Product Upgrades and Posted Prices*

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ABSTRACT

This paper studies the dynamic pricing problem of a durable good monopolist with commitment power, when a new version of the good is expected at some point in the future. The new version of the good is superior to the existing one, bringing a higher flow utility. When the arrival is a stationary stochastic process, the corresponding optimal price path is shown to be constant for both versions of the good, hence there is no delay on purchases and time is not used to discriminate over buyers, which is in line with the literature. However, if the arrival of the new version occurs at a commonly known deterministic date, then the price path may decrease over time, resulting in delayed purchases. For both arrival processes, posted prices is a sub-optimal selling mechanism. The optimal one involves bundling of both versions of the good and selling them only together, which can easily be implemented by selling the initial version of the good with a replacement guarantee.

KEYWORDS: durable goods, product upgrades, commitment, posted prices, dynamic mechanism design.

JEL CLASSIFICATION NUMBERS: D42, D82.

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1 Introduction

The literature on durable good monopolies assumes a population of forward looking buyers with heterogeneous valuations for a unique product that a monopolist sells over time. Buyers, that are strategically timing their purchase decisions, have unit demand for the product, and hence leave the market after purchasing. In this general framework, there are two counterbalancing factors governing the pricing decision of the monopolist. First, to be able to sell the product to the agents with lower valuations, the monopolist must decrease the price of the good over time. Second, customers with high valuations might delay their purchases since they predict that the monopolist will decrease the price over time. Hence, decreasing prices is beneficial as it allows the firm to capture the surplus from the demand of the agents with lower valuations, but at the same time it is costly as it causes a delay in the purchases.

The pioneering paper of the literature, Stokey (1979), shows that, if the firm can commit to a price path before starting to sell, then the optimal price path is constant and equal to the static monopoly price. Consequently, all the purchases take place at the beginning of the sales and so there is no delay. This result is significant as it asserts that the time is not used to discriminate over buyers with different valuations, which would have occurred with a decreasing price path and delayed purchases.

In this paper, we turn our attention to the optimal pricing problem when a new version of the existing good is expected to arrive at some point in the future. In contrast to one of the main assumptions of the existing literature, durable goods, for many real life examples, do not persistently stay in the market. Rather, newer and better functioning versions are taking place of the older ones over time. Technology companies such as Apple, Intel, Samsung, and Microsoft are good examples for this. This situation does not hurt the durability of the product that is replaced as customers can still use it after the new version is launched. However it alters the consumer’s preferences as the newer product might offer more benefits. In such an environment, the structure of the buyers’ incentives would be different than the ones in the classical framework. In particular, a buyer does not necessarily leave the market after purchasing a version of the good. He may rather prefer to stay to purchase the newer version as well when it is launched. Or he may abandon to purchase the current version to purchase the newer one. That is to say that the price path of a version of the good not only affects its own sales, but also the sales of the other versions.

We consider a monopolist which is selling two consecutive versions of the same good, with a
restriction that it only sells the most current version at a given period. We assume that it has full commitment power, and can commit to a price path for both versions of the good at the beginning. The monopolist is initially endowed with the first version and the upgrade will take place over time. The timing of the upgrade is not a choice variable in this paper, rather we take it as exogenously given.

The analysis is divided into two parts depending on the specification of the arrival process of the new version of the good. In the first part, the arrival time is stochastic and follows a Poisson process. The optimal price path is shown to be constant for both versions of the good (at different levels) and the price level for the second version of the good is independent from the realized arrival time. Consequently, there are no delays in any purchases. Any purchase of the first version occurs immediately at the beginning, and any purchase of the second version occurs as soon as it arrives into the market. This result is in line with Stokey (1979) as it also suggests that the time is not used as a tool to discriminate over buyers with different valuations. Stationarity is the main reason for this result to carry over to our setting.

In the second part, we considered the case in which the arrival of the newer version occurs at a commonly known certain date. This non-stationary environment comes with a cost of intractability, and to overcome this, we assume a binary type space for buyers’ valuations. It is shown that, in this case, a decreasing price path is possible for some parameters and this gives rise to delayed purchases. That is to say that the time might be used as a discriminatory tool when the arrival of the new version occurs at a commonly known certain date.

Unlike the existing literature on durable good pricing with full commitment power, in our environment, a sale mechanism with posted prices is not an optimal mechanism. Strictly speaking, the anonymous structure of the posted prices (The sales of the new version of the good cannot be conditioned on the first version sales under the regime of posted prices.), comes at a cost for the monopolist. The optimal selling mechanism, on the other hand, requires the monopolist to bundle both versions of the good and selling them only together. More precisely, in this optimal sales mechanism there exists a group of consumers that are purchasing both versions of the good, and the rest of the consumers does not purchase anything. The resulting allocation of this policy cannot be implemented with posted price path which is anonymous by definition.
1.1 Related Literature

Our paper mainly fits into the literature of durable good pricing with full commitment power. The classic reference is Stokey (1979), which shows that the optimal price path is constant and equal to the static monopoly price.\(^1\) Therefore all the buyers either purchase immediately or leave without any purchase, hence the time is not used by monopolist to discriminate over buyers with different valuations.

In our paper, there are both favoring and contradicting results with this mainstream result. We show that, under some circumstances, the optimal posted price path may be decreasing over time. Some other papers have also shown that the optimal posted price path, under full commitment power, may fluctuate over time. In particular, Board (2008) shows that, in a setting with stochastic population, if there is demand heterogeneity over the incoming population of buyers, then the optimal price path might fluctuate over time. This leads some buyers to delay their purchases, hence time is an effective discriminatory tool in such a situation. Another important paper showing a fluctuating price path is Garrett (2012) in which the flow valuations of buyers are stochastic due to the private circumstances. The environment is stationary, as neither the value distributions of the entering buyers nor the stochastic process governing the valuations are time dependent, yet the optimal price path is fluctuating over time. Like our paper, in Garrett (2012), the posted prices is not an optimal selling mechanism. The optimal mechanism involves selling option contracts to purchase the good at future dates. However, the source of the inefficiency regarding the posted prices is different than the one in this paper.

In addition, our paper is also related to the literature on dynamic auctions.\(^2\) A common feature of this literature is that there is a certain time period \(T\), until when the seller must sell his multi-unit indivisible goods. The buyers are entering into and leaving from the market over time. Pai and Vohra (2008), also Board and Skrzypacz (2010) are good examples. They first define the optimal allocations, which follows a simple index rule, and then show how to implement this via monetary transfers. The main difference of our paper from this literature is that, in ours, the number of goods that the seller can sell is not limited whereas in dynamic auction literature the seller is endowed with a certain number of goods.

There are couple of other topics incorporating some features that are conceptually related\(^2\) Bergemann and Said (2011) is a good survey about dynamic auctions.

\(^1\)Actually optimal price path is not unique, there are infinitely many price paths resulting with the same allocation. The important thing is that the initial price level is equal to the static monopoly price and the price level never falls below that.

\(^2\) Bergemann and Said (2011) is a good survey about dynamic auctions.
to ours. The literature of planned obsolescence is one of them. The monopolist that is selling a single good in a dynamic setting strategically arranges the lifetime of the durable good. There are benefits from producing goods with a shorter lifetime as it leads consumers to repurchase again. Bulow (1986) is a good example of this literature. There is an old literature investigating the effect of vintage capital on the aggregate growth of the economy. The basic story of these papers is that, in each period, the machines of production are being improved because of technological progress. And firms are deciding how much to invest in replacing old machines with the newer ones. An example of this literature is Benhabib and Rustichini (1991). Finally, there is a literature regarding the R&I planning of durable good monopolies, see for example Swan (1970), Fishman and Rob (2000). The main concern of these papers is to understand the product development decisions of the firms given a non-strategic buyers side, which is represented with a demand function on the quality of the durable good.

2 Model

Time, $t \in [0, \infty)$, is continuous and $r$ is the common discount factor. There is a monopoly selling a durable good, an initial version of which exists in the marketplace at the beginning of the time ($t=0$), and a newer version will eventually take the place of the existing one. The process governing the arrival of the newer version will be described later. The firm sells only the most current version of the good at a given time; in other words, when the new version arrives the firm is no longer able to sell the earlier version. The cost of the production is normalized to 0 for both versions of the good.

On the demand side, there is a unit mass of buyers that are heterogeneous in terms of their valuations. They can consume at most one unit of the good at a time, and all of them are having a higher value for the newer version of the good. Moreover, we assume that the ratio of valuations for two versions of the good is same for all buyers. In particular, there is a constant $\beta > 1$ such that, a type $x$ buyer has a flow utility $x$ and a flow utility $\beta x$ from consuming the first and the second versions of the good respectively. Buyers’ types follows a continuously differentiatable distribution function $F(x)$ over the unit interval $[0, 1]$. $F(x)$ has full support and a corresponding continuous density function $f(x)$. The buyers are strategically deciding the time of their purchase(s), and also which version(s) of the good to buy. The good is durable and so a buyer may use it forever after purchasing. However, since the flow utility of the newer version is higher, one may want to replace the old one with the newer one. Therefore, buyers do not necessarily leave the market upon purchase,
unlike the existing literature. On the other hand, version-wise strategic delays of purchases may appear. Precisely, a buyer might prefer to wait for the arrival of the new version rather than buying the current version of the good.

The monopolist commits to a price path for both versions of the good at $t = 0$. The price path is consisting of a price level for the first version of the good for each time until the arrival of the new version and also a price level for the new version of the good for each time after its arrival contingent on its arrival time. Note that a posted price that is defined in this way indirectly implies anonymity: the monopolist has to charge the same price for every buyer that is purchasing at the same. In other words, the possibility of conditioning the new version sales to the buyers’ ownership status of the old version is ruled out. This puts a restriction on the monopolist and hence the resulting optimal posted prices will not be the optimal mechanism. Nevertheless, as a benchmark, we also analyzed the case without anonymity. The arrival process of the new version is modeled in two different ways:

2.1 Stochastic Arrival

The second version of the good arrives stochastically with a Poisson arrival process at rate $\lambda$, the realized arrival time is denoted by $T$. The price path that the firm commits at $t = 0$ is contingent on $T$. More precisely, it is of the form: $\left( \{p_t\}_{t \in [0, \infty)}, \{p^T_t\}_{t \in [0, \infty)} \right)_{T \in [0, \infty)}$. The first term is the single price path for the first good, since the arrival is stochastic it is defined over $t \in [0, \infty)$. The second term is the price path for the newer version. Note that there is a different price path for every possible arrival time. In particular, conditional on the arrival time $T$, the term $p^T_t$ is the price level of the second version of the good at $T + t$, i.e. $t$ period after the arrival $T$.

Each buyer decides whether and when to purchase the first version of the good, also whether and when to purchase the second version of the good for each possible arrival time $T$. More precisely, buyer $x$’s decisions are of the form: $\left( t_x, \{t_{x,T}\}_{T \in (0, \infty)} \right)$. The term $t_x$ is the purchase time of the first version of the good; hence if realized $T$ is less than $t_x$, then it means that the buyer does not buy the first version. Therefore, if $t_x = \infty$, then it means that the buyer never purchases the first version of the good. The term $t_{x,T}$, on the other hand, specifies how long after the realized arrival time $T$, buyer $x$ purchases the second version. Hence, the corresponding calendar time of the purchase is $T + t_{x,T}$. Again if $t_{x,T} = \infty$ then the agent does not purchase the second version if the arrival occurs at $T$.

The utility type $x$ buyer, denoted by $U(x)$:
\[
U(x) = \int_0^{t_x} e^{-\lambda T} \left( \int_{T+t_x,T} e^{-r(T+x,T)} e^{-rT} p_{t_x,T} \right) dT
+ \int_{t_x}^{\infty} e^{-\lambda T} \left( \int_{t_x}^{T+t_x,T} e^{-rT} \beta x dt + \int_{T+t_x,T}^{\infty} e^{-rT} \beta x dt - e^{-rt_x} p_{t_x} - e^{-r(T+t_x,T)} p_{t_x,T} \right) dT
\]

The first line captures the contingencies in which arrival occurs before \( t_x \). The utility in these cases depends only on the timing of the purchase of the newer version. For each arrival time \( T \in [0,t_x] \), the corresponding utility is the expression inside the parenthesis. Then, after weighting them with probability of arrival at \( T \) (i.e with \( e^{-\lambda T} \)), we integrate it to get the expectation. The second line accounts for the arrivals occurring after the purchase of the first version, where the expression inside the parenthesis represents the utility corresponding to a specific arrival time \( T \in [t_x,\infty) \). In each contingency the agent acquires a flow utility \( x \) until the purchase of the new version, and \( \beta x \) afterwards. We also discount the payments and integrate them after weighting with the probability of arrival. Note that, if \( t_x = \infty \), i.e the agent never purchases the first generation of the good, then the second line is irrelevant; similarly if \( t_x = 0 \) then the first line is irrelevant.

The profit of the firm, denoted by \( \Pi \):

\[
\Pi = \int_0^1 e^{-\lambda T} e^{-rT} p_{t_x} f(x) dx + \int_0^{\infty} e^{-\lambda T} \left( \int_0^1 e^{-r(T+t_x,T)} p_{t_x,T} f(x) dx \right) dT.
\]

First and second terms are the corresponding profits from the sales of first and second versions of the good respectively. For the first term, the discounted payment of each type of buyer is integrated over the type space. To discount the payment of type \( x \) buyer (i.e \( p_{t_x} \)), we multiply it by \( e^{-rt_x} \) and also by the probability of the event that the arrival does not occur until \( t_x \), which is \( e^{-\lambda t_x} \). For the second term, the inner integral is the level of profit resulting from a specific arrival time \( T \); and the outside integral takes their expectations over possible arrival times.

### 2.2 Deterministic Arrival

In the second part of the paper, we assume that the arrival occurs at a certain time period \( T \), which is commonly known. In this case, the monopolist commits to a single price path:
\{p_t\}_{t \in [0,\infty)}, \text{ where } p_t \text{ is the price level of the first(second) version of the good at time } t \text{ if } t < T (t \geq T). \text{ To the sequel, we characterize the optimal posted prices for both of these arrival processes.}

3 Optimal Posted Prices: Stochastic Arrival

Before delving into the main concern of the paper, we consider some benchmarks to develop a better understanding of the general framework. First we illustrate the canonical durable good monopoly pricing problem, in which there is no product upgrade. The second benchmark considers the case where there is a product upgrade but the monopolist is not restricted to use posted prices. It can rather use any selling mechanism including the non-anonymous ones.

3.1 Benchmark I: Canonical Durable Good Monopoly

This benchmark is analyzed in Stokey (1979).\(^3\) There is only one version of the durable good staying in the market forever. It is a special case of our model in several directions. For example, we can get this canonical model from ours by assuming that \(\lambda = 0\), i.e. the newer version of the good never arrives; or by assuming \(\beta = 1\), i.e. there is no distinction between the first and the second versions of the good for buyers. The monopolist chooses a unique price path \(\{p_t\}_{t \in [0,\infty)}\), and agents decide the timing of their purchases \(t_x\). Corresponding utility of the agent \(x\) is:

\[
U(x) = \int_{t_x}^{\infty} e^{-rt} x dt - e^{-rt_x} p_{t_x} = e^{-rt_x} \left( \frac{x}{r} - p_{t_x} \right), \quad (1)
\]

and the profit of the firm is:

\[
\Pi = \int_0^1 e^{-rt_x} p_{t_x} f(x) dx.
\]

Since the monopolist has full commitment power, his problem is a mechanism design problem. Thanks to the revelation principle we can restrict attention to the direct mechanisms. In particular, the firm asks agents to report their types, and then decides their allocations, i.e. a

\(^3\)Here unlike the analysis presented in Stokey (1979) we follow the general mechanism design approach. We first characterize the incentive constraints and then rewrite the firm’s problem as an optimal allocation problem.
purchase time \( t_x \), and a payment level \( p_x \) in an incentive compatible way. The payment for the agents that are purchasing the good at the same time must be the same, otherwise truthful reporting would not be incentive compatibility. Therefore, for each allocation time there is a corresponding payment level, i.e. we can denote the payments by \( p_t \). The following Lemma, illustrating the nature of the incentive constraints, is an adapted version of the fundamental IC Lemma corresponding to the durable good pricing framework.

**Lemma 1.** The direct mechanism is incentive compatible iff:

i) \( t_x \) is non-increasing with \( x \).

\[ U(x) = U(0) + \frac{1}{r} \int_0^x e^{-rt} \, dr \]

**Proof.** See appendix. \( \square \)

Lemma 1 states that a higher type will not purchase the good at a later time than a lower type. It also asserts that the derivative of \( U(x) \) is proportional to the effective discount \((\frac{1}{r} e^{-rt})\). Since these conditions are both necessary and sufficient for incentive compatibility, the monopolist’s problem can be written as:

\[
\max_{\{p_t\} \in [0, \infty), \{t_x\} \in [0, 1]} \int_0^1 e^{-rt} p_{tx} f(x) \, dx \quad \text{s.t.} \quad t_x \text{ is non-increasing with } x.
\]

\[ \cdot \quad U(x) = U(0) + \frac{1}{r} \int_0^x e^{-rt} \, dr \quad \forall x \]

We can further simplify the above problem, and get rid of the price terms in it. To this respect, by using equation (1) and Lemma 1 we get:

\[
e^{-rt} p_{tx} = e^{-rt} x \frac{1}{r} - \frac{1}{r} \int_0^x e^{-rt} \, dr. \tag{4}
\]

Therefore, the profit of the firm is equal to:

\[
\Pi = \int_0^1 e^{-rt} p_{tx} f(x) \, dx = \int_0^1 \left( x \frac{1}{r} - \frac{1}{r} \int_0^x e^{-rt} \, dr \right) f(x) \, dx.
\]

After integrating it by parts we get the new form of the problem as:

\[
\max_{\{t_x\} \in [0, 1]} \frac{1}{r} \int_0^1 e^{-rt} \left( x - \frac{1 - F(x)}{f(x)} \right) f(x) \, dx \quad \text{s.t.} \quad t_x \text{ is non-increasing.} \tag{2}
\]

\[ ^4 \text{In optimal mechanism } U(0) = 0, \text{ hence we can omit it.} \]
Note that, this problem consists of only the terms $t_x$'s. Therefore, its solution gives us the optimal allocations, and then by using the incentive constraints we get the corresponding price path inducing the optimal allocations. The term $(x - \frac{1-F(x)}{f(x)})$ is referred as the virtual value of type $x$, and as can be seen the monopolist’s problem boils down to maximization of the integral of the discounted virtual valuations. The optimal solution of the above problem is of the following form:

$$t_x = \begin{cases} 
0 & \text{if } x \in [x^*, 1] \\
\infty & \text{otherwise}
\end{cases}$$

In words, there is a threshold type $x^*$ such that all the buyers above this threshold are acquiring the good immediately, and the rest of the buyers are never getting the good. If the virtual valuation function, $x - \frac{1-F(x)}{f(x)}$, is increasing in $x$, then the threshold $x^*$ would be the minimum value of $x$ at which the virtual value function is equal to 0. This is rather intuitive since the monopolist’s problem boils down to maximization of the integral of discounted virtual valuations. For a given type $x$, if the virtual value is positive(negative) then $t_x = 0 (t_x = \infty)$ and this does not violate the monotonicity constraint. If virtual valuation function is non-monotonic, the optimal allocations follows a cutoff rule with immediate allocations as well. But in this case the monopolist will choose $x^*$ in such a way that the integral of the virtual valuation function above $x^*$ is maximized. This does not mean that all the buyers of a type higher than $x^*$ have a positive virtual valuation though. An important thing to note here is that this allocation rule is exactly the same with the optimal allocation rule of the static monopoly.

Now, we need to find an optimal price path inducing this optimal allocation rule. Since $U(0) = 0$ in an optimal mechanism, $U(x^*) = e^{-rt_x^*}(\frac{x^*}{r} - p_{t_x^*}) = U(0) + \frac{1}{r} \int_{0}^{x^*} e^{-rt} d\tilde{x} = 0$, which requires $p_{t_x^*} = p_0 = \frac{x^*}{r}$. Therefore an optimal price path to implement the optimal allocation is a constant price path at level $\frac{x^*}{r}$. The importance of this price level is that the buyer with type $x^*$ is indifferent between purchasing and not.

The significance of this result is that, for the dynamic setting, the optimal price is constant and equal to the static monopoly price. Therefore the monopolist does not use the time to discriminate over the buyers with heterogeneous valuations.

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5Even though the optimal allocation is unique, there are infinitely many price paths that can implement it. The important thing is to fix the initial price level to $\frac{x^*}{r}$ and always keep it above or equal to the initial level.
3.2 Benchmark II: Product Upgrade without Anonymous Prices

In this benchmark, we have stochastically arriving upgraded good, but the monopolist is not restricted to use the posted prices. More precisely, the monopolist can choose a mechanism in which he can condition the second version sales on the sales of the first version. The direct mechanism, in this case, contains a joint allocation rule both versions of the goods and a corresponding payment for each type of buyer. The payment rule is denoted by \( P(x) \) for a given reported type \( x \) and is to be paid at time \( t = 0 \).\(^6\) Allocations for type \( x \) buyer are \( t_x \) and \( \{ t_{x,T} \}_{T \in [0, \infty)} \), which are defined in the same way as before. Then the utility of the agent \( x \), \( U(x) \), can be written as:

\[
U(x) = Q(x)x - P(x).
\]

where

\[
Q(x) = \int_0^{t_x} e^{-\lambda T} e^{-r(T+t_{x,T})} dT + \int_{t_x}^{\infty} e^{-\lambda T} e^{-rT} \left( \int_{t_x}^{T+t_{x,T}} e^{-rt} dt + \beta \int_{T+t_{x,T}}^{\infty} e^{-rt} dt \right) dT
\]

The term, \( Q(x) \), can be considered as the total allocation that is resulting from the allocations of both versions of the good, and it can only take values from \( [0, \frac{r+\beta \lambda}{r(r+\lambda)}] \). Its maximum value is acquired by arranging \( t_x = 0 \) and \( t_{x,T} = 0 \ \forall T \) (immediate allocation of both versions), and its minimum value is acquired by arranging \( t_x = \infty \) and \( t_{x,T} = \infty \ \forall T \) (no allocation of any versions of the good). Following the same steps as in the previous benchmark we get:

**Lemma 2.** The direct mechanism without anonymity restriction is incentive compatible if and only if:

i) \( Q(x) \) is non-decreasing

ii) \( U(x) = U(0) + \int_0^x Q(\tilde{x}) d\tilde{x} \)

**Proof.** Follows exactly the same steps with the proof of lemma 1. \( \square \)

Then the monopolist’s problem can be written as:

\[
\max_{\{ Q(x) \}_{x \in [0,1]}} \frac{1}{r} \int_0^1 Q(x) \left( x - \frac{1 - F(x)}{f(x)} \right) f(x) dx \quad s.t \quad Q(x) \ is \ non-decreasing \ with \ x.
\]

\(^6\) Any dynamic payment rule with a present value equal to \( P(x) \) would be an equivalent to this payment rule. Hence there is no loss of generality here.
The solution will be analogous to the one of the previous benchmark. In particular, there exists a threshold $x^*$, which is equal to the threshold that is defined in the first benchmark, such that for all the buyers above (below) this threshold the value of $Q(x)$ is maximized (minimized). More precisely, the optimal allocation rule is:

$$
i) t_x = t_{x,T} = 0 \quad \forall x \in [x^*, 1]$$

$$
ii) t_x = t_{x,T} = \infty \quad \forall x \in [0, x^*)$$

In the optimal mechanism there is no buyer acquiring only one version of the good. In other words, the monopolist is bundling two generations of the good and selling them only together. This allocation rule resembles some selling strategies that we observe in real life. For instance, some companies, like Microsoft, offer discounts to their customers in case they have an old version and want to upgrade to a newer one. Here we see an extreme version of this policy in the sense that the price of the second version for those who already own the first version of the good is equal to zero.

The payment scheme inducing this allocation requires all the agents in $[x^*, 1]$ to pay the same amount since all of them have the same $Q(x)$. This payment is equal to $x^* \frac{r + \beta \lambda}{r(r + \lambda)}$, which leaves the marginal agent $x^*$ indifferent between purchasing and not. This mechanism is the optimal selling mechanism. However, it is impossible to implement this allocation rule by using posted prices. To see this, suppose there is a contingent price path that can implement it anonymously. Then, under these prices, marginal return from the second version purchase for agent $x^*$ is larger than or equal to the price at the corresponding time period. But since his marginal benefit is equal to $(\beta - 1)x^*$, the agent $x^* - \epsilon$, for $\epsilon$ sufficiently small, would prefer to purchase the second version as well. Hence we get a contradiction.

### 3.3 Sales With Posted Prices: Anonymity

Here, the focus is on the characterization of the optimal posted prices, which is anonymous by definition. In this case, buyer $x$ that is owning the first version of the good and the buyer $(\beta - 1)x$ that is not owning the first version of the good will have the same marginal benefit from the newer version of the good. Therefore a mechanism corresponding to the posted prices must treat these buyers in the same manner for the allocation of the second version of the good. Therefore a direct mechanism will have some further constraints in this case. For this reason, we rather use a different approach, which we call as ”two-step mechanism”, in which the allocations of each version of the good follows independent reporting stages.
First, we define the following concept:

**Definition 1.** The Effective type of buyer \( x \) at realized arrival time \( T \), is equal to his marginal flow utility from the purchase of the second version of the good. Particularly, it is equal to \( \beta x \) if he does not own the first version and it is equal to \( (\beta - 1)x \) if he does.

The buyers can use at most one unit of the good at a time. Therefore a buyer, owning both versions of the good, uses only the current one as it gives more flow utility. Alternatively, we can think of these two versions of the goods as if they are two separate goods where both can be consumed at the same time, yet the flow utility of the second good is equal to the effective type that we described above. From this point on we exploit this interpretation in our analysis, as it simplifies the exposition. The two-step mechanism is defined as:

**Definition 2.** The two-step mechanism is a mechanism in which buyers are asked to report twice. First, at \( t = 0 \), buyers are asked to report their types. Then the allocations and payments for the first version of the good are decided according to the first step reports. Second, at the realized arrival time \( T \), buyers are asked to report their effective types and the allocations and the payments of the second version of the good are decided according to the second step reports independent from the first step reports.

Finding the optimal mechanism among this class of mechanisms will give us the optimal posted prices that the monopolist can commit. The mechanism structure here is different than a direct mechanism, hence we slightly modify the notation specified earlier. For the second step allocations, contingent on the realized arrival time \( T \), the amount of time after which the effective type \( x \) purchases the second version of the good is denoted by \( t^T_x \) and so the corresponding purchase time is \( T + t^T_x \). For the first step allocations the relevant information is the initial type hence we keep the old notation, where \( t_x \) is the purchase time for type \( x \).

The utility of an agent is has two parts, one for each step of the mechanism. Starting with the second step, contingent on the arrival time \( T \), discounted expected utility of the effective type \( x \) calculated at \( T \), denoted by \( V^T_x \):

\[
V^T_x = \int_{t^T_x}^{\infty} e^{-rt} x dt - e^{-rt^T_x} p^{T}_T = e^{-rt^T_x} \left( \frac{x}{r} - p^{T}_T \right).
\]

\(^7\)Note that the previous notation was \( t_{x,T} \), for type \( x \). We now take the effective types as our basis rather than the initial type.
For the first step, the expected utility of buyer of type $x$, calculated at $t=0$, denoted by $V_x$:

$$V_x = e^{-\lambda t_x} \int_{t_x}^{\infty} e^{-r t} e^{-(r+\lambda) t} d t = e^{-(r+\lambda) t_x} \left( \frac{x}{r} - p_{t_x} \right).$$

Note that this expression reflects the alternative interpretation mentioned earlier. In particular, first version of the good is used forever after the purchase. Total utility of buyer $x$, which is denoted by $U(x)$, can be written as:

$$U(x) = V_x + \int_0^{t_x} e^{-(r+\lambda) T} \lambda V_{\beta x}^T d T + \int_{t_x}^{\infty} e^{-(r+\lambda) T} \lambda V_{(\beta-1)x}^T d T.$$ 

There is a crucial point that we better to stress out here: depending on the realized arrival time the effective type and hence the resulting second step utility of the buyer changes. If the arrival occurs before (after) $t_x$, then the effective type of the agent $x$ is $\beta x$ ($((\beta - 1)x)$ and the corresponding second step utility is $V_{\beta x}^T (V_{(\beta-1)x}^T)$.

The profit of the firms, $\Pi$, has also two components:

$$\Pi = \int_{0}^{1} e^{-r t_x} e^{-\lambda t_x} p_{t_x} f(x) d x + \int_0^{\infty} e^{-(r+\lambda) T} \lambda \Pi^T d T.$$  

The first and second terms are accounting for the expected profits from the first and second steps of the mechanism respectively. The term $\Pi^T$ is the expected profit (calculated at time $T$) conditional on the arrival time $T$, and it is equal to:

$$\Pi^T = \int_{0}^{1} e^{-r t_x} p_{t_x}^T f_T(x) d x.$$ 

The density $f_T(\cdot)$ is the distribution of the effective types at the realized arrival time $T$ and it is depending on the allocation time $t_x$ as well as the realized $T$. To this respect, the monopolist by arranging the allocations of the first step of the mechanism, can also alter the distribution of the buyers’ marginal flow benefits from the newer version of the good.

To characterize the incentive constraints of the buyers, we need to consider both reporting stages separately. Since buyers are forward looking, while reporting in the initial stage they will internalize the effect of their report on the second stage of the mechanism. Therefore, we start our characterization of incentive constraints from the second stage:
**Lemma 3.** The second step of the mechanism is incentive compatible if and only if, \( \forall T \)

\[
i) \ t^T_x \text{ is non-increasing with } x
\]

\[
ii) \ V^T_x = V^T_0 + \frac{1}{r} \int_0^x e^{-rt \tilde{x}} \, d\tilde{x}.
\]

**Proof.** Follows from the same arguments with the proof of Lemma 1. \( \square \)

**Assumption 1.** \( \lambda \leq \frac{r}{\beta - 1} \).

Now we deal with the incentive constraints of the first stage reports. Any deviation from truthful reporting at this stage will also alter the optimal behavior in the second stage as it changes the allocation time and hence the corresponding effective types.

**Lemma 4.** A two-step mechanism with an incentive compatible second stage, is also incentive compatible at the first stage only if:

\[
i) \ t_x \text{ is non-increasing with } x
\]

\[
ii) \ V_x = V_0 + \frac{1}{r} \int_0^x e^{-(r+\lambda)t_x} \, d(\tilde{x}) - \lambda \int_0^x e^{-(r+\lambda)t_x} \frac{\partial t_x}{\partial x} \left( V^T_{\beta \tilde{x}} - V^T_{(\beta-1)\tilde{x}} \right) \, d\tilde{x}.
\]

**Proof.** See appendix \( \square \)

There are four conditions in total, which are defined in lemma 3 and lemma 4, that the optimal two step mechanism needs to satisfy. Two conditions given in Lemma 3 are necessary and sufficient for the second step incentive compatibility, whereas two conditions given in Lemma 4 are just necessary for the first step incentive compatibility.\(^8\) To the sequel, we define the problem of the monopolist by only taking these four conditions into account. As the latter two conditions are not sufficient for the first stage incentive compatibility, the solution of our problem does not necessarily be the optimal solution that we are looking for. However, the solution of our problem is shown to be incentive compatible, therefore it is also the optimal solution that we are looking for. By using lemma 3, and the fact that the monopolists sets \( V^T_0 = 0 \) in an optimal mechanism, we get:

\[
e^{-rt^T_\tilde{x}} \frac{t^T_\tilde{x}}{r} = e^{-rt^T_\tilde{x}} \frac{t^T_\tilde{x}}{r} - \frac{1}{r} \int_0^x e^{-rt \tilde{x}} \, d\tilde{x}.
\]

\(^8\)Our conjecture is that they are also sufficient but since we do not need the sufficiency in the general result we did not show it formally.
Integrating this by parts gives us:

\[ \Pi^T = \int_0^1 e^{-rt_t} p_t f_t(x) dx = \frac{1}{r} \int_0^1 e^{-rt_t} (x - \frac{1 - F_T(x)}{f_T(x)}) f_T(x) dx. \]

By lemma 4, and the fact that \( V_0 = 0 \) in an optimal mechanism, we get:

\[ e^{-(r+\lambda)t_t} p_t = e^{-(r+\lambda)t_t} \frac{x}{r} - \frac{1}{r} \int_0^x e^{-(r+\lambda)t_t} d(\tilde{x}) + \lambda \int_0^x e^{-(r+\lambda)t_t} \frac{\partial t_{\tilde{x}}}{\partial x} (V_{\tilde{x}}^{t_t} - V_{(\beta-1)\tilde{x}}^{t_t}) d\tilde{x}. \]

Integrating this by parts gives us:

\[ \int_0^1 e^{-rt_t} e^{-\lambda t_t} p_t f_t(x) dx = \frac{1}{r} \int_0^1 e^{-(r+\lambda)t_t} (x - \frac{1 - F(x)}{f(x)}) f(x) dx + \lambda \int_0^1 e^{-(r+\lambda)t_t} \frac{\partial t_{x}}{\partial x} (1 - F(x)) \left( V_{\beta x}^{t_t} - V_{(\beta-1)x}^{t_t} \right) dx \]

Therefore the monopolist’s optimization problem is:

\[
\max_{\{t_t\}_{t \in \{0,1\}}, \{t^T_t\}_{t \in \{0,1\}}} \frac{1}{r} \int_0^1 e^{-(r+\lambda)t_t} \left( x - \frac{1 - F(x)}{f(x)} \right) f(x) dx + \lambda \int_0^1 e^{-(r+\lambda)t_t} \frac{\partial t_{x}}{\partial x} (1 - F(x)) \left( V_{\beta x}^{t_t} - V_{(\beta-1)x}^{t_t} \right) dx + \lambda \int_0^\infty e^{-(r+\lambda)T} \left( \int_0^1 e^{-rt_t} \left( x - \frac{1 - F_T(x)}{f_T(x)} \right) f_T(x) dx \right) dT
\]

subject to

\[ \cdot \ t_x \text{ is non-increasing in } x \]
\[ \cdot \ t^T_x \text{ is non-increasing in } x, \ \forall T \in [0, \infty) \] (4)

We know that \( f_T(.) \) is a function of \( \{t_t\}_{t \in \{0,1\}} \) for each realized arrival time \( T \). Therefore, while finding the optimal mechanism, one must take this indirect effect into account. The first line of the objective function is the sum of the discounted virtual valuations corresponding to sales of the first version of the good, and the third line is the analogous of it corresponding to the second version of the good with an important distinction that the virtual valuations are now based on the effective types and their distributions for every possible arrival time \( T \). The second line, which is always non-positive due to the monotonicity of \( t_x \) on \( x \), can be interpreted as the cost of the inter-versional incentives for the monopolist. A buyer, while purchasing the first version of the good, considers the possibility of the arrival of the newer version of the good at the very moment. If he decides to purchase, his effective type and hence his marginal willingness to pay for the newer version will change. These inter-
versional incentives is reflected at the term \((V^t_{\beta x} - V^t_{(\beta-1)x})\) appearing in the second line of above program.

Note that, when \(\lambda \to 0\) the second and the third terms of objective function in (4) is equal to 0, and hence the problem (4) is equivalent to the problem (2). This is rather intuitive, because when \(\lambda\) approaches to 0 there is no upgrade, and hence we get back to the canonical model. Proposition 1 characterizes the optimal allocation rule of the monopolist problem.

**Proposition 1.** The optimal solution to (4) consists of two cutoff values \(x_1\) and \(x_2\) such that:

\[
\begin{align*}
  t_x & = \begin{cases} 
  0 & x \geq x_1 \\
  \infty & x < x_1 
  \end{cases} \\
  t^T_x & = \begin{cases} 
  0 & x \geq x_2 \\
  \infty & x < x_2 
  \end{cases} \quad \forall T.
\end{align*}
\]

**Proof.** See appendix.

To prove this result, we write an auxiliary optimization problem in which the second line of the objective function is omitted. Then we show that the solution to this auxiliary problem is also the solution to the problem 4. This follows from the fact that the second line of the objective function is always non-positive due to the monotonicity of \(t_x\), and its value is maximized, i.e. equal to 0, at the optimal solution of the auxiliary problem.

The important thing to note here is that introducing product upgrades into durable good markets does not alter the main result of the canonical model, in the sense that the monopolist still does not use time to discriminate over buyers. Even though the type of buyers that are purchasing the good is different than the canonical durable good model, we still have immediate allocations for both versions of the good. The optimal allocation may incorporate different scenarios in terms of the version(s) of the goods that each type of buyer is acquiring. The distribution function \(f(x)\) and the parameters \(\beta\), \(\lambda\), and \(r\), are crucial to decide the optimal values of \(x_1\), and \(x_2\), and hence on the individual allocations of each buyer. However, here we do not precisely characterize the values of \(x_1\) and \(x_2\) for a given set of parameters.

The monopolist can induce this allocation rule by setting a constant price level for both versions \(p_1\), and \(p_2\), since allocations are immediate.\(^9\) Note that the price of the newer version of the good is independent from the arrival time \(T\) and is given by \(p^T_1 = p_2 = \frac{x_2}{r} \forall T, t\). At this price level, the buyer with the effective type \(x_2\) is indifferent between purchasing and

\(^9\)Version-wise constant price path is not the only price path to implement the optimal allocation rule. In particular, any non-decreasing contingent price path, satisfying \(p_0 = p_1\), and \(p^T_0 = p_2\), would also induce the optimal allocation rule. A decreasing price path would only be optimal if the optimal allocation were to occur throughout time.
not purchasing the second version, so that \( V_{x_2}^T = 0 \). On the other hand, \( p_1 \) is the price level at which the agent of type \( x_1 \) is indifferent on his purchase decision of the first version. In particular, if \((\beta - 1)x_1 \geq x_2\), then he is indifferent between purchasing both versions of the good and purchasing only the second version. And if \((\beta - 1)x_1 < x_2\), then he is indifferent between purchasing only the first version and purchasing only the second version.

**Remark 1.** As it is mentioned earlier the solution of the optimization problem 4 does not necessarily be the optimal solution of the monopolist’s problem, since Lemma 4 is an “only if” statement. However, the optimal allocation and the price path inducing this optimal allocation is obviously incentive compatible, therefore it is also the solution that we are interested in.

## 4 Deterministic Arrival

Now, the arrival of the newer version of the good occurs at a certain time \( T \) which is commonly known from all of the participants of the market. We will show that, in this non-stationary environment, the monopolist, depending on the values of the parameters, might use time to discriminate over buyers with heterogeneous valuations.

To show the possibility of a decreasing optimal price path, we simplify our model by assuming a binary type space for buyers with types H (High) and L (Low), where \( H > L \). The buyers are still a continuum with a unit mass and the measure of the H-type buyers is equal to \( \mu \in (0, 1) \) while the measure of the L-type buyers is \( 1 - \mu \). The flow utility acquired from the second version of the good is still \( \beta \) times the flow utility acquired from the first version of the good for both type of buyers.

To focus on the price path of the first version of the good, we further assume that the price path for the second version is constant, and hence any purchase of the second version occurs only at the arrival time \( T \). The monopolist commits to a price path: \( \{p_t\}_{t \in [0, T]} \), where \( p_t \) is the price level of the first version at \( t \in [0, T) \), and \( p_T \) is the price for the second version good. The following assumption guarantees that the utility from purchasing only the first good at \( t = 0 \) is higher than the utility from purchasing only the second version at time \( T \).

**Assumption 2.** \( \beta e^{-rT} < 1 \).

---

10. To omit the discount factor \( r \) that appears due to the integration of the flow utilities, say the types are \( h, l \), and we have \( H = \frac{h}{r} \) and \( L = \frac{l}{r} \).

11. We can rather think of this as a restriction so that the markets close down after \( T \).
To understand the incentive of the buyers, consider an arbitrary price path \( \{p_t\}_{t \in [0, T]} \) and say that a buyer finds it optimal to purchase the first version of the good at a time \( t \in [0, T) \) (he may or may not purchase the second version). Then it would also be an optimal decision for this buyer to purchase the first version of the good at \( t \in [0, T) \) in an environment where there is only the first version of the good with the corresponding price path: \( \{p_t\}_{t \in [0, T)} \). To see this, suppose that there exists another time period \( \tilde{t} \in (0, T) \) that strictly dominates purchasing at \( t \). But for this to be correct, this buyer must purchase the second version of the good as well, otherwise we get a direct contradiction. Then by revealed preferences of type-X buyer:

\[
e^{-rt}(X - p_t) > e^{-rt}(X - p_t)
\]

\[
X(e^{-rt} + (\beta - 1)e^{-rT}) - e^{-rt}p_t - e^{-rT}P_T \geq X(e^{-rt} + (\beta - 1)e^{-rT}) - e^{-rt}p_t - e^{-rT}P_T.
\]

However the inequalities above are contradicting with each other, hence our claim is correct. This observation is crucial for the next lemma, which is showing that there exists a critical time period, \( t^* < T \), such that before this \( t^* \) the purchasing decisions for the first version of the good is monotonic with respect to the buyers’ type. More precisely, if L-type buyers purchase the first version of the good at a time \( t < t^{\text{star}} \), then H-type buyers also purchase the first version of the good and their purchase time is not later than \( t \). On the contrary, this monotonicity does not carry out to the purchases occurring after \( t^* \). This is because of the fact that the arrival of the second version of the good becomes closer as time goes on, and the incentive to wait for the newer version of the good becomes strengthened, and these strengthened incentives is stronger for H-type buyers if the arrival time is sufficiently close.

**Lemma 5.** For a given price path \( \{p_t\}_{t \in [0, T]} \),

1. If the L-type buyers purchase both versions of the good then the H-type buyers would also purchase both versions of the good.

2. There exists a critical time period \( t^* \), that is defined by \( e^{-rt^*} = \beta e^{-rT} \), such that if the L-type buyers purchase the first version of the good at a time \( t < t^{\text{star}} \), then H-type buyers also purchase the first version of the good and their purchase time is not later than \( t \).\(^{12}\)

**Proof.** See appendix. \(\square\)

Now we can use this partial monotonicity result given in lemma 5 to show that in an optimal price path the monopolist should allocate the first version of the good to H-type buyers.

\(^{12}\)The existence of this \( t^* \) is guaranteed by the assumption 2
immediately, if the assumption 2 is satisfied. If the monopolist is not allocating the first version of the good to H-type buyers at $t = 0$, then it must be the case that they are only purchasing the second version of the good. Then L-type buyers are either only purchasing the first version of the good after $t^*$, or only purchasing the second version of the good, and both of these allocations are dominated when assumption 2 is satisfied.

**Lemma 6.** In an optimal posted price mechanism, H-type buyers purchase the first version of the good immediately at $t = 0$.

**Proof.** See appendix.

The optimal posted prices depend on the values of $H$ and $L$. In particular, the relation between $(\beta - 1)H$ and $\beta L$ is crucial. When $(\beta - 1)H \geq \beta L$ ($(\beta - 1)H \leq \beta L$) the H-type buyer owning the first version of the good gets more (less) additional utility from the second version purchase compared to a L-type buyer that does not own the first version of the good. From now on we will consider the case $(\beta - 1)H > \beta L$. The analysis of the other case follows from similar arguments.

Proposition 2 lists all of the possible optimal price paths that the monopolist can commit. All of the price paths listed is an optimal one for some subset of parameter values. As usual, the price path inducing the optimal allocation is not unique. However, we say that the prices are constant for the first version of the good as long as all of the purchases occurs at time $t = 0$. In this case time is not used to discriminate over buyers for the sale of the first version of the good. On the other hand, if the optimal allocations for the first version of the good occurs at different time periods for different type of buyers, then the monopolist is using time to discriminate over buyers. In this case, the price path inducing the optimal allocation is decreasing over time. The following proposition shows that for some subset of the parameter space, it is possible to get a decreasing optimal price path contrary to the case with stationary stochastic arrival.

**Proposition 2.** Suppose $(\beta - 1)H > \beta L$ and the assumption 2 is satisfied. Then the optimal posted prices and the corresponding purchase decisions of each type of buyer is one of the following. Moreover, each policy is an optimal policy for a non-empty subset of the parameter space

1) $p_t = H \forall t \in [0, T)$, and $p_T > (\beta - 1)H$. Only H-type buyers purchase the first version of the good, and no one purchases the second version.
2) \( p_t = L \forall t \in [0, T) \), and \( p_T = (\beta - 1)H \). Both type of buyers purchase the first version of the good at \( t = 0 \), and only H-type buyers purchase the second version.

3) \( p_t = \begin{cases} (1 - e^{-rT})H & \forall t \in [0, \bar{t}) \\ L & \forall t \in [\bar{t}, T) \end{cases} \) and \( p_T = (\beta - 1)H \) where \( \bar{t} \) satisfies \( e^{-r\bar{t}} = e^{-rT} \cdot \frac{H}{H-L} \).

Both H-type and L-type buyers purchase the first version of the good at times \( t = 0 \) and \( t = \bar{t} \) respectively. And only H-type buyers purchase the second version.

4) \( p_t = (1 - e^{-rT})H \forall t \in [0, T) \), and \( p_T = \beta L \). Only H-type buyers purchase the first version of the good, and both types purchase the second version.

5) \( p_t = (1 - e^{-rT})L \forall t \in [0, T) \), and \( p_T = (\beta - 1)L \). Both types purchase the first version at \( t = 0 \) and they also purchase the second version of the good.

It is easy to calculate the corresponding profit of each policy for the monopolist. Then we can see that each of these policies is the optimal one for some values of the parameters of the model.\(^{13}\) In other words, for each policy there is a subset of parameters, which are also satisfying the condition \((\beta - 1)H > \beta L\) and assumption 2, such that the policy is optimal. We are particularly interested on the third policy, because it displays a decreasing price path for the first version of the good. In particular, the purchases of the first version of the good occur throughout time and hence the corresponding price path implementing this allocation must be decreasing.

The non-stationary environment, resulting from a certain arrival time for product upgrades, strengthens the ability of the monopolist to sort out the buyers with lower valuations. More precisely, unless the monopolist prefers to omit the second version sales by charging a sufficiently high price \( p_T \), the H-type buyers that are purchasing the first version of the good at \( t = 0 \) has an additional option: purchasing only the second version of the good. Therefore even for the case in which the monopolist allocates only the H-type buyers at \( t = 0 \), H-type buyers have a positive utility, if the sales of the second version of the good is not omitted by the monopolist. This in turn introduces the possibility of allocating the first version of the good to the L-type buyers after \( t = 0 \) without hurting the incentives of the H-type players on their purchases of the first version of the good.\(^{14}\) The way that we defined time period \( \bar{t} \) given in the third policy of proposition 2 exploits this possibility. In particular selling the first version of the good to L-type of buyers at period \( t = \bar{t} \) with a price equal to their

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\(^{13}\) The parameters are \( \mu, r, T, H, L \).

\(^{14}\) This is not possible when there is no product upgrade; because in the canonical model, if the optimal allocation rule allocates the good only to the H-type of buyers, then the optimal price path inducing this allocation would leave 0 utility to the H-type buyers. Hence allocating the good to the L-type of buyers at a lower price would hurt the incentives of the H-type buyers.
maximum willingness to pay does not hurt the incentives of the H-type players. And $\bar{t}$ is the earliest among such time periods. The proof of proposition 2, given in the appendix, follows a backward analysis. We first define the optimal sales of the first version of the good for each value of $p_T$, and then we optimize with respect to $p_T$. Since the main concern of this section is to show the possibility of a decreasing optimal price path, the cumbersome details of the possible anonymous optimal sale mechanism are left to the appendix and discussed in the proof of the proposition.

5 Conclusion

The optimal pricing problem of a durable good monopolist is analyzed. An upgraded and a superior version of the durable good arrives and replaces the existing version of the good. The main assumption is that the sales are anonymous, so that the seller cannot condition the sale of the second version of the good on the sales of the first version. When the arrival of the upgraded version follows a stationary stochastic process, the optimal price path is shown to be constant for both versions of the good, hence all the purchases occurs immediately, i.e. the monopolist does not use the time to discriminate over buyers. On the contrary, when the arrival occurs at a commonly known certain time period, it is shown that, depending on the parameters of the model, the optimal price path might decrease over time, and hence delayed purchases might occur. Hence the time might be a useful discriminatory tool for the monopolist that is endowed with full commitment power.

For both cases the optimal selling mechanism, without the restriction of anonymity, requires the monopolist to bundle both versions of the good and to sell them only together. The corresponding allocations in this case cannot be implemented by posted prices which is anonymous by definition.
Appendix

Proof of Lemma 1: The “only if part” of the statement follows directly from revealed preferences. In other words, if the mechanism is incentive compatible, then the buyer of type $x$ does not want to mimic type $x'$ and vice versa. More precisely take $x > x'$, then:

$$U(x) \geq e^{-rt}x - pt = U(x') + e^{-rt}x' - p't$$

$$U(x') \geq e^{-rt}x' - p't = U(x) - e^{-rt}x - p't$$

Then we get,

$$\frac{e^{-rt}x}{r} \geq \frac{U(x) - U(x')}{x - x'} \geq \frac{e^{-rt}x'}{r}$$

which requires $t_x \leq t_{x'}$. Therefore we get $i)$. Now, since $t_x$ is monotone it is differentiable and continuous almost everywhere. Therefore $e^{-rt}$ is differentiable and continuous a.e. and hence, $\lim_{x' \to x} \frac{e^{-rt}x'}{r} = \frac{e^{-rt}x}{r}$ a.e. We also know that $U(x)$ is continuous and differentiable a.e so by taking the limit of the expression (5), when $x' \to x$, we get

$$\frac{\partial U(x)}{\partial x} = \frac{1}{r} e^{-rt} \text{ a.e.}$$

Hence, $ii)$ follows immediately.

For the “if” part, suppose for a given mechanism conditions $i)$ and $ii)$ are satisfied, and we want to show that this mechanism is incentive compatible. Take any two arbitrary types $x$, and $x'$ and WLOG assume $x > x'$. First, we want to show that $x$ does not want to report his type as $x'$. In other words the following must be true

$$U(x) \geq e^{-rt}x - pt = U(x') + e^{-rt}x' - p't$$

However by $ii)$ we know that

$$U(x) - U(x') = \frac{1}{r} \int_{x'}^{x} e^{-rt} d\bar{x}.$$

Hence, expression (6) boils down to:

$$\int_{x'}^{x} e^{-rt} d\bar{x} \geq e^{-rt} (x - x').$$

But this is correct given monotonicity in $i)$. Similar arguments follow for the reports of $x'$
as well. Hence the statement is true.

**Proof of Lemma 4:** i) Monotonicity: Take arbitrarily two agents of type \( x \), and \( x' \), where \( x > x' \) without loss of generality, we want to show that \( t_x \leq t_{x'} \). Showing that purchasing the good at time \( t > t_{x'} \) is worse then purchasing it at \( t_{x'} \) for agent \( x \) is sufficient to prove monotonicity. To this end, take an arbitrary \( t \) satisfying \( t > t_{x'} \). We know by revealed preferences of agent \( x' \) that:

\[
U(x') \geq e^{-(r+\lambda)t} \left( \frac{x'}{r} - p_t \right) + \int_0^t e^{-(r+\lambda)t} \lambda V_{\beta x'}^T dT + \int_t^\infty e^{-(r+\lambda)t} \lambda V_{(\beta-1)x'}^T dT
\]

Then we get:

\[
\frac{x'}{r} (e^{-(r+\lambda)t} - e^{-(r+\lambda)t}) - (e^{-(r+\lambda)t} p_t) \geq \int_{t_{x'}}^t e^{-(r+\lambda)t} \lambda (V_{\beta x'}^T - V_{(\beta-1)x'}) dT
\]

We want show that the symmetric version of the above expression holds for agent \( x \) as well. Hence we need show

\[
\frac{x - x'}{r} (e^{-(r+\lambda)t_{x'}} - e^{-(r+\lambda)t}) \geq \int_{t_{x'}}^t e^{-(r+\lambda)t} \lambda ((V_{\beta x}^T - V_{(\beta-1)x}) - (V_{\beta x'}^T - V_{(\beta-1)x'})) dT. \tag{7}
\]

Now to show the inequality above is correct we need to consider two cases.

- **Case 1:** \([x' < \frac{\beta-1}{\beta} x]\)

Incentive compatibility in the second step is satisfied by hypothesis. Therefore, by the second condition in Lemma 3 we know that the highest possible value of \((V_{\beta x}^T - V_{(\beta-1)x}) - (V_{\beta x'}^T - V_{(\beta-1)x'})\) can be attained by arranging \( t_z^T = 0 \), for all \( z \in [(\beta - 1)x, \beta x] \), and \( t_z^T = \infty \), for all \( z \in [0, (\beta - 1)x] \). Therefore

\[
((V_{\beta x}^T - V_{(\beta-1)x}) - (V_{\beta x'}^T - V_{(\beta-1)x'})) \leq \frac{x}{r},
\]

which leads to

\[
\int_{t_{x'}}^t e^{-(r+\lambda)t} \lambda ((V_{\beta x}^T - V_{(\beta-1)x}) - (V_{\beta x'}^T - V_{(\beta-1)x'})) dT \leq \frac{\lambda}{r(r+\lambda)} (e^{-(r+\lambda)t_{x'}} - e^{-(r+\lambda)t}).
\]

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However, since \( x' < \frac{\beta - 1}{\beta} x \), and \( \lambda < \frac{r}{\beta - 1} \), we know that

\[
\frac{x - x'}{r} (e^{-(r+\lambda)t} - e^{-(r+\lambda)t_x'}) > x \frac{\lambda}{r(\lambda + \beta)} (e^{-(r+\lambda)t_x'} - e^{-(r+\lambda)t}).
\]

Therefore equation (7) is satisfied and we are done for this case.

• Case 2 : \([x' \geq \frac{\beta - 1}{\beta} x]\)

Again by Lemma 3 the highest possible value of \(((V_{\beta x}^T - V_{(\beta - 1) x}) - (V_{\beta x'}^T - V_{(\beta - 1) x'}))\) can be attained by arranging \( t_z^T = 0 \), for all \( z \in [\beta x', \beta x] \), and \( t_z^T = \infty \), for all \( z \in [0, \beta x') \).

Therefore,

\[
(V_{\beta x}^T - V_{(\beta - 1) x}) - (V_{\beta x'}^T - V_{(\beta - 1) x'}) \leq (x - x') \frac{\beta}{r}.
\]

Hence

\[
\int_{t_x'}^{t_x} e^{-(r+\lambda)T} \lambda ((V_{\beta x}^T - V_{(\beta - 1) x}) - (V_{\beta x'}^T - V_{(\beta - 1) x'})) dT \leq (x - x') \frac{\beta \lambda}{r(\lambda + \beta)} (e^{-(r+\lambda)t_x'} - e^{-(r+\lambda)t}).
\]

However, since \( \lambda < \frac{r}{\beta - 1} \), we know that

\[
\frac{x - x'}{r} (e^{-(r+\lambda)t_x'} - e^{-(r+\lambda)t}) > (x - x') \frac{\lambda \beta}{r(\lambda + \beta)} (e^{-(r+\lambda)t_x'} - e^{-(r+\lambda)t}).
\]

So equation (7) is valid for this case as well. Hence we are done to show monotonicity.

ii) Derivative of \( V_x \): By truthfully reporting, agent get the utility

\[
U(x) = V_x + \int_0^{t_x} e^{-(r+\lambda)T} \lambda V_{\beta x}^T dT + \int_{t_x}^\infty e^{-(r+\lambda)T} \lambda V_{(\beta - 1) x}^T dT.
\]

Now, for a given type \( x' \) with \( x > x' \), what would happen if the agent \( x \) reports his type as \( x' \) at first stage reports? He would be allocated the first version of the good at time \( t_x \) rather than \( t_x' \) where, from from part i), we know that \( t_x \leq t_x' \). This deviation from truth-telling will affect his utility via two different channels. The first channel is a direct effect as he now acquires the first version at a different time.\(^{15}\) The second channel is an indirect effect due to the change on second stage utility.

We know that the second step of the mechanism is incentive compatible and so the agent \( x \) reports his effective type truthfully in the second stage. Because of this, misreporting in

\(^{15}\)If \( t_x = t_x' \) then we do not need to worry about first stage incentive constraints.
the first stage alters the reports of the second stage only if it alters the effective types at the realized arrival time. This happens only if the arrival occurs between \( t_x \) and \( t_{x'} \). In particular, after truthful reporting, the effective type of agent \( x \) would be \((\beta - 1)x\) inside the time interval \((t_x, t_{x'})\), and it would be \(\beta x\) if he deviates and misreports its type as \(x'\). Therefore, the incentive constraint of agent \( x \) preventing him to not to mimic \(x'\) is

\[
U(x) \geq e^{-(r+\lambda)t_x} \left( \frac{x}{r} - p_{t_x} \right) + \int_{t_x}^{t_x'} e^{-(r+\lambda)t} \lambda V_{\beta x}^T dT + \int_{t_x'}^{\infty} e^{-(r+\lambda)t} \lambda V_{(\beta-1)x}^T dT
\]

\[
= V_{x'} + e^{-(r+\lambda)t_{x'}} \left( \frac{x - x'}{r} \right) + \int_{t_x}^{t_x'} e^{-(r+\lambda)t} \lambda V_{\beta x}^T dT + \int_{t_x'}^{\infty} e^{-(r+\lambda)t} \lambda V_{(\beta-1)x}^T dT.
\]

The incentive constraint of the agent \(x'\) preventing him to mimic \(x\) is a symmetric version of the above expression. Then by combining these two inequalities we get

\[
e^{-\frac{(r+\lambda)t_x}{r}} + \int_{t_x}^{t_x'} e^{-(r+\lambda)t} \lambda (V_{\beta x}^T - V_{(\beta-1)x}^T) dT \geq \frac{V_x - V_{x'}}{x - x'} \geq e^{-\frac{(r+\lambda)t_x}{r}} + \int_{t_x}^{t_x'} e^{-(r+\lambda)t} \lambda (V_{\beta x}^T - V_{(\beta-1)x}^T) dT.
\]

First of all, we know \(t_x\) is monotone. Therefore it is continuous almost everywhere, and so when \(x' \to x\), \(e^{-\frac{(r+\lambda)t_x}{r}} \to e^{-\frac{(r+\lambda)t_x}{r}}\), almost everywhere. Moreover, when \(x' \to x\), by using Leibniz Rule, L’Hopital’s Rule, almost everywhere continuity of \(t_x\) and incentive constraints of second step reports which we have proven in previous Lemma 3, we get the following:

\[
\lim_{x' \to x} \int_{t_x}^{t_x'} e^{-(r+\lambda)t} \lambda (V_{\beta x}^T - V_{(\beta-1)x}^T) dT \quad \frac{x - x'}{x - x'} = \lim_{x' \to x} \int_{t_x}^{t_x'} e^{-(r+\lambda)t} \lambda (V_{\beta x}^T - V_{(\beta-1)x}^T) dT
\]

\[
= -\lambda e^{-\frac{(r+\lambda)t_x}{r}} \frac{\partial t_x}{\partial x} (V_{\beta x}^T - V_{(\beta-1)x}^T) \quad \text{a.e}
\]

Therefore we can conclude that

\[
\frac{\partial V_x}{\partial x} = e^{-\frac{(r+\lambda)t_x}{r}} \frac{\partial t_x}{\partial x} (V_{\beta x}^T - V_{(\beta-1)x}^T) \quad \text{a.e}
\]

Then by integrating it we get the result ii).

PROOF OF PROPOSITION 1: For now, rather than the problem in (4), we consider an aux-
iliary problem in which the second term of the objective function is omitted. Precisely

\[
\max_{\{t_x\}_{x \in [0,1]}, \{t^T_x\}_{x \in [0,1], T > 0}} \quad \frac{1}{r} \int_0^1 e^{-(r+\lambda)t_x} \left( x - \frac{1 - F(x)}{f(x)} \right) f(x) dx
\]

\[
+ \frac{\lambda}{r} \int_0^{\infty} e^{-(r+\lambda)T} \left( \int_0^1 e^{-rt_x} \left( x - \frac{1 - F_T(x)}{f_T(x)} \right) f_T(x) dx \right) dT
\]

subject to

- \( t_x \) is non-increasing in \( x \)
- \( t^T_x \) is non-increasing in \( x \), \( \forall T \in [0, \infty) \) (8)

In this problem, the contingent allocation terms \( \{t^T_x\}_{x \in [0,1]} \) for the second version of the good, appear only on the last term of the objective function. Then the optimal \( \{t^T_x\}_{x \in [0,1]} \) for this problem will be similar to the one of the canonical model. Therefore it follows a cutoff rule, where the value of the cutoff is a function of the distribution \( f_T(.) \). Hence we denote the cutoff value by \( x^*(f_T) \), and its value is exactly the same as the value of the cutoff for the static monopoly with distribution \( f_T(.) \). So that for a given \( f_T \) the allocations are of the form:

\[
t^T_x = \begin{cases} 
0 & x \geq x^*(f_T) \\
\infty & x < x^*(f_T)
\end{cases}
\]

On the other hand, the first version allocations \( \{t_x\}_{x \in [0,1]} \) are affecting both lines of the objective function as they alter the distribution functions \( f_T(.) \) of the effective types. If this indirect effect did not exist, then the optimal allocation rule would be the immediate allocation for those agents having a type higher then \( x^* \) i.e the static monopoly allocation. Despite this additional effect, the optimal allocations have a similar structure to the one of the static monopoly in the sense that it also follows a cutoff rule. This is because of the stationary structure of the environment.

**Claim:** There is a cutoff value \( \hat{x} \), depending on the values of \( \lambda, \alpha, r \), such that the optimal solution of the program (8) satisfies:

\[
t_x = \begin{cases} 
0 & x \geq \hat{x} \\
\infty & x < \hat{x}
\end{cases}
\]

**Proof of the Claim:** This is due to the stationary structure resulting from the Poisson arrival process. In particular, if at \( t \neq 0 \) an agent is allocated the first version of the good then it must be the case that the total effect of allocating the first version to this agent on the
objective function is positive. But then it must be positive \( t=0 \) as well since the environment is stationary. Therefore it is better for the monopolist to allocate the good to this agent at the beginning \( t = 0 \). Hence we know that the term \( t_x \) must be either 0 or \( \infty \) for every \( x \). Furthermore, since \( t_x \) is restricted to be monotone with respect to \( x \), optimal solution must incorporate a structure as given above.

Then we have the solution of the problem (8), as:

\[
 t_x = \begin{cases} 
 0 & x \geq \hat{x} \\
 \infty & x < \hat{x}
\end{cases} \\
\forall T, t_x^T = \begin{cases} 
 0 & x \geq x^*(\hat{x}) \\
 \infty & x < x^*(\hat{x})
\end{cases}
\]

Since the allocation of the first version only occurs at \( t = 0 \), the distribution of the effective types is independent of the realized arrival time \( T \), and just depending on the cutoff of the first version allocations. Moreover, the allocation of the second version is same with the static monopoly allocations corresponding to the effective type distribution.

Turning back to the original problem of the monopolist as defined in (4) we know that the second term, which is omitted in the relaxed problem, would be equal to zero under the allocation rule that is specified above. This is because of the fact that \( \frac{\partial t_x}{\partial x} = 0 \) almost everywhere. Furthermore, we also know that the highest possible value of this term is also zero, since \( t_x \) must be non-increasing and hence its derivative is never strictly positive. Therefore the solution of program (8), which is defined as above, is also the solution for the original problem (4) as it is maximizing the second term as well. Then we have \( x_1 = \hat{x} \), and \( x_2 = x^*(\hat{x}) \).

\[\Box\]

**Proof of Lemma 5:** To start with the first part suppose that the L-type buyers purchase both versions, hence \( p_T < (\beta - 1)L \). Then the H-type buyers purchase the second version as well, since \( p_T < (\beta - 1)H \). Moreover, since purchasing the first version conditional on purchasing the second version has a positive return for the L-type buyers, it must have a positive return for the H-type buyers as well hence H-type buyers also purchase the first version.

For the second part, suppose the L-type buyers purchase the first version at \( t < t^* \). Then, to prove the statement, we just need to show that the H-type buyers purchase the first version of the good, thanks to our observation given before. Suppose not to get a contradiction. Then it must be the case that the L-type buyers are only purchasing the first version of the good while H-type buyers are only purchasing the second version, because otherwise if the
L-type buyers were purchasing the second version, then, from the first part of the lemma, the H-type buyers would purchase both versions of the good. Also, if a H-type buyer is not purchasing the second version, then it means that he is not purchasing any versions of the good which would also be a contradiction. Then by the revealed preferences:

\[ e^{-rt}(L - pt) \geq e^{-rT}(\beta L - p_T) \]
\[ e^{-rT}(\beta L - p_T) \geq e^{-rt}(H - pt) \]

Which is a direct contradiction since \( H > L \) and \( e^{-rt} > e^{-rt'} = \beta e^{-rT} \).

\[ \square \]

**Proof of Lemma 6**: Showing that under the optimal price path there must be an agent of some type that is purchasing at \( t = 0 \) would be sufficient to prove this lemma due to the second part of Lemma 5.

Assume that nobody purchases at \( t=0 \) to get a contradiction. There must be a sale of the first version at some time before \( T \), because otherwise, if there is a sale of only the second version good, we would get a contradiction immediately, as the monopolist could deviate and sell only the first version of the good at \( t = 0 \) to the agents that are purchasing the second version. This is better for the firm as it can get a higher discounted payment due to assumption 2.

Denote the earliest time period at which a sale of the first version occurs by \( t \). We want to show that \( t \) is equal to 0. Suppose \( t > 0 \) to get a contradiction. Then there must be a sale of the second version of the good, because otherwise there exists an obvious profitable deviation, which is selling at \( t = 0 \) with the price level \( p_t \). If the agent purchasing the first version at \( t \) also purchases the second version, then it must be a H-type from lemma 5. Then the monopolist can be made better off by changing the price level at \( t = 0 \) so that the H-type is indifferent between purchasing at 0 and \( t \) as that does not alter the incentives of the L-type buyers. We get a similar contradiction for the other case in which the agent purchasing the first version at \( t \) is not purchasing the second version.

\[ \square \]

**Proof of Proposition 2**: To prove the proposition, we first treat the price of the second version of the good as a fixed value. We then find the corresponding optimal price path \( \{p_t\}_{t \in [0,T]} \) of the first version of the good for any given value of \( p_T \) and we finally optimize \( p_T \) at the end. There are 5 cases to consider for \( p_T \).
i) \( \beta H < p_T \).

In this case there is no sale of the second version of the good. We know that in an optimal policy H-type buyers purchase the first version at \( t=0 \), and the maximum amount that they are willing to pay at \( t=0 \) is \( H \). On the other hand, at any time \( t > T \), the L-type buyers are willing to pay at most \( L \) (given that there is no sale of the second version). Therefore, if the monopolist is going to sell the first version of the good to L-type buyers at a time \( t \), then he should arrange the price as \( p_t = L \). However this will affect the incentives of the H-type buyers that are purchasing at \( t = 0 \). Given that \( p_t = L \) for some \( t \), the maximum amount that the H-type buyers are willing to pay at \( t = 0 \), which we denote by \( \bar{p} \), satisfies

\[
H - \bar{p} = e^{-rt}(H - L)
\]

\[
\bar{p} = (1 - e^{-rt})H + e^{-rt}L
\]

And the corresponding profit of the monopolist, when L-type buyers purchase at time \( t \), is

\[
\Pi_t = \mu((1 - e^{-rt})H + e^{-rt}L) + (1 - \mu)e^{-rt}L.
\]

(9)

Note that the expression above is linear in \( e^{-rt} \), hence it is maximized either at \( t = 0 \) or at \( t = \infty \). If \( t = 0 \) is optimal, then both types purchase the good at \( t = 0 \) and the price level is equal to \( L \). On the other hand, if \( t = \infty \) is optimal, then only the H-type buyers purchase the good (at \( t = 0 \)) at price \( H \).

Therefore, for the first case there are two candidates of the optimal policy.

- A1: Sell the first version of the good to agents of both types at \( t=0 \) at a price level \( L \) and have no sales of the second version.

- A2: Sell the first version only to the H-types at \( t=0 \) at a price level \( H \), and have no sales of the second version.

Actually, this is analogous to the result of Stokey (1979) and intuitively follows because if there is no sales of the second version, we turn back to canonical model.

ii) \( (\beta - 1)H < p_T \leq \beta H \).

At the optimal policy, the marginal benefit from the second version of the good is \( (\beta - 1)H \) for H-type buyers since they purchase the first version at \( t = 0 \). Therefore, there is no sale of

\footnote{It is also possible have that any \( t > 0 \) is a maximizer of the expression 9. In such a case restricting \( t \) to be either 0 or \( \infty \) is wlog.}
the second version in this case as well. However, the situation is different than the previous case in the sense that now there is an additional option for H-type buyers. In other words, by purchasing only the second version of the good, they can guarantee a non-negative utility. As a result, the maximum amount that H-type buyers are willing to pay at t = 0 is less than H. Denote this maximum price level by $\bar{p}$, which satisfies:

$$H - \bar{p} = e^{-rT} \beta H - e^{-rT} p_T.$$

$$\bar{p} = (1 - \beta e^{-rT}) H + e^{-rT} p_T.$$

Where the LHS of the first line is the utility from the purchase of the first version of the good, and the RHS is from the purchase of the second version. Similar to the previous case, a candidate optimal policy is selling the first version good at t = 0, to both types of agents at a price $L^{17}$. However, the corresponding policy would be equivalent to A1, so that we do not write it again here.

We can think of another candidate, which is a modified version of the policy A2, that is selling version 1 at t = 0 only to the H-type buyers but now with a payment $\bar{p}$, rather than $H$. However, this policy is strictly dominated since the payment is less than the one of A2.

Finally, in policy A2, there is no sale of the first version to the L-types, which is due to the fact that the monopolist needs to decrease the price level at $t = 0$ (which was equal to H) to be able to sell to the L-type buyers at any time. However, in this case, by departing from the case 1, there may exists a time period earlier than T, at which selling the good to the L-type buyers at the maximum price that they are willing to pay (which is equal to L) does not hurt the incentives of the H-types. Hence it does not require to decrease the price at $t = 0^{18}$, because, now H-type buyers are having a positive utility from the first version purchase at price $\bar{p}$. Nevertheless, even if such a period of time exists, doing any better than both of the policies A1 and A2 is not possible due to the fact that the resulting policy would be equivalent to one of the intermediate policies in the expression 9. Therefore, the firm cannot do any better here in this case.

iii) $\beta L < p_T \leq (\beta - 1)H$.

In this case, H-type buyers purchase the second version of the good, since $p_T$ is always smaller

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17 Note that $L$ is less than $\bar{p}$, since $p_T \leq \beta H$, $\beta e^{-rT} < 1$, and $(\beta - 1)H > \beta L$. Therefore H-type buyers are willing to buy at this price level.

18 We can find such a time period by finding a $t$ so that the H-type buyers are indifferent between purchasing at $t$ at price $L$ and purchasing at time 0 with price $\bar{p}$. Time period $t$ satisfying this indifference condition should be less than T.
then their marginal benefit and L-type buyers do not purchase. The maximum amount that H-type buyers are willing to pay at t=0 for the first version of the good, \( \bar{p} \), in this case satisfies the following:

\[
H - \bar{p} + e^{-rT}((\beta - 1)H - p_T) = e^{-rT}(\beta H - p_T) \\
\bar{p} = (1 - e^{-rT})H
\]

where the LHS of the first line is the utility of the H-type buyers from purchasing both versions of the good, and the RHS is the utility from purchasing only the second version of the good. Note that \( \bar{p} \) is higher than \( L \).\(^{19}\) Hence the L-type buyers are not willing to purchase at \( t=0 \) with price \( \bar{p} \). Then the monopolist should either set a smaller price than \( \bar{p} \) at \( t = 0 \) to be able to sell the L-type buyers at \( t = 0 \), or he can sell it at a later time to them. Note that, at any time \( t \), the highest amount that L-type buyers are willing to pay for the first version of the good is \( L \) as they do not purchase the second version of the good in this case. Like in the previous cases, charging a price level that is equal to \( L \) will affect the incentives of the H-type buyers.

One possible optimal policy here is to sell the first version of the good at \( t=0 \) to both types of buyers at a price \( L \), and set \( p_T = (\beta - 1)H \). Another possibility is to sell the first version of the good to the H-type buyers at a price \( \bar{p} \) at \( t = 0 \), and to the L-type buyers at a later time (before \( T \)) without hurting the incentives of the H-type buyers. As we discussed in case 2, finding such a time period is possible here, because H-type buyers are having a positive utility from purchase of the first version of the good. To find this time period, let’s denote \( p^H_t \) as the price level, which leaves the H-type buyers indifferent between purchasing good at \( t=0 \) with payment \( \bar{p} \), and purchasing at \( t \) with payment \( p^H_t \). In particular

\[
H - \bar{p} = e^{-rt}(H - p^H_t) \\
p^H_t = \frac{(e^{-rt} - e^{-rT})H}{e^{-rt}}
\]

Then we can find the earliest possible time period \( \bar{t} \) at which the firm can sell the first version of the good to the L-type buyers at price \( L \) without hurting the incentives of the H-type by using the equality \( p^H_{\bar{t}} = L \). In particular,

\[
(e^{-r\bar{t}} - e^{-rT})H = e^{-r\bar{t}}L \\
e^{-r\bar{t}} = e^{-rT} \frac{H}{H - L}
\]

\(^{19}\)This is because \( \beta e^{-rT} < 1 \), and \( (\beta - 1)H > \beta L \).
Note that $\bar{t} < T$ is always satisfied due to assumption 2 and $\beta L < (\beta - 1)H$. Hence selling the first version of the good to the L-type buyers at $\bar{t}$ at a price $L$ is a feasible policy. To sum up, we have the following two candidates for this case.

- **A3:** Sell the first version to both type of buyers at $t=0$ at price $L$, and sell the second version to only to the H-type buyers at a price $(\beta - 1)H$.\(^{20}\)

- **A4:** Sell the first version of the good to the H-type buyers at $t=0$ with price $\bar{p} = (1 - e^{-rT})H$, and also sell to L-type buyers at $t = \bar{t}$ with payment $p^H_t = L$. Sell the second version to H-type buyers at price $(\beta - 1)H$.

Note that, in this case there can not be any better policy then these two due to the linearity of the profit function as we have discussed in case 1. For instance, take the policy A4, if it is better to decrease the price at $t=0$ to sell to the L-type buyers earlier than $\bar{t}$, then the monopolist should continue to decrease price level at 0 until it reaches $L$ at which L-type is willing to buy; and this corresponds to the policy A3.

**iv) $(\beta - 1)L < p_T \leq \beta L$.**

In this case, the H-type buyers always purchase the second version of the good while L-type buyers purchase the second version only if they have not purchased the first one. We can easily see that the maximum amount that the H-type buyers are willing to pay at $t = 0$ for the first version of the good is same as in case 3 and so it is equal to $\bar{p} = (1 - e^{-rT})H$.

There are three candidates for the optimal policy in this case. The first one is to sell the first version of the good to both types of buyers at $t=0$ at a price level that leaves the L-type buyers indifferent between purchasing only the first version and purchasing only the second version, and to sell the second version only to the H-type buyers. However, this policy is strictly dominated by A3. In particular, the maximum amount of the payment that L-type buyers are willing to pay at $t = 0$ is less than $L$, and the amount charged for the second version of the good at time $T$ is strictly less than the one of A3. The second policy, is to sell the first version of the good to the H-type buyers at $t=0$ with price $\bar{p} = (1 - e^{-rT})H$, and to the L-type buyers at a later time, and to sell the second version only to the H-type buyers. This is dominated by the policy A4 for the same reason above. Then the final candidate is:

- **A5:** Sell the first version of the good only to the H-type buyers at $t=0$ at the price $\bar{p} = (1 - e^{-rT})H$, and sell the second version of the good to both at the price $\beta L$.

**v) $p_T \leq (\beta - 1)L$.**

\(^{20}\)Note that this policy is strictly dominating the policy A1.
In this case, both types of the buyers purchase the second version of the good regardless of their decision on the first version sales. From the same reasoning as above the maximum amount that the H type buyers are willing to pay at \( t=0 \) is \( \bar{p} = (1 - e^{rT})H \), and he is indifferent between purchasing the first version at \( t = 0 \) with payment \( \bar{p} \) and purchasing at \( t \) with payment \( p^H_t = \frac{(e^{-rt} - e^{-rT})H}{e^{-rt}} \). Similarly, the maximum amount that the L-type buyers can are willing to pay for the first version of good at time \( t \) is \( p^L_t = \frac{(e^{-rt} - e^{-rT})L}{e^{-rt}} \).

Since \( p^L_t < p^H_t \), there does not exist a time period in which the monopolist can sell the first version of the good to the L-type buyers at price \( p^L_t \) without hurting the incentives of the H-type buyers when they are purchasing at \( t = 0 \) with price \( \bar{p} \). Therefore the monopolist must decrease the initial price to be able to sell the L-type agents at any time. Then, again due to the linearity of the monopolist profit, there are two possibilities for the optimal policy in this case. However one of them, which is selling the first version of the good only to the H-type buyers with price \( \bar{p} \) and selling the second version of the good to both types with a price \((\beta - 1)L\) is strictly dominated by the policy A5. Therefore the only option that we are left with is:

- A6: Sell the first version of the good to both type of buyers at \( t=0 \) at a price \( (1 - e^{-rT})L \), and sell the second version of the good to both types of buyers at a price \((\beta - 1)L\).

We have considered all of the possible optimal policies for the monopolist. First note that the policy A1 is strictly dominated by the policy A3, (hence we omit A1). Then the corresponding profit level for each are listed below as follows:

<table>
<thead>
<tr>
<th>Policy</th>
<th>Profit</th>
</tr>
</thead>
<tbody>
<tr>
<td>A2</td>
<td>( \mu H )</td>
</tr>
<tr>
<td>A3</td>
<td>( L + \mu e^{-rT}(\beta - 1)H )</td>
</tr>
<tr>
<td>A4</td>
<td>( \mu[1 + e^{-rT}(\beta - 2)]H + (1 - \mu)e^{-rT} \frac{LH}{H-L} )</td>
</tr>
<tr>
<td>A5</td>
<td>( \mu(1 - e^{-rT})H + e^{-rT}\beta L )</td>
</tr>
<tr>
<td>A6</td>
<td>([1 + e^{-rT}(\beta - 2)]L )</td>
</tr>
</tbody>
</table>

Each of these 5 policies may be the optimal one depending on the values of \( \beta, \mu, H, L \) and \( r \). All the policies except A4 involves immediate allocations like in the stochastic arrival case. Hence time is not used to discriminate over people in those policies. On the contrary, in policy A4 the price of the first version of the good is decreasing over time. As a result, purchase times of agents for the first version of the good are different for L and H-types of buyers. More precisely, the H-type buyers purchase at the beginning, whereas the L-type
buyers purchase at a later time (before $T$). The reason for such a pattern is based on the anonymous structure of the posted prices for the second version sales. The existence of the second version puts a restriction on the amount that a H-type buyer is willing to pay for the first version of the good since it is possible for him to give up from the purchase of the first version of the good and purchase only the second one. As a result there exists a time period so that the monopolist can sell the first version of the good to the L-type buyers without hurting the incentives of the H-type buyers. This is not possible in the canonical durable good monopoly model.
References


