Waiver Exam - ECON 897 Final Exam

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Instructions

- This is a closed-book test. No books or lecture notes are permitted.
- You have 180 minutes to complete the exam, the total score is 180.
- Read the questions carefully, and be sure to answer the question asked.
- You can use all the results covered in all three parts, but make sure the conditions are satisfied.
- Please write legibly.
- Good luck!

1. Let A be a compact set on a metric space M. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence contained in A such that every convergent subsequence of (a_n) converges to the same point a^*. Prove that the sequence (a_n) converges to a^*. **[10 points]**

2. Let $m > 0$ be a fixed real number. Define the *Budget Set* correspondence:

$$\Phi : \mathbb{R}^2_{++} \to \mathcal{P}\big(\mathbb{R}^2\big)$$
$$(p_1, p_2) \mapsto B(p_1, p_2)$$
$$B(p_1, p_2) = \big\{(x, y) \in \mathbb{R}^2 : x \geq 0,\ y \geq 0,\ p_1 x + p_2 y \leq m\big\}$$

   (a) Let $S \subset M$ be a subset of a metric space. Let V be an open set such that $V \cap \bar{S} \neq \emptyset$. Prove that $V \cap S \neq \emptyset$. **[10 points]**

   (b) Let V be an open set such that for some $(p_1, p_2) \in \mathbb{R}_{++}$, $V \cap B(p_1, p_2)$ is not empty. Prove there exists $(x^*, y^*) \in V$ such that $p_1 x^* + p_2 y^* < m$. (Hint: use the result of part (a)) **[5 points]**

   (c) Using (b), prove that Φ is lower hemicontinuous. (Hint: use the continuity of $p_1 x^* + p_2 y^*$) **[10 points]**

3. Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable. Suppose there exists $\epsilon > 0$ such that $f''(x) > \epsilon$ for all $x \in \mathbb{R}$.

   (a) Show $f'(x) = 0$ for some $x \in \mathbb{R}$. (Hint: intermediate and mean value theorems.) **[10 points]**

   (b) Prove that f has an unique global minimum. **[10 points]**

4. Assume $U \subset \mathbb{R}^n$ is convex. Let $x^* \in U$ be a point. Prove the followings are equivalent:

   (a) there is no $x \in U$ such that $x_i > x_i^*$ for all $i = 1, \cdots, n$,

   (b) there exists $\lambda \in \mathbb{R}^n_+ \backslash \{0\}$ such that x^* solves

$$\max_{x\in U}\ \lambda^T x.$$

[20 points]

5. Let A be an $m \times n$ matrix and B be an $n \times l$ matrix. Prove

$$\text{rank}A + \text{rank}B - n \leq \text{rank}AB.$$

[20 points]

6. Suppose that (X, Y) has a continuous distribution with continuous probability density function $f : \mathbb{R}^2 \to \mathbb{R}_+$. Show that $U = X + Y$ has a continuous distribution with probability density function f_U given by

$$f_U(u) = \int_{\mathbb{R}} f(v, u - v)\mathrm{d}v.$$

[10 points]

7. Let $X_1, X_2, \cdots$ be iid with cdf

$$F(x) = \frac{1}{1 + e^{-x}}.$$

Let $Y_n = \max\{X_1, \cdots, X_n\} - \log n$. Show that $\{Y_n\}_n$ converges in distribution. **[15 points]**

8. A firm produces a single output $y \in \mathbb{R}_+$ using inputs $z \in \mathbb{R}^n_+$. Given the production function f, the targeted output $y \in \mathbb{R}^n_+$ and a vector of input prices $w \in \mathbb{R}^n_{++}$, the firm's cost minimization problem (CMP) can be stated as follows:

$$\min_z w \cdot z,$$
$$\text{s.t. } f(z) \geq y \text{ and } z \geq 0,$$

Assume $f : \mathbb{R}^n_+ \to \mathbb{R}$ is continuous and the set $\{z \in \mathbb{R}^n_+ | f(z) \geq y\}$ is nonempty.

(a) Prove that the firm's cost minimization problem has a solution. In what follows, denote the solution correspondence by $z(w)$ for each $w \in \mathbb{R}^n_{++}$. **[10 points]**

(b) Suppose $z^* \in z(w)$ is such that $z^* \neq 0$. Prove $f(z^*) = y$. **[10 points]**

(c) Let $c(w) = w \cdot z^*$ for any $z^* \in z(w)$, which is called the firm's cost function. Prove that $c(w)$ is concave. **[10 points]**

(d) Assume $z(w)$ is single valued for all w. Prove $z_i(w)$ is nonincreasing in w_i for any i and $c(w)$ is nondecreasing in w_i. **[10 points]**

Assume for part (e) and part (f) that f is twice continuously differentiable, $z(w)$ is single valued.

(e) Write down the Kuhn-Tucker first-order conditions for a minimum. Under what conditions on f are these first-order conditions necessary and sufficient for a solution? (Specify the most general conditions you can think of.) **[10 points]**

(f) Under the conditions you give in part (e), calculate the derivative of $z(w)$ with respect to w. (It is enough to express the derivatives in terms of matrices.) **[10 points]**