ECON 897 - Waiver Exam
August 21, 2017

Important: This is a closed-book test. No books or lecture notes are permitted. You have **180** minutes to complete the test. Answer all questions. You can use all the results covered in class, but please make sure the conditions are satisfied. Write your name on each blue book and label each question clearly. Write legibly. Good luck!

1. (5 points) Suppose p_n and q_n are convergent sequences in the metric space (X, d). Show that $d(p_n, q_n)$ is a convergent sequence in $\mathbb{R}$. Hint: Show that $|d(p_n, q_n) - d(p_m, q_m)| \leq d(q_m, q_n) + d(p_m, p_n)$.

2. (15 points) We want to prove that if every closed and bounded subset of a metric space M is compact, then M is complete.

   (a) (3 points) Prove that every Cauchy sequence is bounded.

   (b) (4 points) Prove that if a subsequence of a Cauchy sequence converges then the sequence converges.

   (c) (4 points) Prove that if S is bounded, clS is bounded as well.

   (d) (4 points) Conclude the result.

3. (20 points) Let $f : M \to N$.

   (a) (3 points) Define the preimage of the set $S \subseteq N$ of the function f.

   (b) (4 points) Show that for any set $V \in N$, $(f^{-1}(V))^c = f^{-1}(V^c)$.

   (c) (8 points) Show that $f : M \to N$ is continuous if and only if for all $A \subseteq N$, $f(\bar{A}) \subseteq \overline{f(A)}$. (Hint: Use the open criterion for continuity.)

   (d) (5 points) We say f is closed if for every closed set $K \in M$, we have $f(K)$ is a closed set in N. Show that f is closed if and only if $\overline{f(A)} \subseteq f(\bar{A})$.

4. (20 points) Consider the functions:

$$F : \mathbb{R}_+ \to [0, 1], F \in C^2, F' > 0$$

$$\sigma(v) : \mathbb{R}_+ \to \mathbb{R}_+, \sigma \in C^2, \sigma' > 0$$

$$U_v(b) = (v - b)F(\sigma^{-1}(b))$$

   Define $b^*(v) = \max_{b \in \mathbb{R}_+} U_v(b)$

(a) (5 points) Assume that there exists a unique $b^*(v) > 0$. Write the first order conditions.

(b) (5 points) Show that $b^*(v)$ is increasing in v. [Hint: Use the implicit function theorem].

(c) (5 points) Assume in addition that $\sigma(v)$ is a function such that $\forall v \in [0,1], b^*(v) = \sigma(v)$. Show that this implies:

$$\frac{v - \sigma(v)}{\sigma'(v)} = \frac{F(v)}{F'(v)}, \forall v \in [0,1]$$

(d) (2 points) Show that if $\forall v \in [0,1], b^*(v) = \sigma(v)$ then $\forall v \in [0,1], \sigma(v) \leq v$.

(e) (3 points) Without any of these additional assumptions, show that $b^*(v) \leq v$.

5. (10 points) Let A be an $n \times n$ projection matrix of rank k. Suppose that $\{v_1, ..., v_n\}$ is a basis for $\mathbb{R}^n$ and that every for v_i there exists a λ_i such that $Av_i = \lambda_i v_i$.

   (a) (5 points) Show that $\lambda_i \neq 0$ if and only if $v_i \in Im(A)$.

   (b) (5 points) Show that there are exactly k vectors with $\lambda_i \neq 0$ and $(n-k)$ vectors for which $\lambda_i = 0$. Furthermore, show that $\sum_{i=1}^n \lambda_i = k$.

6. (10 points) Suppose that A is an $m \times n$ matrix associated with a linear map.

$$D = \{Ax | x \in \mathbb{R}^n, x \geq 0\}$$

   (a) (5 points) Show that the set D is convex and closed. [Hint: use continuity of the function $T(x) = Ax$]

   (b) (5 points) Let $b \notin D, b \in \mathbb{R}^m$. By the separating hyperplane theorem:

$$\exists p \in \mathbb{R}^m, p \neq 0, r \in \mathbb{R}, p^T y < r, \forall y \in D, p^T b > r$$

   Show that in this case, we also have $p^T y \leq 0, \forall y \in D$ [Hint: start by showing that $r > 0$]

7. (5 points) Give an example of a concave function with discontinuities. (A well-drawn graph suffices.) Explain your choice of example in a sentence.

8. (10 points) Consider the profit maximization problem:

$$\max_{x \in \mathbb{R}_+} pf(x) - wx$$

   Suppose $f'(x) > 0$ and $f''(x) < 0$, for all x. Characterize the solutions to this problem. Must a solution exist? If a solution exists, must it be unique? When is there a boundary solution? $\big(x \in \mathbb{R}_+,\ \ w, p \in \mathbb{R}_{++},\ \ f(x) : \mathbb{R}_+ \to \mathbb{R}.\big)$

9. (6 points) Let X be an exponential random variable, $f_X(x) = \lambda e^{-\lambda x}$, $x \geq 0$. Compute the pdf of X conditional on the event $X \geq t$, where t is a fixed positive number. Then, express this conditional pdf $f_X(x \,|\, X \geq t)$ in terms of the original pdf $f_X(\cdot)$.

10. (19 points) Suppose birthdays are independent and uniformly distributed over the 365 days of a year.

    (a) (1 point) What is the probability that any two people share a birthday?

    (b) (10 points) Suppose there are n people, let Y_n be the number of pairs that share a birthday.

        i. (3 points) Compute $E[Y_n]$.

        ii. (7 points) Compute $Var(Y_n)$.

        (Hint: Let $X_{i,j} = 1$ if person i and j have the same birthday, and 0 otherwise.)

    (c) (8 points) Let $Z_n = Y_n / \binom{n}{2}$.

        i. (1 point) Argue that the standard laws of large numbers do not apply to Z_n.

        ii. (7 points) Using Chebychev's inequality, argue that for any $\epsilon > 0$:

$$P((E[Z_n] - Z_n)^2 > \epsilon) \to 0$$

        Conclude that Z_n converges in probability.