\hat{\pi}_{t+s|t-1}^{e,(k)} + (1-\theta)\sum_{s=0}^{+\infty}\theta^s[(\omega^{-1}+\xi^{-1})\hat{y}_{t+s}^{e,(k)} - (1+\xi^{-1})\mathbb{E}_{t-1}\eta_{t+s}^z].
\end{aligned}$$

According to the price aggregator,

$$\hat{\pi}_t = \hat{p}_t^a = (1-\theta)\sum_{s=0}^{+\infty}\theta^s\hat{\pi}_{t+s|t-1}^{e,(k)} + (1-\theta)^2\sum_{s=0}^{+\infty}\theta^s[(\omega^{-1}+\xi^{-1})\hat{y}_{t+s}^{e,(k)} - (1+\xi^{-1})\mathbb{E}_{t-1}\eta_{t+s}^z].$$

Proof of Lemma 3

The interest rate forecast satisfies

$$\begin{aligned}\hat{r}^{e,(k)}_{t+s|t-1} =&\rho_r \hat{r}^{e,(k)}_{t+s-1|t-1} + (1-\rho_r)(\phi_\pi \hat{\pi}^{e,(k)}_{t+s|t-1} + \phi_y \hat{y}^{e,(k)}_{t+s|t-1}) + \sigma_r \mathbb{E}_{t-1}\epsilon^r_{t+s|t-1}\\ =&\rho_r^{s+1}\hat{r}_{t-1} + (1-\rho_r)\sum_{\tau=0}^{s}\rho_r^{\tau}(\phi_\pi \hat{\pi}^{e,(k)}_{t+s-\tau|t-1} + \phi_y \hat{y}^{e,(k)}_{t+s-\tau|t-1}) + \sigma_r \sum_{\tau=0}^{s}\rho_r^{\tau}\mathbb{E}_{t-1}\epsilon^r_{t+s-\tau}\\ =&\rho_r^{s+1}\hat{r}_{t-1} + (1-\rho_r)\sum_{\tau=0}^{s}\rho_r^{s-\tau}(\phi_\pi \hat{\pi}_{t+\tau|t-1} + \phi_y \hat{y}_{t+\tau|t-1}) + \sigma_r \sum_{\tau=0}^{s}\rho_r^{s-\tau}\mathbb{E}_{t-1}\epsilon^r_{t+\tau}.\end{aligned}$$

Use the following identity

$$\sum_{s=0}^{+\infty}\gamma^s\sum_{\tau=0}^{s}\rho_r^{s-\tau} = \sum_{s=0}^{+\infty}\sum_{\tau=0}^{s}\gamma^s\rho_r^{s-\tau} = \sum_{\tau=0}^{+\infty}\sum_{s=\tau}^{+\infty}\gamma^s\rho_r^{s-\tau} = \frac{1}{1-\gamma\rho_r}\sum_{\tau=0}^{+\infty}\gamma^{\tau} = \frac{1}{1-\gamma\rho_r}\sum_{s=0}^{+\infty}\gamma^s.$$

The cumulative effect of interest rate forecasts becomes

$$\begin{aligned}&\sum_{s=0}^{+\infty}\gamma^s\hat{r}^{e,(k)}_{t+s|t-1}\\ =&\sum_{s=0}^{+\infty}\gamma^s\left[\rho_r^{s+1}\hat{r}_{t-1} + (1-\rho_r)\sum_{\tau=0}^{s}\rho_r^{s-\tau}(\phi_\pi \hat{\pi}^{e,(k)}_{t+\tau|t-1} + \phi_y \hat{y}^{e,(k)}_{t+\tau|t-1}) + \sigma_r \sum_{\tau=0}^{s}\rho_r^{s-\tau}\mathbb{E}_{t-1}\epsilon^r_{t+\tau}\right]\\ =&\frac{1}{1-\gamma\rho_r}\left[\rho_r\hat{r}_{t-1} + (1-\rho_r)\sum_{s=0}^{+\infty}\gamma^s(\phi_\pi \hat{\pi}^{e,(k)}_{t+s|t-1} + \phi_y \hat{y}^{e,(k)}_{t+s|t-1}) + \sigma_r \sum_{s=0}^{+\infty}\gamma^s\mathbb{E}_{t-1}\epsilon^r_{t+s}\right].\end{aligned}$$

Proof of Proposition 2

Lemma 1 and 3 imply

$$\begin{aligned}
\hat{y}_t^{(k)} =&(1-\gamma)\sum_{s=0}^{+\infty}\gamma^s\hat{y}_{t+1+s|t-1}^{e,(k)} - \omega\sum_{s=0}^{+\infty}\gamma^s(\eta_{t+s}^d + \hat{r}_{t+s|t-1}^{e,(k)} - \hat{\pi}_{t+1+s|t-1}^{e,(k)}) \\
=&-\frac{\omega}{1-\gamma\rho_r}\left[\rho_r\hat{r}_{t-1} + (1-\rho_r)\sum_{s=0}^{+\infty}\gamma^s(\phi_\pi\hat{\pi}_{t+s|t-1}^{e,(k)} + \phi_y\hat{y}_{t+s|t-1}^{e,(k)}) + \sigma_r\sum_{s=0}^{+\infty}\gamma^s\mathbb{E}_{t-1}\epsilon_{t+s}^r\right] \\
&-\frac{\omega}{1-\gamma\rho_d}\eta_t^d + \sum_{s=0}^{+\infty}\gamma^s\left[(1-\gamma)\hat{y}_{t+1+s|t-1}^{e,(k)} + \omega\hat{\pi}_{t+1+s|t-1}^{e,(k)}\right] \\
=&-\frac{\omega}{1-\gamma\rho_r}\left[\rho_r\hat{r}_{t-1} + (1-\rho_r)(\phi_\pi\hat{\pi}_{t|t-1}^{e,(k)} + \phi_y\hat{y}_{t|t-1}^{e,(k)})\right] - \frac{\omega}{1-\gamma\rho_d}\eta_t^d \\
&+(1-\gamma)\sum_{s=0}^{+\infty}\gamma^s\left[1 - \frac{\omega\gamma(1-\rho_r)}{(1-\gamma)(1-\gamma\rho_r)}\phi_y\right]\hat{y}_{t+1+s}^{e,(k)} \\
&+(1-\gamma)\sum_{s=0}^{+\infty}\gamma^s\left[\frac{\omega}{1-\gamma} - \frac{\omega\gamma(1-\rho_r)}{(1-\gamma)(1-\gamma\rho_r)}\phi_\pi\right]\hat{\pi}_{t+1+s}^{e,(k)}.
\end{aligned}$$

The inflation equation is obvious from Lemma 2.

Proof of Proposition 3

Base on the initialization of level-0, we have

$$\begin{aligned}
\hat{y}_{t|t-1}^{e,(1)} =&\hat{y}_{t-1}, \\
\hat{\pi}_{t|t-1}^{e,(1)} =&0.
\end{aligned}$$

This implies that in $H^{e,(1)}$, output is full anchored. Hence,

$$\hat{y}_{t+s|t-1}^{e,(1)} = \hat{y}_{t-1}.$$

Applied this output forecasts in Proposition 2, we get

$$\begin{aligned}\hat{y}^{e,(2)}_{t|t-1} - \hat{y}_{t-1} =& \omega\left(-\frac{1-\rho_r}{1-\gamma\rho_r}\frac{\phi_y}{1-\gamma}\hat{y}_{t-1} - \frac{\rho_r}{1-\gamma\rho_r}\hat{r}_{t-1} - \frac{1}{1-\gamma\rho_d}\eta^d_t\right),\\ \hat{\pi}^{e,(2)}_{t|t-1} =& \kappa\left(\hat{y}_{t-1} - \omega\frac{1-\theta}{1-\theta\rho_z}\eta^z_t\right).\end{aligned}$$

For $k \in [1,2]$, $(\hat{y}^{e,(k)}_{t|t-1}, \hat{\pi}^{e,(k)}_{t|t-1}) = (2-k)(\hat{y}^{e,(1)}_{t|t-1}, \hat{\pi}^{e,(1)}_{t|t-1}) + (k-1)(\hat{y}^{e,(2)}_{t|t-1}, \hat{\pi}^{e,(2)}_{t|t-1})$.

Proof of Corollary 1

When $\eta^d_t = \eta^z_t = 0$ and $k \in [1,2]$, the perceived aggregate law of motion becomes

$$\begin{bmatrix}\hat{y}^{e,(k)}_{t|t-1}\\ \hat{r}^{e,(k)}_{t|t-1}\end{bmatrix} = \begin{bmatrix}1 & -(k-1)\frac{\omega\rho_r}{1-\gamma\rho_r}\\ (k-1)(1-\rho_r)[(1-\theta)(\omega^{-1}+\xi^{-1})\phi_\pi + \phi_y] & \rho_r - (k-1)\frac{\omega\rho_r}{1-\gamma\rho_r}(1-\rho_r)\phi_y\end{bmatrix}\begin{bmatrix}\hat{y}_{t-1}\\ \hat{r}_{t-1}\end{bmatrix}.$$

Denote the perceived law of motion as $\hat{h}^{e,(k)}$, then

$$\begin{bmatrix}\hat{y}^{e,(k)}_{t+s|t-1}\\ \hat{r}^{e,(k)}_{t+s|t-1}\end{bmatrix} = (\hat{h}^{e,(k)})^{s+1}\begin{bmatrix}\hat{y}_{t-1}\\ \hat{r}_{t-1}\end{bmatrix}.$$

Hence, $\hat{y}^{e,(k)}_{t+3|t-1}$ can be obtained directly, and $\hat{\pi}^{e,(k)}_{t+3|t-1}$ can be obtained from

$$\hat{\pi}^{e,(k)}_{t+3|t-1} = (k-1)\kappa\hat{y}^{e,(k)}_{t+2|t-1}.$$

Proof of Corollary 2

Change all time index t to $t+1$ in equation (1) and forecast it based on period $t-1$ information.

$$\hat{y}^{(k)}_{t+1} = (1-\gamma)\sum_{s=0}^{+\infty}\gamma^s\hat{y}^{e,(k)}_{(t+2+s|t)|t-1} - \omega\sum_{s=0}^{+\infty}\gamma^s(\hat{r}^{e,(k)}_{(t+1+s|t)|t-1} - \hat{\pi}^{e,(k)}_{(t+2+s|t)t-1} + \mathbb{E}_{t-1}\eta^d_{t+1+s}) \quad (14)$$

Combine equation (1) and (14), and we can get equation (6).

Proof of Proposition 4

Substituting $\hat{\pi}^{e,(k)}_{t+1+s|t-1} = (k-1)\kappa\hat{y}^{e,(k)}_{t+s|t-1}$ into equation (4) yields equation (7).

Proof of Proposition 5

Set $\rho_r = \phi_\pi = \eta^d = \eta^z = 0$ in equation (7), and we can get

$$\hat{y}^{(k)}_t = -\omega\phi_y\hat{y}^{e,(k)}_{t|t-1} + \left(1 - \frac{\omega\gamma}{1-\gamma}\phi_y\right)(1-\gamma)\sum_{s=0}^{+\infty}\gamma^s\hat{y}^{e,(k)}_{t+1+s|t-1}. \tag{15}$$

Since the only state variable now is $\hat{y}_{t-1}$, and it must converge the 0 in the long run, the law of motion becomes $\hat{y}^{(k)}_t = \hat{\rho}^{(k)}_y\hat{y}_{t-1}$. When $k \in [1, +\infty)\cap\mathbb{N}$, the perceived law of motion becomes $\hat{y}^{e,(k)}_{t+s|t-1} = (\rho^{e,(k)}_y)^{s+1}\hat{y}_{t-1} = (\rho^{(k-1)}_y)^{s+1}\hat{y}_{t-1}$.

Substituting the actual and perceived law of motion back to equation (15) yields equation (8).

Proof of Corollary 3

When $|\rho^{(k-1)}_y| \leq 1$, $\frac{\rho^{(k)}_y}{\rho^{(k-1)}_y}$ is monotonic in $\rho^{(k-1)}_y$. Hence, we only need to check the bounds.

When $\rho^{(k-1)}_y = 1$, eductive stability requires

$$\frac{\rho^{(k)}_y}{\rho^{(k-1)}_y} = 1 - \frac{\omega}{1-\gamma}\phi_y > -1 \implies \gamma < 1 - \frac{1}{2}\omega\phi_y.$$

When $\rho^{(k-1)}_y = -1$, eductive stability requires

$$\frac{\rho^{(k)}_y}{\rho^{(k-1)}_y} = -\omega\phi_y - \left(1 - \frac{\omega\gamma}{1-\gamma}\phi_y\right)\frac{1-\gamma}{1+\gamma} = -\frac{1-\gamma}{1+\gamma} - \frac{\omega}{1+\gamma}\phi_y > -1 \implies \gamma > \frac{1}{2}\omega\phi_y.$$

Once these conditions are satisfied, $|\frac{\rho^{(k)}_y}{\rho^{(k-1)}_y}| \leq M$ for $M = \max\left\{|1 - \frac{\omega}{1-\gamma}\phi_y|, \frac{1-\gamma}{1+\gamma} + \frac{\omega}{1+\gamma}\phi_y\right\}$.

Proof of Corollary 4

$$\hat{\pi}_t^{(k)} = (1-\theta)\sum_{s=0}^{+\infty}\theta^s\hat{\pi}_{t+s|t-1}^{e,(k)} + (1-\theta)^2\sum_{s=0}^{+\infty}\theta^s(\omega^{-1}+\xi^{-1})\hat{y}_{t+s|t-1}^{e,(k)}.$$

$$\hat{\pi}_t^{(+\infty)} = (1-\theta)\sum_{s=0}^{+\infty}\theta^s\hat{\pi}_{t+s}^{(+\infty)} + \kappa(1-\theta)\sum_{s=0}^{+\infty}\theta^s\hat{y}_{t+s}^{(+\infty)}.$$

$$\theta\hat{\pi}_{t+1}^{(+\infty)} = (1-\theta)\sum_{s=1}^{+\infty}\theta^s\hat{\pi}_{t+s}^{(+\infty)} + \kappa(1-\theta)\sum_{s=1}^{+\infty}\theta^s\hat{y}_{t+s}^{(+\infty)}.$$

$$\hat{\pi}_t^{(+\infty)} - \theta\hat{\pi}_{t+1}^{(+\infty)} = (1-\theta)\hat{\pi}_t^{(+\infty)} + \kappa(1-\theta)\hat{y}_t^{(+\infty)}.$$

$$\hat{\pi}_t^{(+\infty)} = \frac{\kappa(1-\theta)}{\theta}\hat{y}_t^{(+\infty)} + \hat{\pi}_{t+1}^{(+\infty)} = \frac{\kappa(1-\theta)}{\theta}\sum_{s=0}^{+\infty}\hat{y}_{t+s}^{(+\infty)}.$$

Proof of Corollary 5

$$\begin{aligned}
\hat{\pi}_t^{(k)} =& (1-\theta)\sum_{s=0}^{+\infty}\theta^s\hat{\pi}_{t+s|t-1}^{e,(k)} + (1-\theta)^2\sum_{s=0}^{+\infty}\theta^s(\omega^{-1}+\xi^{-1})\hat{y}_{t+s|t-1}^{e,(k)} \\
=& (k-1)\kappa(1-\theta)\sum_{s=0}^{+\infty}\theta^s\hat{y}_{t-1+s|t-1}^{e,(k)} + \kappa(1-\theta)\sum_{s=0}^{+\infty}\theta^s\hat{y}_{t+s|t-1}^{e,(k)} \\
=& \kappa(k-1)(1-\theta)\hat{y}_{t-1} + \kappa[1+\theta(k-1)](1-\theta)\sum_{s=0}^{+\infty}\theta^s\hat{y}_{t+s|t-1}^{e,(k)}.
\end{aligned}$$

Proof of Proposition 6

When nominal interest rate becomes zero, it declines π_{SS} compared to the original steady state. As the natural rate of interest changes by $\hat{r}^n$, the interest rate gap becomes $\hat{r}_t = -(\pi_{SS} + \hat{r}^n)$ in the liquidity trap. Neither level-0 nor level-1 households react to expectations on future. Hence, we have

$$\begin{aligned}
\hat{y}^{e,(1)}_{t+s|t-1} =& \hat{y}_{t-1}, \\
\hat{\pi}^{e,(1)}_{t+s|t-1} =& 0, \\
\hat{y}^{e,(2)}_{t+s|t-1} =& \hat{y}_{t-1} + (s+1)\frac{\omega}{1-\gamma}(\hat{\pi}_{SS} + \hat{r}^n), \\
\hat{\pi}^{e,(2)}_{t+s|t-1} =& \kappa \hat{y}^{e,(2)}_{t-1+s|t-1}.
\end{aligned}$$

For $k \in [1,2]$, $(\hat{y}^{e,(k)}_{t|t-1}, \hat{\pi}^{e,(k)}_{t|t-1}) = (2-k)(\hat{y}^{e,(1)}_{t|t-1}, \hat{\pi}^{e,(1)}_{t|t-1}) + (k-1)(\hat{y}^{e,(2)}_{t|t-1}, \hat{\pi}^{e,(2)}_{t|t-1})$.

Proof of Proposition 7

The proof is identical to Proposition 6 except that the cumulative effects of interest rate is in a different form

$$\hat{y}^{e,(2)}_{t+1+s|t-1} = \hat{y}_{t-1} - \omega \sum_{\nu=0}^{s+1} \gamma^{\tau-\nu} \hat{r}^{e,(k)}_{t+\tau|t-1} = \hat{y}_{t-1} - \omega \frac{\gamma^{\tau-s-1} - \gamma^{\tau+1}}{1-\gamma} \hat{r}^{e,(k)}_{t+\tau|t-1}.$$

Derive the forward guidance dynamics

$$\begin{aligned}
\hat{y}^{(k)}_t =& (1-\gamma) \sum_{s=0}^{\tau-1} \gamma^s \hat{y}^{e,(k)}_{t+1+s|t-1} + \omega \sum_{s=0}^{\tau-1} \gamma^s \hat{\pi}^{e,(k)}_{t+1+s|t-1} \\
&+ \gamma^\tau [(1-\gamma)\hat{y}^{e,(k)} + \omega(\hat{\pi}^{e,(k)} - \mathbb{I}_r)](\mathbb{I} - \gamma \hat{h}^{e,(k)})^{-1} \hat{s}^{e,(k)}_{t+1+\tau|t-1}.
\end{aligned}$$

Appendix B: State Space Representation of the Full Model

The standard solution procedure for rational expectations DSGE models cannot be directly applied here. Hence, it is useful to describe how to write the model into a state space form. Use Γ to denote the coefficients in linearized equilibrium objects, and the solution procedure can be briefly described in the following.

1. Solve for $\Gamma^{ca,e}$ without using equilibrium objects.
2. Initialize $(\Gamma^{ys,e,(1)}, \Gamma^{\pi s,e,(1)}, \Gamma^{ws,e,(1)}, \Gamma^{\tau s,e,(1)})$ from level-0, and obtain $\Gamma^{ss,e,(1)}$.
3. Solve for $\Gamma^{cs,e,(1)}$ from the perceived households' problem.
4. Solve for $(\Gamma^{ys,(1)}, \Gamma^{\ell s,(1)}, \Gamma^{ws,(1)}, \Gamma^{\tau s,(1)})$ from the temporary equilibrium.
5. Solve for $\Gamma^{\pi s,(1)}$ from the firms' problem, and obtain $\Gamma^{ss,(1)}$.
6. Use $(\Gamma^{ys,e,(j+1)}, \Gamma^{\pi s,e,(j+1)}, \Gamma^{ws,e,(j+1)}, \Gamma^{\tau s,e,(j+1)}) = (\Gamma^{ys,(j)}, \Gamma^{\pi s,(j)}, \Gamma^{ws,(j)}, \Gamma^{\tau s,(j)})$ to update.
7. Obtain the state space representation.

Step 1: Solve for $\Gamma^{ca,e}$

Log-linearizing the optimality conditions for the constrained households yields

$$\begin{aligned}\omega^{-1}\hat{c}^e(\bar{\zeta}) =& \hat{w}^e - \xi^{-1}\hat{\ell}^e(\bar{\zeta}),\\ \hat{c}^e(\bar{\zeta}) =& \hat{a} + wy_{SS}(\hat{w}^e + \hat{\tau}^e + \hat{\ell}^e(\bar{\zeta})).\end{aligned}$$

$(\Gamma^{ca,e}(\bar{\zeta}), \Gamma^{\ell a,e}(\bar{\zeta}))$ can be obtained from

$$\begin{bmatrix}\omega^{-1} & \xi^{-1}\\ 1 & -wy_{SS}\end{bmatrix}\begin{bmatrix}\Gamma^{ca,e}(\bar{\zeta})\\ \Gamma^{\ell a,e}(\bar{\zeta})\end{bmatrix} = \begin{bmatrix}0\\ 1\end{bmatrix}.$$

The solution is $\Gamma^{ca,e}(\bar{\zeta}) = \frac{1}{1+\xi\omega^{-1}wy_{SS}}$.

The unconstrained households have

$$\begin{aligned}\hat{c}^e(1) =& -\omega(\eta^d + \hat{r}^e - \hat{\pi}^{e\prime}) + \underline{\lambda}\hat{c}^{e\prime}(1) + (1-\underline{\lambda})\hat{c}^{e\prime}(\overline{\zeta}),\\ \hat{a}^{e\prime}(1) =& (R_{SS}/\Pi_{SS})[\hat{a} + wy_{SS}(\hat{w}^e + \hat{\tau}^e + \hat{\ell}^e(1)) - \hat{c}^e(1)],\\ \hat{\ell}^e(1) =& \xi\hat{w}^e - \xi\omega^{-1}\hat{c}^e(1).\end{aligned}$$

$(\Gamma^{ca,e}(1), \Gamma^{\ell a,e}(1), \Gamma^{aa,e}(1))$ satisfy

$$\begin{aligned}\Gamma^{ca,e}(1) =& [\underline{\lambda}\Gamma^{ca,e}(1) + (1-\underline{\lambda})\Gamma^{ca,e}(\overline{\zeta})]\Gamma^{aa,e}(1),\\ \Gamma^{aa,e}(1) =& (R_{SS}/\Pi_{SS})(1 + wy_{SS}\Gamma^{\ell a,e}(1) - \Gamma^{ca,e}(1)),\\ \Gamma^{\ell a,e}(1) =& -\xi\omega^{-1}\Gamma^{ca,e}(1).\end{aligned}$$

This yields a quadratic function for $\Gamma^{ca,e}(1)$

$$(\Pi_{SS}/R_{SS})\Gamma^{ca,e}(1) = [\underline{\lambda}\Gamma^{ca,e}(1) + (1-\underline{\lambda})\Gamma^{ca,e}(\overline{\zeta})][1 - (1 + \xi\omega^{-1}wy_{SS})\Gamma^{ca,e}(1)].$$

Solving $\underline{\lambda}$ from $\Gamma^{ca,e}(1)$ yields

$$\underline{\lambda} = \frac{\Gamma^{ca,e}(\overline{\zeta}) - \Gamma^{ca,e}(1)\frac{(\Pi_{SS}/R_{SS})}{1-(1+\xi\omega^{-1}wy_{SS})\Gamma^{ca,e}(1)}}{\Gamma^{ca,e}(\overline{\zeta}) - \Gamma^{ca,e}(1)}.$$

The notation $\underline{\lambda} = \frac{\lambda_1}{\lambda_1 + (1-\lambda_1)\overline{\zeta}}$ yields

$$\lambda_1 = [1 + (\underline{\lambda}^{-1} - 1)\overline{\zeta}^{-1}]^{-1}.$$

The fraction of hand-to-mouth households λ_{HtM} satisfies

$$\begin{aligned}\lambda_{HtM} =& (1-\lambda_1)/(2-\lambda_1-\lambda_2),\\ \lambda_2 =& 1 - (1-\lambda_1)(1-\lambda_{HtM})/\lambda_{HtM}.\end{aligned}$$

Step 2: Initialize $(\Gamma^{ys,e,(1)}, \Gamma^{\pi s,e,(1)}, \Gamma^{ws,e,(1)}, \Gamma^{\tau s,e,(1)})$ **and** $\Gamma^{ss,e,(1)}$

Specify the level-1 expectations.

$$\begin{aligned}
\Gamma^{ys,e,(1)}\hat{s} =& \hat{y}_-, \\
\Gamma^{\pi s,e,(1)}\hat{s} =& 0, \\
\Gamma^{\ell s,e,(1)}\hat{s} =& \hat{y}_- - \eta^z, \\
\Gamma^{ws,e,(1)} =& \omega^{-1}\Gamma^{ys,e,(1)} + \xi^{-1}\Gamma^{\ell s,e,(1)}, \\
\Gamma^{\tau s,e,(1)} =& wy_{SS}^{-1}\Gamma^{ys,e,(1)} - \Gamma^{\ell s,e,(1)} - \Gamma^{ws,e,(1)}.
\end{aligned}$$

According to the perceived Taylor Rule,

$$\Gamma^{rs,e,(1)} = \rho_r\Gamma^r + (1-\rho_r)(\phi_\pi\Gamma^{\pi s,e,(1)} + \phi_y\Gamma^{ys,e,(1)}).$$

The state variable is $\hat{s} = (\hat{y}_-, \hat{r}_-, \eta^d, \eta^z)$. $\Gamma^{ss,e,(1)}$ can be obtained from $(\Gamma^{ys,e,(1)}, \Gamma^{rs,e,(1)})$ and the exogenous law of motion for (η^d, η^z).

Step 3: Solve for $\Gamma^{cs,e,(1)}$

Recall the optimality conditions of the constrained households

$$\begin{aligned}
\omega^{-1}\hat{c}^e(\bar{\zeta}) =& \hat{w}^e - \xi^{-1}\hat{\ell}^e(\bar{\zeta}), \\
\hat{c}^e(\bar{\zeta}) =& \hat{a} + wy_{SS}(\hat{w}^e + \hat{\tau}^e + \hat{\ell}^e(\bar{\zeta})).
\end{aligned}$$

$(\Gamma^{cs,e,(1)}(\bar{\zeta}), \Gamma^{\ell s,e,(1)}(\bar{\zeta}))$ can be obtained from

$$\begin{bmatrix} \omega^{-1} & \xi^{-1} \\ 1 & -wy_{SS} \end{bmatrix} \begin{bmatrix} \Gamma^{cs,e,(1)}(\bar{\zeta}) \\ \Gamma^{\ell s,e,(1)}(\bar{\zeta}) \end{bmatrix} = \begin{bmatrix} \Gamma^{ws,e,(1)} \\ wy_{SS}(\Gamma^{ws,e,(1)} + \Gamma^{\tau s,e,(1)}) \end{bmatrix}.$$

Recall the optimality conditions of the unconstrained households

$$\begin{aligned}
\hat{c}^{e}(1) =& -\omega(\eta^{d} + \hat{r}^{e} - \hat{\pi}^{e\prime}) + \underline{\lambda}\hat{c}^{e\prime}(1) + (1-\underline{\lambda})\hat{c}^{e\prime}(\overline{\zeta}),\\
\hat{a}^{e\prime}(1) =& (R_{SS}/\Pi_{SS})[\hat{a} + wy_{SS}(\hat{w}^{e} + \hat{\tau}^{e} + \hat{\ell}^{e}(1)) - \hat{c}^{e}(1)],\\
\hat{\ell}^{e}(1) =& \xi\hat{w}^{e} - \xi\omega^{-1}\hat{c}^{e}(1).
\end{aligned}$$

$(\Gamma^{cs,e,(1)}(1), \Gamma^{\ell s,e,(1)}(1), \Gamma^{as,e,(1)}(1))$ satisfy

$$\begin{aligned}
\Gamma^{cs,e,(1)}(1) =& -\omega(\Gamma^{ds} + \Gamma^{rs,e,(1)} - \Gamma^{\pi s,e,(1)}\Gamma^{ss,e,(1)})\\
&+ [\underline{\lambda}\Gamma^{ca,e,(1)}(1) + (1-\underline{\lambda})\Gamma^{ca,e,(1)}(\overline{\zeta})]\Gamma^{as,e,(1)}(1),\\
&+ [\underline{\lambda}\Gamma^{cs,e,(1)}(1) + (1-\underline{\lambda})\Gamma^{cs,e,(1)}(\overline{\zeta})]\Gamma^{ss,e,(1)},\\
\Gamma^{as,e,(1)}(1) =& (R_{SS}/\Pi_{SS})[wy_{SS}(\Gamma^{ws,e,(1)} + \Gamma^{\tau s,e,(1)} + \Gamma^{\ell s,e,(1)}(1)) - \Gamma^{cs,e,(1)}(1)],\\
\Gamma^{\ell s,e,(1)}(1) =& \xi\Gamma^{ws,e,(1)} - \xi\omega^{-1}\Gamma^{cs,e,(1)}(1).
\end{aligned}$$

Eliminate $(\Gamma^{\ell s,e,(1)}(1), \Gamma^{as,e,(1)}(1))$ to obtain a single equation of $\Gamma^{cs,e,(1)}(1)$

$$\begin{aligned}
\Gamma^{cs,e,(1)}(1) =& -\omega(\Gamma^{ds} + \Gamma^{rs,e,(1)} - \Gamma^{\pi s,e,(1)}\Gamma^{ss,e,(1)})\\
&+ [\underline{\lambda}\Gamma^{ca,e,(1)}(1) + (1-\underline{\lambda})\Gamma^{ca,e,(1)}(\overline{\zeta})]\\
&\cdot (R_{SS}/\Pi_{SS})\{wy_{SS}[(1+\xi)\Gamma^{ws,e,(1)} + \Gamma^{\tau s,e,(1)}] - (1+\xi\omega^{-1}wy_{SS})\Gamma^{cs,e,(1)}(1)\},\\
&+ [\underline{\lambda}\Gamma^{cs,e,(1)}(1) + (1-\underline{\lambda})\Gamma^{cs,e,(1)}(\overline{\zeta})]\Gamma^{ss,e,(1)}.
\end{aligned}$$

The solution for $\Gamma^{cs,e,(1)}(1)$ is

$$\begin{aligned}
\Gamma^{cs,e,(1)}(1) =& \{(R_{SS}/\Pi_{SS})wy_{SS}[\underline{\lambda}\Gamma^{ca,e,(1)}(1) + (1-\underline{\lambda})\Gamma^{ca,e,(1)}(\overline{\zeta})][(1+\xi)\Gamma^{ws,e,(1)} + \Gamma^{\tau s,e,(1)}]\\
&+ (1-\underline{\lambda})\Gamma^{cs,e,(1)}(\overline{\zeta})\Gamma^{ss,e,(1)} - \omega(\Gamma^{ds} + \Gamma^{rs,e,(1)} - \Gamma^{\pi s,e,(1)}\Gamma^{ss,e,(1)})\}\\
&\{\mathbb{I} + (R_{SS}/\Pi_{SS})(1+\xi\omega^{-1}wy_{SS})[\underline{\lambda}\Gamma^{ca,e,(1)}(1) + (1-\underline{\lambda})\Gamma^{ca,e,(1)}(\overline{\zeta})] * \mathbb{I} - \underline{\lambda}\Gamma^{ss,e,(1)}\}^{-1}.
\end{aligned}$$

Step 4: Solve for $(\Gamma^{ys,(1)}, \Gamma^{\ell s,(1)}, \Gamma^{ws,(1)}, \Gamma^{\tau s,(1)})$

The temporary equilibrium satisfies

$$\begin{aligned}
\hat{y} =& -\omega(\eta^d + \hat{r}^e - \hat{\pi}^{e\prime}) + \underline{\lambda}\hat{c}^{e\prime}(1) + (1-\underline{\lambda})\hat{c}^{e\prime}(\overline{\zeta}),\\
\hat{\ell} =& \hat{y} - \eta^z,\\
\hat{w} =& \xi^{-1}\hat{\ell} + \omega^{-1}\hat{y},\\
\hat{\tau} =& wy_{SS}^{-1}\hat{y} - \hat{\ell} - \hat{w}.
\end{aligned}$$

$(\Gamma^{ys,(1)}, \Gamma^{\ell s,(1)}, \Gamma^{ws,(1)}, \Gamma^{\tau s,(1)})$ can be obtained from

$$\begin{aligned}
\Gamma^{ys,(1)} =& -\omega(\Gamma^d + \Gamma^{rs,e,(1)} - \Gamma^{\pi s,e,(1)}\Gamma^{ss,e,(1)}) + [\underline{\lambda}\Gamma^{cs,e,(1)}(1) + (1-\underline{\lambda})\Gamma^{cs,e,(1)}(\overline{\zeta})]\Gamma^{ss,e,(1)},\\
\Gamma^{\ell s,(1)} =& \Gamma^{ys,(1)} - \Gamma^z,\\
\Gamma^{ws,(1)} =& \xi^{-1}\Gamma^{\ell s,(1)} + \omega^{-1}\Gamma^{ys,(1)},\\
\Gamma^{\tau s,(1)} =& wy_{SS}^{-1}\Gamma^{ys,(1)} - \Gamma^{\ell s,e,(1)} - \Gamma^{ws,(1)}.
\end{aligned}$$

Step 5: Solve for $\Gamma^{\pi s,(1)}$ and Obtain $\Gamma^{ss,(1)}$

Denote $\beta_f = \Pi_{SS}/R_{SS}$ The linearized Phillips Curve with arbitrary expectations $\widetilde{\mathbb{E}}_t$ is

$$\hat{\pi}_t = (1-\theta)(1-\beta_f\theta)\sum_{s=0}^{\infty}(\beta_f\theta)^s\widetilde{\mathbb{E}}_t(\hat{w}_{t+s} - \eta^z_{t+s}) + (1-\theta)\sum_{s=0}^{\infty}(\beta_f\theta)^s\widetilde{\mathbb{E}}_t\hat{\pi}_{t+s}.$$

The matrix representation for $(\Gamma^{\pi s,(1)}, \Gamma^{rs,(1)})$ is

$$\begin{aligned}
\Gamma^{\pi s,(1)} =& (1-\theta)\left\{[(1-\beta_f\theta)\Gamma^{ws,e,(1)} + \Gamma^{\pi s,e,(1)}](\mathbb{I} - \beta_f\theta\Gamma^{ss,e,(1)})^{-1} - \frac{1-\beta_f\theta}{1-\beta_f\theta\rho_z}\Gamma^z\right\},\\
\Gamma^{rs,(1)} =& \rho_r\Gamma^r + (1-\rho_r)(\phi_\pi\Gamma^{\pi s,(1)} + \phi_y\Gamma^{ys,(1)}).
\end{aligned}$$

$\Gamma^{ss,(1)}$ can be obtained from $(\Gamma^{ys,(1)}, \Gamma^{rs,(1)})$ and the exogenous law of motion for (η^d, η^z).

Step 6: Update Expectations

For $\forall k \in [1, +\infty)$, first update expectations to $[k]$ using

$$(\Gamma^{ys,e,(j+1)}, \Gamma^{\pi s,e,(j+1)}, \Gamma^{ws,e,(j+1)}, \Gamma^{\tau s,e,(j+1)}) = (\Gamma^{ys,(j)}, \Gamma^{\pi s,(j)}, \Gamma^{ws,(j)}, \Gamma^{\tau s,(j)}).$$

Level-k expectations are defined as

$$\text{level-k} = (1 - k + [k]) \cdot \text{level-[k]} + ([k] - k) \cdot \text{level-[k+1]}.$$

Step 7: State Space Representation

The transition equation is

$$\hat{s}_{t+1} = \begin{bmatrix} \hat{y}_t \\ \hat{r}_t \\ \hat{\eta}^d_{t+1} \\ \hat{\eta}^z_{t+1} \end{bmatrix} = \Gamma^{ss,(k)} \begin{bmatrix} \hat{y}_{t-1} \\ \hat{r}_{t-1} \\ \hat{\eta}^d_t \\ \hat{\eta}^z_t \end{bmatrix} + \begin{bmatrix} 0 \\ \sigma_m \epsilon^r_t \\ \sigma_d \epsilon^d_t \\ \sigma_z \epsilon^z_t \end{bmatrix}.$$

The measurement equation is

$$\begin{bmatrix} \hat{y}_{t+1} - \hat{y}_t \\ \hat{p}_{t+1} - \hat{p}_t \\ \hat{r}_t \end{bmatrix} = \begin{bmatrix} \Gamma^{ys} - \Gamma^y \\ \Gamma^{\pi s} \\ \Gamma^r \end{bmatrix} \hat{s}_{t+1}.$$

The expectation equations and ex post counterparts are

$$\begin{bmatrix} \hat{y}^e_{t+4} - \hat{y}_t \\ \hat{p}^e_{t+4} - \hat{p}_t \\ \hat{y}_{t+4} - \hat{y}_t \\ \hat{p}_{t+4} - \hat{p}_t \end{bmatrix} = \begin{bmatrix} \Gamma^{ys,e,(k)} (\Gamma^{ss,e,(k)})^3 - \Gamma^y \\ \Gamma^{\pi s,e,(k)} \sum_{\tau=0}^3 (\Gamma^{ss,e,(k)})^\tau \\ \Gamma^{ys,(k)} (\Gamma^{ss,(k)})^3 - \Gamma^y \\ \Gamma^{\pi s,(k)} \sum_{\tau=0}^3 (\Gamma^{ss,(k)})^\tau \end{bmatrix} \hat{s}_{t+1}.$$