Finite Depth of Reasoning and Monetary Policy

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Abstract

This paper examines the role of households’ expectations for monetary policy. First, I uncover a unique pattern “Asymmetric Reasoning” in growth expectations. Although households do observe output growth from one year ago, they do not use this information to forecast output growth, but use interest rate related information to do so. This can hardly be explained by information frictions, limited attention or adaptive learning. Second, I interpret this pattern as finite depth of reasoning, and formalize it by a level-k equilibrium on top of a small scale New Keynesian model with incomplete markets. Both macroeconomic data and expectations data are used to estimate the model. Third, I find that level-k amplifies the output response to monetary shocks by more than 50% compared to the rational expectations benchmark. The role of “Asymmetric Reasoning” can be viewed as a missing self-stabilizing general equilibrium effect in households’ expectations when households’ level of reasoning is not beyond 2.

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1 Introduction

How does the effectiveness of monetary policy depend on the expectations of the households? This question cannot be addressed under full information rational expectations as it provides no flexibility for expectation formation. Once we deviate from it, however, it is not clear which alternative is most relevant, due to the limited information in most expectation surveys. For instance, the lack of responses in forecasts after a macroeconomic shock is a common feature of sticky information, noisy information, and rational inattention (Coibion, Gorodnichenko and Kamdar, 2017), while the autocorrelation in forecast errors can be produced in addition by adaptive learning (Eusepi and Preston, 2011). My paper highlights a new and important pattern of households’ expectations that does not suffer from this issue. I interpret it as finite depth of reasoning, formalize it by a level-k equilibrium on top of a small scale New Keynesian model with incomplete markets, and explore its implications for monetary policy.

My first contribution is to uncover this pattern of growth expectations using Michigan Survey of Consumers\(^1\) (MSC hereafter). In particular, households do not use output related information to forecast output growth, but use interest rate related information to do it. I call this pattern “Asymmetric Reasoning”.

This pattern has important implications for monetary policy. Under full information rational expectations, an expansionary monetary shock increases the aggregate consumption demand primarily because it increases households’ flow of incomes in multiple periods (Kaplan, Moll and Violante, 2017). Hence, the size of aggregate output response depends on to what extent households anticipate it. For instance, if households over-predict the output responses, they will over-react to monetary shocks.

This pattern is also unique in the sense that it is hardly reconciled with many existing models. For instance, information frictions and limited attention do not explain why households are aware of the output dynamics in the past, but do not use it to forecast future growth, while adaptive learning does not explain why households overlook the historical patterns of output dynamics, but not the co-movement between interest rate and output.

\(^1\)Survey of Professional Forecasters (SPF) contains similar patterns, but is less relevant for households.
My second contribution is to interpret “Asymmetric Reasoning” as a consequence of finite depth of reasoning, and formalize this idea using a level-k New Keynesian model. In the model, level-0 households choose the same consumption expenditure and level-0 firms keep the same prices as before. Level-j agents perceive all others as level-j-1. For non-integer k, expectations are assumed to be the weighted average of two adjacent integer levels. All agents are level-k in reality. I also assume that all mechanical links, including the production technology, aggregate resource constraints, and Taylor Rule are common knowledge to agents at all rationality levels.

This model produces “Asymmetric Reasoning” because the connection between current output and future growth requires higher level of reasoning to understand. Consider a level-2 agent who perceive all others as level-1. This agent does not understand that higher output level is followed by lower output growth because level-1 households do not perceive any inflation or interest rate responses, and hence will not lower their consumption expenditure. In contrast, when interest rate is high, level-1 households will cut their consumption expenditure. Level-2 agents understand it and expect lower output growth in the near future.

In order to implement this idea, I establish a Recursive Level-k Equilibrium, which allows for endogenous state variables. This makes my model quantitative, and different from theoretical works such as García-Schmidt and Woodford (2016); Farhi and Werning (2017); Iovino and Sergeyev (2017). In the equilibrium, perceiving others as one level below is formalized as using their equilibrium objects as expectations to solve a temporary equilibrium Woodford (2013). Subjective expectations can be obtained from biased forecast rules.

The model is estimated using both macroeconomic and expectations data. I first estimate it for each value of k using macroeconomic data, and then choose the parameter k to minimize the distance between model and data in the predictability of growth and inflation forecast errors. The two steps in estimation are not merged as in Milani (2011) for two reasons. First, my goal is to explore the role of expectations for allocations, instead of whether the level-k model can match both. Second, I would like to highlight the role of “Asymmetric Reasoning”, instead of trying to match households’ expectations along all dimensions.

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2Farhi and Werning (2017) has mentioned a similar idea, but did not formalize it.
3Inflation expectations are also used here to add one more discipline.
As $k$ increases from 1 to 2, the estimated persistence of demand wedge increases from 0.61 to 0.95, with a corresponding decline of standard deviations while the estimated persistence of interest rate declines from 0.90 to 0.82 with no change in standard deviations. These results indicate that lower levels of $k$ can generate more internal propagation, while higher levels of $k$ generate more amplification when $k \in [1, 2]$. Propagation helps fit the macroeconomic data, while amplification does not.

My third contribution is to explore the implication of “Asymmetric Reasoning” for monetary policy. I introduce uninsurable idiosyncratic preference shocks and borrowing constraints in a way that does not induce wealth distribution, or change the equilibrium allocations under rational expectations. My goal is to capture the fact that households’ actual planning horizon is shorter than that in a standard representative agent model\(^4\). The frequency of idiosyncratic shocks is calibrated to match the quarterly marginal propensity of consumer (MPC hereafter) of the unconstrained households, which is 7% according to Kaplan and Violante (2014). The model indicates that finite depth of reasoning can amplify the output response to a federal fund rate shock by more than 50%, compared to the rational expectations benchmark.

The amplification can be viewed as a consequence of shutting down a self-stabilizing channel. With rational expectations, households anticipate the rise of output following lower interest rates, as well as the limit of this rise due to further inflation hike and interest rate response. In a level-$k$ New Keynesian model with $k \leq 2$, the second channel is shut down. Households tend to over-predict the response of output, and over-react to monetary shocks.

In the baseline results, $k=1.5$ and MPC=$7.0\%$. Both numbers are curcial for the results. With the same parameters other that $k$, $k=1$ induces a smaller but delayed output response, while $k=2$ induces a slightly more short-lived response. With re-estimated parameters, $k=1$ induces a larger and delayed response, while $k=2$ induces a much smaller response. In all these results, output responses are much larger than the rational expectations benchmark, which indicates that amplification is a robust feature of the level-$k$ New Keynesian model. However, when I reduce the MPC by one half to 3.5\%, the amplification effect almost disappears.

\(^4\)Farhi and Werning (2017) assumes shorter planning horizons directly. That approach is not convinient in my model, becuase the perceived interest rate may deviate from the steady state permanently, which leads to exploding budgets in expectations. Precautionary saving can help rule out this unpleasant feature.
**Related Literature**  In terms of methodology, my paper is related to the literature using temporary equilibrium as an alternative to rational expectations equilibrium. The idea is to impose market clearing conditions only in the current period, and specify how expectations are formed in a particular way. The literature has explored both adaptive learning and eductive learning (Woodford, 2013). My paper follows the tradition of eductive learning.

Quantitative work is fruitful for adaptive learning (Milani, 2007; Eusepi and Preston, 2011), but not for eductive learning. One possible obstacle is the difficulty in obtaining convergence to rational expectations. Evans, Guesnerie and McGough (2015) has demonstrated this issue in a real business cycle model. My model does not have this issue because (1) price is sticky, (2) interest rate response to inflation is slow, and (3) households have short planning horizons.

In terms of research question, my paper is related to the literature on the role of expectations in a New Keynesian model (Milani, 2011; Fuhrer, 2017), as well as the literature on monetary policy transmission (Auclert, 2017; Kaplan et al., 2017). My paper can bridge these two, because (1) households have realistic MPC, (2) expectation formation is modeled in a structural way, and (3) expectations are disciplined by survey data. Instead of interpreting the waves of optimism and pessimism as sentiment shocks (Milani, 2011), I model them as an internal mechanism that could potentially amplify and propagate monetary shocks.

My paper is also related to the literature that re-examines the notion of general equilibrium. Angeletos and Lian (2017) argues that the general equilibrium effects are hard to fully realize in the short-run because it takes time to coordinate expectations, while Farhi and Werning (2017) argues that the lack of deep reasoning only dampens the general equilibrium effects of forward guidance. In my model, the lack of deep reasoning induces the missing self-stabilizing general equilibrium effects, and hence amplifies the effects of monetary policy. In addition, my argument is well supported by expectations data.

The rest of the paper is organized as the follows. Section 2 uncover the “Asymmetric Reasoning” in households’ growth expectations. Section 3 develops the model. Section 4 characterizes the model, and summarizes the main mechanisms. Section 5 shows the estimation of the model. Section 6 explores the implications for monetary policies. Section 7 concludes.
2 Stylized Facts

This section uncovers the “Asymmetric Reasoning” in households’ growth expectations using MSC data. I show that households do observe the output dynamics in the past, but not use it to forecast future growth. In contrast, they understand how future growth is correlated with interest rate related variables. This feature is summarized in a regression showing the predictability of forecast errors by current macroeconomic data, which I will use to discipline the expectations formation of the model.

2.1 Growth Expectations

MSC does not provide direct measures for households’ growth rate expectations. Instead, it ask two related questions. The first one is on the current business conditions compared with a year ago. The second on in the expectation change in business conditions in a year. The answers are qualitative and contains three options: better, worse and do not know. An index is constructed for each based on the distribution of the answers. A brief review of the results are shown in Figure 1.

![Figure 1: Growth Expectations in MSC 1969-2007](image-url)
Figure 1 shows that the one year back evaluations on business conditions are highly correlated to with output growth rate, while the one year ahead expectations are have much smaller variations. After 1985, the one year back evaluations become more volatile than growth rate, while the one year ahead expectations become much less correlated with it. Considering the potential effects of monetary regime switch in early 1980s’, most analysis in this paper will focus on the sample period after 1985. The data after 2007 is also not considered here due to the potential shift of growth trend after that.

Due to this reason, I zoom in the figure, and show the sample period after 1985 in Figure 2. The co-movements between the one year back evaluations and growth rate are very strong, but the co-movements between the one year ahead expectations and it are very weak.

![Figure 2: Growth Expectations in MSC 1985-2007](image)

In order to impute the growth rate expectations from these indexes, I assume that after 1985, households can observe what happens during the last year perfectly, and use this to rescale the index so that it has a unit of output growth rate. This approach will only amplify the co-movement between expectations and growth rate, and hence obtain the upper bound of rationality in growth expectations. The following equation summarizes this idea.
\[
\frac{\Delta \text{expected growth rate}}{\Delta \text{forward index}} = \frac{\Delta \text{ex post growth rate}}{\Delta \text{backward index}}.
\]

I also compare the expectations with realizations in output levels. Figure 3 shows the results. Even though the volatility of growth expectations has been maximized, the correlation between one year ahead expectations and the output level is still very weak.

![Figure 3: Output Expectations in MSC](image)

### 2.2 Asymmetric Reasoning

This subsection examines what predicts growth expectations in more details. I show that households do not use output dynamics in the past to predict future, but do use the interest rate related information. The results is summarized by multiple regression in the Table 1. All data are linearly or log-linear detrended. The first 6 columns of this table compare the predictability of the expected one year output growth with its realized counterpart, while the last column shows the predictability of forecast errors.
**Table 1: Asymmetric Reasoning**

<table>
<thead>
<tr>
<th></th>
<th>$\hat{y}_{t+4} - \hat{y}_t$</th>
<th>$\hat{y}_{t+4} - \hat{y}_t$</th>
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<th>$\hat{y}_{t+4} - \hat{y}_t$</th>
<th>$\hat{y}_{t+4} - \hat{y}_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{y}_t$</td>
<td>-0.018</td>
<td>-0.429***</td>
<td>0.053**</td>
<td>-0.269***</td>
<td>0.073***</td>
<td>-0.187*</td>
</tr>
<tr>
<td>$\hat{r}_t$</td>
<td></td>
<td>-0.148***</td>
<td>-0.194**</td>
<td>-0.105</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\hat{\pi}_t$</td>
<td></td>
<td></td>
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<tr>
<td>$\hat{y}<em>t - \hat{y}</em>{t-4}$</td>
<td>0.031</td>
<td>0.454***</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\hat{r}<em>t - \pi</em>{t+4}$</td>
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<tr>
<td>$\hat{r}<em>t - \pi</em>{t+4}$</td>
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</tr>
<tr>
<td>Obs.</td>
<td>84</td>
<td>84</td>
<td>88</td>
<td>88</td>
<td>88</td>
<td>88</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.014</td>
<td>0.303</td>
<td>0.190</td>
<td>0.143</td>
<td>0.358</td>
<td>0.176</td>
</tr>
</tbody>
</table>

* Standard errors are in the parenthesis. ***,*** denote significance at the 10%, 5% and 1% levels.

“Asymmetric Reasoning” can be summarized by the following patterns. Column 1-2 show that the one year ahead realized growth rate is strongly correlated with the current output level and output growth, while the expected growth rate has almost no correlation. Column 3-6 show that both the ex ante real interest rate and the actual nominal interest rate can predict the expected output growth, and the predictability is no weaker than that for the realized output growth. Column 7 shows the predictability of forecast errors. It shows that households are over-confident on the expected output growth when the current output level and inflation is high, while conditional on interest rate, this over-confidence is insignificant.

**Discussion.** All of these results indicate that households use the interest related information but not output related information to forecast output growth. This pattern does not suffer from the concern that households are less aware of the aggregate output because otherwise, the way how I construct the expected output growth will induce no predictability of growth expectations at all.

I tend to use the predictability of forecast errors to discipline the model because this is the only way to rule out the possibility that some full information rational expectations model may produce similar patterns, because in those models, forecast errors are not predictability regardless of how the models are specified.

Inflation expectations also have similar patterns. I do not show it here because the patterns are less transparent compared with the growth expectations. Still, I will use the predictability of its forecast errors to discipline the model in the empirical part.
3 A Level-k New Keynesian Model

Two frictions are imposed on top of a small scale New Keynesian model. First, households are subject to uninsurable idiosyncratic preference shocks and borrowing constraints, so that their planning horizon can be disciplined. Second, they only have level-k depth of reasoning, so that part of the equilibrium effects are missing in their expectations.

3.1 Households

There are a measure one of infinitely-lived households with a constant discount factor $\beta$, and an aggregate stochastic demand wedge $\eta^d$ multiplied to it. They choose consumption $c$, labor supply $\ell$, and real bond $b$ each period. The instantaneous utility function is $u(c) - v(\ell)$.

**Timing.** Within each period, events happen in the following order:

1. Households inherit observations and expectations from the end of last period.
2. Households observe the current gross inflation rate $\Pi$. The real net wealth $a$ is determined by the last period real bond position $b_-$, the last period nominal gross interest rate $R_-$, and the current gross inflation rate $\Pi$ through $a = b_- R_- / \Pi$.
3. Households are hit by idiosyncratic preference shocks $\zeta \in \{1, \zeta\}$, with $\zeta \geq 1$ and transition probability $\Pr(\zeta | \zeta_-) = \lambda_{\zeta|\zeta}$. This yields an unconditional probability $\Pr(\zeta) = \lambda_{\zeta}$.
4. Households observe the real wage rate $W$ and receive a lump sum transfer of real dividend $D$. They consume $c$ and supply labor $\ell$ to get utility $\zeta [u(c) - v(\ell)]$. The rest of the budget is saved in real bond $b$, with borrowing constraint $b \geq 0$ only for $\zeta = \zeta$.
5. Households observe the aggregate output $Y$ as well as all aggregate shocks $\epsilon = (\epsilon', \epsilon^d, \epsilon^z)$ that drives the current gross nominal interest rate $R$, as well as the next period demand wedge and technology level $(\eta^d', \eta^z')$.
6. Households form new expectations based on these observations.

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5I assume that households do not update expectations until the end of each period using the newly arrived information of $\{\Pi, W, D\}$ to avoid a twoway feedback between current equilibrium outcomes and expectations on future. As a result, expectations are inferred exclusively from the aggregate states. Under full information rational expectations, expectations inferred from the aggregate states are consistent with equilibrium outcomes, hence $\{\Pi, W, D\}$ provide no additional information. In my specification of level-k reasoning, there are ex post forecast errors in $\{\Pi, W, D\}$ but households do not learn from them.
States and Equilibrium Objects. Denote the vector of aggregate states as $S$. Individual states include the real net wealth $a$ and idiosyncratic preference shocks $\zeta$. Level-k households take as given the following equilibrium objects:

1. perceived and actual real wage rate $\{W^{e,(k)}(S), W^{(k)}(S)\}$,
2. perceived and actual real dividend $\{D^{e,(k)}(S), D^{(k)}(S)\}$,
3. perceived gross nominal interest rate $R^{e,(k)}(S, \epsilon')$,
4. perceived gross inflation rate $\Pi^{(k)}(S)$,
5. perceived aggregate law of motion $H^{(k)}(S, \epsilon)$.

Households’ Problems. Households have perceptions on their policy and value functions $\{c^{e,(k)}, \ell^{e,(k)}, b^{e,(k)}, V^{h.e.(k)}\}$ on $(\zeta, a, S)$ for the future. These functions solve

$$V^{h.e.(k)}(\zeta, a, S) = \max_{\{c, \ell, b\}} \left\{ \zeta[u(c) - v(\ell)] + \beta \exp(\eta^d) \cdot \mathbb{E}[V^{h.e.(k)}(\zeta', a', S')|S] \right\}$$

subject to

$$b = -c + W^{e,(k)}(S)\ell + D^{e,(k)}(S) + a,$$

$$b \geq 0 \text{ when } \zeta = \bar{\zeta},$$

$$a' = b \cdot R^{e,(k)}(S, \epsilon')/\Pi^{e,(k)}(S'),$$

$$S' = H^{e,(k)}(S, \epsilon).$$

In the current period, households observe the actual real wage and dividend $\{W^{(k)}(S), D^{(k)}(S)\}$; have continuation values given by the perceived value function $V^{h.e.(k)}$, and solve for their actual policy and value functions $\{c^{(k)}, \ell^{(k)}, b^{(k)}, V^{h.(k)}\}$ on $(\zeta, a, S)$ from

$$V^{h.(k)}(\zeta, a, S) = \max_{\{c, \ell, b\}} \left\{ \zeta[u(c) - v(\ell)] + \beta \exp(\eta^d) \cdot \mathbb{E}[V^{h.e.(k)}(\zeta', a', S')|S] \right\}$$

subject to

$$b = -c + W^{(k)}(S)\ell + D^{(k)}(S) + a,$$

$$b \geq 0 \text{ when } \zeta = \bar{\zeta},$$

$$a' = b \cdot R^{e,(k)}(S, \epsilon')/\Pi^{e,(k)}(S'),$$

$$S' = H^{e,(k)}(S, \epsilon).$$

Remark 1. Since $\{W^{e,(k)}, D^{e,(k)}\} \neq \{W^{(k)}, D^{(k)}\}$, we have $\{c^{e,(k)}, \ell^{e,(k)}, b^{e,(k)}\} \neq \{c^{(k)}, \ell^{(k)}, b^{(k)}\}$ generically. Given state variables, households do not correctly anticipate their decision rules.
Aggregation. With the assumption that bond $b$ is in zero net supply, and $b \geq 0$ binds only when $\zeta = \overline{\zeta}$, an initially degenerate wealth distribution will always induce equilibrium with degenerate wealth distribution. This is formalized in the following lemma.

**Lemma 1.** $\lambda_1 b^{(k)}(1, 0, S) + (1 - \lambda_1) b^{(k)}(\overline{\zeta}, 0, S) = 0 \implies$

1. $b^{(k)}(1, 0, S) = b^{(k)}(\overline{\zeta}, 0, S) = 0$,
2. $\ell^{(k)}(1, 0, S) = \ell^{(k)}(\overline{\zeta}, 0, S)$,
3. $c^{(k)}(1, 0, S) = c^{(k)}(\overline{\zeta}, 0, S) = W^{(k)}(S)\ell^{(k)}(1, 0, S) + D^{(k)}(S)$.

Lemma 1, we know that households behave as if they are representative agents in the equilibrium, so that we can have the following simple aggregation

$$B^{(k)}(S) = b^{(k)}(1, 0, S) = b^{(k)}(\overline{\zeta}, 0, S) = 0,$$
$$L^{(k)}(S) = \ell^{(k)}(1, 0, S) = \ell^{(k)}(\overline{\zeta}, 0, S),$$
$$C^{(k)}(S) = c^{(k)}(1, 0, S) = c^{(k)}(\overline{\zeta}, 0, S).$$

Variety Demand. Each household’s consumption $c$ is made of a measure one of varieties $\{c_j\}$ with $j \in [0, 1]$. Assume Dixit-Stiglitz aggregator\(^6\) for both varieties and their prices

$$c = \left( \int_0^1 c_j \frac{\varepsilon - 1}{\varepsilon} dj \right)^{\frac{1}{\varepsilon - 1}}, \quad P = \left( \int_0^1 p_j^{\frac{1}{1-\varepsilon}} dj \right)^{\frac{1}{1-\varepsilon}},$$

where $\varepsilon > 1$. This yields the individual variety demand and its aggregate counterpart

$$c_j = \left( \frac{p_j}{P} \right)^{-\varepsilon} c, \quad C_j = \left( \frac{p_j}{P} \right)^{-\varepsilon} C.$$

3.2 Firms

There are a measure one of infinitely-lived firms. Firm $j \in [0, 1]$ produces variety $j$ exclusively, using labor input $n_j$ via a linear technology $Y_j = \exp(\eta^2) n_j$. They set prices to attract variety demand $C_j$ and produce $Y_j = C_j$. Profits are discounted by real interest rates.

\(^6\)A market structure of final good producers and intermediate good producers is not necessary here.
Timing. Within each period, events happen in the following order:

1. A random fraction $\theta$ of the firms are drawn to keep their previous prices no changed.
2. Each other firm $j$ sets a price before observing $\{\Pi, W, C_j\}$.
3. Each firm $j \in [0, 1]$ observes $(W, C_j)$, produces $C_j$, and pays $W$ to each unit of labor input.
4. Profits are paid as dividend to households. The aggregate dividend is $D$.
5. Firms observe aggregate output $Y$ and all aggregate shocks $\epsilon$. They form new expectations.

States and Equilibrium Objects. The firms that reset prices choose $p^n = p/P_-$ as the new price over the previous aggregate price. The firm not resetting prices have individual state $p^n _e = p_ - / P_-$. Level-k firms take as given the following equilibrium objects:

1. perceived gross inflation rate $\Pi^e(k)(S)$,
2. perceived real wage rate $W^e(k)(S)$,
3. perceived aggregate output $Y^e(k)(S)$,
4. perceived nominal gross interest rate $R^e(k)(S, \epsilon')$,
5. perceived aggregate law of motion $H^e(k)(S, \epsilon)$.

Firms’ Problems. Policy and value functions $\{p^a(k), V^{a,e}(k), V^{n,e}(k)(p^n, \cdot)\}$ on $S$ solve the problem of the firms that reset prices

$$V^{a,e}(k)(S) = \max_{p^a} \left( \frac{p^a}{\Pi^e(k)(S)} - \frac{W^e(k)(S)}{\exp(\eta^2)} \right) \left( \frac{p^a}{\Pi^e(k)(S)} \right)^{-\epsilon} Y^e(k)(S)$$

$$+ \mathbb{E} \left[ \frac{\Pi^e(k)(S')}{R^e(k)(S, \epsilon')} \left( \theta V^{n,e}(k) \left( \frac{p^n}{\Pi^e(k)(S)} , S' \right) + (1 - \theta) V^{a,e}(k)(S') \right) \middle| S \right]$$

s.t. $S' = H^e(k)(S, \epsilon)$,

as well as the bellman equation of the firms that do not reset prices

$$V^{n,e}(k)(p^n, S) = \left( \frac{p^n}{\Pi^e(k)(S)} - \frac{W^e(k)(S)}{\exp(\eta^2)} \right) \left( \frac{p^n}{\Pi^e(k)(S)} \right)^{-\epsilon} Y^e(k)(S)$$

$$+ \mathbb{E} \left[ \frac{\Pi^e(k)(S')}{R^e(k)(S, \epsilon')} \left( \theta V^{n,e}(k) \left( \frac{p^n}{\Pi^e(k)(S)} , S' \right) + (1 - \theta) V^{a,e}(k)(S') \right) \middle| S \right]$$

s.t. $S' = H^e(k)(S, \epsilon)$.
Aggregation. The aggregate inflation and dividend are determined by

\[ \Pi^{(k)}(S) = \left( \theta + (1 - \theta)p^{a,(k)}(S)^{1-\epsilon} \right)^{\frac{1}{1-\epsilon}}, \]
\[ D^{(k)}(S) = Y^{(k)}(S) - \frac{W^{(k)}(S)Y^{(k)}(S)}{\exp(\eta^z)}. \]

3.3 Monetary Policy, Market Clearing and Aggregate Shocks

Monetary Policy. Denote steady state nominal gross interest rate, gross inflation rate and output as \( \{ R_{SS}, \Pi_{SS}, Y_{SS} \} \). Assume these are common knowledge to households. The monetary authority chooses a nominal gross interest rate following a standard Taylor Rule

\[ R^{(k)}(S, \epsilon') = \left( \frac{R_{SS}}{R_{SS}} \right)^{\rho_r} \left( \frac{\Pi^{(k)}(S)}{\Pi_{SS}} \right)^{(1-\rho_r)\phi_\pi} \left( \frac{Y^{(k)}(S)}{Y_{SS}} \right)^{(1-\rho_r)\phi_y} \exp(\sigma_r \epsilon'). \]

where \( \rho_r \) denotes the level of interest rate smooth, \( (\phi_\pi, \phi_y) \) denotes the response coefficients of nominal interest rate to inflation and output, \( \epsilon' \) denotes the i.i.d. federal fund rate shock, and \( \sigma_r \) denotes the standard deviation of it.

Market Clearing. In each period when agents are making decisions, supply must be equal to demand in product market, labor market and bond market.

\[ Y^{(k)}(S) = C^{(k)}(S), \]
\[ L^{(k)}(S) = Y^{(k)}(S)/\exp(\eta^z), \]
\[ 0 = B^{(k)}(S). \]

Aggregate Shocks. The aggregate shocks \( \epsilon = (\epsilon', \epsilon^d, \epsilon^z) \) and processes \( (\eta^d, \eta^z) \) satisfy

\[ \epsilon' \sim \mathcal{N}(0, 1), \]
\[ \eta^d = \rho_d \eta^d + \sigma_d \epsilon^d, \quad \epsilon^d \sim \mathcal{N}(0, 1), \]
\[ \eta^z = \rho_z \eta^z + \sigma_z \epsilon^z, \quad \epsilon^z \sim \mathcal{N}(0, 1). \]
3.4 Recursive Level-k Equilibrium

This subsection establishes a Recursive Level-k Equilibrium. Both households and firms have level-k depth of reasoning. They perceive all others as one level below. The agents with lower levels are hypothetical and only exist in higher level agents’ expectations. This expectations structure implies that agents do not understand market clearing conditions in their exceptions, so that market clearing conditions are imposed only on the current period when agents are making decisions. Perceiving all others as one level lower is equivalent to using the actual equilibrium objects of this level as perceived equilibrium objects. Therefore, the recursive level-k equilibrium can be established by iterating on the perceived equilibrium objects following the same logic of level-k equilibrium in game theory. The Recursive Level-k Equilibrium nests the definition of level-k equilibrium in Farhi and Werning (2017) as a special case, and allows for a state space representation of the model.

Level-1 Expectations. The level-1 expectations are initialized in the following steps.

1. Level-0 households do not change consumption expenditure, and the level-0 firms do not reset prices. Aggregate labor supply $L^{e,(1)}$ satisfies the production technology.

$$ (Y^{e,(1)}(S), \Pi^{e,(1)}(S), L^{e,(1)}(S)) = (Y_{-}, 1, Y_{-}/\exp(\eta^{2})). $$

2. $W^{e,(1)}$ ensures the optimality of $L^{e,(1)}$, and $D^{e,(1)}$ satisfies the aggregate resource constraint.

$$ (W^{e,(1)}(S), D^{e,(1)}(S)) = (v_{e}(L^{e,(1)}(S))/u_{e}(Y_{-}), Y_{-} - W^{e,(1)}(S)L^{e,(1)}(S)). $$

3. Given $\{\Pi^{e,(1)}, Y^{e,(1)}\}$, $R^{e,(1)}$ is given by Taylor Rule

$$ \frac{R^{e,(k)}(S, \epsilon')}{R_{SS}} = \left( \frac{R_{-}}{R_{SS}} \right)^{\rho_{e}} \left( \frac{\Pi^{e,(k)}(S)}{\Pi_{SS}} \right)^{(1-\rho_{e})\phi_{e}} \left( \frac{Y^{e,(k)}(S)}{Y_{SS}} \right)^{(1-\rho_{e})\phi_{y}} \exp(\sigma_{e}\epsilon'). $$

4. The perceived aggregate law of motion $H^{e,(1)}$ is given by

$$ H^{e,(1)}(S, \epsilon) = (Y^{e,(1)}(S), R^{e,(1)}(S, \epsilon), H^{n}(\eta^{d}, \eta^{z}, \epsilon)). $$
Level-k Updating. For $\forall j \geq 1$ and $j \in \mathbb{N}_+$, expectations are updated according to
\[
(Y^{e,(j)}, \Pi^{e,(j)}, W^{e,(j)}, D^{e,(j)}, R^{e,(j)}, H^{e,(j+1)}) = (Y^{(j)}, \Pi^{(j)}, W^{(j)}, D^{(j)}, R^{(j)}, H^{(j)}).
\]
For $\forall k \geq 1$, and $j \leq k \leq j + 1$, expectations are defined as
\[
(Y^{e,(k)}, \ldots) = (j + 1 - k)(Y^{e,(j)}, \ldots) + (k - j)(Y^{e,(j+1)}, \ldots).
\]
The solution to the model with given expectations gives us the following mapping
\[
T : (Y^{e,(k)}, \ldots) \longrightarrow (Y^{(k)}, \ldots).
\]

Aggregate States. The specification of level-k expectations indicates that the aggregate state can be summarized by $S = (Y_-, R_-, \eta^d, \eta^z)$.

Recursive Level-k Equilibrium.

Definition 1. The Recursive Level-k Equilibrium consists of a set of policy and value functions for households $\{c^{(k)}, \ell^{(k)}, b^{(k)}, V^{h,(k)}\}$, $\{c^{e,(k)}, \ell^{e,(k)}, b^{e,(k)}, V^{h,e,(k)}\}$ on $(\zeta, a, S)$, a set of policy and value functions for firms $\{p^{a,(k)}, V^{a,e,(k)}, V^{n,e,(k)}(p^n, \cdot)\}$ on $S$, a set of actual aggregate objects $\{Y^{(k)}, \Pi^{(k)}, W^{(k)}, D^{(k)}, C^{(k)}, L^{(k)}, B^{(k)}, R^{(k)}\}$ on $S$, and the corresponding perceived ones $\{Y^{e,(k)}, \Pi^{e,(k)}, W^{e,(k)}, D^{e,(k)}, C^{e,(k)}, L^{e,(k)}, B^{e,(k)}, R^{e,(k)}\}$ on $S$, such that
1. Individual policy and value functions solve the corresponding problems.
2. Actual individual decisions are consistent with actual aggregate objects.
3. Monetary policy follows Taylor Rule.
4. Market clearing conditions hold.
5. Perceived aggregate objects are determined by level-k updating.

Definition 2. Replacing the level-k updating with consistency between the actual and perceived objects yields the Rational Expectation Equilibrium.

Lemma 2. When $k \to +\infty$, if the Recursive Level-k Equilibrium converges, it must converge to the Rational Expectation Equilibrium.
4 Understanding the Model

This section characterize the model. First, I demonstrate what level-k expectations look like when $k \in \{1, 2\}$, and when idiosyncratic shocks are absent. Second, I explain how expectation biases affect equilibrium allocations. Whenever applicable, I use CRRA utility function with $\omega$ to denote the intertemporal elasticity of substitution and $\xi$ to denote the frisch elasticity of labor supply.

4.1 Level-k Expectations

Consider a situation when $(\lambda_{11}, \lambda_{12}) = (1, 0)$ so that the idiosyncratic shocks are turned off. In order to make the analysis transparent, I demonstrate all the results in this subsection with log-linearization. Use $(\hat{y}, \hat{r}, \hat{w}, \hat{\ell}, \hat{c}, \hat{\pi})$ to denote the corresponding real aggregate output, inflation rate, federal fund rate, real wage rate, price markup, real aggregate consumption, and aggregate labor supply, respectively. Use $wy_{SS}$ to denote the steady state labor share in net national incomes. Then, the optimization conditions induce the following lemma.

**Lemma 3.** The aggregate consumption and inflation in level-k equilibrium are determined by

$$
\hat{c}_t^{(k)} = \frac{1 - \beta}{1 + wy_{SS}\xi\omega^{-1}} \left\{ wy_{SS}[(1 + \xi)\hat{w}_t^{(k)} + \hat{\tau}_t^{(k)}] + \sum_{s=1}^{\infty} \beta^s wy_{SS}[(1 + \xi)\hat{w}_{t+s|t}^{(k)} + \hat{\tau}_{t+s|t}^{(k)}] \right\} - (\omega\beta) \sum_{s=0}^{\infty} \beta^s (\eta_{t+s} - \hat{\pi}_{t+s|t})^d,
$$

$$
\hat{\pi}_t^{(k)} = (1 - \theta)(1 - \beta\theta) \sum_{s=0}^{\infty} (\beta\theta)^s (\hat{w}_{t+s|t}^{(k)} - \eta_{t+s}) + (1 - \theta) \sum_{s=0}^{\infty} (\beta\theta)^s \hat{\pi}_{t+s|t}^{(k)}.
$$

These two conditions describe the aggregate consumption and inflation functions in terms of both current observed prices and the expectations on various current and future variables. The households need to observe the real wage rate and the real dividend to make consumption and labor supply decisions, while the firms do not need to observe anything when setting prices. These conditions do not depend on how expectations are formed.
Now impose the market clearing conditions, how \((\hat{w}^e, \hat{\pi}^e)\) is formed, production technology, aggregate resources constraints, and the assumption that agents understand Taylor Rule, as in the definition of level-k expectations. We have Lemma 4.

**Lemma 4.** Before specifying the expectations of output and inflation, the level-k equilibrium must satisfy the following two conditions.

\[
\hat{y}^{(k)}_t = (1 - \beta) \sum_{s=0}^{\infty} \beta^s \hat{y}^{e,(k)}_{t+1+s|t} - \omega \sum_{s=0}^{\infty} \beta^s \left\{ \eta^d_{t+s} + \rho_r^{s+1} \hat{r}_{t+1} + (1 - \rho_r) \sum_{\tau=0}^{s} \rho_r^{s-\tau} \left( \phi_{\pi} \hat{\pi}^{e,(k)}_{t+\tau|t} + \phi_y \hat{y}^{e,(k)}_{t+\tau|t} \right) - \hat{\pi}^{e,(k)}_{t+1+s|t} \right\},
\]

\[
\hat{\pi}^{(k)}_t = (1 - \theta) \sum_{s=0}^{\infty} (\beta \theta)^s \hat{\pi}^{e,(k)}_{t+1+s|t} + (1 - \theta)(1 - \beta \theta) \sum_{s=0}^{\infty} (\beta \theta)^s \left[ (\omega^{-1} + \xi^{-1}) \hat{y}^{e,(k)}_{t+s|t} - (1 + \xi^{-1}) \eta^z_{t+s} \right].
\]

These two conditions resemble the pair of beauty contest problems as in Angeletos and Lian (2016), except that the Taylor Rule also plays a role here. Now impose the initial conditions \((y^{e,(1)}, \pi^{e,(1)}) = (\hat{y}_{t-1}, 0)\), and the exogenous process of \((\eta^d, \eta^z)\), so that we can get the level-1 equilibrium conditions. Level-k expectations with \(k \in [1, 2]\) can be derived from it.

**Proposition 1.** Level-k expectations for \(k \in [1, 2]\) are given by

\[
\hat{y}^{e,(k)}_{t|t} - \hat{y}_{t-1} = (k - 1) \left[ -\frac{(1 - \rho_r) \omega \phi_y}{(1 - \beta)(1 - \beta \rho_r)} \hat{y}_{t-1} - \frac{\omega \rho_r}{1 - \beta \rho_r} \hat{r}_{t-1} - \frac{\omega}{1 - \beta \rho_d} \eta^d_{t} \right],
\]

\[
\hat{\pi}^{e,(k)}_{t|t} = (k - 1) \left[ (\omega^{-1} + \xi^{-1})(1 - \theta) \hat{y}_{t-1} - (1 + \xi^{-1})(1 - \theta) \frac{1 - \beta \theta}{1 - \beta \theta \rho_z} \eta^z_{t} \right].
\]

A few lessons can be learned from this proposition. First, when \(\phi_y = 0\), growth expectations are only driven by the last period nominal interest rate and the current demand wedge, while inflation expectations are only driven by the last period output and current technology level. This is also true in the full model so that we have Corollary 1.
Corollary 1. \[
\frac{\partial Y^{e(k)}(S)}{\partial Y_-} = 1, \quad \text{and} \quad \frac{\partial Y^{e(k)}(S)}{\partial \eta^y} = \frac{\partial \Pi^{e(k)}(S)}{\partial R_-} = 0, \quad \text{if } \phi_y = 0 \text{ and } k \in [1, 2].
\]

This result can be viewed as the missing general equilibrium effects in households expectations. More specifically, higher last period output reduces the current growth rate because it raises inflation and interest rate, which in turn slows down the growth rate; technology progress raises output growth through lowering inflation rate and then the interest rate; higher interest rate and demand wedge lower inflation both via lowering output. All these channels require level of reasoning beyond level-2.

In the full model, one year ahead forecast rules are given by

\[
Y^{e(k),1y}(S) = Y^{e(k)}((H^{e(k)})^3(S)),
\]

\[
\Pi^{e(k),1y}(S) = \prod_{\tau=0}^3 \Pi^{e(k)}((H^{e(k)})^{\tau}(S)).
\]

\[
\frac{\partial Y^{e(k)}(S)}{\partial Y_-} = 1
\]
indicates that the one quarter ahead growth expectations are zero. The feature drives the one year ahead growth expectations also towards 0. In estimation, \(\phi_y\) is not equal to zero, but still a small number close to zero. This limits households’ perception of the mean reverting nature of output dynamics. \(\frac{\partial Y^{e(k)}(S)}{\partial R_-}\) does not suffer from this issue and hence can drive up the usefulness of interest rate related information in growth forecast toward what the expectation data suggest.

Second, level-2 expectations are over-shooting. When we move from level-1 to level-2, households start to fully realize the direct effect of interest rate on output, and firms start to realize the direct effect of output on inflation. However, households neglect the reaction of interest rate to output so that they over-predict the persistence of interest rate, while firms neglect the response of output to inflation so that they overpredict the persistent of output. As a result, they anticipate too much growth and inflation responses in level-2. Reasoning beyond level-2 may have a chance to reduce the over-reaction, but induces too much mean reversion in output. Therefore, the theoretical analysis suggests that parameter k is likely to be some number between 1 and 2 in the estimation.
The third lesson is that in growth expectations, higher discount factor $\beta$ increases the response of growth expectations. Since the steady state real interest rate is driven by $\beta$, higher $\beta$ implies lower interest rate, and high present value of future incomes. When interest rate is changed by some fixed amount, the present value of incomes respond more if the original level interest rate is low. In the extreme case when $\beta$ is close to 1, we must have $k$ close to 1 in order to generate realistic output expectations. We need a realistic planning horizon of the households, otherwise, the estimation of level-$k$ will not have an proper economic interpretation.

This result resembles the complementarity between level-$k$ and market incompleteness in Farhi and Werning (2017). However, the question I intend to address is essentially different. They fix parameter $k$, and argue that the market incompleteness that is neutral under rational expectations affects both expectations and allocations under level-$k$ reasoning. In my paper, when market incompleteness increases, $k$ has to increase also in order to fit the expectation data. Therefore, market incompleteness is not used to affect expectations, but to capture the effects of biased expectations.

### 4.2 The Effect of Expectation Biases

**Irrelevance Conditions.** Lemma 1 indicates that the aggregate output is determined by the consumption of the unconstrained households. The particular way of specifying borrowing constraints allows me to derive the following equilibrium condition after imposing the goods market clearing condition. The equilibrium prices will induce the constrained households to make decisions that are exactly identical to the unconstrained households.

**Lemma 5.** In the level-$k$ equilibrium of the full model, we have

$$u'(Y^{(k)}(S_-)) = \beta(\lambda_{11} + (1 - \lambda_{11})\zeta) \cdot \exp(\eta^d) \cdot \mathbb{E}[\left(R^{e,(k)}(S_-, \epsilon) / \Pi^{e,(k)}(S)\right)$$

$$\{\lambda u'(c^{e,(k)}(1, 0, S)) + (1 - \lambda)u'(c^{e,(k)}(\zeta, 0, S))\}|S_-],$$

where $\lambda = \frac{\lambda_{11}}{\lambda_{11} + (1 - \lambda_{11})\zeta}$.

A few irrelevance conditions can be derived from 5, and I summarize them in Proposition 2.
Proposition 2. If parameter $\beta$ is calibrated to match the steady state interest rate, then

1. Under rational expectations, idiosyncratic shocks do not affect equilibrium allocations.
2. Under level-$k$ reasoning, idiosyncratic shocks affect them only through $\lambda$.

The first part of Proposition 2 is true because we can impose $\alpha^{e,(+\infty)}(1,0,S) = Y^{(+\infty)}(S)$ to get an equilibrium Euler equation

$$u'(Y(S)) = \beta(\lambda_{1\|1} + (1 - \lambda_{1\|1})\zeta) \cdot \exp(\eta^d) \cdot \mathbb{E}[(R(S_-, S)/\Pi(S))u'(Y(S))|S_-].$$

This provides an insight similar to the main argument of Werning (2015). It isolates the role of precautionary savings in affecting the aggregate discount factor, without bringing in any side effects. This proposition justifies the use of representative agents in New Keynesian models with rational expectations when redistribution is not a major concern.

The second part Proposition 2 is true for a similar reason as in the first part. It implies that the two parameters $(\lambda_{1\|1}, \zeta)$ only have one degree of freedom to be disciplined. It also implies that $\lambda_{1\|1}$ is irrelevant to the equilibrium allocations. The major role of $\lambda$ is to affect households planning horizons, which is similar to the effect of $\beta$.

There is one situation, in which expectations are biased while households’ planning horizon is still irrelevant. That is the case when $\lambda_{1\|1} = 1$, and the unconstrained households perceived their own consumption to be identical to the average of the others in the future, i.e.

$$u'(Y(S)) = \beta(\lambda_{1\|1} + (1 - \lambda_{1\|1})\zeta) \cdot \exp(\eta^d) \cdot \mathbb{E}[(R^e(S_-, S)/\Pi^e(S))u'(Y^e(S))|S_-].$$

This property has been discussed in Honkapohja, Mitra and Evans (2012) and directly imposed in Milani (2011) and Fuhrer (2017). Here, I will use an example to show that imposing this property, the so call “Euler Equation learning”, will impose very strict restrictions on expectations if we require that households are making optimized decisions.

Example 1. Consider a special case in which $\lambda_{1\|1} = 1$ (representative agents), $\theta = 1$ (fully sticky price), $\phi_y > 0$ (monetary rule with active responding to output), $\epsilon' = 0$ (no monetary shocks). Use $\bar{\mathbb{E}}$ to denote the operator of subjective expectations.
If agents expect their future decisions to be identical to the average of others, then

$$\hat{c}_t = (1 - \beta)\hat{y}_t + (1 - \beta) \sum_{s=1}^{\infty} \beta^s \bar{E}_t \hat{y}_{t+s} - \omega \sum_{s=0}^{\infty} \beta^s \bar{E}_t (r_{t+s} - \eta_{t+s}^d).$$

Again impose $$\bar{E}_t \hat{c}_{t+1} = \bar{E}_t \hat{y}_{t+1},$$ then households understanding that they will make optimization choice in the future will expect

$$\bar{E}_t \hat{y}_{t+1} = (1 - \beta)\bar{E}_t \hat{y}_{t+1} + (1 - \beta) \sum_{s=1}^{\infty} \beta^s \bar{E}_t \bar{E}_{t+1} \hat{y}_{t+1+s} - \omega \sum_{s=0}^{\infty} \beta^s \bar{E}_t \bar{E}_{t+1} (r_{t+1+s} - \eta_{t+1+s}^d).$$

Apply $$\bar{E}_t \bar{E}_{t+1} = \bar{E}_t,$$ and consider an initial value $$\eta_{t-1}^d = 0.$$ Then, we have

$$\bar{E}_t \hat{y}_{t+1} = [1 - (1 + \phi \omega) \beta] \sum_{s=0}^{\infty} \beta^s \bar{E}_t \hat{y}_{t+1+s}. \quad (1)$$

Equation (1) imposes a strong restriction on expectations. For example, Milani (2007) assumes that agents estimate $$\hat{E}_t \hat{y}_t = \alpha_1 + \alpha_2 \hat{y}_{t-1} + \alpha_3 \eta_{t}^d$$ from historical data. When $$\eta_{t}^d = 0,$$ this becomes $$\hat{E}_t \hat{y}_t = \alpha_1 + \alpha_2 \hat{y}_{t-1}.$$ Applying $$\bar{E}_t \bar{E}_{t+1} = \bar{E}_t$$ yields

$$\bar{E}_t \hat{y}_{t+s} = \alpha_1 \frac{1 - \alpha_2^{s+1}}{1 - \alpha_2} + \alpha_2^{s+1} \hat{y}_{t-1}.$$ 

Substituting this back to equation (1) yields $$\alpha_2 = 0$$ or $$\alpha_2 > 1.$$ $$\alpha_2 > 1$$ is implausible, because it implies forward exploding expectations. In another word, it eliminates the backward looking feature of output expectations.\footnote{Milani (2007) requires $$0 < \alpha_2 < 1$$ to get persistent output dynamics from expectations. Hence agents in the model are not making optimized decisions. Honkapohja et al. (2012) argues that agents who observe $$\hat{c}_{t-1} = \hat{y}_{t-1}$$ should also expect that $$\bar{E}_t \hat{c}_{t+1} = \bar{E}_t \hat{y}_{t+1}.$$ This also implicitly implies that agents are not making optimized decisions in many situations.}

The lesson from this example is that when we deviate from rational expectations, it becomes very difficult to ensure: (1) households perceive their own future decisions to be identical to others, (2) households understand that they are making optimized decisions.
Biased Expectation Effects  This subsection will characterize how the equilibrium allocations are affected by the biased expectations when $k \in [1, 2]$, and $(\eta^d, \eta^z, \phi_y) = (0, 0, 0)$.

Proposition 1 yields

\[
\hat{y}_{t|t}^{e,(k)} = \hat{y}_{t-1} - (k - 1)\frac{\omega \rho_r}{1 - \beta \rho_r} \hat{r}_{t-1},
\]

\[
\hat{\pi}_{t|t}^{e,(k)} = (k - 1)(\omega^{-1} + \xi^{-1})(1 - \theta)\hat{y}_{t-1}.
\]

The law of motion for the perceived interest rate is

\[
\hat{\pi}_{t|t}^{e,(k)} = \rho_r \hat{r}_{t-1} + (1 - \rho_r) \phi_\pi (k - 1)(\omega^{-1} + \xi^{-1})(1 - \theta)\hat{y}_{t-1}.
\]

It is easy to verify that the matrix representing the perceived law of motion for $(\hat{y}_{t}, \hat{\pi}_{t})$ have two complex conjugated eigenvalues inside the unit circle. As a result, we have Corollary 2.

**Corollary 2.** When $\lambda_{1|1} = 1$, $k \in [1, 2]$, and $(\eta^d, \eta^z, \phi_y) = (0, 0, 0)$, the perceived output dynamics converge to zero in the long run.

Recall the first half of Lemma 4 with $(\eta^d, \eta^z, \phi_y) = (0, 0, 0)$,

\[
\hat{y}_{t}^{(k)} = (1 - \beta) \sum_{s=0}^{\infty} \beta^s \hat{y}_{t+1+s|t}^{e,(k)} - \omega \sum_{s=0}^{\infty} \beta^s \left\{ \rho_r^{s+1} \hat{r}_{t-1} + \phi_\pi (1 - \rho_r) \sum_{\tau=0}^{s} \rho_r^{s-\tau} \hat{\pi}_{t+\tau|t}^{e,(k)} - \hat{\pi}_{t+1+s|t}^{e,(k)} \right\}.
\]

Rearranging this condition yields the following proposition

**Proposition 3.** When $\lambda_{1|1} = 1$, $k \in [1, 2]$, and $(\eta^d, \eta^z, \phi_y) = (0, 0, 0)$,

\[
\hat{y}_{t}^{(k)} = \omega \frac{\rho_r}{1 - \beta \rho_r} \hat{r}_{t-1} - \omega \frac{\phi_\pi (1 - \rho_r)}{1 - \beta \rho_r} (k - 1)(\omega^{-1} + \xi^{-1})(1 - \theta)\hat{y}_{t-1}
\]

\[
- \omega \beta \left[ \frac{\phi_\pi (1 - \rho_r)}{1 - \beta \rho_r} - 1 \right] (k - 1)(\omega^{-1} + \xi^{-1})(1 - \theta)\hat{y}_{t|t}^{e,(k)}
\]

\[
+ \left\{ 1 - \frac{\omega \beta}{1 - \beta} \left[ \frac{\phi_\pi (1 - \rho_r)}{1 - \beta \rho_r} - 1 \right] (k - 1)(\omega^{-1} + \xi^{-1})(1 - \theta) \right\} (1 - \beta) \sum_{s=0}^{\infty} \beta^s \hat{y}_{t+1+s|t}^{e,(k)}.
\]

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From Proposition 3, we know $\beta$ affects the role of output expectations through two channels. First, when $\beta$ gets smaller, $1 - \frac{\omega \beta (1 - \rho)}{1 - \beta \rho} \left[ \frac{\phi \pi^*}{(1 - \theta)} \right] (k - 1) (\omega^{-1} + \xi^{-1})(1 - \theta)$ gets larger, so that expectations as a whole can play a larger role. Second, since the biases in output expectations are only short lived, $(1 - \beta) \sum_{s=0}^{\infty} \beta^s \hat{y}_{t+1+s|t}$ implies that smaller $\beta$ places larger weight on the expectations in the near future instead of the far future, so that output expectations can also play a larger role.

The role of $k$ should be discussed in two different situations. First, given fixed output expectations, larger $k$ (not beyond 2) will always weaken the role of output expectations. Second, Proposition 1 indicates that larger $k$ also implies large response in households growth expectations. As a result, the effects from large $k$ is actually ambiguous and we need quantitative work to figure the combined effects.

Besides $(\beta, k)$, other parameters such as the intertemporal elasticity of substitution $\omega$, price stickiness $\theta$, and interest rate responses to inflation $\phi \piem$ can all affect the role of output expectations. The role of expectations is increasing in $\theta$ but decreasing in $\omega$ and $\phi \piem$. All these parameters should be disciplined property by macroeconomic data.

In the full model, I will no longer assume that $\beta$ gets smaller, but instead use occasionally binding borrowing constraints to obtain similar effect of short planning horizons. The working channels are similar, but the results are less transparent.
5 Empirical Analysis

5.1 Estimation

I denote by $\Theta \in \mathbb{R}^{15}$ the vector of model parameters. It is convenient to organize the discussion around the following partition, $\Theta = (\Theta_1, \Theta_2, \Theta_3, \Theta_4)$:

$$\begin{align*}
\Theta_1 &= (\omega, \xi, \zeta), \\
\Theta_2 &= (\lambda_{11}, \beta), \\
\Theta_3 &= (\theta, \phi_y, \phi_d, \rho_z, \rho_r, \sigma_d, \sigma_z, \sigma_r), \\
\Theta_4 &= k.
\end{align*}$$

In $\Theta_1$ and $\Theta_2$, $(\omega, \xi) = (0.50, 1/0.72)$ is chosen directly as is standard in literature. $\zeta = 2.00$ is chosen so that households never perceive that they accumulate enough wealth to avoid the binding of borrowing constraint. $\lambda_{11} = 0.957$ is calibrated to match the MPC of unconstrained households (7%) following Kaplan and Violante (2014). $\beta = 0.954$ is inferred from

$$\beta[\lambda_{11} + (1 - \lambda_{11})\zeta](R_{SS}/\Pi_{SS}) = 1.$$ 

These parameters are summarized in the Table 2.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Value</th>
<th>Targets</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega$</td>
<td>0.50</td>
<td>standard in literature</td>
</tr>
<tr>
<td>$\xi$</td>
<td>1/0.72</td>
<td>standard in literature</td>
</tr>
<tr>
<td>$\zeta$</td>
<td>2.00</td>
<td>ensure $b = 0$ for $\zeta = \zeta$ in expectations</td>
</tr>
<tr>
<td>$\lambda_{11}$</td>
<td>0.957</td>
<td>MPC=7% for unconstrained households</td>
</tr>
<tr>
<td>$\beta$</td>
<td>0.954</td>
<td>2% annualized real interest rate in steady state</td>
</tr>
</tbody>
</table>

For each parameter $k$ in $\Theta_4$, $\Theta_3$ is estimated to fit the quarterly growth rate, inflation rate, and federal fund rate during 1985q1-2007q4 by maximal likelihood.
Denote \( \{ \beta_{j,i}^{\text{MSC}}, \beta_{j,i}^{\text{SMM}} \} \) with \( j \in \{g, \pi\} \) and \( i \in \{y, r, \pi\} \) as the coefficients of the predictability regression (Table 4) for MSC data and the simulated model, and define a distance

\[
M = \frac{1}{6} \left[ \sum_{i,j} \left( \beta_{j,i}^{\text{SMM}} - \beta_{j,i}^{\text{MSC}} \right)^2 \right]^{\frac{1}{2}}.
\]

\( k \) is estimated to minimize the \( M \). The following table summarizes the estimation results.

<table>
<thead>
<tr>
<th>( k )</th>
<th>1.5</th>
<th>+( \infty )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \theta )</td>
<td>0.948</td>
<td>0.944</td>
</tr>
<tr>
<td>( \phi_x )</td>
<td>1.624</td>
<td>1.764</td>
</tr>
<tr>
<td>( \phi_y )</td>
<td>0.202</td>
<td>0.221</td>
</tr>
<tr>
<td>( \rho_d )</td>
<td>0.917</td>
<td>0.933</td>
</tr>
<tr>
<td>( \rho_z )</td>
<td>0.807</td>
<td>0.744</td>
</tr>
<tr>
<td>( \rho_r )</td>
<td>0.855</td>
<td>0.868</td>
</tr>
<tr>
<td>100( \sigma_d )</td>
<td>0.163</td>
<td>0.167</td>
</tr>
<tr>
<td>100( \sigma_z )</td>
<td>5.876</td>
<td>5.549</td>
</tr>
<tr>
<td>400( \sigma_r )</td>
<td>0.452</td>
<td>0.449</td>
</tr>
</tbody>
</table>

The values of estimated parameters are close in \( k = 1.5 \) and \( k = +\infty \).

5.2 Model Fit

**Targeted Regression.** The following table demonstrates the fit of the targeted regressions for the estimation of parameter \( k \). These regression examines the predictability of one year ahead growth and inflation forecast errors by current output, interest rate, and inflation rate. The first column shows that households are more over-confident on output growth when the current output level is higher and when current inflation rate is higher. The second column is the regression for level-\( k \) model. The model has similar predictability as the data. The third and forth columns compare the predictability of inflation forecast errors between data and model. The model fit is not as good as that for growth, but still improve upon rational expectations models, which has no predictability in forecast errors at all.
Table 4: Predictability of Forecast Errors

<table>
<thead>
<tr>
<th></th>
<th>$FE_{t+4}^{g, MSC}$</th>
<th>$FE_{t+4}^{g,k=1.5}$</th>
<th>$FE_{t+4}^{π, MSC}$</th>
<th>$FE_{t+4}^{π,k=1.5}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{y}_t$</td>
<td>0.305**</td>
<td>0.450</td>
<td>-0.260**</td>
<td>0.112</td>
</tr>
<tr>
<td></td>
<td>[0.130, 0.481]</td>
<td>[-0.339, -0.181]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\hat{r}_t$</td>
<td>-0.105</td>
<td>-0.023</td>
<td>0.313**</td>
<td>0.039</td>
</tr>
<tr>
<td></td>
<td>[-0.296, 0.087]</td>
<td>[0.227, 0.399]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\hat{π}_t$</td>
<td>0.742**</td>
<td>0.295</td>
<td>-0.370**</td>
<td>-0.291</td>
</tr>
<tr>
<td></td>
<td>[0.392, 1.091]</td>
<td>[-0.528, -0.213]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.287</td>
<td>0.276</td>
<td>0.445</td>
<td>0.197</td>
</tr>
<tr>
<td>obs.</td>
<td>88</td>
<td>10000</td>
<td>88</td>
<td>10000</td>
</tr>
</tbody>
</table>

* Blue numbers are substantially improved compared with RE.

Untargeted Regressions  This section runs similar regressions as in Table 5 to check of the model generated growth expectations. All coefficients are close between model and data except for the mean reverting nature of output. My model still produces some mild reversion in growth expectations because I assume that households completely understand the Taylor Rule so that the interest rate response to output is understood by agents below level-2. If I assume that agents are not aware of this connection as is support by Carvalho and Nechio (2014), then, there will be very little reversion in growth expectations. I choose not to do this in the baseline model just to avoid too many deviations from the standard framework.

Table 5: Predictability of Growth Expectations

<table>
<thead>
<tr>
<th></th>
<th>MSC</th>
<th>Model</th>
<th>MSC</th>
<th>Model</th>
<th>MSC</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{y}_t$</td>
<td>-0.018</td>
<td>-0.138***</td>
<td>0.053**</td>
<td>-0.118***</td>
<td>0.073***</td>
<td>-0.093***</td>
</tr>
<tr>
<td>$\hat{r}_t$</td>
<td>-0.148***</td>
<td>-0.117***</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\hat{r}<em>t - \hat{\pi}</em>{t-4}$</td>
<td>0.031</td>
<td>0.064***</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\hat{r}<em>t - \hat{\pi}</em>{t+4}^{year}$</td>
<td>-0.121***</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Obs.</td>
<td>84</td>
<td>10000</td>
<td>88</td>
<td>10000</td>
<td>88</td>
<td>10000</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.014</td>
<td>0.283</td>
<td>0.190</td>
<td>0.254</td>
<td>0.358</td>
<td>0.271</td>
</tr>
</tbody>
</table>

* Standard errors are in the parenthesis. ***, *** denote significance at the 10%, 5% and 1% levels.

The Overall Fit. I also check the overall fit of the model in figure 4. The top two panels show the fit of expectation data and macroeconomic data corresponding to different value k. The bottom two panels show the fit of expectation data along business cycles by plotting the model generated expectations, the expectation data, and the realized data together.
The distance in predictability regressions is minimized at level-1.5. \( k = +\infty \) provides a benchmark in which forecast errors are not predictable at all. The fit of macroeconomic data is measured by the log-likelihood values. The top right panel shows that level-1.5 model does not improve the fit, while level-1 model fits better. This is one of the reasons why I do not estimate all parameters together. Otherwise, the fit of macroeconomic data will drive the estimation of parameter \( k \) towards \( k = 1 \). Since the goal of my paper is not to check whether level-k model can match both dataset, I choose to target on expectations data, and explore its implications for economic allocations. The 2 panels as the bottom shows that the model generated expectations well capture the variance of expectation data. The model generated inflation expectations lead the expectation data by 1 or 2 quarters, and also increases too strongly during the late 1990s.
5.3 Propagation and Amplification

This subsection compares other estimated parameters for different values of $k$. The results are summarized in Figure 5. As $k$ increases from 1 to 2, the estimated persistence of demand wedge increases from 0.61 to 0.95, with a corresponding decline of standard deviations while the estimated persistence of interest rate declines from 0.90 to 0.82 with no change in standard deviations. These results indicate that lower levels of $k$ can generate more internal propagation, while higher levels of $k$ generate more amplification when $k \in [1, 2]$. Propagation helps fit the macroeconomic data, while amplification does not according to Figure 4.

Figure 5: Other Estimated Parameters for Different $k$
6 Quantitative Results

Baseline Results  Figure 6 summarizes the baseline results. The top left panels show the output, output expectations, and interest rate response to negative one standard deviation of a federal fund rate shock. The response of the output expectations are plotted at each point of time with a one year horizon. Households over-predict the persistence of output dynamics and hence are over-confident in growth when output level is high. The remaining three panels compare the impulse responses of the federal fund rate, output, and inflation between the level-k model, rational expectations model, and VAR results. The rational expectations model fits the size of responses, but not the hump-shaped. The level-k model has the potential to generate a hump-shape, but not large enough compared with data. One of the reasons is that the Taylor Rule assumes that interest rate responds to output, and households understand it. As a result, the mean reverting of output in households’ expectations is still too strong compared with data. In fact, Carvalho and Nechio (2014) has provided some empirical evidence showing that households are not aware of the connection between economic slack and interest rate.

Figure 6: Baseline Results
**Level-k Effects.** Figure 7 summarizes the effects of level-k. I compare the baseline case $k = 1.5$ with $k = 1.0$, $k = 2.0$ and $k = +\infty$ for both re-estimated parameters and the baseline parameters under a -100 basis point federal fund rate shock. The two bottom panels with baseline parameters indicate that the level of rationality has a non-monotonic effect on the size of output responses. As households have higher level of rationality, they understand more about the consequence of an interest rate shock, but also become more aware of the reverting nature of the output dynamics. As a result, their over-confidence of the output dynamics is not increasing monotonically. The impulse responses of inflation is unrealistically too strong under $k = 1.0$. One possible reason is that the model assumes mechanical connection between real output and real wage, and hence too easy to understand for a level-k agents, which does not show up in the expectations data. In the top two panels, the level-k effects are monotonic because the persistence of interest rate is decreasing in $k$ as in Figure 5 to limit the response of output and inflation when $k$ is high. Despite the difference in quantity, all exercises show that level-k can amplify both the output and inflation responses.

Figure 7: Level-k Effects
MPC Effects. Figure 8 summarizes the MPC effects. The baseline case has MPC=7.0% for the unconstrained households. Once this number is cut down by one half, the amplification of level-1.5 on output responses almost disappears. This is partly due to smaller forecast errors and also partly due to the weight households put on forecast errors in short horizons.

![Figure 8: MPC Effects](image)

7 Conclusion

This paper uncovers the feature that households’ reasoning in growth forecast is asymmetric. I interpret it as finite depth of reasoning, and formalize it in a level-k New Keynesian model with incomplete markets. The results indicate that level-k always amplifies the output responses to a federal fund rate shock, and the MPC of the unconstrained households is crucial for these results. The amplification can be viewed as shutting down one self-stabilizing channel of output dynamics in households’ expectations. Level-k also produces some internal propagation, and reduces the reliance on persistence exogenous demand wedge, although this propagation is not large enough to improve the fit to macroeconomic data. The results shed light on the importance of households’ expectations in monetary policy analysis.
This paper opens the Pandora’s Box and directly explores the effects of biased expectations. This strand of literature is still in its infancy stage in the sense that the nature of biases and the corresponding consequences are not well explored. A lot of questions need to be answered.

Is asymmetric reasoning a robust feature of households and firms’ expectations in micro data, as well as for other variables? Is it possible to identify whether the under-response in households and firms’ expectations are due to limited attention or finite depth of reasoning? To what extent does it matter whether expectations are driven by limited attention or finite depth of reasoning? Can we endogenize the depth of reasoning? Is finite depth of reasoning useful to study other type of questions such as financial bubbles? How should we introduce finite depth of reasoning into the competitive equilibrium in a way that properly handles the issue that market participants may be able to infer the economic fundamentals from the prices they observe? How should stabilization policies be designed when only part of the general equilibrium effects are anticipated? I leave all these valuable questions for future research.
References


Appendix A: State Space Representation of the Full Model

The standard solution procedure for rational expectations DSGE models cannot be directly applied here. Hence, it is useful to describe how to write the model into a state space form. Use $\Gamma$ to denote the coefficients in linearized equilibrium objects, and the solution procedure can be briefly described in the following.

1. Solve for $\Gamma^{ca,e}$ without using equilibrium objects.
2. Initialize $(\Gamma^{ys,e,(1)}, \Gamma^{\pi s,e,(1)}, \Gamma^{ws,e,(1)}, \Gamma^{\tau s,e,(1)})$ from level-0, and obtain $\Gamma^{ss,e,(1)}$.
3. Solve for $\Gamma^{cs,e,(1)}$ from the perceived households’ problem.
4. Solve for $(\Gamma^{ys,(1)}, \Gamma^{\ell s,(1)}, \Gamma^{ws,(1)}, \Gamma^{\tau s,(1)})$ from the temporary equilibrium.
5. Solve for $\Gamma^{\pi s,(1)}$ from the firms’ problem, and obtain $\Gamma^{ss,(1)}$.
6. Use $(\Gamma^{ys,e,(j+1)}, \Gamma^{\pi s,e,(j+1)}, \Gamma^{ws,e,(j+1)}, \Gamma^{\tau s,e,(j+1)}) = (\Gamma^{ys,(j)}, \Gamma^{\pi s,(j)}, \Gamma^{ws,(j)}, \Gamma^{\tau s,(j)})$ to update.
7. Obtain the state space representation.

Step 1: Solve for $\Gamma^{ca,e}$

Log-linearizing the optimality conditions for the constrained households yields

$$
\omega^{-1}\hat{c}^e(\zeta) = \hat{w}^e - \xi^{-1}\hat{\ell}^e(\zeta),
\hat{c}^e(\zeta) = \hat{a} + wyss(\hat{w}^e + \hat{\ell}^e + \hat{\ell}^e(\zeta)).
$$

$(\Gamma^{ca,e}(\zeta), \Gamma^{fa,e}(\zeta))$ can be obtained from

$$
\begin{bmatrix}
\omega^{-1} & \xi^{-1} \\
1 & -wyss
\end{bmatrix}
\begin{bmatrix}
\Gamma^{ca,e}(\zeta) \\
\Gamma^{fa,e}(\zeta)
\end{bmatrix} =
\begin{bmatrix}
0 \\
1
\end{bmatrix}.
$$

The solution is $\Gamma^{ca,e}(\zeta) = \frac{1}{1 + \xi \omega^{-1} wyss}.$

A1
The unconstrained households have

\[ \hat{e}^e(1) = -\omega(\eta^d + \hat{\rho}^e - \hat{\pi}^e) + \lambda \hat{e}^e(1) + (1 - \lambda)\hat{e}^e(\zeta), \]

\[ \hat{e}^e(1) = (R_{SS}/\Pi_{SS})[\hat{a} + wy_{SS}(\hat{\omega}^e + \hat{\rho}^e + \hat{e}^e(1)) - \hat{e}^e(1)], \]

\[ \hat{\epsilon}(1) = \xi \hat{\omega}^e - \xi \omega^{-1} \hat{e}^e(1). \]

\((\Gamma^{ca,e}(1), \Gamma^{\ell a,e}(1), \Gamma^{aa,e}(1))\) satisfy

\[ \Gamma^{ca,e}(1) = [A^{ca,e}(1) + (1 - \lambda)A^{ca,e}(\zeta)]A^{aa,e}(1), \]

\[ \Gamma^{aa,e}(1) = (R_{SS}/\Pi_{SS})(1 + wy_{SS}A^{ca,e}(1) - A^{ca,e}(1)), \]

\[ \Gamma^{\ell a,e}(1) = -\xi \omega^{-1}A^{ca,e}(1). \]

This yields a quadratic function for \(A^{ca,e}(1)\)

\[ (\Pi_{SS}/R_{SS})A^{ca,e}(1) = [A^{ca,e}(1) + (1 - \lambda)A^{ca,e}(\zeta)][1 - (1 + \xi \omega^{-1}wy_{SS})A^{ca,e}(1)]. \]

Solving \(\lambda\) from \(A^{ca,e}(1)\) yields

\[ \lambda = \frac{\Gamma^{ca,e}(\zeta) - \Gamma^{ca,e}(1)}{\Gamma^{ca,e}(\zeta) - \Gamma^{ca,e}(1)} \frac{(\Pi_{SS}/R_{SS})}{1 - (1 + \xi \omega^{-1}wy_{SS})A^{ca,e}(1)}. \]

The notation \(\Lambda = \frac{\lambda_1}{\lambda_1 + (1 - \lambda_1)\kappa}\) yields

\[ \lambda_1 = [1 + (\Lambda^{-1} - 1)(\zeta^{-1})^{-1}]^{-1}. \]

The fraction of hand-to-mouth households \(\lambda_{HtM}\) satisfies

\[ \lambda_{HtM} = (1 - \lambda_1)/(2 - \lambda_1 - \lambda_2), \]

\[ \lambda_2 = (1 - \lambda_1)(1 - \lambda_{HtM})/\lambda_{HtM}. \]

A2
**Step 2: Initialize** \((\Gamma^{y_{s,e},(1)}, \Gamma^{\pi_{s,e},(1)}, \Gamma^{w_{s,e},(1)}, \Gamma^{\tau_{s,e},(1)})\) and \(\Gamma^{s,s,e,(1)}\)

Specify the level-1 expectations.

\[
\begin{align*}
\Gamma^{y_{s,e},(1)} \hat{s} &= \hat{y}_- , \\
\Gamma^{\pi_{s,e},(1)} \hat{s} &= 0 , \\
\Gamma^{f_{s,e},(1)} \hat{s} &= \hat{y}_- - \eta^e , \\
\Gamma^{w_{s,e},(1)} &= \omega^{-1}\Gamma^{y_{s,e},(1)} + \xi^{-1}\Gamma^{f_{s,e},(1)} , \\
\Gamma^{\tau_{s,e},(1)} &= \omega y^{ss}_S \Gamma^{y_{s,e},(1)} - \Gamma^{f_{s,e},(1)} - \Gamma^{w_{s,e},(1)} .
\end{align*}
\]

According to the perceived Taylor Rule,

\[
\Gamma^{r_{s,e},(1)} = \rho_r \Gamma^r + (1 - \rho_r)(\phi_n \Gamma^{\pi_{s,e},(1)} + \phi_y \Gamma^{y_{s,e},(1)}) .
\]

The state variable is \(\hat{s} = (\hat{y}_-, \hat{\pi}_-, \eta^d, \eta^e)\). \(\Gamma^{s,s,e,(1)}\) can be obtained from \((\Gamma^{y_{s,e},(1)}, \Gamma^{r_{s,e},(1)})\) and the exogenous law of motion for \((\eta^d, \eta^e)\).

**Step 3: Solve for** \(\Gamma^{c,s,e,(1)}\)

Recall the optimality conditions of the constrained households

\[
\omega^{-1} \hat{c}^{e}(\zeta) = \hat{w}^e - \xi^{-1} \hat{c}^e(\zeta) ,
\]

\[
\hat{c}^{e}(\zeta) = \hat{a} + wy^{ss}_S (\hat{w}^e + \hat{\pi}^e + \hat{c}^e(\zeta)) .
\]

\((\Gamma^{c,s,e,(1)}(\zeta), \Gamma^{f_{s,e},(1)}(\zeta))\) can be obtained from

\[
\begin{bmatrix}
\omega^{-1} & \xi^{-1} \\
1 & -wy^{ss}_S
\end{bmatrix}
\begin{bmatrix}
\Gamma^{c,s,e,(1)}(\zeta) \\
\Gamma^{f_{s,e},(1)}(\zeta)
\end{bmatrix} =
\begin{bmatrix}
\Gamma^{w_{s,e},(1)} \\
wy^{ss}_S (\Gamma^{w_{s,e},(1)} + \Gamma^{\tau_{s,e},(1)})
\end{bmatrix} .
\]
Recall the optimality conditions of the unconstrained households

\[
\check{c}^e(1) = -\omega(\eta^d + \check{\xi}^e - \hat{\xi}^e') + \Delta \check{c}^{e'}(1) + (1 - \Delta)\hat{c}^{e'}(\zeta),
\]
\[
\check{\lambda}^e(1) = \frac{R_{SS}/\Pi_{SS}}{\hat{\lambda} + wy_{SS}(\check{\xi}^e + \hat{\xi}^e(1)) - \check{c}^e(1)},
\]
\[
\hat{\xi}^e(1) = \xi \check{\xi}^e - \xi \omega^{-1} \hat{c}^e(1).
\]

\((\Gamma_{cs,e,(1)}(1), \Gamma_{fs,e,(1)}(1), \Gamma_{as,e,(1)}(1))\) satisfy

\[
\Gamma_{cs,e,(1)}(1) = -\omega(\Gamma_{ds} + \Gamma_{rs,e,(1)} - \Gamma_{rs,e,(1)}\Gamma_{ss,e,(1)})
\]
\[
+ [\Delta \Gamma_{ca,e,(1)}(1) + (1 - \Delta)\Gamma_{ca,e,(1)}(\zeta)]\Gamma_{as,e,(1)}(1),
\]
\[
+ [\Delta \Gamma_{cs,e,(1)}(1) + (1 - \Delta)\Gamma_{cs,e,(1)}(\zeta)]\Gamma_{ss,e,(1)},
\]
\[
\Gamma_{as,e,(1)}(1) = \frac{R_{SS}/\Pi_{SS}}{\hat{\lambda} + wy_{SS}(\Gamma_{ws,e,(1)} + \Gamma_{rs,e,(1)} + \Gamma_{fs,e,(1)}(1)) - \Gamma_{cs,e,(1)}(1)},
\]
\[
\Gamma_{fs,e,(1)}(1) = \xi \Gamma_{ws,e,(1)} - \xi \omega^{-1} \Gamma_{cs,e,(1)}(1).
\]

Eliminate \((\Gamma_{fs,e,(1)}(1), \Gamma_{as,e,(1)}(1))\) to obtain a single equation of \(\Gamma_{cs,e,(1)}(1)\)

\[
\Gamma_{cs,e,(1)}(1) = -\omega(\Gamma_{ds} + \Gamma_{rs,e,(1)} - \Gamma_{rs,e,(1)}\Gamma_{ss,e,(1)})
\]
\[
+ [\Delta \Gamma_{ca,e,(1)}(1) + (1 - \Delta)\Gamma_{ca,e,(1)}(\zeta)]
\]
\[
\cdot \frac{R_{SS}/\Pi_{SS}}{\hat{\lambda} + wy_{SS}((1 + \xi)\Gamma_{ws,e,(1)} + \Gamma_{rs,e,(1)}(1)) - (1 + \xi \omega^{-1} wy_{SS})\Gamma_{cs,e,(1)}(1)},
\]
\[
+ [\Delta \Gamma_{cs,e,(1)}(1) + (1 - \Delta)\Gamma_{cs,e,(1)}(\zeta)]\Gamma_{ss,e,(1)},
\]

The solution for \(\Gamma_{cs,e,(1)}(1)\) is

\[
\Gamma_{cs,e,(1)}(1) = \left\{ \frac{(R_{SS}/\Pi_{SS})wy_{SS}[[\Delta \Gamma_{ca,e,(1)}(1) + (1 - \Delta)\Gamma_{ca,e,(1)}(\zeta)]((1 + \xi)\Gamma_{ws,e,(1)} + \Gamma_{rs,e,(1)})]
\]
\[
+ (1 - \Delta)\Gamma_{cs,e,(1)}(\zeta)\Gamma_{ss,e,(1)} - \omega(\Gamma_{ds} + \Gamma_{rs,e,(1)} - \Gamma_{rs,e,(1)}\Gamma_{ss,e,(1)})
\]
\[
\cdot \left[ I + \frac{(R_{SS}/\Pi_{SS})(1 + \xi \omega^{-1} wy_{SS})[[\Delta \Gamma_{ca,e,(1)}(1) + (1 - \Delta)\Gamma_{ca,e,(1)}(\zeta)] * I - \Delta \Gamma_{ss,e,(1)}]}{-1}.
\]

A4
Step 4: Solve for \((\Gamma_{ys,(1)}, \Gamma_{\ell s,(1)}, \Gamma_{ws,(1)}, \Gamma_{\tau s,(1)})\)

The temporary equilibrium satisfies

\[
\hat{y} = -\omega(\eta^d + \hat{\tau}^e - \hat{\pi}^e_t) + \Delta\hat{e}^e(1) + (1 - \Delta)\hat{e}^e(\bar{\zeta}), \\
\hat{\ell} = \hat{y} - \eta^z, \\
\hat{w} = \xi - 1 \hat{\ell} + \omega - 1 \hat{\pi}^t, \\
\hat{\tau} = wy_{\bar{\zeta}} - \hat{\ell} - \hat{w}.
\]

\((\Gamma_{ys,(1)}, \Gamma_{\ell s,(1)}, \Gamma_{ws,(1)}, \Gamma_{\tau s,(1)})\) can be obtained from

\[
\Gamma_{ys,(1)} = -\omega(\Gamma_{d} + \Gamma_{\ell s,e,(1)} - \Gamma_{\tau s,e,(1)}\Gamma_{ws,e,(1)} + [\Delta\Gamma_{cs,e,(1)}(1) + (1 - \Delta)\Gamma_{cs,e,(1)}(\bar{\zeta})]\Gamma_{ss,e,(1)}, \\
\Gamma_{\ell s,(1)} = \Gamma_{ys,(1)} - \Gamma_{z}, \\
\Gamma_{ws,(1)} = \xi - 1 \Gamma_{\ell s,(1)} + \omega - 1 \Gamma_{ys,(1)}, \\
\Gamma_{\tau s,(1)} = wy_{\bar{\zeta}} - \Gamma_{\ell s,e,(1)} - \Gamma_{ws,(1)}.
\]

Step 5: Solve for \(\Gamma_{\tau s,(1)}\) and Obtain \(\Gamma_{ss,(1)}\)

The linearized Phillips Curve with arbitrary expectations \(\hat{E}_t\) is

\[
\hat{\pi}_t = (1 - \theta)(1 - \beta\theta)\sum_{s=0}^{\infty} (\beta\theta)^s \hat{E}_t(\hat{w}_{t+s} - \eta_{t+s}^z) + (1 - \theta)\sum_{s=0}^{\infty} (\beta\theta)^s \hat{E}_t \hat{\pi}_{t+s}.
\]

The matrix representation for \((\Gamma_{\tau s,(1)}, \Gamma_{rs,(1)})\) is

\[
\Gamma_{\tau s,(1)} = (1 - \theta) \left\{ \left( (1 - \beta\theta)\Gamma_{ws,e,(1)} + \Gamma_{\tau s,e,(1)} \right) (1 - \beta\theta) \Gamma_{ss,e,(1)} \right\}^{-1} - \frac{1 - \beta\theta}{1 - \beta\theta \rho_z} \Gamma_{\tau}.
\]

\[
\Gamma_{rs,(1)} = \rho_r \Gamma_{r} + (1 - \rho_r) \left( \phi_r \Gamma_{\tau s,(1)} + \phi_y \Gamma_{ys,(1)} \right).
\]

\(\Gamma_{ss,(1)}\) can be obtained from \((\Gamma_{ys,(1)}, \Gamma_{\tau s,(1)})\) and the exogenous law of motion for \((\eta^d, \eta^z)\)
Step 6: Update Expectations

For $\forall k \in [1, +\infty)$, first update expectations to $[k]$ using

\[
(\Gamma_{ys,e,(j+1)}, \Gamma_{\pi s,e,(j+1)}, \Gamma_{ws,e,(j+1)}, \Gamma_{r s,e,(j+1)}) = (\Gamma_{ys,(j)}, \Gamma_{\pi s,(j)}, \Gamma_{ws,(j)}, \Gamma_{r s,(j)}).
\]

Level-$k$ expectations are defined as

\[
\text{level-}k = (1 - k + [k]) \cdot \text{level-[}k] + ([k] - k) \cdot \text{level-[}k+1].
\]

Step 7: State Space Representation

The transition equation is

\[
\hat{s}_{t+1} = \begin{bmatrix} \hat{y}_t \\ \hat{r}_t \\ \hat{\eta}^d_{t+1} \\ \hat{\eta}^z_{t+1} \end{bmatrix} = \Gamma_{ss,(k)} \begin{bmatrix} \hat{y}_{t-1} \\ \hat{r}_{t-1} \\ \hat{\eta}^d_t \\ \hat{\eta}^z_t \end{bmatrix} + \begin{bmatrix} 0 \\ \sigma_m \epsilon^t \\ \sigma_d \epsilon^d_t \\ \sigma_z \epsilon^z_t \end{bmatrix}.
\]

The measurement equation is

\[
\begin{bmatrix} \hat{y}_{t+1} - \hat{y}_t \\ \hat{\rho}_{t+1} - \hat{\rho}_t \\ \hat{r}_t \end{bmatrix} = \begin{bmatrix} \Gamma_{ys} - \Gamma^y \\ \Gamma_{\pi s} \\ \Gamma^r \end{bmatrix} \hat{s}_{t+1}.
\]

The expectation equations and ex post counterparts are

\[
\begin{bmatrix} \hat{y}^e_{t+4} - \hat{y}_t \\ \hat{\rho}^e_{t+4} - \hat{\rho}_t \\ \hat{y}_{t+4} - \hat{y}_t \\ \hat{\rho}_{t+4} - \hat{\rho}_t \end{bmatrix} = \begin{bmatrix} \Gamma_{ys,e,(k)}(\Gamma_{ss,e,(k)})^3 - \Gamma^y \\ \Gamma_{\pi s,e,(k)} \sum_{\tau=0}^3 (\Gamma_{ss,e,(k)})^\tau \\ \Gamma_{ys,(k)}(\Gamma_{ss,(k)})^3 - \Gamma^y \\ \Gamma_{\pi s,(k)} \sum_{\tau=0}^3 (\Gamma_{ss,(k)})^\tau \end{bmatrix} \hat{s}_{t+1}.
\]

A6