Endogenous Labor Market Cycles

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Abstract

In a perfectly stationary physical environment of the labor market, moral hazard and optimal termination in long-term contracts can generate two-period and much longer cycles in employment and output, and in other aggregate activities, including the creation and destruction of jobs, and the flows of workers entering and exiting employment. This theory sheds light on the unemployment volatility puzzle which has inspired many discussions in the literature.

Keywords: Endogenous Cycles, Moral Hazard, Long-term Contract, Termination

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1 Introduction

The search and matching model of the labor market introduces unemployment and aggregate worker flows into the baseline real business cycle theory, but faces challenges in generating enough cyclical movements in unemployment. This was pointed out by Shimer (2005), who showed that, with the commonly used parameter values, the volatility in unemployment that a standard Mortensen and Pissarides (2009) model can generate is very small compared to that in the U.S. data. The vacancy-unemployment ratio (the “tightness”) in the U.S. data is 20 times as large as what the standard search model predicts. Pissarides (2009) calls this the unemployment volatility puzzle: standard theory cannot explain why unemployment fluctuates so much in the data.

To resolve this puzzle, the baseline search and matching model has been modified to produce more fluctuations in unemployment. The role of sticky wages has been emphasized. Shimer (2005) and Hall (2005) argue that if wages are sticky, then the economy’s response to a productivity shock should be reflected more in the cyclical movements of the model’s employment and unemployment measures. Following this idea, Kennan (2010) and Moen and Rosen (2011) show that private information with regard to match quality can give rise to wage stickiness which, in turn, increases the responsiveness of unemployment to productivity shocks. Costain and Jansen (2010) show that moral hazard, put in a standard search model, could impose a lower bound on the worker’s share of match surplus which, in turn, may amplify fluctuations in hiring and make the firm’s share of the surplus move procyclically.1 The story, however, has not ended.

After reviewing the empirical evidence on wages in the U.S. data, Pissarides (2009) concludes that the empirical evidence on wage rigidity is not consistent with what is needed for the search model to provide a resolution for the unemployment volatility puzzle. He argues that job creation is affected by wages of new hires, which are volatile both in the data

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1See Rogerson and Shimer (2011) for a review of this literature.
and in the standard search and matching model. Although the model does imply (too) much volatility for wages in ongoing jobs, which is sticky in the data, this cyclicality is irrelevant for job creation. The same argument is also made by Haefke, Sonntag and van Rens (2013) and Kudlyak (2014) who provide direct empirical evidence that wages of newly hired workers, unlike the aggregate wage, are as volatile as labor productivity. And they suggest that in order to replicate these findings in a search model, wages must be rigid in ongoing jobs but flexible with new jobs, but this should not affect job creation and thus cannot help resolve the puzzle.

We aim to shed light on the unemployment volatility puzzle. Relative to the literature, however, we take an entirely different approach. Instead of seeking to produce stronger labor market fluctuations, particularly in the measure of labor-market tightness, as a response to productivity shocks, we show that the labor market has, in itself, a natural and potentially important source of instability to drive macroeconomic dynamics in jobs and unemployment. Specifically, we develop a theory of endogenous cycles in the tightness of the labor market, whose cyclical movements are not only what the theory seeks to explain, but also part of the engine for driving market fluctuations.

The story is based on a dynamic interaction between the optimal employment contract, the endogenous shift in market power between firms and unemployed workers, and the creation and destruction of jobs. There are no productivity shocks in the model. There are no search and matching frictions. Neither are there any exogenously imposed or endogenously derived wage rigidity. Wages are not bargained periodically between the matched worker and firm; they are, along with an optimally designed termination clock, part of the employment contract whose values for the parties are determined in the competitive equilibrium of the economy. Moral hazard and non-negative compensation are the only source of instability. Termination, as an incentive device in the optimal contract, induces the endogenous shift in the tightness of the labor market, which then plays a key role in dividing the surplus of contracting within individual firms. Such a theory, differing in almost all its essential aspects
from standard search and matching models, exhibits strong abilities in generating volatility in jobs and unemployment.

Specifically, the model economy consists of overlapping generations of workers who each live for two periods. Firms are free to enter and exit the market to create new jobs and destroy old. Both short-term and long-term contracts are traded in the labor market. Firms run a stochastic technology that uses worker efforts as inputs. Worker efforts are not observable to the firm. Termination, as an instrument for motivating worker efforts, creates an externality. Suppose the current labor market has a small number of vacant firms relative to that of unemployed workers. Then the firms would have an upper hand in the current market which allows them to offer an equilibrium contract that dictates a low expected utility for new hires, together with a high probability of termination. This high probability of termination, in turn, would generate a large number of vacant firms in the next period, giving them a lower hand over unemployed workers in that period. Consequently, in the next period, the equilibrium contract would entail a lower probability of termination, and hence fewer vacant firms in the period to follow. This completes a two-period cycle. The same mechanism is at the heart of a large set of much more complex cycles, with longer and shorter durations, and the creation and destruction of jobs. In particular, the model displays equilibrium cycles where firms enter the market when the measure of existing vacant firms is relatively small, and exit the market when the measure of existing vacant firms is relatively large. This generates equilibrium fluctuations in the economy’s aggregate employment and unemployment, and in the equilibrium contracts offered.

1.1 More on the literature

Models of financial markets and intermediation, including Bernanke and Gertler (1989), Williamson (1987), Kiyotaki and Moore (1997), and He and Krishnamurthy (2012) among others, emphasize the role of private information for amplifying and propagating aggregate
shocks. A more recent literature shows that in addition to propagating dynamics that originates from other sectors of the economy, the credit market itself can be a source of macroeconomic instability. An earlier contribution is Suarez and Sussman (1997, 2007), where increased output of the product good in a boom period reduces the price of the good and increases the demand for external finance. With moral hazard, this in turn leads to excessive risk taking by the entrepreneurs and a high failure rate in their investment, pushing the economy into a bust. Suarez and Sussman (1997, 2007) produce only two-period cycles. This is in contrast to our model which generates a larger and more complex set of non-stationary equilibria, including cycles with much longer durations and more complex dynamics.

Myerson (2012) models moral hazard in long-term lending relationships for generating endogenous fluctuations. He shows that the aggregation of the dynamics in individual lending contracts can display cycles in the credit market. Cycles are of the same fixed length as that of the life of the banker, and he imposes that the banker is terminated after his investment fails.

In Favara (2012), entrepreneurs must costly evaluate and then select a project which generates verifiable income and non-verifiable private benefits. Investor control encourages entrepreneurs to select projects with higher productivity and lower private benefits, but too much control discourages the undertaking of new projects. In a boom period, wealthier entrepreneurs rely less on external finance for investment and the investor’s incentives to control are weaker. As a result, projects with lower productivity and higher private benefits are selected, paving the way for a subsequent bust.

Banerji and Wang (2013) consider economies with risk averse agents where cycles are driven by the trade-off between incentives and insurance. In a bust period, young agents take more risk, work harder and are more productive, and this leads to an economic boom. In a boom period, young agents are fully insured, they shirk and are less productive, and

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For recent surveys of this literature, see Bernanke, Gertler and Gilchrist (1999) and Gertler and Kiyotaki (2010).
this leads to a recession.

Gu et al. (2013) show that endogenous cycles can arise in credit markets where agents honor debt obligations to avoid exclusion from future credit. Suppose borrowers believe the debt limit tomorrow will increase, but not by too much. Then their utility tomorrow will decrease as competition bids up the price of credit, and they are more willing to renege on current debt which, in turn, makes today’s debt limit lower. Their model is capable of generating a large set of non-stationary equilibria, including self-fulfilling sunspot cycles.\(^3\)

Matsuyama (2013) considers an OG model where old agents choose to invest in a “good” or a “bad” project. Good projects must use labor inputs of young agents who, with the earnings from the supply of their labor, are then able to finance a project next period when they become old. Bad projects require no labor inputs but are associated with a borrowing constraint. In a bust period, old agents, low in net worth, can only finance the good project, which leads to a boom and an improvement in the young agents’ net worth next period. In a boom period, old agents are able to finance the bad project, which results in a deterioration in the young agents’ net worth next period.

To summarize, in addition to being a labor rather than a credit market model, our work differs from the existing theories of endogenous cycles in the essence of the mechanism that generates the fluctuations.

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 characterizes the optimal contracts. Section 4 defines an equilibrium. Section 5 analyzes the stationary equilibria. Section 6 characterizes the model’s equilibrium cycles. Section 7 discusses the multiplicity in the equilibrium of the model. Section 8 compares welfare outcomes in the stationary and non-stationary equilibria. Section 9 concludes the paper.

\(^3\) Cycles in our model are not driven by self-fulfilling expectations. In our model, the agents’ expectations on future market conditions induce actions that work against, rather than reinforce, the cycle. This aspect of our story distinguishes it from models that rely on sunspot or other self-fulfilling mechanisms for generating endogenous fluctuations.
2 Model

Time is discrete and lasts forever: \( t = 1, 2, \ldots \). There is a single perishable good. The economy contains a sequence of overlapping generations of workers. Workers live for two periods, are young in the first and old in the second. We normalize the total measure of workers living at any time to one, with each generation containing \( 1/2 \) units of mass. Workers born in period \( t \) and alive in periods \( t \) and \( t+1 \) maximize

\[
E_t \{(1-\delta)(c_t-a_t)+\delta(c_{t+1}-a_{t+1})\},
\]

where \( E_t \) denotes expectation taken at the beginning of period \( t \); \( c_t \) and \( a_t \) are period \( t \) consumption and effort, respectively; and \( \delta \equiv \beta/(1+\beta) \), where \( \beta \in (0,1) \), is the workers’ discount factor.\(^4\) Consumption must be non-negative: \( c_t \geq 0 \) for all \( t \). For analytical tractability, assume effort takes one of two values, with the low effort normalized to zero. That is, we assume \( a_t \in \{0, \psi\} \) for all \( t \), with \( \psi > 0 \).

The economy also has a positive measure of infinitely lived firms, each of them could employ in each period one worker to produce. There is moral hazard: the effort that the employed worker exerts is observed by the worker herself only. By choosing effort \( a_t \) in period \( t \), the worker produces a publicly observed random output in period \( t \) that is a function of \( a_t \). Let \( \theta_t \in \{\theta_1, \theta_2\} \) denote the realization of this random output, where \( \theta_1 < \theta_2 \). Let

\[
x_i = \text{Prob}\{\theta_t = \theta_i|a = \psi\} \quad \text{and} \quad x'_i = \text{Prob}\{\theta_t = \theta_i|a = 0\}, \quad i = 1, 2.
\]

That is, \( x_i \) is the probability with which output \( \theta_i \) is produced if the worker works, and \( x'_i \) the probability if he shirks. Assume \( x_i, x'_i > 0 \) for all \( i \) and \( x_1 + x_2 = x'_1 + x'_2 = 1 \). Let \( \bar{\theta} = \sum x_i \theta_i \) (\( \bar{\theta} = \sum x'_i \theta_i \)) denote the mean output produced when the worker works (shirks). We assume \( \bar{\theta} - \bar{\theta} \) is large enough so that \( a = 0 \) is never desirable for the firm.

At the start of any period, firms are in three different states respectively: incumbent

\(^4\)The use of \( \delta \) normalizes the worker’s lifetime utility. This normalization is necessary for the specification of the optimal employment contracting problems which are to appear, as firms in the model live an infinite life whereas workers live only for two periods, and both static and long-term contracts must be considered and their values compared.
firms (those currently in the labor market) who are with an old worker retained from the prior period, incumbent firms who are currently vacant, and potential entrants. When the period begins, the incumbent vacant firms decide whether to exit the market, and the potential entrants decide whether to enter the market. Once the entry and exit decisions are made, the labor market opens where vacant firms and unemployed workers trade and enter employment contracts. Then production takes place, the contracts are carried out, and the period ends.

The labor market is organized in two submarkets, for young and old workers, respectively. Each period, vacant firms first decide, competitively, in which submarket to participate, and then offer employment contracts, again competitively, to the unemployed workers in their own submarket.

In order to enter the market an entrant must first make an initial investment, modeled as a fixed entry cost, $C_e/(1 - \beta) > 0$, which is sunk thereafter. In order to exit the market, however, a vacant firm could simply stop the business and leave, not subject to any costs. In order to stay in the market, each firm must incur an operating cost $C_o > 0$ in each period.

The contract between any firm and any old worker is one-period long. A firm and a young worker have the opportunity to enter a long-term contract which can potentially last for two periods, but may specify a condition under which the worker is terminated after a single period. When a worker is terminated, she is free to go back to the labor market to look for new employment opportunities, and the firm is free to go back to the labor market to hire a new worker. As part of the physical environment, we assume that once the worker is fired (i.e., once the contract is terminated), the interaction between the worker and the firm ends. In particular, if a worker is fired by a firm at the end of period $t$, she will not be able to receive payments from the firm in period $t + 1$. We assume that there are no physical costs that either party must incur in the process of any termination.

We assume the firm is fully committed to the terms of any contract it is willing to enter with any worker. Since workers are identical and termination is costly — even though there
are no physical costs associated with termination, without this assumption termination would not occur in equilibrium. The assumption of full commitment also requires that the firm implement any realized outcome of a randomized termination rule. This makes it feasible for the contract to specify a non-degenerate probability of termination. The commitment to the contract from the worker, however, is not assumed.

3 Contracting

Consider the partial problem of optimal contracting for an individual vacant firm, taking as given the current and future states of the market and that the contract must give the worker a given level of expected utility, denoted $w$. Let the period be $t$. A one-period contract offered in this period takes the form of $\sigma_s = \{c_1, c_2\}$, where $c_i (i = 1, 2)$ is the worker’s compensation in output state $i$. We omit the subscript $t$ for simplicity. For any given expected utility $w$ it must deliver to the worker, the firm seeks to maximize its expected value:

$$V_{s,t}(w) \equiv \max_{\sigma_s} \left\{ \left[ (1 - \beta) \sum x_i (\theta_i - c_i - C_o) \right] + \beta V_{t+1} \right\}, \quad (1)$$

subject to:

$$c_1, c_2 \geq 0, \quad (2)$$
$$x_1 c_1 + x_2 c_2 - \psi \geq x'_1 c_1 + x'_2 c_2, \quad (3)$$
$$x_1 c_1 + x_2 c_2 - \psi = w, \quad (4)$$

where $V_{t+1}$ is the maximized value of the firm who is vacant at the start of period $t + 1$, having not made the decision whether to stay in the market. Here, equation (2) is a limited liability condition on compensation; (3) is the incentive compatibility constraint, which says that the worker is better off working than shirking; and (4) is the promise keeping constraint.

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5 The problem of dynamic contracting with moral hazard and endogenous termination is studied in Spear and Wang (2005). This problem is embedded in a general equilibrium environment in Wang (2013) to study, in stationary labor market equilibria, the effects of firing costs on aggregate activity and welfare. The idea that termination can be used as an incentive device to motivate workers’ effort is modeled in Shapiro and Stiglitz (1984) in a theory of efficiency wages.
Notice that constraint (3) implies $c_2 - c_1 \geq \psi/(x_2 - x_2')$. Combining this with constraints (2) and (4) gives

$$w = c_1 + x_2(c_2 - c_1) - \psi \geq w, \text{ where } w \equiv \frac{x_2'\psi}{x_2 - x_2'} > 0.$$ 

That is, $w$ is the minimum $w$ that is consistent with incentive compatibility. When $w \geq w$, the firm’s value from entering the contract is:

$$V_{s,t}(w) = (1 - \beta)(\bar{\theta} - \psi - w - C_o) + \beta V_{t+1}, \forall w \geq w.$$ 

Note that the firm’s expected value of being vacant is $V_{v,t} \equiv -(1 - \beta)C_o + \beta V_{t+1}$. Let $C \equiv \bar{\theta} - \psi - w$. This is the maximum one-period value a firm can obtain from entering a one-period contract. Assume $C > 0$. That is, conditional on staying in the market, in any period the firm is better off entering a short-term contract that gives the worker $w$ than being vacant in that period.

Consider next the problem of optimal long-term contracting. A long-term contract takes the form of $\sigma_l = \{c_{ik}, w_{ik}, p_i, i = 1, 2, k = r, f\}$, where $i$ denotes the state of the first period output; $k = r(f)$ indicates that the worker is retained (fired) after the first period ends; $p_i$ is the probability with which the worker is fired after the first period in state $i$; and $c_{ik}$ and $w_{ik}$ are the worker’s first period compensation and second period expected utility respectively in state $ik$. Note again we drop the subscript $t$ for simplicity.

Since the firm and the worker will not interact with each other after termination, we have $w_{if} = w_{st+1}$, where $w_{st+1}$ denotes the expected utility of any unemployed old worker at the start of period $t + 1$. With this, the value of a firm entering a long-term contract that gives the young worker expected utility $w$ at the beginning of period $t$ is

$$V_{l,t}(w) = \max_{\sigma_l} \left\{ \sum x_i[p_i((1 - \beta)(\theta_i - c_{if} - C_o) + \beta V_{t+1}) + (1 - p_i)((1 - \beta)(\theta_i - c_{ir} - C_o) + \beta V_{s,t+1}(w_{ir}))] \right\},$$

(6)
\[ \text{s.t. } c_{ik} \geq 0, w_{ir} \geq w, 0 \leq p_i \leq 1, i = 1, 2, r = k, f, \]
\[ \sum_i x_i[(1 - p_i)((1 - \delta)c_{ir} + \delta w_{ir}) + p_i((1 - \delta)c_{if} + \delta w_{st+1})] - (1 - \delta)\psi \geq \sum_i x_i'[(1 - p_i)((1 - \delta)c_{ir} + \delta w_{ir}) + p_i((1 - \delta)c_{if} + \delta w_{st+1})], \tag{7} \]
\[ \sum_i x_i[(1 - p_i)((1 - \delta)c_{ir} + \delta w_{ir}) + p_i((1 - \delta)c_{if} + \delta w_{st+1})] - (1 - \delta)\psi = w. \tag{8} \]

Here the inequality \( w_{ir} \geq w \) says that if the worker is retained after one period, then her second period expected utility must be sufficiently high to support incentive compatibility. Equations (7) and (8) are, respectively, the incentive and promise-keeping constraints for the first period.

To solve for the optimal long-term contract, consider

**Assumption 1** \( V_{t+1} - (1 - \beta)(\bar{\theta} - \psi - w_{st+1} - C_o) - \beta V_{t+2} < 0. \)

This assumption essentially postulates that the firm should minimize rather than maximize the probability of termination. To see this, consider an increase in \( p_i \) in the above problem of optimal long-term contracting. This affects the firm's value in two opposite directions. On the one hand, the firm is better off being vacant than staying with the old worker: \( V_{t+1} \geq V_{s,t+1}(w_{ir}) \), an optimality condition which we will derive a bit later (equation(15)). On the other hand, in order to satisfy the promise keeping constraint the firm must raise \( w_{ir} \), the worker's value of retention. The overall marginal cost of termination is thus \( (1 - \beta)(\bar{\theta} - \psi - w_{st+1} - C_o) + \beta V_{t+2} - V_{t+1} \), which is positive by Assumption 1.

Observe, however, that in Assumption 1, \( V_{t+1}, V_{t+2} \) and \( w_{st+1} \) are all endogenous variables of the model. To resolve this difficulty, we use the following strategy: we first solve for the optimal long-term contract under Assumption 1; we then solve for the equilibrium in which Assumption 1 does hold in all periods; we then show that the model does not have equilibria in which Assumption 1 is violated in any period.

Let \( w_{A,t} \equiv (1 - \delta)w + \delta w_{st+1} \). Then \( w_{A,t} < w \) by Assumption 1 and the optimality condition (15).
Proposition 1 Suppose Assumption 1 holds for a given $t$. (i) If $w < w_{A,t}$, then $w$ is not attained by any incentive compatible long-term contract. (ii) If $w \in [w_{A,t}, w)$, then the optimal long-term contract in period $t$ has

$$p^*_1(w) = \frac{w - w}{\delta(w - w_{t+1})}, p^*_2(w) = 0,$$

$$c^*_{1r}(w) = c^*_1(w) = c^*_2(w) = 0,$$

$$w^*_{1r}(w) = w, w^*_{2r}(w) = \frac{1}{\delta} \left( w - (1 - \delta)w + \frac{(1 - \delta)\psi}{x_2 - x_2'} \right).$$

(iii) If $w \geq w$, the optimal long-term contract in period $t$ has

$$p^*_1(w) = p^*_2(w) = 0,$$

$$c^*_{1r}(w) = c^*_1(w) = c^*_2(w) = 0,$$

$$w^*_{1r}(w) = \frac{w - (1 - \delta)w}{\delta}, w^*_{2r}(w) = w^*_{1r}(w) + \frac{(1 - \delta)\psi}{(x_2 - x_2')\delta}.$$

The proof of the proposition, and those of other lemmas and propositions of the paper, are relegated to the appendix.

Proposition 1 illustrates how termination is used optimally as an incentive device in the long-term contract. Remember the expected utility of a retained old worker must be at least $w$ in order for it to be consistent with incentive compatibility, but that of an unemployed old worker is equal to $w_{t+1}$ and $w_{t+1} < w$. This gap between $w_{t+1}$ and $w$ allows termination to impose a punishment on the worker, and a higher $p^*_1$ gives the worker an ex ante larger punishment for producing the low output. Remember also that termination is costly and the contract seeks to minimize the use of it. Start now with a sufficiently high expected utility $w$, with $w \geq w$, that the contract promises to the young worker. Incentives can then be achieved without using termination as an incentive device. Incentives can be obtained simply by giving the worker more compensation (in expected terms) in the state of high output and less in the state of low output. Suppose next that $w$ is reduced to be below $w$. 

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Then a state contingent compensation plan that excludes any termination could no longer be consistent with the expected utility promised — the binding constraint being that the worker’s expected utility be at least $w$ in all states of retention. A positive probability of termination must then be built into the contract. Moreover, the lower is $w$, the higher is the probability of termination that the optimal contract must prescribe. That is, $p_1^*$ increases as $w$ decreases. In addition, $p_1^*$ is increasing in $w_{st+1}$: the lower $w_{st+1}$ is, the more effective termination is as an incentive device, the less it then must be used in the optimal contract, and hence the lower $p_1^*$ is.

Since it is feasible to set $p_1 = p_2 = 1$ in any long-term contract to make it essentially a short-term contract and, as we show in Appendix A.1, if $p_2 = 1$ then the firm can get strictly better off by reducing $p_2$ and $c_{2f}$ simultaneously, we have

$$V_{l,t}(w) > V_{s,t}\left(\frac{w - \delta w_{st+1}}{1 - \delta}\right), \quad \forall w \geq w_{A,t},$$

where note that in order to fulfill an expected utility $w$ promised to a young worker at the beginning of period $t$, a one-period contract must give him an expected utility of $(w - \delta w_{st+1})/(1 - \delta)$ within period $t$. Equation (9) thus says that the firm is strictly better off offering the young worker the optimal long-term contract than the optimal one-period contract. In particular, $V_{l,t}(w_{A,t}) > V_{s,t}(w)$.

It follows easily from Proposition 1 that the value of a firm entering the optimal long-term contract for each $w \geq w_{A,t}$ is given by

$$V_{l,t}(w) = \begin{cases} k_0 - kw, & w \in [w_{A,t}, w), \\ (1 - \beta^2)(\bar{\theta} - \psi - w - C_o) + \beta^2V_{t+2}, & w \in [w, \infty), \end{cases}$$

where

$$k = (1 + \beta)x_1\frac{V_{t+1} - V_{s,t+1}(w)}{w - w_{st+1}} + (1 - \beta^2)x_2 > 0.$$
achieves its maximum value at $w_{A,t}$. The relationship between $V_{s,t}(\cdot)$ and $V_{l,t}(\cdot)$ is illustrated by Figure 1.

To close this section, remember Proposition 1 is derived under Assumption 1. In Proposition 4, which is stated and proved in the appendix, we show that if Assumption 1 is violated for any $t$, then for all $w$ with which the problem that defines $V_{l,t}(w)$ has a solution, it is optimal to set $p_i(w) = 1$ for $i = 1, 2$. This is straightforward to see. Suppose Assumption 1 is violated at $t$, then termination is profitable in period $t$ and the optimal contract must maximize the use of it, setting $p_1^* = p_2^* = 1$. That is, the optimal long-term contract is essentially a short-term contract with which the firm’s value is

$$V_{l,t}(w) = (1 - \beta) \left( \bar{\theta} - \psi - \frac{w - \delta w_{t+1}}{1 - \delta} - C_o \right) + \beta V_{t+1}, \ \forall w \geq w_{A,t}.$$  

4 Equilibrium

4.1 Entry and exit of firms

Before the labor market opens for matching and contracting, potential entrants must decide whether to enter it and incumbent vacant firms must decide whether to exit it, and they do so taking as given their rationally perceived equilibrium dynamics and distributions of the
Consider first an incumbent vacant firm who has not made the decision whether to exit the market. This firm’s value, denoted $V_t$, is

$$V_t = \max\{\tilde{V}_t, 0\}, \quad (11)$$

where $\tilde{V}_t$ denotes the firm’s value once it decides to stay in the market, and 0 is its value if it exits the market.

Let $\eta_t$ denote the measure of firms in the market in period $t$ after all the entry and exit decisions have been made. In equilibrium, if some firms choose to stay in the market in any period $t$, then it must hold that $\tilde{V}_t \geq 0$, for otherwise these firms would do better exiting the market. Similarly, if there is a positive outflow of firms from the market, i.e., $\eta_t < \eta_{t-1}$, then it must hold that $\tilde{V}_t = 0$ — the outflow would continue until leaving the market no longer entails a positive net gain.

For a potential entrant, the net gains from entering the market is $\tilde{V}_t - C_e$. Since firms (potential entrants) are willing to enter the market as long as this value is positive, and there is an infinite supply of them, free entry implies $\tilde{V}_t - C_e \leq 0$, where the equality holds whenever $\eta_t > \eta_{t-1}$, in which case firms must have entered the market until the net gains from entering have been pushed down to zero.

Thus, a condition we will impose on the equilibrium of the model, whose full definition to be given shortly, is

$$0 \leq \tilde{V}_t \leq C_e, \quad \forall t. \quad (12)$$

Then, following from equations (11) and (12), it holds in equilibrium that $V_t = \tilde{V}_t$. From now on, we will drop the use of $\tilde{V}_t$, and let $V_t$ represent a vacant firm’s value both before and after it has made the decision to stay in the market. Thus, condition (12) can be rewritten as

$$0 \leq V_t \leq C_e, \quad \forall t, \quad (13)$$
where $V_t = 0$ if $\eta_t < \eta_{t-1}$ and $V_t = C_e$ if $\eta_t > \eta_{t-1}$.

4.2 The submarkets

The labor market unfolds in two stages. In the first, firms divide themselves into the two submarkets. When choosing which submarket to enter, each individual firm takes as given its rationally perceived fraction of firms entering each submarket and the allocation of values that will subsequently emerge in each submarket. In the second stage, firms and workers interact to produce the equilibrium outcomes in each submarket. A submarket is active if there is a positive measure of firms participating in it, and inactive otherwise.

Consider the equilibrium in an active submarket, taking as given the measures of firms and workers participating in it — determined in the first stage of the game. An equilibrium in this submarket is defined as a collection of matched worker-firm pairs and the expected utilities offered to workers within the pairs such that: (a) no individual worker or firm can benefit from unilaterally deviating from her/its current status in the labor market; and (b) no new worker-firm pair can arise to make both parties weakly better off and at least one party strictly better off.\footnote{This definition of equilibrium is analogous to that of stability in a two-sided matching model with transferable utility. See, for example, Shapley and Shubik (1971). The main departure is that we have a continuum of agents to be matched on each side of the market.}

Observe first that in equilibrium the expected utility offered to the worker must be constant across all worker-firm pairs in any given submarket. To see this, suppose there is an equilibrium where one firm-worker pair offers the worker expected utility $w_H$, and another firm-worker pair offers $w_L$, with $w_H > w_L$. Then the firm offering $w_H$ and the worker receiving $w_L$ can form a new pair to give the worker an expected utility $w \in (w_L, w_H)$ to make both parties strictly better off. This breaks the equilibrium.

In each submarket, if there are more vacant firms than unemployed workers, we assume the unemployed workers are rationed equally among firms. That is, all vacant firms have an equal probability to pair up with an unemployed worker. Similarly, if there are more
unemployed workers than vacant firms, then vacant firms are rationed equally among workers. Let $\eta_{V,y,t}$ and $\eta_{V,o,t}$ denote the measure of vacant firms in the submarkets for young and old workers, respectively. Let $w_{yt}$ and $w_{ot}$ denote the prevailing expected utilities offered to the workers in the submarkets for young and old workers, respectively.

Consider first the submarket for young workers. The vacancy-unemployment ratio — the market tightness — is given by $2\eta_{V,y,t}$ for this submarket. Depending on the value of $2\eta_{V,y,t}$, there are three cases regarding how the contract for a new hire is determined. First, suppose $2\eta_{V,y,t} < 1$. In this case, the market is slack and the firm captures all the surplus from trade, and it holds that $w_{y,t} = w_{A,t}$. Remember $w_{A,t}$ is the minimum $w$ achievable with an incentive compatible long-term contract and so $w_{y,t} \geq w_{A,t}$. But if $w_{y,t} > w_{A,t}$, then a firm and a currently unmatched young worker can form a new match at $w_{A,t}$ and be both strictly better off, violating the equilibrium condition.

Second, suppose $2\eta_{V,y,t} > 1$. Then the market is tight, the firm extracts all the surplus from the contract, and it holds that $w_{y,t} = \bar{w}_t$, where $\bar{w}_t$ is such that the firm is indifferent between entering a long-term contract and being vacant in the period: $V_{t,t}(\bar{w}_t) = V_{v,t}$. In this case, if $w_{y,t} > \bar{w}_t$, then a firm strictly prefers to being vacant than being in a long-term contract; and if $w_{y,t} < \bar{w}_t$, then a worker and a currently unmatched firm can form a new match at any $w \in (w_{y,t}, \bar{w}_t)$ and be both strictly better off.

Third, suppose $2\eta_{V,y,t} = 1$. Then any $w_{y,t} \in [w_{A,t}, \bar{w}_t]$ can be an equilibrium expected utility for the newly hired young worker. In this case, the division of the surplus from contracting is indeterminate. This indeterminacy, however, will be resolved in the upper layer of the equilibrium where firms make optimal choices between the submarkets.

Let $L_t$ denote the measure of all unemployed workers at the beginning of period $t$. Then the measure of unemployed old workers is $L_t - 1/2$, and the tightness of the submarket for old workers is $\eta_{V,o,t}/(L_t - 1/2)$. We then have, similar to what we do for the submarket for young workers, the following for the equilibrium in the submarket for old workers: if $0 < \eta_{V,o,t} < L_t - 1/2$, then $w_{o,t} = \bar{w}$; if $\eta_{V,o,t} > L_t - 1/2$, then $w_{o,t} = \bar{\theta} - \psi$ since $V_{s,t}(\bar{\theta} - \psi) = V_{v,t}$;
and if \( \eta_{o,t}^V = L_t - 1/2 \), then any \( w_{o,t} \in [w, \bar{\theta} - \psi] \) is an equilibrium expected utility for the newly hired old worker.

Move back now to consider the first stage of the labor market where vacant firms choose which submarket to enter. In equilibrium, the division of the vacant firms between the two submarkets must be such that each firm in each submarket has no incentives to switch to the other, given the (equilibrium) choices of all other firms (i.e., given the equilibrium fraction of vacant firms participating in each submarket) and the outcomes that it perceives to arise subsequently in the submarkets. Specifically, let \( \lambda_{o,t}(\lambda_{y,t}) \) be the probability with which a vacant firm in the submarket for old (young) workers is filled. Then the equilibrium condition for the first stage of the market is

\[
V_t = \max \{ \lambda_{o,t} V_{s,t}(w_{o,t}), \lambda_{y,t} V_{l,t}(w_{y,t}) \}. \tag{14}
\]

### 4.3 Three Types of Periods

We focus on equilibria which are non-degenerate (i.e., the labor market does open in each period) and with unemployment. For this, we make the following assumption:

**Assumption 2** \( C < C_o \leq \bar{C} \), where \( \bar{C} \equiv \bar{\theta} - \psi - \frac{w}{(1 + x_2 \beta)}. \)

Here, the inequality \( C_o \leq \bar{C} \) ensues the existence of a non-degenerate equilibrium, with \( \bar{C} \) being the value a firm attains if it hires a young worker and offers her the expected utility \( w_{A,t} \) — the minimum to support incentives in a long-term contract — in any period \( t \) when it is vacant.\(^7\) The inequality \( C < C_o \) implies that a firm’s one-period payoff from entering a short-term contract is negative. This ensues that unemployment exists in all periods in each of the non-degenerate equilibria (i.e., \( 0 < \eta_t < 1 \) for all \( t \)).\(^8\) It also opens the possibility for non-stationary equilibria with entry and exit of firms.\(^9\)

---

\(^7\)In Appendix B.4, we show that in equilibrium it holds that \( V_t \leq \bar{C} \) for all \( t \).

\(^8\)See Proposition 5 in Appendix B.1.

\(^9\)The analysis for the case of \( \bar{C} \geq C_o \) is in Appendix E.
Periods may differ in the tightness of the labor market. We call a period *Type S* if, in that period, the measure of vacant firms is strictly less than that of young workers, $1/2$; *Type T* if the measure of vacant firms is strictly larger than that of young workers; and *Type I* if the measure of vacant firms is equal to $1/2$.

In a Type S period, in equilibrium all firms enter the submarket for young workers which is slack ($\eta_{y,t} < 1/2$) and, from the discussion above, all vacancies are filled with $w_{y,t} = w_{A,t}$. Remember $V_{l,t}(w_{A,t})$ is strictly higher than $V_{s,t}(w)$, the highest expected value a firm could possibly obtain by entering the submarket for old workers. Hence, in equilibrium only the submarket for young workers is active, and the value of a vacant firm is $V_t = V_{l,t}(w_{A,t})$.

In a Type T period, in equilibrium $1/2$ units of firms enter the submarket for young workers ($\eta_{y,t} = 1/2$) and the rest enter the submarket for old workers. Furthermore, $w_{o,t} = w$ and $w_{y,t} = w_{B,t}$, where $w_{B,t}$ is such that $V_{l,t}(w_{B,t}) = V_{s,t}(w)$ (Figure 1). To see this, suppose $\eta_{y,t} < 1/2$. Then $w_{y,t} = w_{A,t}$, which gives firms higher expected values than what they could get from entering the submarket for old workers. So no firms enter the submarket for old workers. However, $\eta_{y,t} < 1/2$ also implies the submarket for old workers is active, a contradiction. Suppose $\eta_{y,t} > 1/2$. Then the firm’s value from entering the submarket for young workers is $V_{y,t} = V_{v,t}$. Remember Assumption 2 implies $\eta_t < 1$. Hence, $\eta_{o,t} < L_t - 1/2$ and $w_{o,t} = w$. By assumption, $V_{s,t}(w) > V_{v,t}$, and so firms should not enter the submarket for young workers, a contradiction to $\eta_{y,t} > 1/2$. Thus in a Type T period it must hold in equilibrium that $\eta_{y,t} = 1/2$. Then since $\eta_t < 1$, we have $\eta_{o,t} < L_t - 1/2$ and $w_{o,t} = w$. Finally, in order for firms to be indifferent between entering the two submarkets, $w_{y,t}$ must be such that $V_{l,t}(w_{y,t}) = V_{s,t}(w)$. The value of a vacant firm is thus given by $V_t = V_{l,t}(w_{B,t}) = V_{s,t}(w)$.

Similarly, in a Type I period in which the measure of vacant firms is equal to that of young workers, in equilibrium $\eta_{y,t} = 1/2$, and the submarket for old workers is inactive. In this case, we define the utility of a newly hired old worker as the limit of that in an active submarket with the measure of firms going to zero, i.e., $w_{o,t} = w$. Finally, for firms to be willing to enter the submarket for young workers, $w_{y,t}$ must be such that $V_{l,t}(w_{y,t}) \geq V_{s,t}(w)$.
Hence, \( w_{y,t} \in [w_{A,t}, w_{B,t}] \).

The above analysis also shows that in equilibrium \( \lambda_{y,t} = \lambda_{o,t} = 1 \), and

\[
V_t \geq V_{s,t}(w). \tag{15}
\]

### 4.4 Equilibrium

Before defining an equilibrium, we first describe the law of motion for the aggregate states of the model. Let \( \hat{p}^*_t \equiv p^*_t(w_{y,t}) \) for simplicity. Let \( \alpha_{y,t} (\alpha_{o,t}) \) denote the job finding probability of an unemployed young (old) worker in period \( t \). Then the evolution of \( L_t, \alpha_{y,t} \) and \( \alpha_{o,t} \) must satisfy

\[
L_t = 1 - \frac{1}{2} \alpha_{y,t-1} [1 - (x_1 \hat{p}^*_{1t-1} + x_2 \hat{p}^*_{2t-1})], \tag{16}
\]

\[
\frac{1}{2} \alpha_{y,t} + (L_t - \frac{1}{2}) \alpha_{o,t} = \eta_t - \frac{1}{2} \alpha_{y,t-1} [1 - (x_1 \hat{p}^*_{1t-1} + x_2 \hat{p}^*_{2t-1})]. \tag{17}
\]

Equation (16) says in particular that the measure of unemployed workers in any period depends on the firm’s firing policy in the prior period. Equation (17) says that in equilibrium the measure of vacant firms in any period is equal to that of unemployed workers who obtain employment in that period. It then follows from equation (16) that the measure of vacant firms in period \( t \) satisfies

\[
\eta_t - \frac{1}{2} \alpha_{y,t-1} [1 - (x_1 \hat{p}^*_{1t-1} + x_2 \hat{p}^*_{2t-1})] = \eta_t - (1 - L_t). \]

Under Assumptions 1 and 2,

**Definition 1** A rational expectations equilibrium of the model is a sequence

\[
\{\eta_t, L_t, \alpha_{o,t}, \alpha_{y,t}, w_{st}, V_t; \sigma^*_{s,t}, \sigma^*_{l,t}, w_{y,t}, w_{o,t}\}_{t \geq 1}
\]

where for all \( t, \)

1. Given \( w_{st+1}, \sigma^*_{s,t} \) solves (1)-(4) and \( \sigma^*_{l,t} \) solves (6)-(8), and
(a) If $\eta_t - (1 - L_t) < \frac{1}{2}$ (the period is Type S),

\[ w_{o,t} = w, \quad w_{y,t} = w_{A,t}, \]
\[ 0 < \alpha_{y,t} < 1, \quad \alpha_{o,t} = 0, \]
\[ V_t = V_{l,t}(w_{y,t}); \]

(18) \hspace{1cm} (19) \hspace{1cm} (20)

(b) If $\eta_t - (1 - L_t) > \frac{1}{2}$ (the period is Type T),

\[ w_{o,t} = w, \quad w_{y,t} = w_{B,t}, \]
\[ \alpha_{y,t} = 1, \quad \alpha_{o,t} > 0, \]
\[ V_t = V_{l,t}(w_{y,t}) = V_{s,t}(w); \]

(21) \hspace{1cm} (22) \hspace{1cm} (23)

(c) If $\eta_t - (1 - L_t) = \frac{1}{2}$ (the period is Type I),

\[ w_{o,t} = w, \quad w_{y,t} \in [w_{A,t}, w_{B,t}], \]
\[ \alpha_{y,t} = 1, \quad \alpha_{o,t} > 0, \]
\[ V_t = V_{l,t}(w_{y,t}) \geq V_{s,t}(w); \]

(24) \hspace{1cm} (25) \hspace{1cm} (26)

2. The expected utility for unemployed old workers at the beginning of period $t$ is given by

\[ w_{*t} = \alpha_{o,t} w_{o,t}; \]

3. The aggregate variables $\eta_t$, $\alpha_{y,t}$, $\alpha_{o,t}$ and $L_t$ satisfy (16) and (17).

4. Firms make optimal entry and exit decisions, i.e., $V_t$ satisfies (13).

In Appendix B.2, Proposition 6, we verify that Assumption 1 holds for any equilibrium defined in Definition 1 and with $0 < \eta_t < 1$ for all $t$. In other words, it is indeed legitimate to focus on equilibria that satisfy Assumption 1. Recall that an increase in the probability of termination affects a firm’s value in two ways. First, termination allows it to go back to the labor market to do better than what the continuation of the contract offers: $V_t \geq V_{s,t}(w)$. 

\[ 21 \]
Second, to compensate the worker for the utility loss that results from the higher probability of termination, the firm must increase his expected utility in the state of retention, \( w_{ir} \), and this costs the firm. With free entry of firms into the market, in equilibrium \( V_t \) can never be sufficiently high to make the benefit of termination dominate its cost.

5 Stationary Equilibria

We say an equilibrium is stationary if in that equilibrium all of the model’s aggregate states are constant in time.

**Proposition 2** The model has a stationary equilibrium for any \((C_o, C_e)\) with \( C_e > 0 \) and \( C_o \in (C, \overline{C}] \). Specifically, there exists \( \eta^-_s(C_o, C_e) \) and \( \eta^+_s(C_o) \), with \( 0 \leq \eta^-_s(C_o, C_e) < \eta^+_s(C_o) < 1 \), such that: (i) If \( C_o + C_e < \overline{C} \), then for each \( \eta \in [\eta^-_s, \eta^+_s] \) there is a stationary equilibrium where all periods are Type I. (ii) If \( C_o + C_e \geq \overline{C} \), then for each \( \eta \in [\eta^-_s, (1+x_2)/2) \) there is a stationary equilibrium where all periods are Type S; and for each \( \eta \in [(1+x_2)/2, \eta^+_s] \) there is a stationary equilibrium where all periods are Type I.

Given Proposition 2, several intuitive results emerge, all straightforward to verify. (a) As \( \eta \) increases from \( \eta^-_s \) to \( \eta^+_s \), the firm’s expected value \( V_t \), which is constant in \( t \), falls from \( C_e \) to 0. (b) Both \( \eta^-_s \) and \( \eta^+_s \) are decreasing in the operating cost \( C_o \), indicating that if the cost to stay in the market is higher, in equilibrium a smaller measure of firms will stay in the market. (c) For any given level of \( C_o \), \( \eta^-_s \) decreases as the entry cost \( C_e \) increases. Obviously, a larger entry cost deters entry and allows the market to support a smaller stock of incumbent firms who each enjoy higher values.

Proposition 2 indicates that the model has no stationary equilibria where both submarkets are active (i.e., all periods are Type T). In any stationary equilibrium, only long-term contracts are traded and only young workers are offered employment. Once laid off, the worker never regains employment. Why is this the case? Suppose in equilibrium the measure of vacant firms is less than or equal to 1/2. Then the analysis in Section 4.3 shows that
no firms would enter the submarket for old workers. Suppose in equilibrium the measure of vacant firms is strictly larger than $1/2$. Then, again from the analysis in Section 4.3, a positive measure of vacant firms must be in the submarket for old workers, where the maximum per period value any firm can achieve, $\bar{\theta} - \psi - w - C_o (\equiv C - C_o)$, is negative, a contradiction.

That a stationary equilibrium never permits an active submarket for old workers has important implications. It is by way of freeing up the market from this constraint the model is able to achieve a larger set of equilibria, some of which displaying cycles in which an active submarket for old workers may emerge. In a non-stationary equilibrium, firms who incur a loss in the current period may, in return, obtain a positive value next period when the market condition improves, and are thus better off staying in the market, rather than exiting it.

6 Cycles

We now study the model’s non-stationary equilibria where the aggregate states of the economy move dynamically in cycles. To streamline notation, call a period Type SI if it is Type S or Type I, and a period Type TI if it is Type T or Type I.

Observe the following law of motion for the measure of vacant firms:

$$\eta^V_{t+1} \equiv \eta_{t+1} - (1 - L_{t+1}) = \eta_{t+1} - \eta_t + \eta_t - \min \{ \eta_t^V, 1/2 \} (1 - x_1 \hat{p}^*_{1t}).$$

That is, the measure of vacant firms in period $t+1$, $\eta^V_{t+1}$, equals the inflow of firms, $\eta_{t+1} - \eta_t$, plus the measure of incumbent firms that have just become vacant, $\eta_t - \min \{ \eta_t^V, 1/2 \} (1 - x_1 \hat{p}^*_{1t})$, which, in turn, equals the measure of active firms in period $t$, $\eta_t$, minus that of those who retain their current worker, $\min \{ \eta_t^V, 1/2 \} (1 - x_1 \hat{p}^*_{1t})$.

As indicated earlier, $\eta^V_t$ plays a key role in defining the condition of the current labor market and thus in generating market dynamics. First, all else equal, a larger number
of vacant firms in the current market means that (weakly) more firms will enter a long-
term contracts with a young worker (a larger \( \min \{ \eta_t - (1 - L_t), \frac{1}{2} \} \)), resulting in a smaller
measure of vacant firms in the next period. Second, when the market has more vacant firms
in the current period, the equilibrium long-term contract is more likely to prescribe a lower
probability of termination, \( \hat{p}_{1t} \), which, in turn, implies a smaller measure of vacant firms in
the next period. This gives us

**Lemma 1** In equilibrium, there cannot be two successive Type TI periods with \( w_{y,t} = w_{B,t} \).

In particular, there cannot be two successive Type T periods in equilibrium. Suppose
both the current and the next periods are Type T. Then competition for young workers
in the current market would imply that termination would never occur with these workers,
resulting in too few vacant firms in the labor market next period to make the period Type
T, a contradiction.

**Lemma 2** In equilibrium, for any \( t \), if both periods \( t - 1 \) and \( t \) are Type SI, then period \( t + 1 \)
is also Type SI.

To illustrate the intuition, consider an equilibrium where there is no entry and exit of
firms. Let periods \( t - 1 \) and \( t \) be Type SI. In these periods, the market condition is favorable
to firms, which allows them to hire young workers at a low \( w \) and fire them with a high
probability. Without entry and exit of firms, vacancies in any period must be from two
sources: those who just fired a young worker and those who employed an old worker last
period. In period \( t + 1 \), the number of vacant firms from the first source is bounded from
above by \( x_1 \hat{p}_{1t} \)/2 (there are no more than 1/2 of vacant firms in period \( t \)). The number of
vacant firms from the second source is restricted by the fact that few firms stayed with an
old worker in period \( t \), given that firms were terminating their young worker in period \( t - 1 \)
with a high probability. As a result, the measure of vacant firms is bounded from above by
1/2, and so period \( t + 1 \) must be Type SI.

Combining Lemmas 1 and 2 gives us
Proposition 3  In equilibrium, there exists \( \hat{t} \geq 0 \) such that for all \( t \geq \hat{t} \), either all periods are Type SI, or Type SI and Type T periods alternate.

Proposition 3 offers directions for where to look for the model’s dynamic equilibria. In the rest of the section, three special classes of the model’s dynamic equilibria are constructed to illustrate how cycles may emerge in the model. The first is a class of two-period cycles that feature entry and exit of firms. The second is a class of equilibria where the stock of firms is constant in time, while other aggregate variables fluctuate in two-period cycles. The third is a class of equilibria that display cycles longer than two periods, in fact the length of each cycle could be any even number. In these equilibria, the first period in each individual cycle is a boom period, followed by a recession whose duration could be viewed as being assigned by a random number generator.

6.1 Example 1: Two-period cycles with entry and exit of firms

The model has a class of equilibria that display two-period cycles where the economy’s aggregate values, including the stock of firms, employment, unemployment, output, as well as the job finding and separation probabilities, rise and fall over the cycle. Specifically, in Proposition 7 in Appendix B.5, we show, by construction, that for all \((C_o, C_e)\) with

\[
0 < C_e \leq \frac{(1 - \beta)(1 + x_2\beta)(\bar{C} - C_o)}{1 - x_2\beta^2},
\]

there is an equilibrium in two-period cycles where all periods are Type I. The first period of the cycle is a boom period. At the start of this period, the market condition is favorable to firms, with the measure of vacant firms less than that of young workers: \( \eta^{(2)} - (1 - L^{(1)}) < 1/2 \). If no new firms enter the market, the value of a vacant firm, who would enter a long-term contract with a young worker, is higher than the cost for any potential firm to enter the market: \( (1 - \beta)(1 + x_2\beta)(\bar{C} - C_o) + x_2\beta^2 C_e > C_e \). This generates an inflow of firms into the market to push the value of the vacant firm down to \( C_e \), where the total measure of
vacant firms in the market is just equal to that of young workers (1 in Figure 2). Note that no firms would enter the submarket for old workers. So the equilibrium labor-market tightness, measured after all the new vacancies are created, is 1 for the submarket for young workers, and 0 for that for old workers. Thus the average job finding probability across the two submarkets is $\frac{1}{2} / \left(\frac{1}{2} + \frac{1}{2}\tilde{p}^*(2)\right)$, where $(1/2) \times \tilde{p}^*(2)$ is the measure of the unemployed old workers in the first period — those newly laid off from a long-term contract. As such, in equilibrium, young workers are all hired at a low starting expected utility $w_{y,t}$, and to be terminated after one period with a high probability ($\tilde{p}^*(1) > \tilde{p}^*(2)$). This will then create a large measure of vacant firms in the next (second) period of the cycle (2 in Figure 2).

The second period of the cycle is a recession. At the start of this period, the market condition has become favorable to workers: $\eta^{(1)} - (1 - L^{(2)}) > 1/2$. If no firms exit the market, then each of the vacant firms would be of a negative value. This generates an outflow of firms, until the remaining vacant firms are indifferent between exiting and staying in the market (3 in Figure 2). After the outflow ends, the average job finding probability across the two submarkets is $\frac{1}{2} / \left(\frac{1}{2} + \frac{1}{2}\tilde{p}^*(1)\right)$, lower than that in the boom period. In the recession, young workers are laid off with a lower probability, generating a small measure of vacant firms in the next period, moving the distribution of the market power in a direction that favors firms (4 in Figure 2).

The boom period features a lower unemployment rate and a higher (average) job finding
The job finding probability is higher for two reasons. First, new jobs are created through an inflow of firms, and that increases the measure of vacancies. Firms flow in because the period starts with a labor market condition favoring vacant firms, which in turn results from a tight labor market in the prior period that generated fewer new vacancies. Second, fewer workers are unemployed at the beginning of a boom period, because the tight labor market in the prior period generated fewer layoffs.

Similarly, in a recession period, the unemployment rate is higher and the average job finding probability is lower. This is because (i) some of the existing jobs have been destroyed, and (ii) the slack labor market in the prior boom period generated more layoffs and hence more unemployed (old) workers in the current period.

6.2 Example 2: Two-period cycles without entry and exit of firms

The model also has a class of equilibria where the market oscillates between Type SI and Type T periods. Here we present a subset of such equilibria where there is a constant stock of firms but the equilibrium wages and job finding and separation probabilities move cyclically in time.

In Proposition 8 in Appendix B.5, we show that if

\[ C < C_o < C + \frac{x_2 \beta^2 w}{1 + \beta} \quad \text{and} \quad C_e \geq \frac{(1 - \beta)(C_o - C)}{\beta}, \]

(27)

then there is an interval \([\eta_c^-, \eta_c^+]\) so that for any \(\eta\) in this interval, there is an equilibrium where \(\eta_t = \eta\) for all \(t \geq 1\) and Type S and Type T periods alternate. A Type S period is a boom period. At the start of such a period, the market condition is in favor of firms, with the equilibrium job finding probability being strictly less than 1 for young workers and 0 for old workers. The average job finding probability is \((\eta - (1 - L^{(1)})) / L^{(1)}\). As such, in

\(^{10}\)In this equilibrium, all unemployed workers are old and the job finding probability for young workers is one.

\(^{11}\)Lemma 8 in Appendix B.5 gives a general characterization of all such equilibria.
equilibrium all young workers are hired at a low \( w_{y,t} \) and to be terminated after one period with a high probability \( (\tilde{\rho}_1^{*(1)} > \tilde{\rho}_1^{*(2)}) \). In addition, in contrast to a Type T period, fewer firms enter long-term contracts with a young worker: 
\[
\frac{1}{2} > \eta - \frac{1}{2} + x_1 \tilde{\rho}_1^{*(2)} / 2.
\]
Both of these help create more vacancies in the next period, which is Type T (1 in Figure 3).

A Type T period is a recession period. At the start of such a period, the market condition favors the workers, with the equilibrium job finding probability being one for young workers and positive for old, and the average job finding probability being \( (\eta - (1 - L^{(2)})) / L^{(2)} \). Since \( L^{(1)} < L^{(2)} \), this job finding probability is lower than that in the prior boom period. Relative to the prior period, in the current period a larger measure of young workers are employed with a long-term contract which prescribes a low probability of termination, generating a smaller measure of vacancies in the next period, which is Type S (2 in Figure 3).

### 6.3 Example 3: Longer cycles

The model is capable of generating a large set of cyclical equilibria, with shorter and longer individual cycles. In this subsection, we construct a set of equilibria in which all individual cycles share the same cyclicality where the economy’s stock of firms and aggregate output rise in the first period of the cycle, fall in the second, and then stay constant in the remaining time of the cycle. In these equilibria, over each individual cycle the economy oscillates between Type S and Type T periods, with firms entering the market when a new cycle begins and
Figure 4: Equilibrium cycles: a numerical example

Exiting when the boom ends. Importantly, these individual cycles need not have the same length. In fact, the length of each individual cycle can be any positive even number.

As described in Proposition 9 in Appendix B.6, in these equilibria, once the length of an individual cycle, $2n$, $n$ being any positive integer, is given, the dynamics over the cycle is determined. That is, what happens in any individual cycle depends, completely and deterministically, on how long the cycle lasts. The equilibrium puts no restrictions on the length of any individual cycle, except that it must be an even number of periods. One way to think of such an equilibrium, therefore, is to view the length of each individual cycle, or the duration of each recession, as being determined ex ante by a random number generator. As such, Proposition 9 essentially depicts a scenario in which all individual cycles exhibit the same cyclicality (that is, starting with a boom which is followed by a recession) but each last for an ex ante randomly determined number of periods.

Figure 5 depicts a numerical example of such an equilibrium, with the relevant parameters summarized in Table 1 in Appendix B.6.3. Notice the regularity in the cyclicality and the randomness in the duration of the recessions the equilibrium generates.
7 Co-existence

In the model, stationary and non-stationary equilibria coexist over at least a subset of the parameter space we consider. This is shown in Figure 5. Specifically, a stationary equilibrium, as described in Proposition 2, exists over the whole parameter space, i.e., for all \((C_o, C_e)\) such that \(C < C_o \leq C\) and \(C_e > 0\). For any \((C_o, C_e)\) in area OAB, the model has an equilibrium in two-period cycles where all periods are Type I, as described in Proposition 7. For any \((C_o, C_e)\) in area OEFG, the model has an equilibrium where Type S and Type T periods alternate, as described in Proposition 8. Finally, for any \((C_o, C_e)\) in area OED, the model displays equilibrium cycles where each individual cycle lasts for an arbitrary even number of periods, as described in Proposition 9.

One parameter of the model that is particularly important for the multiplicity is the entry cost \(C_e\). Remember we have so far focused on the case of \(C_e > 0\). Suppose \(C_e = 0\), then condition (13) implies \(V_t = 0\) for all \(t\), and for each \(C_o\) satisfying Assumption 2 the model has a unique stationary equilibrium with \(\eta_t = \eta_s^+\) for all \(t\) (Proposition 2). In fact, it
can be shown that this is the only equilibrium of the model.\textsuperscript{12} What this says, therefore, is that the zero cost of entry, an assumption often maintained in models that allow for entry and exit of firms, has in this model important implications for the equilibrium dynamics and outcomes the model generates.

8 Output and Welfare

Do stationary equilibria provide better output and/or welfare outcomes than the non-stationary equilibria, or vice versa? Given the multiplicity in both the model’s stationary and non-stationary equilibria, one way to answer this question is to compare the equilibrium which achieves the best output and welfare outcomes among all stationary equilibria with the one that attains the best output and welfare outcomes among all non-stationary equilibria. The “best” stationary equilibrium is simply the one that attains the highest measure of firms, $\eta^+$. This is the stationary equilibrium that offers the highest aggregate output and the maximum expected utility for young workers. To find the “best” non-stationary equilibrium, however, is less trivial. First, in a non-stationary equilibrium, generations differ in lifetime utility and it is not obvious how welfare should be aggregated across generations. Second, the set of all non-stationary equilibria is large.

To narrow the focus, consider the “best” equilibrium among all non-stationary equilibria with two-period cycles described in Proposition 8. Notice that for all $(C_0, C_e)$ satisfying (51), stationary and non-stationary equilibria coexist, and in all these equilibria the economy’s aggregate output is determined by the constant $\eta$. Three observations emerge. First, the maximum output attainable in a stationary equilibrium is higher than what a non-stationary equilibrium can achieve. Second, young workers are better off to be born into a Type T period in the “best” non-stationary equilibrium than in the “best” stationary equilibrium, but they are worse off to be born into a Type S period in the “best” non-stationary equilibrium than in the “best” stationary equilibrium. Third, the average utility of young workers in the

\textsuperscript{12}See Appendix C.
“best” non-stationary equilibrium (in Type T and S periods respectively) is lower than that of young workers in the “best” stationary equilibrium. See Appendix D for details.

Relative to the stationary equilibria, an important aspect of the non-stationary equilibria is that they permit an active submarket for old workers and, therefore, make old workers better off. Remember the one-period payoff for a firm participating in the submarket for old workers is negative which, in a stationary equilibrium, necessarily implies negative firm payoffs in all periods. In a non-stationary equilibrium, however, a firm participating in the submarket for old workers is anticipating a positive value from next period on. This offers an incentive for firms to stay in the market, which, in turn, benefits old workers.

Could the incentives to create employment for old workers in the non-stationary equilibria be strong enough to make welfare higher in a non-stationary equilibrium than in the stationary equilibria? We have so far not been able to find examples that provide a positive answer to this question.

9 Concluding Remarks

This paper shows that moral hazard and termination in long-term contracting can generate endogenous fluctuations in the labor market. We claim that these fluctuations, coming internally from the optimality in labor contracting, may contribute to the explanation of the observed “excess” volatility in employment which standard search models are not able to account for. For analytical tractability, the model is designed to consist of two-period-lived overlapping generations of workers so a contract is at most two-period long. An interesting extension of the analysis is to consider longer worker life and hence longer contract durations. This would render the model suitable for more serious quantitative simulation and even empirical testing. We do not believe, however, that introducing longer contracts and more complex dynamics would comprise the fundamental logic of the paper which, we hope, is
better illustrated with a simple model like the one we have studied.\footnote{The OG structure of the model, however, is essential to our analysis. It has two effects. First, it creates, in each period, a division of the labor market into submarkets where contracts of differential durations are traded. Moral hazard gives longer-term contracts higher values, which then generate competition for the young workers which, in turn, creates the dynamic interaction between the supply of and the demand for long-term jobs, which then determines the optimal termination in employment. Second, it dictates that if more workers enter a long-term contract in any given period, then more vacancies will be released because old workers die to leave the market. The first effect is essential to our story, the second not. That the second effect is not essential can be seen from Example 1 (Section 6.2) where it does not play a role in equilibrium — the measure of firms entering a long contract is constant at 1/2. To summarize, the overlapping generations assumption is important because of the way it works with moral hazard and optimal termination, not because it physically forces old workers to die to create a relation between vacancies across periods.}

References


**Mathematical Appendix: For Online Publication**

A Contracting

We will use the following simplifying notation throughout the appendix.

\[ \theta'_i \equiv \theta_i - C_o, \quad i = 1, 2; \quad \bar{\theta}' \equiv \bar{\theta} - C_o. \]

In particular, \( \theta'_i \) is the firm’s gross output net of the fixed operating cost in state \( i \).

A.1 Proof of Proposition 1

**Lemma 3** For all \( w \) with which the problem that defines \( V_{t,t}(w) \) has a solution, it is optimal to set \( c_{ir} = 0 \) for \( i = 1, 2 \).

**Proof.** Suppose for some \( w \), a solution that defines \( V_{t,t}(w) \) has \( c_{ir} = \Delta > 0 \), for some \( i \). Consider a deviation from this solution, letting \( c'_{ir} = 0 \) and \( w'_{ir} = w_{ir} + \Delta / \beta \). Clearly,
constraints (7) and (8) would continue to hold under this deviation, while the firm’s value
not changed. ■

Given Lemma 3, constraint (7) can be rewritten as:

\[
\left[(1 - p_2)\delta w_{2r} + p_2((1 - \delta)c_{2f} + \delta w_{st+1})\right] - \left[(1 - p_1)\delta w_{1r} + p_1((1 - \delta)c_{1f} + \delta w_{st+1})\right] \geq \frac{(1 - \delta)\psi}{x_2 - x'_2}.
\]

Lemma 4 For all \(w\) with which the problem that defines \(V_{l,t}(w)\) has a solution, it is optimal
to set \(c_{if} = 0\) if \(p_i < 1\), for \(i = 1, 2\).

Proof. Suppose for some \(w\), \(c_{if} = \Delta > 0\) and \(p_i < 1\) for some \(i\). Let \(c'_{if} = 0\) and
\(w'_{ir} = w_{ir} + \frac{p_i\Delta}{(1 - p_i)\beta}\). Then constraints (7) and (8) are still satisfied, and the firm’s value is
not changed. ■

Given Lemma 4, we could then focus on contracts that satisfy \(p_i c_{if} = c_{if}\). It then follows
that the problem of optimal long-term contracting can be restated in the following two
sub-problems.

S1:

\[
G(\Delta_i) = \max_{c_{if}, w_{ir}, p_i} (1 - \beta)\theta'_i - p_i(1 - \beta)c_{if} + p_i\beta V_{t+1} + (1 - p_i)\beta V_{s,t+1}(w_{ir}),
\]

s.t.

\[
c_{if} \geq 0, \ w_{ir} \geq w, \ 0 \leq p_i \leq 1,
\]

\[
(1 - p_i)\delta w_{ir} + p_i((1 - \delta)c_{if} + \delta w_{st+1}) - (1 - \delta)\psi = \Delta_i;
\]

\[
p_i c_{if} = c_{if}.
\]

S2:

\[
V_{l,t}(w) = \max_{\Delta_1, \Delta_2} x_1 G(\Delta_1) + x_2 G(\Delta_2),
\]

s.t.

\[
\Delta_2 - \Delta_1 \geq \frac{(1 - \delta)\psi}{x_2 - x'_2},
\]

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\[ x_1 \Delta_1 + x_2 \Delta_2 = w. \]

We first solve \( S_1 \) with \( c_i^f \) fixed at its optimal level. In other words, we solve

\[
\max_{p_i, w_{ir}} p_i V_{t+1} + (1 - p_i)V_{s,t+1}(w_{ir}),
\]

s.t.

\[
w_{ir} \geq w, \quad 0 \leq p_i \leq 1,
\]

\[
(1 - p_i)w_{ir} + p_i w_{s,t+1} = \delta_i,
\]

\[ p_i c_i^f = c_i^f, \]

where \( \delta_i \equiv (\Delta_i + (1 - \delta)\psi - (1 - \delta)c_i^f)/\delta > 0 \). To solve this problem, in turn we remove the constraint \( p_i c_i^f = c_i^f \) from the constraint set, solve the resulting problem, and then show that the resulting solution does satisfy the omitted constraint \( p_i c_i^f = c_i^f \).

Substituting \( V_{s,t+1}(w) = (1 - \beta)(\bar{\theta} - \psi - w) + \beta V_{t+2} \) into the above problem (i.e., after removing the constraint \( p_i c_i^f = c_i^f \)) gives:

\[
\max_{p_i, w_{ir}} p_i[V_{t+1} - (1 - \beta)(\bar{\theta} - \psi - w_{s,t+1}) - \beta V_{t+2}] + (1 - \beta)(\bar{\theta} - \psi) - (1 - \beta)\delta_i + \beta V_{t+2},
\]

s.t.

\[
w_{ir} \geq w, \quad 0 \leq p_i \leq 1,
\]

\[
(1 - p_i)w_{ir} + p_i w_{s,t+1} = \delta_i.
\]

By Assumption 1, this problem is rewritten as

\[
\min_{p_i, w_{ir}} p_i,
\]

s.t.

\[
w_{ir} \geq w, \quad 0 \leq p_i \leq 1,
\]
(1 - p_i)w_{ir} + p_i w_{st+1} = \delta_i.

Since \( \delta_i \geq (1 - p_i)w + p_i w_{st+1} \geq w_{st+1} \), we have that the solution to the above problem has

(i) If \( \delta_i \geq w \), then \( p_i = 0 \);
(ii) If \( \delta_i \in [w_{st+1}, w) \), then \( w_{ir} = w \) and \( p_i = \frac{w - \delta_i}{w - w_{st+1}} \).

**Lemma 5** For all \( w \) with which the problem that defines \( V_{l,t}(w) \) has a solution, \( p^*_2(w) = 0 \).

**Proof.** Suppose \( p^*_2(w) = 1 \) for some \( w \). Then the incentive constraint can be rewritten as

\[
(1 - \delta)c_{2f} + \delta w_{st+1} - [(1 - p_1)\delta w_{1r} + p_1((1 - \delta)c_{1f} + \delta w_{st+1})] \geq \frac{(1 - \delta)\psi}{x_2 - x'_2}.
\]

This implies \( c_{2f} > 0 \). Consider the following deviation from the optimal contract. Let \( c'_{2f} = c_{2f} - \Delta > 0 \) for some sufficiently small \( \Delta > 0 \). Let \( w'_{2r} = w \) and \( c'_{2r} = c_{2f} \). Let \( \Delta \) and \( p'_2 \) be chosen to satisfy

\[
(1 - p'_2)((1 - \delta)c_{2f} + \delta w) + p'_2((1 - \delta)(c_{2f} - \Delta) + \delta w_{st+1}) = (1 - \delta)c_{2f} + \delta w_{st+1}
\]

and so the incentive and promise-keeping constraints (7) and (8) continue to hold with the deviation. With this deviation, the firm would obtain a net gain of

\[
\epsilon = x_2 \{p'_2(1 - \beta)\Delta + (1 - p'_2)\beta[V_{s,t+1}(w'_{2r}) - V_{t+1}]\}
\]  
\[
> x_2 \{p'_2\Delta + \beta(1 - p'_2)(w_{st+1} - w)\}
\]  
\[
= 0,
\]

where the inequality holds by Assumption 1, a contradiction. Hence it must hold that \( p^*_2(w) < 1 \). And then it is optimal to set \( c_{2f} = 0 \) from Lemma 4. Then it follows from the incentive constraint that

\[
\delta_2 \equiv (1 - p_2)w_{2r} + p_2 w_{st+1} > \frac{\psi}{\beta(x_2 - x'_2)} > w.
\]
Thus $p^*_2(w) = 0$. ■

**Lemma 6** For all $w$ with which the problem that defines $V_{l,t}(w)$ has a solution, it is optimal to set $c_{1f} = 0$.

**Proof.** Suppose $c_{1f} > 0$ for some $w$. Consider a deviation from the optimal contract with $c'_{2f} = 0$ and let $w'_{2r}$ be chosen to satisfy

$$x_2 \delta w'_{2r} = x_2 \delta w_{2r} + x_1 p_1 (1 - \delta) c_{1f},$$

while holding other parts of the contract constant. With this deviation, constraints (7) and (8) continue to be satisfied, and the values for both the worker and the firm remain unchanged. ■

Lemmas 5 and 6 imply $p_i c_{i f} = c_{i f}$ for $i = 1, 2$. Now given Lemmas 3-6, the optimization problem that defines $V_{l,t}(w)$ can be rewritten as

$$V_{l,t}(w) = \max_{p_1, w_{1r}, w_{2r}} \{ (1 - \beta) \tilde{\theta}' + x_1 \beta [p_1 V_{t+1} + (1 - p_1) V_{s,t+1}(w_{1r})] + x_2 \beta V_{s,t+1}(w_{2r}) \},$$

s.t.

$$w_{1r}, w_{2r} \geq \underline{w}, \quad 0 \leq p_1 \leq 1,$$

$$\delta w_{2r} - [(1 - p_1) \delta w_{1r} + p_1 \delta w_{s,t+1}] \geq \frac{(1 - \delta) \psi}{x_2 - x_2'},$$

$$x_1 [(1 - p_1) \delta w_{1r} + p_1 \delta w_{s,t+1}] + x_2 \delta w_{2r} - (1 - \delta) \psi = w.$$  \hspace{1cm} (31)

Substituting the incentive constraint (30) into the promise-keeping constraint (31) yields

$$w \geq \delta w_{s,t+1} + (1 - \delta) \underline{w}.$$  

This implies that $w_{A,t} \equiv \delta w_{s,t+1} + (1 - \delta) \underline{w}$ is the minimum $w$ that an incentive compatible long-term contract can implement.
Now given the above, and following from $V_{s,t+1}(w) = (1 - \beta)(\bar{\theta}' - \psi - w) + \beta V_{t+2}$ and (31), we have

$$V_{l,t}(w) = \max_{p_1, w_{1r}, w_{2r}} \{(1 - \beta^2)(\bar{\theta}' - \psi - w) + \beta^2 V_{t+2} + x_1 \beta p_1 [V_{t+1} - (1 - \beta)(\bar{\theta}' - \psi - w_{s,t+1}) - \beta V_{t+2}]\},$$

subject to (29) and (31). By Assumption 1, this optimization problem is equivalent to

$$\min_{p_1, w_{1r}, w_{2r}} p_1 \text{ s.t. (29)-(31)}.$$ 

With this problem, observe then that the incentive constraint (30) must be binding. Suppose not. Then construct a deviation from the optimal contract by, while holding other parts of the contract constant, letting $w'_{2r} = w_{2r} - \Delta$ for some sufficiently small $\Delta > 0$, and choosing $\Delta$ and $p'_1$ be such that

$$x_1 \delta [(1 - p_1)w_{1r} + p'_1 w_{s,t+1}] + x_2 \delta w'_{2r} = x_1 \delta [(1 - p_1)w_{1r} + p_1 w_{s,t+1}] + x_2 \delta w_{2r}.$$ 

This deviation would not violate the constraints (30) and (31), but since

$$p_1 - p'_1 = \frac{x_2 \Delta}{x_1(w_{1r} - w_{s,t+1})} > 0,$$

it reduces the probability of termination, increasing the value of the firm. So (30) holds as an equality at the optimum.

Substituting next the binding incentive constraint (35) into constraint (31) gives

$$\delta [(1 - p_1)w_{1r} + p_1 w_{s,t+1}] + (1 - \delta)w = w.$$ 

Then if $w \geq w$, we have $p^*_1(w) = 0$ and $w_{1r} = \frac{w - (1 - \delta)w}{\delta}$; if $w \in [w_{A,t+1}, w]$, we have $p^*_1(w) = \frac{w - w}{w - w_{s,t+1}}$ and $w_{1r} = w$. This completes the proof of the proposition.
A.2 Proposition 4 and Proof

**Proposition 4** Suppose Assumption 1 is violated at $t$. Then for all $w$ with which the problem that defines $V_{t,t}(w)$ has a solution, it is optimal to set $p_i(w) = 1$ for $i = 1, 2$.

**Proof.** Suppose not and suppose, with the optimal contract, $p_i(w) < 1$ at some $w$. Then, following from Lemmas 3 and 4 in A.1, it is optimal to set $c_{ir} = c_{if} = 0$. Consider a deviation from the optimal contract, letting $p_i' = 1$ and $c_{if}' = \beta(1 - p_i)(w_{ir} - w_{st+1})$, while holding other parts of the contract constant. Obviously, this deviation does not violate constraints (7) and (8), but the resulting contract would make the firm weakly better off in expected utility, by a non-negative amount of

$$
\epsilon = x_i \beta (1 - p_i) [V_{t+1} - (1 - \beta)(\bar{\theta}' - \psi - w_{st+1}) - \beta V_{t+2}] \geq 0.
$$

This proves the proposition. ■

B Equilibrium

For technical convenience, we define the equilibrium expected utility for a new hire in an inactive submarket to be the limit of that in an active submarket with the measure of firms going to zero. That is, $w_{y,t} = w_{A,t}$ if the submarket for young workers is inactive, and $w_{o,t} = w$ if the submarket for old workers is inactive.

B.1 Proposition 5 and Proof

**Proposition 5** The model does not have any equilibrium in which $\eta_t \geq 1$ for some $t$ if and only if $C_o > C$.

**Proof.** (Sufficiency) Suppose $C_o > C$. Suppose $\eta_t \geq 1$ for some $t$. Consider the two cases (i) $\eta_t > 1$ and (ii) $\eta_t = 1$ separately.
(i) Suppose $\eta_t > 1$. Then in period $t$ there must be a submarket in which there are more vacant firms than unemployed workers. Hence firms’ expected value from entering that submarket is

$$-(1 - \beta)C_o + \beta V_{t+1}.$$ 

In order for the firms to be willing to enter that submarket in the first place, it must be that their expected value from entering the other submarket be at most $-(1 - \beta)C_o + \beta V_{t+1}$. Hence

$$V_t = -(1 - \beta)C_o + \beta V_{t+1}.$$ 

By the free-entry-and-exit condition, $V_t \geq 0$, which implies $V_{t+1} > 0$. Hence no firm exits in period $t + 1$: $\eta_{t+1} \geq \eta_t > 1$. By induction we have $\eta_{\tau} > 1$ for all $\tau > t$. But this implies that $V_t = -C_o < 0$, violating the free-entry-and-exit condition.

(ii) Suppose $\eta_t = 1$. Then there are two cases: (a) there is a positive measure of unemployed old workers in period $t$; and (b) there are no unemployed old workers in period $t$.

(a) Suppose there is a positive measure of unemployed old workers in period $t$. Then there must be a positive measure of firms entering the submarket for old workers, since otherwise a firm’s expected value from entering the submarket for young workers is

$$V_{t,t}(\bar{w}_t) = -(1 - \beta)C_o + \beta V_{t+1},$$

which is strictly less than $V_{s,t}(\bar{w})$, its expected value from entering the submarket for old workers. Hence

$$V_t \leq V_{s,t}(\bar{w}) = (1 - \beta)(C - C_o) + \beta V_{t+1}.$$ 

By the free-entry-and-exit condition, $V_t \geq 0$, which implies $V_{t+1} > 0$ under Assumption 2. Hence no firm exits in period $t + 1$: $\eta_{t+1} \geq \eta_t > 1$. 

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(b) Suppose there are no unemployed old workers in period $t$. This is the case if and only if $1/2$ measure of firms entered long-term contracts with young workers in period $t - 1$ and stay with their retained old workers in period $t$. This implies Assumption 1 holds for $t - 1$:

$$V_t - (1 - \beta)(C - C_o) - \beta V_{t+1} < 0,$$

since otherwise the optimal long-term contract in period $t - 1$ specifies $p_{i,t-1}(w) = 1$ for all $w$ and $i = 1, 2$. Here $w_{st} = w$ since $\eta_t = 1$. By the free-entry-and-exit condition, $V_t \geq 0$, and therefore $V_{t+1} > 0$ and $\eta_{t+1} \geq \eta_t \geq 1$.

By induction, we have $\eta_{t} \geq 1$ for all $t \geq t$ and

$$V_{\tau} \leq (1 - \beta)(C - C_o) + \beta V_{\tau+1}, \quad \forall \tau \geq t.$$ 

Hence $V_t \leq C - C_o < 0$, by Assumption 2, violating the free-entry-and-exit condition.

(Necessity) Suppose $C_o \leq C$. Then for all $C_e > 0$ the model has a stationary equilibrium in which $\eta_t = 1$ for all $t$. In each period, there is a measure 1 of unemployed workers ($L_t = 1$) and a measure 1 of vacant firms. Among the vacant firms, half of them enter the submarket for young workers, and the rest enter the submarket for old workers. Each firm is able to hire one worker via a short-term contract, and will reenter the labor market in the next period with probability 1. The prevailing expected utilities offered to workers in both submarkets are $w_{y,t} = w_{o,t} = w \in [w_\hat{\theta} - \psi - C_o]$ such that $V_t = \hat{\theta} - \psi - w - C_o \leq C_e$. Clearly, firms’ entry and exit decisions are optimal. Each unemployed worker is hired with probability 1 and therefore $w_{st} = w$. It is easy to verify that in the equilibrium described above Assumption 1 is violated for all $t$, and therefore by Proposition 4 the optimal long-term contract is essentially the short-term contract. 

\section*{B.2 Proposition 6 and Proof}

\textbf{Proposition 6} Assumption 1 holds for all $t$ in any equilibrium where $0 < \eta_t < 1$ for all $t$. 

\newpage
Proof. Fix an arbitrary \( t \). We prove the proposition by showing that Assumption 1 holds for \( t - 1 \). There are two cases to consider: (i) Assumption 1 holds for \( t \) and (ii) Assumption 1 is violated at \( t \).

(i) Suppose Assumption 1 holds for \( t \). Then the optimal long-term contract in period \( t \) is described by Proposition 1, and all the equilibrium conditions for \( t \) in Definition 1 must hold. There are two subcases to consider here: Period \( t \) is Type SI and period \( t \) is Type T. If period \( t \) is Type SI, then

\[
V_t = V_{l,t}(w_{y,t}),
\]

\[
\leq (1 - \beta)(1 + x_2\beta)(\bar{\theta}' - \psi) - (1 - \beta)w + x_1\beta V_{t+1} + x_2\beta^2 V_{t+2},
\]

\[
\leq (1 - \beta)(\bar{\theta}' - \psi) - (1 - \beta)(1 - x_2\beta)w + \beta V_{t+1},
\]

\[
< (1 - \beta)(\bar{\theta}' - \psi - w_{st}) + \beta V_{t+1}.
\]

The first inequality holds since \( w_{y,t} \geq w_{A,t} \) in any incentive compatible long-term contract. The second inequality holds since \( V_{t+1} \geq V_{s,t+1}(w) \) by (15). The last inequality holds since \( w_{st} = 0 \) in a Type SI period. If period \( t \) is Type T, then

\[
V_t = (1 - \beta)(\bar{\theta}' - \psi - w) + \beta V_{t+1} < (1 - \beta)(\bar{\theta}' - \psi - w_{st}) + \beta V_{t+1}.
\]

The inequality holds since \( 0 < \eta_t < 1 \).

(ii) Suppose Assumption 1 is violated at \( t \). Then it follows from Proposition 4 that for all \( w \) with which the problem that defines \( V_{l,t}(w) \) has a solution, it is optimal to set \( p_s(w) = 1 \). That is, the optimal long-term contract is essentially the optimal short-term contract. Consider now the labor market equilibrium. The two submarkets are unified since firms are indifferent between contracting with young and old workers. Given \( 0 < \eta_t < 1 \), it is easy to see the prevailing expected utilities offered to workers are

\[
w_{y,t} = w_{A,t} \text{ and } w_{o,t} = w.
\]

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Hence the firm’s expected value in period $t$ is

$$V_t = (1 - \beta)(\bar{\theta'} - \psi - w) + \beta V_{t+1} < (1 - \beta)(\bar{\theta'} - \psi - w_{st}) + \beta V_{t+1},$$

where the strict inequality holds because $0 < \eta_t < 1$.

To summarize, we have shown that in all equilibria with $0 < \eta_t < 1$ for all $t$, Assumption 1 holds for $t - 1$, no matter whether it holds for $t$ or not. This implies that the model does not have any equilibrium in which $0 < \eta_t < 1$ for all $t$ and Assumption 1 is violated at some $t$. ■

B.3 Stationary Equilibria

Proof of Proposition 2.

Let

$$\eta^+_s(C_o) \equiv 1 - \frac{x_1(1 + \beta)(C_o - \bar{C})}{2x_1\beta(C_o - \bar{C}) + 2x_2\beta w},$$

$$\eta^-_s(C_o, C_e) \equiv \begin{cases} 
1 - \frac{x_1(1+\beta)(C_e + C_o - \bar{C})}{2x_1\beta(C_o + C_e - \bar{C}) + 2x_2\beta w} & \text{if } C_e + C_o < \bar{C}, \\
0 & \text{if } C_e + C_o \geq \bar{C}.
\end{cases}$$

Because $C < C_o \leq \bar{C}$ (by Assumption 2) and $C_e > 0$, it is straightforward to verify that $0 \leq \eta^-_s < \eta^+_s < 1$.

To prove the proposition, more specifically we show: The model has a stationary equilibrium for any $(C_o, C_e)$ with $C_e > 0$ and $C_o \in (C, \bar{C}]$.

1. Suppose $C_o + C_e < \bar{C}$. Then for each $\eta \in [\eta^-_s, \eta^+_s]$ the model has a stationary equilibrium in which all periods are Type I, the optimal long-term contract is as described in
Proposition 1(ii), and the equilibrium values of the aggregate variables satisfy

\[ \eta_t = \eta, \quad L_t = \frac{3}{2} - \eta, \]

\[ \alpha_{y,t} = 1, \quad \alpha_{o,t} = 0, \]

\[ w_{o,t} = w, \quad w_{y,t} = w - \frac{2\beta(1 - \eta)(w - w_{s,t+1})}{(1 + \beta)x_1}, \quad w_{s,t+1} = 0, \]

\[ V_t = \bar{\theta} - \psi - \frac{[x_1(1 + \beta) - 2\beta(1 - \eta)]w}{x_1[1 + \beta - 2\beta(1 - \eta)]} - C_o. \]

2. Suppose \( C_o + C_e \geq \overline{C} \). Then

(a) for each \( \eta \in [\eta_s^-, (1 + x_2)/2) \) the model has a stationary equilibrium where all periods are Type S, the optimal long-term contract is as described by Proposition 1(ii), and the equilibrium values of the aggregate variables satisfy

\[ \eta_t = \eta, \quad L_t = 1 - \frac{x_2\eta}{1 + x_2}, \]

\[ \alpha_{y,t} = \frac{2\eta}{1 + x_2}, \quad \alpha_{o,t} = 0, \]

\[ w_{o,t} = w, \quad w_{y,t} = w_{A,t}, \quad w_{s,t+1} = 0, \]

\[ V_t = \bar{\theta} - \psi - \frac{1}{1 + x_2\beta}w - C_o; \]

(b) for each \( \eta \in [(1 + x_2)/2, \eta_s^+] \) the model has a stationary equilibrium where all periods are Type I, as characterized in 1.

First, we claim that there are no stationary equilibria in which all periods are Type T. Suppose not. Then it follows from the analysis in Section 4.2 that

\[ V = (1 - \beta)(C - C_o) + \beta V, \]

and hence

\[ V = C - C_o < 0. \]
where the inequality holds by Assumption 2. But $V < 0$ contradicts the free-entry-and-exit condition.

Next, consider the stationary equilibria where all periods are Type S or Type I. In such equilibria, all vacant firms in each period enter a long-term contract with a young worker. Let $\hat{p}_1^*$ denote the equilibrium probability with which a young worker is fired when her first period output is low. Then the equilibrium values of $L$ and $\alpha_y$ satisfy

$$L = 1 - [\eta - (1 - L)](1 - x_1\hat{p}_1^*),$$ (32)

$$\alpha_y = \frac{\eta - (1 - L)}{\frac{1}{2}}.$$ (33)

Combining (33) and (32) yields

$$\alpha_y = \frac{2\eta}{2 - x_1\hat{p}_1^*}.$$

There are two cases: (i) all periods are Type S ($\alpha_y < 1$), and (ii) all periods are Type I ($\alpha_y = 1$).

(i) Suppose all periods are Type S. Then $w_{y,t} = w_{A,t}$ and $\hat{p}_1^* = 1$ and hence

$$\alpha_y = \frac{2\eta}{1 + x_2},$$

which is strictly less than 1 if and only if $\eta < (1 + x_2)/2$. In this case, the equilibrium value of a firm is given by

$$V = \bar{\theta}' - \psi - \frac{w}{1 + x_2\beta} = \bar{C} - C_o.$$

Remember that the free-entry-and-exit condition requires $0 \leq V_t \leq C_e$. In a stationary equilibrium where all periods are Type S, the first inequality holds if and only if $C_o \leq \bar{C}$, and the second inequality holds if and only if $C_e \geq \bar{C} - C_o$.

(ii) Suppose all periods are Type I. $\alpha_y = 1$ implies $\hat{p}_1^* = 2(1 - \eta)/x_1$. For $\hat{p}_1^*$ to be a well
defined probability,

\[ \eta \in \left[ \frac{x_2 + 1}{2}, 1 \right). \]

In this case, the equilibrium value of the firm is given by

\[
V = \bar{\theta}' - \psi - \frac{[x_1(1 + \beta) - 2\beta(1 - \eta)]w}{x_1[1 + \beta - 2\beta(1 - \eta)]}.
\]

Observe that \( V \) is strictly decreasing in \( \eta \), goes to \( \bar{C} - C_o \geq 0 \) as \( \eta \) goes to \( (x_1 + 1)/2 \), and goes to \( \underline{C} - C_o < 0 \) as \( \eta \) goes to 1. Again the free-entry-and-exit condition requires \( 0 \leq V_t \leq C_e \). Clearly, there exists a unique \( \eta^+(C_o) \in [(x_2 + 1)/2, 1) \) such that \( V \geq 0 \) if and only if \( \eta \leq \eta^+(C_o) \), where

\[
\eta^+(C_o) = 1 - \frac{x_1(1 + \beta)(\bar{\theta}' - \psi - w)}{2x_1\beta(\bar{\theta}' - \psi) - 2\beta w}.
\]

Notice here that \( \eta^+ \) is strictly decreasing in \( C_o \).

Suppose \( C_e \geq \bar{C} - C_o \). Then \( V \leq C_e \) for all \( \eta \geq (x_2 + 1)/2 \). Suppose \( 0 < C_e < \bar{C} - C_o \). There is a unique \( \eta^-(C_o, C_e) \in ((x_2 + 1)/2, 1) \) such that \( V \leq C_e \) when \( \eta \geq \eta^-(C_o, C_e) \), where

\[
\eta^-(C_o, C_e) = 1 - \frac{x_1(1 + \beta)[C_e - (\bar{\theta}' - \psi - w)]}{2x_1\beta[C_e - (\bar{\theta}' - \psi)] + 2\beta w}.
\]

Observe that \( \eta^- \) is strictly decreasing in both \( C_e \) and \( C_o \).

Define \( \eta^-_{s}(C_o, C_e) \equiv 0 \) when \( C_e \geq \bar{C} - C_o \). This proves the proposition.

\[ \blacksquare \]

### B.4 The Bounds on Equilibrium Firm Values

**Lemma 7** The equilibrium value of the firm \( V_t \) satisfies

\[
\bar{\theta}' - \psi - w \leq V_t \leq \bar{\theta}' - \psi - \frac{1}{1 + x_2\beta}w, \quad \forall t.
\]  

(34)
Proof. It follows from (15) that

\[ V_{t-1} \geq (1 - \beta)(\bar{\theta}' - \psi - \bar{w}) + \beta V_t, \]

\[ \ldots \]

\[ \geq (1 - \beta)(\bar{\theta}' - \psi - \bar{w})(1 + \beta + \cdots + \beta^s) + \beta^{s+1}V_{t+s}. \]

Let \( s \) go to infinity and we have \( V_{t-1} \geq \bar{\theta}' - \psi - \bar{w} \) for all \( t \geq 1 \).

To obtain the upper bound on \( V_{t-1} \), we have

\[ V_{t-1} \leq V_{t-1}(w_{A,t-1}) \]

\[ = (1 - \beta)(1 + x_2\beta)(\bar{\theta}' - \psi - \bar{w}) + (1 - \beta)x_2\beta(\bar{w} - w_{t-1}) + x_1\beta V_t + x_2\beta^2V_{t+1}, \]

\[ \leq (1 - \beta)(1 + x_2\beta)(\bar{\theta}' - \psi) - (1 - \beta)\bar{w} + x_1\beta V_t + x_2\beta^2V_{t+1}. \]  

(35)

It follows that

\[ V_{t-1} + x_2\beta V_t \leq (1 - \beta)(1 + x_2\beta)(\bar{\theta}' - \psi) - (1 - \beta)\bar{w} + \beta(V_t + x_2\beta V_{t+1}), \]

\[ \ldots \]

\[ \leq [(1 - \beta)(1 + x_2\beta)(\bar{\theta}' - \psi) - (1 - \beta)\bar{w}](1 + \beta + \cdots + \beta^s) \]

\[ + \beta^{s+1}(V_{t+s} + x_2\beta V_{t+s+1}). \]

Let \( s \) go to infinity to get

\[ V_{t-1} + x_2\beta V_t \leq (1 + x_2\beta)(\bar{\theta}' - \psi) - \bar{w}, \forall t \geq 1. \]

Substitute this into (35) and we have

\[ V_{t-1} \leq (1 - x_2\beta)[(1 + x_2\beta)(\bar{\theta}' - \psi) - \bar{w}] + x_2^2\beta^2V_{t+1}, \]

\[ \ldots \]
\[ \leq (1 - x_2\beta)[(1 + x_2\beta)(\bar{\theta}' - \psi) - w] (1 + x_2^2\beta^2 + \cdots + x_2^{2s}\beta^{2s}) + x_2^{2s+2}\beta^{2s+2}V_{t+2s+1}. \]

Let \( s \) go to infinity to get

\[ V_{t-1} \leq \bar{\theta}' - \psi - \frac{1}{1 + x_2\beta}w, \quad \forall t \geq 1. \]

and therefore the inequality holds. Furthermore, in a Type S or Type I period with \( w_{y,t} < w_{B,t} \), it must hold that

\[ V_t > V_{s,t}(w) \geq \bar{\theta}' - \psi - w, \]

and in a Type T or Type I period with \( w_{y,t} > w_{A,t} \), it must hold that

\[ V_t < V_{l,t}(w_{A,t}) \leq \bar{\theta}' - \psi - \frac{1}{1 + x_2\beta}w. \]

The lemma is proven. \( \blacksquare \)

### B.5 Equilibrium Cycles

**Proof of Lemma 1.** Suppose not and let period \( t - 1 \) and \( t \) be Type TI with \( w_{o,t} = w_{B,t} \). This implies that in period \( t \) we have

\[ V_t = (1 - \beta)(\bar{\theta}' - \psi - w) + \beta V_{t+1}. \]

Substituting this into \( V_{t-1}(w_{B,t-1}) = V_{s,t-1}(w) \) yields \( w_{y,t-1} = w_{B,t-1} = w \). It then follows from Proposition 1(iii) that \( \hat{p}^*_1 t-1 = \hat{p}^*_2 t-1 = 0 \). Thus, the number of vacant firms in period \( t \) is given by \( \eta_t - \frac{1}{2} \), which is strictly less than \( \frac{1}{2} \). This contradicts with our initial assumption that period \( t \) is Type TI. (Note that Assumption 2 is not used in the above proof, and it is clear that Lemma 1 holds whether Assumption 2 holds or not.) \( \blacksquare \)

**Proof of Lemma 2.** Suppose not and let period \( t - 1 \) be Type SI. First, no prospective firm is willing to enter the market in period \( t + 1 \):

\[ V_{t+1} = (1 - \beta)(\bar{\theta}' - \psi - w) + \beta V_{t+2} < V_{t+2} \leq C_e. \quad (36) \]
The first inequality holds because period $t + 2$ is Type SI with $w_{y,t+2} < w_{B,t+2}$ by Lemma 1, and that the left inequality of (34) holds strictly in this case. Hence, $\eta_{t+1} \leq \eta_t$. Then the number of vacant firms in period $t + 1$ satisfies:

$$\eta_{t+1} - [\eta_t - (1 - L_t)](x_2 + x_1(1 - \hat{p}^*_t))]$$

$$= \eta_{t+1} - \eta_t - (\eta_t - (1 - L_{t-1}))(x_2 + x_1(1 - \hat{p}^*_t))]$$

$$\leq x_1 \hat{p}^*_t \eta_t + (1 - x_1 \hat{p}^*_t)(1 - x_1 \hat{p}^*_t)[\eta_t - (1 - L_{t-1})]$$

$$\leq \frac{1}{2} x_1 \hat{p}^*_t + (1 - x_1 \hat{p}^*_t)[\eta_t - (1 - L_{t-1})] \leq \frac{1}{2}$$

The first inequality holds since $\eta_{t+1} \leq \eta_t$, the second inequality holds since period $t$ is Type SI: $\eta_t - (\eta_{t-1} - (1 - L_{t-1}))(1 - x_1 \hat{p}^*_t) \leq \frac{1}{2}$, and the last two inequalities hold because period $t - 1$ is Type SI: $\eta_{t-1} - (1 - L_{t-1}) \leq \frac{1}{2}$. That is, period $t + 1$ is a Type SI period. (Note, as for Lemma 1, that Assumption 2 is not used in the proof of Lemma 2 which in fact holds whether Assumption 2 holds or not.)

**Proposition 7** For all $(C_o, C_e)$ such that

$$0 < C_e \leq \frac{(1 - \beta)(1 + x_2 \beta)(\bar{C} - C_o)}{1 - x_2 \beta^2},$$

the model has an equilibrium in two-period cycles where all periods are Type I, the optimal long-term contract offered is as described in Proposition 1(ii), and the values of the economy’s aggregates, $\{L(j), \alpha^y(j), \alpha^o(j), w^y(j), w^o(j), w^*_j, V(j); j = 1, 2\}$, where $j$ denotes the $j$th period in the cycle, are given as follows.

1. For the first period of the cycle,

$$\eta^{(1)} = 1 - \frac{1}{2} x_1 \hat{p}^*_1(2), \quad L^{(1)} = \frac{1}{2} + \frac{1}{2} x_1 \hat{p}^*_1(2),$$

$$\alpha^{(1)} = 1, \quad \alpha^o(1) = 0,$$

$$w^{(1)}_o = w, \quad w^{(1)}_y = (1 - \delta \hat{p}^*(1))w, \quad w^*_1 = 0.$$
\[ \tilde{V}(1) = C_e. \]

2. For the second period of the cycle,

\[ \eta^{(2)} = 1 - \frac{1}{2}x_1 \hat{p}^{*(1)}, \quad L^{(2)} = \frac{1}{2} + \frac{1}{2}x_1 \hat{p}^{*(1)}, \]

\[ \alpha_y^{(2)} = 1, \quad \alpha_o^{(2)} = 0, \]

\[ w_o^{(2)} = \bar{w}, \quad w_y^{(2)} = (1 - \delta \hat{p}_1^{*(2)}) \bar{w}, \quad w_0^{(2)} = 0, \]

\[ \bar{V}(2) = 0, \]

where

\[ \hat{p}^{*(1)} = \frac{(1 - \beta^2)(\bar{\theta} - \psi - \bar{w} - C_o - C_e)}{x_1 \beta[(1 - \beta)(\bar{\theta} - \psi - \bar{w}/x_1 - C_o) + \beta C_e]}, \]

\[ \hat{p}^{*(2)} = \frac{-(1 - \beta^2)(\bar{\theta} - \psi - \bar{w} - C_o)}{x_1 \beta[C_e - (1 - \beta)(\bar{\theta} - \psi - \bar{w}/x_1 - C_o)]}. \]

**Proof of Proposition 7.** We construct equilibria in which the measure of firms in the market rises in the first period of the cycle and falls in the second. Then the optimality of firm’s entry and exit decisions implies that its expected values measured at the beginning of each period are \( V^{(1)} = C_e \) and \( V^{(2)} = 0 \). By Proposition 1, we have

\[ V^{(1)} = (1 - \beta^2)(\bar{\theta} - \psi - \bar{w}) + \beta^2 V^{(1)} + x_1 \hat{p}_1^{*(1)} \beta \left[ V^{(2)} - (1 - \beta) \left( \bar{\theta} - \psi - \bar{w}/x_1 \right) - \beta V^{(1)} \right]. \]

Since \( V^{(1)} = C_e \) and \( V^{(2)} = 0 \), we have

\[ \hat{p}_1^{*(1)} = \frac{(1 - \beta^2)[C_e - (\bar{\theta} - \psi - \bar{w})]}{-x_1 \beta[(1 - \beta)(\bar{\theta} - \psi - \bar{w}/x_1) + \beta C_e]}. \]

Similarly,

\[ \hat{p}_1^{*(2)} = \frac{-(1 - \beta^2)(\bar{\theta} - \psi - \bar{w})}{x_1 \beta[C_e - (1 - \beta)(\bar{\theta} - \psi - \bar{w}/x_1)]}. \]
It is easy to see that $0 \leq \hat{p}_1^{*(2)} \leq 1$. Notice that $\hat{p}_1^{*(1)} \geq 0$ if and only if
\[
C_e < \frac{(1 - \beta)(\bar{\theta}' - \psi - \frac{w}{x_1})}{\beta},
\]
and $\hat{p}_1^{*(1)} \leq 1$ if and only if
\[
0 < C_e \leq \frac{(1 - \beta)(1 + x_2\beta)(\bar{C} - C_o)}{1 - x_2\beta^2}.
\]
Notice also that
\[
-\frac{(1 - \beta)(\bar{\theta}' - \psi - \frac{w}{x_1})}{\beta} > \frac{(1 - \beta)(1 + x_2\beta)(\bar{C} - C_o)}{1 - x_2\beta^2}
\]
holds if and only if
\[
-(1 - \beta)(1 - x_2\beta^2)(\bar{C} - C_o) + (1 - \beta)(1 - x_2\beta^2)\frac{x_2\bar{w}}{x_1} > \beta(1 - \beta)(1 + x_2\beta)(\bar{C} - C_o) + \beta(1 - \beta)x_2\bar{w},
\]
or
\[
C_o > \bar{C} - \frac{(1 - \beta)x_2\bar{w}}{x_1},
\]
which holds because $C_o \geq \bar{C}$. Hence we have
\[
C_e \leq \frac{(1 - \beta)(1 + x_2\beta)(\bar{C} - C_o)}{1 - x_2\beta^2}.
\]
Such a $C_e$ exists if and only if $C_o < \bar{C}$. Given $\hat{p}_1^{*(1)}$ and $\hat{p}_1^{*(2)}$, it is straightforward to compute the values for $\{L^{(j)}, \alpha_y^{(j)}, \alpha_o^{(j)}, w_y^{(j)}, w_o^{(j)}, w_s^{(j)}; j = 1, 2\}$ and to verify that they are just as given in the proposition. ■

**Lemma 8** Suppose $\{\eta_r\}_{r \geq 1}$ constitutes the path of the measure of firms in an equilibrium where Type SI and Type T periods alternate. Then

1. For any two periods $t + 1$ and $t + 2$, where period $t + 1$ is Type SI and period $t + 2$ is
Type $T$, the following three conditions hold:

(a) $\eta_{t+1} \geq \eta_{t+2}$ and

$$\frac{1 + \beta}{2(1 + x_1\beta)} < \eta_{t+1} < 1;$$

(b) If $\frac{1 + \beta}{2(1 + x_1\beta)} < \eta_{t+1} \leq \frac{2 + x_1\beta}{2(1 + x_1\beta)}$, then period $t+1$ is Type $S$ and

$$\eta_{t+2} > x_2\eta_{t+1} + \frac{x_1\beta + x_1}{2(x_1\beta + 1)};$$

(c) If $\frac{2 + x_1\beta}{2(1 + x_1\beta)} < \eta_{t+1} < 1$, then

$$\eta_{t+2} > 1 - \frac{1 - \eta_{t+1}}{\beta - 2\beta(1 - \eta_{t+1})}$$

when period $t+1$ is Type $S$, and

$$\eta_{t+2} = 1 - \frac{1 - \eta_{t+1}}{\beta - 2\beta(1 - \eta_{t+1})}$$

when period $t+1$ is Type $I$.

2. The equilibrium values of the aggregate variables $\{L_\tau, \alpha_{y,\tau}, \alpha_{o,\tau}, w_{y,\tau}, w_{o,\tau}, w_*, V_\tau\}_{\tau \geq 1}$ can be written as a function of $\{\eta_\tau\}_{\tau \geq 1}$. Specifically,

(a) If period $t+1$ is Type $S$, then

$$L_{t+1} = 1 - \frac{1}{2}(1 - x_1\hat{p}_{1t}^*), \quad L_{t+2} = 1 - x_2[\eta_{t+1} - \frac{1}{2}(1 - x_1\hat{p}_{1t}^*)],$$

$$\alpha_{y,t+1} = 2[\eta_{t+1} - \frac{1}{2}(1 - x_1\hat{p}_{1t}^*)], \quad \alpha_{o,t+1} = 0, \quad \alpha_{y,t+2} = 1, \quad \alpha_{o,t+2} = \frac{\beta(1 - x_1\hat{p}_{1t}^*) - \hat{p}_{1t}^*}{\beta(1 - x_1\hat{p}_{1t}^*)},$$

$$w_{o,t} = w_{o,t+1} = w, \quad w_{y,t} = (1 - \delta\hat{p}_{1t}^*)w, \quad w_{y,t+1} = w_{A,t+1},$$

$$w_{*t+1} = 0, \quad w_{*t+2} = \alpha_{o,t+2}w,$$

$$V_{t+1} = (1 - \beta^2)(\bar{\theta} - \psi - w) + \beta^2V_{t+3} + x_2\beta(1 - \beta)(w - w_{*t+2}),$$
\[ V_{t+2} = (1 - \beta)(\bar{\theta}' - \psi - w) + \beta V_{t+3}, \]

where \( \hat{p}_{t+1}^* \) is the smaller root of

\[
\frac{1}{2} x_1 x_2 (\hat{p}_{t+1}^*)^2 - \left[ \frac{1}{2} x_1 + x_2 (1 - \eta_{t+1}) + x_1 \beta (1 - \eta_{t+2}) \right] \hat{p}_{t+1}^* + (1 - \eta_{t+2}) \beta = 0. \tag{37}
\]

(b) If period \( t + 1 \) is Type I, then

\[
L_{t+1} = \frac{3}{2} - \eta_{t+1}, \quad L_{t+2} = \frac{1}{2} + \frac{(1 - \eta_{t+1})}{\beta (2 \eta_{t+1} - 1)(w - w_{st+2})},
\]

\[
\alpha_{y,t+1} = 1, \quad \alpha_{o,t+1} = 0, \quad \alpha_{y,t+2} = 1, \quad \alpha_{o,t+2} = \frac{w_{st+2}}{w},
\]

\[
w_{o,t} = w_{o,t+1} = w, \quad w_{y,t} = w - \frac{2(1 - \eta_{t+1})w}{x_1 (1 + \beta)}, \quad w_{y,t+1} = w - \frac{2(1 - \eta_{t+1})w}{x_1 (1 + \beta) (2 \eta_{t+1} - 1)},
\]

\[
w_{st+1} = 0, \quad 0 < w_{st+2} < w,
\]

\[
V_{t+1} = (1 - \beta^2)(\bar{\theta}' - \psi - w) + \beta^2 V_{t+3} + \frac{2x_2 (1 - \beta)(1 - \eta_{t+1}) w}{x_1 (2 \eta_{t+1} - 1)},
\]

\[
V_{t+2} = (1 - \beta)(\bar{\theta}' - \psi - w) + \beta V_{t+3}.
\]

In addition, suppose \( \{\eta_{t}\}_{t \geq 1} \) satisfies the three conditions in part 1 and the expected values \( \{V_{t}\}_{t \geq 1} \) from part 2 are such that \( 0 \leq V_t \leq C_e \) where \( V_t = 0 \) if \( \eta_t < \eta_{t-1} \) and \( V_t = C_e \) if \( \eta_t > \eta_{t-1} \). Then \( \{\eta_{t}\}_{t \geq 1} \) constitutes an equilibrium path of the measure of firms in the market.

**Proof of Lemma 8.** It follows from Proposition 1 that \( V_{t+1} \) is given by

\[
V_{t+1} = (1 - \beta^2)(\bar{\theta}' - \psi - w) + \beta^2 V_{t+3} + \hat{p}_{t+1}^* x_1 \beta \left[ V_{t+2} - (1 - \beta) \left( \bar{\theta}' - \psi - \frac{w - x_2 w_{st+2}}{x_1} \right) - \beta V_{t+3} \right]. \tag{38}
\]
Since period $t + 2$ is Type T, we have $V_{t+2} = (1 - \beta)(\bar{\theta}' - \psi - w) + \beta V_{t+3}$. Combining this and equation (38), we have

$$V_{t+1} - (1 - \beta)(\bar{\theta}' - \psi - w) - \beta V_{t+2} = \hat{p}_{1t+1}^* \beta (1 - \beta) x_2 (w - w_{st+2}). \tag{39}$$

Observe also that

$$V_t = (1 - \beta^2)(\bar{\theta}' - \phi - w) + \beta^2 V_{t+2} + \hat{p}_{1t}^* x_1 \beta \left[ V_{t+1} - (1 - \beta) \left( \bar{\theta}' - \psi - \frac{w - x_2 w_{st+1}}{x_1} \right) - \beta V_{t+2} \right]. \tag{40}$$

Since period $t$ is Type T, we have $V_t = (1 - \beta)(\theta' - \bar{\psi} - w) + \beta V_{t+1}$. Combining this and equation (40) gives us

$$\hat{p}_{1t}^* = \frac{\beta [V_{t+1} - (1 - \beta)(\bar{\theta}' - \psi - w) - \beta V_{t+2}]}{x_1 \beta [V_{t+1} - (1 - \beta)(\theta' - \bar{\psi} - w) - \beta V_{t+2} + x_2 \beta (1 - \beta)(w - w_{st+1})]}
= \frac{\hat{p}_{1t+1}^* \beta (w - w_{st+2})}{x_1 \hat{p}_{1t+1}^* \beta (w - w_{st+2}) + w}. \tag{41}$$

where the last equality follows from (39) and that $w_{st+1} = 0$ in a Type SI period.

Now equation (41) can be rewritten as

$$w_{st+2} = \frac{\beta \hat{p}_{1t+1}^* (1 - x_1 \hat{p}_{1t}^*) - \hat{p}_{1t}^*}{\beta \hat{p}_{1t+1}^* (1 - x_1 \hat{p}_{1t}^*)} \frac{w}{w}. \tag{42}$$

And we know that $w_{st+2}$ also satisfies:

$$\frac{w_{st+2}}{w} = \alpha_{o,t} = \frac{\eta_{t+2} - (1 - L_{t+2}) - \frac{1}{2}}{\frac{1}{2} - (1 - L_{t+2})}, \tag{43}$$

where

$$L_{t+2} = 1 - [\eta_{t+1} - (1 - L_{t+1})](1 - x_1 \hat{p}_{1t+1}^*). \tag{44}$$
Substituting (42) into (43) yields

$$\frac{\beta \hat{p}_{tt+1}^r(1 - x_1 \hat{p}_t^r) - \hat{p}_t^r}{\beta \hat{p}_{tt+1}^r(1 - x_1 \hat{p}_t^r)} = \frac{\eta_{t+2} - \frac{1}{2} - \frac{1}{2} \left[ \eta_{t+1} - (1 - L_{t+1}) \right]}{\eta_{t+1} - (1 - L_{t+1})}.$$  \hspace{1cm} (45)

Last, as we argued in the proof of Lemma 2, since period \( t + 2 \) is Type T, it must hold that \( \eta_{t+2} \leq \eta_{t+1} \).

There are two cases to be discussed: (i) period \( t + 1 \) is Type I, and (ii) period \( t + 1 \) is Type S.

(i) Suppose \( t + 1 \) is Type I, or \( \eta_{t+1} - (1 - L_{t+1}) = 1/2 \). Since \( L_{t+1} = 1 - (1 - x_1 \hat{p}_t^r)/2 \), we can write \( \hat{p}_t^r \) as a function of \( \eta_{t+1} \):

$$\hat{p}_t^r = \frac{2(1 - \eta_{t+1})}{x_1} > 0,$$  \hspace{1cm} (46)

where the inequality holds since \( 0 < \eta_{t+1} < 1 \). Hence, by equation (42), \( w_{st+2} < w \).

Substituting (46) and \( \eta_{t+1} - (1 - L_{t+1}) = 1/2 \) into (45) yields

$$\eta_{t+2} = 1 - \frac{1 - \eta_{t+1}}{\beta - 2\beta(1 - \eta_{t+1})},$$  \hspace{1cm} (47)

which implies that \( \eta_{t+2} < \eta_{t+1} \). For \( t + 2 \) to be a Type T period, it must be that \( w_{st+2} > 0 \).

It follows from (42) that \( w_{st+2} > 0 \) if and only if

$$\hat{p}_{1t+1}^r > \frac{\hat{p}_t^r}{\beta(1 - x_1 \hat{p}_t^r)} = \frac{2(1 - \eta_{t+1})}{\beta x_1(2\eta_{t+1} - 1)}.$$  

This implies

$$\frac{2(1 - \eta_{t+1})}{\beta x_1(2\eta_{t+1} - 1)} < 1,$$

or

$$\eta_{t+1} > \frac{2 + \beta x_1}{2(1 + \beta x_1)}. \hspace{1cm} (48)$$
Moreover, the equilibrium values of the variables \( \{ L_\tau, \alpha_{y,\tau}, \alpha_{o,\tau}, w_{y,\tau}, w_{o,\tau}, w^*_\tau, V_\tau \}_{\tau \geq 1} \) can be written as a function of \( \{ \eta_\tau \}_{\tau \geq 1} \). To see this, \( \eta_{t+1} - (1 - L_{t+1}) = 1/2 \) implies

\[
L_{t+1} = \frac{3}{2} - \eta_{t+1}.
\]

By (46), the measure of unemployed workers in period \( t + 2 \) is

\[
L_{t+2} = 1 - x_2[\eta_{t+1} - \frac{1}{2}(1 - x_1 \hat{p}^*_t)] = \frac{1}{2} + \frac{(1 - \eta_{t+1})}{\beta(2\eta_{t+1} - 1)(w - w_{st+2})}.
\]

It follows from (46) and Proposition 1 that

\[
w_{y,t} = \frac{w - \frac{2\beta(1 - \eta_{t+1})w}{x_1(1 + \beta)}}{x_1(1 + \beta)}.
\]

It follows from (42), (46) and Proposition 1 that

\[
w_{y,t+1} = w - \frac{2(1 - \eta_{t+1})}{x_1(1 + \beta)(2\eta_{t+1} - 1)}.
\]

It is then immediate that \( \alpha_{y,t+1} = \alpha_{y,t+2} = 1, \alpha_{o,t+1} = 0, \alpha_{o,t+2} = w_{st+2}/w, w_{st+1} = 0 \) and \( 0 < w_{st+2} < w \).

Notice that both \( w_{st+2} \) and \( \hat{p}^*_{1,t+1} \) are indeterminate. To see this, note that, when \( \hat{p}^*_{1,t} \) is given by (46), for any \( \hat{p}^*_{1,t+1} \) such that \( w_{st+2} \) given by (42) lies in \( (0, w) \), equation (45) holds. Finally, \( V_{t+2} = (1 - \beta)(\tilde{\theta}' - \psi - w) + \beta V_{t+3} \). Substituting this, (42) and (46) into (38) gives us

\[
V_{t+1} = (1 - \beta^2)(\tilde{\theta}' - \psi - w) + \beta V_{t+3} + \frac{2x_2(1 - \beta)(1 - \eta_{t+1})w}{x_1(2\eta_{t+1} - 1)}.
\]

(ii) Suppose \( t + 1 \) is Type S. Then \( \hat{p}^*_{1,t+1} = 1 \) and (45) can be rewritten as

\[
\frac{1}{2}x_1x_2(\hat{p}^*_{1,t})^2 - \frac{1}{2}[x_1 + x_2(1 - \eta_{t+1}) + x_1\beta(1 - \eta_{t+2})]\hat{p}^*_{1,t} + (1 - \eta_{t+2})\beta = 0.
\]

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For $t + 2$ to be a Type T period, it must be that $w_{s,t+2} > 0$, which holds, by (42), if and only if

$$\hat{p}_{t+1}^* > \frac{\hat{p}_{lt}^*}{\beta(1-x_1\hat{p}_{lt}^*)}.$$ 

Since $\hat{p}_{lt+1}^* = 1$, the above inequality can be rewritten as

$$\hat{p}_{lt}^* < \frac{\beta}{x_1\beta + 1}.$$ 

Furthermore, by (42), $w_{s,t+2} < w$ only if $\hat{p}_{lt}^* > 0$. Finally, since period $t+1$ is Type S, we have $\eta_{t+1} - (1 - L_{t+1}) < \frac{1}{2}$ which, given $L_{t+1} = 1 - \frac{1}{2}(1 - x_1\hat{p}_{lt}^*)$, is equivalent to

$$\hat{p}_{lt}^* < \frac{2(1 - \eta_{t+1})}{x_1}.$$  

(50)

Denote the left-hand side of (49) by $g(p_1)$, which is a quadratic function. Then $\eta_{t+1}$ and $\eta_{t+2}$ are consistent with an equilibrium only if there is $p_1 \in P \triangleq (0, \min\{\beta/(1 + x_1\beta), 2(1 - \eta_{t+1})/x_1\})$ such that $g(p_1) = 0$. Note that $g(0) > 0$.

(a) Suppose $\eta_{t+1} > (2 + x_1\beta)/2(1 + x_1\beta)$. Then $P = (0, 2(1 - \eta_{t+1})/x_1)$. It is easy to verify that the axis of symmetry of $g(\cdot)$ is larger than $2(1 - \eta_{t+1})/x_1$ and therefore the relevant root of $g(p_1) = 0$ is the smaller one, which lies in $P$ if and only if

$$g\left(\frac{2(1 - \eta_{t+1})}{x_1}\right) = \beta(1 - \eta_{t+2})(2\eta_{t+1} - 1) - (1 - \eta_{t+1}) < 0.$$ 

This is equivalent to

$$\eta_{t+2} > 1 - \frac{1 - \eta_{t+1}}{\beta - 2\beta(1 - \eta_{t+1})}.$$ 

There exists $\eta_{t+2} \leq \eta_{t+1}$ such that the above inequality holds for all $\eta_{t+1} < 1$. 

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(b) Suppose \( \eta_{t+1} \leq (2 + x_1 \beta)/2(1 + x_1 \beta) \). Then \( P = (0, \beta/(1 + x_1 \beta)) \). It is easy to verify that the axis of symmetry of \( g(\cdot) \) is larger than \( \beta/(1 + x_1 \beta) \) and therefore the relevant root of \( g(p_1) = 0 \) is the smaller one, which lies in \( P \) if and only if

\[
\eta_{t+1} + \frac{1}{2}(1 - \eta_{t+1}) - x_2(1 - \eta_{t+1}) < 0.
\]

This is equivalent to

\[
\eta_{t+2} > x_2 \eta_{t+1} + \frac{x_1 \beta + x_1}{2(x_1 \beta + 1)}.
\]

There exists \( \eta_{t+2} \leq \eta_{t+1} \) such that the above inequality holds if and only if \( \eta_{t+1} > 1 + \beta/2(1 + x_1 \beta) \).

Similarly, the equilibrium values of the variables \( \{L_\tau, \alpha_{y,\tau}, \alpha_{o,\tau}, w_{y,\tau}, w_{o,\tau}, w^{*}_{y,\tau}, V_\tau\}_{\tau \geq 1} \) can be written as a function of \( \{\eta_\tau\}_{\tau \geq 1} \).

Last, it is straightforward to verify the following. Suppose \( \{\eta_\tau\}_{\tau \geq 1} \) satisfies all the conditions in part 1 of the lemma and the expected values \( \{V_\tau\}_{\tau \geq 1} \) given in part 2 of the lemma are such that \( 0 \leq V_t \leq C_e \) where \( V_t = 0 \) if \( \eta_t < \eta_{t-1} \) and \( V_t = C_e \) if \( \eta_t > \eta_{t-1} \). Then \( \{\eta_\tau\}_{\tau \geq 1} \) is an equilibrium path of the measure of the firms in the market. And this proves the lemma.

\[\square\]

**Proposition 8** Suppose

\[
C < C_o < C + \frac{x_2 \beta^2 w}{1 + \beta} \quad \text{and} \quad C_e \geq \frac{(1 - \beta)(C_o - C)}{\beta}.
\]  

There exists an interval \([\eta_c^-, \eta_c^+]\) such that for any \( \eta \) in this interval, the model has an equilibrium where \( \eta_t = \eta \) for all \( t \geq 1 \), Type S and Type T periods alternate, the optimal long-term contract is described by Proposition 1, and the equilibrium values of \( \{L^{(j)}, \alpha_{y,\tau}^{(j)}, \alpha_{o,\tau}^{(j)}, w_{y,\tau}^{(j)}, w_{o,\tau}^{(j)}, w^{*}_{y,\tau}^{(j)}, V^{(j)}; j = 1, 2\} \) are given by

1. In the first period (Type S) of a cycle,

\[
L^{(1)} = \frac{1}{2} + \frac{1}{2} x_1 \beta^{*}(2).
\]
\[\alpha_y^{(1)} = 2\eta - 1 + x_1\hat{p}_1^{* (2)}, \quad \alpha_o^{(1)} = 0,\]
\[w_o^{(1)} = w, \quad w_y^{(1)} = w_A = (1 - \delta)w + \delta w^*_1, \quad w_s^{(1)} = 0,\]
\[\bar{V}^{(1)} = \bar{\theta} - \psi - w + \frac{x_2\beta}{1 + \beta}(w - w_s^{(2)}) - C_o,\]

2. In the second period (Type T) of a cycle,
\[L^{(2)} = 1 - x_2(\eta - \frac{1}{2} + \frac{1}{2}x_1\hat{p}_1^{* (2)}),\]
\[\alpha_y^{(2)} = 1, \quad \alpha_o^{(2)} = \frac{w_s^{(2)}}{w},\]
\[w_o^{(2)} = w, \quad w_y^{(2)} = (1 - \delta\hat{p}_1^{* (2)})w, \quad w_s^{(2)} = w - \frac{\hat{p}_1^{* (2)}}{\beta(1 - x_1\hat{p}_1^{* (2)})}w,\]
\[\bar{V}^{(2)} = \bar{\theta} - \psi - w + \frac{x_2\beta^2}{1 + \beta}(w - w_s^{(2)}) - C_o.\]

Here \(\hat{p}_1^{* (2)}\) is the smaller root of
\[
\frac{1}{2}x_1x_2\hat{p}_1^{2} - \left[\frac{1}{2}x_1 + (x_2 + x_1\beta)(1 - \eta)\right]\hat{p}_1 + (1 - \eta)\beta = 0. \tag{52}
\]

Proof of Proposition 8. Proposition 8 is a direct corollary of Lemma 8. Let \(\eta_t = \eta\) for all \(t \geq 1\). Suppose period \(t + 1\) is Type I. Then, by Lemma 8, we have
\[\eta_{t+2} = 1 - \frac{1 - \eta_{t+1}}{\beta - 2\beta(1 - \eta_{t+1})} < \eta_{t+1},\]
a contradiction to \(\eta_t\) being constant in time. Hence in equilibrium it must be the case that Type S and Type T periods alternate.

We now construct the model’s equilibria in two-period cycles where the first period is Type S and the second Type T. Let \(Z^{(j)}\) denote the value of \(Z\) in the \(j\)th period of a cycle \((j = 1, 2)\). By Lemma 8, we could solve for the equilibrium values of \(\{L^{(j)}, \alpha_y^{(j)}, \alpha_o^{(j)}, w_y^{(j)}, w_o^{(j)}, w_s^{(j)}, V^{(j)}; j =\)

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\[ 1, 2 \} \text{ as a function of } \eta, \text{ where} \]
\[ \frac{1 + \beta}{2(1 + x_1\beta)} < \eta < 1. \]

In particular, we have
\[ \bar{V}^{(1)} = \bar{\theta}' - \psi - w + \frac{x_2\beta}{1 + \beta}(w - w^{(2)}_*), \]
\[ \bar{V}^{(2)} = \bar{\theta}' - \psi - w + \frac{x_2\beta^2}{1 + \beta}(w - w^{(2)}_*), \]

where \( w^{(2)}_* = w - \frac{\bar{p}_1^{* (2)}}{\beta(1-x_1\bar{p}_1^{* (2)})} w \) and \( \bar{p}_1^{* (2)} \) is the smaller root of
\[ \frac{1}{2} x_1 x_2 p_1^2 - \left[ \frac{1}{2} x_1 + (x_2 + x_1\beta)(1 - \eta) \right] p_1 + (1 - \eta)\beta = 0, \quad (52) \]

which can be rewritten as
\[ \frac{1}{2} x_1 x_2 p_1^2 - \frac{1}{2} x_1 p_1 = (1 - \eta)[(x_2 + x_1\beta)p_1 - \beta]. \]

Observe that as \( \eta \) increases the left-hand side of the above equation rotates clockwise around the point \((\beta/(x_2 + x_1\beta), 0)\). Hence \( \bar{p}_1^{* (2)} \) is decreasing in \( \eta \). This implies that \( w^{(2)}_* \) is increasing in \( \eta \) and therefore both \( V^{(1)} \) and \( \bar{V}^{(2)} \) are decreasing in \( \eta \). Next, the free-entry-and exit condition requires
\[ 0 \leq \bar{V}^{(2)} \leq \bar{V}^{(1)} \leq C_e. \]

When \( \eta \) goes to 1, both \( \bar{V}^{(1)} \) and \( \bar{V}^{(2)} \) go to \( C - C_o < 0 \). When \( \eta \) goes to \( (1 + \beta)/(2(1 + x_1\beta)) \), \( w^{(2)}_* \) goes to 0, and therefore \( \bar{V}^{(1)} \) goes to \( \bar{C} - C_o \geq 0 \) and \( \bar{V}^{(2)} \) goes to \( \bar{C} - C_o + x_2\beta^2 w/(1 + \beta) \).

Hence if and only if \( C_o \leq \bar{C} + x_2\beta^2 w/(1 + \beta) \), there exists \( \eta^{+}_e(C_o) \in [(1 + \beta)/(2(1 + x_1\beta)), 1) \) such that \( 0 \leq \bar{V}^{(2)} \leq \bar{V}^{(1)} \) when \( \eta \leq \eta^{+}_e(C_o) \), where
\[ \eta^{+}_e(C_o) = 1 - \frac{\frac{1}{2} x_1 (1 + \beta)(C_o - \bar{C})[(x_2 - x_1)(1 + \beta)(C_o - \bar{C}) - w x_2\beta]}{[x_1(1 + \beta)(C_o - \bar{C}) + x_2\beta w][x_2(1 + \beta)(C_o - \bar{C}) - x_2\beta^2 w]}. \]
Now consider two cases: (i) If \( C_e \geq \bar{C} - C_o \), then \( V^{(2)} \leq V^{(1)} \leq C_e \) for all \( \eta \in [(1 + \beta)/(2(1 + x_1\beta)), 1) \). (ii) If \( 0 < C_e < \bar{C} - C_o \) then there exists \( \eta_e(C_o, C_e) \in [(1 + \beta)/(2(1 + x_1\beta)), 1) \) such that \( V^{(2)} \leq V^{(1)} \leq C_e \) when \( \eta \geq \eta_e(C_o, C_e) \), where

\[
\eta_e(C_o, C_e) = 1 - \frac{1}{2}x_1(1 + \beta)(C_e + C_o - C)(x_2 - x_1)(1 + \beta)(C_e + C_o - C) - w x_2}{x_1(1 + \beta)(C_e + C_o - C) + x_2 w (x_2(1 + \beta)(C_e + C_o - C) - x_2 \beta w)}.
\]

In this case \( \eta_e(C_o, C_e) \leq \eta_e^+(C_o) \) if and only if \( C_e \geq -(1 - \beta)(C - C_o)/\beta \).

Define \( \eta_e^-(C_o, C_e) \equiv (1 + \beta)/(2(1 + x_1\beta)) \) when \( C_e \geq \bar{C} - C_o \). This completes the proof.

\[\blacksquare\]

### B.6 Longer Cycles

Let an individual cycle be \( 2n \) periods long, where \( n \) is any positive integer. Let \( Z^{(j)} \) denote the value of a variable \( Z \) in the \( j \)th period of the cycle, \( j = 1, \ldots, 2n \). Let \( j = 0 \) denote the period preceding the cycle and \( j = 2n + 1 \) the period following the cycle. The equilibria we seek to construct have \( \eta^{(1)} > \eta^{(2)} = \cdots = \eta^{(2n)} \), as described in Proposition 9, which we now state and prove.

**Proposition 9** Suppose

\[
C < C_o < \bar{C} + \frac{x_2 \beta^2 w}{1 + \beta},
\]

and

\[
- \frac{1 - \beta}{\beta} (C - C_o) \leq C_e < (1 - \beta)(\bar{C} - C_o) + \min \left\{ 1, \frac{2(1 - \eta^{(2)})}{x_1} \right\} x_2 \beta(1 - \beta)w,
\]

where

\[
\eta^{(2)}(C_0) = 1 - \frac{1}{2}x_1 x_2 p_1^2 - \frac{1}{2}x_1 p_1, \quad \text{and} \quad p_1 = \frac{\beta(1 + \beta)(\bar{\theta} - \psi - w - C_o)}{x_1(1 + \beta)(\bar{\theta} - \psi - w - C_o) + w x_2 \beta^2}.
\]

Then the model has an equilibrium that consists of an infinite sequence of individual cy-
cles each of which lasting for an even number of periods. In any individual cycle with length $2n$, $n$ being a positive integer, $\eta^{(1)} > \eta^{(2)} = \cdots = \eta^{(2n)}$, Type S and Type T periods alternate, the optimal long-term contract is described by Proposition 1, and the values of \(\{L^{(j)}, \alpha_y^{(j)}, \alpha_o^{(j)}, w_o^{(j)}, w_y^{(j)}, w_o^*, w_y^*, V^{(j)}; j = 1, \cdots, 2n\}\) are given as follow.

1. In any odd period (a Type S period),

\[
L^{(j)} = \frac{1}{2} + \frac{1}{2}x_1\hat{p}_1^{*(j-1)},
\]

\[
\alpha_y^{(j)} = 2\eta^{(j)} - 1 + x_1\hat{p}_1^{*(j-1)}, \quad \alpha_o^{(j)} = 0,
\]

\[
w_o^{(j)} = w, \quad w_y^{(j)} = (1 - \delta)w + \delta w_s^{(j+1)}, \quad w_s^{(j)} = 0,
\]

\[
V^{(1)} = C^e, \quad V^{(j)} = (1 - \beta^2)(\bar{\theta} - \psi - w - C_o) + (1 - \beta)x_2\beta(w - w_s^{(j+1)}) + \beta^2V^{(j+2)}, \quad j \geq 3.
\]

2. In any even period (a Type T period),

\[
L^{(j)} = 1 - x_2(\eta^{(j-1)} - \frac{1}{2} + \frac{1}{2}x_1\hat{p}_1^{*(j-2)}),
\]

\[
\alpha_y^{(j)} = 1, \quad \alpha_o^{(j)} = \frac{w_s^{(j)}}{\hat{w}},
\]

\[
w_o^{(j)} = w, \quad w_y^{(j)} = (1 - \delta\hat{p}_1^{*(j)})w, \quad w_s^{(j)} = w - \frac{\hat{p}_1^{*(j-2)}}{\beta(1 - x_1\hat{p}_1^{*(j-2)})}w,
\]

\[
V^{(2)} = 0, \quad V^{(j)} = (1 - \beta)(\bar{\theta} - \psi - w - C_o) + \beta V^{(j+1)}, \quad j \geq 4,
\]

where \(\hat{p}_1^{*(0)}\) is the smaller root of

\[
\frac{1}{2}x_1x_2p^2 - \left[\frac{1}{2}x_1 + x_2(1 - \eta^{(1)}) + x_1\beta(1 - \eta^{(2)})\right]p + (1 - \eta^{(2)})\beta = 0, \quad (55)
\]

and \(\hat{p}_1^{*(j)} \quad (j = 2, 4, \cdots, 2n - 2)\) is the smaller root of

\[
\frac{1}{2}x_1x_2p^2 - \left[\frac{1}{2}x_1 + x_2(1 - \eta^{(2)}) + x_1\beta(1 - \eta^{(2)})\right]p + (1 - \eta^{(2)})\beta = 0. \quad (56)
\]
**Proof.** After some algebra, one can verify that \( \{ \eta^{(j)} \}_{j=1}^{2n} \) satisfies the conditions given in part 1 of Lemma 8 if and only if

\[
\eta^{(1)} > \eta^{(2)} > \frac{1 + \beta}{2(1 + x_1\beta)}.
\]  

(57)

and

\[
\begin{align*}
\eta^{(1)} &< \frac{\eta^{(2)}}{x_2} - \frac{x_1 + x_1\beta}{2x_2(1 + x_1\beta)} & \text{if } \frac{1 + \beta}{2(1 + x_1\beta)} < \eta^{(2)} \leq \frac{1 + x_2}{2}, \\
\eta^{(1)} &< 1 - \frac{\beta(1 - \eta^{(2)})}{1 + 2\beta(1 - \eta^{(2)})} & \text{if } \eta^{(2)} > \frac{1 + x_2}{2}.
\end{align*}
\]  

(58)

It follows from part 2 of Lemma 8 that the equilibrium values of \( \bar{V}^{(j)} \) can be written as a function of \( \{ \eta_t \}_{t \geq 1} \):

\[
\begin{align*}
V^{(1)} &= (1 - \beta^2)(\bar{\theta}' - \psi - w) + \beta^2V^{(3)} + (1 - \beta)x_2\beta(w - w^{(2)}), \\
V^{(2j)} &= (1 - \beta)(\bar{\theta}' - \psi - w) + \beta V^{(2j+1)}, \quad j = 1, \ldots, n, \\
V^{(2j+1)} &= (1 - \beta^{2n-2j})(\bar{\theta}' - \psi - w) + \beta^{2n-2j}V^{(2n+1)} \\
&\quad + \frac{x_2\beta}{1 + \beta}(1 - \beta^{2n-2j})(w - w^{(4)}), \quad j = 1, \ldots, n - 1,
\end{align*}
\]  

(59)\hspace{1cm}(60)\hspace{1cm}(61)

where

\[
\begin{align*}
w^{(2)} &= w - \hat{p}_1^{*(0)}w/[(1 - x_1\hat{p}_1^{*(0)})], \\
w^{(4)} &= w - \hat{p}_1^{*(2)}w/[(1 - x_1\hat{p}_1^{*(2)})],
\end{align*}
\]  

(62)\hspace{1cm}(63)

where \( \hat{p}_1^{*(0)} \) is the smaller root of (37) with \( \eta_{t+1} = \eta^{(1)} \) and \( \eta_{t+2} = \eta^{(2)} \), and \( \hat{p}_1^{*(2)} \) is the smaller root of (37) with \( \eta_{t+1} = \eta_{t+2} = \eta^{(2)} \). That is, \( \hat{p}_1^{*(0)} \) and \( \hat{p}_1^{*(2)} \) are the smaller roots of the following equations (55) and (56), respectively. Now the optimality of the firm’s entry and exit decisions implies

\[ V^{(1)} = V^{(2n+1)} = C_e, \quad V^{(2)} = 0. \]
Substituting these into (59) and (60) gives us

\[
\bar{\theta}' - \psi - w + \frac{x_2 \beta (1 - \beta)}{1 - \beta^{2n}} (w - w_*^{(2)}) + \frac{x_2 \beta^3 (1 - \beta^{2n-2})}{(1 + \beta)(1 - \beta^{2n})} (w - w_*^{(4)}) = C_e, \tag{64}
\]

\[
\bar{\theta}' - \psi - w + \frac{x_2 \beta^{2n} (1 - \beta)}{1 - \beta^{2n}} (w - w_*^{(2)}) + \frac{x_2 \beta^2 (1 - \beta^{2n-2})}{(1 + \beta)(1 - \beta^{2n})} (w - w_*^{(4)}) = 0, \tag{65}
\]

both equations are a function of \(\{n, \eta^{(1)}, \eta^{(2)}\}\), following from (62)-(56).

In the remainder of this appendix, we show in B.6.1 and B.6.2 that for a positive measure set of the pairs \((C_o, C_e)\) (namely the pairs \((C_o, C_e)\) that satisfy (53) and (54)), there do exist \(n \geq 1, \eta^{(1)}\) and \(\eta^{(2)}\) that solve (64) and (65) subject to (57) and (58). Note that with three variables and two equality constraints, there are likely multiple solutions for \((n, \eta^{(1)}, \eta^{(2)})\) to equations (64) and (65). This explains why our model permits equilibria where individual cycles share the same cyclicity (a boom followed by a number of recession periods), but not the same length, that is, the \(n\) does not need to be constant across all individual cycles. Moreover, when a solution exists, we have \(0 < w_*^{(2)} < w_*^{(4)} < w\) since \(\eta^{(1)} > \eta^{(2)}\), and therefore

\[
0 = V^{(2)} < V^{(3)} < V^{(5)} < \cdots < V^{(2n-1)} < V^{(1)} = C_e,
\]

\[
0 = V^{(2)} < V^{(4)} < \cdots < V^{(2n)} < V^{(2n+1)} = C_e. \tag{66}
\]

That is, the firm’s entry and exit decisions are optimal.

To summarize, any \(\{n \geq 1, \eta^{(1)}, \eta^{(2)}\}\) that solve equations (64) and (65) subject to (57) and (58) would describe a complete cycle in an equilibrium that has the characteristics we imposed in Proposition 9. If in addition the \(\eta^{(1)}\) in one cycle is greater than or equal to the \(\eta^{(2)}\) in the preceding cycle, then the sequence \(\{\eta_t\}_{t \geq 1}\) would constitute an equilibrium path of firm measures by Lemma 8. In particular, this is the case when \(n\) is constant for all individual cycles.

\[\text{When } n = 1, \text{ it is obvious that } 0 = V^{(2)} < V^{(1)} = C_e.\]
B.6.1 Two-Period Cycles

Lemma 9 Suppose \((C_o, C_e)\) satisfies (53) and (54) where the first inequality in (54) holds with equality. Then there exist \(\eta^{(1)}\) and \(\eta^{(2)}\) that solve (64) and (65) subject to (57) and (58) at \(n = 1\).

Proof. When \(n = 1\), (64) and (65) read

\[
V^{(1)} = \bar{\theta}' - \psi - w + \frac{x_2 \beta}{1 + \beta} (w - w^{(2)}_*) = C_e, \tag{66}
\]

\[
V^{(2)} = \bar{\theta}' - \psi - w + \frac{x_2 \beta^2}{1 + \beta} (w - w^{(2)}_*) = 0. \tag{67}
\]

It is easy to see that if there exist \(\eta^{(1)}\) and \(\eta^{(2)}\) that solve (66) and (67) subject to (57) and (58), then \((C_o, C_e)\) satisfies the following conditions:

\[
C < C_o < C + \frac{x_2 \beta^2 w}{1 + \beta}, \tag{68}
\]

\[
C_e = -\frac{1 - \beta}{\beta} (C - C_o). \tag{69}
\]

Next we show that for any pair of \((C_o, C_e)\) satisfying conditions (68) and (69) there exist \(\eta^{(1)}\) and \(\eta^{(2)}\) that solve (66) and (67) subject to (57) and (58). Note that if (69) holds, (66) is equivalent to (67). Hence it suffices to show that for any \(C_o\) satisfying condition (68) there exists \(\eta^{(1)}\) and \(\eta^{(2)}\) that solve (67) subject to (57) and (58). This is equivalent to showing that for all \(p_1 \in (0, \beta/(1 + x_1 \beta))\) there exist \(\eta^{(1)}\) and \(\eta^{(2)}\) such that \(p_1\) is the smaller root of (55).

Fix \(p_1 \in (0, \beta/(1 + x_1 \beta))\). For any \(\eta^{(2)} \in ((1 + \beta)/(2(1 + x_1 \beta)), 1)\), let

\[
\eta^{(1)} = 1 - \frac{\frac{1}{2} x_1 x_2 p_1^2 - [\frac{1}{2} x_1 + x_1 \beta (1 - \eta^{(2)})] p_1 + (1 - \eta^{(2)}) \beta}{x_2 p_1} \in (0, 1).
\]
Then \( p_1 \) is the smaller root of (55). We want to show that there exists \( \eta^{(2)} \) such that \((\eta^{(1)}, \eta^{(2)})\) satisfies conditions (57) and (58). Note first that \( \eta^{(1)} > \eta^{(2)} \) if and only if

\[
\eta^{(2)} > \eta^{(2)} \equiv 1 - \frac{1}{2} x_1 x_2 p_1^2 - \frac{1}{2} x_1 p_1}{(x_2 + x_1 \beta)p_1 - \beta},
\]

where \( \eta^{(2)} \in ((1 + \beta)/(2(1 + x_1 \beta)), 1) \) for \( p_1 \in (0, \beta/(1 + x_1 \beta)) \) as we will show later. Note also that when \( \eta^{(1)} = \eta^{(2)} \), i.e., \( \eta^{(2)} = \eta^{(2)} \), condition (58) holds. By continuity, there exists \( \eta^{(2)} \in (\eta^{(2)}, 1) \) such that \((\eta^{(1)}, \eta^{(2)})\) satisfies conditions (57) and (58).

We complete the proof by showing \( \eta^{(2)} > (1 + \beta)/(2(1 + x_1 \beta)) \). This inequality, after some tedious algebra, can be rewritten as

\[
x_1 x_2 p_1^2 - \left[ (1 + x_1 \beta) - \frac{x_2 \beta (x_1 \beta + x_2)}{1 + x_1 \beta} \right] p_1 + \beta - \frac{x_2 \beta^2}{1 + x_1 \beta} > 0,
\]

where the left-hand side is a a quadratic function of \( p_1 \), denoted by \( g(p_1) \). We claim that \( g(\cdot) \) is strictly decreasing in \((0, \beta/(1 + x_1 \beta))\). Since \( x_1 x_2 > 0 \), it suffices to show that the symmetric axis of \( g(\cdot) \) is greater than \( \beta/(1 + x_1 \beta) \), i.e.,

\[
\frac{(1 + x_1 \beta) - \frac{x_2 \beta (x_1 \beta + x_2)}{1 + x_1 \beta}}{2 x_1 x_2} > \frac{\beta}{1 + x_1 \beta},
\]

or equivalently

\[
x_1 (2x_1 - 1) \beta^2 + (x_1^2 + 2x_2 - 1) \beta + 1 > 0.
\]

Clearly, the above inequality holds when \( x_1 \geq 1/2 \). When \( 0 < x_1 < 1/2 \), the left-hand side of the above inequality is a concave function of \( \beta \), denoted by \( h(\beta) \). Since \( h(0) > 0 \) and \( h(1) > 0 \), \( h(\beta) > 0 \) for all \( \beta \in (0,1) \). Finally, since \( g(\cdot) \) is strictly decreasing in \((0, \beta/(1 + x_1 \beta))\) and \( g(\beta/(1 + x_1 \beta)) = 0 \), we have \( g(p_1) > 0 \) for all \( p_1 \in (0, \beta/(1 + x_1 \beta)) \).
B.6.2 Longer Cycles

Lemma 10 Suppose \((C_0, C_e)\) satisfies (53) and (54) where the first inequality in (54) holds strictly. Then there exist \(n > 1, \eta^{(1)}\) and \(\eta^{(2)}\) that solve (64) and (65) subject to (57) and (58).

Proof. Let \(p^{(2j)}_1\) denote \(p^{*(2j)}_1\) for simplicity, \(j = 0, 1\). For a given pair of \((w_2^*(2), w_4^*)\), we can solve \(\eta^{(1)}\) and \(\eta^{(2)}\) from (64) and (65):

\[
\eta^{(2)}(w_4^*) = 1 - \frac{\frac{1}{2}x_1x_2(p_1^{(2)})^2 - \frac{1}{2}x_1p_1^{(2)}}{(x_2 + x_1\beta)p_1^{(2)} - \beta},
\]

(70)

and

\[
\eta^{(1)} = 1 - \frac{\frac{1}{2}x_1x_2(p_1^{(0)})^2 - \left[\frac{1}{2}x_1 + x_1\beta(1 - \eta^{(2)})\right]p_1^{(0)} + (1 - \eta^{(2)})\beta}{x_2p_1^{(0)}},
\]

(71)

where \(p_1^{(2j-2)} = \beta(w - w_4^{*(2j)})/[x_1\beta(w - w_4^{*(2j)}) + w]\), \(j = 1, 2\). It is easy to see that \(\eta^{(2)}\) is decreasing in \(p_1^{(2)}\). Since \(p_1^{(2)}\) is decreasing in \(w_4^*\), \((w_2^*, w_4^*)\) defined in (70) is increasing in \(w_4^*\).

We now proceed to prove that for any given \((C_0, C_e)\) satisfying conditions (53) and (54) where the first inequality in (54) holds strictly, there exist \(n, w_2^*(2)\) and \(w_4^*\), with \(n > 1\) and \(w_2^*(2) < w_4^*\), that solve equations (64) and (65), and the corresponding \(\eta^{(1)}\) and \(\eta^{(2)}\) given by (70) and (71) satisfy conditions (57) and (58):

\[
\eta^{(1)} > \eta^{(2)} > \frac{1 + \beta}{2(1 + x_1\beta)},
\]

(57)

and

\[
\begin{cases}
\eta^{(1)} < \frac{\eta^{(2)}}{x_2} - \frac{1 + 2x_1\beta}{2(1 + x_1\beta)} & \text{if } \frac{1 + \beta}{2(1 + x_1\beta)} < \eta^{(2)} \leq \frac{1 + x_2}{2}, \\
\eta^{(1)} < 1 - \frac{\beta(1 - \eta^{(2)})}{1 + 2\beta(1 - \eta^{(2)})} & \text{if } \eta^{(2)} > \frac{1 + x_2}{2}.
\end{cases}
\]

(58)
Now denote the left-hand side of equations (65) by $V^{(2)}(n, \eta^{(1)}, \eta^{(2)})$. Since $w^{(2)}_* < w^{(4)}_*$, we have

\[ \bar{\theta}' - \psi - w + \frac{x_2 \beta^2}{1 + \beta} (w - w^{(4)}_*) < V^{(2)}(n, \eta^{(1)}, \eta^{(2)}) < \bar{\theta}' - \psi - w + \frac{x_2 \beta^2}{1 + \beta} (w - w^{(2)}_*). \]

Let $w^{(4)}_*(C_o) \equiv w + (1 + \beta)(\bar{\theta}' - \psi - w)/x_2 \beta^2 \in (0, w)$. Then $\eta^{(2)}(C_o) = \eta^{(2)}(w^{(4)}_*)$. Since $w^{(4)}_* > w^{(4)}_*$ by equation (65) and $\eta^{(2)}$ is increasing in $w^{(4)}_*$, we have $\eta^{(2)} > \eta^{(2)}$, i.e., $\eta^{(1)} > \eta^{(2)}$. Since $\eta^{(2)}$ is a decreasing function of $C_o$ over $(C, C + x_2 \beta^2 w/(1 + \beta))$ and approaches $(1 + \beta)/[2(1 + x_1 \beta)]$ as $C_o$ approaches $C + x_2 \beta^2 w/(1 + \beta)$, inequality (57) holds.

Furthermore, let $w^{(4)}_* = (1 + \varepsilon)w^{(4)}_*$, where $\varepsilon > 0$ is small enough such that $w^{(4)}_* \in (0, w)$. Substituting this into (65) gives

\[ w^{(2)}_* = \frac{1 + \beta}{x_2 \beta^2} (C - C_o) + w - H(\varepsilon, n) < w^{(4)}_*, \tag{72} \]

where

\[ H(\varepsilon, n) = \frac{\varepsilon(1 - \beta^{2n-2})}{x_2 (1 - \beta^2)^2} \left[ \frac{1 + \beta}{x_2 \beta^2} (C - C_o) + w \right]. \]

Observe that for any fixed $n > 1$, $H(\varepsilon, n) \to 0$ as $\varepsilon \to 0$. Let $\varepsilon \propto \beta^n$, then $H(\varepsilon, n) \to +\infty$ and $\varepsilon \to 0$ as $n \to \infty$. Thus for any $\bar{\varepsilon} > 0$ and $H_0 \in (0, +\infty)$, there is $n > 1$ and $0 < \varepsilon < \bar{\varepsilon} < 0$ such that $H(\varepsilon, n) = H_0$.

Using (72) and (65), equation (64) can be rewritten as

\[ -\frac{1 - \beta}{\beta} (C - C_o) + x_2 \beta (1 - \beta) H(\varepsilon, n) = C_e. \]

Let $\hat{C}$ be the $C_o$ such that $\eta^{(2)}(\hat{C}) = (1 + x_2)/2$. Then:

(i) Suppose $\hat{C} < C_o < C + x_2 \beta^2 w/(1 + \beta)$, i.e., $\eta^{(2)} < (1 + x_2)/2$. Then inequality (54) can be simplified to read

\[ -\frac{1 - \beta}{\beta} (C - C_o) < C_e < (1 - \beta)(C - C_o) + x_2 \beta (1 - \beta) w. \]
which is equivalent to

\[ 0 < H(\varepsilon, n) < \frac{1 + \beta}{x_2 \beta^2} (C - C_o) + w. \]

Furthermore, since \( \eta^{(2)}(2) < (1 + x_2)/2 \), there is \( \bar{\varepsilon} > 0 \) such that \( \eta^{(2)}(w_*(\varepsilon)) \leq (1 + x_2)/2 \) for all \( 0 < \varepsilon < \bar{\varepsilon} \). Then inequality (58) can be simplified as follows:

\[ \eta^{(2)} < \eta^{(1)} < \eta^{(2)} \frac{1 + 2x_1 \beta}{2(1 + x_1 \beta)}, \]

\[ \iff 0 < w^{(2)} < w^{(4)}_*, \]

\[ \iff 0 < H(\varepsilon, n) < \frac{1 + \beta}{x_2 \beta^2} (C - C_o) + w. \]

Recall that for any \( \bar{\varepsilon} > 0 \) and \( H_0 \in (0, +\infty) \), there is \( n \) and \( \epsilon \) with \( n > 1 \) and \( 0 < \varepsilon < \bar{\varepsilon} < 0 \) such that \( H(\varepsilon, n) = H_0 \). Thus, when \( \hat{C} < C_o < C + x_2 \beta^2 w/(1 + \beta) \) and \( C_e \) satisfies inequality (54), there is \( n \) and \( \epsilon \) with \( n > 1 \) and \( 0 < \varepsilon < \bar{\varepsilon} < 0 \) such that the corresponding \( (n, w^{(2)}_*, w^{(4)}_*) \) solves equations (65) and (64) and the corresponding \( (\eta^{(1)}, \eta^{(2)}) \) satisfies conditions (57) and (58).

(ii) Suppose \( C < C_o \leq \hat{C} \), i.e., \( \eta^{(2)}(2) \geq (1 + x_2)/2 \). Then inequality (54) can be simplified as

\[ -\frac{1 - \beta}{\beta}(C - C_o) \leq C_e < (1 - \beta)(C - C_o) + \frac{2x_2 \beta(1 - \eta^{(2)})w}{1 + \beta}, \]

which is equivalent to

\[ 0 < H(\varepsilon, n) < \frac{1 + \beta}{x_2 \beta^2} (C - C_o) + \frac{2(1 - \eta^{(2)})w}{1 + \beta}. \]

Furthermore, since \( \eta^{(2)}(2) \geq (1 + x_2)/2 \), \( \eta^{(2)}(w_*(\varepsilon)) > (1 + x_2)/2 \) for all \( \varepsilon > 0 \). Then inequality (58) can be simplified as

\[ \eta^{(2)} < \eta^{(1)} \leq 1 - \frac{\beta(1 - \eta^{(2)})}{1 + 2\beta(1 - \eta^{(2)})}, \]

\[ \iff w - \frac{2(1 - \eta^{(2)})w}{1 + \beta} \leq w^{(2)}_* < w^{(4)}_*, \]

\[ \iff 0 < H(\varepsilon, n) \leq \frac{1 + \beta}{x_2 \beta^2} (C - C_o) + \frac{2(1 - \eta^{(2)})w}{1 + \beta}. \]
Note that $\eta^{(2)}(w_4^{(4)}(\varepsilon)) \rightarrow \eta^{(2)}$ as $\varepsilon \rightarrow 0$. Thus, when $0 < C_o \leq \hat{C}$ and $C_e$ satisfies inequality (54), there is $n > 1$ and $\varepsilon > 0$ such that the so defined $(n, w_2^{(2)}, w_4^{(4)})$ solves equation (65) and (64) and the corresponding $(\eta^{(1)}, \eta^{(2)})$ satisfies conditions (57) and (58).

This completes the proof of the lemma.

\section*{B.6.3 Numerical Example}

\begin{table}[H]
\centering
\caption{A numerical example}
\begin{tabular}{llll}
\hline
\hline
\text{Preferences} & & & \\
\hline
$\beta = 0.9$ & Moral hazard & & \\
\hline
$\theta_1 = 0$ & $\theta_2 = 100$ & $\psi = 20$ & \\
x_1 = 0.4 & x_2 = 0.6 & x'_1 = 0.8 & x'_2 = 0.2 \\
\hline
\text{Costs} & & & \\
$C_e = 0.2328$ & $C_o = 31.8182$ & & \\
\hline
\end{tabular}
\end{table}

\section*{C The Case of $C_e = 0$.}

In this section, we show that if $C_e = 0$, then the model has a unique equilibrium which is stationary.

First, when $C_e = 0$, Type T periods never exist. To see this, note that firms incur a negative period payoff in a Type T period by staying in the market, and their continuation values are zero. Hence firms are better off exiting the market. Thus, by Proposition 3, all periods are Type SI in equilibrium. (The proof of Proposition 3 does not depend on the assumption of $C_e > 0$.) Hence we have for each period $t$,

$$
V_t = V_{t,t}(w_{y,t}) = (1 - \beta^2)(C - C_o + w - w_{y,t}) + \beta^2 V_{t+2} + x_1\beta \frac{w}{w}[V_{t+1} - (1 - \beta)(C - C_o + w) - \beta V_{t+2}],
$$
which implies

$$0 = C - C_o + (w - w_{y,t}) \left[ 1 - \frac{x_1 \beta (1 - \beta) (C - C_o + w)}{(1 + \beta) w} \right].$$

The right-hand side of the above equation is strictly decreasing in $w_{y,t}$, strictly positive if $w_{y,t} = w_{A,t} = (1 - \delta)w$ and strictly negative if $w = \bar{w}$. Hence there exists a unique $w_{y,t} \in (w_{A,t}, \bar{w})$ that solves the above equation. This implies that every period is Type I and therefore $\alpha_{y,t} = 1$. This also proves that $w_{y,t}$ and therefore the corresponding $\hat{p}_{1,t}$ are constant over time. Furthermore, we can solve from (16) and (17) the equilibrium $\eta_t$ and $L_t$, which are constant over time. That is, the model has a unique stationary equilibrium. Finally, it is easy to verify that this is the stationary equilibrium characterized in Proposition 2 with $\eta_t = \eta_s^+.$

D Welfare Comparison

This appendix presents the calculations behind the output and welfare comparisons in Section 8. Remember we are comparing the stationary equilibria in Proposition 2 and the non-stationary equilibria in Proposition 8. Our goal is to compare the aggregate output and worker welfare achieved in the two types of equilibria. Notice that given that $(C_o, C_e)$ satisfies equation (51), the economy has a constant stock of firms in any given stationary or non-stationary equilibrium. Let $\eta_s^+ (\eta_c^+)$ denote the maximum stock of firms that a stationary (non-stationary) equilibrium can support.

We first show that the maximum number of firms (and hence the maximum aggregate output) that a stationary equilibrium can support is larger than what a non-stationary equilibrium can, i.e., $\eta_s^+ > \eta_c^+$. After some algebra, we can show that $\eta_s^+ > \eta_c^+$ if and only if

$$x_1 (1 + \beta) (x_2 + x_1 \beta) (C_o - C_e)^2 + x_1 x_2 \beta \bar{w} (1 + \beta - \beta^2) (C_o - C_e) + (1 - \beta) (x_2 \beta \bar{w})^2 > 0.$$
And it is easy to see that this inequality holds for all $C_o > C$.

Next, we compare worker welfare across the stationary and non-stationary equilibria. Note first that the lifetime expected utility of a worker born in period $t$ is measured by

$$\alpha_{y,t}w_{y,t} + (1 - \alpha_{y,t})\delta w_{*,t+1},$$

which is increasing in $\eta$.

Compare the worker’s expected utility, $w_s$, in a stationary equilibrium when $\eta = \eta_s^+$ with those, $w_{c}^{(j)}$ ($j = 1, 2$), in a two-period cycles where $\eta = \eta_c^+$, and we have $w_c^{(2)} > w_s > w_c^{(1)}$.

In a stationary equilibrium where $\eta = \eta_c^+$, a worker’s expected utility is given by

$$w_s = \left[1 - \frac{2\delta(1 - \eta_c^+)}{x_1}\right] w.$$

In the two-period cycles where $\eta = \eta_c^+$ the expected utility of a worker born in a Type T period is given by

$$w_{c}^{(2)} = (1 - \delta \hat{p}_1^{(2)} w), \text{ with } \hat{p}_1^{(2)} = \frac{(1 + \beta)(C_o - C)}{x_2 \beta w + x_1 (1 + \beta)(C_o - C)}. \quad (73)$$

It is easy to verify that $2(1 - \eta_c^+)/x_1 > \hat{p}_1^{(2)}$ and therefore $w_s < w_{c}^{(2)}$. The expected utility of a worker born in a Type S period is given by

$$w_{c}^{(1)} = (2\eta_c^+ - 1 + x_1 \hat{p}_1^{(2)})(1 - \delta)w + \delta w_s^{(2)},$$

where $\hat{p}_1^{(2)}$ is given by (73) and $w_s^{(2)} = w - (1 + \beta)(C_o - C)/x_2 \beta^2$. We then have

$$w_{c}^{(1)} - w_s = -\frac{x_1(C_o - C)}{x_2 \beta w + x_1 (1 + \beta)(C_o - C)} \frac{x_1(1 + \beta)(C_o - C) + x_2 \beta (1 - \beta)w}{x_2(1 + \beta)(C_o - C) + x_2 \beta^2 w} w - \frac{C_o - C}{x_2 \beta} \frac{(C_o - C)w}{x_2 w + x_1(C_o - C)}$$

$$< 0,$$

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where the last inequality holds for all $C_o > C$.

Finally, we compare the average expected utility of young workers in the non-stationary equilibrium, $(w_c^{(1)} + w_c^{(2)})/2$, with that in the stationary equilibrium, $w_s$:

$$w_c^{(1)} + w_c^{(2)} - 2w_s, \quad \text{where the last inequality holds whenever } C_o > C.$$
Assumption 1 is violated in period $t$, which implies the optimal long-term contract in period $t$ is essentially a short-term contract.

Consider the model’s stationary equilibria in this case. For all $\eta < 1$ and $\eta \in [\eta_s^-, \eta_s^+]$, the model has a stationary equilibrium in which $\eta_t = \eta$ for all $t$ as described in Proposition 2, just as in the case of $C_o > \underline{C}$. In addition, the model has a stationary equilibrium in which $\eta_t = 1$ for all $t$ and Assumption 1 is violated in every period:

**Proposition 11** Suppose $C_o \leq \underline{C}$. Then the model has a stationary equilibrium in which all periods are Type T, the optimal long-term contract is described by Proposition 4, and the equilibrium values of the aggregate variables satisfy

$$
\eta_t = 1, \ L_t = 1, \ \\
\alpha_{y,t} = 1, \ \alpha_{o,t} = 1, \ \\
w_{o,t} = w_{y,t} = w_{*,t} = w \in [w, \bar{\theta} - \psi - C_o], \ \\
V_t = \bar{\theta} - \psi - w - C_o.
$$

**Proof.** The proof is straightforward and omitted here. □

Would the model still be able to generate equilibria with positive entry and exit of firms so that the economy’s stock of firms and aggregate output fluctuate over time? The following proposition rules out such equilibria, by claiming that in equilibrium firms never exit the market. In other words, all equilibria are weakly expansionary, with the the economy’s stock of firms, and employment and output, growing monotonically over time.

**Proposition 12** If $C_o \leq \underline{C}$, then there is no exit of firms in any period in any of the model’s equilibria.

**Proof.** Suppose in some equilibrium there is exit of firms in some period $t$, or $\eta_{t-1} > \eta_t$. Suppose first $C_o < \underline{C}$. Then it must be the case that $\eta_t = 1$, since otherwise $\eta_t < 1$ by
Proposition 10 and thus

\[ V_t \geq (1 - \beta)(C - C_o) + \beta V_{t+1} > 0, \]

a contradiction to the free-entry-and-exit condition which implies \( V_t = 0 \). However, by Proposition 10, we have \( \eta_{t-1} \leq 1 = \eta_t \). That is, there is no exit of firms in period \( t \). A contradiction.

Suppose \( C_o = \underline{C} \). We first argue that there cannot be entry of firms in any period \( t' > t \). To see this, suppose that there is entry of firms in period \( t' > t \) and suppose, without loss of generality, that there is no entry of firms in any period \( \tau \in (t, t') \). Then it holds that \( \eta_\tau \leq \eta_t < \eta_{t-1} \leq 1 \) and firms would get a nonnegative period payoff in any period \( \tau \in [t, t') \). Furthermore, \( V_{\tau} = C_e > 0 \) since there is entry of firms in period \( t' \). Hence it must be the case that \( V_t > 0 \), a contradiction to the free-entry-and-exit condition.

Hence \( \eta_\tau \leq \eta_t < \eta_{t-1} \leq 1 \) for all \( \tau \geq t \) and, by a similar argument as that in the proof of Proposition 6, Assumption 1 holds for all \( \tau \geq t \). Furthermore, \( \eta_\tau < 1 \) for all \( \tau \geq t \) implies that firms earn nonnegative period payoff in every period \( \tau \geq t \). Since \( V_t = 0 \) by the free-entry-and-exit condition, it must be the case firms earn zero payoff in every period \( \tau \geq t \). This is only possible if period \( \tau \geq t \) is Type TI and \( V_\tau = (1 - \beta)(\underline{C} - C_o) + \beta V_{\tau+1} \) for all \( \tau \geq t \). However, by a similar argument as that in the proof of Lemma 1, we can show that period \( t + 1 \) is a Type S period, a contradiction.

To see an example, suppose \( C_e = \bar{C} - C_o \). Then the model has an equilibrium that exhibits growth-like dynamics. Specifically, the equilibrium has \( \eta_t \in [0, (1 + x_2)/2] \) and \( \eta_t \leq \eta_{t+1} \) for all \( t \), the optimal long-term contract is described by Proposition 1(ii), and the equilibrium values of other aggregate variables satisfy: \( L_t = 1 - x_2 \eta_{t-1}/(1 + x_2), \alpha_{y,t} = 2\eta_{t-1}/(1 + x_2), \alpha_{o,t} = 0, w_{o,t} = w, w_{y,t} = w_{A,t} \), and \( V_t = \bar{C} - C_o \).

Note that this “growth-like” equilibrium dynamics should not resemble in any sense the economic growth traditionally understood, usually driven by changes in the fundamentals.
of the economy. What the model suggests is that, in a moral hazard environment, the economy may start from a state in which resources are not fully utilized, but continues to move towards more full utilization of resources, through the expansion of the market.