Mechanism Design with Costly Verification and Limited Punishments

Yunan Li†
University of Pennsylvania

April 4, 2016

Abstract

A principal has to allocate a good among a number of agents, each of whom values the good. Each agent has private information about the principal’s payoff if he receives the good. There are no monetary transfers. The principal can inspect agents’ reports at a cost and penalize them, but the punishments are limited. I characterize an optimal mechanism featuring two thresholds. Agents whose values are below the lower threshold and above the upper threshold are pooled, respectively. If the number of agents is small, then the pooling area at the top of value distribution disappears. If the number of agents is large, then the two pooling areas meet and the optimal mechanism can be implemented via a shortlisting procedure.

Keywords: Mechanism Design, Costly Verification, Limited Punishments

JEL Classification: D82

*I would like to express my gratitude to Rakesh Vohra for his encouraging support since the very beginning of the project. I am also grateful to Tymofiy Mylovanov, Steven Matthews, Mariann Ollar, Mallesh Pai and the participants in seminars at University of Pennsylvania for valuable discussions. All remaining errors are my responsibility. Any suggestion or comment is welcome.

†Author’s addresses: Yunan Li, Economics Department, University of Pennsylvania, Philadelphia, PA, USA, yunanli@sas.upenn.edu.
1 Introduction

In many large organizations scarce resources must be allocated internally without the benefit of prices. Examples include the head of personnel for an organization choosing one of several applicants for a job, venture capital firms choosing which startup to fund and funding agencies allocating a grant among researchers. In these settings the principal must rely on verification of agents’ claims, which can be costly. For example, the head of personnel can verify an job applicant’ claim or monitor his performance once he is hired. A venture capital firm can investigate the competing startups and audit the progress of a startup once it is funded. Furthermore, the principal can punish an agent if the claims are found to be false. For example, the head of personnel can fire an employee or deny a promotion. Venture capitals and funding agencies can cut off funding.

Prior work has examined two extreme cases. In Ben-Porath et al. (2014), verification is costly, but punishment is unlimited in the sense that an agent can be rejected, and does not receive the good. In Mylovanov and Zapechelnyuk (2014), verification is free, but punishment is limited. In this paper I consider a situation with both costly verification and limited punishment. I interpret inspection or verification as acquiring information (e.g., requesting documentation, interviewing an agent, or monitoring an agent at work), which could be costly. Moreover, punishment can be limited because inspection is imperfect or information arrives only after the agent has been hired for a while.

The trade-off faced by the principal is as follows. The principal would like to give the good to an agent whose value is high, but this encourages all agents to exaggerate. The principal can discourage agents from exaggerating by rationing at the bottom/top or by verification and punishment. Rationing at the bottom/top makes it less attractive for a low value agent to misreport as a high value agent but reduces allocative efficiency. Verification and punishment clearly discourage agents from misreporting, but verification is costly. It is not obvious which is the best way to provide incentives.

In the main part of the paper, I consider the symmetric environment and characterize an
optimal symmetric mechanism in this setting. If the number of agents is sufficiently small, the optimal mechanism is an one-threshold mechanism as in Ben-Porath et al. (2014). The optimal allocation rule is efficient at the top of valuation distribution, and involves pooling at the bottom. For intermediate and large number of agents, the optimal allocation rule also involves pooling at the top as in Mylovanov and Zapechelnyuk (2014). Specifically, the optimal mechanism is a two-threshold mechanism. If there are agents whose values are above the upper threshold, one of them is chosen at random. If all agents’ values are below the upper threshold, but some are above the lower threshold, the one with the highest value is chosen. If all agents’ values are below the lower threshold, one of them is chosen at random. For a sufficiently large number of agents, the two thresholds coincide and the optimal mechanism can be implemented using a shortlisting procedure. Agents whose values are above the threshold are shortlisted for sure, and agents whose values are below the threshold are shortlisted with some probability. The principal chooses one agent from the shortlist at random. The fact that the optimal mechanism depends on the number of agents implies that small and large organizations should behave differently.

The distinction between small and intermediate number of agents is absent if either verification is free or punishment is unlimited. This distinction is important since it allows us to establish a connection between Ben-Porath et al. (2014) and Mylovanov and Zapechelnyuk (2014).

In Section 5.1, I study the general model with asymmetric agents. In this setting, threshold mechanisms are still optimal. The analysis, however, is much more complex. Though there is still a unique lower threshold for all agents, different agents may face different upper thresholds. Using this result, I revisit the symmetric environment and characterize the set of all optimal threshold mechanisms. I find that limiting the principal’s ability to punish agents also limit her ability to treat agents differently. In particular, when a one-threshold mechanism is optimal, the set of all optimal threshold mechanisms shrinks as the punishments become more limited. Eventually, the unique optimal mechanism is symmetric. If
the punishments are sufficiently limited so that a two-threshold mechanism or a shortlisting procedure, the principal can still treat agents differently but to the extent that they share the same set of thresholds. The comparison is less clear in this case because the sets of optimal mechanisms are disjoint for different punishments.

Technically, I use tools from linear programming, which allows me to analyze Ben-Porath et al. (2014) and Mylovanov and Zapechelnyuk (2014) in a unified way. It also allows me to get some results on optimal mechanisms in the asymmetric environment with limited punishments, unavailable in Mylovanov and Zapechelnyuk (2014).

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 characterizes symmetric optimal mechanisms when agents are ex ante identical. Section 4 discusses the properties of optimal mechanisms. Section 5.1 studies the general asymmetric environment. Section 5.2 considers a variation of the model in which punishments are independent of allocation.

### 1.1 Other Related Literature

This paper is related to the costly state verification literature. The first contribution in the series is Townsend (1979) who studies a model of a principal and a single agent. In Townsend (1979) verification is deterministic. Border and Sobel (1987) and Mookherjee and Png (1989) generalize it by allowing random inspection. Gale and Hellwig (1985) consider the effects of costly verification in the context of credit markets. These models differ from what I consider here in that in their models there is only one agent and monetary transfers are allowed.

Technically, this paper is related to literature on reduced form implementation — see, e.g., Maskin and Riley (1984), Matthews (1984), Border (1991a) and Mierendorff (2011). The most related one is Pai and Vohra (2014), who also use reduced form implementation and linear programming to derive optimal mechanisms for financially constrained agents.
2 Model

The set of agents is $I \equiv \{1, \ldots, n\}$. There is a single indivisible object to allocate among them. The value to the principal of assigning the object to agent $i$ is $v_i$, where $v_i$ is agent $i$’s private information. I assume $v_i$’s are independently distributed, whose density $f_i$ is strictly positive on $V_i := [u_i, v_i] \subset \mathbb{R}_+$. The assumption that an agent’s value to the principal is always non-negative simplifies some statements but the results can easily extend to include negative values. I use $F_i$ to denote the corresponding cumulative distribution function. Let $V = \prod_i V_i$. Agent $i$ gets a payoff of $b_i(v_i)$ if he receives the object, and 0 otherwise. The principal can inspect agent $i$’s report at a cost $k_i \geq 0$ if agent $i$ receives the object and $k_i^\beta \geq 0$ if agent $i$ does not receive the object. I allow for inspection costs to be different depending on whether an agent gets the object. This is natural in some environments. For example, if the object is a job slot and the private information is about an agent’s ability, then it is easier to inspect an agent who is hired. I assume inspection perfectly reveals an agent’s type and the cost to the agent of providing information is zero. If agent $i$ is inspected, the principal can impose a penalty $c_i(v_i) \geq 0$ if agent $i$ receives the object, and $c_i^\beta(v_i) \geq 0$ if agent $i$ does not receive the object. In Ben-Porath et al. (2014), the principal can inspect an agent whether or not he receives the object, i.e., $k_i = k_i^\beta$. However, the principal can only penalize an agent if he receives the object, i.e., $c_i^\beta = 0$. In Mylovanov and Zapechelnyuk (2014), the principal can only inspect and penalize an agent if he receives the object, i.e., $k_i^\beta = \infty$ and $c_i^\beta = 0$. For the rest of the paper, I follow Ben-Porath et al. (2014) and Mylovanov and Zapechelnyuk (2014) and assume that $c_i^\beta = 0$. The interpretation is that the principal can only penalize an agent by taking the object back possibly after a number of periods (e.g., rejecting a job applicant or firing him after a certain length of employment). In Section 5.2 I discuss to what extent this assumption can be relaxed.

I invoke the Revelation Principle and focus on direct mechanisms in which truth-telling is a Bayes-Nash equilibrium. Clearly, if an agent is inspected, it is optimal to penalize him if and only if he is found to have lied. A direct mechanism is a pair $(p, q)$, where $p_i : V \rightarrow [0, 1]$
denotes the probability $i$ is assigned the object, $q_i : V \rightarrow [0, 1]$ denotes the probability of inspecting $i$ conditional on the object being assigned to agent $i$, and $q_i^\beta : V \rightarrow [0, 1]$ denotes the probability of inspecting $i$ conditional on the object not being assigned to agent $i$. The mechanism is feasible if $\sum_i p_i(v) \leq 1$ for all $v \in V$. The principal’s objective function is

$$
\mathbb{E}_v \left[ \sum_{i=1}^n p_i(v) (v_i - q_i(v)k_i) - (1 - p_i(v))q_i^\beta(v)k_i^\beta \right].
$$

The incentive compatibility constraint for agent $i$ is

$$
\mathbb{E}_{v_{-i}} [p_i(v_i, v_{-i})b_i(v_i)] 
\geq \mathbb{E}_{v_{-i}} \left[ p_i(v'_i, v_{-i}) (b_i(v_i) - q_i(v'_i, v_{-i})c_i(v_i)) - (1 - p_i(v'_i, v_{-i}))q_i^\beta(v'_i, v_{-i})c_i^\beta(v) \right], \forall v_i, v'_i.
$$

Clearly, since $c_i^\beta = 0$, it is optimal to set $q_i^\beta = 0$. Then the principal’s objective function becomes

$$
\mathbb{E}_v \left[ \sum_{i=1}^n p_i(v) (v_i - q_i(v)k_i) \right].
$$

The incentive compatibility constraint for agent $i$ becomes

$$
\mathbb{E}_{v_{-i}} [p_i(v_i, v_{-i})b_i(v_i)] \geq \mathbb{E}_{v_{-i}} \left[ p_i(v'_i, v_{-i}) (b_i(v_i) - q_i(v'_i, v_{-i})c_i(v_i)) \right], \forall v_i, v'_i.
$$

Given $(p, q)$, let $P_i(v_i) = \mathbb{E}_{v_{-i}} [p_i(v_i, v_{-i})]$, and $Q_i(v_i) = \mathbb{E}_{v_{-i}} [p_i(v_i, v_{-i})q_i(v_i, v_{-i})] / P_i(v_i)$ if $P_i(v_i) \neq 0$ and $Q_i(v_i) = 0$ otherwise. The principal’s problem can be written in the reduced form:

$$
\max_{P, Q} \sum_{i=1}^n \mathbb{E}_{v_i} [P_i(v_i) (v_i - Q_i(v_i)k_i)],
$$

subject to

$$
P_i(v_i)b_i(v_i) \geq P_i(v'_i) (b_i(v_i) - Q_i(v'_i)c_i(v_i)), \forall v_i, v'_i, \quad (IC)
$$

$$
0 \leq Q_i(v_i) \leq 1, \forall v_i, \quad (F1)
$$
\[
\sum_i \int_{S_i} P_i(v_i) dF_i(v_i) \leq 1 - \prod_i \left( 1 - \int_{S_i} dF_i(v_i) \right), \forall S_i \subset V_i.
\]  

(F2)

If \( k_i = 0 \), then this is Mylovanov and Zapechelnyuk (2014). If \( c_i(v_i) = b_i(v_i) \), then this is Ben-Porath et al. (2014).

In both Mylovanov and Zapechelnyuk (2014) and Ben-Porath et al. (2014), it is easy to solve for the optimal \( Q_i \). If \( k_i = 0 \), then \( Q(v_i) = 1 \) for all \( v_i \in V_i \). If \( c_i(v_i) = b_i(v_i) \), then the (IC) constraint becomes \( P_i(v_i)b_i(v_i) \geq P_i(v'_i)b_i(v'_i) (1 - Q_i(v'_i)) \) for all \( v_i \) and \( v'_i \). The (IC) constraint holds if and only if

\[
\inf_{v_i} P_i(v_i) \geq P_i(v'_i) (1 - Q_i(v'_i)), \forall v'_i.
\]

Since the principal’s objective function is strictly decreasing in \( Q_i \), it is optimal to set \( Q_i(v_i) = 1 - \varphi_i/P_i(v_i) \) for all \( v_i \in V_i \), where \( \varphi_i := \inf_{v_i} P_i(v_i) \). In general, for \( k_i > 0 \) and \( c_i(v_i) \neq b_i(v_i) \), solving for the optimal \( Q_i \) is hard.

For tractability, I assume \( c_i(v_i) = c_i b_i(v_i) \) for some \( 0 < c_i \leq 1 \). This assumption is natural in some applications. In the job slot example, this assumption is satisfied if an agent gets a private benefit for each period he is employed and the penalty is being fired after a pre-specified number of periods. In the example of venture capital firms or funding agencies, this assumption is satisfied if agents receive funds periodically and the penalty is cutting off funding after certain periods. Furthermore, this assumption allows us to obtain a clear analysis on the interaction between verification cost \( (k) \) and level of punishment \( (c) \). Lastly, this assumption can be relaxed, and the results can easily extend if \( c_i(v_i)/b_i(v_i) \) is minimized at \( v_i \).

\footnote{The (IC) constraint can be rewritten as:}

\[
Q_i(v'_i) \geq \frac{b_i(v_i)}{c_i(v_i)} \left( 1 - \frac{P_i(v_i)}{P_i(v'_i)} \right), \forall v_i, v'_i.
\]

Suppose \( c_i(v_i)/b_i(v_i) \) is minimized at \( v_i \) and \( P_i(v_i) \) is non-decreasing. Then given \( v'_i \), the left-hand side of the above inequality is maximized at \( v_i \). Hence, redefine \( c_i := c_i(v_i)/b_i(v_i) \) and the (IC) constraint holds if and only if (4) holds.
Under the assumption that the penalty, $c_i(v_i)$, is proportional to the private benefit, $b_i(v_i)$, the (IC) constraint becomes $P_i(v_i) \geq P_i(v'_i) \left(1 - c_i Q_i(v'_i)\right)$ for all $v_i$ and $v'_i$. The (IC) constraint holds if and only if

$$\varphi_i \geq P_i(v'_i) \left(1 - c_i Q_i(v'_i)\right), \forall v'_i.$$  \hfill (4)

Since $Q_i(v'_i) \leq 1$, then (4) holds only if

$$(1 - c_i) P_i(v'_i) \leq \varphi_i, \forall v'_i.$$  \hfill (5)

**Remark 1** Note that if $c_i = 1$ as in Ben-Porath et al. (2014), then (5) is satisfied automatically. This explains why there is no pooling at the top in Ben-Porath et al. (2014). In contrast, if $0 < c_i < 1$, then (5) imposes an upperbound on $P_i$ and, as a result, there may be pooling at the top.

For the rest of the paper, I assume that $0 < c_i < 1$. Suppose (5) holds, then it is optimal to set $Q_i(v_i) = (1 - \varphi_i/P_i(v_i))/c_i$ for all $v_i \in V_i$. Substitute this into the principal’s objective function:

$$\sum_{i=1}^{n} \mathbb{E}_{v_i} \left[ P_i(v_i) \left(v_i - \frac{k_i}{c_i}\right)\right] + \frac{\varphi_i k_i}{c_i}.$$  \hfill (6)

For the main part of the paper, I assume $v_i$’s are identically distributed, whose density $f$ is strictly positive on $V = [\underline{v}, \overline{v}] \subset \mathbb{R}_+$. I use $F$ to denote the corresponding cumulative distribution function. In addition, I assume $c_i = c$ and $k_i = k$ for all $i$. The main results of the paper can be extended to environments in which the valuations ($v_i$) of different agents may follow different distributions ($F_i$), and both the punishments ($c_i$) and verification costs ($k_i$) may be different for different agents. I discuss the general asymmetric setting in Section 5.1. In this symmetric setting, there exists an optimal mechanism that is symmetric. Thus, I focus on symmetric mechanisms in Sections 3 and 4. In what follows, I suppress the subscript $i$ whenever the meaning is clear.
3 Optimal Mechanism

3.1 Optimal Mechanism for Fixed \( \varphi \)

Fix \( \varphi = \inf_v P(v) \). I first solve the following problem \((OPT - \varphi)\):

\[
\max_{P,Q} \mathbb{E}_v \left[ P(v) \left( v - \frac{k}{c} \right) + \frac{\varphi k}{c} \right],
\]

subject to

\[
\varphi \leq P(v) \leq \frac{\varphi}{1 - c}, \quad \forall v,
\]

\[
n \int_S P(v) dF(v) \leq 1 - \left( 1 - \int_S dF(v) \right)^n, \quad \forall S \subset V.
\]

Note that \((OPT - \varphi)\) is feasible only if \( \varphi \leq 1/n \). To solve this problem, I approximate the continuum type space with a finite partition, characterize an optimal mechanism in the finite model and take limits. Later on, I show the limiting mechanism is optimal in the continuum model.

3.1.1 Finite Case

Fix an integer \( m \geq 2 \). For \( t = 1, \ldots, m \), let

\[
v^t := v + \frac{(2t - 1)(v - \bar{v})}{2m},
\]

\[
f^t := F\left( v + \frac{t(v - \bar{v})}{m} \right) - F\left( v + \frac{(t-1)(v - \bar{v})}{m} \right).
\]

Consider the finite model in which \( v_i \) can take only \( m \) possible different values, i.e., \( v_i \in \{v^1, \ldots, v^m\} \) and the probability mass function satisfies \( f(v^t) = f^t \) for \( t = 1, \ldots, m \). The corresponding problem of \((OPT - \varphi)\) in the finite model, denoted by \((OPTm - \varphi)\), is given
by:
\[
\max_{P} \sum_{t=1}^{m} f^t P^t \left( v^t - \frac{k}{c} \right) + \frac{\varphi k}{c},
\]
subject to
\[
\varphi \leq P^t \leq \frac{\varphi}{1-c}, \forall t, \quad (IC')
\]
\[
n \sum_{t \in S} f^t P^t \leq 1 - \left( \sum_{t \notin S} f^t \right)^n, \forall S \subset \{1, \ldots, m\}. \quad (F2)
\]

To solve \((OPTm - \varphi)\), I first rewrite it as a polymatroid optimization problem. Define \(G(S) := 1 - \left( \sum_{t \notin S} f^t \right)^n\) and \(H(S) := G(S) - n\varphi \sum_{t \in S} f^t\) for all \(S \subset \{1, \ldots, m\}\). Define \(z^t := f^t(P^t - \varphi)\) for all \(t = 1, \ldots, m\). Clearly, \(P^t \geq \varphi\) if and only if \(z_t \geq 0\) for all \(t = 1, \ldots, m\).

Using these notations, constraint \((F2)\) can be rewritten as
\[
n \sum_{t \in S} z^t \leq H(S), \forall S \subset \{1, \ldots, m\}.
\]

It is easy to verify that \(H(\emptyset) = 0\) and \(H\) is submodular. However, \(H\) is not monotonic.

Define \(\overline{H}(S) := \min_{S' \supseteq S} H(S)\). Then \(\overline{H}(\emptyset) = 0\), and \(\overline{H}\) is non-decreasing and submodular.

Furthermore, by Lemma 2,
\[
\left\{ z \mid z \geq 0, n \sum_{t \in S} z^t \leq H(S), \forall S \right\} = \left\{ z \mid z \geq 0, n \sum_{t \in S} z^t \leq \overline{H}(S), \forall S \right\},
\]

Thus, \((OPTm - \varphi)\) can be rewritten as \((OPTm1 - \varphi)\)
\[
\max_{z} \sum_{t=1}^{m} z^t \left( v^t - \frac{k}{c} \right) + \varphi \sum_{t=1}^{m} f^t v^t,
\]
subject to
\[
0 \leq z^t \leq \frac{c\varphi f^t}{1-c}, \forall t, \quad (IC')
\]
\[ n \sum_{t \in S} z^t \leq \mathcal{H}(S), \forall S \subset \{1, \ldots, m\}. \]  

(F2)

Without the upper-bound on \( z^t \) in (IC'), this is a standard polymatroid optimization problem, and can be solved using the greedy algorithm. With the upper-bound, this is a weighted polymatroid intersection problem and there exist efficient algorithms solving the optima if the weights \( (v^t - k/c) \) are rational. See, for example, Cook et al. (2011) and Frank (2011).

In this paper, I solve the problem using a “guess-and-verify” approach. Though we cannot directly apply the greedy algorithm to \((OPTm1 - \phi)\), it is not hard to conjecture the optimal solution. Intuitively, \( z^t = 0 \) if \( v^t < k/c \). Consider \( v^t \geq k/c \). Since \( \mathcal{H} \) is submodular and the upperbound on \( z^t \) is linear in \( f^t \), the solution found by the greedy algorithm potentially violates the upperbound for large \( t \). Thus, it is natural to conjecture that there exists a cutoff \( \bar{t} \) such that for \( t \) above the cutoff, (IC') binds, i.e., \( z^t = c\phi f^t/(1 - c) \); and for \( t \) below the cutoff, (F2) binds.

Formally, let \( S^t = \{t, \ldots, m\} \) for all \( t = 1, \ldots, m \), and \( S^{m+1} = \emptyset \). If \( \phi \leq (1 - c)/n \), let \( \bar{t} := 0 \); otherwise, there exists a unique \( \bar{t} \in \{1, \ldots, m + 1\} \) such that

\[ \mathcal{H}(S^t) \leq n \sum_{\tau = \bar{t}}^{m} \frac{c\phi f^\tau}{1 - c} \quad \text{and} \quad \mathcal{H}(S^{\bar{t}+1}) > n \sum_{\tau = \bar{t}+1}^{m} \frac{c\phi f^\tau}{1 - c}. \]

Define

\[ z^t := \begin{cases} \frac{c\phi f^t}{1 - c} & \text{if } t > \bar{t} \\ \frac{1}{n} \mathcal{H}(S^\bar{t}) - \sum_{\tau = \bar{t}+1}^{m} \frac{c\phi f^\tau}{1 - c} & \text{if } t = \bar{t} \\ \frac{1}{n} \left[ \mathcal{H}(S^t) - \mathcal{H}(S^{\bar{t}+1}) \right] & \text{if } t < \bar{t} \end{cases} \]

and

\[ \hat{z}^t := \begin{cases} z^t & \text{if } v^t \geq k/c \\ 0 & \text{if } v^t < k/c \end{cases}. \]  

(7)

Then \( \hat{z}^t \) is feasible following from the fact that \( \mathcal{H}(\emptyset) = 0 \), and \( \mathcal{H} \) is non-decreasing and submodular. Furthermore, I can verify the optimality of \( \hat{z}^t \) by the duality theorem:
Lemma 1 \( \hat{z}^t \) defined in (7) is an optimal solution to \((OPTm_1 - \varphi)\).

Let \( \overline{P}^t := \overline{z}^t / f^t + \varphi \) and
\[
\overline{P}_m^t := \begin{cases} 
\overline{P}^t & \text{if } v^t > \frac{k}{c} \\
\varphi & \text{if } v^t < \frac{k}{c}
\end{cases}. \tag{8}
\]

The following corollary directly follows from Lemma 1:

Corollary 1 \( P_m \) defined in (8) is an optimal solution to \((OPTm - \varphi)\).

3.1.2 Continuum Case

I characterize the optimal solution in the continuum case by taking \( m \) to infinity. Let \( v^l \) be such that \( F(v^l)^{n-1} = n\varphi \) and
\[
v^u := \inf \left\{ v \left| 1 - F(v)^n - \frac{n\varphi}{1 - c} [1 - F(v)] \geq 0 \right. \right\}. \tag{9}
\]

Note that if \( \varphi \leq (1 - c)/n \), then \( v^u = v \). Let \( \overline{P} \) be defined as follows: If \( v^l < v^u \), let
\[
\overline{P}(v) := \begin{cases} 
\phi \frac{v^u}{1 - c} & \text{if } v \geq v^u \\
F(v)^{n-1} & \text{if } v^l < v < v^u \\
\varphi & \text{if } v \leq v^l
\end{cases}.
\]

If \( v^l \geq v^u \), let
\[
\hat{v} := \inf \left\{ v \left| 1 - n\varphi F(v) - \frac{n\varphi}{1 - c} [1 - F(v)] \geq 0 \right. \right\} \in [v^u, v^l], \tag{10}
\]

and
\[
\overline{P}(v) := \begin{cases} 
\phi \frac{v}{1 - c} & \text{if } v \geq \hat{v} \\
\varphi & \text{if } v < \hat{v}
\end{cases}.
\]
Finally, let
\[ P^*(v) := \begin{cases} 
P(v) & \text{if } v \geq \frac{k}{c} \\
\varphi & \text{if } v < \frac{k}{c} .
\end{cases} \tag{11} \]

I show in appendix that \( P^* \) is the “pointwise limit” of \( \{P_m\} \). Moreover, \( P^* \) is an optimal solution to \((OPT - \varphi)\).

**Theorem 1** P* defined in (11) is an optimal solution to \((OPT - \varphi)\).

### 3.2 Optimal \( \varphi \)

I complete the characterization of the optimal mechanism by characterizing the optimal \( \varphi \). First, if inspection is sufficiently costly or the principal’s ability to punish an agent is sufficiently limited, then pure randomization is optimal.

**Theorem 2** If \( \overline{v} - k/c \leq \mathbb{E}_v[v] \), then pure randomization, i.e., \( P^{**} = 1/n \) is optimal.

To make the problem more interesting, in what follows, I assume:

**Assumption 1** \( \overline{v} - k/c > \mathbb{E}_v[v] \).

Recall that given \( \varphi \), \( v^l \) is uniquely pinned down by \( F(v^l)^{n-1} = n\varphi \) and \( v^u \) is uniquely pinned down by (9). Define \( v^* \) and \( v^{**} \) by the equations (12) and (13), respectively:

\[ \mathbb{E}_v[v] - \mathbb{E}_v[\max\{v, v^*\}] + \frac{k}{c} = 0, \tag{12} \]
\[ \mathbb{E}_v[v] - \mathbb{E}_v[\min\{v, v^{**}\}] + (1-c) \left[ \mathbb{E}_v[v] - \mathbb{E}_v[\max\{v, v^{**}\}] + \frac{k}{c} \right] = 0. \tag{13} \]

They are well defined under Assumption 1. Furthermore, \( v^{**} > v^* \geq k/c \). Finally, let

\[ v^\natural := \sup \left\{ v \middle| \frac{F(v)^{n-1}(1-F(v))}{1-c} - 1 + F(v)^n \leq 0 \right\} . \tag{14} \]

The optimal mechanism is characterized by the following theorem:
Theorem 3 Suppose Assumption 1 holds. There are three cases.

1. If $F(v^*)^{n-1} \geq n(1-c)$, then the optimal $\varphi^* = F(v^*)^{n-1}/n$ and the following allocation rule is optimal:

$$P^{**}(v) := \begin{cases} F(v)^{n-1} & \text{if } v \geq v^* \\ \varphi^* & \text{if } v < v^* \end{cases}$$

2. If $F(v^*)^{n-1} < n(1-c)$ and $v^{**} \leq \bar{v}$, then the optimal $\varphi^* = (1-c)/n(1-cF(v^{**}))$ and the following allocation rule is optimal

$$P^{**}(v) := \begin{cases} \varphi^* & \text{if } v \geq v^{**} \\ \varphi^* & \text{if } v < v^{**} \end{cases}$$

3. If $F(v^*)^{n-1} < n(1-c)$ and $v^{**} > \bar{v}$, then the optimal $\varphi^*$ is defined by

$$E[v] - E[v^{*}] + (1-c) \left[ E[v] - E[v^{*}] \right] + \frac{k}{c} = 0, \quad (15)$$

and the following allocation rule is optimal:

$$P^{**}(v) := \begin{cases} \varphi^* & \text{if } v \geq v^{u}(\varphi^*) \\ F(v)^{n-1} & \text{if } v^{l}(\varphi^*) < v < v^{u}(\varphi^*) \\ \varphi^* & \text{if } v \leq v^{l}(\varphi^*) \end{cases}$$

To understand the result, consider the following implementation of the optimal mechanism. There are two thresholds. I abuse notation here and denote them by $v^l$ and $v^u$ with $v \leq v^l \leq v^u \leq \bar{v}$. If every agent reports a value below $v^l$, then an agent is selected uniformly at random and receives the good and no one is inspected. If any agent reports a value above $v^l$ but all reports are below $v^u$, then the agent with the highest reported value receives the good, is inspected with some probability (proportional to $1/c$) and is penalized if he is found to have lied. If any agent reports a value above $v^u$, then an agent is selected uniformly at
random among all the agents whose reported value is above $v^u$, receives the good, is inspected with some probability (proportional to $1/c$) and is penalized if he is found to have lied. I call a mechanism a one-threshold mechanism if $v^u = \bar{v}$, a mechanism a shortlisting mechanism if $v^l = v^u$, and a mechanism a two-thresholds mechanism if $v^l < v^u$.

Consider the impact of a reduction in $v^l$. Intuitively, this raises the total inspection cost as well as the allocation efficiency at the bottom of the value distribution. After some algebra, one can verify that the increase in inspection cost is proportional to $k/c$ and the increase in allocation efficiency is proportional to $\mathbb{E}[\min\{v, v^u\}] - \mathbb{E}[v] / (1 - c)$. However, as $v^l$ decreases, agents with low $v$’s get worse off and have stronger incentives to misreport as a high type. Since the principal’s ability to penalize an agent is limited, more pooling at the top, i.e., a lower $v^u$ is required to restore the incentive compatibility. This reduces the allocation efficiency at the top by an amount proportional to $\mathbb{E}[\min\{v, v^u\}] - \mathbb{E}[v] / (1 - c)$. In an optimal mechanism, the marginal gain from a reduction in $v^l$ (proportional to the left-hand side of (16)) must equal the marginal cost (proportional to the right-hand side of (16)):

$$\mathbb{E}[v] - \mathbb{E}[\max\{v, v^l\}] = \frac{\mathbb{E}[\min\{v, v^u\}] - \mathbb{E}[v]}{1 - c} + \frac{k}{c}. \quad (16)$$

This is precisely the case captured by the third part of Theorem 3 (compare (16) with (15)). If the limited punishment constraint does not bind, i.e., $v^u = \bar{v}$, there is no efficiency loss at the top and $[\mathbb{E}[\min\{v, v^u\}] - \mathbb{E}[v]] / (1 - c) = 0$. In this case, (16) becomes (12) ($v^l = v^*$) and the optimal mechanism is characterized by the first part of Theorem 3. If the principal’s ability to punish an agent is sufficiently limited so that $v^u = v^l (= v^{**})$, then (16) becomes (13) and the optimal mechanism is characterized by the first part of Theorem 3.

**Remark 2** $v^*$ is strictly increasing in $k/c$. If $n(1 - c) < 1$, there is no pooling at the top for $k > 0$ sufficiently large. If $k = 0$, then $v^* = v$ and $F(v^*)^{n-1} = 0 < n(1 - c)$ for any $0 < c < 1$. That is, with limited punishment, there is always pooling at the top if verification is free (Mylovanov and Zapechelnyuk (2014)).
4 Properties of Optimal Mechanism

The properties of optimal mechanisms crucially depend on the number of agents \((n)\), the verification cost \((k)\) and the level of punishment \((c)\). Let \(\rho := k/c \geq 0\). I call \(\rho\) the effective verification cost. The effective verification cost, \(\rho\), is strictly decreasing in \(c\). This is because a smaller \(c\) implies a lower level of punishment, which makes inspection less attractive, or essentially more costly.

Let \(n^*(\rho, c) < 1/(1 - c)\) be defined by

\[ F(v^*)^{n^*(\rho, c)-1} = n^*(\rho, c)(1 - c), \]

where \(v^*\) is defined by

\[ \mathbb{E}_v[v] - \mathbb{E}_v[\max\{v, v^*\}] + \rho = 0. \]

Since \(v^*\) is independent of \(n\), by Theorem 3, one-threshold mechanisms are optimal if and only if \(n \leq n^*(\rho, c)\). Intuitively, for fixed \(v^*\), an agent whose type below \(v^*\) gets the object with probability

\[ \varphi^* = \frac{1}{n} F(v^*)^{n-1}, \]

which is strictly decreasing in \(n\). That is, a low type agent gets worse off and has stronger incentive to misreport as a high type agent when the number of agent, \(n\), increases. For \(n\) sufficiently large, the (IC) constraint is violated without pooling at the top.

Since \(v^*\) is strictly increasing in \(\rho\), the left-hand side of (17) is strictly decreasing in \(n\), and the right-hand side of (18) is strictly increasing in \(n\), we have \(n^*\) is strictly increasing in \(\rho\). Fixed \(\rho\), \(v^*\) is independent of \(c\), but the left-hand side of (17) is strictly decreasing in \(c\). Hence, \(n^*\) is strictly increasing in \(c\).

Let \(n^{**}(\rho, c) < 1/(1 - c)\) be defined by

\[ \frac{1 - F(v^{**})^{n^{**}(\rho, c)-1}}{1 - F(v^{**})} = \frac{F(v^{**})^{n^{**}(\rho, c)-1}}{1 - c}, \]
where \( v^{**} \) is given by

\[
E_v[v] - E_v[\min\{v, v^{**}\}] + (1-c) \left( E_v[v] - E_v[\max\{v, v^{**}\}] + \frac{k}{c} \right) = 0.
\]  

(13)

Then \( v^{**} \leq v^\hat{\gamma} \) if and only if \( n \geq n^*(\rho, c) \). By Theorem 3, shortlisting mechanisms are optimal if and only if \( n \geq n^*(\rho, c) \). Intuitively, a low type agent gets worse off and has stronger incentive to misreport as a high type agent when the number of agent, \( n \), increases. Thus, pooling areas at both the bottom and the top of the value distribution are increased to ensure the mechanism is incentive compatible and save the inspection cost. Eventually, for a sufficiently large number of agents, the two pooling areas meet and there is a unique threshold so that all agents whose value above the threshold are pooled together and all agents whose value below the threshold are pooled together.

Recall that \( v^{**} > v^* \). Hence,

\[
\frac{F(v^{**})^{n^*(\rho,c)-1}}{1-c} > \frac{F(v^*)^{n^*(\rho,c)-1}}{1-c} = n^*(\rho,c) \geq \frac{1 - F(v^{**})^{n^*(\rho,c)}}{1 - F(v^{**})}.
\]

Since the left-hand side of (18) is strictly increasing in \( n \), and the right-hand side of (18) is strictly decreasing in \( n \), we have \( n^{**}(\rho, c) > n^*(\rho, c) \). It is easy to see that \( v^{**}(\rho, c) \) is strictly increasing in both \( \rho \) and \( c \), and independent of \( n \). I show in Lemma 4 that if \( n(1-c) < 1 \) then \( v^{\hat{\gamma}} \) is strictly increasing in \( n \) and strictly decreasing in \( c \). Hence \( n^{**}(\rho, c) \) is strictly increasing in both \( \rho \) and \( c \). These results are summarized by the following corollary:

**Corollary 2** Given \( k > 0 \) and \( c \in (0,1) \), there exists \( 0 < n^*(\rho, c) < n^{**}(\rho, c) < 1/(1-c) \) such that the following is true:

1. If \( n \leq n^*(\rho, c) \), then one-threshold mechanisms are optimal; if \( n^*(\rho, c) < n < n^{**}(\rho, c) \), then two-thresholds mechanisms are optimal; if \( n \geq n^{**}(\rho, c) \), then shortlisting mechanisms are optimal.

2. \( n^*(\rho, c) \) and \( n^{**}(\rho, c) \) are strictly increasing in \( \rho \) and \( c \).
3. \( v^*(n, \rho, c) \) is strictly increasing in \( \rho \) and independent of \( n \) and \( c \). \( v^{**} \) is strictly increasing in \( \rho \) and \( c \) and independent of \( n \). If \( n^*(\rho, c) < n < n^{**}(\rho, c) \), then \( v^l(n, \rho, c) \) is strictly increasing in \( n \), \( \rho \) and \( c \) and \( v^u(n, \rho, c) \) is strictly decreasing in \( n \) and strictly increasing in \( \rho \) and \( c \).

The impact of \( k \) is straightforward. As \( k \) increases, inspection becomes more costly. The optimal mechanism sees more pooling at the bottom (measured by \( \phi \)) to save inspection cost. This relaxes the upper-bound on \( P \), which leads to less pooling at the top or no pooling at the top at all. The impact of \( c \) is ambiguous. On the one hand, given the amount of pooling at the bottom, i.e., \( \phi \), an increase in \( c \) lowers the upper-bound on \( P \) (see (5)), which implies more pooling at the top. On the other hand, an increase in \( c \) makes verification more costly. Similar to the case of an increase in \( k \), this change increases the amount of pooling at the bottom, i.e., \( \phi \), and relaxes the upper-bound on \( P \). As a result, there may be less pooling at the top or no pooling at the top at all. The second channel is absent if verification is free, i.e., \( k = 0 \). This is further illustrated by the following numerical example.

**Example 1** Consider a numerical example in which \( v_i \)'s are uniformly distributed on \([0, 1]\). There are \( n = 8 \) agents. The inspection cost is \( k = 0.4 \). I abuse notation a bit and redefine \( v^l = v^u = v^{**} \) if \( v^l > v^u \). Figure 1 plots \( v^l \) and \( v^u \) as functions of \( c \). Observe that the change of \( v^u \) is not monotonic. As \( c \) increases, the pooling area at the top first increases and then decreases.

Finally, a careful examination of (17) and (12) proves the following corollary:

**Corollary 3** \( \lim_{c \to 1} n^*(k/c, c) = \infty \) and \( \lim_{k \to 0} n^*(k/c, c) = 0 \).

Corollary 3 shows that as the principal’s ability to punish an agent becomes unlimited, the model collapses to Ben-Porath et al. (2014) and as the inspection cost diminishes, the model collapses to Mylovanov and Zapechelnyuk (2014).
5 Extensions

5.1 Asymmetric Environment

In this section, I consider the general model with asymmetric agents. Fix $\varphi_i = \inf_{v_i} P_i(v_i)$.

I first solve the following problem ($\text{OPTH} - \varphi$):

$$\max_{P,Q} \sum_{i=1}^{n} E_{v_i} \left[ P_i(v_i) \left( v_i - \frac{k_i}{c_i} \right) \right] + \frac{\varphi_i k_i}{c_i},$$

subject to

$$\varphi_i \leq P_i(v_i) \leq \frac{\varphi_i}{1 - c_i}, \forall v_i, \quad (\text{IC}')$$

$$0 \leq Q_i(v_i) \leq 1, \forall v_i, \quad (\text{F1})$$

$$\sum_i \int_{S_i} P_i(v_i) dF_i(v_i) \leq 1 - \prod_i \left( 1 - \int_{S_i} dF_i(v_i) \right), \forall S_i \subset V_i. \quad (\text{F2})$$
Clearly, \((OPTH - \varphi)\) is feasible only if \(\sum_i \varphi_i \leq 1\). As in the symmetric case, I approximate the continuum type space with a finite partition, solve an optimal mechanism in the finite model and take limits. The following theorem gives an optimal solution to \((OPTH - \varphi)\):

**Theorem 4** There exists \(d^l\) and \(d^u_i\) for \(i = 1, \ldots, n\) such that \(P^*\) defined by

\[
P^*_i(v_i) := \begin{cases} 
\overline{P}_i(v_i) & \text{if } v_i > \frac{k_i}{c_i} \\
\varphi_i & \text{if } v_i < \frac{k_i}{c_i}.
\end{cases}
\tag{19}
\]

where

\[
\overline{P}_i(v_i) := \begin{cases} 
\frac{\varphi_i}{1-c_i} & \text{if } v_i > d^u_i + \frac{k_i}{c_i} \\
\prod_{j \neq i, d^u_j \geq v_i - \frac{k_j}{c_j}} F_j \left(v_i - \frac{k_i}{c_i} + \frac{k_j}{c_j}\right) & \text{if } d^l + \frac{k_i}{c_i} < v_i < d^u_i + \frac{k_i}{c_i} \\
\varphi_i & \text{if } v_i < d^l + \frac{k_i}{c_i}
\end{cases}
\tag{20}
\]

is an optimal solution to \((OPTH - \varphi)\).

Note that threshold mechanisms are still optimal. However, even though there is a unique lower threshold \(d^l\) such that all agents whose “net” value \(v_i - k_i/c_i\) below the threshold are pooled, there can be up to \(n\) distinctive upper thresholds \(d^u_i\) \((i = 1, \ldots, n\). To illustrate how the mechanism works, assume \(d^u_1 = \cdots = d^u_j > d^u_{j+1} = \cdots d^u_n\). If there exists some agent \(i\) \((1 \leq i \leq j)\) whose “net” value \(v_i - k_i/c_i\) is above \(d^u_i\), then one of such agents is selected at random and receives the good. If \(v_i - k_i/c_i < d^u_i\) for all \(1 \leq i \leq j\) but \(v_i - k_i/c_i \geq d^u_{j+1}\) for some \(1 \leq i \leq j\), then the agent with the highest reported “net” value among the first \(j\) agents receives the good. If \(v_i - k_i/c_i < d^u_{j+1}\) for all \(1 \leq i \leq j\) but \(v_i - k_i/c_i \geq d^u_{j+1}\) for some \(j+1 \leq i \leq n\), then one agent is selected at random among all the agents whose reported “net” value is above \(d^u_{j+1}\) and receives the good. If \(v_i - k_i/c_i < d^u_{j+1}\) for all \(i\) but \(v_i - k_i/c_i \geq d^l\) for some \(i\), then the agent with the highest reported “net” value receives the good. If \(v_i - k_i/c_i < d^l\) for all \(i\), then one agent is selected at random and receives the good.
Because of the complication of pooling areas at the top, it is much harder to find an optimal solution to \((OPTH - \varphi)\). Specifically, \(d^u_i\)'s are solved recursively from the largest to the smallest. Furthermore, to characterize the optimal \(\varphi := (\varphi_1, \ldots, \varphi_n)\), without priori knowledge of which set of agents share the same upper threshold, one must consider \(2^n\) different cases.\(^2\) Thus, I leave the full characterization of optimal mechanisms for future research.

Though it is extremely hard to characterize the optimal \(\varphi\) due to the complication of pooling areas at the top, I give a partial characterization of optimal mechanisms when the upperbounds on \(P_i\) do not bind, i.e., \(d^u_i = \pi_i - \frac{k_i}{c_i}\) for all \(i\). By a similar argument to that in the proof of Theorem 2, one can show that pure randomization is optimal if inspection is sufficiently costly or the principal’s ability to punish an agent is sufficiently limited, i.e., 
\[
\bar{v}_i - k_i/c_i \leq \mathbb{E}_{v_i}[v_i] \quad \text{for all } i.
\]
To make the problem interesting, in what follows, I assume:

**Assumption 2** \(\pi_i - k_i/c_i > \mathbb{E}_{v_i}[v_i] \text{ for some } i\).

If \(d^u_i = \pi_i - \frac{k_i}{c_i}\) for all \(i\), then \(d^l \geq \max \{v_j - k_j/c_j\}\) is such that
\[
\sum_{i=1}^{n} \varphi_i F_i \left( d^l + \frac{k_i}{c_i} \right) = \prod_{i=1}^{n} F_i \left( d^l + \frac{k_i}{c_i} \right).
\]
Lemma 17 shows that there exists a unique \(d^l\) satisfying the above equation. Note that unless \(\varphi_i = 0\) for all \(i\), we have \(d^l > \max \{v_j - k_j/c_j\}\). Clearly, in optimum, \(\varphi_i > 0\) for some \(i\). Hence, \(d^l > \max \{v_j - k_j/c_j\}\). Let \(d^*_{i} (i = 1, \ldots, n)\) be defined by
\[
\mathbb{E}_{v_i}[v_i] - \mathbb{E}_{v_i} \left[ \max \left\{ v_i, d^*_{i} + k_i/c_i \right\} \right] + \frac{k_i}{c_i} = 0,
\]
and \(d^{*} := \max_i d^*_{i}\).

---
\(^2\)Assume, without loss of generality, that \(d^u_1 \geq \cdots \geq d^u_n\). If there are \(i\) distinctive upper thresholds, then there are \(C^n_i\) possibilities. In total, there are \(\sum_{i=1}^{n} C^n_i = 2^n\) possibilities to consider.
Theorem 5 Suppose Assumption 2 holds. If

$$\sum_{i=1}^{n} (1 - c_i) \prod_{j \neq i} F_j \left( \overline{v}_j - \frac{k_i}{c_i} + \frac{k_j}{c_j} \right) F_i \left( d^{*} + \frac{k_i}{c_i} \right) \leq \prod_{i=1}^{n} F_i \left( d^{*} + \frac{k_i}{c_i} \right),$$

then the set of optimal \( \overrightarrow{\varphi} \) is the convex hull of

$$\left\{ \begin{array}{l}
\overrightarrow{\varphi} \\
\varphi_i^{*} = \frac{\prod_{i=1}^{n} F_i \left( d^{*} + \frac{k_i}{c_i} \right) - \sum_{i=\neq i^{*}} (1 - c_i) \prod_{j \neq i} F_j \left( \overline{v}_j - \frac{k_i}{c_i} + \frac{k_j}{c_j} \right) F_i \left( d^{*} + \frac{k_i}{c_i} \right)}{F_i^{*} \left( d^{*} + \frac{k_i^{*}}{c_i} \right)}
\end{array} \right. .$$

For each optimal \( \overrightarrow{\varphi}^{*} \), the following allocation rule is optimal:

$$P^{**}_{i}(v_i) := \begin{cases} 
\prod_{j \neq i} F_j \left( v_i - \frac{k_i}{c_i} + \frac{k_j}{c_j} \right) & \text{if } v_i \geq d^i + \frac{k_i^{*}}{c_i} \\
\varphi_i^{*} & \text{if } v_i < d^i + \frac{k_i^{*}}{c_i} 
\end{cases} .$$

5.1.1 Symmetric Environment Revisited

In this section, I revisit the symmetric environment. In the symmetric environment, an optimal mechanism must satisfy: \( d^1 = \cdots = d^n \). To understand the intuition behind this result, note first that in the symmetric environment \( d^i \geq d_j \) only if \( \varphi_i \geq \varphi_j \). Assume, without loss of generality, that \( d^1 \geq \cdots \geq d^n \). Consider, for simplicity, the case in which \( d^1 > d^2 > d^3 \), which implies that \( \varphi_1 > \varphi_2 \). We can construct a new mechanism in which \( \varphi_1^{*} = \varphi_2^{*} = \sum_{i=1}^{2} \varphi_i / 2 \) and \( \varphi_i = \varphi_i^{*} \) for all \( i \geq 3 \). In this new mechanism, agents 1 and 2 share the same upper threshold \( d^{*} \in (d^2, d^1) \) and the upper thresholds of the other agents remain the same. The new mechanism improves the principal’s value by allocating the good between agents 1 and 2 more efficiently when their “net” values, \( v_i - k_i / c_i \), lie between \( (d^2, d^1) \).

This property of optimal mechanisms facilitates our analysis of optimal \( \overrightarrow{\varphi} \). Theorem 6 below characterizes the set of all optimal \( \overrightarrow{\varphi} \). Let \( v^*, v^{**} \) and \( v^{*\#} \) be defined by (12), (13) and (14), respectively.

Theorem 6 Suppose Assumption 1 holds. There are three cases.
1. If $F(v^*)^{n-1} \geq n(1-c)$, then the set of optimal $\varphi$ is the convex hull of

$$\{ \varphi \mid \varphi_i = F(v^*)^{n-1} - (n-1)(1-c), \varphi_j = 1 - c \forall j \neq i^*, i^* \in I \}.$$ 

For each optimal $\varphi^*$, the following allocation rule is optimal:

$$P^{**}_i(v_i) := \begin{cases} F(v_i)^{n-1} & \text{if } v_i \geq v^* \\ \varphi_i^* & \text{if } v_i < v^* \end{cases}.$$ 

2. If $F(v^*)^{n-1} < n(1-c)$ and $v^{**} \leq v^*$, then the set of optimal $\varphi$ is the convex hull of

$$\left\{ \varphi \mid \varphi_{ij} = (1-c)F(v^{**})^{j-1} \text{ if } j \leq h - 1, \varphi_{ih} = \frac{1-c}{1-cF(v^{**})} - \sum_{j=1}^{h-1}(1-c)F(v^{**})^{j-1}, \varphi_{ij} = 0 \text{ if } j \geq h + 1 \text{ and } (i_1, \ldots, i_n) \text{ is a permutation of } (1, \ldots, n) \right\},$$

where $1 \leq h \leq n$ is such that

$$\frac{1 - F(v^{**})^{h-1}}{1 - F(v^{**})} \leq \frac{1}{1 - cF(v^{**})} < \frac{1 - F(v^{**})^h}{1 - F(v^{**})}.$$

For each optimal $\varphi^*$, the following allocation rule is optimal:

$$P^{**}_i(v_i) := \begin{cases} \frac{\varphi_i^*}{1-c} & \text{if } v_i \geq v^{**} \\ \varphi_i^* & \text{if } v_i < v^{**} \end{cases}.$$ 

3. If $F(v^*)^{n-1} < n(1-c)$ and $v^{**} > v^*$, then the set of optimal $\varphi$ is the convex hull of

$$\{ \varphi \mid \varphi_{ij} = (1-c)F(v^u(\varphi^*))^{j-1} \forall j \text{ and } (i_1, \ldots, i_n) \text{ is a permutation of } (1, \ldots, n) \},$$

where $\varphi^*$ is defined by (15) and, for each $\varphi$, $v^i$ is such that $F(v^i)^{n-1} = \varphi$ and $v^u$ is
defined by (9). For each optimal $\varphi^*$, the following allocation rule is optimal:

$$P^*_i(v_i) := \begin{cases} \frac{\varphi^*_i}{1-c} & \text{if } v_i \geq v^u(\varphi^*) \\ F(v_i)^{n-1} & \text{if } v^l(\varphi^*) < v_i < v^u(\varphi^*) \\ \varphi^*_i & \text{if } v_i \leq v^l(\varphi^*) \end{cases}$$

Theorem 6 illustrates how limited punishment restricts the principal’s ability to treat agents differently. Suppose $F(v^*)^{n-1} \geq n(1-c)$, i.e., the upperbounds do not bind. In Ben-Porath et al. (2014) ($c = 1$), there is a class of optimal mechanisms called favored-agent mechanisms. In a favored-agent mechanism, there exists a favored-agent $i^*$ whose $\varphi^*_i = F(v^*)^{n-1}$ and $\varphi_i = 0$ for any other agent $i$. However, if $c < 1$, then in an optimal mechanism no agent can receive the good with a probability lower than $1 - c$ since otherwise some upperbounds on $P_i$ bind. Fix the ratio of $k/c$ so that $v^*$ remains the same, the optimal set of $\varphi$ shrinks as $c$ becomes smaller. When $F(v^*)^{n-1} = n(1-c)$, the unique optimal $\varphi^*$ is such that $\varphi^*_1 = \cdots = \varphi^*_n$.

Suppose $F(v^*)^{n-1} < n(1-c)$, then the principal can again treat agents differently but to the extent that they share the same upper threshold. Assume, without loss of generality, that an agent with a smaller index is more favored by the principal in terms of a larger $\varphi_i$. Then, in an optimal mechanism, the first $h$ agents cannot be favored too much in the sense that $\sum_{i=1}^{h} \varphi_i \leq (1 - c) \sum_{i=1}^{h} F(v^u)^{i-1}$ for all $h = 1, \ldots, n$.

### 5.2 Punishments Independent of Allocation

In this section, I consider a variation of the model in which I assume $k_i^\beta < \infty$ and $c_i^\beta > 0$. This means that the principal can acquire information about an agent and penalize the agent even if he does not receive the object. In general, given $p_i(v)$, it is optimal for the principal to

$$\min_{q_i(v),q_i^\beta(v)} p_i(v)q_i(v)k_i + (1 - p_i(v))q_i^\beta(v)k_i^\beta$$
subject to

\[ p_i(v)q_i(v)c_i(v_i) + (1 - p_i(v))q_i^\beta(v)c_i^\beta(v_i) = c_i(v), \]  

(22)

where \( c_i(v) \leq p_i(v)c_i(v_i) + (1 - p_i(v))c_i^\beta(v_i) \) is the expected punishment. There are three cases. If \( k_i/c_i(v_i) < k_i^\beta/c_i^\beta(v_i) \), then it is optimal to set \( q_i(v) = \min\{c_i(v)/p_i(v)c_i(v_i), 1\} \) and \( q_i^\beta(v) = \max\{0, (c_i(v) - p_i(v)c_i(v_i))/(1 - p_i(v))c_i^\beta(v_i)\} \). Similarly, if \( k_i/c_i(v_i) > k_i^\beta/c_i^\beta(v_i) \), then it is optimal to set \( q_i^\beta(v) = \min\{1, c_i(v)/(1 - p_i(v))c_i^\beta(v_i)\} \) and \( q_i(v) = \max\{0, (c_i(v) - (1 - p_i(v))c_i^\beta(v_i))/p_i(v)c_i(v_i)\} \). Finally, if \( k_i/c_i(v_i) = k_i^\beta/c_i^\beta(v_i) \), then any \( q_i(v) \) and \( q_i^\beta(v) \) satisfying (22) is optimal.

For simplicity, I assume that \( k_i^\beta = k_i \) and \( c_i^\beta = c_i \). The results can readily extend to more general cases, e.g., \( k_i^\beta > k_i \) and \( c_i^\beta < c_i \). Given \((p, q)\), let \( P_i(v_i) := \mathbb{E}_{v_{-i}}[p_i(v_i, v_{-i})] \) and \( \hat{P}(v_i) := \mathbb{E}_{v_{-i}}[p_i(v_i, v_{-i})q_i(v_i, v_{-i}) + (1 - p_i(v_i, v_{-i}))q_i^\beta(v_i, v_{-i})] \). The principal’s problem can be written in the reduced form:

\[ \max_{P, Q} \sum_{i=1}^n \mathbb{E}_{v_i} \left[ P_i(v_i)v_i - \hat{P}(v_i)k_i \right], \]

subject to

\[ P_i(v_i)b_i(v_i) \geq P_i(v'_i)b_i(v_i) - \hat{P}_i(v'_i)c_i(v_i), \forall v_i, v'_i, \]

(23)

\[ 0 \leq \hat{P}_i(v_i) \leq 1, \forall v_i, \]

(24)

\[ \sum_i \int_{S_i} P_i(v_i)dF_i(v_i) \leq 1 - \prod_i \left( 1 - \int_{S_i} dF_i(v_i) \right), \forall S_i \subset V_i. \]

(25)

For tractability, I assume \( c_i(v_i) = c_i b_i(v_i) \). The (IC) constraint becomes \( P_i(v_i) \geq P_i(v'_i) - \hat{P}(v'_i)c_i \) for all \( v_i \) and \( v'_i \). This constraint holds if and only if

\[ \varphi_i \geq P_i(v'_i) - \hat{P}(v'_i)c_i, \forall v'_i. \]
Since \( \hat{P}_i(v'_i) \leq 1 \), (23) holds only if

\[
P_i(v'_i) \leq \varphi_i + c_i, \forall v'_i. \tag{24}
\]

Suppose (24) holds, then it is optimal to set \( \hat{P}_i(v_i) = (P_i(v_i) - \varphi_i)/c_i \) for all \( v_i \in V_i \). Substituting this into the principal’s objective function yields:

\[
\sum_{i=1}^{n} \mathbb{E}_{v_i} \left[ P_i(v_i) \left( v_i - \frac{k_i}{c_i} \right) \right] + \frac{\varphi_i k_i}{c_i}. \tag{6}
\]

Note that, given \( \{\varphi_i\} \), the principal’s objective function is as same as that in the case that \( c_i^{\beta} = 0 \). The only difference is the upperbound on \( P_i \). If the upperbound does not bind in the original problem, i.e., \( \varphi_i/(1 - c_i) \geq 1 \), then \( \varphi_i + c_i \geq 1 \) which implies that the upperbound does not bind in the new problem either. The reverse is also true. If the upperbound binds in the original problem, i.e., \( \varphi_i/(1 - c_i) < 1 \), then the new upperbound is larger:

\[
\varphi_i + c_i - \frac{\varphi_i}{1 - c_i} = c_i \left( 1 - \frac{\varphi_i}{1 - c_i} \right) > 0.
\]

Hence, any feasible solution to the new problem is also feasible in the original problem. This can be seen more clearly if we revisit (23). In (23), we have \( \hat{P}_i(v'_i) \leq 1 \). In contrast, if \( c_i^{\beta} = 0 \) which implies that \( q_i^{\beta} = 0 \), then we have \( \hat{P}_i(v'_i) \leq P_i(v'_i) \leq 1 \). This is intuitive since allowing the principal to penalize an agent even if he does not receive the object clearly relaxes the principal’s problem.

For simplicity, as in the main part of the paper, I assume \( v_i \)’s are identically distributed and \( c_i = c \) and \( k_i = k \) for all \( i \). Without loss of generality, I can focus on symmetric mechanisms. In what follows, I suppress the subscript \( i \) whenever the meaning is clear.

Fix \( \varphi = \inf_{v} P(v) \). I first solve the following problem \((OPTI - \varphi)\):

\[
\max_{P,Q} \mathbb{E}_v \left[ P(v) \left( v - \frac{k}{c} \right) \right] + \frac{\varphi k}{c},
\]

26
subject to
\[ \varphi \leq P(v) \leq \varphi + c, \forall v, \quad (IC') \]
\[ n \int_S P(v) dF(v) \leq 1 - \left( 1 - \int_S dF(v) \right)^n, \forall S \subset V. \quad (F2) \]

Note that \((OPTI - \varphi)\) is feasible only if \(\varphi \leq 1/n\). Let \(v^l\) be such that \(F(v^l)^{n-1} = n\varphi\) and redefine
\[ v^u := \inf \{ v \mid 1 - F(v)^n - n(\varphi + c)[1 - F(v)] \geq 0 \}. \quad (25) \]

Redefine \(\overline{P}\) as follows: If \(v^l < v^u\), let
\[ \overline{P}(v) := \begin{cases} \varphi + c & \text{if } v \geq v^u \\ F(v)^{n-1} & \text{if } v^l < v < v^u \\ \varphi & \text{if } v \leq v^l \end{cases}. \]

If \(v^l \geq v^u\), let
\[ \hat{v} := \inf \{ v \mid 1 - n\varphi F(v) - n(\varphi + c)[1 - F(v)] \geq 0 \} \in [v^u, v^l], \quad (26) \]
and
\[ \overline{P}(v) := \begin{cases} \varphi + c & \text{if } v \geq \hat{v} \\ \varphi & \text{if } v < \hat{v} \end{cases}. \]

Finally, redefine
\[ P^*(v) := \begin{cases} \overline{P}(v) & \text{if } v \geq \frac{k}{c} \\ \varphi & \text{if } v < \frac{k}{c} \end{cases}. \quad (27) \]

Then we have the following theorem:

**Theorem 7** \(P^*\) defined in (27) is an optimal solution to \((OPTI - \varphi)\).

The proof is similar to that of Theorem 1 and neglected here. I complete the charac-
terization of the optimal mechanism by characterizing the optimal $\varphi$. First, if inspection is sufficiently costly or the principal’s ability to punish an agent is sufficiently limited, then pure randomization is optimal. In particular, Theorem 2 still applies here. To make the problem more interesting, in what follows, I assume Assumption 1 holds.

Recall that given $\varphi$, $v^l$ is uniquely pinned down by $F(v^l)^{n-1} = n\varphi$ and $v^u$ is uniquely pinned down by (25). Let $v^*$ be defined by (12) and redefine $v^{**}$ by

$$E_v[v] - E_v[\min\{v, v^{**}\}] + E_v[v] - E_v[\max\{v, v^{**}\}] + \frac{k}{c} = 0. \quad (28)$$

They are well defined under Assumption 1. Furthermore, $v^{**} > v^* \geq k/c$. Finally, redefine

$$v^\natural := \sup \left\{ v \mid (F(v)^{n-1} + nc) (1 - F(v)) - 1 + F(v)^n \leq 0 \right\}. \quad (29)$$

The optimal mechanism is characterized by the following theorem:

**Theorem 8** Suppose Assumption 1 holds. There are three cases.

1. If $F(v^*)^{n-1} \geq n(1-c)$, then the optimal $\varphi^* = F(v^*)^{n-1}/n$ and the following allocation rule is optimal:

   $$P^{**}(v) := \begin{cases} 
   F(v)^{n-1} & \text{if } v \geq v^* \\
   \varphi^* & \text{if } v < v^*
   \end{cases}. \quad (30)$$

2. If $F(v^*)^{n-1} < n(1-c)$ and $v^{**} \leq v^\natural$, then the optimal $\varphi^* = (1-nc)/n(1-nc+ncF(v^{**}))$ and the following allocation rule is optimal:

   $$P^{**}(v) := \begin{cases} 
   \varphi^* + c & \text{if } v \geq v^{**} \\
   \varphi^* & \text{if } v < v^{**}
   \end{cases}. \quad (30)$$

3. If $F(v^*)^{n-1} < n(1-c)$ and $v^{**} > v^\natural$, then the optimal $\varphi^*$ is defined by

   $$E_v[v] - E_v[\min\{v, v^u(\varphi^*)\}] + E_v[v] - E_v[\max\{v, v^l(\varphi^*)\}] + \frac{k}{c} = 0, \quad (30)$$
and the following allocation rule is optimal:

\[
P^{**}(v) := \begin{cases} 
\varphi^* + c & \text{if } v \geq v^u(\varphi^*) \\
F(v)^{n-1} & \text{if } v^l(\varphi^*) < v < v^u(\varphi^*) \\
\varphi^* & \text{if } v \leq v^l(\varphi^*) \end{cases}
\]

The proof is similar to that of Theorem 3 and neglected here. Note that the optimal mechanism obtained here is very similar to what obtained if the principal can only penalize an agent who receives the object, but has different thresholds for pooling when the limited punishments constraint is binding.

6 Conclusion

In this paper, I study the problem of a principal who have to allocate a good among a number of agents. There are no monetary transfers. The principal can inspect agents’ reports at a cost and penalize them, but the punishments are limited. I characterize an optimal mechanism in this setting. This paper includes Ben-Porath et al. (2014) and Mylovanov and Zapechelnyuk (2014) as special cases and bridges their connections. For a small number of agents, the optimal mechanism only involves pooling area at the bottom of value distribution as in Ben-Porath et al. (2014). As the number of agents increase, pooling at the top is required to guarantee the incentive compatibility as in Mylovanov and Zapechelnyuk (2014).

A Omitted Proofs

A polymatroid is a polytope of type

\[
P(f) := \left\{ x \in \mathbb{R}^E \middle| x \geq 0, \sum_{e \in A} x_e \leq f(A) \text{ for all } A \subset E \right\},
\]

where $E$ is a finite set and $f : 2^E \to \mathbb{R}_+$ is a submodular function.
Lemma 2 There exists a monotone and submodular function $g : 2^E \to \mathbb{R}_+$ with $g(\emptyset) = 0$ and $P(f) = P(g)$.

Proof. Let $g(\emptyset) := 0$ and $g(X) := \min_{A \supset X} f(A)$ for $X \neq \emptyset$. Let $X \subset Y \subset E$. If $X = \emptyset$, then $g(X) = 0 \leq g(Y)$. If $X \neq \emptyset$, since $A \supset Y$ implies that $A \supset X$, we have

$$g(X) = \min_{A \supset X} f(A) \leq \min_{A \supset Y} f(A) = g(Y).$$

That is, $g$ is monotone. Let $e \in E \setminus Y$. I want to show that

$$g(Y \cup \{e\}) - g(Y) \leq g(X \cup \{e\}) - g(X).$$

Recall that $g(\emptyset) = 0 \leq \min_A f(A)$. It suffices to show that $g$ is submodular, i.e.,

$$\min_{C \supset Y \cup \{e\}} f(C) + \min_{D \supset X} f(D) \leq \min_{A \supset X \cup \{e\}} f(A) + \min_{B \supset Y} f(B).$$

Let $A^* \in \arg\min_{A \supset X \cup \{e\}} f(A)$ and $B^* \in \arg\min_{B \supset Y} f(B)$. Then $A^* \cup B^* \supset Y \cup \{e\}$ and $A^* \cap B^* \supset X$. Hence

$$\min_{A \supset X \cup \{e\}} f(A) + \min_{B \supset Y} f(B) = f(A^*) + f(B^*)$$

$$\geq f(A^* \cup B^*) + f(A^* \cap B^*)$$

$$\geq \min_{C \supset Y \cup \{e\}} f(C) + \min_{D \supset X} f(D),$$

where the first inequality holds since $f$ is submodular. Finally, I want to show that $P(f) = P(g)$. Since $f(A) \geq g(A)$ for all $A \subset E$, we have $P(g) \subset P(f)$. Suppose there exists $x \in P(f)$ and $x \notin P(g)$. Then there exists $A \neq \emptyset$ such that $\sum_{e \in A} x_e > g(A)$. By construction, there exists $B \supset A$ such that $g(A) = f(B)$. However, then we have $\sum_{e \in B} x_e \geq \sum_{e \in A} x_e > g(A) = f(B)$, a contradiction to that $x \in P(f)$. Hence $P(f) = P(g)$. ■
Proof of Lemma 1. First, since $H(\emptyset) = 0$, and $H$ is non-decreasing and submodular, $\hat{z}^t$ is feasible. Next, I show that $\hat{z}^t$ is optimal.

I begin by characterizing $H$. Clearly, there exists a unique $t \in \{1, \ldots, m\}$ such that

$$\frac{1}{n} \left( \sum_{\tau=1}^{t-1} f^\tau \right)^{n-1} < \varphi \leq \frac{1}{n} \left( \sum_{\tau=1}^{t} f^\tau \right)^{n-1}.$$

It is easy to verify that\(^3\)

$$H(S^t) = \begin{cases} 
1 - (\sum_{\tau=1}^{t-1} f^\tau)^n - n\varphi \sum_{\tau=t}^{m} f^\tau & \text{if } t > \bar{t}, \\
1 - n\varphi & \text{if } t \leq \bar{t}.
\end{cases} \tag{32}$$

Let $\Delta(t) := H(S^t) - n \sum_{\tau=t}^{m} \frac{c\varphi f^\tau}{1 - c}$. It is easy to see that $\Delta(t)$ is concave in $t$, and $\Delta(m+1) = 0$.

Recall that if $\Delta(1) = 1 - n\varphi/(1 - c) \geq 0$, then $\bar{t} = 0$; otherwise, there exists a unique $\bar{t} \in \{1, \ldots, m+1\}$ such that

$$H(S^\bar{t}) \leq n \sum_{\tau=1}^{\bar{t}} \frac{c\varphi f^\tau}{1 - c} \text{ and } H(S^{\bar{t}+1}) > n \sum_{\tau=\bar{t}+1}^{m} \frac{c\varphi f^\tau}{1 - c}.$$

Consider the dual to problem $(OPTm_1 - \varphi)$, denoted by $(DOPTm_1 - \varphi)$,

$$\min_{\lambda, \beta, \mu} \sum_{t=1}^{m} \frac{c\varphi f^t \lambda^t}{1 - c} + \sum_{S} \beta(S)H(S) + \varphi \sum_{t=1}^{m} f^tv^t,$$

subject to

$$v^t - \frac{k}{c} - \lambda^t + \mu^t - n \sum_{S \ni t} \beta(S) \geq 0, \forall t,$$

$$\lambda \geq 0, \beta \geq 0, \mu \geq 0.$$

\(^3\)This result can be seen as a corollary of Lemmas 8 and ?? in the asymmetric case.
Let $\hat{z}$ be define in (7), and $(\hat{\lambda}, \hat{\beta}, \hat{\mu})$ be the corresponding dual variables. Let $t^0$ be such that $v^t \geq k/c$ if and only if $t \geq t^0$.

**Case 1:** $v^\bar{t} < \frac{k}{c}$ or $\bar{t} < t^0$.

In this case, we have

\[
\hat{z}^t := \begin{cases} 
\frac{c\varphi f^t}{1-c} & \text{if } t > \bar{t} \\
0 & \text{if } t \leq \bar{t} 
\end{cases}
\]

Let $\hat{\beta}(S) = 0$ for all $S$. If $v^t < k/c$, let $\hat{\lambda}^t = 0$ and $\hat{\mu}^t = k/c - v^t > 0$; if $v^t \geq k/c$, let $\hat{\mu}^t = 0$ and $\hat{\lambda}^t = v^t - k/c \geq 0$. It is easy to see this is a feasible solution to $(DOPTm1 - \varphi)$, and the complementary slackness conditions are satisfied. Finally, the dual objective is equal to

\[
\sum_{t=t^0}^{m} \frac{c\varphi f^t}{1-c} \left(v^t - \frac{k}{c}\right) + \varphi \sum_{t=1}^{m} f^tv^t.
\]

By the duality theorem, $\hat{z}$ is an optimal solution to $(OPTm1 - \varphi)$.

**Case 2:** $v^\bar{t} \geq \frac{k}{c}$ or $\bar{t} \geq t^0$.

\[
\hat{z}^t := \begin{cases} 
\frac{c\varphi f^t}{1-c} & \text{if } t > \bar{t} \\
\frac{1}{n} \overline{H}(S^\bar{t}) - \sum_{r=t+1}^{m} \frac{c\varphi f^r}{1-c} & \text{if } t = \bar{t} \\
\frac{1}{n} \left[\overline{H}(S^t) - \overline{H}(S^{t+1})\right] & \text{if } t^0 \leq t < \bar{t} \\
0 & \text{if } t < t^0
\end{cases}
\]

Let $\hat{\beta}(S) > 0$ if $S = S^t$ for $t^0 \leq t \leq \bar{t}$; and $\hat{\beta}(S) = 0$ otherwise. If $t < t^0$, let $\hat{\lambda}^t = 0$ and $\hat{\mu}^t = k/c - v^t \geq 0$. If $t^0 \leq t \leq \bar{t}$, let $\hat{\lambda}^t = \hat{\mu}^t = 0$, $\hat{\beta}(S^t) = (v^t - v^{t-1})/n$ for $t > t^0$ and $\hat{\beta}(S^{t^0}) = (\vartheta^{t^0} - k/c)/n$. If $t > \bar{t}$, let $\hat{\lambda}^t = v^t - v^\bar{t}$ and $\hat{\mu}^t = 0$. It is easy to see this is a feasible solution to $(DOPTm1 - \varphi)$, and the complementary slackness conditions are
satisfied. Finally, the dual objective is equal to the primal objective:

\[
\frac{1}{n} \mathcal{H}(S^0) (v^0 - \frac{k}{c}) + \sum_{t=0+1}^T \frac{1}{n} \mathcal{H}(S^t) (v^t - v^{t-1}) + \sum_{t=t+1}^m \frac{c \varphi f^t}{1 - c} \left( v^t - \frac{k}{c} \right) + \varphi \sum_{t=1}^m f^t v^t.
\]

By the duality theorem, \( \hat{z} \) is an optimal solution to \( \text{OPTm}1 - \varphi \).

**Lemma 3** An optimal solution to \( \text{OPT} - \varphi \) exists.

**Proof.** Let \( \mathcal{D} \) denote the set of feasible solutions, i.e., solutions satisfying (IC') and (F2). Consider \( \mathcal{D} \) as a subset of \( L_2 \), the set of square integrable functions with respect to the probability measure corresponding to \( F \). Topologize \( L_2 \) with its weak\(^*\), or \( \sigma(L_2, L_2) \), topology. It is straightforward to verify that \( \mathcal{D} \) is \( \sigma(L_2, L_2) \) compact. See, for example, Border (1991b).

Let \( V(\varphi) := \sup_{P \in \mathcal{D}} \mathbb{E}_v \left[ P(v) \left( v - \frac{k}{c} \right) \right] + \frac{\varphi k}{c} \). Let \( \{P_m\} \) be a sequence of feasible solutions to \( \text{OPT} - \varphi \) such that

\[
\int P_m(v) \left( v - \frac{k}{c} \right) dF(v) + \frac{\varphi k}{c} \to V(\varphi).
\]

By Helly’s selection theorem, after taking subsequences, I can assume there exists \( P \) such that \( \{P_m\} \) converges pointwise to \( P \). Since \( \mathcal{D} \) is \( \sigma(L_2, L_2) \) compact, after taking subsequences again, I can assume \( P \in \mathcal{D} \) such that \( \{P_m\} \) converges to \( P \) in \( \sigma(L_2, L_2) \) topology. Since \( v - k/c \in L_2 \), the weak convergence of \( \{P_m\} \) implies that

\[
\int P(v) \left( v - \frac{k}{c} \right) dF(v) + \frac{\varphi k}{c} = V(\varphi).
\]

**Proof of Theorem 1.** Let \( \{P_m\} \) be the sequence of optimal solutions to \( \text{OPTm}1 - \varphi \) defined in Corollary 1. I can extend \( P_m \) to \( V \) by setting

\[
P_m(v) := P_m^t \quad \text{for} \quad v \in \left[ \bar{v} + \frac{(t-1)(\bar{v} - \underline{v})}{m}, \underline{v} + \frac{t(\bar{v} - \underline{v})}{m} \right], \quad t = 1, \ldots, m.
\]
Recall that $\overline{H}$ is given by (32). Thus, there are three cases. If $\overline{t} > t$, then

$$
\mathcal{P}^t := \begin{cases} 
\frac{\varphi}{1-c} & \text{if } t > \overline{t} \\
\frac{1}{n} \frac{1}{n} \left( \sum_{r=1}^{t} f^r \right)^n - \frac{1}{n} \sum_{r=\overline{t}+1}^{m} \frac{\varphi f^t}{1-c} & \text{if } t = \overline{t} \\
\frac{1}{n} \frac{1}{n} \left( \sum_{r=1}^{t} f^r \right)^n \frac{1}{n} \left( \sum_{r=1}^{\overline{t}} f^r \right)^n & \text{if } t < \overline{t} < t \\
\frac{1}{n} \frac{1}{n} \left( \sum_{r=1}^{t} f^r \right)^n - \varphi \sum_{r=\overline{t}+1}^{1} f^r & \text{if } t = t \\
\varphi & \text{if } t < \overline{t} 
\end{cases}
$$

If $\overline{t} = t$, then

$$
\mathcal{P}^t := \begin{cases} 
\frac{\varphi}{1-c} & \text{if } t > \overline{t} \\
\frac{1}{n} \frac{1}{n} \left( \sum_{r=1}^{t} f^r \right)^n - \frac{1}{n} \sum_{r=t+1}^{m} \frac{\varphi f^t}{1-c} & \text{if } t = \overline{t} \\
\varphi & \text{if } t < \overline{t} 
\end{cases}
$$

If $\overline{t} < t$, then

$$
\mathcal{P}^t := \begin{cases} 
\varphi & \text{if } t \geq \overline{t} \\
\varphi & \text{if } t < \overline{t} 
\end{cases}
$$

It is easy to see that $\{P_m\}$ converges pointwise to $P^*$, which is a feasible solution to $(OPT - \varphi)$.

To show the optimality of $P^*$, let $\hat{P}$ be an optimal solution to $(OPT - \varphi)$, which exists by Lemma 3. Define $\hat{P}_m$ be such that

$$
\hat{P}^t_m := \frac{1}{f^t} \int_{v + \frac{(t-1)(\overline{v} - v)}{m}}^{v + \frac{t(\overline{v} - v)}{m}} \hat{P}(v) dF(v) \text{ for } t = 1, \ldots, m,
$$

and it can be extended to $V$ by setting

$$
\hat{P}_m(v) := \hat{P}^t_m \text{ for } v \in \left[ v + \frac{(t-1)(\overline{v} - v)}{m}, v + \frac{t(\overline{v} - v)}{m} \right], t = 1, \ldots, m.
$$
By the Lebesgue differentiation theorem, \( \{\hat{P}_m\} \) converges pointwise to \( \hat{P} \). It is easy to verify that \( \hat{P}_m \) defined on \( \{v^1, \ldots, v^m\} \) is a feasible solution to \((OPT_m - \phi)\). Hence

\[
\sum_{t=1}^{m} f^t \hat{P}_m^t \left( v^t - \frac{k}{c} \right) + \frac{\phi k}{c} \leq \sum_{t=1}^{m} f^t P_m^t \left( v^t - \frac{k}{c} \right) + \frac{\phi k}{c}
\]

By the dominated convergence theorem,

\[
\sum_{t=1}^{m} f^t \hat{P}_m^t \left( v^t - \frac{k}{c} \right) = \int_V \hat{P}_m(v) \left( v - \frac{k}{c} \right) dF(v) \rightarrow \int_V \hat{P}(v) \left( v - \frac{k}{c} \right) dF(v)
\]

and

\[
\sum_{t=1}^{m} f^t P_m^t \left( v^t - \frac{k}{c} \right) = \int_V P_m(v) \left( v - \frac{k}{c} \right) dF(v) \rightarrow \int_V P^*(v) \left( v - \frac{k}{c} \right) dF(v).
\]

Hence, \( P^* \) is optimal, i.e.,

\[
\int_V P^*(v) \left( v - \frac{k}{c} \right) dF(v) = \int_V \hat{P}(v) \left( v - \frac{k}{c} \right) dF(v).
\]

\[\blacksquare\]

**Lemma 4** Suppose \((1-c)/n \leq \phi \leq \min\{1/n, 1-c\}\). Then \( v^l \geq v^u \) if and only if \( v^l \leq v^\natural \), where \( v^\natural \) is defined by (14). Furthermore, if \( n(1-c) < 1 \) then \( v^\natural \) is strictly increasing in \( n \) and strictly decreasing in \( c \).

**Proof.** Since \((1-c)/n \leq \phi \leq \min\{1/n, 1-c\}\), \( v^l \) and \( v^u \) satisfies:

\[
\frac{1 - F(v^u)^n}{1 - F(v^u)} = \frac{F(v^l)^{n-1}}{1-c}.
\]

Define

\[
\Delta(v) := \frac{F(v)^{n-1}(1 - F(v))}{1-c} - 1 + F(v)^n.
\]

35
Then $\Delta(v) = -1 < 0$ and $\Delta(\bar{v}) = 0$. Then

$$\Delta'(v) = \frac{F(v)^{n-2}f(v)}{1-c}[-cnF(v) + n - 1].$$

Clearly, the term in the brackets is strictly decreasing in $v$. Moreover, $\Delta'(v) = n - 1 > 0$ and $\Delta'(\bar{v}) = n(1 - c) - 1$.

If $n(1 - c) \geq 1$, then $\Delta'(v) \geq 0$ for all $v$. Hence $\Delta(v)$ is non-decreasing, and therefore $\Delta(v) \leq 0$ for all $v$. Hence

$$\frac{1 - F(v^u)^n}{1 - F(v^u)} = \frac{F(v^l)^{n-1}}{1 - c} \leq \frac{1 - F(v^l)^n}{1 - F(v^l)},$$

which implies $v^l \geq v^u$.

If $n(1 - c) < 1$, then there exists $v^\flat$ such that $\Delta'(v) > 0$ for $v \in [v, v^\flat]$ and $\Delta'(v) < 0$ for $v \in [v^\flat, \bar{v}]$. Hence $\Delta(v)$ in strictly increasing in $[v, v^\flat]$, and strictly decreasing in $[v^\flat, \bar{v}]$. Hence there exists a unique $v^\sharp \in (v, \bar{v})$ such that $\Delta(v) \leq 0$ if and only if $v \leq v^\sharp$. By (33), this implies that $v^l \geq v^u$ if and only if $v^l \leq v^\sharp$. Finally, for any $v$, $\Delta(v)$ is strictly decreasing in $n$, and strictly increasing in $c$. Hence $v^\sharp$ is strictly increasing in $n$, and strictly decreasing in $c$. ■

**Lemma 5** $Z(\varphi) \leq Z_1(v^l(\varphi))$.

**Proof.** Fix $\varphi$ and the corresponding $v^l$. Note that $Z_1(v^l)$ is attained by the following allocation rule

$$P_1(v) := \begin{cases} F(v)^{n-1} & \text{if } v \geq \max \left\{ v^l, \frac{k}{c} \right\} \\ \varphi & \text{if } v < \max \left\{ v^l, \frac{k}{c} \right\} \end{cases}.$$

It is easy to see that $P_1 - P^*$ is non-decreasing, and

$$\int_{v^l}^{\bar{v}} P_1(v) dF(v) = \int_{v^l}^{\bar{v}} P^*(v) dF(v) = \frac{1}{n}.$$
Moreover, \( v - k/c \) is non-decreasing. Hence by Lemma 1 in Persico (2000), \( Z(\varphi) \leq Z_1(v^l(\varphi)) \).

Lemma 6 If \( \varphi \leq 1 - c \), then \( Z(\varphi) \leq Z_2(\hat{v}(\varphi)) \).

Proof. Fix \( \varphi \) and the corresponding \( \hat{v} \). Note that \( Z_2(\hat{v}) \) is attained by the following allocation rule

\[
P_2(v) := \begin{cases} 
\frac{\varphi}{1-c} & \text{if } v \geq \max\left\{ \hat{v}, \frac{k}{c} \right\} \\
\varphi & \text{if } v < \max\left\{ \hat{v}, \frac{k}{c} \right\}
\end{cases}
\]

It is easy to see that \( P_2 - P^* \) is non-decreasing, and

\[
\int_{\underline{v}}^{\overline{v}} P_2(v) dF(v) = \int_{\underline{v}}^{\overline{v}} P^*(v) dF(v) = \frac{1}{n},
\]

Moreover, \( v - k/c \) is non-decreasing. Hence by Lemma 1 in Persico (2000), \( Z(\varphi) \leq Z_1(v^l(\hat{\varphi})) \).

Proof of Theorem 3. First, if \( \varphi \leq (1 - c)/n \), then \( v^u = \hat{v} = \underline{v} \), and

\[
P^*(v) := \begin{cases} 
\frac{\varphi}{1-c} & \text{if } v \geq \frac{k}{c} \\
\varphi & \text{if } v < \frac{k}{c}
\end{cases}
\]

The principal’s objective is

\[
\frac{c\varphi}{1-c} \int_{\underline{v}}^{\overline{v}} \left( v - \frac{k}{c} \right) dF(v) + \varphi \int_{\underline{v}}^{\overline{v}} v dF(v),
\]

which is strictly increasing in \( \varphi \). Hence, in optimum, \( \varphi \geq (1 - c)/n \).

Given \( \varphi \), let \( Z(\varphi) \) denote the principal’s payoff. Suppose \( \varphi \geq 1 - c \) or \( F(v^l)^{n-1} \geq n(1-c) \).

Then \( v^u = \overline{v} \), and the principal’s payoff is \( Z(\varphi) = Z_1(v^l(\varphi)) \), where

\[
Z_1(v^l) := \int_{\max\{v^l, \hat{v}\}}^{\overline{v}} \left( v - \frac{k}{c} \right) F(v)^{n-1} dF(v)
\]
\[
+ \frac{1}{n} F(v')^{n-1} \int_{v}^{\max \{v', \frac{k}{c} \}} (v - \frac{k}{c}) \, dF(v) + \frac{1}{n} F(v')^{n-1} \frac{k}{c}.
\]

If \( v' < k/c \), then \( Z_1(v') \) is strictly increasing in \( v' \). If \( v' \geq k/c \), then

\[
Z_1'(v') = \frac{n-1}{n} F(v')^{n-2} f(v') \left\{ \mathbb{E}_v[v] - \mathbb{E}_v[\max\{v, v'\}] + \frac{k}{c} \right\}.
\]

Clearly, the term inside the braces is strictly decreasing in \( v' \). Recall that \( v^* \geq k/c \) is defined by (12). Then \( Z_1'(v') \geq 0 \) if and only if \( v' \leq v^* \). Hence, \( Z_1 \) achieves its maximum at \( v' = v^* \).

I show in Lemma 5 in appendix that for any \( \varphi \) and the corresponding \( v' \), we have

\[ Z(\varphi) \leq Z_1(v'(\varphi)) \leq Z_1(v^*). \]

Thus, if \( F(v^*)^{n-1} \geq n(1-c) \), it is optimal to set \( v' = v^* \). This proves the first part of Theorem 3.

If \( F(v^*)^{n-1} < n(1-c) \), then in optimum \( \varphi \leq 1-c \) or \( F(v')^{n-1} \leq n(1-c) \). Since \( (1-c)/n \leq \varphi \leq 1/n \), there is a one-to-one correspondence between \( \hat{v} \) and \( \varphi \). Given \( \varphi \), \( \hat{v}(\varphi) \) is uniquely pinned down by

\[ 1 - n \varphi F(\hat{v}) = \frac{n \varphi}{1-c} [1 - F(\hat{v})] = 0. \]

If \( \varphi \) is such that \( v' \geq v^* \), then \( Z(\varphi) = Z_2(\hat{v}(\varphi)) \), where

\[
Z_2(\hat{v}) := \frac{1-c}{n(1-cF(\hat{v}))} \int_{v}^{\max \{\hat{v}, \frac{k}{c} \}} (v - \frac{k}{c}) \, dF(v)
\]

\[ + \frac{1}{n(1-cF(\hat{v}))} \int_{\max \{\hat{v}, \frac{k}{c} \}}^{v} (v - \frac{k}{c}) \, dF(v) + \frac{1-c}{n(1-cF(\hat{v}))} \frac{k}{c}. \]

If \( \hat{v} < k/c \), then \( Z_2(\hat{v}) \) is strictly increasing in \( \hat{v} \). If \( \hat{v} \geq k/c \), then

\[
Z_2'(\hat{v}) = \frac{cf(\hat{v})}{n(1-cF(\hat{v}))^2} \left\{ \mathbb{E}_v[v] - \mathbb{E}_v[\min\{v, \hat{v}\}] + (1-c) \left[ \mathbb{E}_v[v] - \mathbb{E}_v[\max\{v, \hat{v}\}] + \frac{k}{c} \right] \right\}.
\]
Clearly, the term inside the braces is strictly decreasing in $\hat{v}$. Recall that $v^** > v^* \geq k/c$ is defined by (13). Since $Z_2(\hat{v}) \geq 0$ if and only if $\hat{v} \leq v^**$, $Z_2$ achieves its maximum at $\hat{v} = v^**$. I show in Lemma 6 in the appendix that for any $\varphi \leq 1 - c$ and the corresponding $\hat{v}$, we have

$$Z(\varphi) \leq Z_2(\hat{v}(\varphi)) \leq Z_2(v^**).$$

This, together with Lemma 4, proves the second part of Theorem 3.

Suppose $F(v^*)^{n-1} < n(1 - c)$ and $v^** > v^u$. Then

$$Z(\varphi) = \varphi \int_v^{\max\{v', v^u\}} \left( v - \frac{k}{c} \right) dF(v) + \int_{\max\{v', v^u\}}^{\max\{v^u, v^*\}} \left( v - \frac{k}{c} \right) F(v)^{n-1} dF(v)$$

$$+ \frac{\varphi}{1 - c} \int_{\max\{v^u, v^*\}}^{v^*} \left( v - \frac{k}{c} \right) dF(v) + \frac{\varphi k}{c}.$$

If $\varphi$ is such that $v^l < k/c$, then $Z_2(\hat{v})$ is strictly increasing in $\varphi$. If $v^l \geq k/c$, then

$$Z'(\varphi) = \frac{1}{1 - c} \left[ \mathbb{E}_v[v] - \mathbb{E}_v[\min\{v, v^u\}] \right] + \mathbb{E}_v[v] - \mathbb{E}_v[\max\{v, v^l\}] + \frac{k}{c}.$$

Since both $v^l$ and $v^u$ are strictly increasing in $\varphi$, $Z'(\varphi)$ is strictly decreasing in $\varphi$. Let $\varphi^*$ be such that

$$\mathbb{E}_v[v] - \mathbb{E}_v[\min\{v, v^u(\varphi^*)\}] + (1 - c) \left[ \mathbb{E}_v[v] - \mathbb{E}_v[\max\{v, v^l(\varphi^*)\}] + \frac{k}{c} \right] = 0.$$  \hspace{1cm} (15)

Compare (15) with (13) and (12), and it is easy to see that $v^u(\varphi^*) > v^** > v^l(\varphi^*) > v^*$. Since $Z'(\varphi) \geq 0$ if and only if $\varphi \leq \varphi^*$, $Z$ achieves its maximum at $\varphi = \varphi^*$. This proves the third part of Theorem 3. ■

**Proof of Theorem 2.** If $\bar{v} - k/c \leq \mathbb{E}_v[v]$, then $Z_1(v)$ is strictly increasing in $v^l$ and achieves its maximum when $v^l = \bar{v}$. By Lemma 5, $Z(\varphi) \leq Z_1(v^l(\varphi)) \leq Z_1(\bar{v})$. Note that $Z_1(\bar{v})$ can be achieved via pure randomization. This completes the proof. ■
Proof of Corollary 2. The analysis in Section 4 has proved most results of Corollary 2. What is left to prove is that if \( n^*(\rho, c) < n < n^{**}(\rho, c) \), then \( v'(n, \rho, c) \) is strictly increasing in \( n, \rho \) and \( c \) and \( v^u(n, \rho, c) \) is strictly decreasing in \( n \) and strictly decreasing in \( \rho \) and \( c \).

If \( n^*(\rho, c) < n < n^{**}(\rho, c) \), then \( v' \) and \( v^u \) satisfies (33). It is easy to see that \( v^u \) is strictly increasing in \( v' \) and vice versa. Let

\[
\Delta_i(v', n, \rho, c) := \mathbb{E}_v[v] - \mathbb{E}_v[\min\{v, v^u\}] + (1 - c) \left[ \mathbb{E}_v[v] - \mathbb{E}_v[\max\{v, v^l\}] + \frac{k}{c} \right],
\]

where \( v^u \) is a function of \( v' \), \( n \) and \( c \) defined by (33). We have

\[
\begin{align*}
\frac{\partial \Delta_i}{\partial v'} &= -[1 - F(v^u)] \frac{\partial v^u}{\partial v'} - (1 - c) F(v') < 0, \\
\frac{\partial \Delta_i}{\partial n} &= -[1 - F(v^u)] \frac{\partial v^u}{\partial n} > 0, \\
\frac{\partial \Delta_i}{\partial \rho} &= 1 - c > 0, \\
\frac{\partial \Delta_i}{\partial c} &= - \left[ \mathbb{E}_v[v] - \mathbb{E}_v[\max\{v, v^l\}] + \frac{k}{c} \right] > 0.
\end{align*}
\]

Hence, by the implicit function theorem, we have \( \partial v'/\partial n > 0 \), \( \partial v'/\partial \rho > 0 \) and \( \partial v'/\partial c > 0 \).

To see that \( \partial v^u/\partial n < 0 \), let

\[
\Delta(v^u, v', n) := \frac{F(v')^{n-1}(1 - F(v^u))}{1 - c} - 1 + F(v^u)^n.
\]

Then

\[
\begin{align*}
\frac{\partial \Delta}{\partial v^u} &= \left[ - \frac{F(v')^{n-1}}{1 - c} + n F(v^u)^{n-1} \right] F(v^u) = \left[ - \frac{1 - F(v^u)^n}{1 - F(v^u)} + n F(v^u)^{n-1} \right] F(v^u) < 0, \\
\frac{\partial \Delta}{\partial n} &= \frac{F(v')^{n-1}[1 - F(v^u)] \log F(v')}{1 - c} + F(v^u)^n \log F(v^u) < 0.
\end{align*}
\]

Hence, \( \partial v^u/\partial n = - (\partial \Delta/\partial n)/(\partial \Delta/\partial v^u) < 0 \).
Let
\[ \Delta_u(v^u, n, \rho, c) := \mathbb{E}_v[v] - \mathbb{E}_v[\min\{v, v^u\}] + (1 - c) \left[ \mathbb{E}_v[v] - \mathbb{E}_v[\max\{v, v^l\}] + \frac{k}{c} \right], \]
where \( v^l \) is a function of \( v^u, n \) and \( c \) defined by (33). We have
\[
\begin{align*}
\frac{\partial \Delta_u}{\partial v^u} &= -[1 - F(v^u)] - (1 - c)F(v^l)\frac{\partial v^l}{\partial v^u} < 0, \\
\frac{\partial \Delta_u}{\partial n} &= -(1 - c)F(v^l)\frac{\partial v^l}{\partial n} < 0, \\
\frac{\partial \Delta_u}{\partial \rho} &= 1 - c > 0, \\
\frac{\partial \Delta_u}{\partial c} &= - \left[ \mathbb{E}_v[v] - \mathbb{E}_v[\max\{v, v^l\}] + \frac{k}{c} \right] > 0.
\end{align*}
\]
Hence, by the implicit function theorem, we have \( \partial v^u/\partial n < 0, \partial v^u/\partial \rho > 0 \) and \( \partial v^u/\partial c > 0 \).
To see that \( \partial v^l/\partial n > 0 \), note that
\[
\frac{\partial \Delta}{\partial v^l} = \frac{(n - 1)F(v^l)^{n-2}f(v^l)[1 - F(v^u)]}{1 - c} > 0.
\]
Hence, \( \partial v^l/\partial n = - (\partial \Delta/\partial n)/(\partial \Delta/\partial v^l) > 0. \)

**B Asymmetric Environment**

**B.1 Finite Case**

Let \( \mathcal{D} := \bigcup_i [v_i - k_i/c_i, \bar{v}_i - k_i/c_i] \). Let \( d := \inf \mathcal{D} \) and \( \bar{d} := \sup \mathcal{D} \). Fix an integer \( m \geq 2 \).

For \( t = 1, \ldots, m \), let
\[
\begin{align*}
d^t &= d + \frac{(2t - 1)(\bar{d} - d)}{2m}, \\
f^t_i &= F_i \left( \frac{d + t(\bar{d} - d)}{m} + \frac{k_i}{c_i} \right) - F_i \left( \frac{d + (t - 1)(\bar{d} - d)}{m} + \frac{k_i}{c_i} \right), i = 1, \ldots, n.
\end{align*}
\]
Consider the finite model in which $v_i - k_i/c_i$ can take only $m$ possible different values, i.e., $v_i - k_i/c_i \in \{d_1, \ldots, d^m\}$ and the probability mass function satisfies $f_i(d^t) = f_i^t$ for $t = 1, \ldots, m$. It is possible that $f_i^t = 0$ for some $t$. The corresponding problem of $(OPTH - \varphi)$ in the finite model, denoted by $(OPTH m - \varphi)$, is given by:

$$\max \sum_{i=1}^{n} \left[ \sum_{t=1}^{m} f_i^t d^t + \frac{\varphi_i k_i}{c_i} \right],$$

subject to

$$\varphi_i \leq P_i^t \leq \frac{\varphi_i}{1 - c_i}, \forall t,$$  \hspace{1cm} (IC')

$$\sum_{i=1}^{n} \sum_{t \in S_i} f_i^t P_i^t \leq 1 - \prod_{i=1}^{n} \sum_{t \notin S_i} f_i^t, \forall S_i \subset \{1, \ldots, m\}. \hspace{1cm} (F2)$$

Define $H(S) := 1 - \prod_{i=1}^{n} \sum_{t \notin S_i} f_i^t - \sum_{i=1}^{n} \sum_{t \in S_i} \varphi_i f_i^t$ for all $S := (S_1, \ldots, S_n)$ and $S_i \subset \{1, \ldots, m\}$. Define $\overline{H}(S) := \min_{S' \supset S} H(S')$ for all $S$. Let $z_i^t := f_i^t (P_i^t - \varphi_i)$ for all $i$ and $t$.

By Lemma 2, $(OPTH m - \varphi)$ can be rewritten as $(OPTH m1 - \varphi)$

$$\max \sum_{i=1}^{n} \sum_{t=1}^{m} z_i^t d^t + \sum_{i=1}^{n} \varphi_i \left( \sum_{t=1}^{m} f_i^t d^t + \frac{k_i}{c_i} \right),$$

subject to

$$0 \leq z_i^t \leq \frac{c_i \varphi_i f_i^t}{1 - c_i}, \forall i, \forall t,$$  \hspace{1cm} (IC')

$$\sum_{i=1}^{n} \sum_{t \in S_i} z_i^t \leq \overline{H}(S), \forall S \subset \{1, \ldots, m\}^n. \hspace{1cm} (F2)$$

Note that if $f_i^t = 0$, then $z_i^t = 0$ by definition and therefore (IC') is satisfied automatically.

Lemma 7 proves a useful property of $H$. Lemma 8 characterizes $\overline{H}$.

**Lemma 7** If $H(S) < 1 - \sum_{i=1}^{n} \varphi_i$ and $S' \subset S$, then $H(S') \leq H(S)$.
Proof. Consider \( S = (S_1, \ldots, S_n) \). We have

\[
H(S) - 1 + \sum_{i=1}^{n} \varphi_i = \sum_{i=1}^{n} \varphi_i \sum_{t \notin S_i} f_i^t - \prod_{i=1}^{n} \sum_{t \notin S_i} f_i^t.
\]

Let \( S_i^{\text{supp}} := \{ t | f_i^t > 0 \} \). If \( S_i = S_i^{\text{supp}} \) for some \( i \), then \( \sum_{t \notin S_i} f_i^t = 0 \) and therefore \( H(S) \geq 1 - \sum_{i=1}^{n} \varphi_i \). Hence, \( H(S) < 1 - \sum_{i=1}^{n} \varphi_i \) implies that \( S_i \neq S_i^{\text{supp}} \) or \( \sum_{t \notin S_i} f_i^t > 0 \) for all \( i \).

Thus, \( \varphi_i \leq \prod_{j \neq i} \sum_{t \notin S_j} f_j^t \) for all \( i \). Let \( S' := (S_1, \ldots, S_i-1, S_i \setminus \{ t \}, S_i+1, \ldots, S_n) \). Then

\[
H(S) - H(S') = f_i^t \left( \prod_{j \neq i} \sum_{t \notin S_j} f_j^t - \varphi_i \right) \geq 0.
\]

Hence, \( H(S') \leq H(S) \). By induction, \( H(S') \leq H(S) \) for all \( S' \subset S \). 

Lemma 8 \( \overline{H}(S) = \min \{ H(S), 1 - \sum_{i=1}^{n} \varphi_i \} \).

Proof. Recall that \( \overline{H}(S) = \min_{S' \supset S} H(S) \). Since \( S^1 \supset S \) and \( H(S^1) = 1 - \sum_{i=1}^{n} \varphi_i \), we have \( \overline{H}(S) \leq 1 - \sum_{i=1}^{n} \varphi_i \).

Suppose \( H(S) \leq 1 - \sum_{i=1}^{n} \varphi_i \). Let \( S'' \supset S \). If \( H(S'') \geq 1 - \sum_{i=1}^{n} \varphi_i \), then \( H(S) \leq 1 - \sum_{i=1}^{n} \varphi_i \). If \( H(S'') < 1 - \sum_{i=1}^{n} \varphi_i \), then \( H(S) \leq H(S'') \) by Lemma 7. Hence, \( \overline{H}(S) = H(S) \).

Suppose \( H(S) > 1 - \sum_{i=1}^{n} \varphi_i \). I claim that \( \overline{H}(S) = 1 - \sum_{i=1}^{n} \varphi_i \). Suppose not, then there exists \( S'' \supset S \) such that \( H(S'') < 1 - \sum_{i=1}^{n} \varphi_i \). Then, by Lemma 7, \( H(S) \leq H(S'') < 1 - \sum_{i=1}^{n} \varphi_i \), a contradiction. Hence, \( \overline{H}(S) = 1 - \sum_{i=1}^{n} \varphi_i \).

Algorithm 1 below describes an algorithm that finds a feasible solution to \( (OPTHm - \varphi) \).

In order to describe the mechanism, I first introduce some notations. Let \( S_i := \{ t, \ldots, m \} \) and \( S_i^{m+1} := \emptyset \) for all \( i \) and \( t \) and \( S' := \{ t, \ldots, m \}^n \) and \( S'^{m+1} := \emptyset \) for all \( t \). Define \( S+(t,i) := (S_1, \ldots, S_i-1, S_i \cup \{ t \}, S_{i+1}, \ldots, S_n) \) and \( S-(t,i) := (S_1, \ldots, S_i-1, S_i \setminus \{ t \}, S_{i+1}, \ldots, S_n) \).

**Algorithm 1** Let \( \mathcal{I}_0^m := \{ i | f_i^m = 0 \} \) and \( \overline{\mathcal{I}}_0^m := \emptyset \) for all \( i \in \mathcal{I}_0^m \). Define \( \mathcal{I}_i^m \subset \mathcal{I} \setminus \mathcal{I}_0^m \), \( \{ \pi_i^m, \ldots, \pi_{i}^{m,n} \} \), \( \{ S_i^m, \ldots, S_i^{m,n} \} \) and \( \overline{\mathcal{I}}_i^m \) for all \( i \notin \mathcal{I}_0^m \) recursively as follows.
1. Let $I_m^0 = \emptyset$ and $\nu = 1$.

2. If $I_m^1 = \mathcal{I} \setminus I_m^0$, then go to step 5. Otherwise, let $\iota = 1$ and go to step 2.

3. If there exists $\mathcal{I}' \neq \emptyset$ such that $|\mathcal{I}'| = \iota$, $\mathcal{I}' \cap (I_m^0 \cup I_m^1) = \emptyset$ and

$$\mathcal{H} \left( S + \sum_{i \in \mathcal{I}'} (m, i) \right) - \mathcal{H}(S) \leq \sum_{i \in \mathcal{I}'} \frac{c_i \phi_i f_i^m}{1 - c_i},$$

where $S_j = S_j^m$ if $j \in I_m^1$ and $S_j = S_j^{m+1}$ otherwise, then let $z_m^i \leq c_i \phi_i f_i^m / (1 - c_i)$ for $i \in \mathcal{I}'$ be such that

$$\sum_{i \in \mathcal{I}'} z_m^i = \mathcal{H} \left( S + \sum_{i \in \mathcal{I}'} (m, i) \right) - \mathcal{H}(S).$$

Let $\pi^{m, \nu} := \mathcal{I}'$ and $S^{m, \nu} := S$. Redefine $\nu$ as $\nu + 1$ and $I_m^1$ as $\mathcal{I}' \cup I_m^1$ and go to step 2. If there does not exist such an $\mathcal{I}'$, go to step 4.

4. If $\iota < n - |I_m^0 \cup I_m^1|$, redefine $\iota$ as $\iota + 1$ and go to step 3. If $\iota = n - |I_m^0 \cup I_m^1|$, go to step 5.

5. Let $n^m := \nu - 1$ and $z_m^i := c_i \phi_i f_i^m / (1 - c_i)$ for all $i \in \mathcal{I} \setminus (I_m^0 \cup I_m^1)$.

Let $1 \leq t \leq m - 1$. Suppose we have defined $I_0^\tau$, $I_1^\tau$, $\{\pi^{\tau,1}, \ldots, \pi^{\tau,n^\tau}\}$, $\{S^{\tau,1}, \ldots, S^{\tau,n^\tau}\}$ and $z_i^\tau$ for all $i$ for all $\tau \geq t + 1$. Let $I_0^t := \{i | f_i^t = 0\}$ and $z_0^t := 0$ for all $i \in I_0^t$. Define $I_1^t \subset \mathcal{I} \setminus I_0^t$, $\{\pi^{t,1}, \ldots, \pi^{t,n^t}\}$, $\{S^{t,1}, \ldots, S^{t,n^t}\}$ and $z_i^t$ for all $i \notin I_0^t$ recursively as follows.

1. Let $I_1^t := \emptyset$ and $\nu = 1$.

2. If $I_1^t = \mathcal{I} \setminus I_0^t$, then go to step 5. Otherwise, let $\iota = 1$ and go to step 2.

3. If there exists $\mathcal{I}' \neq \emptyset$ such that $|\mathcal{I}'| = \iota$, $\mathcal{I}' \cap (I_0^t \cup I_1^t) = \emptyset$ and

$$\min_{t_j} \mathcal{H} \left( S + \sum_{i \in \mathcal{I}'} (t, i) \right) - \sum_{j=1}^n \sum_{\tau \in S_j} z_j^\tau \leq \sum_{i \in \mathcal{I}'} \frac{c_i \phi_i f_i^t}{1 - c_i},$$

44
where $\mathcal{S} = (\mathcal{S}_1^{t_1}, \ldots, \mathcal{S}_n^{t_n})$ with $t_j \geq t$ if $j \in \pi^{t,1} \cup \cdots \cup \pi^{t,\nu-1}$, $t_j = t+1$ if $j \in \mathcal{T}'$ and $t_j \geq t+1$ otherwise, then let $\mathcal{Z}_i^t \leq c_i \varphi_i f_i^t / (1 - c_i)$ for $i \in \mathcal{T}'$ be such that

$$\sum_{i \in \mathcal{T}'} \mathcal{Z}_i^t = \min_{t_j} H \left( \mathcal{S} + \sum_{i \in \mathcal{T}'} (m, i) \right) - \sum_{i=1}^n \sum_{\tau \in S_i} \mathcal{Z}_i^\tau.$$

Let $\pi^{t,\nu} := \mathcal{T}'$ and $\mathcal{S}_{t,\nu}$ as a minimizer of the right-hand side of the above equation such that there is no $\mathcal{S} \supseteq \mathcal{S}^{t,\nu}$ which is also a minimizer. Redefine $\nu$ as $\nu + 1$ and $\mathcal{T}_1^t$ as $\mathcal{T}' \cup \mathcal{T}_1^t$ and go to step 2. If there does not exist such an $\mathcal{T}'$, go to step 4.

4. If $i < n - |\mathcal{T}_0^t \cup \mathcal{T}_1^t|$, redefine $i$ as $i + 1$ and go to step 3. If $i = n - |\mathcal{T}_0^t \cup \mathcal{T}_1^t|$, go to step 5.

5. Let $n^t := \nu - 1$ and $\mathcal{Z}_i^t := c_i \varphi_i f_i^t / (1 - c_i)$ for all $i \in \mathcal{T} \setminus (\mathcal{T}_0^t \cup \mathcal{T}_1^t)$.

Let $\mathcal{Z}$ be a solution found by Algorithm 1. I first prove that $\mathcal{Z}$ is a feasible solution to $(OPTHm1 - \varphi)$. For each $i$ and $t$, let $\overline{P}_i^t := \mathcal{Z}_i^t / f_i^t + \varphi_i$ if $f_i^t > 0$ and $\overline{P}_i^t = 0$ otherwise. Then $\mathcal{Z}$ is a feasible solution to $(OPTHm1 - \varphi)$ if and only if $\overline{P}$ is a feasible solution to $(OPTHm - \varphi)$. Lemmas 10 and 11 below prove that $\overline{P}$ is non-decreasing. By Theorem 2 in Mierendorff (2011), $\overline{P}$ is a feasible solution to $(OPTHm - \varphi)$ if and only if for all $t_1, \ldots, t_n \in \{1, \ldots, m\}$

$$\sum_{i=1}^n \sum_{t \in S_i} \overline{P}_i^t \leq H(\mathcal{S}),$$

where $\mathcal{S} = (\mathcal{S}_1^{t_1}, \ldots, \mathcal{S}_n^{t_n})$. By construction, this is true if

**Lemma 9** For all $t_1, \ldots, t_n \in \{1, \ldots, m\}$,

$$\sum_{i=1}^n \sum_{t \in S_i} \mathcal{Z}_i^t \leq H(\mathcal{S}), \quad (34)$$

where $\mathcal{S} = (\mathcal{S}_1^{t_1}, \ldots, \mathcal{S}_n^{t_n})$.

**Proof.** For each $t$, let $\pi^{t,0} := \emptyset$ and $\pi^{t,n^t+1} := \mathcal{T} \setminus (\mathcal{T}_0^t \cup \mathcal{T}_1^t)$. Suppose $\mathcal{S} \subset \mathcal{S}_m$, i.e., $t_i \geq m$ for
all i. By construction, we have

\[ \sum_{i \in \pi^{m,1}} z_{i}^m = H\left( \emptyset + \sum_{i \in \pi^{m,1}} (m, i) \right), \]

\[ \sum_{i \in I'} z_{i}^m \leq \sum_{i \in I'} c_i \varphi_i f_i^m \leq H\left( \emptyset + \sum_{i \in I'} (m, i) \right), \forall I' \subseteq \pi^{m,1}. \]

Thus, (34) holds if \( t_i = m + 1 \) for all \( i \notin \pi^{m,1} \). Suppose we have shown that (34) holds if \( t_i = m + 1 \) for all \( i \notin \pi^{m,1} \cup \cdots \cup \pi^{m,\nu-1} \) and \( \nu \geq 2 \). Suppose \( t_i = m + 1 \) for all \( i \notin \pi^{m,1} \cup \cdots \cup \pi^{m,\nu} \). Let \( S' := \emptyset + \sum_{i \in \pi^{m,1} \cup \cdots \cup \pi^{m,\nu-1}} (m, i) \) and \( I' := \{ i \in \pi^{m,\nu} | t_i = m \} \). By construction, we have for all, \( I' \subset \pi^{m,\nu} \),

\[ \sum_{i \in I'} z_{i}^m = H\left( S' + \sum_{i \in I'} (m, i) \right) - H(S') \text{ if } I' = \pi^{m,\nu} \text{ and } \nu \leq n^m, \]

\[ \sum_{i \in I'} z_{i}^m \leq \sum_{i \in I'} c_i \varphi_i f_i^m \leq H\left( S' + \sum_{i \in I'} (m, i) \right) - H(S') \text{ if } I' \subsetneq \pi^{m,\nu} \text{ or } \nu = n^m + 1. \]

Since \( S - \sum_{i \in \pi^{m,\nu} \cup \pi^{m,1}} (m, i) \subset S' \), we have

\[ \sum_{i=1}^{n} \sum_{t \in S'_i \setminus S_i} z_{i}^t = \sum_{i=1}^{n} \sum_{t \in S'_i} z_{i}^t - \sum_{i \in \pi^{m,\nu}} \sum_{t \in S_i} z_{i}^t \]

\[ \geq H(S') - H\left( S - \sum_{i \in \pi^{m,\nu}} (m, i) \right) \]

\[ \geq H\left( S' + \sum_{i \in I'} (m, i) \right) - H(S), \]

where the last inequality holds since \( H \) is submodular. Hence,

\[ \sum_{i=1}^{n} \sum_{t \in S_i} z_{i}^t = \sum_{i=1}^{n} \sum_{t \in S'_i} z_{i}^t - \sum_{i=1}^{n} \sum_{t \in S'_i \setminus S_i} z_{i}^t + H\left( S' + \sum_{i \in I'} (m, i) \right) - H(S') \]

\[ \leq H(S') - H\left( S' + \sum_{i \in I'} (m, i) \right) + H(S) + H\left( S' + \sum_{i \in I'} (m, i) \right) - H(S') \]

\[ = H(S). \]
Suppose $S \subset S^{t+1} + \sum_{i \in \pi^{t+1}_t \cup \ldots \cup \pi^t_i} (t, i)$ for $t \leq m - 1$ and $1 \leq i \leq n^t + 1$. Let $I' := \{ i \in \pi^{t+1}_i | t = t \}$ and $S' := S - \sum_{i \in I'} (t, i)$. By construction, we have

$$\sum_{i \in I'} z_{t+1}^i \leq H \left( S' + \sum_{i \in I'} (t, i) \right) - \sum_{i=1}^n \sum_{\tau \in S'_t} z_{t+1}^\tau = H(S) - \sum_{i=1}^n \sum_{\tau \in S'_t} z_{t+1}^\tau.$$ 

Hence, (34) holds. $\blacksquare$

**Lemma 10** Suppose $f_i^t, f_{i+1}^t > 0$. Then $z_{t+1}^i \in I_{t+1}^i$ implies that $z_t^i \in I_t^i$.

**Proof.** Suppose $f_i^t, f_{i+1}^t > 0$ and $z_{t+1}^i \in I_{t+1}^i$. Then there exists $S$ with $S_j \subset S_{j+1}^t$ for all $j$ and $t + 1 \in S_i$ such that

$$\sum_{j=1}^n \sum_{\tau \in S_j} z_{t+1}^\tau = H(S).$$

Suppose $H(S) < 1 - \sum_{j=1}^n \varphi_j$. Since, by Lemma 9,

$$\sum_{j \neq i} \sum_{\tau \in S_j} z_{t+1}^\tau + \sum_{\tau \in S_i \setminus \{t+1\}} z_t^i \leq H(S - (t + 1, i)),$$

we have

$$\frac{c_i \varphi_i f_{i+1}^t}{1 - c_i} \geq z_{t+1}^i \geq H(S) - H(S - (t + 1, i)) = f_{i+1}^t \left( \prod_{j \neq i} f_{i+1}^j - \varphi_i \right),$$

where the last equality holds by Lemmas 7 and 8. This implies that $\prod_{j \neq i} \sum_{\tau \in S_j} f_{j+1}^\tau \leq \frac{\varphi_i}{1 - c_i}$. Hence,

$$z_t^i \leq H(S + (t, i)) - \sum_{j=1}^n \sum_{\tau \in S_j} z_{t+1}^\tau \leq H(S + (t, i)) - H(S) = f_i^t \left( \prod_{j \neq i} f_{i+1}^j - \varphi_i \right) \leq \frac{c_i \varphi_i f_t^t}{1 - c_i}.$$
Suppose $H(S) \geq 1 - \sum_{j=1}^{n} \varphi_j$, then by Lemmas 7 and 8, $\overline{H}(S) = \overline{H}(S+(t,i)) = 1 - \sum_{j=1}^{n} \varphi_j$. Hence,

$$z^t_i \leq \overline{H}(S + (t,i)) - \sum_{j=1}^{n} \sum_{\tau \in S_j} z^\tau_j \leq \overline{H}(S + (t,i)) - \overline{H}(S) = 0 \leq \frac{c_i \varphi_i f^t_i}{1 - c_i}.$$ 

Hence, $z^t_i \in I^t_i$. ■

**Lemma 11** $P^t_i$ is non-decreasing in $t$ on $\{t | f^t_i > 0\}$.

**Proof.** Suppose $f^t_i, f^{t+1}_i > 0$. Suppose $z^{t+1}_i \notin I^{t+1}_i$, then $P^t_i \leq \frac{\varphi_i}{1 - c_i} = P^{t+1}_i$. Suppose $z^{t+1}_i \in I^{t+1}_i$. Then there exists $S$ with $S_j \subset S^{t+1}_j$ for all $j$ and $t+1 \in S_i$ such that

$$\sum_{j=1}^{n} \sum_{\tau \in S_j} z^\tau_j = \overline{H}(S).$$

Suppose $H(S) < 1 - \sum_{j=1}^{n} \varphi_j$. By the proof of Lemma 10, we have

$$\overline{P}^t_i \leq \prod_{j \neq i} \sum_{\tau \in S_j} f^\tau_j \leq \overline{P}^{t+1}_i.$$ 

Suppose $H(S) \geq 1 - \sum_{j=1}^{n} \varphi_j$. By the proof of Lemma 10, we have $\overline{P}^t_i = \varphi_i \leq \overline{P}^{t+1}_i$. ■

By Lemmas 9 and 11 and Theorem 2 in Mierendorff (2011), $\overline{P}$ is a feasible solution to $(OPTHm - \varphi)$, or equivalently, $\overline{z}$ is a feasible solution to $(OPTHm1 - \varphi)$. Let

$$\tilde{z}^t_i := \begin{cases} 
z^t_i & \text{if } d^t_i \geq 0 \\
0 & \text{if } d^t_i < 0
\end{cases}. \quad (35)$$

Clearly, $\tilde{z}$ is also a feasible solution to $(OPTHm1 - \varphi)$. As I prove below in Lemma 14, $\tilde{z}$ is an optimal solution to $(OPTHm1 - \varphi)$. Before doing that, I first prove some useful
properties of $S^{t,\nu}$ and $\overline{z}$.

**Lemma 12** $S^{t,1} \supset S^{t+1,n^{t+1}} + \sum_{i \in \pi^{t+1,1}i} (t+1,i)$ for $1 \leq t \leq m-1$; and $S^{t,\nu+1} \supset S^{t,\nu} + \sum_{i \in \pi^{t,\nu}} (t,i)$ for $1 \leq t \leq m$.

**Proof.** By construction, $S^{m,\nu+1} \supset S^{m,\nu} + \sum_{i \in \pi^{m,\nu}} (m,i)$. Let $t \leq m-1$ and $I' = \pi^{t,1}$. Let $S := S^{t+1,n^{t+1}} + \sum_{i \in \pi^{t+1,1}i} (t+1,i)$. Then $\sum_{j=1}^{n} \sum_{\tau \in S_j} z_{j,\tau} = \overline{H}(S)$. Suppose $S^{t,1} \not\supset S$. Let $S' := S \cup S^{t,1}$. Then $S'_j = S'_j$ for some $t_j \geq t+1$ for all $j$. By Lemma 9, we have

$$\sum_{j=1}^{n} \sum_{\tau \in S'_j \setminus S^{t,1}_j} z_{j,\tau} = \sum_{j=1}^{n} \sum_{\tau \in S_j} z_{j,\tau} - \sum_{j=1}^{n} \sum_{\tau \in S^{t,1}_j \cap S_j} z_{j,\tau} \geq \overline{H}(S) - \overline{H}(S \cap S^{t,1}).$$

Hence,

$$\overline{H} \left( S' + \sum_{i \in I'} (t,i) \right) - \sum_{j=1}^{n} \sum_{\tau \in S'_j} z_{j,\tau} = \overline{H} \left( S^{t,1} + \sum_{i \in I'} (t,i) \right) + \sum_{j=1}^{n} \sum_{\tau \in S^{t,1}_j} z_{j,\tau} \leq \left[ \overline{H} \left( S' + \sum_{i \in I'} (t,i) \right) - \overline{H}(S) \right] - \left[ \overline{H} \left( S^{t,1} + \sum_{i \in I'} (t,i) \right) - \overline{H}(S \cap S^{t,1}) \right] \leq 0,$$

where the last inequality holds since $\overline{H}$ is sub-modular. A contradiction. Hence, $S^{t,1} \supset S^{t+1,n^{t+1}} + \sum_{i \in \pi^{t+1,1}i} (t+1,i)$. By a similar argument, one can show that $S^{t,\nu+1} \supset S^{t,\nu} + \sum_{i \in \pi^{t,\nu}} (t,i)$ for all $t \leq m-1$. ■
By Lemmas 7, 8 and 12, there exists \( t \) and \( \nu \) such that

\[
H \left( S^t, \nu + \sum_{i \in \pi^t, \nu} (t, i) \right) = \begin{cases} 
1 - \sum_i \varphi_i & \text{if } \nu \geq \nu^*, t \geq \tilde{t} \\
H \left( S^t, \nu + \sum_{i \in \pi^t, \nu} (t, i) \right) < 1 - \sum_i \varphi_i & \text{otherwise}
\end{cases}
\]  \tag{36}

By a similar argument to that in Lemma 10, we have

**Lemma 13** If \( t > \tilde{t} \) or \( t = \tilde{t} \) and \( i \notin \pi^{t,1} \cup \cdots \cup \pi^{t,\nu} \), then \( \tilde{z}_i = 0 \).

**Lemma 14** \( \hat{z} \) defined in (35) is an optimal solution to \((OPTH_m 1 - \varphi)\).

**Proof.** Consider the dual to problem \((OPTH_m 1 - \varphi)\), denoted by \((DOPTH_m 1 - \varphi)\),

\[
\min \lambda, \beta, \mu \sum_{i=1}^{m} \sum_{t=1}^{m} \lambda_i^t f_i^t - c_i = \sum_{t=1}^{m} \beta(S)H(S) + \sum_{i=1}^{n} \varphi_i \left( \sum_{t=1}^{m} f_i^t - \mu \right),
\]

subject to

\[
d^t - \lambda_i^t + \mu_i^t - \sum_{S_i \ni t} \beta(S) \geq 0 \text{ if } f_i^t > 0, \forall i, \forall t,
\]

\[
\lambda \geq 0, \mu \geq 0, \beta \geq 0.
\]

Let \( \hat{z} \) be defined by (35) and \( \hat{\beta}, \hat{\lambda}, \hat{\mu} \) be the corresponding dual variables. Let \( t^0 \) be such that \( d^0 \geq 0 \) if and only if \( t \geq t^0 \).

**Case 1:** \( \tilde{t} < t^0 \) or \( \tilde{d} < 0 \). Let \( \hat{\beta} \left( S^{t,n^t} + \sum_{i \in \pi^{t,n^t}} (t, i) \right) \geq 0 \) for \( t \geq t^0 \) and \( \hat{\beta}(S) = 0 \) otherwise. Let \( \hat{\mu}_i^t = 0 \) if \( t \geq t^0 \) and \( \hat{\lambda}_i^t = 0 \) if \( t < t^0 \). (i) If \( t < t^0 \), then \( \hat{\beta}(S) = 0 \) for all \( S \) such that \( S_i \ni t \). Hence, \( \hat{\mu}_i^t = -d^t \geq 0 \). (ii) If \( t = t^0 \), let \( \hat{\beta} \left( S^{t,n^t} + \sum_{j \in \pi^{t,n^t}} (t, j) \right) = d^t \geq 0 \). If \( i \in T_1^t \), let \( \hat{\lambda}_i^t = 0 \). If \( i \notin T_0^t \cup T_1^t \), let \( \hat{\lambda}_i^t = d^t \geq 0 \). (iii) If \( t > t^0 \), let \( \hat{\beta} \left( S^{t,n^t} + \sum_{j \in \pi^{t,n^t}} (t, j) \right) = d^t - d^{t-1} \geq 0 \). If \( i \in T_1^t \), let \( \hat{\lambda}_i^t = 0 \). If \( i \notin T_0^t \cup T_1^t \), let \( \hat{\lambda}_i^t = d^t - d^* \geq 0 \) where \( t^* = \inf\{ t' > t | S^{t',n^t'} \ni \} \). Thus, \( (\hat{\lambda}, \hat{\mu}, \hat{\beta}) \) is a feasible solution to \((DOPTH_m 1 - \varphi)\) and the complementary slackness conditions are satisfied. Finally, it
is easy to verify that the dual objective is equal to the primal objective. By the duality theorem, \( \hat{z} \) is an optimal solution to \( OPTHm_1 - \varphi \).

**Case 2:** \( t^0 \geq t \) or \( d^t \geq 0 \). Let \( \hat{\beta} \left( S^{t,n^t} + \sum_{i \in \pi^{t,n^t}} (t,i) \right) \geq 0 \) for \( t \geq t^0 \) and \( \hat{\beta} (S) = 0 \) otherwise. Let \( \hat{\mu}_i = 0 \) if \( t \geq t^0 \) and \( \hat{\lambda}_i = 0 \) if \( t < t^0 \). (i) If \( t < t^0 \), then \( \hat{\beta} (S) = 0 \) for all \( S \) such that \( S_i \subseteq t \). Hence, \( \hat{\mu}_i = -d^t \geq 0 \). (ii) If \( t^0 \leq t \leq t^0 \), let \( \hat{\beta} \left( S^{t,n^t} + \sum_{i \in \pi^{t,n^t}} (t,i) \right) = d^t - d^t - 1 \geq 0 \) if \( t > t^0 \) and \( \hat{\beta} \left( S^{t,n^t} + \sum_{i \in \pi^{t,n^t}} (t,i) \right) = d^0 \) if \( t = t^0 \). If \( i \in \mathcal{I}_1 \), let \( \hat{\lambda}_i = 0 \). Note that by Lemma 13, \( i \in \mathcal{I}_1 \) for all \( i \not\in \mathcal{I}_0 \). (iii) If \( t > t^0 \), let \( \hat{\beta} \left( S^{t,n^t} + \sum_{j \in \pi^{t,n^t}} (t,j) \right) = d^t - d^t - 1 \geq 0 \). If \( i \in \mathcal{I}_1 \), let \( \hat{\lambda}_i = 0 \). If \( i \not\in \mathcal{I}_0 \cup \mathcal{I}_1 \), let \( \hat{\lambda}_i = d^t - d^t \geq 0 \) where \( t^* = \inf \{ t' > t^0 | S^{t',n'} \subseteq i \} \). Thus, \( (\hat{\lambda}, \hat{\mu}, \hat{\beta}) \) is a feasible solution to \( (DOPTHm_1 - \varphi) \) and the complementary slackness conditions are satisfied. Finally, it is easy to verify that the dual objective is equal to the primal objective. By the duality theorem, \( \hat{z} \) is an optimal solution to \( OPTHm_1 - \varphi \). ■

Finally, let

\[
P_{i}^{m,t} := \begin{cases} 
P_{i}^{m,t} & \text{if } d^t \geq 0 \\
\varphi & \text{if } d^t < 0 
\end{cases}
\]  

(37)

The following corollary directly follows from Lemma 14:

**Corollary 4** \( P_m \) defined in (37) is an optimal solution to \( OPTHm - \varphi \).

Before moving on to the continuum case, I prove the following two lemmas which is useful in characterizing the limit of \( P_m \).

**Lemma 15** Suppose \( S^{t,\nu} = (S_1^{t_1}, \ldots, S_n^{t_n}) \). Then \( t_i^* = t \) if \( i \in \pi^{t_1} \cup \ldots \cup \pi^{t,\nu - 1} \), \( t_i^* = t + 1 \) if \( i \in \mathcal{I}_{t+1}^1 \cup \pi^{t,\nu} \setminus (\pi^{t_1} \cup \ldots \cup \pi^{t,\nu - 1}) \) and \( t_i^* \in \{ t + 1, m + 1 \} \) otherwise. Furthermore, for \( h \not\in \mathcal{I}_{t+1}^1 \cup \pi^{t_1} \cup \ldots \cup \pi^{t,\nu} \), we have

1. If \( \frac{\varphi_h}{1-c_h} - \prod_{i \not\in h} \sum_{t=1}^{t_i^* - 1} f_i^t \geq 0 \), then \( t_h^* = t + 1 \).
2. If \( \frac{\varphi_h}{1-c_h} - \prod_{i \not\in h} \sum_{t=1}^{t_i^* - 1} f_i^t < 0 \) and \( \overline{H} (S_i^{t,\nu} + \sum_{i \not\in \pi^{t,\nu}} \pi (t,i)) < 1 - \sum_{i=1}^{n} \varphi_i \), then \( t_h^* = m + 1 \).
3. If \( \frac{\varphi_h}{1-c_h} - \prod_{i \not\in h} \sum_{t=1}^{t_i^* - 1} f_i^t < 0 \) and \( \overline{H} (S_i^{t,\nu} + \sum_{i \not\in \pi^{t,\nu}} \pi (t,i)) = 1 - \sum_{i=1}^{n} \varphi_i \), then \( t_h^* = t + 1 \).
Proof. By construction, \( t_i^* = t + 1 \) if \( i \in \pi^{t,\nu} \). By Lemma 12, \( t_i^* = t \) if \( i \in \pi^{t,1} \cup \cdots \cup \pi^{t,\nu - 1} \) and \( t_i^* = t + 1 \) if \( i \in \mathcal{I}_1^{t+1} \setminus (\pi^{t,1} \cup \cdots \cup \pi^{t,\nu - 1}) \). If \( t = m \), then, by construction, \( t_i^* = m + 1 \) for \( i \notin \pi^{t,1} \cup \cdots \cup \pi^{t,\nu - 1} \).

Let \( t \leq m - 1 \). For the ease of notation, let \( \mathcal{I}' = \pi^{t,\nu} \) and \( \mathcal{S} = (S_1^{\nu}, \ldots, S_n^{\nu}) \) is such that \( t_i = t \) if \( i \in \pi^{t,1} \cup \cdots \cup \pi^{t,\nu - 1} \), \( t_i = t + 1 \) if \( i \notin \mathcal{I}_1^{t+1} \cup \pi^{t,\nu - 1} \) and \( t_i = t + 1 \) otherwise. Fix \( h \notin \mathcal{I}_1^{t+1} \cup \pi^{t,1} \cup \cdots \cup \pi^{t,\nu} \) and \( t_i \) for all \( i \neq h \). Define

\[
\Delta(t_h) := \prod \left( S + \sum_{i \in \mathcal{I}'} (t, i) \right) - \sum_{i=1}^n \sum_{\tau \in S_i} \tau.
\]

There exists \( t \leq t^* \leq m + 1 \) such that if \( t + 1 \leq t_h \leq t^* \), then \( \Delta(t_h) = 1 - \sum_{i=1}^n \varphi_i - \sum_{i=1}^n \sum_{\tau \in S_i} \tau \); and if \( t^* < t_h \leq m + 1 \), then

\[
\Delta(t_h) = 1 - \left( \prod_{i \notin \mathcal{I}', i \neq h} f_i^{t_h} \right) \left( \prod_{i \notin \mathcal{I}', i \neq h} f_i^{t-1} \right) - \sum_{i=1}^m \sum_{\tau = t_i}^m f_i^{t_h} \varphi_i - \sum_{i=1}^m \sum_{\tau = t_i}^m f_i^{t-1} \varphi_i - \sum_{i=1}^m \sum_{\tau = t_i}^m \tau.
\]

Recall that \( \mathcal{Z}_h^{t_h} = c_h \varphi_h f_i^{t_h} / (1 - c_h) \) for all \( h \geq t + 1 \). If \( t_h < t^* \), we have \( \Delta(t_h + 1) - \Delta(t_h) = c_h \varphi_h f_i^{t_h} / (1 - c_h) \geq 0 \). Hence, \( \Delta(t + 1) \leq \Delta(t_h) \) for all \( t_h \leq t^* \). If \( t_h > t^* \), we have

\[
\Delta(t_h + 1) - \Delta(t_h) = f_i^{t_h} \left( \varphi_h / (1 - c_h) - \left( \prod_{i \notin \mathcal{I}', i \neq h} f_i^{t_h} \right) \left( \prod_{i \notin \mathcal{I}', i \neq h} f_i^{t-1} \right) \right).
\]

If \( t_h = t^* \), we have

\[
\Delta(t_h + 1) - \Delta(t_h) \geq f_i^{t_h} \left( \varphi_h / (1 - c_h) - \left( \prod_{i \notin \mathcal{I}', i \neq h} f_i^{t_h} \right) \left( \prod_{i \notin \mathcal{I}', i \neq h} f_i^{t-1} \right) \right).
\]

Hence, if \( \varphi_h / (1 - c_h) - \left( \prod_{i \notin \mathcal{I}', i \neq h} \sum_{\tau = t_i}^{t - 1} f_i^{t} \right) \left( \prod_{i \notin \mathcal{I}', i \neq h} \sum_{\tau = t_i}^{t - 1} f_i^{t-1} \right) \geq 0 \), then \( \Delta(t_h + 1) \geq \Delta(t_h) \) for all \( t_h \geq t^* \). Furthermore, since \( \Delta(t + 1) \leq \Delta(t_h) \) for all \( t_h \leq t^* \), we have \( \Delta(t + 1) \leq \Delta(t_h) \) for all \( t_h \leq t^* \).
for all $t_h \geq t + 1$, hence $t^*_h = t + 1$.

If \( \frac{\varphi_h}{1 - c_h} - \left( \prod_{i \notin \mathcal{T}, i \neq h} \sum_{\tau = 1}^{t_i - 1} f^\tau_i \right) \left( \prod_{i \in \mathcal{T}} \sum_{\tau = 1}^{t_i - 1} f^\tau_i \right) < 0 \), then $\Delta(t_h + 1) \leq \Delta(t_h)$ for all $t_h > t^*$. Hence, $\Delta(m + 1) \leq \Delta(t_h)$ for all $t_h > t^*$. Recall that $\Delta(t + 1) \leq \Delta(t_h)$ for all $t_h \leq t^*$. Hence, $t^*_h \in \arg \min \{ \Delta(t + 1), \Delta(m + 1) \}$. □

**Lemma 16** Suppose $S^{t, \nu} = (S^{t_1}_1, \ldots, S^{t_{\nu}}_{\nu})$ and $h \notin I^{t + 1}_1 \cup \pi^{t, 1} \cup \ldots \cup \pi^{t, \nu}$, we have $t^*_h = t + 1$ implies that $h \in I^1_t$.

**Proof.** Suppose $\overline{H} (S^{t, \nu} + \sum_{i \in \pi^{t, \nu}} (t, i)) = 1 - \sum_{i=1}^{n} \varphi_i$, then by Lemma 13, $h \in I^1_t$. Suppose $\overline{H} (S^{t, \nu} + \sum_{i \in \pi^{t, \nu}} (t, i)) < 1 - \sum_{i=1}^{n} \varphi_i$. By Lemma 15, $\frac{\varphi_h}{1 - c_h} \geq \prod_{i \neq h} \sum_{\tau = 1}^{t^*_i - 1} f^\tau_i$. Hence,

\[
\overline{z}^h \leq \overline{H} \left( S^{t, \nu} + \sum_{i \in \pi^{t, \nu}} (t, i) + (t, h) \right) - \overline{H} \left( S^{t, \nu} + \sum_{i \in \pi^{t, \nu}} (t, i) \right) \\
\leq f^t_h \left( \prod_{i \neq h} \sum_{\tau = 1}^{t^*_i - 1} f^\tau_i - \varphi_h \right) \\
\leq f^t_h \left( \frac{\varphi_h}{1 - c_h} - \varphi_h \right) = \frac{c_h \varphi_h f^t_h}{1 - c_h}.
\]

Hence, $h \in I^1_t$. □

### B.2 Continuum Case

I characterize an optimal solution in the continuum case by taking $m$ to infinity. Let $I^{m, t}_1$ denote $I^1_t$ and $\overline{t}^m$ be defined by (36) when $\mathcal{D}$ is discretized by $m$ grid points. Clearly, if $i \in I^{m, t}_1$ then $i \in \overline{I}^{2m,2t-1}_1$. Let $\overline{t}_i^m := \max \{ t | i \in I^{m, t}_1 \}$ and $\overline{\theta}_i^m := d + (\overline{t}_i - 1)(\overline{\theta} - d) / m$. Then the sequence of $\{ \overline{d}_i^{2\kappa} \}$ is non-decreasing and bounded from above by $\overline{d}$. Hence, the sequence converges and let $d_i^{2\kappa} := \lim_{\kappa \to \infty} \overline{d}_i^{2\kappa}$ denote its limit. For each $\kappa$, let $d^{2\kappa} := \frac{d + (\overline{t}^{2\kappa} - 1)(\overline{\theta} - d)}{2\kappa}$, which is bounded. After taking subsequences, we can assume $\{ d^{2\kappa} \}$ converges and let
\( d^i = \lim_{i \to \infty} g^{2^x} \) denote its limit. Let

\[
\bar{P}_i(v_i) := \begin{cases} \frac{\varphi_i}{1-c_i} & \text{if } v_i > d_i^u + \frac{k_i}{c_i} \\ \prod_{j \neq i, d_j^u \geq v_i} F_j \left( v_i - \frac{k_j}{c_j} + \frac{k_i}{c_i} \right) & \text{if } d_i^l + \frac{k_i}{c_i} < v_i < d_i^u + \frac{k_i}{c_i}. \\ \varphi_i & \text{if } v_i < d_i^l + \frac{k_i}{c_i}. \end{cases}
\]

Finally, let

\[
P_i^*(v_i) := \begin{cases} \bar{P}_i(v_i) & \text{if } v_i > \frac{k_i}{c_i} \\ \varphi_i & \text{if } v_i < \frac{k_i}{c_i}. \end{cases}
\]

I show below that \( P^* \) is the pointwise limit of \( P^m \) and an optimal solution to \((OPTH - \varphi)\).

**Proof of Theorem 4.** We can extend \( \bar{P}_i^m \ (P_i^m) \) to \([v_i, \pi_i] \) by setting, for each \( t = 1, \ldots, m \),

\[
\bar{P}_i^m(v_i) := \bar{P}_i^{m,t}(P_i^m(v_i) := P_i^{m,t}) \text{ for } v_i \in \left[ d + \frac{(t-1)(d-d)}{m} + \frac{k_i}{c_i}, d + \frac{t(d-d)}{m} + \frac{k_i}{c_i} \right].
\]

I show that, after taking subsequences, \( \bar{P}_i^m \) converges to \( \bar{P}_i \) pointwise.

First, by construction, \( \bar{P}_i^{2^x}(v_i) = \varphi_i \) for all \( v_i < d_i^{2^x} + \frac{k_i}{c_i} \), we have \( \lim_{i \to \infty} \bar{P}_i^{2^x}(v_i) = \bar{P}_i(v_i) \) for all \( v_i < d_i^l + \frac{k_i}{c_i} \). Similarly, by construction, \( \bar{P}_i^{2^x}(v_i) = \frac{\varphi_i}{1-c_i} \) for all \( v_i > d_i^{2^x} + \frac{k_i}{c_i} \), we have \( \lim_{i \to \infty} \bar{P}_i^{2^x}(v_i) = \bar{P}_i(v_i) \) for all \( v_i > d_i^l + \frac{k_i}{c_i} \).

Suppose \( d_i^l < v_i - \frac{k_i}{c_i} < d_i^u \). Assume without loss of generality that \( d_i^l \geq \cdots \geq d_n^l \geq d^l \). If \( d_i^u = d^l \), then we are done. Assume for the rest of the proof that \( d_i^u > d^l \). Let \( d_{n+1}^u := d^l \). Consider \( v_i \) such that \( d_i^u \geq d_j^u > v_i - \frac{k_j}{c_j} > d_{j+1}^u \) for some \( j \geq i \). For \( m \) sufficiently large, there exists \( t \) such that

\[
d_{j+1}^u < d + \frac{(t-1)(d-d)}{m} < d + \frac{t(d-d)}{m} < v_i - \frac{k_i}{c_i} < d + \frac{(t+1)(d-d)}{m} < d_i^u.
\]

Hence, by construction, we have \( I_i^{m,t} = I_i^{m,t+1} = \{1, \ldots, j\} \). By Lemmas 15 and 16, there exists \( S = (S_1^1, \ldots, S_n^1) \) such that \( t_i = t + 1, t_h \in \{t, t + 1\} \) if \( h \leq j \) and \( h \neq i, t_h = m + 1 \).
if \( h > j \), and

\[
 f^t_i \left( P^m_i - \varphi_i \right) = \overline{H} (S + (t, i)) - \overline{H} (S).
\]

In particular,

\[
 f^t_i \left( P^m_i - \varphi_i \right) 
\leq \overline{H} (S' + (t, i)) - \overline{H} (S')
\]

\[
 = f^t_i \left( \prod_{h \leq j, h \neq i}^t \sum_{\tau = 1}^t f_h^\tau - \varphi_i \right),
\]

where \( S' = (S_1^{t+1}, \ldots, S_j^{t+1}, S_{j+1}^{m+1}, \ldots, S_n^{m+1}) \); and

\[
 f^t_i \left( P^m_i - \varphi_i \right) 
\geq \overline{H} (S'' - (t, i)) - \overline{H} (S')
\]

\[
 = f^t_i \left( \prod_{h \leq j, h \neq i}^{t-1} \sum_{\tau = 1}^t f_h^\tau - \varphi_i \right),
\]

where \( S'' = (S_1^t, \ldots, S_j^t, S_{j+1}^{m+1}, \ldots, S_n^{m+1}) \). Hence,

\[
 \prod_{h \leq j, h \neq i}^{t-1} \sum_{\tau = 1}^t f_h^\tau \leq P^m_i \leq \prod_{h \leq j, h \neq i}^t \sum_{\tau = 1}^t f_h^\tau.
\]

Take \( m = 2^\kappa \) to infinity and we have \( \lim_{\kappa \to \infty} P^2^\kappa_i (v_i) = \overline{P}_i (v_i) \).

It follows that, after taking subsequences, \( P^m_i \) converges to \( P_i^* \) pointwise. \( P^* \) is feasible by a similar argument to that in the proof of Lemma 3 and optimal by a similar argument to that in the proof of Theorem 1.

### B.3 Optimal One-threshold Mechanism

**Lemma 17** There exists a unique \( d^\ell \geq \max \{ v_j - k_j / c_j \} \) such that

\[
 \sum_{i=1}^n \varphi_i F_i \left( d^\ell + \frac{k_i}{c_i} \right) = \prod_{i=1}^n F_i \left( d^\ell + \frac{k_i}{c_i} \right). \quad (38)
\]
Proof. If $\varphi_i = 0$ for all $i$, then $d^l = \max \{v_j - k_j/c_j\}$ is the unique solution to (38). Assume for the rest of the proof that $\varphi_i > 0$ for some $i$. Let

$$\Delta(d^l) := \sum_{i=1}^n \varphi_i F_i \left( d^l + \frac{k_i}{c_i} \right) - \prod_{i=1}^n F_i \left( d^l + \frac{k_i}{c_i} \right).$$

Then

$$\Delta'(d^l) = \sum_{i=1}^n f_i \left( d^l + \frac{k_i}{c_i} \right) \left[ \varphi_i - \prod_{j \neq i} F_j \left( d^l + \frac{k_j}{c_j} \right) \right].$$

If $\Delta(d^l) \leq 0$, then $\varphi_i \leq \prod_{j \neq i} F_j \left( d^l + \frac{k_j}{c_j} \right)$ for all $i$ and the strict inequality holds for some $i$, which implies $\Delta'(d^l) < 0$. Since $\Delta_l \left( \max \{v_j - k_j/c_j\} \right) \geq 0$ and $\Delta_l \left( \max \{v_j - k_j/c_j\} \right) = \sum_i \varphi_i - 1 \leq 0$, there exist a unique $d^l$ satisfying (38). \(\blacksquare\)

Proof of Theorem 5. Let $\Phi(d^l, d^u_1, \ldots, d^u_n) \subset \{ \varphi \mid \sum \varphi_i \leq 1 \}$ denote the feasible set of $\varphi$ given $d^l$ and $d^u_1, \ldots, d^u_n$. I often abuse notation and use $\Phi$ to denote the feasible set when its meaning is clear. Fix $d^l > \max \{v_j - k_j/c_j\}$ and $d^u_i = v_i - k_i$. Then $\varphi$ is feasible if and only if

$$\sum_{i=1}^n \varphi_i F_i \left( d^l + \frac{k_i}{c_i} \right) = \prod_{i=1}^n F_i \left( d^l + \frac{k_i}{c_i} \right),$$

$$\prod_{j \neq i} F_j \left( v_i - \frac{k_i}{c_i} + \frac{k_j}{c_j} \right) \leq \frac{\varphi_i}{1 - c_i}, \forall i.$$

Then the set of feasible $\varphi$, $\Phi$, is non-empty if and only if

$$\sum_{i=1}^n (1 - c_i) F_i \left( d^l + \frac{k_i}{c_i} \right) \prod_{j \neq i} F_j \left( v_i - \frac{k_i}{c_i} + \frac{k_j}{c_j} \right) \leq \prod_{i=1}^n F_i \left( d^l + \frac{k_i}{c_i} \right)$$

Suppose $\Phi$ is non-empty. It is not hard to see that $\Phi$ is convex. Since the objective function is linear in $\varphi$ and the feasible set is convex, given $d^l$ and $d^u$, there is an optimal $\varphi$ which is an extreme point.
Clearly, $\varphi$ is an extreme point of $\Phi$ if and only if there exists $i^*$ such that

$$
\sum_{i=1}^{n} \varphi_i F_i \left( d^l + \frac{k_i}{c_i} \right) = \prod_{i=1}^{n} F_i \left( d^l + \frac{k_i}{c_i} \right),
$$

$$
\varphi_j = (1 - c_j) \prod_{i\neq j} F_i \left( \bar{v}_j - \frac{k_j}{c_j} + \frac{k_i}{c_i} \right), \forall j \neq i^*.
$$

For ease of notation, let $i^* = 1$. Let $\bar{\varphi}_j := (1 - c_j) \prod_{i\neq j} F_i \left( \bar{v}_j - \frac{k_j}{c_j} + \frac{k_i}{c_i} \right)$ for all $j$. Then the principal’s payoff is given by $Z_{1,i^*}(d^l)$ defined as follows:

$$
Z_{1,1}(d^l) := \sum_{i=1}^{n} \int_{\varphi_i}^{\varphi_i} \left( v_i - \frac{k_i}{c_i} \right) \prod_{j\neq i} F_j \left( v_i - \frac{k_i}{c_i} + \frac{k_j}{c_j} \right) dF_i(v_i)
$$

$$
+ \sum_{i \neq 1} \int_{\varphi_i}^{\varphi_i} \left( v_i - \frac{k_i}{c_i} \right) \bar{\varphi}_i dF_i(v_i)
$$

$$
+ \int_{\varphi_i}^{\varphi_i} \left( v_1 - \frac{k_1}{c_1} \right) \prod_{i=1}^{n} F_i \left( d^l + \frac{k_i}{c_i} \right) - \sum_{i \neq 1} F_i \left( d^l + \frac{k_i}{c_i} \right) \bar{\varphi}_i dF_1(v_1)
$$

$$
+ \sum_{i \neq 1} \bar{\varphi}_i \frac{k_i}{c_i} + \frac{k_1}{c_1} \prod_{i=1}^{n} F_i \left( d^l + \frac{k_i}{c_i} \right) - \sum_{i \neq 1} F_i \left( d^l + \frac{k_i}{c_i} \right) \bar{\varphi}_i.
$$

If $d^l < 0$, it is not hard to show that $Z_{1,1}$ is strictly increasing in $d^l$. If $d^l \geq 0$, then, after some algebra, we have

$$
Z'_{1,1}(d^l) = \left[ \sum_{i \neq 1} \int_{\varphi_i}^{\varphi_i} \left( v_1 - d^l - \frac{k_1}{c_1} \right) dF_1(v_1) + \frac{k_1}{c_1} \right].
$$

Since $\bar{\varphi}_i \leq \prod_{j \neq i} F_j \left( d^l + \frac{k_j}{c_j} \right)$, the first-term in the above equation is strictly positive. The second-term is strictly decreasing in $d^l$. Let $d_1^*$ be such that

$$
\int_{\varphi_1}^{d_1^* + \frac{k_1}{c_1}} \left( v_1 - d^* - \frac{k_1}{c_1} \right) dF_1(v_1) + \frac{k_1}{c_1} = 0.
$$

(39)
Then $Z'_{1,1}(d^l) > 0$ if $d^l < d^*_1$ and $Z'_{1,1}(d^l) < 0$ if $d^l > d^*_1$. Hence, $Z_{1,1}(d^l)$ achieves its maximum at $d^l = d^*_1$.

Suppose $d^*_1 \geq d^*_2$. Note that by the same argument as that in Lemma 17, we have

$$\Phi \left( d^*_2, \bar{v}_1 - \frac{k_1}{c_1}, \ldots, \bar{v}_n - \frac{k_n}{c_n} \right) \neq \emptyset$$

then $\Phi \left( d^*_1, \bar{v}_1 - \frac{k_1}{c_1}, \ldots, \bar{v}_n - \frac{k_n}{c_n} \right) \neq \emptyset$. Suppose both $\Phi \left( d^*_2, \bar{v}_1 - \frac{k_1}{c_1}, \ldots, \bar{v}_n - \frac{k_n}{c_n} \right)$ and $\Phi \left( d^*_1, \bar{v}_1 - \frac{k_1}{c_1}, \ldots, \bar{v}_n - \frac{k_n}{c_n} \right)$ are non-empty. Then

$$Z_{1,1}(d^l) - Z_{1,2}(d^l)$$

$$= \left[ \prod_{i=1}^{n} F_i \left( d^l + \frac{k_i}{c_i} \right) - \sum_{i=1}^{n} F_i \left( d^l + \frac{k_i}{c_i} \right) \bar{\varphi}_i \right] \cdot \left\{ \frac{1}{F_1 \left( d^l + \frac{k_1}{c_1} \right)} \int_{\mathbb{E}_1} \left( v_1 - \frac{k_1}{c_1} \right) dF_1(v_1) + \frac{k_1}{c_1} \right\} - \frac{1}{F_2 \left( d^l + \frac{k_2}{c_2} \right)} \int_{\mathbb{E}_2} \left( v_1 - \frac{k_2}{c_2} \right) dF_2(v_2) + \frac{k_2}{c_2} \right\}$$

If $d^l = d^*_2$, then by definition we have

$$Z_{1,1}(d^*_2) - Z_{1,2}(d^*_2)$$

$$= \left[ \prod_{i=1}^{n} F_i \left( d^*_2 + \frac{k_i}{c_i} \right) - \sum_{i=1}^{n} F_i \left( d^*_2 + \frac{k_i}{c_i} \right) \bar{\varphi}_i \right] \left\{ \frac{1}{F_1 \left( d^*_2 + \frac{k_1}{c_1} \right)} \int_{\mathbb{E}_1} \left( v_1 - \frac{k_1}{c_1} \right) dF_1(v_1) + \frac{k_1}{c_1} \right\} - d^*_2$$

$$\geq \left[ \prod_{i=1}^{n} F_i \left( d^*_2 + \frac{k_i}{c_i} \right) - \sum_{i=1}^{n} F_i \left( d^*_2 + \frac{k_i}{c_i} \right) \bar{\varphi}_i \right] (d^*_2 - d^*_2) = 0,$$

where the last inequality holds since $d^*_1 \geq d^*_2$ and the inequality holds strictly if $d^*_1 > d^*_2$. Hence, $Z_{1,1}(d^*_1) \geq Z_{1,2}(d^*_2)$ and the inequality holds strictly if $d^*_1 > d^*_2$.

Let $d^{*} := \max_i d^*_i$. If

$$\sum_{i=1}^{n} \left( 1 - c_i \right) \prod_{j \neq i} F_j \left( \bar{v}_j - \frac{k_j}{c_j} + \frac{k_j}{c_j} \right) F_i \left( d^* + \frac{k_i}{c_i} \right) \leq \prod_{i=1}^{n} F_i \left( d^{*} + \frac{k_i}{c_i} \right),$$

then $\Phi \left( d^{*}, \bar{v}_1 - \frac{k_1}{c_1}, \ldots, \bar{v}_n - \frac{k_n}{c_n} \right)$ is feasible, this completes the proof. ■
B.4 Symmetric Environment Revisited

Fix \( \varphi \) and let \( d_i^u \) and \( d_i^l \) be the associated optimal thresholds. Assume, without loss of generality, \( d_1^u \geq \cdots \geq d_n^u \geq d_i^l \). Let \( 1 \leq \xi_1 < \cdots < \xi_L \leq n \) be such that \( d_1^u = \cdots = d_{\xi_1}^u \), \( d_{\xi_i}^u > d_{\xi_{i+1}}^u = \cdots = d_{\xi_{L+1}}^u \) for \( i = 1, \ldots, L-1 \) and \( d_{\xi_L}^u > d_{\xi_{L+1}}^u = \cdots = d_n^u = d_i^l \). Note that in the symmetric environment only if \( d_i^u \geq d_j^u \) only if \( \varphi_i \geq \varphi_j \). The proof of Theorem 6 uses the following properties of \( d_i^u \) and \( d_i^u \):

**Lemma 18** If \( \frac{\varphi_i}{1-c} \geq 1 \) for all \( i \leq \xi_1 \), then \( d_{\xi_1}^u = \sigma - \frac{k}{c} \); otherwise \( \frac{\varphi_i}{1-c} < 1 \) for all \( i \leq \xi_1 \) and \( d_{\xi_1}^u \) satisfies

\[
1 - F \left( \frac{d_{\xi_1}^u}{c} + \frac{k}{c} \right) = \sum_{i=1}^{\xi_1} \frac{\varphi_i}{1-c} \left[ 1 - F \left( \frac{d_{\xi_1}^u}{c} + \frac{k}{c} \right) \right],
\]

\[
1 - F(v)^{\xi_1} \leq \sum_{i=1}^{\xi_1} \frac{\varphi_i}{1-c} [1 - F(v)] \text{ if } v \geq \frac{d_{\xi_1}^u}{c} + \frac{k}{c},
\]

\[
\frac{\varphi_i}{1-c} \geq F \left( \frac{d_{\xi_1}^u}{c} + \frac{k}{c} \right)^{\xi_i-1}, \forall i = 1, \ldots, \xi_1.
\]

For \( i = 1, \ldots, L-1 \), \( \frac{\varphi_i}{1-c} < 1 \) for \( \xi_i + 1 \leq i \leq \xi_{i+1} \) and \( d_{\xi_{i+1}}^u \) satisfies

\[
F \left( \frac{d_{\xi_{i+1}}}{c} + \frac{k}{c} \right) - F \left( \frac{d_{\xi_{i+1}}}{c} + \frac{k}{c} \right) = \sum_{i=\xi_{i+1}}^{\xi_{i+1}} \frac{\varphi_i}{1-c} \left[ 1 - F \left( \frac{d_{\xi_{i+1}}}{c} + \frac{k}{c} \right) \right],
\]

\[
F(v)^{\xi_i} - F(v)^{\xi_{i+1}} \leq \sum_{i=\xi_{i+1}}^{\xi_{i+1}} \frac{\varphi_i}{1-c} [1 - F(v)] \text{ if } v \geq \frac{d_{\xi_{i+1}}}{c} + \frac{k}{c},
\]

\[
\frac{\varphi_i}{1-c} \geq F \left( \frac{d_{\xi_{i+1}}}{c} + \frac{k}{c} \right)^{\xi_i-1}, \forall i = \xi_i + 1, \ldots, \xi_{i+1}.
\]

Finally, \( d_i^l \) satisfies

\[
F \left( \frac{d_i^l + k}{c} \right) = \sum_{i=1}^{n} \varphi_i F \left( \frac{d_i^l + k}{c} \right) + \sum_{i=\xi_{L+1}}^{n} \frac{\varphi_i}{1-c} \left[ 1 - F \left( \frac{d_i^l + k}{c} \right) \right],
\]

\[
F \left( d_i^l + \frac{k}{c} \right) \leq \sum_{i=1}^{n} \varphi_i F \left( d_i^l + \frac{k}{c} \right) + \sum_{i=\xi_{L+1}}^{n} \frac{\varphi_i}{1-c} \left[ 1 - F \left( d_i^l + \frac{k}{c} \right) \right] \text{ if } v \leq d_i^l + \frac{k}{c}.
\]
The arguments used to prove Lemma 18 is similar to that used to show $\overline{P}_i^{m}$ converges to $\overline{P}_i$ if $d^l < v_i - \frac{k}{c_i} < d^u_i$ and are neglected here.

**Proof of Theorem 6.** The first part of the theorem directly follows from Theorem 5. Assume, for the rest of the proof, that $F(v^*)^{n-1} < n(1 - c)$. Consider an optimal $\overline{\varphi}$ and let $d^u_i$ and $d^l$ be the associated optimal thresholds. Assume, without loss of generality, $d^u_1 \geq \cdots \geq d^u_n \geq d^l$. Let $\xi_i (i = 1, \ldots, L)$ be defined as in the beginning of this subsection.

First, I show that $L = 1$. Suppose, to the contradiction, that $L \geq 2$. Suppose $d^u_{\xi_2} < 0$, then the principal’s objective function is strictly increasing in $\varphi_i$ for $i > \xi_2$. Hence, in optimum, it must be that $d^u_{\xi_2} \geq 0$. Let

$$\varphi^*_1 = \frac{1}{\xi_2} \sum_{j=1}^{\xi_2} \varphi_j, \text{ for all } i = 1, \ldots, \xi_2,$$

and $\varphi^*_i = \varphi_i$ for all $i > \xi_2$. Then $d^u_1 = \ldots = d^u_{\xi_2}$ and $d^u_i = d^u_i$ for all $i > \xi_2$. There are two cases: (1) $\varphi_i < 1 - c$ for all $i \leq \xi_1$ and (2) $\varphi_i \geq 1 - c$ for all $i \leq \xi_1$.

**Case 1:** $\varphi_i < 1 - c$ for all $i \leq \xi_1$. In this case, $\varphi^*_1 < 1 - c$. Then $d^u_{\xi_2}$ is defined by

$$\left[ 1 - F\left( d^u_{\xi_2} + \frac{k}{c} \right) \right] \sum_{i=1}^{\xi_2} \frac{\varphi^*_i}{1 - c} = 1 - F\left( d^u_{\xi_2} + \frac{k}{c} \right) \frac{\xi_2}{1 - c}.$$

Hence, $d^u_{\xi_2} < d^u_{\xi_2} < d^u_{\xi_1}$. Let $Z(\overline{\varphi})$ denote the principal’s payoff given $\overline{\varphi}$. Then

$$Z(\overline{\varphi}) = Z(\overline{\varphi})$$

$$\sum_{i=1}^{\xi_2} \int_{d^u_{\xi_2} + \frac{k}{c}}^{\overline{\varphi}} \left( v_i - \frac{k}{c} \right) \frac{\varphi^*_i}{1 - c} d F(v_i) + \int_{d^u_{\xi_2} + \frac{k}{c}}^{d^u_{\xi_2} + \frac{k}{c}} \left( v_i - \frac{k}{c} \right) F(v_i)^{1-1} d F(v_i) - \sum_{i=\xi_1 + 1}^{\xi_2} \int_{d^u_{\xi_2} + \frac{k}{c}}^{\overline{\varphi}} \left( v_i - \frac{k}{c} \right) \frac{\varphi^*_i}{1 - c} d F(v_i) - \sum_{i=1}^{\xi_1} \int_{d^u_{\xi_2} + \frac{k}{c}}^{d^u_{\xi_1} + \frac{k}{c}} \left( v_i - \frac{k}{c} \right) F(v_i)^{1-1} d F(v_i) + \int_{d^u_{\xi_1} + \frac{k}{c}}^{d^u_{\xi_1} + \frac{k}{c}} \left( v_i - \frac{k}{c} \right) \frac{\varphi^*_i}{1 - c} d F(v_i)$$

60
\[
\int_{d_{\xi_1}^u + \frac{k}{c}}^{d_{\xi_2}^u + \frac{k}{c}} (v - \frac{k}{c}) \xi_2 F(v)^{\xi_2-1} dF(v) + \int_{d_{\xi_2}^u + \frac{k}{c}}^{\pi} \left( v - \frac{k}{c} \right) \sum_{i=1}^{\xi_2} \frac{\varphi_i}{1-c} dF(v)
\]

\[
- \int_{d_{\xi_1}^u + \frac{k}{c}}^{d_{\xi_2}^u + \frac{k}{c}} (v - \frac{k}{c}) \left( \sum_{i=\xi_1+1}^{\xi_2} \frac{\varphi_i}{1-c} + \xi_1 F(v)^{\xi_1-1} \right) dF(v) - \int_{d_{\xi_1}^u + \frac{k}{c}}^{\pi} \left( v - \frac{k}{c} \right) \sum_{i=1}^{\xi_2} \frac{\varphi_i}{1-c} dF(v)
\]

\[
= d_{\xi_1}^u \left[ F \left( d_{\xi_1}^u + \frac{k}{c} \right) \sum_{i=1}^{\xi_1} \frac{\varphi_i}{1-c} - F \left( d_{\xi_2}^u + \frac{k}{c} \right) \xi_1 \right] + d_{\xi_2}^u \left[ F \left( d_{\xi_2}^u + \frac{k}{c} \right) \xi_2 - F \left( d_{\xi_2}^u + \frac{k}{c} \right) \sum_{i=\xi_1+1}^{\xi_2} \frac{\varphi_i}{1-c} \right]
\]

\[
- d_{\xi_2}^u \left[ F \left( d_{\xi_2}^u + \frac{k}{c} \right) \sum_{i=\xi_1+1}^{\xi_2} \frac{\varphi_i}{1-c} - F \left( d_{\xi_2}^u + \frac{k}{c} \right) \xi_2 \right]
\]

where the third equality holds since \( \sum_{i=1}^{\xi_2} \varphi_i = \sum_{i=1}^{\xi_2} \varphi_i^* \) and the last equality holds by integration by parts. Since \( d_{\xi_2}^u \) satisfies (40),

\[
1 - F \left( d_{\xi_1}^u + \frac{k}{c} \right) = \left[ 1 - F \left( d_{\xi_2}^u + \frac{k}{c} \right) \right] \sum_{i=1}^{\xi_1} \frac{\varphi_i}{1-c}
\]

\[
1 - F(v)^{\xi_1} < [1 - F(v)] \sum_{i=1}^{\xi_1} \frac{\varphi_i}{1-c}, \forall v < d_{\xi_1}^u + \frac{k}{c},
\]

and

\[
1 - F \left( d_{\xi_2}^u + \frac{k}{c} \right) = \left[ 1 - F \left( d_{\xi_2}^u + \frac{k}{c} \right) \right] \sum_{i=\xi_1+1}^{\xi_2} \frac{\varphi_i}{1-c} + 1 - F \left( d_{\xi_2}^u + \frac{k}{c} \right) \xi_1
\]

\[
1 - F(v)^{\xi_2} > [1 - F(v)] \sum_{i=\xi_1+1}^{\xi_2} \frac{\varphi_i}{1-c} + 1 - F(v)^{\xi_1}, \forall v > d_{\xi_2}^u + \frac{k}{c}.
\]
we have

\[ Z(\overline{\varphi}^*) - Z(\overline{\varphi}) > (d_{\xi_1} - d_{\xi_2}^u) \left( \sum_{i=1}^{\xi_1} \frac{\varphi_i}{1-c} - 1 \right) - (d_{\xi_2}^u - d_{\xi_2}) \sum_{\xi_1+1}^{\xi_2} \frac{\varphi_i}{1-c} \]

\[ + \int_{d_{\xi_2}^u + \frac{k}{c}}^{d_{\xi_2}^* + \frac{k}{c}} \sum_{i=\xi_1+1}^{\xi_2} \frac{\varphi_i}{1-c} dv - \int_{d_{\xi_2}^u + \frac{k}{c}}^{d_{\xi_2}^* + \frac{k}{c}} \left( \sum_{i=1}^{\xi_1} \frac{\varphi_i}{1-c} - 1 \right) dv = 0, \]

a contradiction.

**Case 2:** \( \varphi_i \geq 1 - c \) for all \( i \leq \xi_1 \). If \( \varphi_i^* \geq 1 - c \), then \( d_{\xi_2}^u = \overline{a} \). In this case, we have

\[ Z(\overline{\varphi}^*) - Z(\overline{\varphi}) \]

\[ = \sum_{i=1}^{\xi_2} \int_{d_{\xi_2}^u + \frac{k}{c}}^{v} \left( v_i - \frac{k}{c} \right) F(v_i)^{\xi_2-1} dF(v_i) - \sum_{i=\xi_1+1}^{\xi_2} \int_{d_{\xi_2}^u + \frac{k}{c}}^{v} \left( v_i - \frac{k}{c} \right) \frac{\varphi_i}{1-c} dF(v_i) \]

\[ - \sum_{i=1}^{\xi_1} \int_{d_{\xi_2}^u + \frac{k}{c}}^{v} \left( v_i - \frac{k}{c} \right) F(v_i)^{\xi_1-1} dF(v_i) \]

\[ = \int_{d_{\xi_2}^u + \frac{k}{c}}^{v} \left( v - \frac{k}{c} \right) \xi_2 F(v)^{\xi_2-1} dF(v) - \int_{d_{\xi_2}^u + \frac{k}{c}}^{v} \left( v - \frac{k}{c} \right) \left( \sum_{i=\xi_1+1}^{\xi_2} \frac{\varphi_i}{1-c} + \xi_1 F(v)^{\xi_1-1} \right) dF(v) \]

\[ = \left( v - \frac{k}{c} \right) \left[ F(v)^{\xi_2} - F(v) \sum_{i=\xi_1+1}^{\xi_2} \frac{\varphi_i}{1-c} - F(v)^{\xi_1} \right] \bigg|_{d_{\xi_2}^u + \frac{k}{c}}^{v} \]

\[ - \int_{d_{\xi_2}^u + \frac{k}{c}}^{v} \left[ F(v)^{\xi_2} - F(v) \sum_{i=\xi_1+1}^{\xi_2} \frac{\varphi_i}{1-c} - F(v)^{\xi_1} \right] dv, \]

where the last equality follows integration by parts. Since

\[ \left[ 1 - F \left( d_{\xi_2}^u + \frac{k}{c} \right) \right] \sum_{i=\xi_1+1}^{\xi_2} \frac{\varphi_i}{1-c} + 1 - F \left( d_{\xi_2}^u + \frac{k}{c} \right)^{\xi_1} = 1 - F \left( d_{\xi_2}^u + \frac{k}{c} \right)^{\xi_2}, \]

\[ [1 - F(v)] \sum_{i=\xi_1+1}^{\xi_2} \frac{\varphi_i}{1-c} + 1 - F(v)^{\xi_1} < 1 - F(v)^{\xi_2}, \forall v > d_{\xi_2}^u + \frac{k}{c}, \]
we have

\[
Z(\overline{\varphi}^*) - Z(\overline{\varphi}) > - \left( \overline{\varphi} - \frac{k}{c} - d^{\xi_2}_{\xi_2} \right) + \sum_{i=1}^{\xi_2} \frac{\varphi_i}{1 - c} + \int_{d^{\xi_2}_{\xi_2}}^{\overline{\varphi}} \sum_{i=1}^{\xi_2} \frac{\varphi_i}{1 - c} dv = 0,
\]

a contradiction. If \( \varphi_1^* < 1 - c \), then let \( d^{u*}_1 = \cdots = d^{u*}_{\xi_2} \) be defined by (40). Note that if \( \xi_1 = 1 \) and \( \varphi_1/(1 - c) = 1 \), then the new mechanism using \( \overline{\varphi}^* \) coincides with the old mechanism using \( \overline{\varphi} \). In this case, we can redefine \( d^i_1 := d^{\xi_2}_{\xi_2} \) without changing the mechanism. Except for this case, we can show, by a similar argument to that in Case 1, that \( Z(\overline{\varphi}^*) - Z(\overline{\varphi}) > 0 \), a contradiction.

Hence, by induction, we have \( L = 1 \). For ease of notation, let \( h := \xi_1 \). Next, we show that \( j = 0 \) or \( 1 \). Suppose, to the contradiction, that \( 0 < j < n \). Suppose \( d^i < 0 \), then the principal’s objective function is strictly increasing in \( \varphi_i \) for \( i > j \). Hence, in optimum, it must be that \( d^i \geq 0 \). Let

\[
\varphi_i^* = \frac{1}{n} \sum_{j=1}^{n} \varphi_j, \text{ for all } i = 1, \ldots, n.
\]

**Case 1:** \( \varphi_i < 1 - c \) for all \( i \leq j \). In this case, \( \varphi_1^* < 1 - c \). Let \( d^{u*}_1 = \cdots = d^{u*}_n \) be defined by

\[
1 - F \left( d^{u*}_j + \frac{k}{c} \right)^n = \sum_{i=1}^{n} \frac{\varphi_i^*}{1 - c} \left[ 1 - F \left( d^{u*}_j + \frac{k}{c} \right) \right].
\] (41)

Then \( d^{u*}_j < d^{u*}_n \). Let \( d^* \) be defined by

\[
F \left( d^* + \frac{k}{c} \right)^n = \sum_{i=1}^{n} \varphi_i^* F \left( d^* + \frac{k}{c} \right). \quad (42)
\]

Then \( d^i < d^* \). There are two subcases to consider: (i) \( d^i \leq d^{u*}_j \) and (ii) \( d^i > d^{u*}_j \).

(i) Suppose \( d^i \leq d^{u*}_j \). Then

\[
Z(\overline{\varphi}^*) - Z(\overline{\varphi})
\]
\[
= \sum_{i=1}^{n} \left[ \int_{v_i}^{d_{n}^* + \frac{k}{c}} (v_i - \frac{k}{c}) \varphi_i^* dF(v_i) + \int_{d_{n}^* + \frac{k}{c}}^{d_{n+1}^* + \frac{k}{c}} (v_i - \frac{k}{c}) F(v_i) \, dF(v_i) + \int_{d_{n+1}^* + \frac{k}{c}}^{\sigma} (v_i - \frac{k}{c}) \varphi_i^* \frac{dF(v_i)}{1-c} \right] \\
- \sum_{i=1}^{n} \int_{v_i}^{d_{i+1}^* + \frac{k}{c}} (v_i - \frac{k}{c}) \varphi_i dF(v_i) - \sum_{i=j+1}^{n} \int_{d_{j}^* + \frac{k}{c}}^{\sigma} (v_i - \frac{k}{c}) \varphi_i \frac{dF(v_i)}{1-c} \\
- \sum_{i=1}^{j} \int_{d_{i}^* + \frac{k}{c}}^{d_{j}^* + \frac{k}{c}} (v_i - \frac{k}{c}) F(v_i) \, dF(v_i) + \int_{d_{j}^* + \frac{k}{c}}^{\sigma} (v_i - \frac{k}{c}) \varphi_i \frac{dF(v_i)}{1-c} \\
= \int_{d_{n}^* + \frac{k}{c}}^{\sigma} \left( \frac{v - \frac{k}{c}}{1-c} \right) \sum_{i=1}^{n} \varphi_i dF(v) + \int_{d_{n}^* + \frac{k}{c}}^{d_{n+1}^* + \frac{k}{c}} \left( \frac{v - \frac{k}{c}}{1-c} \right) nF(v) \, dF(v) + \int_{d_{n+1}^* + \frac{k}{c}}^{\sigma} \left( \frac{v - \frac{k}{c}}{1-c} \right) \sum_{i=1}^{n} \varphi_i \, dF(v) \\
- \int_{d_{n}^* + \frac{k}{c}}^{d_{n+1}^* + \frac{k}{c}} \left( \frac{v - \frac{k}{c}}{1-c} \right) \sum_{i=1}^{n} \varphi_i \, dF(v) - \int_{d_{n+1}^* + \frac{k}{c}}^{\sigma} \left( \frac{v - \frac{k}{c}}{1-c} \right) jF(v) \, dF(v) \\
= \int_{d_{n}^* + \frac{k}{c}}^{\sigma} F(v) \sum_{i=1}^{n} \varphi_i \, dv - \int_{d_{n+1}^* + \frac{k}{c}}^{\sigma} F(v) \, dv - \int_{d_{n}^* + \frac{k}{c}}^{d_{n+1}^* + \frac{k}{c}} F(v) \sum_{i=1}^{n} \varphi_i \, dv \\
+ \int_{d_{n+1}^* + \frac{k}{c}}^{\sigma} F(v) \sum_{i=1}^{n} \varphi_i \frac{dF(v)}{1-c} + F(v) j \, dv,
\]

where the second equality holds since \( \sum_{i=1}^{n} \varphi_i \) = \( \sum_{i=1}^{n} \varphi_i \), and the last equality holds by integration by parts. Since \( d_{j}^* \) satisfies (41), \( d_i^* \) satisfies (42),

\[
1 - F \left( d_{i}^* + \frac{k}{c} \right) = \sum_{i=j+1}^{n} \varphi_i \frac{1}{1-c} \left[ 1 - F \left( d_{j}^* + \frac{k}{c} \right) \right],
\]

\[
1 - F (v) j \leq \sum_{i=j+1}^{n} \varphi_i \frac{1}{1-c} \left[ 1 - F (v) \right], \forall v < d_{j}^*,
\]

and

\[
1 - F \left( d_i^* + \frac{k}{c} \right) + \sum_{i=j+1}^{n} \varphi_i \frac{1}{1-c} \left[ 1 - F \left( d_i^* + \frac{k}{c} \right) \right] + \sum_{i=1}^{n} \varphi_i F \left( d_i^* + \frac{k}{c} \right) = 1,
\]

64
\[
1 - F(v)^j + \sum_{i=j+1}^{n} \frac{\varphi_i}{1-c} [1 - F(v)] + \sum_{i=1}^{n} \varphi_i F(v) < 1, \forall v > d^d
\]

\[
F(v)^j - F(v)^n > [1 - F(v)] \sum_{i=j+1}^{n} \frac{\varphi_i}{1-c}, \forall v > d^d = d^d_{j+1},
\]

we have

\[
Z(\vec{\varphi}^*) - Z(\vec{\varphi})
\]

\[
> d^d_j \left( \sum_{i=1}^{j} \frac{\varphi_i}{1-c} - 1 \right) + d^u_j^* \left( 1 - \sum_{i=1}^{n} \frac{\varphi_i}{1-c} \right) + d^d \sum_{i=j+1}^{n} \frac{\varphi_i}{1-c}
\]

\[
- \int_{d^d + \frac{k}{c}}^{d^d + \frac{k}{c} + \frac{k}{c}} \sum_{i=j+1}^{n} \frac{\varphi_i}{1-c} dv - \int_{d^d + \frac{k}{c}}^{d^d + \frac{k}{c} + \frac{k}{c}} \sum_{i=j+1}^{n} \frac{\varphi_i}{1-c} dv - \int_{d^d_{j+1} + \frac{k}{c}}^{d^d_{j+1} + \frac{k}{c} + \frac{k}{c}} \left( \sum_{i=1}^{j} \frac{\varphi_i}{1-c} - 1 \right) dv = 0,
\]

a contradiction.

(ii) Suppose \(d^d > d^u_j^*\). In this case, redefine \(d^d = d^u_j^*\) such that

\[
\sum_{i=1}^{n} \left[ \frac{\varphi^*_i}{1-c} \left( d^d + \frac{k}{c} \right) + \frac{\varphi^*_i}{1-c} \left( 1 - F \left( d^d + \frac{k}{c} \right) \right) \right] = 1.
\]

Then \(d^d < d^d = d^d_{j+1} < d^d_{j-1}\). By a similar argument to that in case (ii), we can show that \(Z(\vec{\varphi}^*) - Z(\vec{\varphi}) > 0\), a contradiction.

**Case 2:** \(\varphi_i \geq 1 - c\) for all \(i \leq j\). If \(\varphi_1^* \geq 1 - c\), then let \(d^u_1^* = \ldots d^u_n^* = \overline{d}\) and \(d^d\) be defined by (42). By a similar argument to that in Case 1, we can show that \(Z(\vec{\varphi}^*) - Z(\vec{\varphi}) > 0\), a contradiction.

If \(\varphi_1^* < 1 - c\), then let \(d^u_1^* = \ldots d^u_n^* = \overline{d}\) be defined by (41) and \(d^d\) be defined by (42). Note that if \(j = 1\) and \(\varphi_1/(1-c) = 1\), then the new mechanism using \(\vec{\varphi}^*\) coincides with the old mechanism using \(\vec{\varphi}\). In this case, we can redefine \(d^u_1 := d^d\) without changing the mechanism. Except for this case, we can show, by a similar argument to that in Case 1, that \(Z(\vec{\varphi}^*) - Z(\vec{\varphi}) > 0\), a contradiction.

Hence, \(j = 0\) or \(n\).
Case 1: \( j = 0 \). In this case,

\[
P^*(v) = \begin{cases} 
\frac{\varphi_i}{1-c} & \text{if } v \geq d^l + \frac{k}{c} \\
\varphi_i & \text{if } v < d^l + \frac{k}{c}
\end{cases}
\]

is an optimal mechanism given \( \overline{\varphi} \). Furthermore, \( \overline{\varphi} \) and \( d^l \) must satisfy

\[
\left[1 - F\left(d^l + \frac{k}{c}\right)\right] \sum_{j=1}^{i} \frac{\varphi_i}{1-c} \leq 1 - F\left(d^l + \frac{k}{c}\right)^i, \forall i \leq n, \tag{43}
\]

\[
F\left(d^l + \frac{k}{c}\right) \sum_{i=1}^{n} \varphi_i + \left[1 - F\left(d^l + \frac{k}{c}\right)\right] \sum_{i=1}^{n} \frac{\varphi_i}{1-c} = 1. \tag{44}
\]

In particular, (43) holds for \( i = n \), which implies

\[
\sum_{i=1}^{n} \frac{\varphi_i}{1-c} \leq \frac{1 - F\left(d^l + \frac{k}{c}\right)^n}{1 - F\left(d^l + \frac{k}{c}\right)}. \]

Substituting this into (44) yields

\[
F\left(d^l + \frac{k}{c}\right)^{n-1} \leq \sum_{i=1}^{n} \varphi_i \leq \frac{(1-c) \left[1 - F\left(d^l + \frac{k}{c}\right)^n\right]}{1 - F\left(d^l + \frac{k}{c}\right)}. \]

By the proof of the second part in Theorem 3, \( j = 0 \) is optimal if \( v^{**} \leq v^\natural \) in which case the optimal \( d^l = d^l_1 = \cdots = d^l_n = v^{**} - \frac{k}{c} \). The set of optimal \( \overline{\varphi} \) is given by \( \Phi(d^l, d^l_1, \ldots, d^l_n) \).

Clearly, \( \varphi \in \Phi \) if and only if \( \overline{\varphi} \) satisfies conditions (43) and (44). Since \( v^{**} \leq v^\natural \) implies that

\[
1 \leq \frac{1}{1-cF(v^{**})} \leq \frac{1 - F(v^{**})^n}{1 - F(v^{**})},
\]

there exists \( 1 \leq h \leq n \) such that

\[
\frac{1 - F(v^{**})^{h-1}}{1 - F(v^{**})} \leq \frac{1}{1-cF(v^{**})} < \frac{1 - F(v^{**})^h}{1 - F(v^{**})}.
\]

66
Hence, for all $i > h$ (43) holds if (44) holds. Given this, it is easy to see that the set of optimal $\overrightarrow{\varphi}$ is the convex hull of

$$\left\{ \overrightarrow{\varphi} \mid \begin{array}{l}
\varphi_{ij} = (1-c)F(v^{**})^{i-1} \text{ if } j \leq h - 1, \\
\varphi_{ih} = \frac{1-c}{1-cF(v^{**})} - \sum_{j=1}^{h-1}(1-c)F(v^{**})^{j-1}, \\
\varphi_{ij} = 0 \text{ if } j \geq h + 1 \text{ and } (i_1, \ldots, i_n) \text{ is a permutation of } (1, \ldots, n)
\end{array} \right\}. $$

**Case 2:** $j = n$. In this case, let $d^u := d^u_1 = \cdots = d^u_n$. In this case,

$$P^*(v) = \left\{ \begin{array}{ll}
\frac{\varphi_i}{1-c} & \text{if } v \geq d^u + \frac{k}{c} \\
F(v)^{n-1} & \text{if } d^l + \frac{k}{c} < v < d^u + \frac{k}{c} \\
\varphi_i & \text{if } v \leq d^l + \frac{k}{c}
\end{array} \right..$$

Furthermore, $\overrightarrow{\varphi}$, $d^l$ and $d^u$ must satisfy

$$\left[ 1 - F \left( d^u + \frac{k}{c} \right) \right] \sum_{j=1}^{i} \frac{\varphi_j}{1-c} \leq 1 - F \left( d^u + \frac{k}{c} \right)^i, \forall i \leq n - 1, \quad (45)$$

$$\left[ 1 - F \left( d^u + \frac{k}{c} \right) \right] \sum_{i=1}^{n} \frac{\varphi_i}{1-c} = 1 - F \left( d^u + \frac{k}{c} \right)^n, \quad (46)$$

$$F \left( d^l + \frac{k}{c} \right) \sum_{i=1}^{n} \varphi_i = F \left( d^l + \frac{k}{c} \right)^n. \quad (47)$$

(46) and (47) imply that $d^l$ and $d^u$ satisfy

$$\frac{1 - F \left( d^u + \frac{k}{c} \right)^n}{1 - F \left( d^u + \frac{k}{c} \right)^{n-1}} = \frac{F \left( d^l + \frac{k}{c} \right)^{n-1}}{1-c}. $$

By the proof of the third part in Theorem 3, $j = n$ is optimal if $v^{**} > v^*$ in which case the optimal $d^l = v^l(\varphi^*) - \frac{k}{c}$ and the optimal $d^u_1 = \cdots = d^u_n = v^u(\varphi^*) - \frac{k}{c}$. The set of optimal $\overrightarrow{\varphi}$ is given by $\Phi(d^l, d^u_1, \ldots, d^u_n)$. Clearly, $\overrightarrow{\varphi} \in \Phi$ if and only if $\overrightarrow{\varphi}$ satisfies conditions (45)-(47).

It is easy to see that $\Phi$ is the convex hull of

$$\left\{ \overrightarrow{\varphi} \mid \varphi_{ij} = (1-c)F(v^u(\varphi^*))^{j-1} \forall j \text{ and } (i_1, \ldots, i_n) \text{ is a permutation of } (1, \ldots, n) \right\}. $$
References


