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PIER Working Paper 17-010

Voting as a signal of education

by

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May 2017

Abstract

Since the chance of swaying the outcome of an election by voting is usually very small, it cannot be that voters vote solely for that purpose. So why do we vote? One explanation is that smarter or more educated voters have access to better information about the candidates, and are concerned with appearing to have better information about the candidates through their choice of whether to vote or not. If voting behavior is publicly observed then more educated voters may vote to signal their education, even if the election itself is inconsequential and the cost of voting is the same across voters. I explore this explanation with a model of voting where players are unsure about the importance of swaying the election and high type players receive more precise signals. I introduce a new information ordering, a weakening of Blackwell's order, to formalize the notion of information precision. Once voting has occurred, players visit a labor market and are paid the expected value of their type, conditioning only on their voting behavior. I find that in very large games, voter turnout and the signaling return to voting remains high even though the chance of swaying the election disappears and the cost of voting is the same for all types. I explore generalizations of this model, and close by comparing the stylized features of voter turnout to the features of the model.

Keywords: Voting, signaling.

JEL Classification Numbers: D72, D80

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1 Introduction

Why do voters vote in large elections? The natural answer, suggested by political scientists such as Downs (1957), is that voters vote because they want to influence the outcome of the election. But the chance of swaying a large election (the 'pivot probability') is so small[1] that it seems implausible that it plays any significant role. This is the 'paradox of voting'.

In this paper, I consider the following resolution to the paradox: It may be that some voters are better informed (perhaps because they are better educated) than others about the quality of the two candidates; and further, it may be that voters have a preference for choosing to vote if doing so leads others to believe that they are better informed about the quality of the candidates, even though better informed and worse informed voters have the same *ex-post* cost of voting and value from a candidate's election. Imagine, for example, that some voters are well-educated, and others are not. The well-educated voters read about politics, and have a good idea of the relative worth of the various candidates. Hence, the expected return to their vote is higher than to that of a uneducated voter, since they are more likely to make the right decision. By itself, this would be insufficient to compel them to vote, since the chance of swaying the election is relatively small. But they show up on election day and vote, because they have preferences for appearing to be well-informed. The uninformed voters are aware that, if they vote, they will be perceived as well-informed, but they remain at home, because the gains from appearing to be well-informed do not quite outweigh the reduced value they expect their vote will bring.[2]

This paper contributes to the literature on the paradox of voting in two ways. First, it provides an alternate, informational channel through which voters may be motivated to vote,

[1] About 1 in 60 million in the US, as estimated by Gelman et al. (2009).

[2] That there is **some** signaling value to voting, regardless of whether it is driven through the described channel, is difficult to dispute, see for example the well-established fact that self-reported voter participation is always significantly higher than actual voter participation, suggesting that many individuals lie in an attempt to 'look good' when the pollster calls.

one which is consistent with the stylized facts of the voting literature, as well as the intuition that people vote because they wish to signal that they are well-informed. Second, it resolves a paradox from the literature on uncertain voters (Feddersen and Pesendorfer, 1996), which is that such models predict large increases in voter participation as rates of education have risen in the US. If anything, voter participation has fallen. This paper applies an insight of the uncertain voter literature (that a better informed voter may receive, on average, a higher payoff from voting) to a model with players who have concerns for signaling that they are better informed. In the literature on uncertain voters, the gain from voting comes from one's **absolute** level of information, but when players wish to signal that they have better informed, the gain from voting comes from one's level of information **relative** to others, and hence, voter participation rates in the model in this paper are driven not by the absolute quality of information, but in the relative dispersion of heterogeneity in information across the population.

1.1 Prior literature

A model of voter turnout was analyzed formally by Ledyard (1984), who found that while it is possible to sustain high turnout equilibria in a voting game, such equilibria seem intuitively implausible, depending on 'knife-edge' constructions. For example, it may be an equilibrium for 1 million people to vote, 500,000 for either candidate, so that each voter's vote is pivotal. This intuition was formalized by Palfrey and Rosenthal (1985) and Myerson (2000), who observed that high turnout equilibria do not survive the introduction of uncertainty about game parameters, specifically, population. Some attempts have been made to salvage the idea that high turnout can be sustained if voters care only about swaying the election by considering correlated equilibria (Pogorelskiy, 2014). But any such explanation must rely on large pivot probabilities, which empirically we do not observe (Gelman et al., 2009).

Empirical observations do however motivate other explanations. One robust empirical finding is that voters tend to be wealthier and better educated on average.[3] Posner (1998) proposes that wealthier voters have a lower cost of voting, and therefore voting serves as a credible signal (in the sense of Spence (1973)) of their wealth. A related explanation is that more educated voters are also more 'civically minded', meaning they are more likely to benefit from cooperation with others, and that they vote to signal their civic mindedness. Funk (2005, 2010) was the first to analyze this explanation formally as well as empirically, showing that Swiss cantons in which the cost of voting was lowered through the introduction of mail-in voting paradoxically saw reductions in voter turnout. Other empirical studies[4] reach similar conclusions.

Aytimur et al. (2014) expands on the concept of 'civic-mindedness' by formally modeling a second stage, after voting publicly occurs, in which players match with other members of their community. High type voters place a higher value on such matches, and on swaying the outcome of the election. Then even in games with a large population, high voter turnout is supported even though the chance of swaying the election is minuscule.

It is unclear in these models what special role voting plays—beyond providing a costly action that players can use as a signal. In particular, the same results would hold in this setting if, instead of having the option to vote in an election, players had the option to wait in line for an hour at the high school gym, walk into a booth, put checkmarks on a piece of paper, then hand it to a volunteer, who would immediately throw it away. Why would we use voting, which has important real world consequences, as a signaling device when so many other signaling devices are available? One explanation (Posner, 1998) appeals to Schelling (1980)'s concept of 'focal points' to argue that some actions fall naturally into the role

[3] All stylized facts about voting discussed here are taken from Wolfinger and Rosenstone (1980) and Leighley and Nagler (2013).

[4] Kousser and Mullin (2007), also see chapter 4 in Leighley and Nagler (2013) for a summary of the empirical research on this question in the political science literature.

of signaling device. Another explanation (Pesendorfer (1995), who applies the concept to think about fashion trends) is that signaling devices do not arise by chance, rather, they are strategically created by 'norm entrepreneurs' (Sunstein, 1995). Both stories, of focal points and of norm entrepreneurs, are compelling explanations for why voting might serve as a signaling device. For the former explanation, in a democracy voting is a regular, somewhat costly occurance, and so a natural focal point; for the latter explanation, regular voting is viewed as an important element of a well-functioning democracy, so that civic leaders may wish to encourage its role as a signaling device.[5]

In this paper, I focus on a third channel through which voter may serve as a signaling device, which is an informational channel. If some voters have better information about the importance of voting, then their expected returns to voting are higher. The 'uncertain voter' literature suggests that better educated voters may receive more accurate signals of which candidate is the right one, and so derive a higher utility from voting (Feddersen and Pesendorfer, 1996; Matsusaka, 1995). At the same time, perhaps voters simply intrinsically care about voting for the 'right' candidate, or alternatively that some voters ('rule utilitarians') care not only about their own well-being, but the well-being of the group to which they belong (Coate and Conlin, 2004; Feddersen and Sandroni, 2006). Empirically, this model is successful in fitting the stylized features and demographics of voter turnout (Degan and Merlo, 2011). However, the model also make a strong prediction, which to my knowledge is both undiscussed in the literature and falsified by available evidence, namely, that the rise in average educational attainment across the US over the past century should have resulted in a corresponding rise in voter turnout—something we have not observed.[6]

[5] Or even civically-minded private actors, such as when Facebook made available an 'I Voted' sticker to users.

[6] For example, in the last US election, approximately 75% of those with bachelor's degrees or higher voted, 50% of those with a high school voted, and 40% of those with less than a high school education voted. Since 1940, the percentage of the population with a bachelor's degree has approximately quintupled, from 5% to 25%, while the percentage with high school educations has nearly tripled, from 20% to 60%. A back-of-the-envelope calculation suggests that if education had a causal impact on voting turnout then it

Certainly it may be true that other factors were at play over the past century. Perhaps voters became increasingly dissatisfied with the political process, or perhaps they perceive candidates as closer ideologically, making voting less worthwhile. However, there is not much evidence in the US that potential voters have become more dissatisfied with the political process, or that potential voters perceive the ideologies of candidates to be converging.[7]

This paper presents a model of voting which connects the uncertain voter literature to the signaling literature, and in the process provides an explanation for why voting might serve as a signaling device. In the model, a group of players must choose between two candidates. Players do not know their value of swinging the election, but each player observes some private information. I show that even if we depart from the previous literature on signaling in voting and assume that the cost and expected value of voting is homogeneous across players, there is still a role for signaling if players differ in the strength of their private signals, and if players with more precise signals are valued more highly—in the model, by firms who pay a wage conditioned on their observed voting behavior and equilibrium strategies. But other reasonable stories for these gains could include participating in a marriage market or forming connections with other players as in Aytimur et al. (2014).

In such games, high turnout is supported even as the number of voters grows very large and the pivot probability disappears. Furthermore, high-turnout equilibria are the only equilibria for large games. Intuitively, this is because players with more informative signals are *ex-ante* more likely to vote. This generates a positive signaling value to voting. Even as the pivot probability disappears, the effective cost (net of the signaling value) becomes very low, inducing large numbers of players to vote while still separating by type.

should have contributed to a 10 percentage point increase in voter turnout. The direct explanation for this discrepancy is that the correlation between education and turnout has eroded over the past half century, enough to outweigh the increase in education.

[7] Leighley and Nagler (2013) examines self-reported measures of the perceived importance of US elections. The trend is toward perceiving presidential candidates to be more opposed, rather than less, while alienation shows no discernible trend. Other surveys (`http://www.people-press.org/2014/06/12/political-polarization-in-the-american-public`) find that political polarization is increasing.

In Section 2, I describe the model. In Section 3.2, I prove that high-turnout equilibria are the only equilibria in large games using a measure of informativeness I call max-min informativeness. In Section 3.3, I additionally assume that the quality of player's information can be ordered using a rotation order and use this to prove stronger results, including a description of how to construct limiting equilibria. In Section 3.4, I relax the assumption that costs are the same across players. I show that if the set of possible costs is discrete then the results still hold, but that the results are not robust if costs are allowed to be a continuous random variable. A rough numerical exercise, however, establishes that even when costs are allowed to be continuous, for reasonable values of N the contribution of signaling to voter turnout can still be significant. Section 4 concludes.

2 Model

A set of n players each have the option of participating in a majority rules election. n is not known, rather, it is the realization of a random variable drawn from a Poisson distribution with mean N. That is, this is a Poisson game (Myerson, 1998) of voting.

Once n is drawn, an unobserved state of the world $\omega^n \in \Omega^n$ is drawn from a distribution $H(\Omega^n)$. The ith component of ω^n belongs to the finite set $\Omega \subset \mathbb{R}$, and should be interpreted as the value to player i of swaying the election, from her least favored candidate to her most favored candidate. I assume without loss of generality that that the elements of ω are ordered from least to most, $\omega = \{\omega_1, \ldots, \omega_{|\omega|}\}$, that $\omega_1 = 0$, and that $H(\omega^n)$ is symmetric across players for all $n \geq 0$.

Each player has a private type taking values in $T = Y \times S \times \mathcal{C}$. Y is an interval $[\underline{\gamma}, \overline{\gamma}]$, while $S = \{s_1, s_2, \ldots, s_{|S|}\}$, and $\mathcal{C}$ are finite discrete sets. $\gamma \in Y$ represents the precision of a player's information, $s \in S$ represents a player's private information, and $C \in \mathcal{C}$ represents her preferred candidate. Without loss of generality, assume $\mathcal{C} = \{A, B\}$.[8] [9]

Often in games of incomplete information, the player's index i is implicitly assumed to be part of her type. In a Poisson game we explicitly drop this assumption. A player does not know her index, otherwise, she would have private information about the size of the game, namely, that $n \geq i$. A player's type will be denoted (γ, s, C), and her private valuation ω.

γ and C are drawn identically and independently across players from a distribution $F(\gamma, C)$. I assume that γ and C are independent, so that signal precision and candidate preferences are not correlated, and I assume that the support of $F(\gamma)$ includes at least 2 elements, $\{\underline{\gamma}, \overline{\gamma}\}$. The signal s is drawn according to $\mathrm{P}_\gamma(s \mid \omega)$. The set of signal distributions $\{\mathrm{P}_\gamma(\cdot \mid \omega)\}_\gamma$ defines the information structure of the game as well as a joint distribution $\mathrm{P}(\gamma, s, \omega)$. I assume that $\mathrm{P}_\gamma(s \mid \omega)$ is continuous in γ and that it has full support on S.

Once a player observes her type, she has the option of voting, $v = 1$, or not voting, $v = 0$. If she votes, she incurs some loss $k > 0$ and votes for her preferred candidate.[10] The candidate who receives the majority of the votes wins, and in the event of a tie a coin toss decides the outcome.

If candidate C wins, then every player with valuation ω whose type specifies that she prefers C receives a payoff of ω. The players who do not prefer C receive a payoff normalized to zero.[11]

After the election has concluded, there is a second stage to the game. As in Spence (1973)'s canonical paper on signaling, I think of this stage of the game as one in which players

[8] Allowing for more than two candidates complicates the expressions for the pivot probability, but not in an interesting way.

[9] The Y dimension of the type space is continuous, but not the S dimension. For technical reasons, the proof of the main result of the paper does not generalize to the case in which S is infinite.

[10] In a Poisson game, voting for one's own candidate is a strictly dominant strategy, conditional on voting, so for simplicity we focus strategically only on the decision to vote.

[11] This normalization is not without loss of generality, since ω is required to be non-negative. That is, we rule out the case in which players are wrong about which candidate they prefer, and only consider the case in which players are uncertain about the magnitude of their preference for one candidate over another. Similar results may be obtained in the more general model, but in a less tractable way.

are paid a wage in a competitive job market. (Some alternate, equally valid interpretations include a stage where players form local connections in their community or enter the marriage market, as in Aytimur et al. (2014). Further, it may simply be that players care about how other players perceive them.) Since the market is competitive, players are paid their expected marginal worth to the firm, which I assume is the expected value of the component γ of the player's type, conditioning only on equilibrium strategies and the player's voting choice.[12] Imagine that an employee shows up for work with an 'I Voted' sticker, impressing her coworkers. We need not interpret the reward as originating with a firm, however, imagine that neighbors think highly of someone who is observed to be going to the polls, or a prospective date is impressed by political activity.

A strategy profile of the game is denoted $\sigma(\gamma, s, C)$. It takes values in $[0, 1]$ and denotes the probability that a voter of type γ, s, C votes. Strategy profiles are implicitly taken to be symmetric across players. This is a consequence of the fact that a player does not learn her own index.

A player's *ex-post* payoff under strategy profile σ if her favored candidate wins is

$$U^W(\omega, \sigma, v) = \omega + \mathrm{E}[\gamma \mid \sigma, V] - kv. \tag{1}$$

If instead her favored candidate loses she receives

$$U^L(\omega, \sigma, v) = \mathrm{E}[\gamma \mid \sigma, v] - kv. \tag{2}$$

Note that, technically speaking, neither (1) nor (2) are proper payoffs of a game of incomplete information, since both condition on strategies. This is instead an example of a *psychological*

[12]Note that since γ is an arbitrary index of the precision of the player's private information, it may be preserved under increasing transformations, hence, this assumption is a normalization and without loss of generality.

game (Geanakoplos et al., 1989). As in Spence (1973), however, this distinction is not a large one, since the analysis of this game is equivalent to that of one in which we explicitly model the stage where players are paid a wage conditioning on equilibrium strategies.

Consistent with prior literature, I call ω the **electoral value** from voting. It is the direct reward a player receives from having her favored candidate win. I call $\mathrm{E}[\gamma \mid \sigma, v]$ the **signaling value** from voting. It is the reward a player recieves from swaying the perceptions of others about her type through her voting decision.

A voter is pivotal if she makes a tie or breaks a tie. In that event, she either sways the election toward a $\frac{1}{2}$ chance of her opponent winning the tie, or she sways the election from a $\frac{1}{2}$ chance of her opponent winning the tie to her opponent winning for sure. In either case, the expected difference is $\frac{1}{2}\mathrm{E}[\omega \mid \gamma, s]$. A standard result is that *ex ante*, a player only considers her payoff in the event she is pivotal, since this is the only event in which her vote makes a difference. The $\frac{1}{2}$ term is a constant and so without loss of generality can be thought of as included in ω. This observation allows us to write the best reply condition for a player of type (γ, s, C) to vote as

$$\underbrace{\mathrm{E}[\omega \mathrm{P}(\text{piv} \mid \omega) \mid t]}_{\text{Marginal electoral benefit of voting}} + \underbrace{\mathrm{E}[\gamma \mid v=1, \sigma] - \mathrm{E}[\gamma \mid v=0, \sigma]}_{\text{Marginal signaling benefit of voting}} - \underbrace{k}_{\text{Cost of voting}} \geq 0, \tag{BR}$$

where $\mathrm{P}(\text{piv} \mid \omega)$ denotes the probability that a player is pivotal in the game, conditional on a valuation of ω. I denote the marginal electoral benefit of voting by $U_E(t)$, and the marginal signaling benefit of voting by U_S. The voting condition then becomes

$$U_E(t) + U_S - k \geq 0, \tag{3}$$

making it explicit that the electoral value from signaling depends on the player's type, while the signaling value from voting is independent of the player's type.

I rule out some trivial cases through the following assumptions:

Assumptions

A1 $\overline{\gamma} - \mathrm{E}[\gamma] > k$.

A2 $\exists \gamma \in Y$ s.t. $\mathrm{E}_{\gamma}[\omega \mid s] > k$.

Assumption (A1) says that signaling is important enough to players so that if they could convince the firm that they have the highest value of γ as opposed to the *ex ante* expectation of γ, then that alone would be worth the cost of voting. It rules out equilibria where trivially only the highest type votes.

Assumption (A2) says that there is at least one player type who, if she knew she could decide the election, would vote. It rules out equilibria in which trivially no player votes because costs are too high.

The definition of equilibrium is standard.

Definition A strategy profile σ is an **equilibrium** iff

1. The best response condition, (BR), holds for $\sigma(t) = 1$. It holds with equality for $\sigma(t) \in (0, 1)$. The reverse inequality holds when $\sigma(t) = 0$.

2. The marginal signaling benefit of voting and the marginal electoral benefit of voting are consistent with Bayes rule wherever possible.

For a fixed type distribution, I consider games with a large population. Formally, fix a type distribution, and denote the sequence of games obtained only by varying the parameter N, which is the average number of players in the game, as $\{\Gamma_N\}_{N=1}^{\infty}$. Denote a corresponding sequence of equilibria of these games by $\{\sigma_N\}_{N=1}^{\infty}$. We will be interested in analyzing the property of equilibria of such games in the limit as N grows large.

2.1 Assumptions on the information structure

The parameter γ orders the quality of information of players. In this section, I discuss formally the meaning of that statement.

For technical reasons, I assume $|S| \geq |\omega|$, that is, there are at least as many signals as states, and I assume that there is at least one type $\gamma \in [\underline{\gamma}, \overline{\gamma})$ for which the information matrix whose ijth component is given by $\mathrm{P}(\omega_j \mid s_i)$ has rank $|\omega|$. I assume without loss of generality that $\mathrm{E}_\gamma[\omega \mid s]$ is increasing in s for all γ.

I assume that γ orders P_γ with an order I call max-min informativeness, which can be thought of as a weakening of Blackwell's ordering. (For a comparison of max-min informativeness to other information orderings, see Lemma 3, in the appendix.) Intuitively, max-min informativeness captures the following idea: Imagine two Bayesians, Alice and Bob, forming beliefs over some state ω based on some individual specific private information. Alice does not know what Bob knows, but say Alice knows Bob will never believe that the value of the state variable is larger than $\overline{\omega}$. Then if Bob has 'better' information about the value of the state variable than Alice, I claim that Alice should never rationally believe that the value of the state variable is larger than $\overline{\omega}$. That is, the maximum possible dispersion of conditional expectations should be increasing in the informativeness of the signal. Stated differently, it is the best-informed players who should have the potentially most extreme beliefs on the value of the state variable.

Formally,

Definition If $\mathrm{P}_\gamma, \mathrm{P}_{\gamma'}$ satisfy the following conditions, then $\mathrm{P}_{\gamma'}$ is **max-min more informative** than P_γ:

$$\max_s \mathrm{E}_\gamma[g(\omega) \mid s] \leq (<) \max_s \mathrm{E}_{\gamma'}[g(\omega) \mid s] \tag{4}$$

$$\min_s \mathrm{E}_\gamma[g(\omega) \mid s] \geq (>) \min_s \mathrm{E}_{\gamma'}[g(\omega) \mid s], \tag{5}$$

for all $g(\omega)$ (for all $g(\omega)$ such that $\mathrm{E}_{\gamma'}[g(\omega) \mid s]$ is non-constant over s).

Max-min informativeness formalizes the intuition that more informed players potentially have more extreme beliefs. It will play a crucial role in the proof of the main result. The intuition behind its role suggests that it is an information ordering which may be useful more generally in games in which players have reputational concerns for appearing to be well informed. An objection is that, in actuality, we may tend to think of less informed voters as being the ones likely to hold more extreme beliefs. The stereotype is that the less informed tend to think of the world 'in black and white', while the well informed tend to see the nuances. A discussion of this idea is beyond the scope of this paper. However, I present the following two observations: First, max-min informativeness is a weakening of Blackwell's ordering, so that this criticism applies equally well to Blackwell's ordering. Since Blackwell's ordering itself is relatively weak, this suggests that to the extent that this criticism holds, the Bayesian framework is not well-suited to the analysis of environments in which there are reputational returns to appearing to be well informed. Second, in general, if there is value to appearing to be well informed, and the well informed are more likely to be certain in their beliefs, then it is not hard to imagine that less informed players face a strong incentive to exaggerate the strength of their convictions. This suggests, at least, that we should not place too much stock in the observation that in reality it seems to be the less well informed who are more certain of their beliefs, since reporting that one holds extreme beliefs is 'cheap talk' and so unlikely to be informative about the actual strength of any convictions that one may hold.

2.2 Firm beliefs

In the canonical signaling game, off-equilibrium actions and the beliefs of firms in response to such actions are important in potentially enforcing equilibrium actions. For example, consider a game where $\gamma \sim U[0,1]$, and $c = 0.1$. Consider a strategy profile which specifies that all players vote. Say firms have pessimistic beliefs: upon seeing no voting, they assume that the player is of type $\underline{\gamma}$. Then in equilibrium $U_S = \mathrm{E}[\gamma \mid v] = \frac{1}{2}$. Since $U_S > k$, voting is a strictly dominant strategy. These sorts of equilibria exist, but are not the focus of this paper. Therefore, I restrict attention to equilibria in which, if all or none of the players are voting, $U_S = 0$, which suffices to eliminate them as equilibrium candidates.

3 Analysis

This section has five parts. In Section 3.1, I introduce some preliminary results and notation. In Section 3.2, I prove the basic result that voting percentages remain large (meaning, bounded away from zero), even in games with large populations. In Section 3.3, I prove a stronger result under the assumptions that γ orders P_γ with a rotation order and that valuations are independently and identically distributed across players, and I explicitly compute the limiting equilibrium. I exploit these assumptions to establish comparative statics results about the game. In Section 3.4, I relax the assumption that costs are the same across players and allow it to instead be an independently and identically distributed random variable. I show that previous results hold when costs belong to a finite set, but not when costs are allowed to be continuous. I perform some numerical calculations, however, to demonstrate that the contribution of signaling concerns to voter turnout may still be significant even when costs are continuously distributed (but that still, as population grows large, voter turnout disappears.)

3.1 Useful facts and notation

Given a strategy profile σ, let V denote the expected percentage of players voting in equilibrium. Let V_γ denote the expected percentage of players with type component γ voting in equilibrium. Formally,

$$V_\gamma := \sum_{s,C} \mathrm{P}_\gamma(s,C)\sigma(\gamma,s,C) \tag{6}$$

$$V := \int_\gamma V_\gamma dF(\gamma). \tag{7}$$

A useful fact about Poisson games which I will use in future propositions is that as N grows large, if the expected percentage of players voting is bounded away from zero, then the probability that a player will be pivotal disappears, **independently** of the strategy profile:

Lemma 1. *Let $\{\sigma^N\}_{N=1}^{\infty}$ be a series of strategy profiles on the corresponding sequence of games, $\{\Gamma^N\}_{n=1}^{\infty}$. Let $\{V^N\}_{N=1}^{\infty}$ denote the corresponding expected percentage of players voting, as defined by (7), such that the sequence $\{V^N\}_{N=1}^{\infty}$ is bounded from below by zero. Then*

$$\lim_{N\to\infty} P_{\sigma^N}(piv) = 0. \tag{8}$$

The proof is in the appendix. Lemma 1 rules out the intuitively implausible, high-turnout, knife-edge equilibria of Palfrey and Rosenthal (1985), by showing that no strategy profile (equilibrium, or otherwise) can support high pivot probabilities, and hence, in equilibrium, voter turnout must vanish as N grows large.

Finally, I list some of the useful features of Poisson games derived in Myerson (1998) and Myerson (2000):

1. Every player in a game with average population N believes that the number of other players in the game is a Poisson random variable with mean N.

2. If the probability that an event occurs for any arbitrary player is p, then the total

number of players for whom that event occurs is a Poisson random variable with mean pN. For example, the number of players who prefer candidate A is a Poisson random variable with mean $F(A)N$.

3. If players of type C vote with probability p_C, then the pivot probability for players who prefer candidate A may be explicitly written as

$$\sum_{k=0}^{\infty} \frac{e^{-\rho_A p_A N}(\rho_A p_A N)^k}{k!} \frac{e^{-\rho_B p_B N}(\rho_B p_B N)^k}{k!} \left(1 + \frac{\rho_B p_B N}{k+1}\right).$$

Note that the pivot probability is always strictly positive.

3.2 Voter turnout in large games

The next proposition states the general result for this model, that in large games the voting percentages must be bounded away from zero.

The bulk of the proof is concerned with the difficulty that player valuations may be correlated. Intuitively, imagine a player who sees a signal which leads her to believe her electoral value from swaying the election, ω, is very high. Should she therefore also infer from that signal that she has a high marginal electoral value to voting? On one hand, she believes ω to be very high. But on the other hand, if the player's valuations are sufficiently correlated, she also believes that other players believe ω to be high, and so perhaps she believes that voter turnout will be high. Then $\mathrm{P}(\text{piv} \mid \omega)$ is decreasing in ω, and $\omega\mathrm{P}(\text{piv} \mid \omega)$ may be non-monotonic.

Working directly with $\mathrm{P}(\text{piv})$, the unconditional probability that a player is pivotal, is complicated, and $\omega\mathrm{P}(\text{piv} \mid \omega)$ promises to be even more so. (For completeness, the full expression is contained in the appendix.) The proof of Proposition 1 instead exploits the property of max-min informativeness to argue that, regardless of the specific shape of

$\omega\mathrm{P}(\mathrm{piv} \mid \omega)$, if voting turnout were to disappear, then eventually high γ type players would be the only players voting, and so the signaling return would be very high, inducing all players to vote.

Proposition 1. *Let $\{\sigma^N\}_{N=1}^{\infty}$ be a sequence of equilibria of games, each with expected population N. Denote the corresponding voting percentages (7) and (6) in these games by V^N, V_γ^N, and the marginal electoral and signaling benefits by U_S^N, U_E^N. Then*

1. *U_S^N and U_E^N converge to U_S^∞, U_E^∞, satisfying*

$$U_S^\infty = k$$

$$U_E^\infty = 0.$$

   The pivot probability $P_{\sigma^N}(piv)$ converges to 0.

2. *Voting percentages are bounded away from zero,*

$$\liminf_{N\to\infty} V^N > 0. \tag{9}$$

The proof is contained in Appendix A. The intuition of the result is best seen by considering what would happen if voter turnout, as a percentage of the total population, were to disappear as N grew large. In essence, the electoral return to voting is dependent on the **absolute** number of voters, while the signaling return to voting is dependent on the **relative** number of voters of each type voting. If the percentage of players voting vanishes, eventually, the only voters voting are the very highest type voters, a consequence of the max-min assumption. So, the signaling return to voting must become large as the percentage of players voting vanishes. In the limit, eventually only the highest type of voter ever votes in equilibrium—and so, voting is associated with a signaling value of $\overline{\gamma}$. Assumption A1 then implies that every player in the game strictly prefers to vote regardless of their perception of the electoral value of their vote, and so it cannot be that in equilibrium the percentage of voters vanishes.

To further illustrate the intuition behind this result, I now prove a stronger result with stronger assumptions.

3.3 Constructing equilibria

Assume that the player's valuation of the importance of the election are independent and identically distributed,

$$H(\omega^n) = \prod_{i=1}^{n} H(\omega), \tag{10}$$

and γ orders P_γ using a rotation order.

The importance of the rotation order and independence assumption is that it simplifies the problem of finding the expected electoral value and the strategy profiles, as summarized by the following lemma.

Lemma 2. *Say that γ orders F with a rotation order, and say valuations are independently and identically distributed. Then the probability a voter attaches to being pivotal is independent of the value the voter attaches to swaying the election, i.e.,*

$$E_\gamma[\omega P(piv \mid t) \mid s] = E_\gamma[\omega \mid s] P(piv \mid C). \tag{11}$$

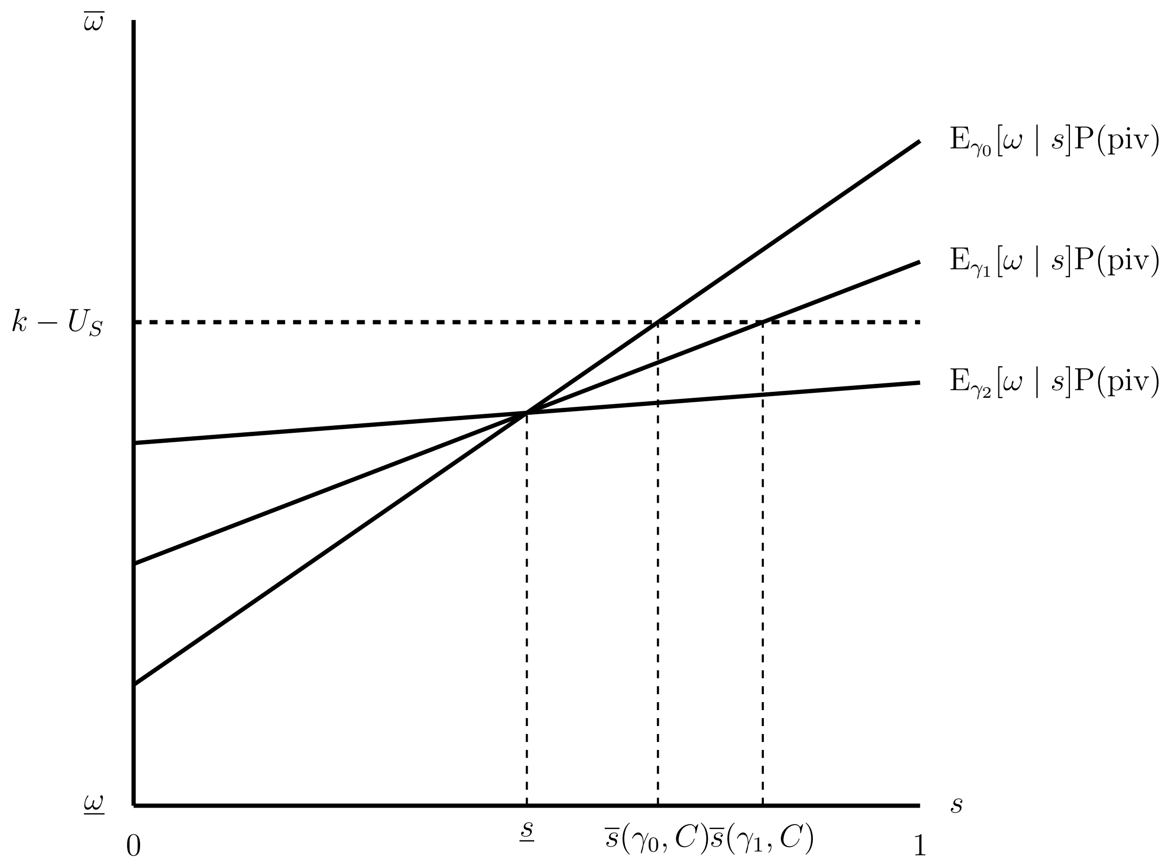

Figure 1: An example of the conditional distributions induced by a rotation order.

Note: Here, $\gamma_0 > \gamma_1 > \gamma_2$. *Ex-ante*, at the given level of $k - U_S$, $1 - \overline{s}(\gamma_0, C)$ percent of players of type γ_0, C are expected to vote, $1 - \overline{s}(\gamma_1, C)$ percent of players of type γ_1, C are expected to vote, and 0 percent of players of type γ_2 are expected to vote.

Furthermore, strategy profiles in equilibrium are characterized by cutoffs $\underline{s}(\gamma, C)$*, such that*

$$\sigma(\gamma, s, C) \in \begin{cases} \{0\} & s < \underline{s}(\gamma, C) \\ [0,1] & s = \underline{s}(\gamma, C) \\ \{1\} & s > \underline{s}(\gamma, C) \end{cases}, \tag{12}$$

and these cutoffs have the feature that $\underline{s}(\gamma, C)$ *is monotone in* γ*.*

The proof is in the appendix. Lemma 2 is a great simplification of the problem, since otherwise, the electoral value $\mathrm{E}_\gamma[\omega \mathrm{P}(\mathrm{piv} \mid t) \mid s]$ may be very complicated. It establishes that when signals are independent, the expected marginal electoral value is proportional to the expected value of swaying the election and the pivot probability is independent of a player's private information.

Proposition 2. *Again, let* $\{\sigma^N\}_{N=1}^{\infty}$ *be a sequence of equilibria of games with expected population* N*. Denote the corresponding voting patterns in these games by* $V^N, V_\gamma^N, U_S^N, U_E^N$*. If player's valuations are independently and identically distributed, and* γ *orders* F *according to a rotation order, then*

1. $\sigma^N, V^N, V_\gamma^N, U_S^N, U_E^N$ *converge to* $\sigma^\infty, V^\infty, V_\gamma^\infty, U_S^\infty, U_E^\infty$*. The pivot probability* $P(piv \mid \sigma_N)$ *converges to 0. (Convergence.)*

2. $V^\infty > 0$*,* $U_S^\infty = k$*, and* $U_E^\infty = 0$*. (Voter turnout remains high in large games.)*

3. V_γ^∞ *is non-decreasing in* γ*. (High types vote more.)*

4. V^∞ *and* V_γ^∞ *are decreasing in* k*, and* U_S^∞ *is increasing in* k*. (A higher cost of voting leads to less voting from all types. More low types than high types stop voting as costs increase.)*

5. *There exist information structures F, F' such that $E_F[\gamma] > E_{F'}[\gamma]$, but voting participation is lower in equilibrium under F then F'. (Even though voting is positively correlated with γ, increasing average γ doesn't necessarily increase voter turnout.)*

The proof is in the appendix. Here, I discuss the intuition. Parts 1 and 2 are consequences of Proposition 1. The monotonicity results, parts 3 and 4, are a straightforward consequences of the assumption of a rotation order. (See Figure 1 for a graphical intuition.) The final part of the proposition, 5, establishes that it is possible for the average level of informedness to increase, and for voting participation to fall. The intuition for this result is analogous to that in a Spence signaling model, in which, for example, increasing the prior probability of the high type may reduce levels of education (the extreme example being the case in which there is a prior probability of 1 that a player is the high type, and hence no education may be supported in equilibrium).

Results 1, 2, and 3 of Proposition 2 are unsurprising. 4 states that, as costs increase, all types are less likely to vote, but low types are proportionally even less likely to vote than high types. This occurs because, in order to outweigh the increase in costs and maintain voter participation, in equilibrium the signaling returns to voting must increase, and so the relative proportion of high types voting must also increase.

Of particular interest is 5, which is a formalization of the claim that this model provides a resolution to the paradox that, counter to what is suggested by models of uncertain voters, voter participation in the United States has fallen over the past century even as education has greatly increased. Informally, voter participation is driven by the dispersion of the quality of information, not the absolute quality of information available to players, so that it is possible to increase the quality of all the player's information, but, by reducing the spread in quality, reduce the signaling return to voting, and so, in turn, reduce voter participation.

3.4 Heterogeneous costs

Here, we allow costs to be stochastically drawn across individuals. Importantly, costs are not allowed to depend on the player's type, otherwise, signaling is supported through standard channels as in Spence (1973). The interesting case is when costs are required to be independently and identically distributed across agents. So, say that a player's type now includes their cost, that is, $t = (\gamma, s, C, k)$, and k is drawn independently and identically across players from some set K.

When K consists of a finite number of elements, we have the following result, analogous to Proposition 1:

Proposition 3. *Let $\{\sigma^N\}_{N=1}^{\infty}$ be a sequence of equilibria of games, each with expected population N. Say K is a finite set. Then*

1. *U_E^N converges to $U_E^{\infty} = 0$. The limit points of $\{U_S^N\}_{N=1}^{\infty}$ is a subset of K, and the pivot probability $P_{\sigma^N}(piv)$ converges to 0.*

2. *Voting percentages are bounded away from zero,*

$$\liminf_{N\to\infty} V^N > 0. \tag{13}$$

Proposition 3 differs from Proposition 1 in the behavior of U_S^N, the marginal signaling return to voting. In Proposition 1, the marginal signaling return to voting approached the cost of voting. This was a necessary consequence of the fact that, in order to support separation of types in voting behavior in equilibrium, the signaling return to voting (which is common knowledge across types) had to approach the cost of voting, since the pivot probability (and so the marginal electoral return to voting) vanishes as N grows large. In Proposition 3, the possible equilibrium values for the marginal signaling return to voting must be close to the elements of K, and for the same reason. The proof is otherwise identical to that of Proposition 1 and so omitted.

On the other hand, voter turnout disappears when costs are allowed to be drawn from a continuous distribution. Assume that k is drawn according to some density function with full support on $[0, \infty)$. A more general result holds for more general continuous distributions, but requires a more careful treatment of the support of costs which is here omitted for simplicity.

Proposition 4. *Let $\{\sigma^N\}_{N=1}^{\infty}$ be a sequence of equilibria of games, each with expected population N. Say K is drawn according to some CDF F_k with support on $[0, \infty)$. Then U_E^N, U_S^N, and V^N all converge to 0.*

Proof. Again, via Lemma 1, as N grows large, the marginal electoral return to voting disappears, $U_E^N \xrightarrow{N} 0$, and the maximum possible spread in the distribution of the electoral return to voting,

$$\left(\max_t U_E^N(t)\right) - \left(\min_t U_E^N(t)\right) \xrightarrow{N} 0, \tag{14}$$

Define

$$\kappa_1^N := \min_t U_E^N(t)$$
$$\kappa_0^N := \max_t U_E^N(t).$$

Re-stating (14), we have $\kappa_1^N - \kappa_0^N \xrightarrow{N} 0$. The best response condition, (BR), implies that, by construction, every type with $k < \kappa_1^N$ will vote, **independently** of the precision γ of their private information, while every type with $k > \kappa_0^N$ will not vote, again independently of γ. With players whose cost falls in $[\kappa_1^N, \kappa_0^N]$, the maximum signaling return to voting is generated by the strategy in which only the highest types, $\overline{\gamma}$, vote, and so the signaling return to voting is at most

$$U_S^N \leq \overbrace{\left(F_k(\kappa_1^N) - F_k(\kappa_0^N)\right)}^{\text{Mass of players who vote conditional on } \gamma, s} \times \overbrace{\left(\overline{\gamma} - \mathrm{E}[\gamma]\right)}^{\text{Maximum marginal signaling return}}$$
$$\xrightarrow{N} 0,$$

That is, as $\kappa_1^N \to \kappa_0^N$, the expected γ type of a voter eventually equals the expected γ type of a non-voter, since, in the limit, players base their voting decision only on the cost of voting, which is independent of γ, s, and so $U_S^N \xrightarrow{N} 0$. But then, the optimality condition (BR) implies that, as $N \to \infty$, eventually the only players voting are those for whom $k = 0$, which, since k is continuously distributed, is a measure zero set of voters, and so $V^N \xrightarrow{N} 0$. $\square$

Proposition 4 states that when costs are continuously distributed, voter turnout disappears in the limit, which runs counter to the message of Proposition 1. One interpretation of Proposition 4 is that it demonstrates the extent to which Proposition 1 is a knife-edge result, relying as it does on a single cost (or, as in Proposition 3, a discrete set of costs) for all players. But Proposition 1 is a limiting result, intended to illustrate the intuition that introducing signaling concerns about the quality of one's information may increase voter turnout significantly. The limiting result, that voter turnout does not disappear, may be sensitive to the assumption that costs take at most a finite number of values. But I now show via numerical example that, for reasonable finite values of N, the signaling return to voting may be quite substantial, even when costs are allowed to be continuous.

3.5 Numerical example with continuous costs and finite N

To that end, assume either candidate is favored with equal probability, that $\omega = \{0, 1\}$, each valuation equally likely and independently distributed across players, and that there are two sorts of voters: Half learn nothing about the value of voting, and so attach an expected worth to the value of swaying the election of 0, and the other half learn the value of swaying the election perfectly, so that half learn that $\omega = 1$, and the other half learn that $\omega = 0$. Let the first, uninformed group of voters be represented by $\underline{\gamma} = 0$, and the second, informed group of voters be represented by $\overline{\gamma} = 1$. Costs, we assume, are uniformly distributed on the interval $(\underline{k}, 1]$, with $\underline{k} > 0$, so that a type which learns that $\omega = 0$ will never vote. We assume that, if players are indifferent between voting or not, they break the tie in favor of not voting (this is without loss of generality, since indifference is a zero-probability event).

First, consider the case in which there is no signaling concern. Since player's valuations are independent, the probability any player attaches to being pivotal is independent of his valuation, and hence, is the same across all players. An uninformed player has an expectation on ω of $\frac{1}{2}$, and so votes if and only if

$$\frac{1}{2}\mathrm{P}(\mathrm{piv}) \geq k,$$

hence, the probability that an uninformed voter votes, is

$$V_{\underline{\gamma}} = \begin{cases} 0 & \frac{1}{2}\mathrm{P}(\mathrm{piv}) \leq \underline{k} \\ \left(\frac{1}{2}\mathrm{P}(\mathrm{piv}) - \underline{k}\right)/(1-\underline{k}) & \frac{1}{2}\mathrm{P}(\mathrm{piv}) > \underline{k}. \end{cases}$$

Similarly, informed players who learn that $\omega = 1$ vote with probability

$$V_{\overline{\gamma},\omega=1} = \begin{cases} 0 & \mathrm{P}(\mathrm{piv}) \leq \underline{k} \\ \left(\mathrm{P}(\mathrm{piv}) - \underline{k}\right)/(1-\underline{k}) & \mathrm{P}(\mathrm{piv}) > \underline{k}. \end{cases}$$

In equilibrium, at least one player type votes with positive probability, since otherwise, every voter is pivotal with probability 1. Hence, the pivot probability must at least satisfy

$\mathrm{P(piv)} > \underline{k}$, and the probability that an informed player who learns that $\omega = 1$ is

$$V_{\overline{\gamma},\omega=1} = \left(\mathrm{P(piv)} - \underline{k}\right)/(1-\underline{k}),$$

while for informed players who learn that $\omega = 0$ never vote,

$$V_{\overline{\gamma},\omega=0} = 0.$$

The unconditional probability that a player votes, V, can therefore be written as

$$V = \underbrace{\left(\tfrac{1}{2}\mathrm{P(piv)} - \underline{k}\right)/(1-\underline{k})}_{\text{Uninformed voters}} \chi_{\mathrm{P(piv)}>\underline{k}} + \tfrac{1}{2}\underbrace{\left(\mathrm{P(piv)} - \underline{k}\right)/(1-\underline{k})}_{\text{Informed voters with } \omega=1}. \tag{15}$$

The probability that an informed voter votes is higher than the probability that an uninformed voter votes (see Figure **??**, it is important that F be convex for this to be true.) Using the Poisson nature of the game, we can compute the pivot probability explicitly as a function of the unconditional probability that another player votes as

$$\mathrm{P(piv)} = \sum_{j=0}^{\infty} \left(\frac{e^{-VN/2}(VN/2)^j}{j!} \right)^2 \left(1 + \frac{VN/2}{j+1} \right). \tag{16}$$

Equations (15) and 16 may be jointly solved numerically. When, e.g., $\underline{k} = 0.1$, then the expected voter turnout when $N = 100000$ (as in Myerson (2000)) is 35 voters, and the pivot probability is slightly higher than 0.1 (as it must be, in order to induce roughly 35 of the informed players who believe $\omega = 1$ to vote).

When there are signaling concerns, the marginal signaling return to voting, U_S, is given by

$$U_S = \frac{\overline{\gamma} V_{\overline{\gamma}}}{\overline{\gamma} V_{\overline{\gamma}} + \underline{\gamma} V_{\underline{\gamma}}} - \frac{\overline{\gamma}(1 - V_{\overline{\gamma}})}{\overline{\gamma}(1 - V_{\overline{\gamma}}) + \underline{\gamma}(1 - V_{\underline{\gamma}})}.$$

The unconditional probability that a player votes is then (note that informed players who believe $\omega = 0$ never vote, since if they vote then all players strictly prefer to vote, which cannot be supported in equilibrium)

$$V = \underbrace{\left(\tfrac{1}{2}\mathrm{P(piv)} + U_S - \underline{k}\right)/(1 - \underline{k})}_{\text{Uninformed voters}} \chi_{\frac{1}{2}\mathrm{P(piv)} + U_S > \underline{k}} + \tfrac{1}{2} \underbrace{\left(\mathrm{P(piv)} + U_S - \underline{k}\right)/(1 - \underline{k})}_{\text{Informed voters with } \omega = 1}.$$

Again, when $\underline{k} = 0.1$, $N = 100000$, and, now, $\underline{\gamma} = 0$, $\overline{\gamma} = 100$, we may solve this system of equations numerically to yield an expected voter turnout of $\approx 29,000$, and a marginal signaling return of ≈ 0.5. The pivot probability is now approximately 0.0001%. Different specifications yield different results, this exercise is intended only to demonstrate that, even with continuously distributed costs, very high levels of voter participation (around 1 in 3 voting with signaling concerns, versus 35 in 100,000) may be sustained even when the pivot probability is very low.

4 Conclusion

The model presented here demonstrates that by introducing reputational concerns for appearing to be well-informed into a model of uncertain voters, high levels of voter turnout may be sustained, even in large games when the chance of swaying the election is very small. I conclude here by asking what sort of empirical evidence would support or refute the thesis that voters rationally vote to signal their education to others.

In particular, how this behavior is empirically distinguished from, say, simply a taste for voting, or a model of voting as signaling in the sense of Spence (1973), is an interesting question. Taking education to be a proxy for informedness, a novel prediction of the model is that there should be a high degree of correlation between one's own education, one's own voting behavior, and the education of one's immediate neighbors. If, for example, one lives in a neighborhood comprised entirely of well-educated neighbors, then one has no signaling incentive to vote, and voting participation should be low. Similarly, neighborhoods comprised entirely of uneducated neighbors would have low voting participation rates, it should be the diverse neighborhoods with a mix of well educated and uneducated people with the highest rates voting participation. This sort of phenomenon would identify this model from models in which individuals, say, simply have a taste for voting, but to the extent that education is correlated with wealth, and wealth correlated with the cost of voting, it makes the same predictions as a model of voting as Spencian signaling. Instead, the model predicts that neighborhoods diverse in wealth, but not education, should have low voting participation rates, while neighborhoods diverse in education, but not wealth, should have high voter participation rates.

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A Omitted proofs

Lemma 1. *Let* $\{\sigma^N\}_{N=1}^{\infty}$ *be a series of strategy profiles on the corresponding sequence of games,* $\{\Gamma^N\}_{n=1}^{\infty}$*. Let* $\{V^N\}_{N=1}^{\infty}$ *denote the corresponding expected percentage of players voting, as defined by* (7)*, such that the sequence* $\{V^N\}_{N=1}^{\infty}$ *is bounded from below by zero. Then*

$$\lim_{N\to\infty} P_{\sigma^N}(piv) = 0. \tag{8}$$

Proof. Consider first the probability that the vote is tied, and denote this event T for 'tie'. Since $G(N)$ is Poisson, the number of people who vote for candidate C is a Poisson random variable with mean NV_C, and so this can be written explicitly as

$$\mathrm{P}(T \mid \sigma_N) = \sum_{k=0}^{\infty} \frac{e^{-NV_A}(NV_A)^k}{k!} \frac{e^{-NV_B}(NV_B)^k}{k!}. \tag{17}$$

Then re-arranging terms yields

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{e^{-NV_A}(NV_A)^k}{k!} \frac{e^{-NV_B}(NV_B)^k}{k!} &= \sum_{k=0}^{\infty} \frac{e^{-N(V_A+V_B)}(N^2V_AV_B)^k}{k!^2} \\
&= e^{-N(V_A+V_B)} \sum_{k=0}^{\infty} \frac{(N^2V_AV_B)^k}{k!^2}.
\end{aligned}$$

The infinite series is a Bessel function of order one at $N\sqrt{V_AV_B}$, denoted $I_0(N\sqrt{V_AV_B})$. For large values of x, the following is an approximation to I_0:

$$I_0(x) \approx \frac{e^x}{\sqrt{2\pi x}}.$$

Therefore, for N large and at least one of V_A, V_B positive, the probability of a tie is approximated by

$$\mathrm{P}(T \mid \sigma_N) \approx \frac{1}{2\sqrt{\pi N \sqrt{V_A V_B}}} e^{-N(V_A + V_B - 2\sqrt{V_A V_B})}. \tag{18}$$

As N grows large V is bounded from below, so at least one of V_A or V_B must be bounded from below, so that $V_A + V_B - 2\sqrt{V_A V_B} \geq B_0 > 0$ is bounded from below for all N. So

$$\lim_{N\to\infty} \mathrm{P}(T \mid \sigma_N) \leq \lim_{N\to\infty} \frac{1}{\sqrt{N\sqrt{V_A V_B}}} e^{-NB_0} = 0,$$

which establishes the result for the event T.

We also must consider the case in which the player's preferred candidate is one vote behind. Similar reasoning establishes that the probability of this event also vanishes. The pivot probability is the sum of the probabilities for the event there is a tie and the event a player's preferred candidate is one vote behind, so the pivot probability vanishes as N grows large. □

Lemma 3. *Let P_{γ_0} and P_{γ_1} be two information structures. Then, if P_{γ_1} is strictly more informative than P_{γ_1} in the Blackwell sense, it is more informative in the max-min sense.*

Proof. First, say that P_{γ_1} is strictly more informative than P_{γ_0} in the Blackwell sense. Blackwell's theorem implies that the signal s under γ_0 is a garbling of the signal s under γ_1, which we can write as

$$\mathrm{P}_{\gamma_0}(s \mid \omega) = \sum_{s'} h(s \mid s')\mathrm{P}_{\gamma_1}(s' \mid \omega). \tag{19}$$

Here $h(s \mid s')$ is some kernel mass function. Then we have for all signals s,

$$\begin{aligned}
\sum_{\omega} g(\omega) \mathrm{P}_{\gamma_0}(\omega \mid s) &= \sum_{\omega} g(\omega) \frac{\mathrm{P}_{\gamma_0}(s \mid \omega) \mathrm{P}_{\gamma_0}(\omega)}{\mathrm{P}_{\gamma_0}(s)} \\
&= \sum_{\omega} \frac{g(\omega) \mathrm{P}_{\gamma_0}(\omega)}{\mathrm{P}_{\gamma_0}(s)} \sum_{s'} h(s \mid s') \mathrm{P}_{\gamma_1}(s' \mid \omega) \\
&= \sum_{s'} h(s \mid s') \frac{\mathrm{P}_{\gamma_1}(s')}{\mathrm{P}_{\gamma_0}(s)} \sum_{\omega} g(\omega) \mathrm{P}_{\gamma_1}(\omega \mid s') \\
&= \sum_{s'} h(s' \mid s) \sum_{\omega} g(\omega) \mathrm{P}_{\gamma_1}(\omega \mid s') \\
&\leq \sum_{\omega} g(\omega) \mathrm{P}_{\gamma_1}(\omega \mid s_{\gamma_1}), \qquad (20)
\end{aligned}$$

where here s_{γ_1} denotes the signal which maximizes the expectation of $g(\omega)$ for type γ_1. (20) shows that the maximum expectation is at least weakly increasing in γ. Similar reasoning shows that the minimum expectation is weakly decreasing.

Now say that $\mathrm{E}_{\gamma_1}[g(\omega) \mid s]$ is non-constant. Then since $h(s' \mid s)$ has full support on s', the weak inequality in (20) becomes a strict inequality. $\square$

Proposition 1. *Let $\{\sigma^N\}_{N=1}^{\infty}$ be a sequence of equilibria of games, each with expected population N. Denote the corresponding voting percentages (7) and (6) in these games by V^N, V_γ^N, and the marginal electoral and signaling benefits by U_S^N, U_E^N. Then*

1. *U_S^N and U_E^N converge to U_S^∞, U_E^∞, satisfying*

$$U_S^\infty = k$$

$$U_E^\infty = 0.$$

The pivot probability $P_{\sigma^N}(piv)$ converges to 0.

2. Voting percentages are bounded away from zero,

$$\liminf_{N\to\infty} V^N > 0. \tag{9}$$

Proof. First, we show (9) by contradiction. Fix a sequence of equilibria as described in the proposition. Assume to the contrary there is an infinite increasing subsequence of $\mathbb{N}$, denoted $\{N_i\}_{i=1}^{\infty}$, with the property that V^{N_i} converges to zero. For simplicity, take $N_i = i$, that is, assume that we have a sequence of equilibria with the property that $V^N \vec{N} 0$. (Analysis of the more general case proceeds similarly.)

Pick a candidate, C, and let

$$\begin{aligned}
g^N(\omega) &:= \omega \mathrm{P}_{\sigma^N}(\text{piv} \mid \omega, C) \\
R^N &:= \{\gamma \mid \exists s \text{ s.t. (BR) holds for } (\gamma, s, C)\} \\
\underline{\gamma}_N &:= \inf R^N.
\end{aligned}$$

$g^N(\omega)$ is the marginal electoral value of voting conditioning on the value of swaying the election to a given voter. R^N is the γ type components of players for whom some signal exists for which they at least weakly prefer to vote. $\underline{\gamma}_N$ is the lowest such type component. (By continuity, $\underline{\gamma}_N \in R^N$.)

The following facts are useful and follow directly from the assumption of max-min informativeness. They state that if a player with some level of information precision, γ, prefers to vote after seeing some signal, then a player with better information also must at prefer to vote at some signal.

Lemma 4. *R^N is an interval, $[\underline{\gamma}_N, \overline{\gamma}]$.*

Proof. Say a player of type (γ, s, C) weakly prefers to vote. Then $\forall \gamma' > \gamma$,

$$
\begin{aligned}
0 &\leq \mathrm{E}_\gamma[g^N(\omega) \mid s] + U_S^N - k \\
&\leq \mathrm{E}_\gamma[g^N(\omega) \mid \overline{s}] + U_S^N - k \\
&\leq \mathrm{E}_{\gamma'}[g^N(\omega) \mid \overline{s}] + U_S^N - k,
\end{aligned}
$$

by max-min informativeness. If (γ, s_γ) strictly prefers to vote, then the first inequality is strict and the second result follows. $\square$

Now consider the limit points of $\{\underline{\gamma}_N\}_{N=1}^\infty$. To derive a contradiction, we will consider four possible sets in which the limit points may lay, and show that our assumption that voting percentages disappear implies that the limit points lay in none of these sets, which therefore implies that the set of limit points of $\{\underline{\gamma}_N\}_{N=1}^\infty$ is empty. However, $\{\underline{\gamma}_N\}_{N=1}^\infty$ lays in a compact space, $[\underline{\gamma}, \overline{\gamma}] \cup \infty$, and so it does have limit points, which provides the contradiction to our assumption. These four sets are $\{\underline{\gamma}\}, (\underline{\gamma}, \overline{\gamma}), \{\overline{\gamma}\}$, and $\{\infty\}$.

1. First, consider the set $(\underline{\gamma}, \overline{\gamma})$. We show that $\{\underline{\gamma}_N\}_{N=1}^\infty$ has no limit points in this set by contradiction. Say there were an infinite subsequence indexed by $\{N_i\}$ such that $\lim_{i\to\infty} \underline{\gamma}_{N_i} = \underline{\gamma}_\infty \in (\underline{\gamma}, \overline{\gamma})$. For simplicity and without loss of generality, take $N_i = i$.

   Fix some N and consider a player with type component $\underline{\gamma}_N$. By construction, there is a signal, $s_{\underline{\gamma}_N}$, at which this player is indifferent between voting and not. By Lemma 4, all players with type components (γ, s_γ, C), where $\gamma \geq \underline{\gamma}_N$ must at least weakly prefer to vote.

   Can it be in addition that there are players who *strictly* prefer to vote with γ type components bounded away from $\overline{\gamma}$ for large N? If so, in the notation of Lemma 4, we must have $\underline{\gamma}_N < \overline{\gamma}$, by continuity. Lemma 4 then implies that there is a positive measure of γ types who strictly prefer to vote bounded away from 0 for large N. Call this bound B_0. Then for large N, the *ex-ante* probability of voting occurring satisfies

$$V^N \geq \sum_{s \in S} \mathrm{P}(s) B_0 \, ds \geq \min_s \mathrm{P}(s) B_0 > 0,$$

which contradicts the assumption that $V^N \to 0$.

So for large N, the measure of type components γ for which some player with that type component strictly prefers to vote must approach zero. More precisely, for every interval $[\underline{\gamma}_N, \overline{\gamma} - \varepsilon], \varepsilon > 0$, there exists an N so that every player with type component $\gamma \in [\underline{\gamma}_N, \overline{\gamma} - \varepsilon]$ weakly prefers not to vote. (It is an interval by Lemma 4.)

Pick ε small enough so that this is a positive measure set. All such players weakly prefer not to vote:

$$\mathrm{E}_\gamma[g^N(\omega) \mid s_\gamma] \leq k - U_S^N \, \forall \gamma \in [\underline{\gamma}_N, \overline{\gamma} - \varepsilon], s.$$

In addition, by the construction of $\underline{\gamma}_N$, after seeing s_γ the player of type (γ, s_γ) is indifferent:

$$\mathrm{E}_\gamma[g^N(\omega) \mid s_\gamma] = k - U_S^N \, \forall \gamma \in [\underline{\gamma}_N, \overline{\gamma} - \varepsilon],$$

or, by the definition of s_γ,

$$\max_s \mathrm{E}_\gamma[g^N(\omega) \mid s] = k - U_S^N \, \forall \gamma \in [\underline{\gamma}_N, \overline{\gamma} - \varepsilon].$$

That is, the upper bound on player's conditional expectation of $g(\omega)$ is constant across $\gamma \in [\underline{\gamma}_N, \overline{\gamma} - \varepsilon]$. The definition of max-min informativeness then implies that for all signals,

$$\mathrm{E}_\gamma[g^N(\omega) \mid s] = k - U_S^N \, \forall \gamma \in [\underline{\gamma}_N, \overline{\gamma} - \varepsilon], s \in S$$

and so

$$\min_s \mathrm{E}_\gamma[g^N(\omega) \mid s] = k - U_S^N \, \forall \gamma \in [\underline{\gamma}_N, \overline{\gamma} - \varepsilon]. \tag{21}$$

Recall from the definition of max-min informativeness that $\min_s \mathrm{E}_\gamma[g^N(\omega) \mid s]$ is weakly decreasing in γ. So equation (21) implies that $\mathrm{E}_\gamma[g^N(\omega) \mid s]$ is constant across s for all types with $\gamma \in [\underline{\gamma}, \overline{\gamma} - \varepsilon]$.

Intuitively, the maximum possible dispersion of beliefs for players with type component less than $\underline{\gamma}_N$ is 'pinched shut' by the dispersion of players with a higher type component. More precisely, for $\gamma < \underline{\gamma}_N$ and any $\hat{s}$, we have

$$\begin{aligned}
\mathrm{E}_\gamma[g^N(\omega) \mid \hat{s}] &\geq \min_s \mathrm{E}_\gamma[g^N(\omega) \mid s] \\
&\geq \min_s \mathrm{E}_{\underline{\gamma}_N}[g^N(\omega) \mid s] \\
&= \max_s \mathrm{E}_{\underline{\gamma}_N}[g^N(\omega) \mid s] \\
&= k - U_S^N,
\end{aligned}$$

so that all types below $\underline{\gamma}_N$ at weakly prefer to vote. $\underline{\gamma}_N$ was constructed so that there could not be any such types, therefore, $\underline{\gamma}_N = \underline{\gamma}$. This proves by contradiction that we cannot have any limit points in $(\underline{\gamma}, \overline{\gamma})$.

2. Now consider whether there might be a limit point in $\{\underline{\gamma}\}$. Assume there is an infinite subsequence $\{N_i\}_{i=1}^{\infty}$ of $\{\underline{\gamma}_N\}_{N=1}^{\infty}$ so that $\lim_{i \to \infty} \underline{\gamma}_{N_i} = \underline{\gamma}$. For simplicity and without

loss of generality, assume that $N_i = i$. Previous reasoning established that for large N, all types with $\gamma \in [\underline{\gamma}, \overline{\gamma} - \varepsilon]$ and any signal were indifferent between voting and not, so that

$$\mathrm{E}_\gamma[g(\omega) \mid s] = k - U_S^N \, \forall \gamma, s.$$

Does such a function $g(\omega)$ exist? If it did, then it would solve the system of linear equations

$$\sum_{\omega \in \omega} \mathrm{P}_\gamma(\omega \mid s) g(\omega) = k - U_S^N, \forall s, \tag{22}$$

$$g(0) = 0. \tag{23}$$

This is a system of $|S| + 1$ equations in $|\omega|$ unknowns. Recall that $|S| \geq |\omega|$, and there is at least one type component $\gamma \in (\underline{\gamma}, \overline{\gamma})$ whose information structure has rank $|\omega|$. This implies that no such solution exists. (See lemma 6 in the appendix for a proof.)

Therefore (22) has no solution and so by contradiction, $\underline{\gamma}_N$ cannot have a limit point at $\underline{\gamma}$.

3. Now consider whether there might be a limit point in $\{\overline{\gamma}\}$. Assume there is an infinite subsequence $\{N_i\}_{i=1}^{\infty}$ of $\{\underline{\gamma}_N\}_{N=1}^{\infty}$ so that $\lim_{i \to \infty} \underline{\gamma}_{N_i} = \overline{\gamma}$. For simplicity and without loss of generality, assume that $N_i = i$. Eventually, only high types vote, and they vote with vanishing probability. This implies that the signaling return to voting approaches

$$\lim_{N \to \infty} U_S^N = \overline{\gamma} - \mathrm{E}[\gamma] > k,$$

by assumption. But then eventually $U_S^N > k$, and so every type strictly prefers to vote, a contradiction.

4. This leaves as our only remaining case the one in which $\lim_{N\to\infty} \underline{\gamma}_N = \infty$. Since we cannot have finite $\underline{\gamma}_N$ which satisfy $\underline{\gamma}_N > \overline{\gamma}$, this implies that $\underline{\gamma}_N$ is infinite in finite time, which by definition means that eventually all players strictly prefer not to vote. But then $\mathrm{P}(\mathrm{piv}) = 1$ independently of type and $U_S = 0$. By assumption, there is a type γ, s for whom $\mathrm{E}_\gamma[\omega \mid s] > k$, and so who strictly prefers to vote, so again we have a contradiction.

   Therefore $\underline{\gamma}_N$ has no limit points. But as previously argued, this is not possible. So by contradiction we've established that V^N is bounded away from 0 and so established equation (9).

Lemma 1 then implies that the pivot probabilities converge to zero, and so the expected marginal electoral return to voting converges to zero.

To see that the expected marginal signaling return to voting converges to k, note that $U_S^N \leq k$—otherwise, all players would strictly prefer to vote, and we would have $U_S^N = 0$. Can it be that $U_S^N \leq k - \varepsilon < k$? If so, as $U_E^N \to 0$, eventually the incentive constraints of all types would satisfy

$$U_E^N(t) + U_S^N - k \leq U_E^N(t) - \varepsilon < 0,$$

so all types would strictly prefer not to vote, and as we have already seen that cannot be an equilibrium. $\square$

Lemma 2. *Say that γ orders F with a rotation order, and say valuations are independently and identically distributed. Then the probability a voter attaches to being pivotal is independent of the value the voter attaches to swaying the election, i.e.,*

$$E_\gamma[\omega P(piv \mid t) \mid s] = E_\gamma[\omega \mid s] P(piv \mid C). \tag{11}$$

Furthermore, strategy profiles in equilibrium are characterized by cutoffs $\underline{s}(\gamma, C)$, such that

$$\sigma(\gamma, s, C) \in \begin{cases} \{0\} & s < \underline{s}(\gamma, C) \\ [0,1] & s = \underline{s}(\gamma, C) \\ \{1\} & s > \underline{s}(\gamma, C) \end{cases}, \tag{12}$$

and these cutoffs have the feature that $\underline{s}(\gamma, C)$ is monotone in γ.

Proof. Equation (11) follows from the independence assumptions:

$$\begin{aligned} \mathrm{E}_\gamma[\omega \mathrm{P}(\text{piv} \mid t) \mid s] &= \mathrm{E}_\gamma[\omega \mathrm{P}(\text{piv} \mid C) \mid t] \\ &= \mathrm{E}_\gamma[\omega \mid s] \mathrm{P}(\text{piv} \mid C), \end{aligned}$$

since by assumption the valuations of other players are independent of s, γ, and the event piv only depends on the valuations of the other players.

Then the left hand side of (BR) could be written

$$\frac{1}{2}\mathrm{E}_\gamma[\omega \mid s]\mathrm{P}(\text{piv} \mid C) + U_S - c,$$

which since $\mathrm{E}_\gamma[\omega \mid s]$ is strictly increasing in s establishes the existence of cutoffs that characterize the equilibrium strategy profile. Recall that there exists a signal $\underline{s}$ such that $\mathrm{E}_\gamma[\omega \mid \underline{s}]$ is constant across γ. Then if $k - U_S > \mathrm{E}_\gamma[\omega \mid \underline{s}]$, and type (γ, s, C) votes, then $s > \underline{s}$ and so type (γ', s, C), where $\gamma' > \gamma$, also prefers to vote because

$$k - U_S \leq \mathrm{E}_\gamma[\omega \mid s]\mathrm{P}(\text{piv} \mid C) < \mathrm{E}_{\gamma'}[\omega \mid s]\mathrm{P}(\text{piv} \mid C).$$

In this case, V_γ is increasing.

Similarly, if $k - U_S < \mathrm{E}_\gamma[\omega \mid \underline{s}]$, type γ, s prefers not to vote, and $\gamma' < \gamma$,

$$k - U_S \geq \mathrm{E}_\gamma[\omega \mid s]\mathrm{P}(\text{piv} \mid C) > \mathrm{E}_{\gamma'}[\omega \mid s]\mathrm{P}(\text{piv} \mid C).$$

In this case, V_γ is decreasing.

Finally, if $k - U_S = \mathrm{E}_\gamma[\omega \mid \underline{s}]$, then all types who see $s > \underline{s}$ strictly prefer to vote, and all types who see $s < \underline{s}$ strictly prefer not to vote, independently of γ, so V_γ is constant. □

Proposition 2. *Again, let $\{\sigma^N\}_{N=1}^{\infty}$ be a sequence of equilibria of games with expected population N. Denote the corresponding voting patterns in these games by $V^N, V_\gamma^N, U_S^N, U_E^N$. If player's valuations are independently and identically distributed, and γ orders F according to a rotation order, then*

1. *$\sigma^N, V^N, V_\gamma^N, U_S^N, U_E^N$ converge to $\sigma^\infty, V^\infty, V_\gamma^\infty, U_S^\infty, U_E^\infty$. The pivot probability P(piv $\mid \sigma_N$) converges to 0. (Convergence.)*

2. *$V^\infty > 0$, $U_S^\infty = k$, and $U_E^\infty = 0$. (Voter turnout remains high in large games.)*

3. *V_γ^∞ is non-decreasing in γ. (High types vote more.)*

4. *V^∞ and V_γ^∞ are decreasing in k, and U_S^∞ is increasing in k. (A higher cost of voting leads to less voting from all types. More low types than high types stop voting as costs increase.)*

5. *There exist information structures F, F' such that $E_F[\gamma] > E_{F'}[\gamma]$, but voting participation is lower in equilibrium under F then F'. (Even though voting is positively correlated with γ, increasing average γ doesn't necessarily increase voter turnout.)*

Proof. For this proof, we construct limiting equilibria in a manner similar to Aytimur et al. (2014). This construction is simpler if we re-imagine s to be a continuous signal distributed according to $F_\gamma(s \mid \omega)$ which is *ex-ante* distributed uniformly on $[0,1]$. This is a more general construction (Lehmann, 1988) if F is allowed to be constant. [13] But for simplicity I take $F_\gamma(s \mid \omega)$ to be strictly increasing and continuous. A similar proof holds for the discrete case, but requires the use of cumbersome correspondences to handle mixed strategies, and so is omitted in favor of a proof which illustrates the key ideas.

For further simplicity, consider the case where preferences for the two candidates are symmetric, so that $\rho_A = \rho_B = \frac{1}{2}$. This implies that the pivot probability is independent of which candidate a player prefers. Accordingly I stop conditioning on the candidate for the remainder of the proof.

A similar proof to the one for Proposition 1 establishes parts 1 and 2 of Proposition 2. We are now interesting in constructing limiting strategies, that is, we are interested in

$$\sigma^\infty := \lim_{N\to\infty} \sigma^N.$$

By Lemma 2, $\{\sigma^N\}_{N=1}^\infty$ is a sequence of cutoff strategies; hence, σ^∞, if it exists, is also a cutoff strategies. To find the limiting cutoff strategies, we find cutoffs consistent with equilibrium-type behavior which generate some signaling value. We know that the signaling value must converge to k in the limit, therefore, σ^∞ must be a strategy profile which delivers a marginal signaling return of k. If it is the case that, for marginal signaling returns close to k, the strategy profile which delivers that marginal signaling value is unique, then the limit of $\{\sigma^N\}_{N=1}^\infty$ must exist and σ^∞ must be the unique strategy profile which delivers . Pick some candidate signaling value U_S^0, fix some pivot probability p, and let $\underline{s}_p(\gamma)$ denote the

[13] We can construct a continuous signal from a discrete signal in the following way: First, number the signal space as $S = \{1, 2, \ldots, |S|\}$. Second, split the interval $[0,1]$ into $|S|$ distinct intervals, $[0, 1/|S|), [2/|S|, 3/|S|), \ldots, [(|S|-1)/|S|, 1]$. Our transformed signal s' will be constructed by first observing the realization of s, then drawing uniformly from $[(s-1)/|S|, s/|S|)$. s' is equivalent to s but has a continuous piecewise linear cumulative distribution function. Now, transform the signal again to s'' so that $s'' = F(s' \mid \omega)$.

solutions to

$$\mathrm{E}_\gamma[\omega \mid \underline{s}_p(\gamma)]p - k + U_S^0 = 0. \tag{24}$$

(If no such solution exists because (24) is strictly negative for all s, then let $\underline{s}_p(\gamma) = 1$. If (24) is strictly positive for all s, then let $\underline{s}_p(\gamma) = 0$.) The cutoffs $\underline{s}_p(\gamma)$ are the best response strategies to the signaling value U_S^0.

Using $\underline{s}_p(\gamma)$ we can then construct the corresponding signaling return to voting, assuming $\underline{s}_p(\gamma)$ is the strategy profile played. It is a straightforward application of Bayes' rule:

$$\begin{aligned}
U_S^1(U_S^0) &:= \mathrm{E}[\gamma \mid \underline{s}_p, v] - \mathrm{E}[\gamma \mid \underline{s}_p, \neg v] \\
&= \frac{1}{\mathrm{P}(v)} \int_\gamma (1 - \underline{s}_p(\gamma))\, dF(\gamma) - \frac{1}{\mathrm{P}(\neg v)} \int_\gamma \underline{s}_p(\gamma)\, dF(\gamma) \\
&= \frac{1}{\mathrm{P}(v)} \int_\gamma 1 - \frac{\underline{s}_p(\gamma)}{\mathrm{P}(\neg v)}\, dF(\gamma).
\end{aligned}$$

We are now interested in the behavior of the inverse of U_S^1 near k:

Lemma 5. *For all p, there exists a unique candidate signaling level U_p^* so that $U_S^1(U_p^*) = k$, satisfying $U_p^* = (1-p)k + pU_1^*$.*

Proof. First, consider $U_S^0 = \overline{U} := k - p\mathrm{E}_{\overline{\gamma}}[\omega \mid s = 1] + \varepsilon$, for some small ε. (ε is necessary because the firm's beliefs may be discontinuous for $\varepsilon = 0$ and so U_S^0 is set valued at $k - p\mathrm{E}_{\overline{\gamma}}[\omega \mid s = 1]$.)

Since γ orders F_γ via a rotation order, the player of type $(\overline{\gamma}, s = 1)$ has the highest expectation of all the types of the marginal electoral value to voting. So (BR) for all types (γ, s), $\gamma < \overline{s}$, satisfies

$$p\mathrm{E}_\gamma[\omega \mid s] + \overline{U} - k = p\left(\mathrm{E}_\gamma[\omega \mid s] - \mathrm{E}_{\overline{\gamma}}[\omega \mid s = 1] - \varepsilon\right) \underbrace{< 0}_{\text{For } \varepsilon \text{ small.}}$$

That is, for ε small only types close to $\overline{\gamma}$ vote, and so

$$\lim_{\varepsilon \to 0} U_S^1(\overline{U}) = \overline{\gamma} - \mathrm{E}[\gamma] > k.$$

Similar reasoning for $U_S^0 = \underline{U} = k - p\mathrm{E}_{\overline{\gamma}}[\omega \mid s = 0] - \varepsilon$ establishes that

$$\lim_{\varepsilon \to 0} U_S^1(\underline{U}) = \mathrm{E}[\gamma] - \overline{\gamma} < -k.$$

U_S^1 is continuous on $[\underline{U}, \overline{U}]$ for $\varepsilon > 0$. This follows by Lemma 2 and the assumption that $F(s \mid \omega)$ is continuous. The intermediate value theorem then implies that there is at least one signaling level U_p^*.

To see that it is also unique, I claim that $U_S^1(U_S^0)$ is decreasing.

That $U_p^* = (1-p)k + pU_1^*$ follows because if $\underline{s}_1(\gamma), U_1^*$ solves (24) for $p = 1$, then

$$\begin{aligned}
\mathrm{E}_\gamma[\omega \mid \underline{s}_1(\gamma)]p - k + U_p^* &= \mathrm{E}_\gamma[\omega \mid \underline{s}_1(\gamma)]p - k + \big((1-p)k + pU_1^*\big) \\
&= p\big(\mathrm{E}_\gamma[\omega \mid \underline{s}_1(\gamma)] - k + U_1^*\big) \\
&= 0.
\end{aligned}$$

Note that this also shows that $\underline{s}_p(\gamma)$ is constant across p. □

I claim that $\underline{s}_1(\gamma)$ is the limiting strategy profile in equilibrium. This follows because $U_S^N \to k$, and $\underline{s}_1(\gamma)$ is the unique strategy profile which delivers $U_S = k$. By continuity it must be the limiting strategy profile in equilibrium.

For example, say that the signal structure is given by a linear experiment, so that conditional expectations are linear:

$$\mathrm{E}_\gamma[\omega \mid s] = \gamma\left(s - \frac{1}{2}\right) + \frac{1}{2}, \gamma \sim U[1/2, 1].$$

For fixed U_S, the corresponding cutoffs are given by solving

$$\gamma\left(\underline{s}(\gamma)-\frac{1}{2}\right)+\frac{1}{2}=k-U_S,$$

which implies $\underline{s}(\gamma)=\frac{1}{2}+\frac{k-U-\frac{1}{2}}{\gamma}$. This establishes the result. □

Lemma 6. *If type γ's information matrix has rank $|\omega|$, then no solution to the linear problem given by (22) and (23) exists.*

Proof. Denote a player with type γ's information matrix as $\mathcal{P}=(\mathrm{P}_\gamma(s_i \mid \omega_j))_{ij}$, and let $M=|S|$ and $T=|\omega|$. If $\mathcal{P}$ has rank $|\omega|$, then the matrix of conditional probabilities $A=(\mathrm{P}_\gamma(\omega_j \mid s_i))_{ij}$ also has rank $|\omega|$, because

$$A=\underbrace{\begin{pmatrix}\frac{1}{\mathrm{P}_\gamma(s_1)} & 0 & \cdots & 0\\ 0 & \frac{1}{\mathrm{P}_\gamma(s_2)} & & 0\\ \vdots & & \ddots & 0\\ 0 & 0 & \cdots & \frac{1}{\mathrm{P}_\gamma(s_M)}\end{pmatrix}}_{\text{Rank } |S|>|\omega|}\mathcal{P}\underbrace{\begin{pmatrix}\mathrm{P}(\omega_1) & 0 & \cdots & 0\\ 0 & \mathrm{P}(\omega_2) & & 0\\ \vdots & & \ddots & 0\\ 0 & 0 & \cdots & \mathrm{P}(\omega_T)\end{pmatrix}}_{\text{Rank } |\omega|}.$$

Consider the linear problem

$$\sum_{i=1}^{T}\mathrm{P}(\omega_i \mid s)g(\omega_i)=k-U_S\,\forall s.$$

Since A has rank $|\omega|$ any solution is unique, and since P is a probability measure that solution is $g(\omega_i)=k-U_S\,\forall i$. But $g(\omega_i)=k-U_S$ does not satisfy $g(0)=0$, because if $k-U_S=0$ then all players strictly prefer to vote.

□