Efficient Mechanisms with Information Acquisition

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Abstract

This paper studies the design of ex ante efficient mechanisms in situations where a single item is for sale, and agents have positively interdependent values and can covertly acquire information at a cost before participating in a mechanism. I find that when interdependency is low and/or the number of agents is large, the ex post efficient mechanism is also ex ante efficient. In cases of high interdependency and/or a small number of agents, ex ante efficient mechanisms discourage agents from acquiring excessive information by introducing randomization to the ex post efficient allocation rule in areas where the information’s precision increases most rapidly.

Keywords: Auctions, Mechanism Design, Information Acquisition, Efficiency

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1 Introduction

Most literature on mechanism design assumes that the amount of information possessed by agents is exogenous. In many important settings, however, this assumption does not apply. For example, in auctions for offshore oil and gas leases in the U.S., companies use seismic surveys to collect information about the tracts offered for sale before participating in the auctions. Another example is the sale of financial or business assets, in which buyers perform due diligence to investigate the quality and compatibility of the assets before submitting offers. In other words, in these settings the information held by agents is endogenous. Moreover, it is costly to acquire information. In the example of U.S. auctions for offshore oil and gas leases (see Haile et al. (2010)), companies can choose to conduct two-dimensional (2-D) or three dimensional (3-D) seismic surveys. 3-D surveys can produce more accurate information, and thus were used in 80% of wells drilled in the Gulf of Mexico by 1996. However, this number was only 5% in 1989 when 3-D surveys were more expensive than 2-D surveys.\footnote{For instance, it costs $1 million to examine a 50 square mile 3-D seismic survey in 1990, while this number was less than $100, 000 in 2000.} Similarly, the legal and accounting costs of performing due diligence often amount to millions of dollars in the sale of a business asset (see Quint and Hendricks (2013) and Bergemann et al. (2009)).

Furthermore, the incentives for agents to acquire information depend on the design of a mechanism. This can be seen from earlier studies that compare first price auctions with second price auctions in terms of the incentives they provide for agents to collect information in advance (among them Matthews (1984a), Stegeman (1996) and Persico (2000)). More recently, Bergemann and Välimäki (2002) consider the socially optimal information acquisition in the context of general mechanism design. They focus on mechanisms that implement the ex post efficient allocations given acquired private information, and find that ex ante efficient information acquisition can be achieved if agents have independent private values. However, if agents’ values are interdependent, then ex post efficient mechanisms will result in socially
sub-optimal information acquisition. In a follow-up paper, Bergemann et al. (2009) study the equilibrium level of information acquisition when agents face binary information decisions, and find that ex post efficient mechanisms result in excessive information acquisition in equilibrium. In summary, there is a conflict between the provision of ex ante efficient incentives to acquire information and the ex post efficient use of information. The question regarding how to design an ex ante efficient mechanism to balance the two trade-offs remains open.

This paper studies the design of ex ante efficient mechanisms in the sale of a single object when agents’ values are positively interdependent. The true value of the object to each agent is initially unknown. Before participating in a mechanism agents can simultaneously and independently decide how much information to acquire. Information acquisition is costly and the information choice of each agent is his private information. In this paper, I assume that the information structures are supermodular ordered. Ganuza and Penalva (2010) first introduce the notion of “supermodular precision”, and Shi (2012) later uses it when studying revenue-maximizing mechanisms with endogenous information.

In the main body of the paper, I focus on symmetric mechanisms and symmetric equilibria in which agents acquire the same amount of information before participating in a mechanism. Firstly, I show that the social planner never withholds the item in an ex ante efficient mechanism. Intuitively, whenever the social planner withholds the object, she can also allocate it randomly. By doing so, the social surplus from the allocation increases while an agent’s ex ante incentive to acquire information remains unaffected. Though intuitive, the proof of the result is non-trivial because of the presence of the non-standard information acquisition constraint. In addition, this result facilitates the analysis by allowing me to work with the interim allocation rule directly.

Secondly, I show that it is socially optimal for agents to acquire no more information than when the ex post efficient mechanism is used. This is consistent with Bergemann and Välimäki (2002) and Bergemann et al. (2009). For any given information choice satisfying the
above condition, I fully characterize mechanisms that implement this choice and maximize ex ante social surplus. An ex ante efficient mechanism discourages agents from excessive information acquisition by sometimes randomly allocating the item. Specifically, an ex ante efficient interim allocation rule randomizes in areas in which the accuracy of an agent’s posterior estimate can be significantly improved if an additional piece of information is acquired. In special cases in which the improvements are the same for all possible posterior estimates, i.e., the information structures are uniformly supermodular ordered, it is optimal to allocate the object uniformly at random with some probability.

To illustrate this result, consider a simple example where an agent’s type can take four possible values: $v_{1L} < v_{1H} < v_{2L} < v_{2H}$. Initially, each agent only knows whether $v \in \{v_{1L}, v_{1H}\}$ or $v \in \{v_{2L}, v_{2H}\}$. Agents have access to the following information technology. By incurring a cost $c > 0$, an agent with type $v_{1H}$ (or $v_{1L}$) can observe a high (or low) signal with probability $1/2 + p$ and a low (or high) signal with probability $1/2 - p$, and an agent with type $v_{2H}$ (or $v_{2L}$) can observe a high (or low) signal with probability $1/2 + \varepsilon p$ and a low (or high) signal with probability $1/2 - \varepsilon p$, where $p \in (0, 1/2)$ and $\varepsilon \in [0, 1]$. If $\varepsilon = 1$, then the signal is uniformly informative over the entire support. If $\varepsilon < 1$, then agents receive more precise information over $\{v_{1L}, v_{1H}\}$ than $\{v_{2L}, v_{2H}\}$. Intuitively, allocating the object randomly over $\{v_{1L}, v_{1H}\}$ can more effectively lower the value of information to agents as well as their incentives to collect information. This is demonstrated in the extreme case of $\varepsilon = 0$ in which randomization over $\{v_{2L}, v_{2H}\}$ has no impact on the value of information. This intuition carries over to cases in which agents have a continuum of types and access to a continuum of information choices.

Methodologically, when characterizing optimal mechanisms, I use an approach first proposed by Reid (1968) and later introduced into the mechanism design problem by Mierendorff (2009). My proof, however, is not a straightforward modification of Mierendorff (2009). In Mierendorff (2009), the interim allocation rule is discontinuous at one known point. In my model, the interim allocation rule could be discontinuous at most countably many unknown
points.

I also study how the socially optimal information choice is affected by model primitives such as the interdependency of agents’ values and the number of agents. In general, it is difficult to solve the optimal information choice analytically. Hence, I restrict attention to the special case in which the information structures are uniformly supermodular ordered. Under this assumption, I show that the optimal level of information gathering decreases as the interdependency of agents’ values increases, and gathering no information is optimal in the case of pure common value. Furthermore, when the ex post efficient mechanism is used, the amount of information acquired by each agent diminishes as the number of agents increases to infinity. As a result, the ex post efficient mechanism is also ex ante efficient for a large number of agents.

Lastly, I study ex ante efficient mechanisms without imposing symmetry restrictions. As in the symmetric case, the social planner never withholds the item in an ex ante efficient mechanism. Since characterizing optimal mechanisms in the general setting is extremely hard, I restrict attention to the special case in which the information structures are uniformly supermodular ordered. Under this assumption, I provide conditions under which the socially optimal information choices are the same for all agents and there exists a symmetric ex ante efficient mechanism.

This paper is related to the literature of mechanism design with endogenous information. Earlier papers focus on the comparison between first- and second-price auctions. For example, Matthews (1984a) considers a first-price auction with pure common values, and examines how an increase in the number of agents affects the information acquisition. Stegeman (1996) finds that both auctions lead to identical and, more importantly, efficient incentives for information acquisition when agents’ values are private and independent. In contrast, Persico (2000) finds that first-price auctions provide stronger incentive for agents to acquire information than second-price auctions do when their values are affiliated. The two most closely related papers are Bergemann and Välimäki (2002) and Bergemann et al. (2009).
Both study the efficiency of information acquisition by agents when the ex post efficient mechanism is used. Instead, I focus on the design of ex ante efficient mechanisms.

This paper is also related to papers that study the revenue-maximizing mechanisms with endogenous information acquisition. The most closely related paper is Shi (2012) who considers the sale of a single asset when buyers have independent private values and who, before the auction, can simultaneously and independently decide how much information to acquire. He finds that the optimal reserve price is always below the standard monopoly price to encourage information acquisition. Several other papers consider the case where the seller can control the timing of information acquisition (see, for example, Levin and Smith (1994), Ye (2004), and Crémer et al. (2009)).

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 contains the main results. More specifically, Section 3.2 characterizes optimal symmetric mechanisms for each given information choice and Section 3.3 studies the socially optimal information choice. Section 4 examines ex ante efficient mechanisms without imposing symmetry restrictions. Section 5 concludes. All omitted proofs are relegated to appendix.

2 Model

There are $n$ agents, indexed by $i \in \{1, \cdots, n\}$, who compete for a single object. Each agent $i$ has a payoff-relevant type $\theta_i$, which is unknown ex ante. The true value of the object to agent $i$ is

$$v_i(\theta) \equiv \theta_i + \gamma \sum_{j \neq i} \theta_j,$$

where $\theta \equiv (\theta_1, \ldots, \theta_n)$ and $\gamma \in [0, 1]$ is a measure of interdependence.\footnote{Under this specification, the range of possible valuations for agent $i$ depends on the number of agents. An alternative normalized specification is given by

$$v_i(\theta) \equiv (1 - \gamma)\theta_i + \frac{\gamma}{n - 1} \sum_{j \neq i} \theta_j.$$}

Each agent $i$ has a quasi-linear utility. His payoff is $q_i v_i(\theta) - t_i$ if he gets the object with probability $q_i \in [0, 1]$.
and pays $t_i \in \mathbb{R}$.

Initially, each agent $i$ only knows that $\theta_i$'s are independently distributed with a common prior distribution $F$ and support $[\theta, \bar{\theta}] \subset \mathbb{R}_+$. $F$ has a positive and continuous density $f$. Each agent $i$ can covertly acquire a costly signal $x_i$ about his type $\theta_i$ by selecting a joint distribution of $(x_i, \theta_i)$ from a family of joint distributions $G(x_i, \theta_i|\alpha_i)$, indexed by $\alpha_i \in \mathcal{A} \equiv [\alpha, \bar{\alpha}] \subset \mathbb{R}$. For each $\alpha_i \in \mathcal{A}$, we also refer the joint distribution $G(\cdot, \cdot|\alpha_i)$ as an information structure. Let $g$ denote the density function associated with $G$. For all $\alpha_i \in \mathcal{A}$, $G(\cdot, \cdot|\alpha_i)$ admits the same marginal distribution of $\theta_i$, i.e., $\int_{\mathbb{R}} g(x_i, \theta_i|\alpha_i)dx_i = f(\theta_i)$ for all $\theta_i \in [\theta, \bar{\theta}]$. Throughout the paper, I assume that $\mathbb{E}[\theta|x, \alpha]$ is strictly increasing in $x$ for all $\alpha \in \mathcal{A}$, i.e., a higher signal results in a higher conditional expectation.\(^3\) A signal with a higher $\alpha_i$ is more precise, in a sense which I formally define below. Let $C(\alpha_i)$ denote the cost of acquiring a signal with precision $\alpha_i$. As is standard in the literature, I assume that $C$ is non-negative, strictly increasing, twice continuously differentiable and strictly convex. Furthermore, $C(\alpha) = C'(\alpha) = 0$.

\section{2.1 Information Order}

Let $G(x|\alpha) \equiv \int G(x|\theta, \alpha)dF(\theta)$ denote the marginal distribution of $x$ given $\alpha$. I define a new signal by applying the probability integral transformation on the original signal: $s \equiv G(x|\alpha)$. The transformed signal is uniformly distributed on $[0, 1]$.\(^4\) Clearly, the transformed signal has the same informational content as the original signal. Furthermore, since any two transformed signals have the same marginal distribution, their realizations are directly comparable. Therefore, I will henceforth work with the transformed signal $s_i$ directly. Let

\(^3\)For each $\alpha \in \mathcal{A}$, let $G(\theta|x, \alpha)$ denote the conditional distribution of $\theta$ given $x$. Then one sufficient condition for this is to assume that $G(\theta|x, \alpha)$ have the monotone likelihood ratio property.

\(^4\) $s$ is uniform on $[0, 1]$ only if $G(x)$ is continuous and strictly increasing. This can be assumed without loss of generality. If $G$ has a discontinuity at $z$, where $\mathbb{P}(\tilde{x} = z) = p$, $x$ can be transformed into $x^*$, which has a continuous and strictly increasing distribution function using the following construction proposed in Lehmann (1988): $x^* = x$ for $x < z$, $x^* = x + pU$ if $x = z$, where $U$ is uniform on $(0, 1)$, and $x^* = x + p$ for $x > z$.\(^7\)
\(^{(s, \alpha)} \equiv \mathbb{E}[\theta|s, \alpha]\) be the conditional expectation of \(\theta\) given \(s\) and \(\alpha\). Then \(w(s, \alpha)\) is strictly increasing in \(s\). Let \(H(w|\alpha) \equiv \mathbb{P}(w(s, \alpha) \leq w)\) denote the distribution of \(w(s, \alpha)\) with density \(h(w|\alpha)\). I assume that \(H(w|\alpha)\) and \(h(w|\alpha)\) are twice continuously differentiable. Throughout the paper, I impose the following assumption on information structures:

**Assumption 1** The information structures are supermodular ordered, i.e.,

\[-\frac{H_a(w|\alpha)}{h(w|\alpha)}\] is strictly increasing in \(w\) on \([w(0, \alpha), w(1, \alpha)]\) for all \(\alpha\).

To better understand Assumption 1, note that \(w_a(s, \alpha) = -H_a(w(s, \alpha)|\alpha)/h(w(s, \alpha)|\alpha)\) which is strictly increasing in \(s\). Thus, \(w(s, \alpha)\) is supermodular in \((s, \alpha)\). Furthermore, I show in the appendix that Assumption 1 implies the following property of \(w(s, \alpha)\):

**Lemma 1** Suppose the information structures are supermodular. Then for all \(s, s' \in (0, 1)\) and \(s > s'\),

\[w(s, \alpha) - w(s', \eta) > w(s', \alpha) - w(s, \eta).\] (1)

Intuitively, if \(s\) contains little information about \(\theta\), then \(w(s, \alpha)\) does not vary much as \(s\) changes and its distribution concentrates around \(\mathbb{E}[\theta]\). As \(s\) becomes more informative about \(\theta\), \(w(s, \alpha)\) varies more as \(s\) changes and its distribution becomes more dispersed. Formally, if \(\alpha > \alpha'\) then \(w(s, \alpha)\) is strictly larger than \(w(s, \alpha')\) in the dispersive order.\(^5\) Ganuza and Penalva (2010) first develop an information order called “supermodular precision” based on this notion of dispersion. Shi (2012) also imposes this assumption for some of his results.

For some results of the paper, I strengthen Assumption 1 to that the information structures are uniformly supermodular ordered. The meaning of “uniformly” will become clear through the characterization of optimal mechanisms in Section 3.2.

\(^5\)(See Ganuza and Penalva (2010)) Let \(Y\) and \(Z\) be two real-valued random variables with distributions \(F\) and \(G\), respectively. We say \(Y\) is greater than \(Z\) in the dispersive order if for all \(q, p \in (0, 1)\) and \(q > p\),

\[F^{-1}(q) - F^{-1}(p) \geq G^{-1}(q) - G^{-1}(p).\]
Assumption 2 The information structures are uniformly supermodular ordered, i.e., for all $\alpha$ and all $w \in [w(0, \alpha), w(1, \alpha)]$ there exists a positive function $b : A \to \mathbb{R}^+$ such that

$$\frac{-H_\alpha(w|\alpha)}{h(w|\alpha)} = \frac{w - \mu}{b(\alpha)}.$$

Uniformly supermodular-ordered information structures include the following two commonly used information technologies in the literature:  

Example 1 (The Linear Experiments) Let $\theta_i$'s be independently distributed with a common prior $F$ and mean $\mu$. Agent $i$ can obtain a costly signal $x_i$ about $\theta_i$ at a cost. Consider the following information structure, which is called “truth-or-noise” in Lewis and Sappington (1994), Johnson and Myatt (2006) and Shi (2012). With probability $\alpha_i \in [0, 1]$, $x_i$ is equal to his true type $\theta_i$, and with probability $1 - \alpha_i$, $x_i$ is an independent draw from $F$. Since the marginal distribution of $x_i$ is $F$, the transformed signal is $s_i \equiv F(x_i)$. Then the posterior estimate of an agent who chooses $\alpha_i$ and receives $s_i$ is $w(s_i, \alpha_i) = \alpha_i F^{-1}(s_i) + (1 - \alpha_i)\mu$. Therefore we have

$$\frac{-H_{\alpha_i}(w_i|\alpha_i)}{h(w_i|\alpha_i)} = \frac{w_i - \mu}{\alpha_i}.$$

It is clear that the information structures are uniformly supermodular ordered.

Example 2 (Normal Experiments) Let $\theta_i$'s be independently distributed with a normal distribution: $\theta_i \sim N(\mu, 1/\beta)$. Agent $i$ can observe a costly signal $x_i = \theta_i + \varepsilon_i$, where $\varepsilon_i \sim N(0, 1/\alpha_i)$. Since the marginal distribution of $x_i$ is also normal, i.e., $x_i \sim N(\mu, (\beta + \alpha_i)/\beta \alpha_i)$, the transformed signal is $s_i \equiv \Phi \left( \sqrt{\beta \alpha_i} (x_i - \mu)/\sqrt{\beta + \alpha_i} \right)$. Then the posterior estimate of an agent who chooses $\alpha_i$ and receives $s_i$ is

$$w(s_i, \alpha_i) = \mu + \frac{\sqrt{\alpha_i} \Phi^{-1}(s_i)}{\sqrt{\beta (\alpha + \beta)}}.$$

\footnote{See, for example, Ganuza and Penalva (2010) and Shi (2012).}
Therefore we have

\[- \frac{H_{\alpha_i(w_i|\alpha_i)}}{h(w_i|\alpha_i)} = \frac{\beta(w_i - \mu)}{2\alpha_i(\alpha_i + \beta)}.\]

It is clear that the information structures are uniformly supermodular ordered.

### 2.2 Timing

The game proceeds in the following way: The social planner announces a mechanism. After observing the mechanism, each agent \(i\) simultaneously chooses an information structure \(\alpha_i\) and observes the realized signal \(s_i\). Both \(\alpha_i\) and \(s_i\) are agent \(i\)'s private information. Then each agent decides whether to participate in the mechanism. Each participating agent reports his private information. Based on their reports, an allocation and payments are implemented according to the announced mechanism.

I assume that the payoff structure, the timing of the game, the information structures and the prior distribution \(F\) are common knowledge. The solution concept is Bayesian Nash equilibrium.

### 2.3 Mechanisms

The private information of agent \(i\) is two-dimensional, including the choice of information structure \(\alpha_i\) and the realized signal \(s_i\). However, similar to Biais et al. (2000), Szalay (2009) and Shi (2012), the usual difficulties inherent in multi-dimensional mechanism design problem do not arise here, since the posterior estimate, \(w(s_i, \alpha_i)\), summarizes all the private information needed to compute agent \(i\)'s valuation of the object:

\[
\mathbb{E}_{\theta}[v_i(\theta)|\alpha_i, s_i] = w(s_i, \alpha_i) + \gamma \sum_{j \neq i} \mathbb{E}[\theta_j].
\]

Furthermore, the seller cannot screen agents with the same posterior estimate but different choices of information structures. Hence, one can appeal to the Revelation Principle and
focus on direct mechanisms in which agents report their posterior estimates directly. For ease of notation, I use \( w_i \) to denote \( w_i(s_i, \alpha_i) \) and \( w \equiv (w_1, \ldots, w_n) \). A direct mechanism is a pair \((q, t)\), where, given the reported vector of posterior estimates \( w \), \( q_i(w) \) denotes the probability agent \( i \) is allocated the object, and \( t_i(w) \) denotes agent \( i \)'s payment. Define agent \( i \)'s interim allocation rule as

\[
Q_i(w_i) \equiv E_{w_{-i}}[q_i(w_i, w_{-i})|\alpha_{-i}],
\]

where \( \alpha_{-i} \) are his opponents' information structures. Then the interim utility of agent \( i \) who has a posterior estimate \( w_i \) and reports \( w_i' \) is

\[
U_i(w_i, w_i') \equiv w_i Q_i(w_i') + E_{w_{-i}} \left[ \gamma \sum_{j \neq i} \theta_j q_i(w_i', w_{-i}) - t_i(w_i', w_{-i}) \right| \alpha_{-i}].
\]

Note that \( Q_i(w_i) \) and \( U_i(w_i, w_i') \) also depend on \( \alpha_{-i} \), and I suppress the dependence for ease of notation.

I now describe the constraints that the social planner's mechanism must satisfy. First, the mechanism must satisfy the individual rationality constraint (IR), i.e.,

\[
U_i(w_i) \equiv U_i(w_i, w_i) \geq 0, \ \forall w_i \in [\theta, \bar{\theta}],
\]

so that the agents are willing to participate in the mechanism. Second the mechanism must satisfy the Bayesian incentive compatibility constraint (IC), i.e.,

\[
U_i(w_i) \geq U_i(w_i, w_i'), \ \forall w_i, w_i' \in [\theta, \bar{\theta}],
\]

so that truth-telling is a Bayesian Nash equilibrium. By the standard argument (see, for example, Myerson (1981)), the (IC) constraint holds if and only if

\( Q_i(w_i) \) is non-decreasing in \( w_i \), \hspace{1cm} \text{(MON)}
and $U_i(w_i)$ is absolutely continuous and satisfies the following envelope condition

$$U_i(w_i) = U_i(\theta) + \int_{\theta}^{w_i} Q_i(\tilde{w}_i) d\tilde{w}_i. \quad (3)$$

Since the social planner’s goal is to maximize social surplus and transfers between agents and the social planner do not affect social surplus, we can guarantee that the (IR) constraint is satisfied by making sufficiently large lump sum transfers to agents. Hence, we can safely ignore the (IR) constraint for the remainder of the paper.

Lastly, since the information structure chosen by an agent is unobservable, the mechanism must also satisfy the information acquisition constraint (IA): no agent stands to gain by deviating from the equilibrium choice $\alpha_i^*$, i.e.,

$$\alpha_i^* \in \arg\max_{\alpha_i} \mathbb{E}_w \left[ q_i(w) \left( w_i + \gamma \sum_{j \neq i} \mathbb{E}[\theta_j] \right) - t_i(w) \left| \alpha_i = \alpha_j^* \forall j \neq i \right. \right] - C(\alpha_i).$$

Using the envelope condition (3) and the fact that the support of $w_i$ is $[w(0, \alpha_i), w(1, \alpha_i)]$, agent $i$’s payoff from choosing $\alpha_i$ can be rewritten as

$$\mathbb{E}_w \left[ q_i(w) \left( w_i + \gamma \sum_{j \neq i} \mathbb{E}[\theta_j] \right) - t_i(w) \left| \alpha_i = \alpha_j^* \forall j \neq i \right. \right] - C(\alpha_i)$$

$$= U_i(w(0, \alpha_i)) + \int_{w(0, \alpha_i)}^{w(1, \alpha_i)} [1 - H(w_i|\alpha_i)] Q_i(w_i) dw_i - C(\alpha_i).$$

The social planner chooses mechanism $(q, t)$ and a vector of recommendations of information structures $\alpha^*$ to maximize social surplus,

$$\max_{\alpha^*: (q, t)} \mathbb{E} \left[ \sum_i (w_i + \gamma \sum_{j \neq i} w_j) q_i(w) \left| \alpha_i = \alpha_i^* \right. \right] - \sum_i C(\alpha_i^*),$$

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subject to the (MON), (IA) and the feasibility constraint (F), i.e.,

\[0 \leq q_i(w) \leq 1, \quad \sum_i q_i(w) \leq 1, \forall w. \quad (F)\]

We say a mechanism \((q, t)\) is ex ante efficient or optimal if there exists \(\alpha^*\) such that \(\alpha^*\) and \((q, t)\) solve the social planner’s problem.

3 Efficient Mechanisms

In this section, I restrict attention to mechanisms that treat all agents symmetrically\(^7\) as well as symmetric equilibria in which all agents acquire the same information structure, i.e., \(\alpha_i^* = \alpha^*\) for all \(i\). The symmetric assumption significantly simplifies the analysis. However, this assumption may result in a loss of generality. Section 4 presents a study of ex ante efficient mechanisms without imposing symmetry restrictions, and provides conditions under which the socially optimal information choices are the same for all agents and there exists a symmetric ex ante efficient mechanism. Note that when \(q\) is symmetric, the corresponding interim allocation rule \(Q_i\) is independent of \(i\). From hereon, I drop the subscript \(i\) from \(Q, w\) and \(\alpha\) whenever the meaning is clear.

I start solving the social planner’s problem by replacing the (IA) constraint by a one-sided first-order necessary condition. In an earlier paper, Bergemann and Välimäki (2002) show that if the social planner adopts the ex post efficient mechanism, then agents tend to acquire more information than the socially desired level. This suggests that an ex ante efficient mechanism would sacrifice some allocation efficiency to discourage agents from acquiring information. Thus, I hypothesize that to ensure the (IA) constraint holds in an ex ante efficient mechanism, it suffices to ensure that no agent has the incentive to choose \(\alpha_i > \alpha^*\):

\(^{7}\)The formal definition of symmetric mechanisms can be found in the appendix.
for all $\alpha_i > \alpha^*_i$,

$$U_i(w(0, \alpha^*_i)) + \int_{w(0, \alpha^*_i)}^{w(1, \alpha^*_i)} [1 - H(w_i|\alpha^*_i)]Q_i(w_i)dw_i - C(\alpha^*_i)$$

$$\geq U_i(w(0, \alpha_i)) + \int_{w(0, \alpha_i)}^{w(1, \alpha_i)} [1 - H(w_i|\alpha_i)]Q_i(w_i)dw_i - C(\alpha_i).$$

This implies the following one-sided first-order condition:

$$\mathbb{E} \left[ -\frac{H_{\alpha_i}(w_i|\alpha^*_i)}{h(w_i|\alpha^*_i)}Q_i(w_i) \bigg| \alpha_i = \alpha^*_i \right] \leq C'(\alpha^*_i). \quad (\text{IA}')$$

The left-hand side of the above inequality is agent $i$’s marginal benefit from acquiring $\alpha^*_i$, and the right-hand side is the marginal cost. In Lemma 7 in the appendix, I show that for any non-decreasing $Q_i$, an agent’s marginal benefit from acquiring information is non-negative. Subsequently, I consider the relaxed problem of the social planner by replacing the (IA) constraint by the (IA') constraint. I later show that (IA') holds with equality when $\alpha^*$ is chosen optimally. The first-order approach is valid if the second-order condition of the agents’ optimization problem is satisfied. Appendix A.3 provides sufficient conditions that ensure the first-order approach is valid.

The (IA') constraint is easier to work with than the original (IA) constraint. However, it is still nonstandard and prevents me from solving the social planner’s problem directly as in Myerson (1981). To overcome this difficulty, I focus on reduced form auctions. Formally,

**Definition 1** We say that $q$ implements $Q : [\theta, \bar{\theta}] \to [0, 1]$ and $Q$ is the reduced form of $q$ if $q$ satisfies (2) and (F) for all $w \in [\theta, \bar{\theta}]$. $Q$ is implementable if $q$ exists implementing $Q$.

One important prior result I use in this paper is the necessary and sufficient condition of Maskin and Riley (1984) and Matthews (1984b) which characterizes the interim allocation rules implementable by symmetric mechanisms. By Theorem 1 in Matthews (1984b), any
non-decreasing function $Q : [\underline{\theta}, \overline{\theta}] \rightarrow [0, 1]$ is implementable if and only if it satisfies

$$Y(w) \equiv \int_{\underline{\theta}}^{\overline{\theta}} \left[ H(z|\alpha^*)^{n-1} - Q(z) \right] h(z|\alpha^*) dz \geq 0, \forall w \in [\underline{\theta}, \overline{\theta}]. \quad (F')$$

Thus, given (MON), we can replace (F) by (F'). Note that the support of $w$ is $W \equiv [w(0, \alpha^*), w(1, \alpha^*)] \subset [\underline{\theta}, \overline{\theta}]$. Therefore, (F') imposes no restriction on $Q$ in $[\underline{\theta}, \overline{\theta}] \setminus W$.

Now all the constraints are expressed as functions of the interim allocation rule $Q$, while the objective function, i.e., the expected social surplus, is not. The expected social surplus is the summation of the payoffs of all agents. In the case of independent private value ($\gamma = 0$), agent $i$’s valuation of the object and his winning probability are independent conditional on his private information. Thus, each agent’s expected payoff is completely determined by $Q$. This is not true in general when $\gamma > 0$, as other agents’ private information affects his winning probability and his value. However, this is true for a socially optimal mechanism, which never withholds the object:

**Theorem 1** Suppose $\alpha^*$ and $(q, t)$ solve the relaxed problem of the social planner, then

$$\sum_i q_i(w) = 1 \text{ for almost all } w \in W^n. \quad (4)$$

Using Theorem 1 and Law of iterated expectations, we can rewrite the social planner’s objective function as a function of $Q$:

$$\mathbb{E} \left[ \sum_i (w_i + \gamma \sum_{j \neq i} w_j) q_i(w) \bigg| \alpha_i = \alpha^* \ \forall i \right] = \sum_i \mathbb{E} \left[ (1 - \gamma) w_i Q(w_i) | \alpha_i = \alpha^* \right] + n\gamma \mathbb{E}[\theta].$$

Since the second term, $n\gamma \mathbb{E}[\theta]$, is a constant, we ignore it from hereon. To summarize, the social planner’s relaxed problem, denoted by $(P')$, can be rewritten as:

$$\max_{\alpha^*, Q} \mathbb{E} \left[ (1 - \gamma) w Q(w) | \alpha^* \right] - C(\alpha^*),$$
subject to

\[ Y(w) = \int_{\theta}^{\bar{\theta}} [H(z|\alpha^*)]^{n-1} - Q(z)h(z|\alpha^*)dz \geq 0, \quad \forall w \in [\theta, \bar{\theta}]. \]  \hspace{1cm} (F')

\[ Q(w) \text{ is non-decreasing in } w, \]  \hspace{1cm} (MON)

\[ \mathbb{E} \left[ -\frac{H_{\alpha}(w|\alpha^*)}{h(w|\alpha^*)} Q(w) \bigg| \alpha^* \right] \leq C'(\alpha^*). \]  \hspace{1cm} (IA')

In addition to being instrumental in solving the social planner’s problem, the result of Theorem 1 has some inherent economic interest. Obviously, the efficient mechanism never withholds the object when information is exogenous. This is not obvious when information is endogenous, since, by withholding the object occasionally, the social planner can discourage agents from information acquisition, which may improve efficiency ex ante. However, intuitively, whenever the social planner withholds the object, she can also allocate it randomly. By doing so, the social surplus from the allocation increases while an agent’s ex ante incentive to acquire information remains unaffected.

Though intuitive, the proof of Theorem 1 is non-trivial. This is because the resulting mechanism, by simply randomizing the object whenever it is withheld, is likely to violate the (IC) and/or (IA') constraints. To illustrate this, let \( A \) be a set of types such that \( \sum_i q_i(w) < 1 \) whenever \( w \in A^\cap \), and suppose there exist \( (w, \bar{w}) \cap A = \emptyset \) such that \( \inf A < w < \bar{w} < \sup A \).

If we simply redefine \( q \) such that it remains unchanged outside \( A^\cap \) and \( \sum_i q_i(w) = 1 \) whenever \( w \in A^\cap \), then the resulting \( Q \) remains unchanged for \( w \in (w, \bar{w}) \) but increases for all \( w \in A \). If we allocate the object too often to agents whose types are in \( [\theta, w] \cap A \), the resulting \( Q \) will no longer be non-decreasing and therefore violate the (IC) constraint. If we allocate the object too often to agents whose types are in \( [w, \bar{\theta}] \cap A \), this will increase an agent’s marginal benefit from acquiring information and lead to a violation of the (IA') constraint. Thus to ensure the new \( q \) generating a higher social surplus while satisfying all the constraints, one needs to adjust \( q \) not only for \( w \) inside \( A^\cap \), but also for \( w \) outside \( A^\cap \). Section 3.1 contains the proof of this result.
3.1 Proof of Theorem 1

I prove Theorem 1 by proving the following two lemmas. Observe first that if \( \alpha_i = \alpha^* \) for all \( i \), then (4) is violated if and only if \( Y(w(0, \alpha^*)) > 0 \). Then

**Lemma 2** Suppose \( \alpha_i = \alpha^* \) for all \( i \). Let \( Q \) be any interim allocation rule satisfying (\( F' \)), (MON), (IA') and \( Y(w(0, \alpha^*)) > 0 \), then there exists \( \hat{Q} \) satisfying (\( F' \)), (MON) and (IA') such that

\[
\hat{Q}(w) \geq Q(w), \forall w \in W, \tag{5}
\]

and "" holds for a set of \( w \) with positive measure.

The intuition behind the proof of Lemma 2 can be illustrated by Figure 1. If \( Q \) satisfies the assumptions of Lemma 2, then one can construct a \( \hat{Q} \) by increasing \( Q \) at the lower end of its domain as in Figure 1. Clearly, the resulting \( \hat{Q} \) is non-decreasing and, furthermore, implementable if the change is sufficiently small. It remains to verify that \( \hat{Q} \) also satisfies the (IA') constraint. Intuitively, agents have fewer incentives to acquire information if the outcome is less sensitive to a change in their private information, which is the case when they face a less steep allocation rule. I show in Lemma 7 that if \( \hat{Q} \) is less steep than \( Q \) in the sense that it differs from \( Q \) by a non-increasing function (as in Figure 1), then for any \( \alpha \), \( \hat{Q} \) gives agents a smaller marginal benefit of acquiring information. Hence, \( \hat{Q} \) satisfies the (IA') constraint as \( Q \).

The gap between Lemma 2 and Theorem 1 is that social surplus is not just a function of \( Q \) when \( \gamma > 0 \). We need to show that, for any ex-post allocation rule \( q \) implementing \( Q \), we can find a \( \hat{q} \) that implements \( \hat{Q} \) and yields higher social surplus. This is the result of Lemma 3.

**Lemma 3** Suppose two implementable allocation rules \( Q \) and \( \hat{Q} \) satisfy (5). Let \( q \) be an ex-post allocation rule that implements \( Q \), then there exists an ex-post allocation rule \( \hat{q} \) that
The proof of Lemma 3 relies on the following technical lemma. Let $m$ denote the probability measure on $W$ corresponding to $H(w_i|\alpha^*)$, then

**Lemma 4** Let $Q : W \rightarrow [0,1]$ be implementable and $\rho : W^n \rightarrow [0,1]$ be a symmetric measurable function, then there exists a symmetric ex post allocation rule $q$ implementing $Q$ such that $\sum_i q_i(w) \geq \rho(w)$ for almost all $w \in W^n$ if and only if for each measurable set $A \subset W$, the following inequality is satisfied

$$
\int_A Q(w_i)dm(w_i) \geq \frac{1}{n} \int_{A^n} \rho(w)dm^n(w). \tag{6}
$$

It is clear that inequality (6) is necessary. Suppose there exists an allocation rule $q$ such that $\sum_i q_i(w) \geq \rho(w)$ for almost all $w \in W^n$ implementing $Q$. For any measurable set $A \subset W$, the probability of an agent whose type is in $A$ getting the object, $n \int_A Q(w_i)dm(w_i)$, must exceed the probability of some agent getting the object when all agents’ types are in $A$, $\int_{A^n} \sum_i q_i(w)dm^n(w)$, which by assumption is greater than $\int_{A^n} \rho(w)dm^n(w)$. A similar argument to that in the proof of Proposition 3.1 in Border (1991) proves that inequality (6)
is also sufficient for such a $q$ to exist and implement $Q$. With Lemma 4 in hand, it is easy to prove Lemma 3.

**Proof of Lemma 3.** Consider two implementable allocation rules $Q$ and $\hat{Q}$ satisfying (5). Let $q$ be a symmetric ex-post allocation rule that implements $Q$. Define $\rho : W^n \to [0, 1]$ by $\rho(w) \equiv \sum_i q_i(w)$. Then

$$\int_A \hat{Q}(w_i)dm(w_i) \geq \int_A Q(w_i)dm(w_i) \geq \frac{1}{n} \int_A \rho(w)dm^n(w).$$

By Lemma 4 there exists an allocation rule $\hat{q}$ implementing $\hat{Q}$ such that $\sum_i \hat{q}_i(w) \geq \rho(w) = \sum_i q_i(w)$ for almost all $w \in W^n$. Hence,

$$\mathbb{E} \left[ \sum_i (w_i + \gamma \sum_{j \neq i} w_j)\hat{q}_i(w) \right| \alpha_i = \alpha^* \forall i]$$

$$= \sum_i \mathbb{E} \left[ (1 - \gamma)w_iQ(w_i) \right| \alpha_i = \alpha^*] + \mathbb{E} \left[ \left( \gamma \sum_i w_i \right) \left( \sum_i \hat{q}_i(w) \right) \right| \alpha_i = \alpha^* \forall i]$$

$$> \sum_i \mathbb{E} \left[ (1 - \gamma)w_iQ(w_i) \right| \alpha_i = \alpha^*] + \mathbb{E} \left[ \left( \gamma \sum_i w_i \right) \left( \sum_i q_i(w) \right) \right| \alpha_i = \alpha^* \forall i]$$

$$= \mathbb{E} \left[ \sum_i (w_i + \gamma \sum_{j \neq i} w_j)q_i(w) \right| \alpha_i = \alpha^* \forall i]$$.

This completes the proof. ■

### 3.2 Efficient Mechanisms for Fixed $\alpha^*$

I solve $(P')$ in two steps. In this section, I solve the following sub-problem for each $\alpha^* \in A$, denoted by $(P'-\alpha^*)$:

$$V(\alpha^*) \equiv \max_Q \mathbb{E} [wQ(w) | \alpha^*] \text{ subject to (F'), (MON) and (IA')}.$$ 

In Section 3.3, I will solve $\alpha^* \in \text{argmax}_{\alpha \in A} (1 - \gamma) V(\alpha) - C(\alpha)$. 

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Fix $\alpha^*$. If $\alpha^*$ is such that
\[
\mathbb{E} \left[ -\frac{H_\alpha(w|\alpha^*)}{h(w|\alpha^*)} H(w|\alpha^*)^{n-1} \bigg| \alpha^* \right] \leq C'(\alpha^*), \tag{7}
\]
then clearly the ex post efficient mechanism, i.e., $Q(w) = H(w|\alpha^*)^{n-1}$ for all $w$, solves $(P' - \alpha^*)$. Thus, in the remainder of this section, I assume $\alpha^*$ is such that (7) is violated.

I first present an informal argument to derive the optimal solution to $(P' - \alpha^*)$. If we ignore the monotonicity constraint (MON), then we can use the following Lagrangian relaxation to get an intuition of the optimal solution:
\[
\mathcal{L} \equiv \int_{w(0,\alpha^*)}^{w(1,\alpha^*)} \varphi^{\lambda_X}(H(w|\alpha^*))Q(w)h(w|\alpha^*)dw + \lambda_X C'(\alpha^*), \tag{8}
\]
where $\lambda_X > 0$ is the Lagrangian multiplier associated with (IA') and $\varphi^{\lambda_X}(t, \alpha^*)$ is defined by
\[
\varphi^{\lambda_X}(t, \alpha^*) \equiv H^{-1}(t|\alpha^*) + \lambda_X \frac{H_\alpha(H^{-1}(t|\alpha^*)|\alpha^*)}{h(H^{-1}(t|\alpha^*)|\alpha^*)}, \quad t \in [0, 1].
\]
Note that since $H(\cdot|\alpha)$ is strictly increasing, $\varphi^{\lambda_X}(\cdot, \alpha^*)$ is strictly increasing (or decreasing) if and only if $\varphi^{\lambda_X}(H(\cdot|\alpha^*), \alpha^*)$ is strictly increasing (or decreasing). If $\varphi^{\lambda_X}(H(\cdot|\alpha^*), \alpha^*)$ is strictly increasing, then the allocation rule $Q$ that maximizes $\mathcal{L}$ is non-decreasing and, more importantly, ex post efficient: $Q(w) = H(w|\alpha^*)^{n-1}$ for all $w$. However, this contradicts the assumption that condition (7) is violated. Hence, the Lagrangian multiplier $\lambda_X$ must be such that $\varphi^{\lambda_X}(H(\cdot|\alpha^*), \alpha^*)$ is not strictly increasing.

Suppose $\varphi^{\lambda_X}(H(\cdot|\alpha^*), \alpha^*)$ is non-decreasing but not strictly increasing. Let $Q|_{(w, \bar{w})}$ denote the restriction of $Q$ to an interval $(w, \bar{w})$. Intuitively, if $\varphi^{\lambda_X}(H(\cdot|\alpha^*), \alpha^*)$ is strictly increasing on an interval $(w, \bar{w})$, then it is optimal to choose $Q|_{(w, \bar{w})} = H(\cdot|\alpha^*)^{n-1}$ so that the allocation rule is ex post efficient on $(w, \bar{w})$: if $\varphi^{\lambda_X}(H(\cdot|\alpha^*), \alpha^*)$ is constant on an maximally chosen interval $(w, \bar{w})$, then any non-decreasing function $Q|_{(w, \bar{w})}$ that satisfies $H(w|\alpha^*)^{n-1} \leq Q(w) \leq H(\bar{w}|\alpha^*)^{n-1}$ for all $w \in (w, \bar{w})$ and $\int_{w}^{\bar{w}}[H(w|\alpha^*)^{n-1} - Q(w)]h(w|\alpha^*)dw = 0$ is optimal. More
formally, any non-decreasing implementable $Q$ maximizes $\mathcal{L}$ subject to $(F')$ and (MON) if and only if $Y(w(0, \alpha^*)) = 0$ and $Y(\cdot) > 0$ on $(\underline{w}, \overline{w})$ implies $\varphi^{\lambda_X}(H(\cdot|\alpha^*), \alpha^*)$ is constant on this interval. Suppose, in addition, that $Q$ satisfies $(IA')$ with equality, then $Q$ solves $(\mathcal{P}'-\alpha^*)$. This can be further illustrated by the following example.

**Example 3 (The Linear Experiments)** Consider the information structures in Example 1. Recall that in this example we have

$$\frac{-H_{\alpha_i}(w_i|\alpha_i)}{h(w_i|\alpha_i)} = \frac{w_i - \mu}{\alpha_i}. $$

Hence,

$$\varphi^{\lambda_X}(H(w|\alpha^*), \alpha^*) = w - \lambda_X \frac{w - \mu}{\alpha^*}. $$

Suppose $\alpha^*$ is such that condition (7) is violated. Then it must be that $\lambda_X \geq \alpha^*$ since otherwise $\varphi^{\lambda_X}(\cdot, \alpha^*)$ is strictly increasing. If $\lambda_X > \alpha^*$, then $\varphi^{\lambda_X}(\cdot, \alpha^*)$ is strictly decreasing. Hence, the $Q$ that maximizes $\mathcal{L}$ and satisfies the monotonicity constraint is constant. However, in this case $(IA')$ constraint does not bind, a contradiction to $\lambda_X > 0$. Hence, in optimum $\lambda_X = \alpha^*$ and $\varphi^{\lambda_X}(\cdot, \alpha^*) = \mu$, which is constant. Hence, any feasible monotonic allocation rule $Q$ satisfying condition (4) maximizes $\mathcal{L}$. If $Q$ also satisfies $(IA')$, then it solves $(\mathcal{P}'-\alpha^*)$. In particular, there exists $\xi \in [0, 1]$ such that

$$\hat{Q}(w) = \xi H(w|\alpha^*)^{n-1} + (1 - \xi) \frac{1}{n}$$

solves $(\mathcal{P}'-\alpha^*)$.

More generally, the arguments in Example 3 can be readily extended to prove Proposition 1.

**Proposition 1** Suppose the first-order approach is valid, and Assumption 2 holds. Suppose, in addition, that $\alpha^*$ is such that (7) is violated. Let $\lambda_X = b(\alpha^*)$ and $Q$ be a non-decreasing
implementable allocation rule. Then \( Q \) solves \((P' - \alpha^*)\) if and only if \( Y(w(0, \alpha^*)) = 0 \) and \( Q \) satisfies \((IA')\) with equality.

If Assumption 2 does not apply, then \( \varphi^{\lambda X}(\cdot, \alpha^*) \) is not monotonic. In this case, an optimal solution can be obtained by ironing \( \varphi^{\lambda X}(\cdot, \alpha^*) \) in the following procedure first introduced by Myerson (1981). For each \( t \in [0, 1] \), define

\[
J^{\lambda X}(t, \alpha^*) \equiv \int_0^t \varphi^{\lambda X}(\tau, \alpha^*) d\tau.
\]

Let \( \mathcal{J}^{\lambda X} \) denote the convex hull of \( J^{\lambda X} \), defined by

\[
\mathcal{J}^{\lambda X}(t, \alpha^*) \equiv \min \{ \beta J(t_1, \alpha^*) + (1 - \beta) J(t_2, \alpha^*) | t_1, t_2 \in [0, 1], \beta t_1 + (1 - \beta) t_2 = t \}.
\]

Since \( \mathcal{J}^{\lambda X}(\cdot, \alpha^*) \) is convex, it is continuously differentiable virtually everywhere. Define \( \overline{\varphi}^{\lambda X}(\cdot, \alpha^*) \) as follows. For each \( t \in (0, 1) \) such that \( \partial \mathcal{J}^{\lambda X}(t, \alpha^*)/\partial t \) exists, let \( \overline{\varphi}^{\lambda X}(t, \alpha^*) \equiv \partial \mathcal{J}^{\lambda X}(t, \alpha^*)/\partial t \). Then extend \( \overline{\varphi}^{\lambda X}(\cdot, \alpha^*) \) to \([0, 1]\) by right continuity. Then \( \overline{\varphi}^{\lambda X}(\cdot, \alpha^*) \) is non-decreasing. Note also that if \( J^{\lambda X}(H(w|\alpha^*), \alpha^*) < J^{\lambda X}(H(w', \alpha^*), \alpha^*) \) for some \( w' \in (w, \omega) \) implies that \( Q \) is constant on \((w, \omega)\). By a standard argument (see, e.g., Myerson (1981) and Toikka (2011)), an optimal \( Q \) must satisfy the following pooling property:

\( J^{\lambda X}(H(w|\alpha^*), \alpha^*) < J^{\lambda X}(H(w|\alpha^*), \alpha^*) \) for all \( w \in (w, \omega) \) implies that \( Q \) is constant on \((w, \omega)\). Therefore any non-decreasing implementable \( Q \) maximizes \( L \) subject to \((F')\) and \((MON)\) if and only if (i) \( Y(w(0, \alpha^*)) = 0 \); (ii) \( Q \) satisfies the pooling property; and (iii) \( Y(\cdot) > 0 \) on \((w, \omega)\) implies \( \overline{\varphi}^{\lambda X}(H(\cdot|\alpha^*), \alpha^*) \) is constant on this interval. Suppose, in addition, that \( Q \) satisfies \((IA')\) with equality, then \( Q \) solves \((P' - \alpha^*)\).

It remains to determine the multiplier \( \lambda_X \). In order to do so I first define two extreme allocation rules satisfying conditions (i)-(iii) given above. Define \( Q^+(\cdot, \lambda_X) \) as follows. If \( J^{\lambda X}(H(w|\alpha^*), \alpha^*) > J^{\lambda X}(H(w|\alpha^*), \alpha^*) \) for \( w \in (\omega, \omega) \) with \( \omega < \omega \) and let \((w, \omega)\) be chosen
maximally, then let

\[ Q^+(w, \lambda_X) \equiv \frac{1}{n}[H(\bar{w}|\alpha^*)^n - H(w|\alpha^*)^n]}{H(\bar{w}|\alpha^*) - H(w|\alpha^*)}, \quad \forall w \in (\bar{w}, \underline{w}). \]

Otherwise, let \( Q^+(w, \lambda_X) \equiv H(w|\alpha^*)^{n-1} \). Define \( Q^-(\cdot, \lambda_X) \) as follows. If \( \varphi^\lambda_X(\cdot|\alpha^*), \alpha^* \) is constant on \((w, \underline{w})\) with \( w < \underline{w} \) and let \((\bar{w}, \underline{w})\) be chosen maximally, then let

\[ Q^-(w, \lambda_X) \equiv \frac{1}{n}[H(\bar{w}|\alpha^*)^n - H(w|\alpha^*)^n]}{H(\bar{w}|\alpha^*) - H(w|\alpha^*)}, \quad \forall w \in (\bar{w}, \underline{w}). \]

Otherwise, let \( Q^-(z, \lambda_X, \alpha^*) \equiv H(z|\alpha^*)^{n-1} \). Clearly, \( Q^+ \) and \( Q^- \) are non-decreasing, implementable and satisfy conditions (i)-(iii). I demonstrate in Appendix A.2.2 that \( Q^+ \) is the “steepest” allocation rule and \( Q^- \) is the “least steep” allocation rule among all non-decreasing implementable \( Q^\prime \)’s satisfying conditions (i)-(iii) in the following sense: for all non-decreasing implementable \( Q^\prime \)’s satisfying conditions (i)-(iii),

\[
\mathbb{E}\left[ -\frac{H_\alpha(w|\alpha^*)}{h(w|\alpha^*)} Q^+(w, \lambda_X) \bigg| \alpha^* \right] \geq \mathbb{E}\left[ -\frac{H_\alpha(w|\alpha^*)}{h(w|\alpha^*)} Q(w) \bigg| \alpha^* \right] \geq \mathbb{E}\left[ -\frac{H_\alpha(w|\alpha^*)}{h(w|\alpha^*)} Q^-(w, \lambda_X) \bigg| \alpha^* \right].
\]

Hence, there exists a non-decreasing implementable \( Q \) satisfying conditions (i)-(iii) and (IA’) with equality if and only if \( \lambda_X \) is such that

\[
\mathbb{E}\left[ -\frac{H_\alpha(w|\alpha^*)}{h(w|\alpha^*)} Q^+(w, \lambda_X) \bigg| \alpha^* \right] \geq C'(\alpha^*) \geq \mathbb{E}\left[ -\frac{H_\alpha(w|\alpha^*)}{h(w|\alpha^*)} Q^-(w, \lambda_X) \bigg| \alpha^* \right]. \tag{9}
\]

Lemma 5 proves that such a \( \lambda_X \) exists and is unique:

**Lemma 5** Suppose the first-order approach is valid and Assumption 1 holds. Suppose, in addition, that \( \alpha^* \) is such that (7) is violated. There exists a unique \( \lambda_X > 0 \) such that inequality (9) holds.

The main result of this section is Theorem 2, which demonstrates that the allocation rules I have derived informally are indeed optimal solutions of \((P'\alpha^*)\):

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Theorem 2 Suppose the first-order approach is valid and Assumption 1 holds. Suppose, in addition, that $\alpha^*$ is such that (7) is violated. Let $\lambda_X > 0$ be such that inequality (9) holds and $Q$ be a non-decreasing implementable allocation rule. Then $Q$ solves ($P'\text{-}\alpha^*$) if and only if $Y(w(0, \alpha^*)) = 0$, and $Q$ satisfies (IA') with equality and the following two properties:

1. (pooling property) If $J^{\lambda_X}(H(w|\alpha^*), \alpha^*) > J^{\lambda_X}(H(w|\alpha^*), \alpha^*)$ for all $w \in (\underline{w}, \bar{w})$ with $\underline{w} < \bar{w}$ and let $(\underline{w}, \bar{w})$ be chosen maximally, then $Q$ is constant on $(\underline{w}, \bar{w})$.

2. If $Y(w) > 0$ for all $w \in (\underline{w}, \bar{w})$ with $\underline{w} < \bar{w}$ and let $(\underline{w}, \bar{w})$ be chosen maximally, then $\phi^{\lambda_X}(H(\cdot|\alpha^*), \alpha^*)$ is constant on $(\underline{w}, \bar{w})$.

My interpretation of the optimal pooling areas is as follows: Optimally, pooling occurs where $\phi^{\lambda_X}(H(\cdot|\alpha^*), \alpha^*)$ is not strictly increasing, i.e.,

$$
\frac{w_{s,\alpha}(s, \alpha^*)}{w_s(s, \alpha^*)} = \frac{\partial}{\partial w} \left[ -\frac{H_{\alpha}(w(s, \alpha^*)|\alpha^*)}{h(w(s, \alpha^*)|\alpha^*)} \right] \geq \frac{1}{\lambda_X}
$$

Recall that if an information structure is more precise, then $w(s, \alpha)$ changes more dramatically as $s$ changes, i.e., $w_s(s, \alpha)$ is larger. Hence, one can interpret $w_s(s, \alpha)$ as a local measure of the information structure’s precision around $s$. Then, $w_{s,\alpha}(s, \alpha)/w_s(s, \alpha)$ is the percentage change of the information structure’s precision around $s$ as $\alpha$ increases. Intuitively, the most effective way to discourage agents from acquire information is to introduce randomization to where the information structure’s precision increases most. Assumption 2 implies that $w_{s,\alpha}(s, \alpha)/w_s(s, \alpha)$ is a constant, i.e., when $\alpha$ increases, the information structure becomes more precision uniformly over $[0, 1]$.

Though intuitive, the proof of Theorem 2 is difficult because of the presence of both the non-standard constraint (IA’) and the monotonicity constraint. In this paper, I use the following approach first proposed by Reid (1968) and later introduced into the mechanism design problem by Mierendorff (2009). I first solve ($P'\text{-}\alpha^*$) under an additional restriction,
that $Q$ is Lipschitz continuous with global Lipschitz constant $K$:

$$|Q(w) - Q(w')| \leq K|w - w'|, \forall w, w' \in W.$$ 

Denote the above maximization problem by $(\mathcal{P}_K^{K^*})$. I show that the solutions of $(\mathcal{P}_K^{K^*})$ converge to that of $(\mathcal{P}^{K^*})$ as $K \to \infty$. This allows one to characterize the solutions of $(\mathcal{P}^{K^*})$.

The proof is not a straightforward modification of Mierendorff (2009). Let $Q$ and $Q^K$ denote the solutions to $(\mathcal{P}_K^{K^*})$ and $(\mathcal{P}^{K^*})$, respectively. In Mierendorff (2009), $Q$ is discontinuous at exactly one point, and it can be shown that for $K$ sufficiently large, the slope of $Q^K$ is equal to $K$ only in a neighborhood around the discontinuity point. In my model, however, $Q$ could be discontinuous at most countably many points. If $Q$ is discontinuous at $w$, then it is possible that every neighborhood of $w$ includes another discontinuity point. Hence, it is non-trivial to characterize $Q$ as the limit of $Q^K$.

Lastly, given Theorem 2, it is straightforward that there exists an optimal $Q$ which takes the following relatively simple form:

**Corollary 1** There exists $\xi \in [0, 1]$ such that

$$\hat{Q}(w_i) \equiv \xi Q^+(w_i, \lambda_X) + (1 - \xi)Q^-(w_i, \lambda_X)$$

solves $(\mathcal{P}^{K^*})$.

### 3.3 Optimal $\alpha^*$

Given optimal solutions of $(\mathcal{P}^{K^*})$, one can now study the optimal information choices, i.e., $\alpha^* \in \text{argmax}_{\alpha \in \Lambda} \pi^*(\alpha) \equiv (1 - \gamma)\bar{V}(\alpha) - C(\alpha)$. I show in Appendix A.2.2 that

$$\pi^*(\alpha) = (1 - \gamma) \int_{w(0,\alpha)}^{w(1,\alpha)} \frac{\pi^X(H(w|\alpha), \alpha)H(w|\alpha)^{\alpha - 1}h(w|\alpha)dw + (1 - \gamma)\lambda_X C'(\alpha) - C(\alpha),}{\lambda_X}$$

where, as demonstrated in Theorem 2, $\lambda_X$ depends on $\alpha$ in a complex way. In general, it is hard to solve the optimal $\alpha^*$. In this section, I first give a condition that the optimal $\alpha^*$
must satisfy and then solve it under Assumption 2.

In Appendix A.3, I show that if the second-order condition of the agents’ optimization problem is satisfied, then

$$\int_{w(0,\alpha)}^{w(1,\alpha)} -H_\alpha(w|\alpha)H(w|\alpha)^{n-1}dw - C'(\alpha) \text{ is non-increasing in } \alpha.$$

(10)

Let

$$\hat{\alpha} \equiv \inf \{ \alpha \in \mathbb{A} | (7) \text{ holds for } \alpha \}.$$

(11)

Then $\hat{\alpha}$ is independent of $\gamma$ and $\lim_{n \to \infty} \hat{\alpha} = \alpha$. By (10), inequality (7) holds for all $\alpha > \hat{\alpha}$. I claim the socially optimal $\alpha^* \leq \hat{\alpha}$. Note first that for all $\alpha > \hat{\alpha}$, the solution to $(P' - \alpha^*)$ is $Q(w) = H(w|\alpha)^{n-1}$. Hence, the average social surplus is

$$\pi^s(\alpha) = (1 - \gamma) \int_{w(0,\alpha)}^{w(1,\alpha)} wH(w|\alpha)^{n-1}h(w|\alpha)dw - C(\alpha).$$

Taking derivative with respect to $\alpha$ gives

$$\pi^{st}(\alpha) = (1 - \gamma) \int_{w(0,\alpha)}^{w(1,\alpha)} -H_\alpha(w|\alpha)H^{n-1}(w|\alpha)dw - C'(\alpha).$$

Since $C'(\alpha)$ is strictly increasing, $\pi^{st}(\alpha)$ is strictly decreasing by (10). By construction, $\pi^{st}(\hat{\alpha}) = -\gamma C'(\hat{\alpha}) \leq 0$. Hence, $\pi^{st}(\alpha) < 0$ for all $\alpha > \hat{\alpha}$ and at optimum $\alpha^* \leq \hat{\alpha}$. This result is summarized in the following proposition:

**Proposition 2** Suppose the first-order approach is valid and Assumption 1 holds. At optimum, $\alpha^* \leq \hat{\alpha}$, where $\hat{\alpha}$ is such that (7) holds with equality if $\alpha = \hat{\alpha}$.

Proposition 2 states the social planner will not encourage agents to acquire more information than they do under the ex post efficient mechanism, which is not surprising given the results of Bergemann et al. (2009). This also implies that (IA’) always holds with equality when $\alpha^*$ is chosen optimally. Hence, it is sufficient to consider the one-sided first-order
To obtain further results about the socially optimal information choice $\alpha^*$, I assume that Assumption 2 holds for the rest of this section. In this case, the average social surplus is

$$\pi_s(\alpha) = (1 - \gamma) \left[ \frac{\mu}{n} + b(\alpha)C''(\alpha) \right] - C'(\alpha), \text{ for } \alpha \in [\underline{\alpha}, \hat{\alpha}],$$

Then

$$\pi_s'(\alpha) = [(1 - \gamma)b'(\alpha) - 1]C'(\alpha) + (1 - \gamma)b(\alpha)C''(\alpha).$$

Hence, $\pi_s'(\alpha) \leq 0$ if and only if $1 - C'(\alpha)/[b'(\alpha)C'(\alpha) + b(\alpha)C''(\alpha)] \leq \gamma$. Assume that $C''(\alpha)/[b'(\alpha)C'(\alpha) + b(\alpha)C''(\alpha)]$ is strictly increasing. Then there exists a unique $\alpha^{\circ} \in \mathbb{A}$ such that $\pi_s'(\alpha) \leq 0$ if and only if $\alpha \geq \alpha^\circ$. Furthermore, $\alpha^\circ$ is strictly decreasing in $\gamma$ and independent of $n$, and $\lim_{n \to \infty} \alpha^\circ = \underline{\alpha}$. Then the socially optimal information choice is $\alpha^* = \min\{\alpha^\circ, \hat{\alpha}\}$. This result is summarized by the following proposition:

**Proposition 3** Suppose the first-order approach is valid and Assumption 2 holds. Suppose, in addition, $C'(\alpha)/[b'(\alpha)C'(\alpha) + b(\alpha)C''(\alpha)]$ is strictly increasing. At optimum, $\alpha^*(n, \gamma) = \min\{\alpha^\circ(\gamma), \hat{\alpha}(n)\}$, where $\alpha^\circ$ is strictly decreasing in $\gamma$, $\lim_{n \to \infty} \alpha^\circ = \underline{\alpha}$ and $\lim_{n \to \infty} \hat{\alpha} = \underline{\alpha}$.

Proposition 3 implies that the ex post efficient mechanism is also ex ante efficient if the level of interdependency is low. As the level of interdependency increases, the socially optimal information choice decreases, and an ex ante efficient mechanism introduces more randomization into the allocation rule to discourage agents from acquiring information. Proposition 3 also implies that the ex post efficient mechanism is also ex ante efficient for a sufficiently large number of agents. Intuitively, when there is a large number of agents, the incentive for each agent to acquire information is already small and there is no need for the social planner to further discourage them from acquiring information by distorting the allocation rule.
Example 4 (The Linear Experiments) Assume that $F(\theta) = \theta$ with support $[0, 1]$. Assume that the cost function (used in Persico (2000)) is of the form

$$C(\alpha) = K(\alpha - \alpha)^2, \quad \forall \alpha \in [\alpha, 1],$$

where $0 < \alpha < 1$ and $K > 0$. Assume $8K\alpha \geq 1$. Then, as I demonstrate in Appendix A.3, the first-order approach is valid. In this case $\hat{\alpha}$ is such that

$$2K(\alpha - \alpha) = \frac{n - 1}{2n(n + 1)}.$$

Note that the left-hand side of the above equation is strictly increasing in $\alpha$; and the right-hand side is strictly decreasing in $n$ for $n \geq 2$ and converges to 0 as $n$ goes to infinity. Hence, $\hat{\alpha}$ is strictly decreasing in $n$ and goes to $\alpha$ as $n$ grows to infinity. Finally,

$$\pi^s(\alpha) = 2K [\gamma\alpha - (2\gamma - 1)\alpha].$$

If $\gamma \leq \frac{1}{2}$, then $\pi^s(\alpha) \geq 0$ and therefore $\alpha^* = \hat{\alpha}$. If $\gamma > \frac{1}{2}$, then $\pi^s(\alpha)$ is strictly decreasing and therefore

$$\alpha^* = \min \left\{ \frac{\gamma\alpha}{2\gamma - 1}, \hat{\alpha} \right\}.$$

Thus, $\alpha^* = \hat{\alpha}$ for $\gamma$ sufficiently small and/or $n$ sufficiently large, the ex post efficient mechanism is also ex ante efficient. For $\gamma$ sufficiently large and/or $n$ sufficiently small, the optimal $\alpha^*$ is strictly decreasing in $\gamma$ and goes to $\alpha$ when $\gamma$ increases to 1.

4 Efficient Asymmetric Mechanisms

In this section, I study ex ante efficient mechanisms without imposing symmetry restrictions. First, I show that the result in Theorem 1 is still valid for general asymmetric mechanisms, i.e., an ex ante efficient mechanism never withholds the object. Second, I derive
ex ante efficient mechanisms under Assumption 2 and present conditions under which the socially optimal information choices are the same for all agents and there exists a symmetric ex ante efficient mechanism.

As in the symmetric case, I consider the relaxed problem of the social planner by replacing the (IA) constraint by the (IA′) constraint and focus on reduced form auctions. Let \( Q \equiv (Q_1, \ldots, Q_n) \), where \( Q_i : [\theta, \theta] \to [0, 1] \) is non-decreasing for all \( i \). By Theorem 3 in Mierendorff (2011), \( Q \) is implementable if and only if it satisfies

\[
\sum_{i=1}^{n} \int_{w_i}^{w(1, \alpha_i^*)} Q_i(z_i) dH(z_i|\alpha_i^*) \leq 1 - \prod_{i=1}^{n} H(w_i|\alpha_i^*), \quad \forall w \in \prod_{i=1}^{n} [w(0, \alpha_i^*), w(1, \alpha_i^*)].
\]

(AF′)

Thus, given the (MON) constraint, we can replace constraint (F) by constraint (AF′). Finally, as in the symmetric case, an ex ante efficient mechanism never withholds the object:

**Theorem 3** Suppose \( \alpha^* \) and \( (q, t) \) solve the relaxed problem of the social planner, then

\[
\sum_{i} q_i(w) = 1 \text{ for almost all } w \in \prod_{i=1}^{n} [w(0, \alpha_i^*), w(1, \alpha_i^*)].
\]

(12)

Using Theorem 3 and Law of iterated expectations, we can rewrite the social planner’s objective function as a function of \( Q \):

\[
\mathbb{E} \left[ \sum_{i} (w_i + \gamma \sum_{j \neq i} w_j) q_i(w) \middle| \alpha_i = \alpha_i^* \forall i \right] = \sum_{i} \mathbb{E} \left[ (1 - \gamma) w_i Q_i(w_i) \middle| \alpha_i = \alpha_i^* \right] + n\gamma \mathbb{E}[\theta].
\]

As in the symmetric case, we ignore the second term, \( n\gamma \mathbb{E}[\theta] \), which is a constant, from hereon. To summarize, the social planner’s relaxed problem, denoted by \( (\mathcal{P}') \), becomes:

\[
\max_{\alpha^*, Q} \mathbb{E} \left[ \sum_{i} (1 - \gamma) w_i Q_i(w_i) \middle| \alpha_i = \alpha_i^* \forall i \right] - \sum_{i} C(\alpha_i^*),
\]
subject to

\[ \sum_{i=1}^{n} \int w_i \, Q_i(z_i) \, dH(z_i|\alpha_i^*) \leq 1 - \prod_{i=1}^{n} H(w_i|\alpha_i^*), \forall \mathbf{w} \in \prod_{i=1}^{n} [w(0, \alpha_i^*), w(1, \alpha_i^*)]. \]  \tag{AF'}

\[ Q_i(w_i) \text{ is non-decreasing in } w_i, \quad \text{(MON)} \]

\[ \mathbb{E} \left[ -\frac{H_{\alpha_i}(w_i|\alpha_i^*)}{h(w_i|\alpha_i^*)} Q_i(w_i) \, \bigg| \alpha_i = \alpha_i^* \right] \leq C'(\alpha_i^*). \]  \tag{IA'}

As in the symmetric case, I solve \((P')\) in two steps. First, for each \(\alpha^* \in \mathbb{A}^n\), I solve the following sub-problem, denoted by \((P' - \alpha^*)\):

\[ V(\alpha^*) \equiv \max \mathbb{E} \left[ \sum_i w_i Q_i(w_i) \right| \alpha^* \text{ subject to } (AF'), (MON) \text{ and } (IA'), \]

Second, I solve \(\alpha^* \in \arg \max_{\alpha \in \mathbb{A}^n} \pi^*(\alpha) \equiv (1 - \gamma)V(\alpha) - \sum_i C(\alpha_i).

Fix \(\alpha^*\). If \(\alpha^*\) is such that

\[ \mathbb{E} \left[ -\frac{H_{\alpha_i}(w_i|\alpha_i^*)}{h(w_i|\alpha_i^*)} \prod_{i \neq j} H(w_j|\alpha_j^*) \bigg| \alpha^* \right] \leq C'(\alpha_i^*), \forall i, \tag{13} \]

then clearly the ex post efficient mechanism, i.e., \(Q_i(w_i) = \prod_{j \neq i} H(w_i|\alpha_i^*)\) for all \(i\) and all \(w\), solves \((P' - \alpha^*)\). Assume \(\alpha^*\) is such that \((13)\) is violated.

Suppose there exists \(0 < k \leq n\) such that the \((IA')\) constraints bind for the first \(k\) agents. Then we can ignore the \((IA')\) constraints for the last \(n - k\) agents. Let \(\lambda_i\) denote the Lagrangian multiplier associated with the \((IA')\) constraint for agent \(i \ (i \leq k)\). By a similar argument to that in Section 3.2, we have \(\lambda_i = b(\alpha_i^*)\). The Lagrangian relaxation becomes

\[ \mathcal{L} = \sum_{i \leq k} \int w_i Q_i(w_i) \, h(w_i|\alpha_i) \, dw_i + \sum_{i > k} \int w_i Q_i(w_i) \, h(w_i|\alpha_i) \, dw_i + \sum_{i \leq k} b(\alpha_i^*) C'(\alpha_i^*). \]

Then, for \(i > k\), it is optimal to set \(Q_i(w_i) = \prod_{j > k, j \neq i} H(w_i|\alpha_j^*)\) if \(w_i > \mu\) and \(Q_i(w_i) = 0\) if
\(w_i < \mu\). Hence,

\[
V(\alpha^*) = \mu \prod_{i > k} H(\mu|\alpha_i^*) + \sum_{i > k} \int_0^{\mu} w_i \prod_{j > k, j \neq i} H(w_i|\alpha_j^*) h(w_i|\alpha_i) dw_i + \sum_{i \leq k} b(\alpha_i^*) C'(\alpha_i^*). \\
= \bar{\theta} - \int_0^{\bar{\theta}} \prod_{i > k} H(w|\alpha_i^*) dw,
\]

where the second line follows from integration by parts. Finally, (IA') holds for \(i > k\) if and only if

\[
\int_0^{w(1,\alpha_i^*)} -H_{\alpha_i}(w_i|\alpha_i^*) \prod_{j > k, j \neq i} H(w_i|\alpha_j^*) dw_i \leq C'(\alpha_i^*). \tag{14}
\]

Consider an agent \(i\) \((i > k)\). I show that if \(\alpha^*\) is chosen optimally, then (14) holds with equality. Suppose to the contrary that (14) holds with strict inequality, then

\[
\frac{\partial \pi^*(\alpha^*)}{\partial \alpha_i} = -(1 - \gamma) \int_0^{w(1,\alpha_i^*)} H_{\alpha_i}(w_i|\alpha_i^*) \prod_{j > k, j \neq i} H(w_i|\alpha_j^*) dw_i - C'(\alpha_i^*) < -\gamma C'(\alpha_i^*) \leq 0,
\]

a contradiction to the optimality of \(\alpha_i^*\). Hence, (14), i.e., the (IA') constraints hold with equality for all \(i > k\). By Assumption 2, we have

\[
\pi^*(\alpha) = (1 - \gamma) \left[ \mu + \sum_{i=1}^{n} b(\alpha_i) C'(\alpha_i) \right] - \sum_{i} C(\alpha_i).
\]

Let \(\alpha^o \in \arg\max_{\alpha \in \mathcal{A}} (1 - \gamma) b(\alpha) C'(\alpha) - C(\alpha)\). Clearly, if \(\alpha^o \leq \hat{\alpha}\), where \(\hat{\alpha}\) is define by (11), then the socially optimal information choices are \(\alpha_i^o = \alpha^o\) for all \(i\). This result is summarized by the following proposition:

**Proposition 4** Suppose the first-order approach is valid and Assumption 2 holds. Suppose, in addition, \(\alpha^o \leq \hat{\alpha}\) where \(\alpha^o \in \arg\max_{\alpha \in \mathcal{A}} (1 - \gamma) b(\alpha) C'(\alpha) - C(\alpha)\) and \(\hat{\alpha}\) is define by (11). Then then the socially optimal information choices are the same for all agents: \(\alpha_i^* = \alpha^o\) for all \(i\).

Note that if the assumptions in Proposition 4 are satisfied, then there exists a symmetric
ex ante efficient mechanism.

5 Conclusion

I have studied ex ante efficient mechanisms in an auction setting in which agents have positively interdependent values and information is endogenous. Specifically, I assume agents are initially uncertain about the value of the object on sale, and they are able to pay a cost to acquire information about this value before participating in a mechanism. In an earlier paper, Bergemann and Välimäki (2002) find that using the ex post efficient mechanism will lead to ex ante over investment in information by agents. This suggests the potential gain in ex ante efficiency by adjusting the ex post efficient mechanism in a way to discourage agents from gathering information.

In the main body of the paper I restrict attention to symmetric mechanisms and symmetric equilibria in which all agents acquire the same amount of information. First, I demonstrate an ex ante efficient mechanism never withholds the object. Intuitively, whenever the object is withheld, one can instead allocate it randomly among agents. This improves the allocation efficiency without giving agents additional incentive to acquire information. This result also allows me to write the social planner’s problem as a function of interim allocation rule, which is more convenient to work with.

Second, I fully characterize ex ante efficient mechanisms. When the interdependence is low and/or the number of agents is large, the ex-post efficient mechanism is also ex ante efficient. When the interdependence is high and/or the number of agents is small, an ex ante efficient mechanism involves randomization. Specifically, an ex ante efficient interim allocation rule randomizes in areas in which the accuracy of an agent’s posterior estimate can be significantly improved if an additional piece of information in acquired.

In this paper, I assume all agents simultaneously acquire information prior to the auction. One interesting direction for future research is to allow for the possibility of sequential
information acquisition. It is likely that the efficiency can be improved if agents are asked to acquire information in turn, and one’s information acquisition decision can depend on the signals received by those who take action earlier. Another interesting direction for future research is to consider the impact of initial private information. In this paper I only considered static mechanisms in which agents only report their private information once. In general, one can consider a dynamic mechanism in which agents report their private information both before and after acquiring information.

A Appendix

Proof of Lemma 1. Note that by construction $H(w(s,\alpha)|\alpha) = s$. Taking derivative of both sides of the equation with respect to $\alpha$ yields

$$h(w(s,\alpha)|\alpha)w_\alpha(s,\alpha) + H_\alpha(w(s,\alpha)|\alpha) = 0.$$ 

Hence,

$$-\frac{H_\alpha(w(s,\alpha)|\alpha)}{h(w(s,\alpha)|\alpha)} = w_\alpha(s,\alpha).$$

Suppose $-H_\alpha(w|\alpha)/h(w|\alpha)$ is strictly increasing in $w$. Then $w_\alpha(s,\alpha)$ is strictly increasing in $s$. Hence, for all $s, s' \in (0, 1)$, $s' > s$ and $\alpha' > \alpha''$ we have

$$w(s',\alpha') - w(s',\alpha'') = \int_{\alpha''}^{\alpha'} w_\alpha(s',\alpha)d\alpha$$

$$> \int_{\alpha''}^{\alpha'} w_\alpha(s,\alpha)d\alpha$$

$$= w(s',\alpha') - w(s',\alpha'').$$

\[\square\]

Lemma 6 Let $(c,d)$ be an interval of the real line, and $J$ and $H$ be two non-decreasing
functions. Assume that for some measure $m$ on $\mathbb{R}$ we have

$$\int_{c}^{d} H(w) \, dm(w) = 0.$$  

Then $\int_{c}^{d} H(w) J(w) \, dm(w) \geq 0$.

**Proof.** This lemma is a corollary of Lemma 1 in Persico (2000).

**Lemma 7** Suppose $J : [\theta, \bar{\theta}] \to \mathbb{R}$ is non-decreasing on $[w(0, \alpha_i), w(1, \alpha_i)]$, then

$$\left[ - \frac{H_{\alpha_i}(w_i|\alpha_i)}{h(w_i|\alpha_i)} J(w_i) \right] \alpha_i \geq 0,$$

where the equality holds if $J$ is constant.

**Proof.** Since $J$ and $-\frac{H_{\alpha_i}(w_i|\alpha_i)}{h(w_i|\alpha_i)}$ are non-decreasing on $[w(0, \alpha_i), w(1, \alpha_i)]$, it suffices to show that

$$\int_{w(0, \alpha_i)}^{w(1, \alpha_i)} H_{\alpha_i}(w_i|\alpha_i) \, dw_i = 0.$$  

(17)

On the one hand since

$$\int_{w(0, \alpha_i)}^{w(1, \alpha_i)} H(w_i|\alpha_i) \, dw_i = w_i H(w_i|\alpha_i) \bigg|_{w(0, \alpha_i)}^{w(1, \alpha_i)} - \int_{w(0, \alpha_i)}^{w(1, \alpha_i)} w_i \, dH(w_i|\alpha_i),$$

$$= w(1, \alpha_i) - E[\theta].$$

Taking derivative with respect to $\alpha_i$ yields

$$\frac{\partial}{\partial \alpha_i} \int_{w(0, \alpha_i)}^{w(1, \alpha_i)} H(w_i|\alpha_i) \, dw_i = w_{\alpha_i}(1, \alpha_i).$$

(18)

On the other hand, by the chain rule, we have

$$\frac{\partial}{\partial \alpha_i} \int_{w(0, \alpha_i)}^{w(1, \alpha_i)} H(w_i|\alpha_i) \, dw_i = w_{\alpha_i}(1, \alpha_i) + \int_{w(0, \alpha_i)}^{w(1, \alpha_i)} H_{\alpha_i}(w_i|\alpha_i) \, dw_i.$$  

(19)
Comparing (18) and (19) proves (17). By Lemma 6, inequality (16) holds. If \( J \) is constant, the equality holds by (17). ■

**Proof of Lemma 2.** Define \( \hat{w}^b \equiv \sup \{ w_i | Y(w_i') > 0 \forall w(0, \alpha^*) \leq w'_i \leq w_i \} \). By the continuity of \( Y \), we have \( Y(\hat{w}^b) = 0 \) and \( \hat{w}^b > w(0, \alpha^*) \). There are four cases to consider.

**Case I:** Suppose there exists \( w_i' \in (w(0, \alpha^*), \hat{w}^b) \) such that \( Q \) is discontinuous at \( w_i' \).

Let \( Q(w_i'^+) \) denote the right-hand limit of \( Q \) at \( w_i' \), and \( Q(w_i'^-) \) the corresponding left-hand limit. Let \( 0 < \varepsilon \leq \min \left\{ \min_{w(0, \alpha^*) \leq w_i \leq w_i'} \frac{Y(w_i)}{H(w_i' | \alpha^*)}, Q(w_i'^+) - Q(w_i'^-) \right\} \). Let \( \hat{Q} \) be such that \( \hat{Q}(w_i) = Q(w_i) \) for \( w_i \leq w(0, \alpha^*) \) and for \( w_i > w(0, \alpha^*) \)

\[
\hat{Q}(w_i) = Q(w_i) + \varepsilon \chi_{\{w_i \leq w_i'\}},
\]

where \( \chi_{\{w_i \leq w_i'\}} \) is an indicator function. By construction, \( \hat{Q}(w) \geq Q(w) \) for all \( w \in W \) and the inequality holds strictly on a positive measure set. We now verify that \( \hat{Q} \) satisfies (MON), (IA') and (F'). Clearly, \( \hat{Q} \) is non-decreasing, i.e., satisfies (MON). Since \( \chi_{\{w_i \leq w_i'\}} \) is non-increasing on \([w(0, \alpha^*), w(1, \alpha^*)]\), by Lemma 7, we have

\[
\mathbb{E} \left[ -\frac{H_{\alpha_i}(w_i | \alpha^*)}{h(w_i | \alpha^*)} \hat{Q}(w_i) \bigg| \alpha_i = \alpha^* \right] = \mathbb{E} \left[ -\frac{H_{\alpha_i}(w_i | \alpha^*)}{h(w_i | \alpha^*)} Q(w_i) \bigg| \alpha_i = \alpha^* \right] + \varepsilon \mathbb{E} \left[ -\frac{H_{\alpha_i}(w_i | \alpha^*)}{h(w_i | \alpha^*)} \chi_{\{w_i \leq w_i'\}} \bigg| \alpha_i = \alpha^* \right] \leq C'(\alpha^*) + 0 = C'(\alpha^*).
\]

Hence, \( \hat{Q} \) satisfies (IA'). Finally, let

\[
\hat{Y}(w_i) \equiv \int_{w_i}^{\hat{w}^b} \left[ H(z | \alpha^*)^{n-1} - \hat{Q}(z) \right] h(z | \alpha^*) dz.
\]

If \( w_i \leq w_i' \), then \( \hat{Y}(w_i) = Y(w_i) - \varepsilon [H(w_i' | \alpha^*) - H(w_i | \alpha^*)] \geq Y(w_i) - \varepsilon H(w_i' | \alpha^*) \geq 0 \). If \( w_i > w_i' \), then \( \hat{Y}(w_i) = Y(w_i) \geq 0 \). Hence, \( \hat{Q} \) satisfies (F').

**Case II:** Suppose \( Q \) is continuous on \([w(0, \alpha^*), \hat{w}^b]\). We first show that there exists
Note that if \( w_i \in (w(0, \alpha^*), w^b) \) such that \( Q(w_i') < Q(w^b) \). Suppose to the contrary that \( Q(w_i) = Q(w^b) \)
for all \( w_i \in (w(0, \alpha^*), w^b) \). If \( Q(w^b) \geq H(w^b) |\alpha^*|^{n-1}, \) then \( Y(w(0, \alpha^*)) = \int_{w(0, \alpha^*)}^{w^b} [H(z|\alpha^*|^{n-1}) - Q(z)] h(z|\alpha^*) dz < 0, \) a contradiction. If \( Q(w^b) < H(w^b) |\alpha^*|^{n-1}, \) then, by the continuity of \( Q \) and \( H \), there exists \( \delta > 0 \) such that \( Q(w_i) < H(w_i|\alpha^*|^{n-1} \) for all \( w_i \in [w^b, w^b + \delta] \). Moreover,

\[
0 = Y(w^b) = \int_{w^b}^{w^b + \delta} [H(z|\alpha^*|^{n-1} - Q(z)] h(z|\alpha^*) dz + Y(w^b + \delta) > Y(w^b + \delta),
\]
a contradiction. Thus, there exists \( w_i' \in (w(0, \alpha^*), w^b) \) such that \( Q(w_i') < Q(w^b) \).

By the continuity of \( Q \), there exists \( w''_i \in (w_i', w^b) \) such that \( Q(w_i'') = \frac{1}{2} (Q(w_i') + Q(w^b)) \).

Let \( 0 < \varepsilon \leq \min \left\{ \min_{w(0, \alpha^*) \leq w_i \leq w''_i} \frac{Y(w_i)}{H(w_i|\alpha^*)}, Q(w''_i) - Q(w_i') \right\} \). Let

\[
\hat{Q}(w_i) \equiv \begin{cases} 
\max\{Q(w_i') + \varepsilon, Q(w_i)\} & \text{if } w_i > w_i', \\
Q(w_i) + \varepsilon & \text{if } w(0, \alpha^*) < w_i \leq w_i', \\
Q(w_i) & \text{if } w_i \leq w(0, \alpha^*). 
\end{cases}
\]

Note that if \( w_i \geq w_i'' \) then \( Q(w_i) \geq Q(w_i'') \geq Q(w_i') + \varepsilon \). Hence, \( \hat{Q}(w_i) = Q(w_i) \) for \( w_i \geq w_i'' \). By construction, \( \hat{Q}(w) \geq Q(w) \) for all \( w \in W \) and the inequality holds strictly on a positive measure set. We now verify that \( \hat{Q} \) satisfies (MON), (IA') and (F'). Clearly, \( \hat{Q} \) is non-decreasing, i.e., satisfies (MON). It is easy to verify that \( \hat{Q} - Q \) is non-increasing on \( [w(0, \alpha^*), w(1, \alpha^*)] \) and therefore \( \hat{Q} \) satisfies (IA') by Lemma 7. Finally, if \( w_i \geq w_i'' \), then \( \hat{Y}(w_i) = Y(w_i) \). If \( w_i < w_i'' \), then

\[
\hat{Y}(w_i) = \int_{w_i}^{w_i''} [H(z|\alpha^*|^{n-1} - \hat{Q}(z)] h(z|\alpha^*) dz,
\]

\[
= Y(w_i) - \int_{w_i}^{w_i''} [\hat{Q}(z) - Q(z)] h(z|\alpha^*) dz,
\]

\[
\geq Y(w_i) - \varepsilon [H(w_i'')|\alpha^*| - H(w_i'|\alpha^*|],
\]

\[
\geq Y(w_i) - \varepsilon H(w_i'')|\alpha^*| \geq 0.
\]
Hence, \( \hat{Q} \) satisfies (F').

**Case III:** Suppose \( Q \) is continuous on \([w(0, \alpha^*), w^b]\) and \( Q(w^b^-) < H(w^b|\alpha^*)^{n-1} \). Define \( R(w_i) \equiv Y(w_i)/(H(w^b|\alpha^*) - H(w_i|\alpha^*)) \) for \( w_i < w^b \). Then by L'Hopital's rule,

\[
\lim_{w_i \to w^b^-} R(w_i) = H(w^b|\alpha^*)^{n-1} - Q(w^b^-) > 0.
\]

Let \( 0 < \varepsilon \leq \min \{ \inf_{w(0, \alpha^*) \leq w_i < w^b} R(w_i), Q(w^b^+) - Q(w^b^-) \} \). Let \( \hat{Q} \) be such that \( \hat{Q}(w_i) \equiv Q(w_i) \) for \( w_i \leq w(0, \alpha^*) \) and \( \hat{Q}(w_i) = Q(w_i) + \varepsilon \chi_{\{w_i < w^b\}} \) for \( w_i > w(0, \alpha^*) \). By construction, \( \hat{Q}(w) \geq Q(w) \) for all \( w \in W \) and the inequality holds strictly on a positive measure set. One can verify that \( \hat{Q} \) satisfies (MON) and (IA') following the arguments in Case I. Finally, if \( w_i < w^b \), then \( \hat{Y}(w_i) = Y(w_i) - \varepsilon [H(w^b|\alpha^*) - H(w_i|\alpha^*)] \geq Y(w_i) - R(w_i) [H(w^b|\alpha^*) - H(w_i|\alpha^*)] = 0 \). If \( w_i \geq w^b \), then \( \hat{Y}(w_i) = Y(w_i) \geq 0 \). Hence, \( \hat{Q} \) satisfies (F').

**Case IV:** Suppose \( Q \) is continuous on \([w(0, \alpha^*), w^b]\) and \( Q(w^b^-) \geq H^{n-1}(w^b|\alpha^*) \). We first show that \( Q(w^b^-) = H^{n-1}(w^b|\alpha^*) \). Suppose to the contrary that \( Q(w^b^-) > H^{n-1}(w^b|\alpha^*) \). Then by the continuity of \( Q \) and \( H \) on \([w(0, \alpha^*), w^b]\), there exists \( \delta > 0 \) such that \( Q(w_i) > H^{n-1}(w_i|\alpha^*) \) for all \( w_i \in (w^b - \delta, w^b) \). Then

\[
Y(w^b - \delta) = \int_{w^b - \delta}^{w^b} [H(z|\alpha^*)^{n-1} - Q(z)]h(z|\alpha^*)dz + Y(w^b) < 0,
\]

a contradiction. Hence, \( Q(w^b^-) = H^{n-1}(w^b|\alpha^*) \). Second, we show that there exists \( w'_i \in (w(0, \alpha^*), w^b) \) such that \( Q(w'_i) < Q(w^b^-) \). Suppose to the contrary that \( Q(w_i) = Q(w^b^-) \) for all \( w_i \in (w(0, \alpha^*), w^b) \), then \( Y(w(0, \alpha^*)) = \int_{w(0, \alpha^*)}^{w^b} [H^{n-1}(z|\alpha^*) - Q(z)]h(z|\alpha^*)dz + Y(w^b) < 0 \), a contradiction. Hence, there exists \( w'_i \in (w(0, \alpha^*), w^b) \) such that \( Q(w'_i) < Q(w^b^-) \). The rest of the proof follows that of Case II. \( \blacksquare \)
A.1 Proof of Lemma 4

The proof follows from a similar argument to that in Border (1991). Before proceeding to the proof, I first introduce some notations and definitions. Let $\sigma_{i,j} : W^n \to W^n$ denote the function that interchanges the $i$th and the $j$th coordinates, i.e.,

$$\sigma_{i,j}(w_1, \ldots, w_n) = (w_1, \ldots, w_{i-1}, w_j, w_{i+1}, \ldots, w_{j-1}, w_i, w_{j+1}, \ldots, w_n), \quad \forall w = (w_1, \ldots, w_n),$$

and $\sigma_i \equiv \sigma_{1,i}$. We say an allocation rule $q$ is symmetric if $q_1$ is such that $q_1(w) = q_1(\sigma_{i,j}(w))$ for all $i, j \neq 1$, $q_i(w) = q_1(\sigma_i(w))$ and $\sum_i q_i(w) \leq 1$ for all $w$. We say a mechanism $(q, t)$ is symmetric if its allocation rule $q$ is symmetric.

Let $\mathcal{D}_1$ denote the set of symmetric allocation rules $q$ satisfying $\sum q_i(w) \geq \rho(w)$ for almost all $w \in W^n$. Let $\mathcal{D}$ collect all $Q : W \to [0, 1]$ such that there exists $q \in \mathcal{D}_1$ that implements $Q$. Let $\langle f, g \rangle$ denote $\int f(w)g(w)dm(w)$. Define for each measurable set $A \subset W$

$$B(A) \equiv \frac{1}{n} \int_{A^n} \rho(w)dm^n(w),$$

and

$$B(A) \equiv \frac{m^n \left( \cup_j \sigma_j(A \times W^{n-1}) \right)}{n}.$$

I break the proof down into several lemmas. The first lemma states that condition (6) is necessary.

**Lemma 8** For all measurable $A \subset W$, all $Q \in \mathcal{D}$

$$\langle \chi_A, Q \rangle \geq B(A).$$

**Proof.** Since $Q \in \mathcal{D}$, there exists a symmetric allocation rule $q$ satisfying $\sum q_i(w) \geq \rho(w)$
for almost all \( w \in W^n \) and implementing \( Q \). For each \( i \),

\[
\langle \chi_A, Q \rangle = \int_A Q(w_i)dm(w_i),
\]
\[
= \int_{\sigma(A \times W^{n-1})} q_i(w)dm^n(w),
\]
\[
\geq \int_{A^n} q_i(w)dm^n(w). \tag{20}
\]

Hence,

\[
n\langle \chi_A, Q \rangle \geq \sum_i \int_{A^n} q_i(w)dm^n(w),
\]
\[
= \int_{A^n} \sum_i q_i(w)dm^n(w),
\]
\[
\geq \int_{A^n} \rho(w)dm^n(w). \tag{21}
\]

That is, \( \langle \chi_A, Q \rangle \geq B(A) \). Inequality (21) holds with equality if \( \sum_i q_i(w) = \rho(w) \) for all \( w \in A^n \). \( \square \)

Each \( \langle \chi_A, \cdot \rangle \) defines a function on \( \mathcal{D} \), and according to Lemma 8, this function is bounded below by \( B(A) \). We now show that this bound is achieved by a hierarchical reduced form.

**Lemma 9** Let \( A_1, \ldots, A_L, A_{L+1}, \ldots, A_{L+K} \) be a partition of \( W \), and define the hierarchical auctions generated by \( A_1, \ldots, A_L, A_{L+1}, \ldots, A_{L+K} \), denoted \( q^{A_1, \ldots, A_{L+K}} \), by

\[
q_i^{A_1, \ldots, A_{L+K}}(w) = \begin{cases} 
\frac{1}{\# \{j : w_j \in A_k\}} & \text{if } k = 1, \ldots, L, w_i \in A_k \text{ and } w_j \notin A_1 \cup \cdots \cup A_{k-1} \forall j \neq i, \\
\frac{\rho(w)}{\# \{j : w_j \in A_k\}} & \text{if } k = L + 1, \ldots, L + K, w_i \in A_k \text{ and } w_j \notin A_1 \cup \cdots \cup A_{k-1} \forall j \neq i, \\
0 & \text{otherwise.}
\end{cases}
\]

Let \( Q^* \) be the reduced form of the hierarchical auction \( q^{A_1, \ldots, A_{L+K}} \). For each \( j = 1, \ldots, L + K \), set \( F_j = A_1 \cup \cdots \cup A_j \); then for each \( j = 1, \ldots, L \),

\[
\langle \chi_{F_j}, Q^* \rangle = B(F_j); \tag{22}
\]
and for each \( j = L, \ldots, L + K - 1 \),

\[
\langle \chi_{F_j^c}, Q^* \rangle = B(F_j^c).
\]  

(23)

**Proof.** Condition (22) follows from Lemma 5.2 in Border (1991). For each \( i \), if \( w \in \sigma_i(F_j^c \times W^{n-1}) \setminus (F_j^c)^n \), then \( w_i \not\in F_j \) and there exists \( i \neq i \) such that \( w_i \in F_j \). By definition, \( q_i^{A_1, \ldots, A_{L+K}}(w) = 0 \). Hence, inequality (20) holds with equality if \( A = F_j^c \). Furthermore, simple calculations imply that \( \sum_{i=1}^n q_i^{A_1, \ldots, A_{L+K}} \) equals to \( \rho \) on \( (F_j^c)^n \) for each \( j = L, \ldots, L + K - 1 \). Hence, inequality (21) holds with equality if \( A = F_j^c \) for each \( j = L, \ldots, L + K - 1 \), and so

\[
\langle \chi_{F_j^c}, Q^* \rangle = B(F_j^c).
\]

This proves (23).  

Lemma 8 shows that each \( Q \in \mathcal{D} \) satisfies condition (6). The next lemma shows that if a simple function separates \( Q \) from \( \mathcal{D} \), then \( Q \) violates (6) on some measurable set of types.

**Lemma 10** Let \( \overline{Q} : W \to [0, 1] \) be implementable and suppose the simple function \( \sum_{j=1}^{L+K} \beta_j \chi_{A_j} \) separates \( \overline{Q} \) from \( \mathcal{D} \). That is, for all \( Q \in \mathcal{D} \),

\[
\left\langle \sum_{j=1}^{L+K} \beta_j \chi_{E_j}, \overline{Q} \right\rangle < \left\langle \sum_{j=1}^{L+K} \beta_j \chi_{E_j}, Q \right\rangle.
\]

(24)

Then for some measurable \( A \subset W \), \( \langle \chi_A, Q \rangle < B(A) \).

**Proof.** Without loss of generality, take the \( E_j \)'s to a partition of \( W \) and numbered so that \( \beta_1 > \cdots > \beta_K > 0 \geq \beta_{K+1} > \cdots > \beta_{K+L} \). To see that at least one \( \beta_j > 0 \) consider a symmetric allocation rule \( \overline{q} \) implementing \( \overline{Q} \). Let \( \overline{p}(w) = \sum_i \overline{q}_i(w) \), which is invariant to permutations of \( w \). Define an allocation rule \( \overline{q} \) by

\[
q_i(w) = \begin{cases} 
\frac{\overline{q}_i(w)}{\overline{p}(w)} & \text{if } \overline{p}(w) > 0, \\
\frac{1}{n} & \text{if } \overline{p}(w) = 0,
\end{cases}
\]

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and $q_i(w) = q_1(\sigma_i(w))$. Clearly, $q$ is symmetric. Moreover, $\sum_i q_i(w) = \sum_i (1/n) = 1 \geq \rho(w)$ for $w$ such that $\rho(w) = 0$ and

$$\sum_i q_i(w) = \sum_i \frac{q_1(\sigma_i(w))}{\rho(\sigma_i(w))},$$

$$= \sum_i \frac{q_i(w)}{\rho(w)},$$

$$= \frac{1}{\rho(w)} \sum_i q_i(w),$$

$$= 1 \geq \rho(w),$$

for $w$ such that $\rho(w) > 0$. Let $Q$ be the reduced form of $q$, then $Q \in \mathcal{D}$ and $Q \geq \overline{Q}$. This implies that at least one $\beta_j > 0$.

Let $A_j = E_{K+L+1-j}$ for $j = 1, \ldots, K + L$ and $Q^*$ be the reduced form of hierarchical auctions generated by $A_1, \ldots, A_{L+K}$. If for any $k = 1, \ldots, K$, we have $\langle \chi_{E_1 \cup \cdots \cup E_k}, \overline{Q} - Q^* \rangle < 0$, then we are done, since

$$\langle \chi_{E_1 \cup \cdots \cup E_k}, Q^* \rangle = \langle \chi_{(A_1 \cup \cdots \cup A_{L+K-k})^c}, Q^* \rangle,$$

$$= B((A_1 \cup \cdots \cup A_{L+K-k})^c),$$

$$= B(E_1 \cup \cdots \cup E_k),$$

where the second equality holds by Lemma 9. Suppose to the contrary that

$$\langle \chi_{E_1 \cup \cdots \cup E_k}, \overline{Q} - Q^* \rangle \geq 0$$

(25)

for all $k = 1, \ldots, K$. To simplify notation, let $Q_j^* = \langle \chi_{E_j}, Q^* \rangle$ and $\overline{Q}_j = \langle \chi_{E_j}, \overline{Q} \rangle$. Then (24) implies that

$$\sum_{j=1}^{L+K} \beta_j (Q_j^* - \overline{Q}_j) > 0.$$
Taking the $j = 1$ term to the right and dividing by $\beta_1 > 0$ yields

$$
\sum_{j=2}^{L+K} \frac{\beta_j}{\beta_1} (Q^*_j - \overline{Q}_j) > \overline{Q}_1 - Q^*_1 \geq 0,
$$

where the second inequality follows from (25) for $k = 1$. If $K > 1$, then $\alpha_2 > 0$, so $\beta_1/\beta_2 > 1$, and multiplying the left-hand side of the above inequality by $\beta_1/\beta_2$ strengthens it. Then, taking the $j = 2$ term to the right and dividing by $\beta_2 > 0$ yields:

$$
\sum_{j=3}^{L+K} \frac{\beta_j}{\beta_2} (Q^*_j - \overline{Q}_j) > (\overline{Q}_1 - Q^*_1) + (\overline{Q}_2 - Q^*_2) \geq 0,
$$

where the second inequality follows from (25) for $k = 2$. Continue in this fashion until arriving at

$$
\sum_{j=K+1}^{L+K} \frac{\beta_j}{\beta_K} (Q^*_j - \overline{Q}_j) > (\overline{Q}_1 - Q^*_1) + \ldots + (\overline{Q}_K - Q^*_K) \geq 0, \quad (26)
$$

where the second inequality follows from (25) for $k = K$. Note that if $L = 0$, then the left-hand side of (26) is equal to 0, a contradiction. Suppose that $L \geq 1$. For $k = 1, \ldots, L$

$$
\sum_{j=L+K+1-k}^{L+K} Q^*_j = \langle \chi_{E_{L+K+1-k} \cup \ldots \cup E_{L+K}}, Q^* \rangle,
$$

$$
= \langle \chi_{A_1 \cup \ldots \cup A_k}, Q^* \rangle,
$$

$$
= \bar{B}(A_1 \cup \ldots \cup A_k),
$$

where the second equality holds by Lemma 9. For $k = 1, \ldots, L$,

$$
\sum_{j=L+K+1-k}^{L+K} \overline{Q}_j = \langle \chi_{E_{L+K+1-k} \cup \ldots \cup E_{L+K}}, \overline{Q} \rangle
$$

$$
= \langle \chi_{A_1 \cup \ldots \cup A_k}, \overline{Q} \rangle
$$

$$
\leq \bar{B}(A_1 \cup \ldots \cup A_k),
$$

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where the last equality holds by Proposition 3.1 in Border (1991). Hence, $\sum_{j=L+K+1-k}^{L+K} (Q_j^* - \overline{Q}_j) \geq 0$ for $k = 1, \ldots, L$. Since $\beta_K > 0 \geq \beta_{K+1} > \cdots > \beta_{K+L}$, the left-hand side of (26) satisfies

$$\sum_{j=K+1}^{L+K} \frac{\beta_j}{\beta_K} (Q_j^* - \overline{Q}_j) + \frac{\beta_{K+2} - \beta_{K+1}}{\beta_K} \sum_{j=K+2}^{L+K} (Q_j^* - \overline{Q}_j) + \cdots + \frac{\beta_{K+L} - \beta_{K+L-1}}{\beta_K} (Q_{K+L}^* - \overline{Q}_{K+L}) \leq 0,$$

a contradiction. Hence, $\langle \chi_{E_1 \cup \cdots \cup E_k}, Q \rangle < \langle \chi_{E_1 \cup \cdots \cup E_k}, Q^* \rangle = B(E_1 \cup \cdots \cup E_k)$ for some $k = 1, \ldots, K$, completing the proof of the lemma. ■

To complete the proof of Theorem 4, we must show that if $\overline{Q} \notin \mathcal{D}$, then it is separated from $\mathcal{D}$ by a simple function. This will be accomplished using a separating hyperplane argument, after establishing some topological preliminaries. The set $\mathcal{D}$ of implementable functions is clearly convex. Consider $\mathcal{D}$ as a subset of $L_\infty (m)$, the set of $m$-essentially bounded measurable functions on $W$ and $\mathcal{D}_1$ as a subset of $L_\infty (m^n)^n$. For $1 \leq p < \infty$, let $L_p (m)$ denote the set of measurable functions whose absolute value raised to the $p$-th power has finite integral. For brevity denote $L_p (m^n)^n$ by $L_p^n$ and $L_p (m)$ by $L_p$.

Since $L_\infty$ is the dual of $L_1$ under the duality $\langle f, g \rangle = \mathbb{E}_{w_i} [f(w_i)g(w_i)|\alpha_i = \alpha^*]$, topologize $L_\infty$ with its weak*, or $\sigma(L_\infty, L_1)$, topology. Similarly give $L_\infty^n$ its $\sigma(L_\infty^n, L_1^n)$ topology.

Equation (2) with $\alpha_j = \alpha^*$ for all $j \neq i$ defines a function $\Lambda : \mathcal{D}_1 \to \mathcal{D}$ mapping each $q$ to its reduced form. The next lemma describes the main topological results.

**Lemma 11**

\begin{align*}
\mathcal{D}_1 & \text{ is } \sigma(L_\infty^n, L_1^n) \text{ compact.} \tag{27} \\
\Lambda : \mathcal{D}_1 \to \mathcal{D} & \text{ is } \sigma(L_\infty^n, L_1^n), \sigma(L_\infty, L_1) \text{ continuous.} \tag{28} \\
\mathcal{D} & \text{ is } \sigma(L_\infty, L_1) \text{ compact.} \tag{29}
\end{align*}
Proof. Since $\mathcal{D}_1$ is a subset of the unit ball of $L^n_\infty$ under the norm given by $||q|| = \max_i ||q_i||_{\infty}$, by the Banach-Alaoglu theorem, for (27) we need only prove that $\mathcal{D}_1$ is $\sigma(L^n_\infty, L^n_1)$ closed. Let $q''$ be a net in $\mathcal{D}_1$ converging in the $\sigma(L^n_\infty, L^n_1)$ topology to $q$. First we verify that $q_i(w) = q_1(\sigma_i(w))$. Observe that $\sigma_i : W^n \to W^n$ is measurable, $\sigma_i = (\sigma_i)^{-1}$, and $m^n \circ \sigma_i = m^n$. Since each $q'' \in \mathcal{D}_1$, for any $f \in L^n_1$, we have

$$\int_{W^n} f(w)q''_i(w)dm^n(w) = \int_{W^n} f(w)q'_1(\sigma_i(w))dm^n(w),$$

$$= \int_{W^n} f(\sigma_i(w))q''_i(w)dm^n(w).$$

Taking limits on each side yields

$$\int_{W^n} f(w)q_i(w)dm^n(w) = \int_{W^n} f(\sigma_i(w))q_1(w)dm^n(w).$$

Transforming variables on the right-hand side, we get

$$\int_{W^n} f(w)q_i(w)dm^n(w) = \int_{W^n} f(w)q_1(\sigma_i(w))dm^n(w).$$

Since $f$ is arbitrary, conclude that $q_i = q_1 \circ \sigma_i$. Similar arguments show that $q_1(w) = q_1(\sigma_{i,j}(w))$ for all $i, j \neq 1$ and $\sum_i q_i \geq \rho$, and so $q \in \mathcal{D}_1$. This completes the proof of (27).

To prove continuity of $\Lambda$, let $q'' \to q$ in the $\sigma(L^n_\infty, L^n_1)$ topology. Let $f \in L^1_n$ and define $\tilde{f} \in L^n_1$ by $\tilde{f}(w) = f(w_1)$. Then

$$\langle \Lambda q'', f \rangle = \int_{W^n} \tilde{f}(w)q''_i(w)dm^n(w).$$

Since $q'' \to q$, the right-hand side converges to $\int \tilde{f}q_idm^n = \langle \Lambda q, f \rangle$. Since $f$ is arbitrary, $\Lambda q'' \to \Lambda q$ in the $\sigma(L_\infty, L_1)$ topology, proving (28).

This shows that $\mathcal{D}$ is compact, since continuous images of compact sets are compact. ■

Lemma 8 shows that the condition (6) is necessary for implementability. To see the con-
verse, suppose \( Q \notin \mathcal{D} \). Since \( \mathcal{D}_1 \) is \( \sigma(L_\infty^n, L_1^n) \) compact and convex, a separating hyperplane theorem implies that there exists a nonzero \( f \in L_1 \) satisfying \( \langle f, Q \rangle > \max\{ \langle f, Q \rangle : Q \in \mathcal{D} \} \).

Since simple functions are norm dense in \( L_1 \), we may take \( f \) to be simple. By Lemma 10 then \( \langle \chi_A, Q \rangle > B(A) \) for some measurable \( A \subset W \), violating the feasibility inequality (6).

Thus we have shown that if \( Q \) satisfies (6) for all measurable \( A \subset W \) then \( Q \in \mathcal{D} \subset L_\infty^n \).

Strictly speaking, an element of \( L_\infty^n \) is not a measurable function, but an equivalent class of measurable functions, where two functions are equivalent if they differ only on a set of measure zero. That is, we have shown that if \( Q \) satisfies (6) for all measurable \( A \subset W \) then there exists a symmetric allocation rule \( q^* \) satisfying \( \sum_i q^*_i(w) \geq \rho(w) \) whose reduced form \( Q^* \) agrees with \( Q \) almost everywhere. The following argument shows that we can modify \( q^* \) on a set of measure zero to implement \( Q \).

Let \( A^* \equiv \{ w_i : Q(w_i) \neq Q^*(w_i) \} \) and define \( q_1 : W \to [0, 1] \) by

\[
q_1(w) = \begin{cases} 
q^*_1(w) & \text{if } w \in (W\setminus A^*)^n, \\
Q(w_1) & \text{if } w \in A^* \times (W\setminus A^*)^{n-1}, \\
0 & \text{otherwise.} 
\end{cases} 
\]  

(30)

Then, for all \( w_1 \in W \),

\[
\int_{W^{n-1}} q_1(w_1, w_2, \ldots, w_n)dm^{n-1}(w_2, \ldots, w_n) = Q(w_1). \]  

(31)

To see this, first note that if \( w_1 \notin A^* \), then \( q_1 \) agrees with \( q_1^* \) unless some \( w_2, \ldots, w_n \) belongs to \( A^* \), a \( m^{n-1} \)-measure zero event. Since \( q^* \) implements \( Q^* \) and \( Q^* \) agrees with \( Q \) outside \( A^* \), (31) holds. If \( w_1 \in A^* \), then \( q_1(w) = Q^*(w_1) \) unless some \( w_2, \ldots, w_n \) belongs to \( A^* \), a \( m^{n-1} \)-measure zero event, and therefore (31) holds.

Finally, we show that setting \( q_i(w) = q_1(\sigma_i(w)) \) yields a symmetric allocation rule \( q : W^n \to [0, 1]^n \) satisfying \( \sum_i q_i(w) \geq \rho(w) \) for almost all \( w \) and implementing \( Q \). Clearly, \( q \) is symmetric, \( \sum_i q_i(w) \geq \rho(w) \) for almost all \( w \) and by (31) it implements \( Q \). It remains
to show that \( q \) is feasible. If \( w \in (W\setminus A^*)^n \), then \( q(w) = q^*(w) \) by construction and \( \sum_i q_i(w) = \sum_i q_i^*(w) \leq 1 \). If \( w \notin (W\setminus A^*)^n \), there are two cases. If more than one of \( w_1, \ldots, w_n \) belongs to \( A^* \), then \( q_1(w) = 0 \) by construction and hence each \( q_i(w) = 0 \). If exactly one of \( w_1, \ldots, w_n \) belongs to \( A^* \), say \( w_i \in A^* \), then \( q_i(w) = Q(w_i) \leq 1 \) and \( q_j(w) = 0 \) for \( i \neq j \). This shows that the modified \( q \) is feasible and completes the proof.

A.2 Solving \((P'-\alpha^*)\)

Recall that the sub-problem \((P'-\alpha^*)\) is

\[
V(\alpha^*) \equiv \max_Q \mathbb{E}\left[wQ(w)\middle|\alpha^*\right],
\]

subject to

\[
Y(w) \equiv \int_w^\bar{\theta} [H(z|\alpha^*)]^{n-1} - Q(z)h(z|\alpha^*)dz \succeq 0, \ \forall w \in [\theta, \bar{\theta}].
\]  

\[
Q(w) \text{ is non-decreasing in } w, \tag{MON}
\]

\[
\mathbb{E}\left[-\frac{H_\alpha(w|\alpha^*)}{h(w|\alpha^*)}Q(w)\middle|\alpha^*\right] \leq C'(\alpha^*). \tag{IA'}
\]

To simplify notation, denote \( w(0,\alpha^*) \) by \( \underline{w} \), \( w(1,\alpha^*) \) by \( \overline{w} \), \( h(z|\alpha^*) \) by \( h(z) \), \( H(z|\alpha^*) \) by \( H(z) \) and \( H_\alpha(z|\alpha^*) \) by \( H_\alpha(z) \). Let \( X(w) \equiv \int_0^w H_\alpha(z)Q(z)dz \). Then this is a control problem with state variables \( X \), \( Y \) and \( Q \), and a control variable \( a \geq 0 \). The evolution of the state variables is governed by

\[
X'(w) = H_\alpha(w)Q(w), \tag{32}
\]

\[
Y'(w) = -[H(w)]^{n-1} - Q(w)h(w), \tag{33}
\]

\[
Q'(w) = a(w), \tag{34}
\]

where the last equality holds if \( Q(w) \) is differentiable at \( w \). The non-negativity constraint
for $a$ guarantees that $Q$ is non-decreasing. This implies some regularity on $Q$, but still leaves some problems to apply control theory directly. First, we have to allow for (upward) jumps in the state variable $Q$. Second, $Q$ is not guaranteed to be piecewise continuous and piecewise continuously differentiable.

These problems can be circumvented by solving the maximization problem under the restriction that $Q$ is Lipschitz continuous with global Lipschitz constant $K$:

$$Q \in \mathcal{L}^K = \{Q : W \to [0, 1] \mid |Q(z) - Q(z')| \leq K|z - z'| \ \forall z, z' \in [0, 1]\}.$$

We define the maximization problem $(P^{K-\alpha^*})$ as $(P^{\prime-\alpha^*})$ subject to an additional constraint $Q \in \mathcal{L}^K$. It will be shown that optimal solutions of $(P^{K-\alpha^*})$ converge to an optimal solution of $(P^{\prime-\alpha^*})$ as $K \to \infty$. Using Lipschitz functions is convenient to show existence because $\mathcal{L}^K$ is sequentially compact.

We call a $Q$ that satisfies (MON), (F$'$) and (IA$'$) a solution of $(P^{\prime-\alpha^*})$ and a $Q$ that maximizes $\mathbb{E}[wQ(w) | \alpha^*]$ subject to (MON), (F$'$) and (IA$'$) an optimal solution of $(P^{\prime-\alpha^*})$. Similarly, we call a $Q \in \mathcal{L}^K$ that satisfies (MON), (F$'$) and (IA$'$) a solution of $(P^{K-\alpha^*})$ and a $Q \in \mathcal{L}^K$ that maximizes $\mathbb{E}[wQ(w) | \alpha^*]$ subject to (MON), (F$'$) and (IA$'$) an optimal solution of $(P^{K-\alpha^*})$. The following proof of existence is based on Mierendorff (2009).

Lemma 12

1. An optimal solution of $(P^{\prime-\alpha^*})$ exists.

2. For every $K > 0$, an optimal solution of $(P^{K-\alpha^*})$ exists.

Proof.

1. Let $\{Q^\nu\}$ be a sequence of solutions of $(P^{\prime-\alpha^*})$ such that

$$\int_{\mathcal{W}} zQ^\nu(z)h(z)dz \to V(\alpha^*).$$

By Helly’s selection theorem, there exists a subsequence $\{Q^{\nu_\infty}\}$ and a non-decreasing function $Q$ such that $Q^{\nu_\infty}$ converges pointwise to $Q$. Let $\mathcal{D}$ collect all $Q : W \to [0, 1]$
that satisfies (F'), (MON) and (IA'). Consider $\mathcal{D}$ as a subset of $L_2(m)$. Recall that $m$ is the probability measure on corresponding to $H(z)$. Then $\mathcal{D}$ is $\sigma(L_2, L_2)$ compact by a proof similar to that of Lemma 11. Therefore, after taking subsequences again, $Q^{\nu_\infty}$ converges to $Q$ in $\sigma(L_2, L_2)$ topology and $Q \in \mathcal{D}$. Since $z \in L_2(m)$ and $h \in L_2(m)$, the weak convergence of $\{Q^{\nu_\infty}\}$ implies that

$$\int_{\mathbb{R}} zQ(z)h(z)dz = V(\alpha^*).$$

2. Let $\{Q^{\nu}\}$ be a sequence of solutions of $(\mathcal{P}^{K-\alpha^*})$ such that

$$\int_{\mathbb{R}} zQ^{\nu}(z)h(z)dz \to V^K(\alpha^*).$$

After taking subsequences we can assume that $Q^{\nu}$ converges to $Q$ pointwise and in $\sigma(L_2, L_2)$ topology, and $Q \in \mathcal{D}$ as in part 1. Since $Q^{\nu} \in L^K$, for all $z, z' \in W$

$$|Q(z) - Q(z')| = \lim_{\nu \to \infty} |Q^{\nu}(z) - Q^{\nu}(z')| \leq K|z - z'|.$$

Hence $Q \in L^K$.

The next step is to show that the optimal solutions of $(\mathcal{P}^{K-\alpha^*})$ converges to an optimal solution of $(\mathcal{P}'-\alpha^*)$. The proof is based on Reid (1968) and Mierendorff (2009).

**Lemma 13** Let $\{Q^{\nu}\}$ be a sequence of optimal solutions of $(\mathcal{P}^{K-\alpha^*})$ where $K^{\nu} \to \infty$ as $\nu \to \infty$. Then there exists a solution $Q$ of $(\mathcal{P}'-\alpha^*)$ and a subsequence $Q^{\nu_\infty}$ such that $Q^{\nu_\infty}$ converges to $Q$ for almost every $z \in W$. Furthermore, this solution is optimal, i.e.,

$$\int_{\mathbb{R}} zQ(z)h(z)dz = V(\alpha^*).$$

**Proof.** After taking a subsequence, we can assume that $Q^{\nu}$ converges pointwise to a solution
\(Q\) of \((P' - \alpha^*)\) (see the proof of Lemma 12). To show optimality of \(\hat{Q}\), let \(Q\) be an optimal solution of \((P' - \alpha^*)\). We can extend \(Q\) to \(\mathbb{R}\) by setting \(Q(z) \equiv 0\) for \(z < w\) and \(Q(z) \equiv 1\) for \(z > w\). Define \(Q_d : \mathbb{R} \to [0, 1]\) as

\[
Q_d(z) = \frac{1}{d} \int_{z-d}^{z} Q(\zeta) d\zeta.
\]

By the Lebesgue differentiation theorem (see, e.g., Theorem 3.21 in Folland (1999)), \(Q_d(z) \to Q(z)\) for almost every \(z \in W\) as \(d \to 0\). Since \(Q\) is non-decreasing and \(Q(z) \in [0, 1]\), \(Q_d\) is non-decreasing and \(Q_d(z) \in [0, 1]\). Furthermore, \(Q_d \in L^1\): For all \(z > z'\),

\[
0 \leq Q_d(z) - Q_d(z') = \frac{1}{d} \left( \int_{z-d}^{z} Q(\zeta) d\zeta - \int_{z'-d}^{z'} Q(\zeta) d\zeta \right) = \frac{1}{d} \left( \int_{z'}^{z} Q(\zeta) d\zeta - \int_{z'-d}^{z-d} Q(\zeta) d\zeta \right) \leq \frac{1}{d} \int_{z'}^{z} Q(\zeta) d\zeta \leq \frac{1}{d} (z - z').
\]

Finally, \(Q_d\) is feasible: For all \(w \in W\),

\[
\int_{w}^{w} \left[ H(z)^{n-1} - Q_d(z) \right] h(z) dz = \int_{w}^{w} \left[ H(z)^{n-1} - \frac{1}{d} \int_{z-d}^{z} Q(\zeta) d\zeta \right] h(z) dz \geq \int_{w}^{w} \left[ H(z)^{n-1} - Q(z) \right] h(z) dz \geq 0,
\]

where the first inequality holds since \(Q\) is non-decreasing, and the second inequality holds since \(Q\) satisfies \((F')\). Next, define \(\tilde{Q}_d \equiv Q_d\) if \(-\int_{w}^{w} H_{\alpha}(z)Q_d(z) dz \leq C'(\alpha^*)\) and otherwise \(\tilde{Q}_d = \beta_d Q_d + (1 - \beta_d)/n\), where

\[
\beta_d = \frac{C'(\alpha^*)}{-\int_{w}^{w} H_{\alpha}(z)Q_d(z) dz}.
\]

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Then by Lemma 7, $-\int \omega H_{\alpha}(z)\bar{Q}_d(z)dz \leq C'(\alpha^*)$. Thus $\bar{Q}_d$ is a solution of $\mathcal{P}^{\frac{1}{2}-\alpha^*}$. Since $\beta_d \to 0$, $\bar{Q}_d \to Q$ almost everywhere as $d \to 0$. By the dominated convergence theorem,

$$\int \omega zQ^d(z)h(z)dz \to \int \omega zQ(z)h(z)dz,$$

and

$$\int \omega zQ^\nu(z)h(z)dz \to \int \omega \hat{Q}(z)h(z)dz.$$

Let $d_\nu = 1/K_\nu$. Then for all $\nu$,

$$\int \omega zQ^{d_\nu}(z)h(z)dz \leq \int \omega zQ^\nu(z)h(z)dz.$$

Hence,

$$\int \omega z\hat{Q}(z)h(z)dz = \int \omega zQ(z)h(z)dz.$$

\[\Box\]

**A.2.1 Solution in the class $L^K$**

The problem $(\mathcal{P}^{K-\alpha^*})$ can be summarized as follows:

$$\max_{X,Y,Q,a} \int \omega zQ(z)h(z)dz,$$
subject to

\[ X'(z) = H_{a_1}(z)Q(z), \]

\[ Y'(z) = -[H(z)^{n-1} - Q(z)]h(z), \]

\[ Q'(z) = a(z), \]

\[ X(w) = 0, \quad X(\overline{w}) \geq -C'(\alpha^*), \]

\[ Y(w) = 0, \quad Y(\overline{w}) = 0, \]

\[ Q(w) \geq 0, \quad Q(\overline{w}) \leq 1, \]

\[ 0 \leq a(z) \leq K, \]

\[ Y(z) \geq 0. \]

We say that some property holds virtually everywhere if the property is fulfilled at all \( z \) except for a countable number of \( z \)'s. We use the following abbreviation for "virtually everywhere": v.e. By Theorem 6.7.15 in Seierstad and Sydsæter (1987), we have

**Lemma 14** Let \((X,Y,Q,a)\) be an admissible pair that solves \((P^K-\alpha^*)\). Then there exist a number \( \lambda_0 \), vector functions \((\lambda_X, \lambda_Y, \lambda_Q)\) and \((\eta_a, \eta_a)\) and a non-decreasing function \( \eta_Y \), all

\[ 0 \leq a(z) \leq K, \]

\[ Y(z) \geq 0. \]
having one-sided limits everywhere, such that the following condition holds:

\[
\lambda_0 = 0 \text{ or } \lambda_0 = 1, \quad (40)
\]

\[
(\lambda_0, \lambda_X(z), \check{\lambda}_Y(z), \lambda_Q(z), \eta_Y(w) - \eta_Y(w)) \neq 0, \forall z, \quad (41)
\]

\[
\lambda_Q(z)a(z) \geq \lambda_Q(z)a, \forall a \in (0, K), \text{v.e.} \quad (42)
\]

\[
\lambda_Q(z) - \eta_a(z) + \eta_a(z) = 0. \quad (43)
\]

\[
\eta_Y(z) \text{ is constant on any interval where } Y(z) > 0. \quad (44)
\]

\[
\lambda_X(z) \text{ and } \lambda_Q(z) \text{ are continuous.} \quad (45)
\]

\[
\lambda'_X(z) = 0, \text{ v.e.} \quad (46)
\]

\[
\lambda'_Q(z) = - \left[ \lambda_0 z + \lambda_X(z) \frac{H_{\alpha}(z)}{h(z)} + \check{\lambda}_Y(z) \right] h(z) + \eta_Y(z)h(z), \text{ v.e.} \quad (47)
\]

\[
\check{\lambda}_Y(z) + \eta_Y(z) \text{ is continuous,} \quad (48)
\]

\[
\check{\lambda}'_Y(z) + \eta'_Y(z) = 0, \text{ v.e.} \quad (49)
\]

\[
\lambda_X(w) \geq 0 (= 0 \text{ if } X(w) > -C'(\alpha^*)) \quad (50)
\]

\[
\lambda_Q(w) \leq 0 (= 0 \text{ if } Q(w) < 1), \quad (51)
\]

\[
\lambda_Q(w) \leq 0 (= 0 \text{ if } Q(w) > 0). \quad (52)
\]

\[
\eta_a(z) \geq 0 (= 0 \text{ if } a(z) > 0), \quad (53)
\]

\[
\eta_a(z) \geq 0 (= 0 \text{ if } a(z) < K). \quad (54)
\]

In what follows, I assume that \((X, Y, Q, a)\) is an admissible pair that solves \((\mathcal{P}^K-\alpha^*)\) and \((X, Y, Q, a, \lambda_0, \lambda_X, \check{\lambda}_Y, \lambda_Q, \eta_a, \eta_Y)\) satisfy the conditions in Lemma 14.

Since \(\lambda_X\) is continuous and \(\lambda'_X(z) = 0\) virtually everywhere, \(\lambda_X(z)\) is constant on \([w, \bar{w}]\), denoted by \(\lambda_X\). Then (50) is equivalent to

\[
\lambda_X \geq 0 (= 0 \text{ if } X(w) > -C'(\alpha^*)).
\]

Similarly, since \(\check{\lambda}_Y + \eta_Y\) is continuous and \(\check{\lambda}'_Y(z) + \eta'_Y(z) = 0\) virtually everywhere, \(\check{\lambda}_Y(z) + \eta_Y(z) = 0\) virtually everywhere,
$\eta_Y(z)$ is constant on $[w, \overline{w}]$. We can assume without loss of generality that $\tilde{\lambda}_Y(z) + \eta_Y(z) = 0$ for all $z \in [w, \overline{w}]$. Let $\lambda_Y \equiv 2\tilde{\lambda}_Y$. Then $\eta_Y = -\lambda_Y/2$ and condition (44) is equivalent to

$$\lambda_Y(z) \text{ is constant on any interval where } Y(z) > 0,$$

and (47) is equivalent to

$$\lambda_Q'(z) = - \left[ \lambda_0 z + \lambda_X(z) \frac{H_{\alpha_i}(z)}{h(z)} + \lambda_Y(z) \right] h(z), \text{ v.e.}$$

Furthermore, $\eta_Y$ is non-decreasing if and only if $\lambda_Y$ is non-increasing. Since $\lambda_Y$ has one-sided limits everywhere, we can assume without loss of generality that $\lambda_Y(w) = \lim_{z \to w} \lambda_Y(z)$ and $\lambda_Y(\overline{w}) = \lim_{z \to \overline{w}} \lambda_Y(z)$. Finally, (42), (43), (53) and (54) can be simplified to for virtually all $z \in (w, \overline{w})$: If $0 < a(z) < K$, then $\lambda_Q(z) = \overline{\eta}_a(z) = \eta_a(z) = 0$. If $a(z) = 0$, then $\overline{\eta}_a(z) = 0$ and $-\eta_a(z) = \lambda_Q(z) \leq 0$. If $a(z) = K$, then $\eta_a(z) = 0$ and $\overline{\eta}_a(z) = \lambda_Q(z) \geq 0$.

**Lemma 15** $\lambda_Q(w) = \lambda_Q(\overline{w}) = 0$.

**Proof.** By the transversality condition (52), $\lambda_Q(w) \leq 0$ and equality holds if $Q(w) > 0$. Suppose to the contrary that $\lambda_Q(w) < 0$. Then $Q(w) = 0$. By continuity there exists $\delta > 0$ such that $\lambda_Q(z) < 0$ for all $z \in (w, w + \delta)$. Hence $a(z) = 0$ for all $z \in (w, w + \delta)$. This implies that $Q(z) = 0$ for all $z \in (w, w + \delta)$. Let $z \in (w, w + \delta)$, then

$$0 = Y(w) = \int_w^z H(\zeta)^{n-1} h(z) dz + Y(z) > Y(z),$$

a contradiction. Hence $\lambda_Q(w) = 0$. A similar argument proves that $\lambda_Q(\overline{w}) = 0$. ■

**Lemma 16** (Non-triviality) $\lambda_0 = 1$.

**Proof.** Suppose to the contrary that $\lambda_0 = 0$. Then

$$\lambda_Q'(z) = - \left[ \lambda_X \frac{H_{\alpha_i}(z)}{h(z)} + \lambda_Y(z) \right] h(z), \text{ v.e.}$$
Hence,

\[
\lambda_Q(w) = \lambda_Q(w) - \int_w^\infty \left[ \lambda_X \frac{H_{\alpha_i}(z)}{h(z)} + \lambda_Y(z) \right] h(z) \, dz,
\]

\[
= \lambda_Q(w) - \int_w^\infty \lambda_Y(z) h(z) \, dz.
\]

Since \( \lambda_Q(w) = \lambda_Q(\overline{w}) = 0 \), we have

\[
\int_w^\infty \lambda_Y(z) h(z) \, dz = 0.
\]

Then it must be that \( \lambda_Y(w) \geq 0 \) and \( \lambda_Y(\overline{w}) \leq 0 \). Suppose \( \lambda_X > 0 \). Then since \( H_{\alpha_i}(z)/h(z) \) is strictly decreasing and \( \lambda_Y(z) \) is non-increasing, \( \lambda_Q(H^{-1}(\cdot)) \) is strictly convex. Hence \( \lambda_Q(z) < 0 \) for all \( z \in (w, \overline{w}) \) and therefore \( a(z) = 0 \) for all \( z \in (w, \overline{w}) \). That is, \( Q \) is constant. However, when \( Q \) is constant \( X(\overline{w}) > -C'(\alpha^*) \), a contradiction to that \( \lambda_X > 0 \). Suppose \( \lambda_X = 0 \). Then

\[
\lambda_Q(z) = -\int_w^z \lambda_Y(\zeta) \, d\zeta.
\]

Suppose \( \lambda_Y(w) = \lambda_Y(\overline{w}) = 0 \), then \( \lambda_Y(z) = 0 \) for all \( z \in (w, \overline{w}) \) and therefore \( \lambda_Q(z) = 0 \) for all \( z \in (w, \overline{w}) \). Then

\[
(\lambda_0, \lambda_X(z), \lambda_Y(z), \lambda_Q(z), \eta_Y(\overline{w}) - \eta_Y(w)) = 0, \quad \forall z.
\]

A contradiction. Hence \( \lambda_Y(w) > 0 \) and \( \lambda_Y(\overline{w}) < 0 \). Thus \( \lambda_Q(z) < 0 \) for all \( z \in (w, \overline{w}) \) and therefore \( Q \) is constant over \( z \in (w, \overline{w}) \). This implies that \( \lambda_Y \) is constant over \( z \in (w, \overline{w}) \), a contradiction to that \( \lambda_Y(w) > 0 \) and \( \lambda_Y(\overline{w}) < 0 \).

To summarize, we have the following corollary:

**Corollary 2** Let \((X, Y, Q, a)\) be an admissible pair for \((\mathcal{P}^K-\alpha^*)\). If \((X, Y, Q, a)\) is optimal, then there exist a constant \( \lambda_X \), a continuous and piecewise continuously differentiable func-
tion \( \lambda_Q \), and a non-increasing function \( \lambda_Y \) such that the following holds:

\[
\lambda_X \geq 0 \quad (= 0 \text{ if } X(\bar{w}) > -C'(\alpha^*)) \tag{55}
\]

\[
\lambda_Q(z) = - \left[ z + \lambda_X \frac{H_{\alpha_i}(z)}{h(z)} + \lambda_Y(z) \right] h(z), \quad \text{v.e.} \tag{56}
\]

\( \lambda_Y \) is constant on any interval \( Y(z) > 0 \). \tag{57}

\[
\lambda_Q(\bar{w}) \leq 0 \quad (= 0 \text{ if } Q(\bar{w}) < 1) \tag{58}
\]

\[
\lambda_Q(w) \leq 0 \quad (= 0 \text{ if } Q(w) > 0). \tag{59}
\]

\[
a(z) = \begin{cases} 
0 & \text{if } \lambda_Q(z) \leq 0, \\
\in [0, K] & \text{if } \lambda_Q(z) = 0, \quad \text{v.e.} \\
K & \text{if } \lambda_Q(z) \geq 0.
\end{cases} \tag{60}
\]

For each \( \lambda_X \geq 0 \) and \( t \in [0, 1] \), define

\[
\varphi^{\lambda_X}(t) \equiv H^{-1}(t) + \lambda_X \frac{H_{\alpha_i}(H^{-1}(t))}{h(H^{-1}(t))},
\]

and

\[
J^{\lambda_X}(t) \equiv \int_0^t \varphi^{\lambda_X}(t) d\tau.
\]

For each \( w \in W \), define

\[
m_Y(w) \equiv -\int_w^\bar{w} \lambda_Y(z) h(z) dz.
\]

It follows from (56) that for \( z, \bar{z} \in W \)

\[
\lambda_Q(\bar{z}) = \lambda_Q(z) - \int_z^{\bar{z}} \left[ z + \lambda_X \frac{H_{\alpha_i}(z)}{h(z)} + \lambda_Y(z) \right] h(z) dz,
\]

\[
= \lambda_Q(z) - \int_z^{\bar{z}} \left[ \varphi^{\lambda_X}(H(z)) + \lambda_Y(z) \right] h(z) dz.
\]

When \( z = \bar{w} \), we have

\[
\lambda_Q(\bar{z}) = \lambda_Q(w) - J^{\lambda_X}(H(\bar{z})) + m_Y(\bar{z}).
\]
Hence, for virtually all \(z \in (w, \bar{w})\),

\[
a(z) = \begin{cases} 
0 & \text{if } \lambda_Q(w) + m_Y(z) \leq J^{\lambda x}(H(z)), \\
[0, K] & \text{if } \lambda_Q(w) + m_Y(z) = J^{\lambda x}(H(z)), \\
K & \text{if } \lambda_Q(w) + m_Y(z) \geq J^{\lambda x}(H(z)).
\end{cases}
\]

**Lemma 17 (Reid)** Suppose \(\lambda_Q(w) + m_Y(H^{-1}(t)) = J^{\lambda x}(t)\) for \(t \in \{\underline{t}, \bar{t}\}\). Let \(a, b \in \mathbb{R}\) and \(l(t) = a + bt\). If \(J^{\lambda x}(t) \geq l(t)\) for all \(t \in [\underline{t}, \bar{t}]\), then

\[
\lambda_Q(w) + m_Y(H^{-1}(t)) \geq l(t), \ \forall t \in [\underline{t}, \bar{t}].
\]

**Proof.** Suppose that \(\lambda_Q(w) + m_Y(H^{-1}(t)) < l(t)\) for some \(t \in (\underline{t}, \bar{t})\). Then there exists \(\varepsilon > 0\) and \(\bar{t} < t_1 < t_2 < \bar{t}\) such that \(\lambda_Q(w) + m_Y(H^{-1}(\tau)) < l(\tau) - \varepsilon\) for \(\tau \in (t_1, t_2)\), and

\[
\begin{align*}
\lambda_Q(w) + m_Y(H^{-1}(t_1)) &= l(t_1) - \varepsilon, \\
\lambda_Q(w) + m_Y(H^{-1}(t_2)) &= l(t_2) - \varepsilon.
\end{align*}
\]

This implies that \(\lambda_Y(H^{-1}(\tau)) = -m_Y'(H^{-1}(\tau))\) cannot be constant on \((t_1, t_2)\). On the other hand, \(\lambda_Q(w) + m_Y(H^{-1}(\tau)) < l(\tau) - \varepsilon < J^{\lambda x}(t)\) for \(\tau \in (t_1, t_2)\). Hence \(a(H^{-1}(\tau)) = 0\) for \(\tau \in (t_1, t_2)\), which implies that \(Y(H^{-1}(\tau)) > 0\) over the interval \((t_1, t_2)\). To see this, note that \(Y'(z) = Q(z) - H^{n-1}(z)\) is strictly decreasing if \(Q\) is constant. Thus, \(Y\) is strictly concave on \((H^{-1}(t_1), H^{-1}(t_2))\). For any \(\tau \in (t_1, t_2)\) there exists \(\lambda \in (0, 1)\) such that \(H^{-1}(\tau) = \lambda H^{-1}(t_1) + (1 - \lambda)H^{-1}(t_2)\). Then \(Y(H^{-1}(\tau)) > \lambda Y(H^{-1}(t_1)) + (1 - \lambda)Y(H^{-1}(t_2)) \geq 0\). By (57), \(Y(H^{-1}(\tau)) > 0\) on \((t_1, t_2)\) implies that \(\lambda_Y(H^{-1}(\tau))\) is constant on \((t_1, t_2)\), a contradiction. \(\blacksquare\)

An immediate implication of the Lemma is that \(\lambda_Q(w) + m_Y(H^{-1}(t)) \geq J^{\lambda x}_{[\underline{t}, \bar{t}]}(t)\), where \(J^{\lambda x}_{[\underline{t}, \bar{t}]}(t)\) denotes the convex hull of \(J^{\lambda x}\) restricted to \([\underline{t}, \bar{t}]\):

\[
J^{\lambda x}_{[\underline{t}, \bar{t}]}(t) \equiv \min \left\{ \beta J(t_1) + (1 - \beta)J(t_2) | t_1, t_2 \in [\underline{t}, \bar{t}], \beta t_1 + (1 - \beta)t_2 = t \right\}.
\]
Furthermore, $\lambda_Q(w) + m_Y(H^{-1}(t))$ is convex because $\lambda_Y$ is non-increasing. This yields the following corollary:

**Corollary 3** If $\lambda_Q(w) + m_Y(H^{-1}(t)) \leq J^\lambda(t)$ for all $t \in [t, \bar{t}]$, with equality at the endpoints of the interval, then $\lambda_Q(w) + m_Y(H^{-1}(t)) = J^\lambda_{[t, \bar{t}]}(t)$ for all $t \in [t, \bar{t}]$.

For each $t \in [0, 1]$, let $\bar{J}^\lambda(t) \equiv J^\lambda_{[0, 1]}(t)$. Since $\bar{J}^\lambda(t)$ is convex, it is continuously differentiable virtually everywhere. Define $\varphi^\lambda(t)$ as follows. For all $t \in (0, 1)$ such that $d\bar{J}^\lambda(t)/dt$ exists, let $\varphi^\lambda(t) \equiv d\bar{J}^\lambda(t)/dt$, and extend $\varphi^\lambda(t)$ to all $[0, 1]$ by right continuity. Note that $\varphi^\lambda(t)$ is non-decreasing. Combining Lemma 15 and Lemma 17 yields:

**Corollary 4** For all $t \in [0, 1]$,

$$\lambda_Q(w) + m_Y(H^{-1}(t)) \geq \bar{J}^\lambda(t).$$

**Lemma 18 (interior solution)** Suppose $a(z) \in (0, K)$ for $z \in (\underline{z}, \bar{z})$, then $\lambda_Y(z) = -\varphi^\lambda(H(z))$ for virtually every $z \in (\underline{z}, \bar{z})$.

**Proof.** If $a(z) \in (0, K)$ for $z \in (\underline{z}, \bar{z})$, then $\lambda_Q(w) + m_Y(z) = J^\lambda(H(z))$ for virtually every $z \in (\underline{z}, \bar{z})$. Differentiating this with respect to $z$ yields for virtually every $z \in (\underline{z}, \bar{z})$: 

$$-\lambda_Y(z)h(z) = \varphi^\lambda(H(z))h(z),$$

i.e. $-\lambda_Y(z) = \varphi^\lambda(H(z))$ since $h(z) > 0$. ■

**Lemma 19 (constant $Q$)** Suppose $a(z) = 0$ on $(\underline{z}, \bar{z})$ with $\underline{z} < \bar{z}$ and let $(\underline{z}, \bar{z})$ be chosen maximally. Then

$$\lambda_Q(z) = 0,$$

$$\lambda_Q(w) + m_Y(z) = J^\lambda(H(z)).$$
for \( z = \bar{z} \) if \( z > w \) and \( z = \bar{z} \) if \( z < \bar{w} \). Furthermore,

\[
\varphi^X(H(\bar{z})) + \lambda_Y(\bar{z}^-) \geq 0, \text{ if } \bar{z} > w, \\
\varphi^X(H(\bar{z})) + \lambda_Y(\bar{z}^+) \leq 0, \text{ if } \bar{z} < \bar{w}.
\]

**Proof.** Since \( a(z) = 0 \) on \((\bar{z}, \bar{z})\), then

\[
\lambda_Q(w) + m_Y(z) \geq J^X(H(z)), \text{ v.e. } z \in (\bar{z}, \bar{z}).
\]

Suppose \( \bar{z} > w \) and let \( S_- \equiv \{ z < \bar{z} \mid a(z) > 0 \} \). Since \((\bar{z}, \bar{z})\) is chosen maximally, \( Q(z) < Q(\bar{z}) \) for all \( z < \bar{z} \). Furthermore, since \( Q \) is absolutely continuous, \( S_- \cap [\bar{z} - \delta, \bar{z}] \) has positive measure for every \( \delta > 0 \). Hence, there exists a sequence \( \{ z_k \} \in S_- \) converging to \( \bar{z} \) with \( \lambda_Q(w) + m_Y(z_k) \geq J^X(H(z_k)) \) for all \( k \). By continuity, \( \lambda_Q(w) + m_Y(z) = J^X(H(z)) \) if \( \bar{z} > w \). A similar argument proves that \( \lambda_Q(w) + m_Y(\bar{z}) = J^X(H(\bar{z})) \) if \( \bar{z} < \bar{w} \).

Furthermore, then for virtually all \( z \in S_- \)

\[
0 = \lambda_Q(\bar{z})
= \lambda_Q(z) - \int_{\bar{z}}^\bar{z} [\varphi^X(H(\zeta)) + \lambda_Y(\zeta)] h(\zeta) d\zeta
\geq -\int_{\bar{z}}^\bar{z} [\varphi^X(H(\zeta)) + \lambda_Y(\zeta)] h(\zeta) d\zeta.
\]

Thus, there exists a sequence \( \{ z_k \} \in S_- \) converging to \( \bar{z} \) such that

\[
\int_{z_k}^{\bar{z}} [\varphi^X(H(\zeta)) + \lambda_Y(\zeta)] h(\zeta) d\zeta \geq 0, \forall k.
\]

Hence,

\[
\varphi(H(\bar{z})) + \lambda_Y(\bar{z}^-) \geq 0, \text{ if } \bar{z} > w.
\]
A similar argument proves that

\[ \varphi(H(z)) + \lambda_Y(z^+) \leq 0, \text{ if } z < w. \]

Lemma 19 implies that there cannot be an interval where \( Q \) is constant if \( \varphi^\lambda_X(H(\cdot)) \) is strictly increasing.

**Lemma 20 \((a(z) = K)\)** Suppose \( a(z) = K \) on \((\underline{z}, \overline{z})\) with \( \underline{z} < \overline{z} \) and let \((\underline{z}, \overline{z})\) be chosen maximally. Then

\[ \lambda_Q(z) = 0, \]
\[ \lambda_Q(w) + m_Y(z) = J^\lambda_X(H(z)). \]

for \( z = \underline{z} \) if \( z > w \) and \( z = \overline{z} \) if \( z < w \). Furthermore,

\[ \varphi(H(\underline{z})) + \lambda_Y(\underline{z}^-) \leq 0, \text{ if } \underline{z} > w, \]
\[ \varphi(H(\overline{z})) + \lambda_Y(\overline{z}^+) \geq 0, \text{ if } \overline{z} < w, \]

and

\[ \varphi(H(\underline{z})) + \lambda_Y(\underline{z}^+) \leq 0, \text{ if } \underline{z} = w, \]
\[ \varphi(H(\overline{z})) + \lambda_Y(\overline{z}^-) \geq 0, \text{ if } \overline{z} = w. \]

**Proof.** The proofs for the case \( \underline{z} > w \) and the case \( \overline{z} < w \) are very similar to that of Lemma 19 and neglected here. We now show that the third inequality holds for \( \underline{z} = w \). Note that
by the transversality condition, $\lambda_Q(w) \leq 0$. For virtually all $z \in (w, \bar{z})$,

\[
0 \leq \lambda_Q(z) = \lambda_Q(w) - \int_w^z [\varphi^{\lambda X}(H(\zeta)) + \lambda_Y(\zeta)] h(\zeta) d\zeta \
\leq -\int_w^z [\varphi^{\lambda X}(H(\zeta)) + \lambda_Y(\zeta)] h(\zeta) d\zeta,
\]

i.e.,

\[
\int_w^z [\varphi(H(\zeta)) + \lambda_Y(\zeta)] h(\zeta) d\zeta \leq 0, \text{ v.e. } z \in (w, \bar{z}).
\]

Let $z$ goes to $w$ and this proves the third inequality for $z = w$. A similar argument proves the fourth inequality for $z = \bar{w}$. ■

**Lemma 21** If $K > \overline{K} \equiv \max_{z \in W} (n - 1)H(z)^{n-2}h(z)$, then

\[
\lambda_X \leq \overline{\lambda}_X = \left[ \min_{z \in \overline{W}} \frac{\partial}{\partial z} \left[ -\frac{H_{\alpha l}(z)}{h(z)} \right] \right]^{-1}.
\]

**Proof.** Suppose to the contrary that $\lambda_X > \overline{\lambda}_X$. Then $\varphi^{\lambda X}(H(z))$ is strictly decreasing. Suppose there exists an interval $(\breve{z}, \bar{z})$ such that $a(z) \in (0, K)$ for $z \in (\breve{z}, \bar{z})$. Then by Lemma 18 $\lambda_Y(z) = -\varphi^{\lambda X}(H(z))$ for virtually every $z \in (\breve{z}, \bar{z})$. Thus, $\lambda_Y$ is strictly increasing on $(\breve{z}, \bar{z})$, a contradiction. Since $a$ is piecewise continuous by assumption, $a(z) \in \{0, K\}$ for almost every $z \in W$.

Suppose there exists an interval $(\breve{z}, \bar{z})$ such that $a(z) = K$ on $(\breve{z}, \bar{z})$ and let $(\breve{z}, \bar{z})$ be chosen maximally. Then $Y''(z) = Q(z) - H(z)^{n-1}$ which is strictly increasing since $K > \max_{z \in W} (n - 1)H(z)^{n-2}h(z)$, and therefore $Y(z)$ is strictly convex on $[\breve{z}, \bar{z}]$. This implies that $Y(z) > 0$ on $[\breve{z}, \bar{z}]$ except at most one point. Suppose $Y(z) > 0$ for all $z \in (\breve{z}, \bar{z})$. Then $\lambda_Y$ is constant on $(\breve{z}, \bar{z})$, and $\lambda_Q(H^{-1}(t)) = \lambda_Q(\breve{z}) - \int_{H(\breve{z})}^{H(\bar{z})} [\varphi^{\lambda X}(\tau) + \lambda_Y(H^{-1}(\tau))] d\tau$ is strictly convex on $(H(\breve{z}), H(\bar{z}))$. By Lemma 19 and the transversality condition, $\lambda_Q(\breve{z}) \leq 0$ and $\lambda_Q(\bar{z}) \leq 0$. Then the strict convexity of $\lambda_Q(H^{-1}(t))$ implies that $\lambda_Q(z) < 0$ for all $z \in (\breve{z}, \bar{z})$. However, $a(z) = K$ on $(\breve{z}, \bar{z})$ implies that $\lambda_Q(z) \geq 0$ for virtually every $z \in (\breve{z}, \bar{z})$,
a contradiction. Hence, there exists a unique \( z_0 \in (\underline{z}, \overline{z}) \) such that \( Y(z_0) = 0 \), and therefore \( Y(\underline{z}) > 0 \) and \( Y(\overline{z}) > 0 \). Since \( Y(w) = Y(\overline{w}) = 0 \) we have \( w < \underline{z} < \overline{z} < \overline{w} \). Note that this also implies that \( \lambda_Y \) is constant on a neighborhood of and therefore continuous at \( z \in \{\underline{z}, \overline{z}\} \).

By Lemma 20, we have

\[
\varphi^{\lambda_X}(H(\underline{z})) + \lambda_Y(\underline{z}) \leq 0,
\]

\[
\varphi^{\lambda_X}(H(\overline{z})) + \lambda_Y(\overline{z}) \geq 0.
\]

Hence,

\[
\lambda_Y(\underline{z}) \leq -\varphi^{\lambda_X}(H(\underline{z})) < -\varphi^{\lambda_X}(H(\overline{z})) \leq \lambda_Y(\overline{z}),
\]

where the second inequality holds since \( \varphi^{\lambda_X} \) is strictly decreasing and \( H \) is strictly increasing. However, this is a contradiction to that \( \lambda_Y \) is non-increasing. Hence, \( a(z) = 0 \) for almost all \( z \in W \).

Since \( Q \) is absolutely continuous, this implies that \( Q \) is constant on \( W \). However, by Lemma 7, \( X(\overline{w}) = 0 > -C'(\alpha^*) \) when \( Q \) is constant on \( W \), which implies that \( \lambda_X = 0 \), a contradiction to the supposition that \( \lambda_X > \bar{\lambda}_X > 0 \). Hence, \( \lambda_X \leq \bar{\lambda}_X \).

**A.2.2 Proof of Theorem 2**

Let \( Q^\nu \) be a sequence of optimal solutions of \( \mathcal{P}^{K^\nu} \) where \( K < K^\nu \to \infty \) and let \( Q^\infty \) denote the almost everywhere limit of the sequence. Denote the joint variables in these solutions by \( \lambda^\nu \), and let \( \lambda_X^\infty = \lim_{\nu \to \infty} \lambda_X^\nu \). By Lemma 13, \( Q^\infty \) is an optimal solution. I show that any \( Q \) satisfying condition in Theorem 2 yields the same expected social surplus as \( Q^\infty \).

**Lemma 22** Let

\[
\varphi^\nu(t) \equiv H^{-1}(t) + \lambda^\nu_X \frac{H_x(H^{-1}(t))}{h(H^{-1}(t))} \quad \text{and} \quad J^\nu(t) \equiv \int_0^t \varphi^\nu(\tau)d\tau,
\]

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and
\[ \varphi^{\infty}(t) \equiv H^{-1}(t) + \lambda^\infty X \frac{H\alpha(H^{-1}(t))}{h(H^{-1}(t))} \quad \text{and} \quad J^{\infty}(t) \equiv \int_0^t \varphi^{\infty}(\tau) d\tau. \]

Then the sequence $\varphi^\nu$ is uniformly convergent with limit $\varphi^\infty$, the sequence $J^\nu$ is uniformly convergent with limit $J^\infty$, and the sequence $\varphi'^\nu$ is uniformly convergent with limit $\varphi'^\infty$.

**Proof.** Let $\gamma_1 \equiv \max_{z \in W} |H\alpha(z)/h(z)| > 0$,

\[ \gamma_2 \equiv \max_{z \in W} \left| \frac{\partial}{\partial z} \left[ -\frac{H\alpha(z)}{h(z)} \right] \right| > 0, \]

and $\gamma_3 \equiv \max_{z \in W} 1/h(z) > 0$. Here $\gamma_1$, $\gamma_2$ and $\gamma_3$ are well defined since $H$ and $h$ are twice continuously differentiable and $W$ is compact.

\[ |\varphi^\nu(t) - \varphi^\infty(t)| = |\lambda^\nu X - \lambda^\infty X| \left| \frac{H\alpha(H^{-1}(t))}{h(H^{-1}(t))} \right| \leq \gamma_1 |\lambda^\nu X - \lambda^\infty X| \to 0, \]

as $\nu \to \infty$. Hence, the sequence $\varphi^\nu$ is uniformly convergent with limit $\varphi^\infty$.

\[ |J^\nu(t) - J^\infty(t)| = \int_0^t |\varphi^\nu(\tau) - \varphi^\infty(\tau)| d\tau \]
\[ \leq t \gamma_1 |\lambda^\nu X - \lambda^\infty X| \]
\[ \leq \gamma_1 |\lambda^\nu X - \lambda^\infty X| \to 0, \]

as $\nu \to 0$. Hence, the sequence $J^\nu$ is uniformly convergent with limit $J^\infty$.

\[ |\varphi'^\nu(t) - \varphi'^\infty(t)| = |\lambda^\nu X - \lambda^\infty X| \left| \frac{\partial}{\partial z} \left[ \frac{H\alpha(H^{-1}(t))}{h(H^{-1}(t))} \right] \right| \frac{1}{h(H^{-1}(t))} \leq \gamma_2 \gamma_3 |\lambda^\nu X - \lambda^\infty X| \to 0, \]

as $\nu \to \infty$. Hence, the sequence $\varphi'^\nu$ is uniformly convergent with limit $\varphi'^\infty$. 

**Lemma 23** Suppose $J^\infty(H(z)) > J^\infty(H(\tilde{z}))$ for $z \in (\tilde{z}, \overline{z})$ with $\tilde{z} < \overline{z}$ and let $(\tilde{z}, \overline{z})$ be chosen maximally. Then $Q^\infty$ is constant on $(\tilde{z}, \overline{z})$.

**Proof.** For each $0 < \delta < (\overline{z} - \tilde{z})/2$, let $\varepsilon(\delta) \equiv \min_{z \in [\tilde{z} + \delta, \overline{z} - \delta]} \{J^\infty(H(z)) - J^\infty(H(\tilde{z}))\}$. Then
\(\varepsilon(\delta)\) is non-increasing in \(\delta\) and converges to zero as \(\delta\) converges to zero. Fix \(\delta_0 > 0\). Let \(\varepsilon_0 \equiv \frac{1}{4}\varepsilon(\delta_0) > 0\). There exist \(0 < \delta_1 < \delta_2 < \delta_0\) such that \(\varepsilon(\delta_1) = \varepsilon_0\) and \(\varepsilon(\delta_2) = 2\varepsilon_0\). Since the sequence \(J^\nu\) is uniformly convergent with limit \(J^\infty\), there exists \(\nu\) such that for all \(\nu > \nu\),

\[
\begin{align*}
J^\nu(H(z)) - \bar{J}^\nu(H(z)) &\geq \frac{7\varepsilon_0}{2} \text{ if } z \in [\bar{z} + \delta_0, \bar{z} - \delta_0], \\
J^\nu(H(z)) - \bar{J}^\nu(H(z)) &\geq \frac{\varepsilon_0}{2} \text{ if } z \in [\bar{z} + \delta_1, \bar{z} - \delta_1], \\
J^\nu(H(z)) - \bar{J}^\nu(H(z)) &\leq \frac{5\varepsilon_0}{2} \text{ if } J^\infty(H(z)) - \bar{J}^\infty(H(z)) \leq 2\varepsilon_0.
\end{align*}
\]

We begin by prove the second inequality. Since the sequence \(J^\nu\) is uniformly convergent with limit \(J^\infty\), there exists \(\nu\) such that for all \(\nu > \nu\), \(|J^\nu(t) - J^\infty(t)| < \varepsilon_0/8\) for all \(t \in [0, 1]\). Let \(t \in [H(z + \delta_1), H(z - \delta_1)]\). Then, by construction, \(J^\infty(t) - \bar{J}^\infty(t) > \varepsilon_0\). Hence, there exists \(\beta, t_1, t_2 \in [0, 1]\) such that \(\beta t_1 + (1 - \beta)t_2 = t\) and \(\beta J^\infty(t_1) + (1 - \beta)J^\infty(t_2) < J^\infty(t) - 3\varepsilon_0/4\). Then

\[
\begin{align*}
\bar{J}^\nu(t) &\leq \beta J^\nu(t_1) + (1 - \beta)J^\nu(t_2) \\
&\leq \beta J^\infty(t_1) + \beta |J^\nu(t_1) - J^\infty(t_1)| + (1 - \beta)J^\infty(t_2) + (1 - \beta)|J^\nu(t_2) - J^\infty(t_2)| \\
&\leq \beta J^\infty(t_1) + (1 - \beta)J^\infty(t_2) + \frac{\varepsilon_0}{8} \\
&\leq J^\infty(t) - \frac{3\varepsilon_0}{4} + \frac{\varepsilon_0}{8} \\
&\leq J^\nu(t) + |J^\nu(t) - J^\infty(t)| - \frac{5\varepsilon_0}{8} \\
&\leq J^\nu(t) + \frac{\varepsilon_0}{8} - \frac{5\varepsilon_0}{8} \\
&= J^\nu(t) - \frac{\varepsilon_0}{2}.
\end{align*}
\]

Hence, \(J^\nu(H(z)) - \bar{J}^\nu(H(z)) \geq \varepsilon/2\) for all \(z \in [\bar{z} + \delta_1, \bar{z} - \delta_1]\). A similar argument proves the first inequality. To show the third inequality, let \(z\) be such that \(J^\infty(H(z)) - \bar{J}^\infty(H(z)) \leq 2\varepsilon_0\).
For any $\beta, t_1, t_2 \in [0, 1]$ such that $\beta t_1 + (1 - \beta) t_2 = t$, we have

\[
\begin{align*}
\beta J^\nu(t_1) + (1 - \beta) J^\nu(t_2) & \geq \beta J^\infty(t_1) - \beta |J^\nu(t_1) - J^\infty(t_1)| + (1 - \beta) J^\infty(t_2) - (1 - \beta) |J^\nu(t_2) - J^\infty(t_2)| \\
& \geq \beta J^\infty(t_1) + (1 - \beta) J^\infty(t_2) - \frac{\epsilon_0}{8} \\
& \geq J^\infty(t) - \frac{\epsilon_0}{8} \\
& = J^\infty(t) - [J^\infty(t) - J^\infty(t)] - \frac{\epsilon_0}{8} \\
& \geq J^\infty(t) - 2\epsilon_0 - \frac{\epsilon_0}{8}, \\
& \geq J^\nu(t) - |J^\nu(t) - J^\infty(t)| - 2\epsilon_0 - \frac{\epsilon_0}{8}, \\
& \geq J^\nu(t) - \frac{\epsilon_0}{8} - 2\epsilon_0 - \frac{\epsilon_0}{8}, \\
& \geq J^\nu(t) - \frac{5\epsilon_0}{2}.
\end{align*}
\]

Hence,

\[J^\nu(t) \equiv \min\{\beta J^\nu(t_1) + (1 - \beta) J^\nu(t_2) | \beta, t_1, t_2 \in [0, 1] \text{ and } \beta t_1 + (1 - \beta) t_2 = t\} \geq J^\nu(t) - \frac{5\epsilon_0}{2}.\]

Since $\epsilon(\delta_1) = \epsilon_0$ and $\epsilon(\delta_2) = 2\epsilon_0$, by continuity

\[
m_\delta \equiv \min\left\{m(\{z \in [z + \delta_1, z + \delta_2] | J^\infty(H(z)) - J^\infty(H(z)) \leq 2\epsilon_0\}) , \right. \\
\left. m(\{z \in [z - \delta_2, z - \delta_1] | J^\infty(H(z)) - J^\infty(H(z)) \leq 2\epsilon_0\}) \right\} > 0.
\]

Fix $\nu > \nu$ such that $K_\nu > 1/m_\delta$. Suppose there exists $(b_1, b_2) \subset (z + \delta_0, z - \delta_0)$ such that $a^\nu(z) > 0$. Then $\lambda_\nu(b_1), \lambda_\nu(b_2) \geq 0$. Since $\lambda_\nu^\nu$ is non-increasing, we have $\lambda_\nu^\nu(b_2) \leq \lambda_\nu^\nu(b_1)$. Note that $J^\nu$ is linear and therefore $\bar{\varphi}^\nu$ is constant on $(z, z)$. Hence, we have either $-\lambda_\nu^\nu(b_2) \geq \bar{\varphi}^\nu(H(z))$ for all $z \in (z, z)$ or $-\lambda_\nu^\nu(b_1) \leq \bar{\varphi}^\nu(H(z))$ for all $z \in (z, z)$. Assume without loss of generality that $-\lambda_\nu^\nu(b_2) \geq \bar{\varphi}^\nu(H(z))$ for all $z \in (z, z)$. For any $z \in [z - \delta_2, z - \delta_1]$ with
\[ J^\infty(H(z)) - \overline{J}^\infty(H(z)) \leq 2\varepsilon_0, \]

we have

\[
\lambda_\nu^\prime Q(z) = \lambda_\nu^\prime Q(b_2) - \int_{b_2}^z \left[ \lambda_\nu^\prime(\zeta) + \varphi^\nu(H(\zeta)) \right] h(\zeta) d\zeta \\
\geq \int_{b_2}^z \overline{\varphi}^\nu(H(\zeta)) h(\zeta) d\zeta - \int_{b_2}^z \varphi^\nu(H(\zeta)) h(\zeta) d\zeta \\
= \overline{J}^\nu(H(z)) - \overline{J}^\nu(H(b_2)) - J^\nu(H(z)) + J^\nu(H(b_2)) \\
= J^\nu(H(b_2)) - \overline{J}^\nu(H(b_2)) - [J^\nu(H(z)) - \overline{J}^\nu(H(z))] \\
\geq \frac{7\varepsilon_0}{2} - \frac{5\varepsilon_0}{2} = \varepsilon_0 > 0.
\]

That is, \( a^\nu(z) = K_\nu \) for almost every \( z \in [z - \delta_2, z - \delta_1] \) with \( J^\infty(H(z)) - \overline{J}^\infty(H(z)) \leq 2\varepsilon_0 \).

However, this is a contradiction to that \( K_\nu > 1/m_\delta \) since \( 0 \leq Q \leq 1 \).

Hence, \( a^\nu(z) = 0 \) for almost every \( z \in [z - \delta_0, z + \delta_0] \) for \( \nu \) sufficiently large. Let \( \nu \) goes to infinity and we have \( Q^\infty \) is constant on \([z - \delta_0, z + \delta_0] \). Since this is true for any \( \delta > 0 \), we have that \( Q^\infty \) is constant on \((z, z) \).

**Lemma 24** The sequence \( J^\nu \) is uniformly convergent with limit \( J^\infty \).

**Proof.** Since the sequence \( J^\nu \) is uniformly convergent with limit \( J^\infty \), for any \( \varepsilon > 0 \) there exists \( \overline{\nu} > 0 \) such that for all \( \nu > \overline{\nu} \), \( |J^\infty(t) - J^\nu(t)| \leq \varepsilon \) for all \( t \in [0, 1] \). Fix \( t \in [0, 1] \). Let \( t_1, t_2, \beta \in [0, 1] \) be such that \( \beta t_1 + (1 - \beta)t_2 = t \). Then for any \( \nu > \overline{\nu} \)

\[
J^\infty(t) \leq \beta J^\infty(t_1) + (1 - \beta)J^\infty(t_2) \\
\leq \beta J^\nu(t_1) + (1 - \beta)J^\nu(t_2) + \beta|J^\infty(t_1) - J^\nu(t_1)| + (1 - \beta)|J^\infty(t_2) - J^\nu(t_2)| \\
\leq \beta J^\nu(t_1) + (1 - \beta)J^\nu(t_2) + \varepsilon
\]

Hence,

\[
J^\infty(t) \leq \min \{ \beta J^\nu(t_1) + (1 - \beta)J^\nu(t_2) | \beta t_1 + (1 - \beta)t_2 = t \} + \varepsilon = J^\nu(t) + \varepsilon.
\]

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Similarly, we can show that \( J'(t) \leq \overline{J}'(t) + \varepsilon \). Hence, \( |\overline{J}'(t) - J'(t)| \leq \varepsilon \). Since this holds for any \( t \in [0, 1] \), we have that the sequence \( J' \) is uniformly convergent with limit \( \overline{J}' \). □

**Lemma 25** Suppose \( Y(z) > 0 \) for all \( z \in (z, \bar{z}) \) with \( \bar{z} < z \) and let \( (\bar{z}, \overline{z}) \) be chosen maximally. Then \( \overline{\varphi}' \) is constant on \( (H(z), H(\bar{z})) \).

**Proof.** Suppose to the contrary that \( \overline{\varphi}'(H(\bar{z})) > \overline{\varphi}'(H(\bar{z})) \). Since \( \overline{\varphi} \) is non-decreasing and right-continuous, there exists \( \delta > 0 \) such that \( \overline{\varphi}'(H(\bar{z} - \delta)) > \overline{\varphi}'(H(\bar{z} + \delta)) \) for all \( \delta \in (0, \overline{\delta}) \). Fix \( \delta \in (0, \min\{\overline{\delta}/2, (\overline{z} - z)/4\}) \). Since \( \overline{J}' \) is convex and \( \overline{\varphi}' \) is non-constant on \( (\bar{z} + \delta, \bar{z} - \delta) \), we have

\[
\overline{J}'(H(z)) < \overline{J}'(H(\bar{z} + \delta)) + [z - (\bar{z} + \delta)] \frac{\overline{J}'(H(\bar{z} - \delta)) - \overline{J}'(\bar{z} + \delta)}{\overline{z} - \bar{z} - 2\delta}, \quad \forall z \in (\bar{z} + \delta, \bar{z} - \delta).
\]

Let

\[
\varepsilon_1 \equiv \min_{z \in [\bar{z} + 2\delta, \bar{z} - 2\delta]} \left\{ \overline{J}'(H(\bar{z} + \delta)) + [z - (\bar{z} + \delta)] \frac{\overline{J}'(H(\bar{z} - \delta)) - \overline{J}'(\bar{z} + \delta)}{\overline{z} - \bar{z} - 2\delta} - \overline{J}'(H(z)) \right\} > 0,
\]

\[
\varepsilon_2 \equiv \min_{z \in [\bar{z} + \delta, \bar{z} - \delta]} Y'(z) > 0,
\]

and

\[
M \equiv 2 \max_{z \in [\bar{z}, \overline{z}]} |\overline{\varphi}'(H(z))| > 0.
\]

Since the sequence \( J' \) is uniformly convergent with limit \( \overline{J}' \), the sequence \( Y' \) is uniformly convergent with limit \( \overline{Y}' \) and the sequence \( \varphi' \) is uniformly convergent with limit \( \overline{\varphi}' \), there exists \( \overline{\eta} \) such that for \( \nu > \overline{\eta} \), \( |Y'(z) - Y'(z)| < \varepsilon_2/2 \) for all \( z \in W \), \( |\overline{J}'(t) - \overline{J}'(t)| \leq \varepsilon_1/8 \) for all \( t \in [0, 1] \) and \( |\overline{\varphi}'(t) - \overline{\varphi}'| \leq M/2 \) for all \( t \in [0, 1] \). Then for all \( \nu > \overline{\eta} \) and \( z \in [\bar{z} + 2\delta, \bar{z} - 2\delta] \).
we have
\[
\mathcal{J}^\nu(H(z + \delta)) + [z - (z + \delta)] \frac{\mathcal{J}^\nu(H(z - \delta)) - \mathcal{J}^\nu(H(z + \delta))}{z - z - 2\delta} - \mathcal{J}^\nu(H(z)) \\
\geq \mathcal{J}^\infty(H(z + \delta)) - \left| \mathcal{J}^\infty(H(z + \delta)) - \mathcal{J}^\nu(H(z + \delta)) \right| - \left| \mathcal{J}^\nu(H(z)) - \mathcal{J}^\nu(H(z)) \right| \\
+ \frac{z - (z + \delta)}{z - z - 2\delta} \left[ \mathcal{J}^\infty(H(z - \delta)) - \left| \mathcal{J}^\infty(H(z - \delta)) - \mathcal{J}^\nu(H(z - \delta)) \right| - \left| \mathcal{J}^\infty(H(z + \delta)) - \mathcal{J}^\nu(H(z + \delta)) \right| \right] \\
\geq \mathcal{J}^\infty(H(z + \delta)) + [z - (z + \delta)] \frac{\mathcal{J}^\infty(H(z - \delta)) - \mathcal{J}^\infty(H(z + \delta))}{z - z - 2\delta} - \mathcal{J}^\infty(H(z)) - \frac{\varepsilon_1}{2}.
\]

For all \( \nu > \nu \) and \( z \in [z + \delta, z - \delta] \) we have
\[
Y^\nu(z) \geq Y^\infty(z) - |Y^\nu(z) - Y^\infty(z)| \geq \frac{\varepsilon_2}{2} > 0.
\]

For all \( \nu > \nu \) and \( z \in [z + \delta, z - \delta] \), we have
\[
\varphi^\nu(H(z)) \leq \varphi^\infty(H(z)) + |\varphi^\nu(H(z)) - \varphi^\infty(H(z))| \leq \frac{M_1}{2} + \frac{M_1}{2} = M_1.
\]

Finally, let \( \varepsilon_3 \equiv \min_{z \in [z, \bar{z}]} h(z) > 0 \) and
\[
M_2 \equiv \left| \frac{\mathcal{J}^\nu(H(z - \delta)) - \mathcal{J}^\nu(H(z + \delta))}{z - z - 2\delta} \right| > 0.
\]

Fix \( \nu > \nu \) such that \( K_\nu > 1/\min\{\varepsilon_1/8, \varepsilon_3M, \varepsilon_1/8M_2, \delta\} \). Since \( Y^\nu(z) > 0 \) on \([z + \delta, \bar{z} - \delta]\), \( \lambda_Y \) is constant and therefore \( m_Y \) is affine on \([z + \delta, \bar{z} - \delta]\). By Corollary 4, we have \( m_Y^\nu(z) \geq \mathcal{J}^\nu(H(z)) \) for all \( z \in W \). In particular, \( m_Y^\nu(z + \delta) \geq \mathcal{J}^\nu(H(z + \delta)) \) and \( m_Y^\nu(z - \delta) \geq \mathcal{J}^\nu(H(z - \delta)) \). Furthermore, \( m_Y \) is affine on \([z + \delta, \bar{z} - \delta]\). Hence, for all \( z \in [z + \delta, \bar{z} - \delta] \),
\[
m_Y^\nu(z) \geq \mathcal{J}^\nu(H(z + \delta)) + [z - (z + \delta)] \frac{\mathcal{J}^\nu(H(z - \delta)) - \mathcal{J}^\nu(z + \delta)}{z - z - 2\delta}.
\]

Suppose \( J^\nu(H(z_0)) = \mathcal{J}^\nu(H(z_0)) \) for some \( z_0 \in [z + 2\delta, \bar{z} - 2\delta] \). Then for all \( z \in \)
Corollary 5 Suppose \( \varphi^\infty(H(z)) \) is constant on \( (z, \bar{z}) \) with \( z < \bar{z} \) and let \( (z, \bar{z}) \) be chosen maximally. Then \( Y^\infty(z) = Y^\infty(\bar{z}) = 0 \), i.e.,

\[
\int_{\bar{z}}^{z} [H(\zeta)^{n-1} - Q^\infty(\zeta)] h(\zeta)d\zeta = 0.
\]

Proof. This is an immediate corollary of Lemma 25. Suppose to the contrary \( Y^\infty(\bar{z}) > 0 \). Then by Lemma 25, \( \varphi^\infty(H(\cdot)) \) is constant on a neighborhood of \( \bar{z} \), a contradiction. Hence \( Y^\infty(\bar{z}) = 0 \). Similarly, \( Y^\infty(z) = 0 \).
Let $Q$ be a non-decreasing implementable allocation rule that satisfies the following two pooling properties:

1. If $J^\infty(H(z)) > J^\infty(H(\bar{z}))$ for $z \in (\underline{z}, \bar{z})$ with $\underline{z} < \bar{z}$ and let $(\underline{z}, \bar{z})$ be chosen maximally, then $Q$ is constant on $(\underline{z}, \bar{z})$.

2. If $Y(z) > 0$ for all $z \in (\underline{z}, \bar{z})$ with $\underline{z} < \bar{z}$ and let $(\underline{z}, \bar{z})$ be chosen maximally. Then $\varphi^\infty$ is constant on $(H(\underline{z}), H(\bar{z}))$.

Then

$$
\int_\underline{z}^\bar{z} \left[ z + \lambda_\infty \frac{H_{\alpha_i}(z)}{h(z)} \right] Q(z)h(z)dz + \lambda_\infty C'(\alpha^*)
= \int_\underline{z}^\bar{z} \varphi^\infty(H(z))Q(z)dH(z) + \lambda_\infty C'(\alpha^*)
= \int_\underline{z}^\bar{z} \left[ \varphi^\infty(H(z)) - \overline{\varphi^\infty}(H(z)) \right] Q(z)dH(z) + \int_\underline{z}^\bar{z} \overline{\varphi^\infty}(H(z))Q(z)dH(z) + \lambda_\infty C'(\alpha^*)
= \int_\underline{z}^\bar{z} \overline{\varphi^\infty}(H(z))Q(z)dH(z) - \int_\underline{z}^\bar{z} \left[ J^\infty(H(z)) - J^\infty(H(\bar{z})) \right] dQ(z) + \lambda_\infty C'(\alpha^*)
= \int_\underline{z}^\bar{z} \overline{\varphi^\infty}(H(z))Q(z)dH(z) + \lambda_\infty C'(\alpha^*)
= \int_\underline{z}^\bar{z} \overline{\varphi^\infty}(H(z))H(z) \alpha_i - \int_\underline{z}^\bar{z} Y(z)\overline{\varphi^\infty}(H(z)) + \lambda_\infty C'(\alpha^*)
= \int_\underline{z}^\bar{z} \overline{\varphi^\infty}(H(z))H(z) \alpha_i - \int_\underline{z}^\bar{z} Y(z)d\overline{\varphi^\infty}(H(z)) + \lambda_\infty C'(\alpha^*)
= \int_\underline{z}^\bar{z} \overline{\varphi^\infty}(H(z))H(z) \alpha_i - \lambda_\infty C'(\alpha^*)
= \int_\underline{z}^\bar{z} \lambda_\infty C'(\alpha^*)
.$$  

The second line follows from the definition of $\varphi^\infty$. The third line follows from integration by parts. The fourth line holds since $Q$ satisfies the first pooling property. The fifth line follows from the definition of $Y$. The sixth line follows from integration by parts. The seventh line holds since $Q$ satisfies the second pooling property. Suppose in addition that

$$
\int_\underline{z}^\bar{z} \frac{H_{\alpha_i}(z)}{h(z)} Q(z)h(z)dz = -C'(\alpha^*)
.$$  

(61)
Then

\[
\int_w^\infty zQ(z)h(z)dz = \int_w^\infty \left[ z + \lambda_X^\infty \frac{H_\alpha(z)}{h(z)} \right] Q(z)h(z)dz + \lambda_X^\infty C'(\alpha^*) = \int_w^\infty \varphi^\infty(H(z))H(z)^{n-1}dH(z) + \lambda_X^\infty C'(\alpha^*).
\]

That is, all non-decreasing implementable allocation rules \(Q\) satisfying the two pooling properties and equality (61) generate the same expected surplus. By Lemmas 23 and 25, \(Q^\infty\) satisfies the two pooling properties. Furthermore, \(Q^\infty\) satisfies equality (61). Hence any non-decreasing implementable allocation rule \(Q\) satisfying the two pooling properties and equality (61) solves problem \((P-\alpha^*)\).

Define \(Q^+\) in the following way. If \(J^\infty(H(z)) > \overline{J}^\infty(H(z))\), let

\[
Q^+(z) = \frac{1}{n}[H(z)^n - H(z)^n] \frac{1}{H(z) - H(z)}.
\]

where \((z, \overline{z})\) with \(z < z < \overline{z}\) is the maximal interval on which \(J^\infty(H(\cdot)) > \overline{J}^\infty(H(\cdot))\).

Otherwise, let \(Q^+(z) \equiv H(z)^{n-1}\). Define \(Q^-\) in the following way. If there exists \((z, \overline{z})\) with \(z < z < \overline{z}\) on which \(\varphi(H(\cdot))\) is constant and let \((z, \overline{z})\) be chosen maximally, then let

\[
Q^-(z) = \frac{1}{n}[H(z)^n - H(z)^n] \frac{1}{H(z) - H(z)}.
\]

Otherwise, let \(Q^-(z) \equiv H(z)^{n-1}\). Note that all \(Q^+, Q^-\) and \(Q^\infty\) are implementable, non-decreasing and satisfy the two pooling properties. Hence for \(Q \in \{Q^+, Q^-, Q^\infty\}\)

\[
\int_w^\infty \left[ z + \lambda_X^\infty \frac{H_\alpha(z)}{h(z)} \right] Q(z)h(z)dz + \lambda_X^\infty C'(\alpha^*) = \int_w^\infty \varphi^\infty(H(z))H(z)^{n-1}dH(z) + \lambda_X^\infty C'(\alpha^*).
\]

(62)

Next we show that \(Y^+ \leq Y^\infty \leq Y^-\). Let \(S^+ = \{z \in W | Y^+(z) > 0\}\), \(S^- = \{z \in W | Y^-(z) > 0\}\)
$0$ and $S = \{ z \in W | Y^\infty(z) > 0 \}$. By construction,

$$S^+ \equiv \cup \{ (z, \bar{z}) | J^\infty(H(z)) > H^\infty(z) \} \forall z \in (z, \bar{z})$$

and

$$S^- \equiv \cup \{ (z, \bar{z}) | \varphi^\infty(H(\cdot)) \text{ is constant on } (z, \bar{z}) \}.$$ 

It follows from Lemma 23 and Lemma 25 that $S^+ \subset S \subset S^-$. If $z \notin S^-$, then $Y^+(z) = Y^\infty(z) = Y^-(z) = 0$. If $z \in S^- \setminus S$, then $Y^+(z) = Y^\infty(z) = 0 < Y^-(z)$. Consider $z \in S \subset S^-$, then there exists $(z, \bar{z})$ with $z < z < \bar{z}$ such that $\varphi^\infty(H(\cdot))$ is constant on $(z, \bar{z})$. Let $(z, \bar{z})$ be chosen maximally, then by Corollary 5, $Y^\infty(z) = Y^\infty(\bar{z}) = 0$. For any $z \in (z, \bar{z})$,

$$Y^-(z) - Y^\infty(z) = \int_{z}^{\bar{z}} [Q^\infty(\zeta) - Q^-((\zeta)] dH(\zeta).$$

Then for any $t \in (H(z), H(\bar{z}))$,

$$[Y^-(H^{-1}(t)) - Y^\infty(H^{-1}(t))]' = Q^-(H^{-1}(t)) - Q^\infty(H^{-1}(t)),$$

which is non-increasing on $(H(z), H(\bar{z}))$ since $Q^\infty$ is non-decreasing and $Q^-$ is constant on $(z, \bar{z})$ by construction. Hence $Y^-(H^{-1}(t)) - Y^\infty(H^{-1}(t))$ is concave on $(H(z), H(\bar{z}))$. Since

$$Y^-(z) - Y^\infty(z) = 0$$

and $Y^-(z) - Y^\infty(\bar{z}) = 0$, we have $Y^-(z) - Y^\infty(z) \geq 0$ for all $z \in (z, \bar{z})$. Thus $Y^-(z) - Y^\infty(z) \geq 0$ for all $z \in S$. If $z \in S \setminus S^+$, then $Y^-(z) \geq Y^\infty(z) \geq 0 = Y^+(z)$.

Finally, consider $z \in S^+ \subset S$, it suffices to show that $Y^+(z) \leq Y^\infty(z)$. By construction there exists $(z, \bar{z})$ with $z < z < \bar{z}$ such that $\tilde{J}^\infty(H(z)) < J^\infty(H(z))$ for all $z \in (z, \bar{z})$. Let $(z, \bar{z})$ be chosen maximally, then by construction $Y^+(z) = Y^+(\bar{z}) = 0$. For any $z \in (z, \bar{z})$

$$Y^+(z) - Y^\infty(z) = \int_{z}^{\bar{z}} [Q^\infty(\zeta) - Q^-((\zeta)] dH(\zeta) - Y^\infty(z).$$
Then for any \( t \in (H(\bar{z}), H(\bar{z})) \),

\[
[Y^+(H^{-1}(t)) - Y^\infty(H^{-1}(t))]' = Q^+(H^{-1}(t)) - Q^\infty(H^{-1}(t)),
\]

which is constant on \((H(\bar{z}), H(\bar{z}))\) since \( Q^\infty \) is constant on \((\bar{z}, \bar{z})\) by Lemma 23 and \( Q^- \) is constant on \((\bar{z}, \bar{z})\) by construction. Hence \( Y^+(H^{-1}(t)) - Y^\infty(H^{-1}(t)) \) is affine on \((H(\bar{z}), H(\bar{z}))\).

Since \( Y^+(\bar{z}) = 0 \leq Y^\infty(\bar{z}) \) and \( Y^+(\bar{z}) = 0 \leq Y^\infty(\bar{z}) \), we have \( Y^+(z) - Y^\infty(z) \leq 0 \) for all \( z \in (\bar{z}, \bar{z}) \). Thus \( Y^+(z) - Y^\infty(z) \leq 0 \) for all \( z \in S^+ \).

Furthermore, for any implementable allocation rule \( Q \), we have

\[
\int_{\bar{w}}^w zQ(z)H(z)\,dz = \int_{\bar{w}}^w zY'(z)dz + \int_{\bar{w}}^w zH(z)^n-1H(z)\,dz = \int_{\bar{w}}^w zH(z)^n-1H(z)\,dz - \int_{\bar{w}}^w Y(z)dz.
\]

Hence

\[
\int_{\bar{w}}^w zQ^+(z)H(z) \geq \int_{\bar{w}}^w zQ^\infty(z)H(z) \geq \int_{\bar{w}}^w zQ^-(z)H(z).
\]

Since \( \lambda_X^\infty > 0 \), combining this and (62) yields

\[
X^+(\bar{w}) \leq -C'(\alpha^*) \leq X^-(\bar{w}).
\]

Since \( X \) is linear in \( Q \), there exists \( \rho \in [0,1] \) such that \( \hat{Q} = \rho Q^+(1-\rho)Q^- \) is implementable, non-decreasing and

\[
\int_{\bar{w}}^w \frac{H_{\alpha}(z)}{h(z)}\hat{Q}(z)h(z)dz = -C'(\alpha^*).
\]

Then both \( Q^\infty \) and \( \hat{Q} \) yield the same expected surplus.

**Lemma 26** Let \( t \in (0,1) \). If \( J(t) = J(t) \), then \( J \) is continuously differentiable at \( t \) with
derivative $\varphi(t) = \varphi(t)$ and $\varphi'(t) \geq 0$. Furthermore, $J(t) = \overline{J}(t)$ if and only if

$$J(\tau) \geq (\tau - t)\varphi(t) + J(t), \forall \tau \in [0, 1].$$  \hspace{1cm} (64)

**Proof.** Suppose $J(t) = \overline{J}(t)$. Suppose to the contrary that $\overline{J}$ is not continuously differentiable at $t$, then $\varphi(t^-) < \varphi(t^+)$. Then either $\varphi(t) > \varphi(t^-)$ or $\varphi(t) < \varphi(t^+)$. Assume without loss of generality that $\varphi(t) < \varphi(t^+)$. Since $\varphi$ is continuous and $\varphi$ is non-decreasing, there exists $\delta > 0$ such that $\varphi(\tau) < \varphi(t^+) \leq \varphi(\tau)$ for all $\tau \in (t, t + \delta)$. Then

$$J(t + \delta) = J(t) + \int_t^{t+\delta} \varphi(\tau)d\tau < \overline{J}(t) + \int_t^{t+\delta} \varphi(\tau)d\tau = \overline{J}(t + \delta),$$

a contradiction. Hence $\overline{J}$ is continuously differentiable at $t$. It follows from a similar argument that $\overline{\varphi}(t) = \varphi(t)$ with $\varphi'(t) \geq 0$. Furthermore, for all $\tau \in [0, 1],

$$J(\tau) \geq \overline{J}(\tau) \geq (\tau - t)\varphi(t) + \overline{J}(t) = (\tau - t)\varphi(t) + J(t),$$

where the second inequality holds since $\overline{J}$ is convex.

Suppose (64) holds. Then $(\tau - t)\varphi(t) + J(t)$ is a convex function below $J$. Since $\overline{J}$ is the greatest convex function below $J$, we have

$$\overline{J}(\tau) \geq (\tau - t)\varphi(t) + J(t) \forall \tau \in [0, 1].$$

When $\tau = t$, we have $\overline{J}(t) \geq J(t)$. Hence $\overline{J}(t) = J(t)$. $\blacksquare$

**Lemma 27** Let $\lambda'_x > \lambda_x$. Suppose $\overline{\varphi}^\lambda_x$ is constant on $(\underline{t}, \overline{t})$ with $\underline{t} < \overline{t}$ and let $(\underline{t}, \overline{t})$ be chosen maximally. Then there exists $\delta > 0$ such that $J^{\lambda_x}(t) > \overline{J}^{\lambda_x}(t)$ for all $t \in (\underline{t} - \delta, \overline{t} + \delta).

**Proof.** Fix $t \in (\underline{t}, \overline{t})$. Let $\beta \in (0, 1)$ be such that $\beta \underline{t} + (1 - \beta)\overline{t} = t$. It suffices to show that

$$J^{\lambda_x}(t) > \beta J^{\lambda_x}(\underline{t}) + (1 - \beta)J^{\lambda_x}(\overline{t}).$$
Since $\varphi^\lambda_x$ is constant on $(t, \bar{t})$, we have

$$J^\lambda_x(t) \geq J^\lambda_x(t)$$

$$= \beta J^\lambda_x(t) + (1 - \beta) J^\lambda_x(\bar{t})$$

$$= \beta J^\lambda_x(t) + (1 - \beta) J^\lambda_x(\bar{t}),$$

where the last line holds since $J^\lambda_x(t) = J^\lambda_x(t)$ and $J^\lambda_x(\bar{t}) = J^\lambda_x(\bar{t})$. To see that $J^\lambda_x(t) = J^\lambda_x(t)$, suppose to the contrary that $J^\lambda_x(t) > J^\lambda_x(t)$. Then $\varphi^\lambda_x(t)$ is constant in a neighborhood of $t$. A contradiction to that $(t, \bar{t})$ is chosen maximally. A similar argument proves that $J^\lambda_x(\bar{t}) = J^\lambda_x(\bar{t})$. Furthermore,

$$0 \leq J^\lambda_x(t) - \beta J^\lambda_x(t) - (1 - \beta) J^\lambda_x(\bar{t})$$

$$= \int_0^t H^{-1}(\zeta)d\zeta - \beta \int_0^t H^{-1}(\zeta)d\zeta - (1 - \beta) \int_0^\bar{t} H^{-1}(\zeta)d\zeta$$

$$+ \lambda_x \left[ \int_0^t \frac{H_\alpha(H^{-1}(\zeta))}{h(H^{-1}(\zeta))} d\zeta - \beta \int_0^t \frac{H_\alpha(H^{-1}(\zeta))}{h(H^{-1}(\zeta))} d\zeta - (1 - \beta) \int_0^\bar{t} \frac{H_\alpha(H^{-1}(\zeta))}{h(H^{-1}(\zeta))} d\zeta \right].$$

Since $H^{-1}(\cdot)$ is strictly increasing, $\int_0^t H^{-1}(\zeta)d\tau$ is strictly convex in $t$ and therefore

$$\int_0^t H^{-1}(\zeta)d\zeta - \beta \int_0^t H^{-1}(\zeta)d\zeta - (1 - \beta) \int_0^\bar{t} H^{-1}(\zeta)d\zeta < 0.$$

Hence

$$\int_0^t \frac{H_\alpha(H^{-1}(\zeta))}{h(H^{-1}(\zeta))} d\zeta - \beta \int_0^t \frac{H_\alpha(H^{-1}(\zeta))}{h(H^{-1}(\zeta))} d\zeta - (1 - \beta) \int_0^\bar{t} \frac{H_\alpha(H^{-1}(\zeta))}{h(H^{-1}(\zeta))} d\zeta > 0.$$
Hence, we have
\[
J^\lambda_X(t) - \beta J^\lambda_X(t) - (1 - \beta) J^\lambda_X(t)
\]
\[
= \int_0^t H^{-1}(\zeta) d\zeta - \beta \int_0^t H^{-1}(\zeta) d\zeta - (1 - \beta) \int_0^\tau H^{-1}(\zeta) d\zeta
\]
\[
+ \lambda_X \left[ \int_0^t \frac{H_\alpha(H^{-1}(\zeta))}{h(H^{-1}(\zeta))} d\zeta - \beta \int_0^t \frac{H_\alpha(H^{-1}(\zeta))}{h(H^{-1}(\zeta))} d\zeta - (1 - \beta) \int_0^\tau \frac{H_\alpha(H^{-1}(\zeta))}{h(H^{-1}(\zeta))} d\zeta \right]
\]
\[
> \int_0^t H^{-1}(\zeta) d\zeta - \beta \int_0^t H^{-1}(\zeta) d\zeta - (1 - \beta) \int_0^\tau H^{-1}(\zeta) d\zeta
\]
\[
+ \lambda_X \left[ \int_0^t \frac{H_\alpha(H^{-1}(\zeta))}{h(H^{-1}(\zeta))} d\zeta - \beta \int_0^t \frac{H_\alpha(H^{-1}(\zeta))}{h(H^{-1}(\zeta))} d\zeta - (1 - \beta) \int_0^\tau \frac{H_\alpha(H^{-1}(\zeta))}{h(H^{-1}(\zeta))} d\zeta \right]
\]
\[
\geq J^\lambda_X(t) - \beta J^\lambda_X(t) - (1 - \beta) J^\lambda_X(t)
\]
\[
\geq 0.
\]

Consider \( \bar{t} \). Since \( J^\lambda_X(\bar{t}) \) is constant on \((\bar{t}, \bar{t})\) with \( J^\lambda_X(\bar{t}) = \bar{J}^\lambda_X(t) \) and \( J^\lambda_X(\bar{t}) = \bar{J}^\lambda_X(\bar{t}) \), by Lemma 26, we have
\[
\bar{J}^\lambda_X(\bar{t}) = J^\lambda_X(\bar{t})
\]
\[
\geq (\bar{t} - t) \varphi^\lambda_X(t) + J^\lambda_X(t)
\]
\[
=(\bar{t} - t) \varphi^\lambda_X(t) + \bar{J}^\lambda_X(t) = \bar{J}^\lambda_X(\bar{t}).
\]

Hence, \( J^\lambda_X(\bar{t}) = (\bar{t} - t) \varphi^\lambda_X(t) + J^\lambda_X(t) \). Thus,
\[
\int_\bar{t}^\tau H^{-1}(\tau) d\tau - (\bar{t} - t) H^{-1}(t) = \lambda_X \left[ (\bar{t} - t) \frac{H_\alpha(H^{-1}(\bar{t}))}{h(H^{-1}(\bar{t}))} - \int_\bar{t}^\tau \frac{H_\alpha(H^{-1}(\tau))}{h(H^{-1}(\tau))} \right].
\]

Since \( H^{-1}(t) \) is strictly increasing, the left-hand side of the above equality is strictly positive. Hence for \( \lambda' > \lambda > 0 \) we have
\[
\int_\bar{t}^\tau H^{-1}(\tau) d\tau - (\bar{t} - t) H^{-1}(t) < \lambda' \left[ (\bar{t} - t) \frac{H_\alpha(H^{-1}(\bar{t}))}{h(H^{-1}(\bar{t}))} - \int_\bar{t}^\tau \frac{H_\alpha(H^{-1}(\tau))}{h(H^{-1}(\tau))} \right],
\]

\[75\]
i.e.,

\[
J^{X^t}(\bar{t}) < (\bar{t} - t)\varphi^{X^t}(t) + J^{X^t}(t).
\]

By Lemma 26, \(J^{X^t}(t) > J^{X^t}(\bar{t})\).

A similar argument proves that \(J^{X^t}(\bar{t}) > J^{X^t}(\bar{t})\). By continuity, there exists \(\delta > 0\) such that \(J^{X^t}(t) > J^{X^t}(\bar{t})\) for all \(t \in (t - \delta, \bar{t} + \delta)\).

\begin{lemma}
Let \([\underline{z}, \bar{z}] \subset [\underline{w}, \bar{w}]\) with \(\underline{z} < \bar{z}\), and \(z^0 \in (\underline{z}, \bar{z})\). Suppose \(Q : [\underline{\theta}, \bar{\theta}] \to [0, 1]\) and \(\bar{Q} : [\underline{\theta}, \bar{\theta}] \to [0, 1]\) satisfying that
\[
\int_{\underline{z}}^{\bar{z}} Q(z)h(z)dz = \int_{\underline{z}}^{\bar{z}} \bar{Q}(z)h(z)dz,
\]
and
\[
Q(z) \geq \bar{Q}(z) \text{ if } z > z^0, \text{ and } Q(z) \leq \bar{Q}(z) \text{ if } z < z^0.
\]

Then
\[
\int_{\underline{z}}^{\bar{z}} -\frac{H_{\alpha_i}(z)}{h(z)}[Q(z) - \bar{Q}(z)]h(z)dz \geq 0,
\]
where the inequality holds strictly if the set \(\{z \in [\underline{z}, \bar{z}]|Q(z) \neq \bar{Q}(z)\}\) has a positive measure.

\end{lemma}

\begin{proof}
Since \(-\frac{H_{\alpha_i}(w_i)}{h(w_i)}\) is strictly increasing in \(w_i\), and \(Q\) and \(\bar{Q}\) satisfy (66), we have
\[
\int_{\underline{z}}^{\bar{z}} \left[-\frac{H_{\alpha_i}(z)}{h(z)} + \frac{H_{\alpha_i}(z^0)}{h(z^0)}\right][Q(z) - \bar{Q}(z)]h(z)dz \geq 0,
\]
where the inequality holds strictly if the set \(\{z \in [\underline{z}, \bar{z}]|Q(z) \neq \bar{Q}(z)\}\) has a positive measure.

This implies inequality (67) by (65).

By Lemmas 27 and 28 we have
\[
\int_{\underline{w}}^{\bar{w}} -\frac{H_{\alpha_i}(z)}{h(z)}Q^+(z, \lambda_X^t)h(z)dz < \int_{\underline{w}}^{\bar{w}} -\frac{H_{\alpha_i}(z)}{h(z)}Q^-(z, \lambda_X^t)h(z)dz.
\]

Hence, \(\lambda_X\) is unique.
A.3 Sufficient Conditions for the First-order Approach

In this section I provide sufficient conditions for the first-order approach to be valid. Let \( \pi(\alpha) \) denote an agent \( i \)'s payoff from choosing \( \alpha \) given mechanism \( (q, t) \) and \( \alpha_j = \alpha^* \) for all \( j \neq i \). Then

\[
\pi(\alpha_i) = U(w(0, \alpha_i)) + \int_{w(0, \alpha_i)}^{w(1, \alpha_i)} [1 - H(w_i|\alpha_i)] Q(w_i)dw_i - C(\alpha_i),
\]

where \( Q \) is defined by (2) for \( \alpha_j = \alpha^* \) for all \( j \neq i \). Then

\[
\pi'(\alpha_i) = \int_{w(0, \alpha_i)}^{w(1, \alpha_i)} -H(\alpha_i|\alpha)Q(w_i)dw_i - C'(\alpha_i).
\]

A sufficient condition for the first-order approach to be valid is that \( \pi'(\alpha_i) \) is strictly decreasing. Note that this also implies (10) holds for \( Q(w_i) = H(w_i|\alpha)^{n-1} \). If the support of the conditional expectation \( [w(0, \alpha_i), w(1, \alpha_i)] \) is invariant, then \( \pi'(\alpha_i) \) is strictly decreasing if \( -H(\alpha_i|\alpha_i) \) has the single-crossing property in \( (\alpha_i; w_i) \) and \( C'(\alpha_i) \) is strictly decreasing. Suppose \( C \) is twice continuously differentiable, then

\[
\pi''(\alpha_i) = \int_{w(0, \alpha_i)}^{w(1, \alpha_i)} -\frac{\partial^2 H(w_i|\alpha)}{\partial \alpha_i^2} Q(w_i)dw_i - H(\alpha_i(w(1, \alpha_i)|\alpha_i))w_{\alpha_i}(1, \alpha_i)Q(w(1, \alpha_i))
\]

\[
+ H(\alpha_i(w(0, \alpha_i)|\alpha_i))w_{\alpha_i}(0, \alpha_i)Q(w(0, \alpha_i)) - C''(\alpha_i),
\]

\[
\leq \int_{w(0, \alpha_i)}^{w(1, \alpha_i)} -\frac{\partial^2 H(w_i|\alpha)}{\partial \alpha_i^2} Q(w_i)dw_i - H(\alpha_i(w(1, \alpha_i)|\alpha_i))w_{\alpha_i}(1, \alpha_i)Q(w(1, \alpha_i)) - C''(\alpha_i).
\]

The inequality holds since \( H(\alpha_i(w(0, \alpha_i)|\alpha_i) \geq 0 \) and \( w_{\alpha_i}(0, \alpha_i) \leq 0 \) by Assumption 1. The following proposition from Shi (2012) gives sufficient condition for \( \pi''(\alpha) < 0 \) for the two leading examples.

**Proposition 5 (Shi (2012))** Sufficient conditions for first order approach:

- In the linear experiments, if \( \alpha_iC''(\alpha_i) \geq f(\theta)(\theta - \mu)^2 \) for all \( \alpha_i \), then \( \pi''(\alpha_i) < 0 \) when either \( F(\theta) \) is convex of \( F(\theta) = \theta^b \) (\( b > 0 \)) with support \([0, 1]\).
• In the normal experiments, \( \pi''(\alpha_i) < 0 \) if \( \sqrt{\beta^3}/[\alpha_i^3(\alpha_i + \beta)^5] < 2\sqrt{2\pi C''(\alpha_i)} \) for all \( \alpha_i \).

### A.4 Efficient Asymmetric Mechanisms

#### A.4.1 Proof of Theorem 3

As in the symmetric case, I prove Theorem 3 by proving the following two lemmas. Define

\[
Y(w) \equiv 1 - \prod_{i=1}^{n} H(w_i|\alpha_i^*) - \sum_{i=1}^{n} \int_{w_i}^{w(1,\alpha_i^*)} Q_i(z_i) dH(z_i|\alpha_i^*), \forall w \in \prod_{i=1}^{n} [w(0, \alpha_i^*), w(1, \alpha_i^*)].
\]

Observe that if \( \alpha_i = \alpha_i^* \) for all \( i \), then (12) is violated if and only if \( \inf_{w-i} Y(w(0, \alpha^*), w_{-i}) > 0 \) for some \( i \). Then

**Lemma 29** Suppose \( \alpha_i = \alpha_i^* \) for all \( i \). Let \( Q \) be any interim allocation rule satisfying (AF'), (MON), (IA') and \( \inf_{w-i} Y(w(0, \alpha^*), w_{-i}) > 0 \) for some \( i \), then there exists \( \hat{Q} \) satisfying (F'), (MON) and (IA') such that \( \hat{Q}_j = Q_j \) for \( j \neq i \) and

\[
\hat{Q}_i(w_i) \geq Q_i(w_i), \forall w_i \in [w(0, \alpha_i^*), w(1, \alpha_i^*)],
\]

and "\( \geq \)" holds for a set of \( w_i \) with positive measure.

**Proof.** Define \( Y_i(w_i) \equiv \inf_{w-i} Y(w) \) for all \( w_i \in [w(0, \alpha_i^*), w(1, \alpha_i^*)] \). By Theorem 3 in Milgrom and Segal (2002), \( Y_i \) is differentiable and \( Y_i'(w_i) = -h(w_i|\alpha_i^*) \prod_{j \neq i} H(w^*_j|\alpha_j^*) + Q_i(w_i)h(w_i|\alpha_i^*) \) where \( w^*_i \) is such that \( Y(w_i, w^*_i) = Y_i(w_i) \) for all \( w_i \in (w(0, \alpha_i^*), w(1, \alpha_i^*)) \).

By assumption, there exists \( i \) such that \( Y_i(w(0, \alpha_i^*)) > 0 \). Define \( w^b \equiv \sup \{w_i | Y_i(w'_i) > 0 \forall w(0, \alpha_i^*) \leq w'_i \leq w_i \} \). By the continuity of \( Y_i \), we have \( Y_i(w^b) = 0 \) and \( w^b > w(0, \alpha_i^*) \). There are four cases to consider.

**Case I:** Suppose there exists \( w'_i \in (w(0, \alpha_i^*), w^b) \) such that \( Q_i \) is discontinuous at \( w'_i \). Let \( Q_i(w^i_+) \) denote the right-hand limit of \( Q_i \) at \( w'_i \), and \( Q_i(w^i_-) \) the corresponding left-hand
limit. Let \( 0 < \varepsilon \leq \min \left\{ \min_{w(0, \alpha^*) \leq w_i \leq w'_i} \frac{Y_i(w_i)}{H(w'_i | \alpha^*_i)}, Q_i(w'_i^+) - Q_i(w'_i^-) \right\} \), and \( \hat{Q}_i \) be such that \( \hat{Q}_i(w_i) \equiv Q_i(w_i) \) for \( w_i \leq w(0, \alpha^*) \) and for \( w_i > w(0, \alpha^*) \)

\[
\hat{Q}_i(w_i) \equiv Q_i(w_i) + \varepsilon \chi_{\{w_i \leq w'_i\}},
\]

where \( \chi_{\{w_i \leq w'_i\}} \) is the indicator function. Let \( \hat{Q}_j \equiv Q_j \) for all \( j \neq i \). By construction, \( \hat{Q}_i(w) \geq Q_i(w) \) for all \( w_i \in W_i \) and the inequality holds strictly on a positive measure set. By a similar argument to that in the proof of Lemma 2, \( \hat{Q}_i \) satisfies (MON) and (IA'). We now verify that \( \hat{Q} \) satisfies (AF'). If \( w_i \leq w'_i \), then \( \hat{Y}(w_i, w_{-i}) = Y(w_i, w_{-i}) - \varepsilon [H(w'_i | \alpha^*) - H(w_i | \alpha^*)] \geq Y(w_i, w_{-i}) - \varepsilon H(w'_i | \alpha^*_{-i}) \geq 0 \) for all \( w_{-i} \). If \( w_i > w'_i \), then \( \hat{Y}(w_i, w_{-i}) = Y(w_i, w_{-i}) \geq 0 \) for all \( w_{-i} \). That is, \( \hat{Q} \) satisfies (AF').

**Case II:** Suppose \( Q_i \) is continuous on \( [w(0, \alpha^*_i), w^b] \). We first show that there exists \( w'_i \in (w(0, \alpha^*_i), w^b) \) such that \( Q_i(w'_i) < Q_i(w^b) \). Suppose to the contrary that \( Q_i(w_i) = Q_i(w^b) \) for all \( w_i \in (w(0, \alpha^*_i), w^b) \). Let \( w^b_{-i} \) be such that \( Y(w^b, w^b_{-i}) = Y_i(w^b) = 0 \). If \( Q_i(w^b) \geq \prod_{j \neq i} H(w_j^* | \alpha^*_j) \), then \( Y(w(0, \alpha^*_i), w^b_{-i}) = Y(w^b, w^b_{-i}) + \int_{w(0, \alpha^*_i)}^{w^b} \left[ \prod_{j \neq i} H(w_j^* | \alpha^*_j) - Q_i(z) \right] h(z | \alpha^*_i) dz < 0 \), a contradiction. If \( Q_i(w^b) < \prod_{j \neq i} H(w_j^* | \alpha^*_j) \), then, by the continuity of \( Q_i \) and \( H \), there exists \( \delta > 0 \) such that \( Q_i(w_i) < \prod_{j \neq i} H(w_j^* | \alpha^*_j) \) for all \( w_i \in [w^b, w^b + \delta] \). Moreover,

\[
0 = Y(w^b, w^b_{-i}) = \int_{w^b}^{w^b + \delta} \left[ \prod_{j \neq i} H(w_j^* | \alpha^*_j) - Q_i(z) \right] h(z | \alpha^*_i) dz + Y(w^b + \delta, w^b_{-i}) > Y(w^b + \delta, w^b_{-i}),
\]

a contradiction. Thus there exists \( w'_i \in (w(0, \alpha^*_i), w^b) \) such that \( Q_i(w'_i) < Q_i(w^b) \).

By the continuity of \( Q_i \), there exists \( w''_i \in (w'_i, w^b) \) such that \( Q_i(w''_i) = \frac{1}{2} (Q_i(w'_i) + Q_i(w^b)) \). Let \( 0 < \varepsilon \leq \min \left\{ \min_{w(0, \alpha^*) \leq w_i \leq w''_i} \frac{Y_i(w_i)}{H(w'_i | \alpha^*_i)}, Q_i(w''_i) - Q_i(w'_i) \right\} \). Let
\( \hat{Q}_j \equiv Q_j \) for \( j \neq i \) and

\[
\hat{Q}_i(w_i) = \begin{cases} 
\max\{Q_i(w_i') + \varepsilon, Q_i(w_i)\} & \text{if } w_i > w_i', \\
Q_i(w_i) + \varepsilon & \text{if } w(0, \alpha_i^+) < w_i \leq w_i', \\
Q_i(w_i) & \text{if } w_i \leq w(0, \alpha_i^+). 
\end{cases}
\]

Note that if \( w_i \geq w_i'' \) then \( Q_i(w_i) \geq Q_i(w_i'') \geq Q_i(w_i') + \varepsilon \). Thus, \( \hat{Q}_i(w_i) = Q_i(w_i) \) for \( w_i \geq w_i'' \). By construction, \( \hat{Q}_i(w_i) \geq Q_i(w_i) \) for all \( w_i \in W_i \) and the inequality holds strictly on a positive measure set. By a similar argument to that in the proof of Lemma 2, \( \hat{Q}_i \) satisfies (MON) and (IA'). We now verify that \( \hat{Q} \) satisfies (AF'). If \( w_i \geq w_i'' \), then

\[
\hat{Y}(w_i, w_{-i}) = Y(w_i, w_{-i}) \geq 0 \quad \text{for all } w_{-i}. \]

If \( w_i < w_i'' \), then for all \( w_{-i} \),

\[
\hat{Y}(w_i, w_{-i}) = Y(w_i, w_{-i}) - \int_{w_i}^{w_i''} \left[ \hat{Q}_i(z) - Q_i(z) \right] h_i(z|\alpha_i^*) \, dz,
\]

\[
\geq Y(w_i, w_{-i}) - \varepsilon [H(w_i''|\alpha_i^*) - H(w_i|\alpha_i^*)],
\]

\[
\geq Y(w_i, w_{-i}) - \varepsilon H(w_i''|\alpha_i^*) \geq 0.
\]

Hence, \( \hat{Q} \) satisfies (AF').

**Case III:** Let \( w_{-i}^* \) be such that \( Y(w_b, w_{-i}^*) = Y_i(w_b) = 0 \). Suppose \( Q_i \) is continuous on \([w(0, \alpha_i^+), w_b]\) and \( Q_i(w_b^-) < \prod_{j \neq i} H(w_j^*|\alpha_j^*) \). Define \( R(w_i) \equiv Y_i(w_i)/(H(w_b^*|\alpha_i^*) - H(w_i|\alpha_i^*)) \) for \( w_i < w_b \). Then by Theorem 3 in Milgrom and Segal (2002) and L'Hopital's rule,

\[
\lim_{w_i \to w_b^-} R(w_i) = \prod_{j \neq i} H_j(w_j^*|\alpha_j^*) - Q_i(w_b^-) > 0.
\]

Let \( 0 < \varepsilon \leq \min\{\inf_{w(0, \alpha_i^+)} R(w_i), Q_i(w_b^+) - Q_i(w_b^-)\} \). Let \( \hat{Q}_j \equiv Q_j \) for all \( j \neq i \), \( \hat{Q}_i(w_i) \equiv Q_i(w_i) \) for \( w_i \leq w(0, \alpha_i^+) \) and \( \hat{Q}_i(w_i) \equiv Q_i(w_i) + \varepsilon \chi_{\{w_i < w_b\}} \) for \( w_i > w(0, \alpha_i^+) \).

By construction, \( \hat{Q}_i(w_i) \geq Q_i(w_i) \) for all \( w_i \in W_i \) and the inequality holds strictly on a positive measure set. One can verify that \( \hat{Q}_i \) satisfies (MON) and (IA') by an argument.
similar to that in the proof of Lemma 2. Finally, if \( w_i < w^b \), then \( \hat{Y}(w_i, w_{-i}) = Y(w_i, w_{-i}) - \varepsilon[H(w^b|\alpha_i^*) - H(w_i|\alpha_i^*)] \geq Y(w_i, w_{-i}) - R(w_i)[H(w^b|\alpha^*) - H(w_i|\alpha^*)] = 0 \) for all \( w_{-i} \). If \( w_i \geq w^b \), then \( \hat{Y}(w_i, w_{-i}) = Y(w_i, w_{-i}) \geq 0 \) for all \( w_{-i} \). Hence, \( \hat{Q} \) satisfies (AF').

**Case IV:** Let \( w^*_{-i} \) be such that \( Y(w^b, w^*_{-i}) = Y_i(w^b) = 0 \). Suppose \( Q_i \) is continuous on \([w(0, \alpha_i^*), w^b]\) and \( Q_i(w^b^-) = \prod_{j \neq i} H(w_j^*|\alpha_j^*) \). We first show that \( Q_i(w^b^-) = \prod_{j \neq i} H(w_j^*|\alpha_j^*) \).

Suppose to the contrary that \( Q_i(w^b^-) > \prod_{j \neq i} H(w_j^*|\alpha_j^*) \). Then by the continuity of \( Q_i \) and \( H \) on \([w(0, \alpha^*), w^b]\), there exists \( \delta > 0 \) such that \( Q_i(w) > \prod_{j \neq i} H(w_j^*|\alpha_j^*) \) for all \( w \in (w^b - \delta, w^b) \). Then

\[
Y(w^b - \delta, w_{-i}) = \int_{w^b - \delta}^{w^b} \left[ \prod_{j \neq i} H(w_j^*|\alpha_j^*) - Q_i(z) \right] h(z|\alpha_i^*)dz < 0,
\]

a contradiction. Hence, \( Q_i(w^b^-) = \prod_{j \neq i} H(w_j^*|\alpha_j^*) \). Second, we show that there exists \( w'_i \in (w(0, \alpha_i^*), w^b) \) such that \( Q_i(w'_i) < Q_i(w^b^-) \). Suppose to the contrary that \( Q_i(w_i) = Q_i(w^b^-) \) for all \( w_i \in (w(0, \alpha^*), w^b) \), then \( Y(w(0, \alpha^*), w_{-i}) = \int_{w(0, \alpha^*)}^{w^b} \left[ \prod_{j \neq i} H(w_j^*|\alpha_j^*) - Q_i(z) \right] h(z|\alpha_i^*)dz < 0 \), a contradiction. Hence, there exists \( w'_i \in (w(0, \alpha_i^*), w^b) \) such that \( Q_i(w'_i) < Q_i(w^b^-) \). The rest of the proof follows that of Case II. ■

**Lemma 30** Suppose two implementable allocation rules \( Q \) and \( \hat{Q} \) satisfy (5). Let \( q \) be an ex-post allocation rule that implements \( Q \), then there exists an ex-post allocation rule \( \hat{q} \) that implements \( \hat{Q} \) and satisfies

\[
\mathbb{E} \left[ \sum_i (w_i + \gamma \sum_{j \neq i} w_j)\hat{q}_i(w) \mid \alpha_i = \alpha^* \forall i \right] > \mathbb{E} \left[ \sum_i (w_i + \gamma \sum_{j \neq i} w_j)q_i(w) \mid \alpha_i = \alpha^* \forall i \right].
\]

The proof of Lemma 30 relies on the following technical lemma. For each \( i \), let \( m_i \) denote the probability measure on \([w(0, \alpha_i^*), w(1, \alpha_i^*)]\) corresponding to \( H(w_i|\alpha_i^*) \), then

**Lemma 31** Let \( Q \) be implementable and \( \rho : \prod_i [w(0, \alpha_i^*), w(1, \alpha_i^*)] \to [0, 1] \) be a measurable function. Then there exists an ex-post allocation rule \( q \) implementing \( Q \) such that \( \sum_i q_i(w) \geq \rho(w) \). 

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\( \rho(w) \) for almost all \( w \in \prod_i [w(0, \alpha_i^*), w(1, \alpha_i^*)] \) if and only if for each measurable set \( A = (A_1, \ldots, A_n) \) where \( A_i \subset [w(0, \alpha_i^*), w(0, \alpha_i^*)] \) for each \( i \), the following inequality is satisfied

\[
\sum_i \int_{A_i} Q(w_i) dm_i(w_i) \geq \int_A \rho(w) dm_1(w_1) \ldots dm_n(w_n).
\] (69)

The proof of Lemma 4 can be extended to prove Lemma 31 in a similar fashion to that of Mierendorff (2011) and is neglected here. With Lemma 31 in hand, the proof of Lemma 3 can be readily extended to prove Lemma 30 and is neglected here. Theorem 3 follows immediately given Lemmas 29 and 30.

References


