Abstract

We analyze inﬁnitely repeated prisoners’ dilemma games with imperfect private monitoring, and construct sequential equilibria where strategies are measurable with respect to players’ beliefs regarding their opponents’ continuation strategies. We show that, when monitoring is almost perfect, the symmetric e¢ cient outcome can be approximated in any prisoners’ dilemma game, while every individually rational feasible payo¤ can be approximated in a class of prisoner dilemma games. We also extend the approximate e¢ ciency result to n-player prisoners’ dilemma games and to prisoner’s dilemma games with more general information structure. Our results require that monitoring be su¢ ciently accurate but do not require very low discounting.

1 Introduction

We analyze a class of inﬁnitely repeated prisoners’ dilemma games with imperfect private monitoring and discounting. The main contribution of this paper is to construct “belief-based” strategies, where a player’s continuation strategy is a function only of her beliefs about her opponent’s continuation strategy. This simpliﬁes the analysis considerably — in the two-player case, we explicitly construct sequential equilibria, enabling us to invoke the one-step deviation
principle of dynamic programming. By doing so, we prove that one can approximate the symmetric efficient payoffs in any prisoners' dilemma game provided that the monitoring is sufficiently accurate. Furthermore, for a class of prisoners' dilemma games, one can approximate every individually rational feasible payoff. Our efficiency results also generalize to the $n$ player case, where we show that the symmetric efficient payoff can similarly be approximated.

These results are closely related to an important paper by Sekiguchi [13], who shows that one can approximate the efficient payoff in two-player prisoners' dilemma games provided that the monitoring is sufficiently accurate. Sekiguchi's result applies for a class of prisoners' dilemma payoffs, and relied on the construction of a Nash equilibrium which achieves approximate efficiency. The results in this paper can be viewed as an extension and generalization of the approach taken in Sekiguchi's paper.

Our substantive results are also related to those obtained in recent papers by Piccione [12] and Ely and Välimäki [5], which adopt a very different approach. The current paper (and Sekiguchi's) utilizes initial randomization to ensure that a player's beliefs adjust so that she has the incentive to punish or reward her opponent(s) as is appropriate. The Piccione-Ely-Välimäki approach on the other hand relies on making each player indifferent between her different actions at most information sets, so that her beliefs do not matter. We defer a more detailed discussion of the two approaches to the concluding section of this paper.

The rest of this paper is as follows. Section 2 constructs sequential equilibria which approximate the efficient outcome in the two-player case, while section 3 approximates the set of individually rational feasible payoffs in this case. Section 4 shows that the efficiency result can be generalized to $n$ player prisoners' dilemma games. The final section concludes.

2 Approximating the Efficient Payoffs

<table>
<thead>
<tr>
<th></th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>D</td>
<td>1+g</td>
<td>0</td>
</tr>
</tbody>
</table>

We consider the prisoners' dilemma with the stage game payoffs given above, where the row indicates the player's own action and the column indicates her opponent's action. Players only observe their own actions, and also observe a private signal which is informative about their opponent's action. This signal belongs to the set $S = \{c; d\}$, where $c$ (resp. $d$) is more likely when the opponent plays $C$ (resp. $D$). The signalling structure is assumed symmetric, in the sense that the probability of errors does not depend on the action played. Given any action profile $a = (a_1; a_2)$, $a_1 \in A = \{C\; ; \; D\}$, the probability that exactly one player receives a wrong signal is $\sigma > 0$, and the probability that both receive

\[^{1}\text{Obara [11] found the same kind of strategy independently, but used it for repeated games with imperfect public monitoring to construct a sequential equilibrium which pareto-dominates perfect public equilibria in simple repeated partnership games.}\]
Wrong signals is \( x > 0 \): Players maximize the expected sum of stage game payoffs discounted at rate \( \varepsilon \). We also assume that at the end of each period, players observe the realization of a public randomization device uniformly distributed on the unit interval.

Our approach is closely related to Sekiguchi's [13]: we show that one can construct a mixed trigger strategy sequential equilibrium which achieves partial cooperation. With public randomization or by "dividing up the game" as in Ellison [4], one can modify the strategy appropriately in order to approximate full cooperation. Our approach involves the construction of a "belief-based" strategy, i.e., a strategy which is a function of the player's beliefs about his opponent's continuation strategy. This results in a major simplification as compared to the more conventional notion of a strategy which is a function of the private information of the player.

We begin by defining partial continuation strategies. In any period \( t \); define the partial continuation strategy \( \frac{1}{t} \) as follows: play \( D \) at period \( t \); and at period \( t + 1 \) play \( \frac{1}{t} \) if the realized outcomes in period \( t \) are \( (D_c) \) or \( (D_d) \); define the partial continuation strategy \( \frac{1}{t} \) as follows: in any period \( t \) play \( C \); at period \( t + 1 \) play \( \frac{1}{t} \) if the realized outcomes in period \( t \) is \( (C_c) \); and play \( \frac{1}{t} \) if the realized outcome at \( t \) is \( (C_d) \). We call \( \frac{1}{t} \) and \( \frac{1}{t} \) a partial continuation strategy since each of these fully specifies the player's actions in every subsequent period at every information set that arises given that he conforms to the strategy.

In consequence, the (random) path and payoffs induced by any pair of partial continuation strategies is well defined. However, a partial continuation strategy does not specify the player's actions in the event that she deviates from the strategy at some information set. This is deliberate, since our purpose is to construct the full strategies that constitute a sequential equilibrium. Note also that for any player \( i \); only the partial continuation strategy of player \( j \) is relevant when computing \( i \)'s payoffs in any equilibrium.

Let \( V_{ab}(\varepsilon; \tau; \gamma) \); \( a, b \in \{C, D\} \) denote the repeated game payoff of \( \frac{1}{t} \) against \( \frac{1}{t} \) — these payoffs are well defined since the path induced by each pair is well defined. We have that \( V_{DD} > V_{CD} \) for all parameter values. Furthermore, if \( \varepsilon > \frac{1}{1 + \gamma} \) and \( (\varepsilon + \gamma) \) is sufficiently small, then \( V_{CC} > V_{DC} \): Suppose that player \( i \) believes that her opponent is playing either \( \frac{1}{t} \) or \( \frac{1}{t} \); and is playing \( \frac{1}{t} \) with probability \( \tau \). Then the difference between the payoff from playing \( \frac{1}{t} \) and the payoff from playing \( \frac{1}{t} \) is given by

\[
\xi V^{(1)}(\varepsilon; \tau; \gamma) = V_{CC} - V_{CD} + \tau(1 - \tau)(V_{DD} - V_{CD})
\]

Hence \( \xi V^{(1)} \) is increasing and linear in \( \tau \) and there is a unique value, \( \tau \); at which it is zero. Suppose now that at \( t = 1 \) both players are restricted to choosing between \( \frac{1}{t} \) and \( \frac{1}{t} \): There is a mixed equilibrium of the restricted game, where each player plays the strategy \( \frac{1}{t} \) which plays \( \frac{1}{t} \) with probability \( \tau \) and \( \frac{1}{t} \) with probability \( \tau \). Call this partial mixed strategy \( \frac{1}{t} \). Note that \( \tau \) increases to 1 as we decrease \( \varepsilon \) towards its lower bound \( \frac{1}{1 + \gamma} \). Let \( \varepsilon \) be such that \( \varepsilon \geq \frac{1}{2} \).
For future reference we emphasize that equation (1) applies to any period — if a player believes that her opponent’s continuation strategy is \( \frac{1}{3} \) with probability \( \frac{1}{2} \) and \( \frac{2}{3} \) with probability \( \frac{1}{2} \); then she prefers \( \frac{1}{3} \) to \( \frac{2}{3} \) if \( \frac{1}{2} > \frac{1}{4} \) and prefers \( \frac{2}{3} \) to \( \frac{1}{3} \) if \( \frac{1}{2} < \frac{1}{4} \). Note also that if a player’s opponent begins at \( t = 1 \) with a strategy in \( \frac{1}{3} \); \( \frac{2}{3} \) g; her continuation strategy also belongs to this set, since \( \frac{2}{3} \) induces only \( \frac{1}{3} \); while \( \frac{1}{3} \) may induce either \( \frac{1}{3} \) or \( \frac{2}{3} \); depending upon the private history that the opponent has observed.

We define the following four belief revision operators. Starting with any initial belief \( \frac{1}{2} \); we can define a player’s new beliefs when she takes action \( a \) and receives signal \( \beta \): Her new belief (i.e. the probability that \( j \)’s continuation strategy is \( \frac{1}{3} \)) will be given by \( \tilde{A}_c(a\beta) (\frac{1}{2}) \):

\[
\tilde{A}_c(a\beta) (\frac{1}{2}) = \frac{1}{(1 - \frac{2}{3} \beta + \frac{1}{3} \beta)} \frac{(1 - \frac{2}{3} \beta + \frac{1}{3} \beta)}{(1 - \frac{2}{3} \beta + \frac{1}{3} \beta)}
\]

\[
\tilde{A}_d(a\beta) (\frac{1}{2}) = \frac{1}{(1 - \frac{2}{3} \beta + \frac{1}{3} \beta)} \frac{(1 - \frac{2}{3} \beta + \frac{1}{3} \beta)}{(1 - \frac{2}{3} \beta + \frac{1}{3} \beta)}
\]

\[
\tilde{A}_c(a\beta) (\frac{1}{2}) = \frac{1}{(1 - \frac{2}{3} \beta + \frac{1}{3} \beta)} \frac{(1 - \frac{2}{3} \beta + \frac{1}{3} \beta)}{(1 - \frac{2}{3} \beta + \frac{1}{3} \beta)}
\]

\[
\tilde{A}_d(a\beta) (\frac{1}{2}) = \frac{1}{(1 - \frac{2}{3} \beta + \frac{1}{3} \beta)} \frac{(1 - \frac{2}{3} \beta + \frac{1}{3} \beta)}{(1 - \frac{2}{3} \beta + \frac{1}{3} \beta)}
\]

Starting with any initial belief \( \frac{1}{2} \) at the beginning of the game, a player’s belief at any private history, i.e. after an arbitrary sequence \( H(a_1, a_2, \ldots, a_t) \); can be computed by iterated application of the appropriate belief operators. Let \( \Psi (\frac{1}{2}) \) be the set of possible beliefs, i.e. \( \frac{1}{2} \geq \frac{1}{2} \geq \frac{1}{2} \geq \frac{1}{2} \) and \( \frac{1}{2} \) be a (full) strategy, which is defined at every information set, i.e. after arbitrary private histories. Clearly, \( \frac{1}{2} \) is a best response to \( \frac{1}{2} \) after every private history if and only if it is optimal to play \( \frac{1}{2} \) at every belief \( \frac{1}{2} \); \( \frac{1}{2} \); \( \frac{1}{2} \); \( \frac{1}{2} \); i.e. at all possible beliefs given the initial belief \( \frac{1}{2} \).

We now examine the properties of these belief operators. First, each is a strictly increasing function: The operator \( \tilde{A}_c \) has an interior fixed point at \( \frac{1}{2} \) and \( \tilde{A}_c(a\beta) (\frac{1}{2}) \) as \( \frac{1}{2} \) ? \( \frac{1}{2} \). The value of \( \frac{1}{2} \) depends upon \( (\frac{1}{2}, \beta) \) in the following way

\[
\frac{1}{2} (\frac{1}{2}, \beta) = \frac{1}{(1 - \frac{2}{3} \beta + \frac{1}{3} \beta)} \frac{(1 - \frac{2}{3} \beta + \frac{1}{3} \beta)}{(1 - \frac{2}{3} \beta + \frac{1}{3} \beta)}
\]

We shall assume that \( \max f (\frac{1}{2}, \beta) = \frac{1}{2} \); which in turn entails that \( \frac{1}{2} > \frac{1}{2} \).
We now show $\hat{A}_{ai}(1) < 1$ for each of the other three belief operators, $A_{Ci}; A_{Di}$ and $A_{Di}$, provided that $\max f''; g < \frac{1}{3}$; To verify this, take any typical expression from (3)-(5), and divide by $\frac{3}{4}$; This yields " (or ») in the numerator, while the denominator is strictly larger since it is a convex combination of (" + ») and (1 i »):

Since $1 < \mu$ $\hat{A}_{ai}(1) < \mu$ for any belief operator, this immediately implies that if $\frac{1}{2} < \mu$ $\hat{A}_{ai}(1) < \mu$; This follows from the fact that the initial belief $p$ is strictly less than $\mu$ and since we have demonstrated that no point $1 \hat{A}_{ai}(1)$ is the image of any $1 \cdot \mu$ under any belief operator.

Hence, provided that initial beliefs are given by $\frac{1}{2} < \mu$ it suffices to define our belief based strategy for beliefs in the set $[0; \mu]$; Let $\frac{1}{2}; [0; \mu]$ ! $fC; D; \frac{1}{2}g$ be defined as follows: $\frac{1}{2}fC(1) = C$ if $1 2 (\frac{1}{2} \mu)$ and $\frac{1}{2}d(1) = D$ if $1 2 [0; \frac{1}{2}];$ If $1 = \frac{1}{2} fC(1) = \frac{1}{2} i.e.$ $\frac{1}{2}$ plays $C$ with probability $\frac{1}{2}$ and $D$ with probability $1; \frac{1}{2}$ Hence the pair $(\frac{1}{2}; \mu)$, i.e. $\frac{1}{2}$ in conjunction with an initial belief $\frac{1}{2} \mu$ specifies an action at every possible belief, and hence a complete strategy.

The advantage of this specification is that a player’s continuation strategy is specified even at information sets which arise due to a player’s deviating from $\frac{1}{2}$ in the past. The belief based strategy $(\frac{1}{2} \mu)$ is realization equivalent to the partial strategy $\frac{1}{2}$ if it induces the same probability distribution over actions at every private history. This reduces to the following condition:

De nition 1 $(\frac{1}{2} \mu)$ is realization equivalent to $\frac{1}{2}$ if $1 2 [\frac{1}{2} \mu]$ $[A_{C}(1)] > \frac{1}{2}$ and $A_{C}(1) < \frac{1}{2}$ and $1 2 [0; \frac{1}{2}]; [A_{D}(1)] < \frac{1}{2}$ and $A_{D}(1) < \frac{1}{2}$

Lemma 2 If $\frac{1}{2} < \mu$ $(\frac{1}{2}; \mu)$ is realization equivalent to $\frac{1}{2}$

Proof. To verify that $1 2 [\frac{1}{2} \mu]$ $A_{C}(1) > \frac{1}{2} \mu$ recall that $A_{C}(1) > 1$ if $1 < \mu$ so that $A_{C}(1) > \frac{1}{2}$ for any $k$: To verify $1 2 [\frac{1}{2} \mu]$ $A_{C}(1) < \frac{1}{2}$ it suffices to verify that $A_{C}(1) < \frac{1}{2}$ since $A_{C}$ is strictly increasing:

$$A_{C}(\mu) = \frac{\text{"(1)} 3\text{"} 2\text{»}}{\text{"(1)} 3\text{"} 2\text{»} + (1 \text{"} 1 \text{»})} < \frac{\text{"(1)} 3\text{"} 2\text{»}}{2 \text{»}} = \frac{1}{2} < \frac{1}{2} \quad (7)$$

$1 \cdot \frac{1}{2} A_{C}(1) < \frac{1}{2}$ and $1 \cdot \frac{1}{2}$ follow from the fact already established that $A_{C}$ and $A_{D}$ lie below the 45 line.

Note that if $\text{"(1)} 3\text{"} 2\text{»}$ and $\text{"(1)} 3\text{"} 2\text{»}$ we can select $\pm \frac{1}{2}$ so that $\frac{1}{2} \pm \text{"(1)} 3\text{"} 2\text{»}$ this follows from the fact that $\frac{1}{2} \pm \text{"(1)} 3\text{"} 2\text{»}$ ! 1 as $\pm \frac{1}{2}$ $\frac{1}{2}$ and $\text{"(1)} 3\text{"} 2\text{»}$ ! 0; while $\frac{1}{2} \pm \text{"(1)} 3\text{"} 2\text{»}$ ! 0 if $\pm$ 1 and $\text{"(1)} 3\text{"} 2\text{»}$ ! 0: Henceforth we shall assume that $\pm$ is such that $\frac{1}{2} 2 (\frac{1}{2}; \mu)$ so that $(\frac{1}{2} \mu)$ is consistent.

We have therefore established that the pair $(\frac{1}{2} \mu)$ defines a full strategy which is behaviorally equivalent to $\frac{1}{2}$

Proposition 3 If $\frac{1}{2} < \frac{1}{2} < \mu$ $(\frac{1}{2} \mu)$; the strategy pro le where each player plays $(\frac{1}{2} \mu)$ is a sequential equilibrium.
Proof. Note rst that if \( i = \frac{1}{2} \) a player is indifferent between playing \( \frac{3}{4} \) and \( \frac{1}{2} \); and hence a one-step deviation from playing \( \frac{1}{2} \) is not pro. table. Since the payo s from playing \( \frac{3}{4} \) and \( \frac{1}{2} \) are equal at belief \( \frac{1}{2} \) one may also, for the purposes of computing payo s, use \( \frac{3}{4} \) or \( \frac{1}{2} \) as is computationally convenient in the event of belief \( \frac{1}{2} \).

Consider rst the case when \( i > \frac{1}{2} \): one-step deviation from \( \frac{1}{2} \) is to play \( D \); and to continue with \( \frac{1}{2} \) in the next period. The following sub-cases arise:

a) Suppose that \( \Delta_{Dc}^{(1)} < \frac{1}{4} \) and \( \Delta_{Dd}^{(1)} > \frac{1}{4} \). In this case, a one-step deviation from \( \frac{1}{2} \) is to play \( \frac{1}{2} \); whereas \( \frac{1}{4} \) is \( \frac{3}{4} \): However, (1) establishes that in this case \( \frac{3}{4} \) is preferable to \( \frac{1}{2} \); and hence a one-step deviation from \( \frac{1}{2} \) is unpro. table.

b) Suppose that \( \Delta_{Dc}^{(1)} > \frac{1}{4} \) and \( \Delta_{Dd}^{(1)} > \frac{1}{4} \). In this case, a one-step deviation is to play \( D \) today and continue with \( \frac{1}{2} \) if \( Dc \) is reached, and to continue with \( \frac{3}{4} \) if \( Dd \) is reached. Let \( \xi \) be the payo difference between the equilibrium strategy and the one-step deviation. Note that the one step deviation differs from \( \frac{1}{2} \) only at the information set \( Dd \); at this information it continues by playing \( \frac{3}{4} \) whereas \( \frac{1}{2} \) continues with \( \frac{1}{2} \): Hence we can write \( \xi \) as the payo difference between \( \frac{3}{4} \) and \( \frac{1}{2} \) minus the payo difference between \( \frac{3}{4} \) and \( \frac{1}{2} \): conditional on \( Dd \) being reached, as follows:

\[
\xi \mathcal{V}^{(1)} = \xi \mathcal{V}^{(1)}_i \quad \mathcal{D}^{(1)} \{ \xi \} + \{1 - \xi \} \{1 \} \{1 \} \}
\]

Equation (1) shows that this implies that \( \xi \mathcal{V}^{(1)} > \xi \mathcal{V}(\Delta_{Dd}^{(1)}) \). Since the coe cient multiplying \( \xi \mathcal{V}(\Delta_{Dd}^{(1)}) \) is strictly less than one, this implies that \( \xi \mathcal{V}^{(1)} > 0 \); hence if \( i > \frac{1}{2} \) a one-step deviation is unpro. table.

c) Finally, we establish that \( \Delta_{Dc}^{(1)} < \frac{1}{4} \) \( \Delta \) so that no other sub-case need be considered. Evaluating \( \Delta_{Dc}^{(1)} \) at the upper bound \( \Delta \) we have

\[
\Delta_{Dc}^{(1)} = \frac{\{1 \} \{3 \} \{2 \}}{(1 \) \{3 \} \{2 \}) + (\) \{\} \}
\]

where the last step follows from the assumption that \( max \{, \} < \frac{1}{2} \).

Consider now the case when \( i < \frac{1}{2} \): In this case, a one-step deviation from \( \frac{1}{2} \) is to play \( C \) today, and to continue with \( \frac{3}{4} \) to \( \frac{3}{4} \) if \( \Delta_{Cc}^{(1)} > \frac{1}{4} \) but to continue with \( \frac{1}{2} \) if \( \Delta_{Cc}^{(1)} < \frac{1}{4} \) (Note that \( i < \frac{1}{4} \) \( \Delta_{Cc}^{(1)} < \frac{1}{4} \) so the continuation strategies do not differ in this event.) In the rst sub-case, the one-step deviation from \( \frac{1}{2} \) corresponds to playing \( \frac{1}{2} \), and (1) establishes that in this case \( \frac{1}{2} \) is preferable to \( \frac{3}{4} \); and hence a one-step deviation from \( \frac{1}{2} \) is unpro. table. In the second sub-case, the one-step deviation differs from \( \frac{1}{2} \) only at the information set \( Cc \) — it plays \( \frac{1}{2} \) at this information set rather than \( \frac{1}{2} \): Let \( \xi \mathcal{V}^{(1)} \) denote the payo difference between the one-step deviation and the equilibrium strategy \( \frac{1}{2} \): We have

\[
\xi \mathcal{V}^{(1)} = \xi \mathcal{V}^{(1)}_i \quad \mathcal{D}^{(1)} \{ \xi \} + \{1 - \xi \} \{1 \} \{1 \} \}
\]
Since $\frac{1}{4} > \hat{A}_C(\theta) = 1$, $\xi V(1) < \xi V(\hat{A}_C(1)) < 0$: Also, the coefficient multiplying $\xi V(\hat{A}_C(1))$ is less than 1 which establishes that $\xi V(1) > 0$.

We have therefore established that if a player's opponent $j$ plays the strategy $\frac{1}{2}$ (which randomizes between $\frac{1}{6}$ and $\frac{1}{3}$), it is optimal for player $i$ to play $\frac{1}{2}$ with initial belief $\frac{1}{4}$. However, $(\frac{1}{2} \frac{1}{4})$ is consistent and behaviorally equivalent to the strategy $\frac{1}{4}$ Hence the profile where both players play $(\frac{1}{2} \frac{1}{4})$ is a sequential equilibrium.

Under what conditions is there a pure strategy sequential equilibrium where both players begin in period one by playing $\frac{1}{2}$ with probability one? The above analysis also permits an answer to this question, with the difference that the initial belief $^\theta = 1$ rather than $\frac{1}{4}$. To ensure that it is always optimal to cooperate after receiving good signals, we require that $\hat{A}_C(\theta) > \frac{1}{4} + 8k$, which will be satisfied as long as $\theta < \frac{1}{4}$. Additionally, it must be optimal to switch to playing $D$ on receiving a bad signal, i.e. we must have $\hat{A}_C(\theta) < \frac{1}{4}$ for $\theta = 1$ or $\theta = \hat{A}_C(1)$ for some $k$: Hence it is necessary and sufficient that

$$\hat{A}_C(1) = \frac{"}{\mu} < \mu$$

(11)

This requires that the $\hat{A}_C$ function always lies below $\mu$ which requires the inequality

$$\frac{2}{\mu} < \nu(1 \ 3)$$

(12)

This inequality will be satisfied if $\nu$ is sufficiently small relative to $\mu$, i.e. if signals are sufficiently "positively" correlated. It is easily verified that this inequality cannot be satisfied if signals are independent or "negatively" correlated so that the equilibrium must be in mixed strategies.

Note that $\frac{1}{4}$ plays a dual role in the construction of the mixed strategy equilibrium. On the one hand it is the randomization probability in the first period, and on the other hand, it is simply a number which defines the threshold at which behavior changes. These roles are obviously distinct, as is apparent from our discussion of the pure strategy equilibrium. This distinction is particularly relevant when we discuss the folk theorem in the following section.

With the construction of the mixed equilibrium, one can approximate full cooperation by using one of two devices. If a public randomization device is available, then it is immediate that the equilibrium payoff set is monotonically increasing (in the sense of set inclusion) in $\pm$- given any $\hat{d}$ $\pm$ players may simply re-start the game with probability $m = \frac{\hat{d}}{2}$. In the absence of such public randomization, one may use the construction introduced by Ellison [4] (see also, Sekiguchi [13]), of dividing the game into $n$ separate repeated games, thereby reducing the discount factor.

Note 2: The conditions for the optimality of playing $D$ once a player has played $D$ are as before, and hence will also be satisfied.
Lemma 4 Let $\pm_0 < \pm_1 < 1$; and let there be a symmetric strategy profile which is a sequential equilibrium of the repeated game for any $\pm_2$ ($\pm_0; \pm_1$); yielding payoff $v(\pm)$, $v$ for any $\pm_2$ ($\pm_0; \pm_1$). There exists $\hat\pm < 1$ such that the repeated game has a symmetric sequential equilibrium with payoff greater than $v$ for any $\pm, \hat\pm$. If a public randomization device is available and $(v_1; v_2)$ is a sequential equilibrium payoff for some $\pm_2$ $(0, 1)$; it is also an equilibrium payoff for any $\pm > \pm_0$.

Proof. For the proof of the first part of this lemma, see Ellison [4]. To prove the second part, let $\hat\zeta$ be the strategy profile giving the required payoff given $\pm$. Given $\pm_0$; let $m = \frac{\pm_0}{\hat\zeta}$. Players play a sequence of games: they begin with the strategy profile $\hat\zeta$; if the sunspot in any period $\hat\zeta > m$; they play a new game and re-start with $\hat\zeta$.

Proposition 5 For any $\varphi < 1$; there exists a symmetric sequential equilibrium with payoff greater than $\varphi$ if $\hat\varphi$ is sufficiently small, provided that either (i) $\pm$ is sufficiently close to 1 or (ii) $\pm > \frac{\varphi}{1 + g}$ and a public randomization device is available.

Proof. Proposition 3 implies that if $\hat\varphi$ is sufficiently small, so that $\hat\mu(\hat\varphi)$ is close to 1; we have an open interval of values of $\hat\mu$ such that $\frac{1}{2} + \frac{1}{2}$ is a sequential equilibrium. In this range, $\frac{1}{2} - \hat\varphi$ is a strictly decreasing function of $\pm$ and hence if $\hat\varphi$ are sufficiently small, there is an open interval of values of $\pm$ such that $\frac{1}{2} - \hat\varphi$ is a sequential equilibrium. Since $\hat\varphi$ are close to zero, we can select this interval of values of $\frac{1}{2}$ close to 1; so that the payoff in any such equilibrium is greater than $\varphi$: Part (i) of the proposition then follows from the first part of lemma 4. If a public randomization device is available, let $(\hat\varphi, \hat\varphi)$! $(0, 0)$ and $\hat\varphi$! $\frac{\varphi}{1 + g}$; so that $\frac{1}{2}$! 1: The equilibrium payoff tends to one. Lemma 4 ensures that this result holds for all $\pm > \frac{\varphi}{1 + g}$.

Although it is common to allow for vanishing discounting in proving folk theorems in repeated games, it is worth pointing out that in order to obtain approximate efficiency, such vanishing discounting is not required if we have a public randomization device. In the absence of such a randomization device, one does require vanishing discounting, essentially due to an "integer" problem.

2.1 Generalizing the information structure.

We now show that the above construction also extends for a more general information structure. Let $\mathcal{S}$ be a common finite set of signals observed by the players, and assume that the marginal distribution of $p$ has full support, i.e. for any action profile $a$: $P(!_i = !_j a) = \frac{1}{2}$, $P(!_i !_j | a) > 0$ for all $!_i 2$; $i = 1; 2$: Our assumptions on the signal structure are as follows:

1. Assume that the set $\mathcal{S}$ can be partitioned into the set of good signals $\mathcal{S}_c$
and the set of bad signals \( - d \); where the likelihood ratios satisfy

\[
P(\text{i} = \text{a}; D) < P(\text{i} = \text{d}; D) \quad \text{for every } c 2 - c; d 2 - d \text{ and for any action } a_1 2 f C; Dg \text{ taken by player 1.}
\]

Although we focus on player 1, the same conditions and results also hold for player 2.

2. Assume that \( - c \leq - c \text{ is } 1 \text{ i } " \text{ evident given the action profile } (C); \) where " is a small number. This ensures that if player 1 receives any good signal, and her prior belief assigns high probability to the action profile \( (C); \) then player 1 assigns high probability to the event that her opponent has also received a good signal. Note that this assumption is consistent with signals being independent conditional upon the action profile — under independence, if \( P(\text{i} = 2 - c; C) > 1 \text{ i } " \); then \( - c \leq - c \text{ is } 1 \text{ i } " \text{ evident given } (C);^3

3. Let \( \min P(\text{i} = \text{a}; D) \) and \( \max P(\text{i} = \text{a}; D) \) be bounded below by some number \( \gamma > 0 \) independent of "; assume also that \( \max P(\text{i} = 2 - c; D); 1 \text{ i } " \cdot \gamma > 0 \); again independent of ".

Note that assumptions 1-3 above are consistent with the signals being independent conditional upon the strategy profile. Also note that this information structure can be quite different from almost perfect monitoring.

We now show that under these assumptions, beliefs evolve in such a way that they are always above the initial belief \( \frac{1}{2} \)as long as a player plays C and receives good signals in \( - c \); but they fall below \( \frac{1}{2} \)whenever a player receives a bad signal and they continue to stay below \( \frac{1}{2} \)when a player plays D.

When player 1 plays C and receives some signal \( c 2 - c \); the updated belief is given by

\[
\hat{A}_{c, c}(1) = \frac{1}{1} \frac{P(\text{i} = \text{a}; D) - c 2 - c; C)}{P(\text{i} = \text{a}; D) + (1 \text{ i } - 1) P(\text{i} = \text{d}; C)}
\]

The fixed point of this mapping will be close to 1 for every \( c 2 - c \) if \( \frac{1}{P(\text{i} = \text{a}; D) - c 2 - c; C)} = P(\text{i} = 2 - c; C); 1 \text{ is } \gamma \text{ large enough for any } c 2 - c \): This is greater than \( 1 \text{ i } " \text{ since } - c \leq - c \text{ is } 1 \text{ i } " \text{ evident.}

The second condition is that a player should switch to playing \( \frac{1}{2} \) when he receives a bad signal. For each \( c 2 - c \); we can define the associated fixed point of the mapping (14), just as in (6). Let \( \mu \) denote the largest such fixed point (in the set \( - c \)) and let \( \mu \) denote the smallest such fixed point. A sufficient condition for our construction is that \( \hat{A}_{c,d}(1) < \mu \) for any \( d 2 - d \); in this case one can

\[^3\text{Mailath and Morris [8] introduce and use such conditions on the signal structure. Given their focus on almost public monitoring (where signals are correlated), they also assume a similar condition for } - d \leq - d.\]
select $\frac{1}{2} \left( \hat{A}_{C;D}[\hat{\mu}] \right)$ so that switching to $\frac{1}{2}$ is optimal: Since $\mu_1 > 0$; and $\mu_5 \geq \max_{C_2 \in C} \frac{P(1 = c_1 | 2 \in C)}{P(1 = c_1 | C)}$; we need that $\hat{A}_{C;D}[\max_{C_2 \in C} \frac{P(1 = c_1 | 2 \in C)}{P(1 = c_1 | C)}]$ is bounded away from 1 independent of $\mu$.

A straightforward application of the belief operators shows that:

$$\hat{A}_{C;D}[\hat{\mu}] \geq \frac{1}{1 + \frac{1}{2}}$$

for all $d \in D$; which is bounded away from 1 by assumption 2 above.

Finally, we need to ensure that $\hat{A}_{D;1}(\hat{\mu})$ remains low, so that a player continues with playing $\frac{1}{2}$: If $\mu_1 > 0$; it is easy to verify that $\hat{A}_{D;1}(\hat{\mu}) < 1$. If $\mu_1 = 0$; then $p(1 = 2 \in C | D)$ is less than $\mu_1 > 0$; and bounded away from 1 for any $c$ by assumption 3: Hence it is optimal to continue with $\frac{1}{2}$ once a player starts playing $D$.

Summarizing the above arguments and checking sequential rationality, we have the following theorem:

**Proposition 6** Given $\mu > 0$; there exists $\varepsilon > 0$ such that for any $\mu < \varepsilon$, our mixed trigger strategy ($\frac{1}{2}$) = C if $\mu_1 > 0$; and $\frac{1}{2}$ = D if $\mu_1 = 0$; is a sequential equilibrium. If $\mu > 0$ we can approximate the efficiency outcome provided that either i) $\mu > 0$ or ii) $\mu < \frac{1}{1+g}$ and a public randomization device is available.

### 3 Approximating Any Individually Rational Feasible Payoff

We now build on the construction of the previous section and show how to approximate any individually rational feasible payoff. We shall consider a prisoners' dilemma game where the symmetric efficient payoff is given by the profile (C;C) (rather than by a convex combination of (C;D) and (D;C)); and we also assume that there are only two signals, c and d. We also assume in this section that a public randomization device is available. The key step is to approximate the payoff $\left( \frac{1}{1+g} \right)$, which is player 1's maximal payoff within the set of individually rational and feasible payoffs. Since the payoff (1,1) has already been approximated in the previous section, and (0,0) is a stage game equilibrium payoff, one can then use public randomization to approximate any individually rational feasible payoff.

It might be useful to outline the basic construction and to explain the complications that arise. The basic idea in approximating the extremal asymmetric payoff is that play begins in the asymmetric phase where player 1 plays D and player 2 randomizes, playing C with a high probability, $\hat{A}$. This asymmetric phase continues or ends, depending upon the realization of a public randomization device. Thus player 1's per-period payoff in the asymmetric phase is approximately $1 + g$ while player 2's per-period payoff is approximately $\frac{1}{2}$.
Since the latter is less than the individually rational payoff for player 2, he must be rewarded for playing C. To ensure this, when the asymmetric phase ends, both player’s continuation strategies depend upon their private information. Player 1 continues with \( \frac{1}{2} \) if he has observed the signal \( c \) in the last period (i.e. if his information is \( Dc \)) and continues with \( \frac{1}{2} \) if she has observed \( d \) (i.e. if his information is \( Dd \)). This ensures that player 2 is rewarded for playing C in the asymmetric phase. Similarly, player 2 continues with \( \frac{1}{2} \) if his private information is \( Cd \); the information set which is most likely when he plays C; and continues with \( \frac{1}{2} \) if his private information is \( Dd \). Hence, if the noise is small, player 2’s continuation payoff when the asymmetric phase ends is approximately 1 if he has played C in the previous period and approximately zero if he has played D.

Hence if \( \pm \) is large relative to \( l \) (\( \pm > \frac{1}{l+1} \)), we can, by choosing the value of the sunspot appropriately, make player 2 indifferent between C and D in the asymmetric phase. The payoffs in this equilibrium converge to \( \left( \frac{1+g+1}{l+1}, 0 \right) \) as the noise vanishes.

However, one must also verify that the players find it optimal to play \( \frac{1}{2} \) and \( \frac{1}{2} \), as appropriate, at each information set after the end of the asymmetric phase. A complication arises here, as compared to the previous section, since player 1 does not randomize in the asymmetric phase, i.e. she plays D for sure. (Indeed, she cannot play C with positive probability, since in that case her payoff in the asymmetric phase is bounded above by 1 and hence cannot approximate \( 1 + g \)).

When player 2 receives the signal \( c \), he knows that there has been at least one error in signals, and his beliefs about player 1’s continuation strategy depend upon the relative probability of one (”) versus two errors (”). In other words, his continuation strategy at the information sets \( Cd \) and \( Dc \) depends upon the correlation structure of signals. Since player 2’s continuation strategy depends upon the correlation structure, this implies that player 1’s beliefs also depend upon the correlation structure.

We adopt two alternative approaches to handle this problem. First, we show that if signals are positively correlated, so that the probability of two errors is at least as large as the probability of one error, then one can approximate the asymmetric payoff, without any restriction upon payoffs. Second, we show that one does not need such positive correlation of signals provided that one can choose \( \pm \) so that \( \frac{1}{4} \frac{\pm}{l} \) sufficiently close to one. This result applies to any prisoners’ dilemma game where \( g \), \( l \) — in any such game one can approximate the asymmetric payoff arbitrarily closely. However, this second approach does not work if \( l > g \), since in this case one cannot have \( \frac{1}{4} \frac{\pm}{l} \) ! 1: The reason for this is the for \( \frac{1}{4} \) to be close to 1, we must have \( \frac{1}{4} \frac{\pm}{l} \). However, in the asymmetric phase, player 2 incurs a loss of \( l \) by playing C; whereas his continuation payoff gain is no more than 1: Hence player 2 will be willing to play C in the asymmetric phase only if \( \pm > \frac{1}{\frac{1}{1+1}} \). Hence if \( l > g \), one cannot have

---

4This argument is more general and implies that one cannot have a folk theorem in completely mixed strategies for stage games with non-degenerate payoffs. Let \( \psi \) be the supremum payoff of player 1 in any equilibrium where player 1 randomizes in every period at every information set. Since \( \psi \cdot \left( I; \min a_1, \max a_2, u_1(a_1, a_2)g + \Delta \right) \); this implies \( \psi \cdot \min a_1, \max a_2, u_1(a_1, a_2)g \)
close to 1 since ± is bounded away from \( \frac{a}{1+a} \).

We make the following assumption for this section:

Assumption A: Either A1: \( \gg \) or A2: \( g \gg 1 \) and \( (1_{ij} \gg (1_{ij} 3"_{ij} 2) > ^3 \).

Note that A1 is a relatively strong assumption that signals are positively correlated, but does not require any assumption on payoffs. On the other hand, A2 requires an assumption on payoffs but is a mild assumption about the relative probability of errors. It is always satisfied if signals are positively correlated, or independent. In the independent signal case, the left hand side is a term of order " whereas the right hand side is a term of order \( ^3 \); Hence A2 is satisfied even if signals are negatively correlated provided that they are not too highly so.

We now define the players' strategies more precisely. In any period \( t_{ij} 1 \) in the asymmetric phase, player 1 plays D for sure, while player 2 randomizes between C and D; choosing C with a constant probability \( \hat{A} \) which is close to 1. At the end of period, players observe the realization, \( \hat{A}_{ij} 1 \), of a sunspot which is uniformly distributed on \([0;1] \). If \( \hat{A}_{ij} 1 > 1 \) \( i \) \( j \); both players continue in the asymmetric phase for the next period. If \( \hat{A}_{ij} 1 > 1 \) \( i \) \( j \); the asymmetric phase ends for both players, and is never reached again. In this case, players' continuation strategies (i.e. their states) depend upon the realization of their private information at date \( t_{ij} 1 \) (i.e. players ignore their private information from previous dates). Let \( \hat{a}_{ij} 1 \) denote the player's private information realization at date \( t_{ij} 1 \): Player 1 continues with \( \hat{C} \) if \( \hat{a}_{ij} 1 = D_{ij} \); if \( \hat{a}_{ij} 1 = D_{ij} \); she continues in period t with \( \hat{C} \). Player 2's continues with \( \hat{C} \) if \( \hat{a}_{ij} 1 = C_{ij} \); and continues with \( \hat{D} \) if \( \hat{a}_{ij} 1 = D_{ij} \). If \( \hat{a}_{ij} 1 = \hat{C} \) \( \hat{C} \) \( \hat{C} \) \( \hat{D} \); player 2 continues with \( \hat{C} \) if \( \hat{1}_{ij} (\hat{a}_{ij} 1) > 1_{ij} \hat{C} \) \( \hat{D} \) and with \( \hat{D} \) if \( \hat{1}_{ij} (\hat{a}_{ij} 1) \leq 1_{ij} \hat{C} \) \( \hat{D} \).

Our analysis proceeds as follows. First, we show that player 2 is willing to randomize in the asymmetric phase provided that \( \hat{A} \) is appropriately chosen, and that the payoffs associated with this class of equilibria tend to \((1_{ij} 1+1_{ij}, 0)\) as the noise vanishes. Subsequently, we shall demonstrate that all players are choosing optimally at every information set.

Write \( W_{ij} (D) \) for the payoff of player 2 in the asymmetric phase given that he plays D; and \( W_{ij} (C) \) for the payoff in the asymmetric phase from playing C. Since \( W_{ij} (D) = W_{ij} (C) = W_{ij} \); we have

\[
W_{ij} (D) = \hat{\delta} (1_{ij} \hat{C}) W_{ij} + \hat{\delta} (1_{ij} \hat{D}) \]

where \( \hat{\delta} (1_{ij} \hat{D}) \) is the expected payoff to player 2 conditional on the fact that the asymmetric phase has ended and that he has played D. Similarly, letting

\[^5\]

We show that any strategy which plays C in the asymmetric phase is dominated, and hence we need not define precisely the optimal continuation strategy after playing C. The existence of an optimal continuation strategy follows from the same argument as in Sekiguchi [13]. Since player 1 never plays C in the asymmetric phase, his continuation after his own deviation does not affect player 2's incentives.
Let $V_2(C)$ be the expected payoF to 2 conditional on the fact that the asymmetric phase has ended and that he has played C; we have

$$W_2(C) = (1 + \frac{1}{\mu}(1) + \frac{1}{\mu}1)W_2 + \pm V_2(C)$$ (17)

Clearly, $V_2(D) = 0$ as (";») ! (0; 0): We now show that $V_2(C) = 0$ as (";») ! (0; 0): Let $V_2(Cd)$ (resp. $V_2(Cc)$) denote the continuation payoF at the end of the asymmetric phase, conditional on Cd (resp. Cc): Since player 1 plays D for sure in the asymmetric phase, we have

$$V_2(C) = (1 + "i »)V_2(Cd) + (» + »)V_2(Cc)$$ (18)

Hence it suc ces to establish that $V_2(Cd) = 0$ as (";») ! (0; 0); Write $1_2(Cd)$ for the probability that player 1's continuation strategy is $\frac{1}{\mu}$; given that $a_i = 1 = Cd$: Since $\frac{1}{2}(Cd)$, $\frac{1}{1+\frac{1}{\mu}}1_2(Cd) = 0$: Hence from equation (1)$V_2(Cd) = V_{Cc}$; where $V_{Cc} = 0$ as (";») ! (0; 0): Hence if " + » is suc ciently small and $\pm > \frac{1}{1+\frac{1}{\mu}}$, there exists a value of $\mu$ which equates $W_2(C)$ and $W_2(D)$. Further, as (" + ») ! 0, if $1_2(\mu)$, and player 2's payoF converges to zero.

If $A = 1$, player 1's per-period payoF tends to $(1 + g)$ in the asymmetric phase, and 1 in the cooperative phase. By substituting for the limiting value of $\mu$, which is $\frac{1}{2}$, we see that player 1's payoF converges to $1 + g$; (We shall establish later that $A = 1$).

We now verify that each player plays optimally at each information set in this equilibrium. In the asymmetric phase, this is so for player 2 by construction, since he is indifferent between C and D: It is easy to see that player 1 also plays optimally in the asymmetric phase, since she is choosing her one shot best response.6

Consider now the transition to the cooperative phase, i.e. the player's actions in the rst period after the sunspot signals at the end of the asymmetric phase. Since players only condition on their private information in the previous period, we may focus on this alone. Player 1 has two possible information sets, (Dc) and (Dd); whereas player 2 has four possible information sets. Let $1_1(\mu)$ denote the probability assigned by player i to his opponent's continuation strategy being $\frac{1}{\mu}$; given that i is at information set $\mu$.

As in the previous section, we shall assume that $\max \frac{1}{\mu} > g < \frac{1}{\mu}$; Furthermore, as in the previous section, we assume that $\frac{1}{2} + \frac{1}{\mu}$ - this assumption does not imply any restrictions upon g or $l$: However, if we invoke the assumption $g \leq l$ in A2, then we may also choose $\frac{1}{2}$ to be arbitrarily close to

---

6It is possible that playing C in the asymmetric phase increases player 1's continuation payoF in the cooperative phase. However, it is easy to see that such an increase can never offset the loss from playing C: A simple proof is as follows. If playing C in the asymmetric phase is optimal for 1, then playing C in every period in the asymmetric phase is also optimal. The overall payoF of this strategy is approximately 1 if the noise is small, whereas the payoF of player 1 in the equilibrium tends to $1 + g$, which is strictly greater.
its upper bound. We shall also assume that $A_2(\frac{1}{2}u(\cdot;\cdot))$: Since $\mu > 1$ as $(\cdot;\cdot) = (0;0)$ we can also have $A_1 = 1$

Consider first the beliefs of player 2: Let $\hat{\beta}_2(\cdot)$ denote the probability assigned by 2 to the event that 1's continuation strategy is $\frac{3}{8}$: Since player 1 plays $\frac{3}{8}$ at $Dc$ and $\frac{3}{8}$ at $Dd$; and since player 1 does not play C in the asymmetric phase, we have

$$\hat{\beta}_2(Dc) = \frac{1}{2} > \mu$$

Since $\frac{1}{2} < \mu$, it is optimal to continue with $\frac{3}{8}$ today at information set $Cd$. Further, we have

$$\hat{\beta}_{cd}(\hat{\beta}_2(Dc)) = \frac{(\frac{1}{2} \cdot 2^n \cdot \frac{\cdot}{\cdot} \cdot \frac{\cdot}{\cdot})}{(\frac{1}{2} \cdot 2^n \cdot \frac{\cdot}{\cdot} \cdot \frac{\cdot}{\cdot}) + (\frac{1}{2} \cdot 2^n \cdot \frac{\cdot}{\cdot} \cdot \frac{\cdot}{\cdot})} < \frac{1}{2}$$

Hence it is optimal for player 2 to switch to the defection phase if he receives the signal $Cd$ at any date in the future.

At $Dd$, we have

$$\hat{\beta}_2(Dd) = \frac{\cdot}{\cdot} \cdot \frac{\cdot}{\cdot}$$

This is clearly less than $\frac{1}{2}$ since $\max f^e > \frac{3}{8}$, so that it is optimal to continue with $\frac{3}{8}$:

Consider next the beliefs of player 2 at $(Cc)$ and $(Dc)$; i.e. at the information sets where player 2 knows that there has been at least one error in the signals.

$$\hat{\beta}_2(Dc) = \frac{\cdot}{\cdot} \cdot \frac{\cdot}{\cdot}$$

$$\hat{\beta}_2(Cc) = \frac{\cdot}{\cdot} \cdot \frac{\cdot}{\cdot}$$

Recall that player 2 plays $\frac{3}{8}$ at least at one of these information sets, since the above probabilities cannot be both greater than $\frac{1}{4}$ since this is greater than one-half. Hence there are three possibilities: either both $\hat{\beta}_2(Dc)$ and $\hat{\beta}_2(Cc)$ are less than $\frac{1}{4}$ or exactly one of these is greater than $\frac{1}{4}$: Now if $\hat{\beta}_2(\cdot) < \frac{1}{4}$ at any information set, it is optimal to continue with $\frac{3}{8}$ today, and at every future date. Hence it remains to verify the case when $\hat{\beta}_2(\cdot) = \frac{1}{4}$

Suppose that $\frac{\cdot}{\cdot} \cdot \frac{\cdot}{\cdot} > \frac{1}{4}$ so that player 2 plays $\frac{3}{8}$ at $Dc$ if $\frac{\cdot}{\cdot} \cdot \frac{\cdot}{\cdot} \cdot \mu$; lemma 2 verifies that it is optimal to continue with $\frac{3}{8}$ in this case. Hence consider the case where $\frac{\cdot}{\cdot} \cdot \frac{\cdot}{\cdot} > \mu$: We have that $\hat{\beta}_2 > \mu$. Hence the case $\hat{\beta}_2(\cdot) = \frac{1}{4}$: Further, since $\hat{\beta}_{cd}$ is an increasing function, it suces to verify that $\hat{\beta}_{cd}(\frac{\cdot}{\cdot} \cdot \frac{\cdot}{\cdot}) < \frac{1}{4}$ since this implies that $\hat{\beta}_{cd}(\frac{\cdot}{\cdot} \cdot \frac{\cdot}{\cdot}) < \frac{1}{4}$ for $\hat{\beta}_c(\frac{\cdot}{\cdot} \cdot \frac{\cdot}{\cdot})$ for any $k$.
\[ \mathcal{A}_{cd}(\pi^{""\rightarrow}) = \frac{\pi^{""\rightarrow}}{\pi^{\rightarrow\rightarrow} + \pi^{\rightarrow\leftarrow}} \] (24)

This is less than \( \frac{1}{2} \) if \( \text{maxf}^{""\rightarrow} \) is less than \( \frac{1}{6} \). Hence player 2's continuation strategy is optimal at \( Dc \).

Finally, we consider the case where that player 2 plays \( \frac{3}{6} \) at \( Cc \), i.e. when \( \pi^{\rightarrow\rightarrow} \) is less than \( \frac{1}{4} \). Note that in this case A1 is violated. Hence we assume A2, which ensures that we can make \( \pi^{\rightarrow\rightarrow} \) arbitrarily close to its upper bound \( \mu \) by selecting \( \epsilon \) sufficiently close to \( \frac{3}{6} \). We can find a value of \( \epsilon \) such that \( \mathcal{A}_{cd}(\pi^{""\rightarrow}) < \frac{1}{4} \) provided that \( \mathcal{A}_{cd}(\pi^{""\rightarrow}) \) is less than the upper bound for \( \frac{3}{6} \).

\[ \mathcal{A}_{cd}(\pi^{""\rightarrow}) = \frac{\pi^{""\rightarrow} - \pi^{\rightarrow\rightarrow}}{\pi^{\rightarrow\rightarrow} + \pi^{\rightarrow\leftarrow}} < \mu \] (25)

It is easily verified that the inequality above is ensured by condition A2.

Consider now the beliefs of player 1: Her beliefs will depend upon player 2's strategy, which in turn depends upon the parameters of the signal distribution, and as we have seen, there are three possible cases.

Consider first the case where player 2 plays \( \frac{3}{6} \) only at information set \( Cc \):

\[ \hat{A}_{1}(Dc) = \frac{A(1 \rightarrow 2 \rightarrow \rightarrow)}{A(1 \rightarrow \rightarrow \rightarrow) + (1 \rightarrow A)(\pi^{\rightarrow\rightarrow})} \] (26)

Note that the expression is such that \( \hat{A}_{1}(Dc) = \mathcal{A}_{cc}(\hat{A}) \), where \( \mathcal{A}_{cc} \) is the belief revision operator defined in the previous section. Hence it follows that if \( \hat{A} < \frac{3}{6} \mu \), it follows that \( \mathcal{A}_{cc}(\hat{A}) > \frac{3}{6} \mu \); and hence it is optimal for player 1 to continue with \( \frac{3}{6} \) at every information set.

Consider 1's beliefs at \( (Dd) \): Once again, it is easy to verify that \( \hat{A}_{1}(Dd) = \mathcal{A}_{cd}(\hat{A}) \); and since \( \hat{A} < \mu \) it is optimal to continue with \( \frac{3}{6} \) at this information set.

Consider next the case where \( \frac{3}{6}(Cd) = \frac{3}{6}(Cc) = \frac{3}{6} \) and \( \frac{3}{6}(Dd) = \frac{3}{6}(Dc) = \frac{3}{6} \). In this case, assumption A2 applies, so that we may choose \( \frac{3}{6} \) close to its upper bound. We have

\[ \hat{A}_{1}(Dc) = \frac{A(1 \rightarrow 2 \rightarrow \rightarrow)}{A(1 \rightarrow \rightarrow \rightarrow) + (1 \rightarrow A)(\pi^{\rightarrow\rightarrow})} \] (27)

If \( \hat{A} > \frac{3}{6} \) then \( \hat{A}_{1}(Dc) > \frac{1}{4} \) so that it is optimal to start by playing \( \frac{3}{6} \) in this case. To see that player 1 will find it optimal to switch to \( \frac{3}{6} \) on receiving a bad signal, note that requires

\[ \mathcal{A}_{cd}(\hat{A}_{1}(Dc)) = \frac{\hat{A}^{""}}{\pi^{\rightarrow\rightarrow}} < \frac{1}{4} \] (28)
Now if \( 1 \cdot (Dc) \cdot \mu \) lemma 2 has verified that a player who begins with \( \frac{1}{2} \) will switch to \( \frac{1}{3} \) on receiving signal \( Cd \) in any subsequent period. If \( 1 \cdot (Dc) > \mu \) it suffices to verify that \( \bar{A}_{C_D} (\bar{1} \cdot (Dc)) < \mu \) which is the upper bound for \( \bar{A} \). This yields the condition

\[
\bar{A} < \frac{(\bar{n} + \bar{m}) \bar{u}}{\bar{u}}
\]  

(29)

Since \( \bar{A} < \mu \) this condition is also satisfied.

Finally, we consider the case where \( \frac{1}{2} (Cd) = \frac{1}{3} (Dc) = \frac{1}{3} (Cc) = \frac{1}{3} (Cd) \):

\[
1 \cdot (Dc) = \frac{\bar{A} (1_i \cdot 2^n \cdot i \cdot \bar{u}) + (1_i \cdot \bar{A}) \bar{u}}{\bar{A} (1_i \cdot " \cdot i \cdot \bar{u}) + (1_i \cdot \bar{A}) (" \cdot \bar{u})} < \frac{1_i \cdot 2^n \cdot i \cdot \bar{u}}{1_i \cdot " \cdot i \cdot \bar{u}}
\]  

(30)

Hence it suffices to evaluate \( \bar{A}_{C_D} \) at the upper bound, which yields

\[
\bar{A}_{C_D} (1 \cdot (Dc)) < \frac{1}{2}
\]  

(31)

Hence \( \bar{A}_{C_D} (1 \cdot (Dc)) < \frac{1}{2} \) for every value of \( \bar{u} \).

We have therefore proved that the payoffs \( \frac{1 + \bar{g} + \bar{l}}{1 + \bar{g}} \circ 0 \) (and obviously the payoffs \( 0 ; \frac{1 + \bar{g} + \bar{l}}{1 + \bar{g}} \)) can be approximated under assumption A provided that \( \pm > \max \phi \frac{\bar{g}}{1 + \bar{g}} ; \frac{\bar{g}}{1 + \bar{g}} \) and provided that \( \bar{n} \) and \( \bar{m} \) are sufficiently small. The payoffs \( 1 ; 1 \) has been approximated under a weaker set of assumptions \( \pm > \frac{\bar{g}}{1 + \bar{g}} \) and \( \bar{n} \) and \( \bar{m} \) sufficiently small), and the payoff \( 0 ; 0 \) is a static Nash payoff. Since any payoff individually rational feasible payoff is a convex combination of these payoffs, and can be achieved via public randomization, we have proved the following theorem.

Theorem 7 Assume that Assumption A is satisfied, and players observe a public randomization device, then for any individually rational feasible payoff vector \( u = (u_1; u_2) \) and any number \( \beta > 0 \), there exist \( " \bar{(} \beta) \bar{>} 0 \); \( \beta \bar{(} \beta) \bar{>} 0 \) such that there exists a sequential equilibrium with payoffs within \( \beta \) distance of \( u \) provided that \( " \bar{(} \beta) \bar{;} \) and \( \beta \bar{(} \beta) \bar{;} \) are sufficiently small. The payoff \( 1 ; 1 \) has been approximated under a weaker set of assumptions \( \pm > \frac{\bar{g}}{1 + \bar{g}} \) and \( \bar{n} \) and \( \bar{m} \) sufficiently small), and the payoff \( 0 ; 0 \) is a static Nash payoff. Since any payoff individually rational feasible payoff is a convex combination of these payoffs, and can be achieved via public randomization, we have proved the following theorem.

This result is most closely related to those obtained in a paper by Piccione [12], who also analyzes the prisoners' dilemma with imperfect private monitoring. Our results differ, both in terms of substance and in the techniques/strategies used. Piccione's substantive results are that full cooperation can always be approximated, and further, any individually rational feasible payoff can be approximated in a class of prisoners' dilemma games, i.e. for games where \( 1 \cdot g \). The "folk theorem" condition A in the present paper is, in a sense, the opposite of Piccione's condition. More recently, Ely and Välimäki [5] have considerably simplified the technique used in Piccione, and generalized the folk theorem obtained there. We shall discuss the differences between the approach of the present paper and the approach of Piccione and Ely-Välimäki in the concluding section.
4 The n-player case

In this section, we extend the approximate efficiency result to the n-player case. Let \( N = \{1; 2; \ldots; n\} \) be the set of players and \( G \) be the stage game played by those players. The stage game \( G \) is as follows. Player \( i \) chooses an action \( a_i \) from the action set \( A_i = \{fC; D; g\} \). Actions are not observable to the other players and taken simultaneously. A n-tuple action profile is denoted by \( A = (a_1; a_2; \ldots; a_n) \). An action profile of all players but player \( i \) is denoted by \( A_i^\complement \) and \( a_i = a_i^\complement \) for all \( i \in N \).

Each player receives an \( n \)-tuple private signal profile about all the other players’ actions. Let \( \Pi_i = \{!_i; 1; \ldots; !_i; n\} \) be a generic signal received by player \( i \) where \( !_i \) stands for player \( i \)'s signal about player \( j \)'s action: A generic signal profile is denoted by \( \Pi = (\Pi_1; \Pi_2; \ldots; \Pi_n) \). All players have the same payoff function \( u \). Player \( i \)'s payoffs \( u(a_i; \Pi) \) depend on her own action \( a_i \) and private signal \( \Pi_i \). Other players’ actions affect a player \( i \)'s payoffs only through the distribution over the signal which player \( i \) receives. The distribution conditional on \( a \) is denoted by \( p(!ja) \). It is assumed that \( p(!ja) \) are full support distributions, that is, \( p(!ja) > 0 \) \( \forall \) \( a \). The space of a set of full support distributions \( \mathbb{P}(\Pi) \) is denoted by \( \mathbb{P} \).

We also introduce the perfectly informative signal distribution \( \mathbb{P}_0 = f\mathbb{P}_0(!ja)g_{2A}^i \), where, for any a 2 A; \( p_0(!ja) = 1 \) if \( !_i = a_i \) for all \( i \). The whole space of the information structure \( \mathbb{P} \) is endowed with the Euclidean norm.

Since we are interested in the situation where information is almost perfect, we restrict attention mainly to a subset of \( \mathbb{P} \) where information is almost perfect. Information is almost perfect when every person’s signal profile is equal to the actual action profile with probability larger than \( 1/n \) for some small number \( n \).

To sum up, the space of the information structure we deal with in this section is the following subset of \( \mathbb{P} \):

\[
P_\ast = \mathbb{P}(\Pi) g_{2A}^i \quad \text{if } \exists i, j \text{ such that } p(!ja) > 0, \quad \text{and } 8a_i, p(!ja) = 1
\]

(32)

and we use \( p_\ast \) for a generic element of \( P_\ast \).

A player’s realized payoff only depends on the number of bad signals \( d \) that a player receives. Let \( d(!_i) \) be the number of \( d \) contained in \( !_i \); then, \( u(a_i; !_i) = u(a_i; !_i) \) if \( d(!_i) = d(!_i) \) for any \( a_i \). Let \( u(a_i; D; c) \) be the payoff of player \( i \) when \( a_i = D \) and \( c \). The deviation gain when \( d \) defections are observed is \( g(\mu) = u(D; c) - u(D; c) \); which is strictly positive for all \( \mu \). The largest deviation gain and the smallest deviation gain is \( g \) and \( g \) respectively, where \( g = \max_k g(k) \) and \( g = \min_k g(k) \).

We impose the following symmetry assumption on \( p_\ast \):
Player, but this dependence is not shown explicitly as it is obvious. Let only one player without loss of generality. Let measure also depends on the initial level of mixture between players. Then a probability measure is derived conditional on the realization of the private history with a common discount factor \( \mu \). This allows us to treat all the players in a symmetric way and to focus on only one player without loss of generality. Let \( U \) be the expected payoff of player \( i \) when \( \mu \) players are playing \( D \). The payoffs \( U \), \( C \), \( D \), and \( N \) are normalized to 1 and 0 respectively for all \( i \): It is assumed that \( (1; \ldots ; 1) \) is the symmetric efficient stage game payoffs.

The stage game \( G \) is repeated indefinitely by \( n \) players, who discount their payoffs with a common discount factor \( \pm 2 \) \( (0;1) \). Time is discrete and denoted by \( t = 1; 2; \ldots \). Player \( i \)’s private history is \( h^t_i = (\ldots ; a^t_i ; \ldots ) \). Each \( h^t_i \) being a mapping from \( H^t_i \) to probability measures on \( A_i \) and denote \( \pi_t(h^t_i) \) for \( \pi_t(h^t_i) \) discounted average payoffs for player \( i \) is \( V(\pi_t; p) = (1_j \; \pm \; \cdots \; \cdots \; \cdots \; \cdots ) = \sum_{t=1}^{\infty} \pi_t(h^t_i) [u((a^t_i; !^t_i)] \), where the probability measure on \( H^t_i \) is generated by \( (\pi_t) \).

For this \( n \)-player repeated prisoner’s dilemma, \( \pi_c \) and \( \pi_D \) are defined as the partial continuation strategies which are realization equivalent to the following grim trigger strategy and permanent defection respectively:

\[
\frac{1}{2}
\begin{align*}
\pi_c(h^t_i) &= C & \text{if } h^t_i = ((C; c); \ldots ; (C; c)) \\
\pi_D(h^t_i) &= D & \text{otherwise}
\end{align*}
\]

where \( c = (C; \ldots ; C) \).

This grim trigger strategy is the harshest one among all the variations of grim trigger strategies in the \( n \) player case. Players using \( \pi_c \) switch to \( \pi_D \) as soon as they observe any signal pro.\( \ast \)e which is not fully cooperative. When player \( i \) is mixing \( \pi_c \) and \( \pi_D \), with probability \( (1_j ; \cdots ) \); that strategy is denoted by \( (1_j ; \pi_c + (1_j ; \cdots ) \pi_D) \).

Suppose that either \( \pi_c \) or \( \pi_D \) is chosen in the first period by all players. Let \( \mu \) be the number of players using \( \pi_D \) as a continuation strategy among \( n \) players. Then a probability measure \( \pi(h^t_i; p) \) on the space \( \mathcal{E} = f0; 1; \ldots ; n \) is derived conditional on the realization of the private history \( h^t_i \). Clearly, this measure also depends on the initial level of mixture between \( \pi_c \) and \( \pi_D \) by every player, but this dependence is not shown explicitly as it is obvious. Let \( U \) be the space of such probability measures, which is an \( n \) \( 1 \) dimensional simplex.

In the two player case, a player’s strategy is represented as a function of belief, using the fact that the other player is always playing either \( \pi_c \) or \( \pi_D \) on and on the equilibrium path. Note that the space of the other players’ “types” is much larger. However, there is a convenient way to summarize relevant information. We classify \( \mathcal{E} \) into two sets; \( f0 \) and \( f1; \ldots ; n \) \( 1 \); that is, the state no
one have ever switched to $\frac{3}{\theta}$ and the state where there is at least one player who has already switched to $\frac{3}{\theta}$. Player $i$'s conditional subjective probability that no player has started using $\frac{3}{\theta}$ is denoted by $A^{1}_{i, i}$ given $1_{i, i} \cap 2_{U}$. The reason why we just focus on this number is that the exact number of players who are playing $\frac{3}{\theta}$ does not make much difference to what will happen in the future given that everyone is playing $\frac{3}{\theta}$. As soon as someone starts playing $\frac{3}{\theta}$, every other player starts playing $\frac{3}{\theta}$ with very high probability from the very next period on by the assumption of almost perfect monitoring. What is important is not how many players have switched to $\frac{3}{\theta}$; but whether anyone has switched to $\frac{3}{\theta}$ or not.

Finally, let $V(\frac{3}{\theta}; \mu; p; \theta)$ be player $i$'s discounted average payoff when $\mu$ other players are playing $\frac{3}{\theta}$ and $n_{i} \cup i$ other players are playing $\frac{3}{\theta}$: We need the following notations:

$$V_{i, i}^{1/\theta; \mu; i} : p; \theta = \mathbb{E}^{1_{i, i}} V(\frac{3}{\theta}; \mu; p; \theta)$$

(34)

$$4V(\mu; p_{0}; \theta) = V(\frac{3}{\theta}; \mu; p_{0}; \theta) + V(\frac{3}{\theta}; \mu; p_{0}; \theta)$$

(35)

$$g_{i, i}^{1/\theta; \mu} : p = \max_{\mu=0}^{n_{i}} U_{i, i}^{D} : D_{i}^{\theta} : p_{i}^{\theta} U_{i, i}^{C} : D_{i}^{\theta} : p_{i}^{\theta}$$

(36)

### 4.1 Belief Dynamics and Best Response

For the two player case, the equilibrium strategy was described as a mapping from $U$ to $4 A_{i}$. The equilibrium strategy we will construct here has a similar structure except that belief lies in a larger space. It has the following expression:

$$\frac{1}{2}^{1}_{i, i} : \theta = \begin{cases} C & \text{if } 1_{i, i} \cap 2_{U}^{C} \\ D & \text{if } 1_{i, i} \cap 2_{U}^{D} \\ \frac{1}{4} & \text{if } 1_{i, i} \cap 2_{U}^{C} \\ \frac{1}{4} & \text{if } 1_{i, i} \cap 2_{U}^{D} \\ \frac{1}{4} & \text{if } 1_{i, i} \cap 2_{U}^{C} \\ \frac{1}{4} & \text{if } 1_{i, i} \cap 2_{U}^{D} \end{cases}$$

(36)

where $U^{C}; U^{D}$ are mutually exclusive and exhaustive sets in $U$; and $\frac{1}{4}$ means playing $C$ with probability $\frac{1}{4}$ and playing $D$ with probability $\frac{3}{4}$.

In order to verify that $\frac{1}{2}$ is a Nash equilibrium and achieves the approximate efficient payoff, we strengthen the path dominance argument used in Sekiguchi [13] instead of appealing to the one-shot deviation argument used in the previous sections. The argument is divided into several steps. First step is to give an almost complete characterization of the unique optimal action as a function of belief. As a next step, we analyze the dynamics of belief by introducing natural assumptions on the information structure when players are playing either $\frac{3}{\theta}$ or $\frac{3}{\theta}$. The third step is to check consistency of this strategy profile, that is, to check if players are actually playing either $\frac{3}{\theta}$ or $\frac{3}{\theta}$ by following $\frac{1}{2}^{1}_{i, i}$.

Once it is established that $\frac{1}{2}$ is a Nash equilibrium, then we can use the fact that there exists a sequential equilibrium which is realization equivalent to a
Nash equilibrium if the game is of non-observable deviation. Finally, as in the two-player case, we can use a public randomization device or divide the original repeated game to component repeated games to implement the same payoff for large $\pm$

After we prove the existence of sequential equilibrium which is realization equivalent to $\frac{1}{2}1_{i,i}$ and approximates the efficient outcome, we show, with one more assumption, that $\frac{1}{2}1_{i,i}$ itself is actually a sequential equilibrium. The unique optimal action is indeed shown to have exactly the same form as $\frac{1}{2}1_{i,i}$ for a certain range of $\pm$ if monitoring is almost perfect.

Before analyzing the unique optimal action, we first extend one property which holds in the two-player case to the $n_i$ player case. In the two-player case, the difference in payoffs by $\frac{1}{2}1$ and $\frac{1}{2}0$ is linear and there is a unique $\frac{1}{2}(\pm;\gamma)$ where a player is indifferent between $\frac{1}{2}1$ and $\frac{1}{2}0$ with perfect monitoring. When the number of players is more than two; the corresponding object $V\frac{1}{2}1_{i,i}: p_i; \pm; V\frac{1}{2}0_{i,i}: p_i; \pm$ is a slightly more complex. Even when players randomize independently and symmetrically playing $\frac{1}{2}1$ with probability $1$ and $\frac{1}{2}0$ with probability $(1 - \frac{1}{2})$, that is, $1_{i,i}(\mu) = \sum_{\mu=0}^{1} p_i(n_i - 1)\mu(n_i - 1)^{\mu - 1}0^{n_i - 1 - \mu}$ for $\mu = 0; \cdots; n_i - 1$; it is a degree polynomial of $\mu$ in $[0;1]$. However, if $\pm > \frac{1}{1 + \gamma}0$; it is possible to show that there exists a unique $2(0;1)$ such that $V\frac{1}{2}1_{i,i}: p_i; \pm; V\frac{1}{2}0_{i,i}: p_i; \pm = 0$.

**Lemma 8** If $\pm > \frac{1}{1 + \gamma}0$; there exists a unique $2(0;1)$ such that

$$\sum_{\mu=0}^{1} p_i(n_i - 1)\mu(n_i - 1)^{\mu - 1}0^{n_i - 1 - \mu} V(\mu; p_i; \pm) = 0$$

**Proof.** Let $f(\mu) = \sum_{\mu=0}^{1} p_i(n_i - 1)\mu(n_i - 1)^{\mu - 1}0^{n_i - 1 - \mu} V(\mu; p_i; \pm)$: Since $f(\mu) > 0$; $f(0) < 0$ and $f$ is continuous, existence of such $\mu$ is guaranteed. To show uniqueness, we prove $f(\mu) = 0$: $\sum_{\mu=0}^{1} p_i(n_i - 1)\mu(n_i - 1)^{\mu - 1}0^{n_i - 1 - \mu} 4 V(\mu; p_i; \pm) = 0$.

$$\sum_{\mu=0}^{1} p_i(n_i - 1)\mu(n_i - 1)^{\mu - 1}0^{n_i - 1 - \mu} = \sum_{\mu=0}^{1} p_i(n_i - 1)\mu(n_i - 1)^{\mu - 1}0^{n_i - 1 - \mu} 4 V(\mu; p_i; \pm)$$

Since $4 V(\mu; p_i; \pm) < 0$ for $\mu$, $1; \sum_{\mu=0}^{1} p_i(n_i - 1)\mu(n_i - 1)^{\mu - 1}0^{n_i - 1 - \mu} 4 V(\mu; p_i; \pm) = 0$.

Let $\frac{1}{2}(\pm; p_0)$ be this level of mixture where players are indifferent between $\frac{1}{2}1$ and $\frac{1}{2}0$ and monitoring is perfect, and denote the associated belief on $\mu$ by $1_{i,i}$. 

20
The following lemma extends a useful property in the two player case to the n player case.

Lemma 9 \( \frac{1}{2} (\pm p_0) \neq 1 \) as \( \pm \frac{q(0)}{1 + q(0)} \)

Proof. See Appendix

When monitoring is almost perfect, \( V^{1/2} ; 1 \), \( 1 \text{ i} : p ; 1 \) is very close to \( V^{1/2} ; 1 \text{ i} : p_0 ; 1 \). Actually, it is easy to see that the former converges to the latter uniformly in \( 1 \) as \( \frac{1}{2} \to 0 \). Hence, we can find \( \frac{1}{2} (\pm p_0) \) in the neighborhood of \( \frac{1}{2} (\pm p_0) \) when monitoring is almost perfect.

Now we derive the unique optimal action as a function of \( 1 \text{ i} \): If monitoring is perfect, then \( V^{1/2} ; 1 \text{ i} : p_0 ; 1 \) is the unique optimal action for this belief \( 1 \text{ i} \). We show that the unique optimal action with almost perfect monitoring is almost the same. So, this result is essentially the maximum theorem in the sense that the optimal choice is "continuous". As a first step, the following lemma shows that \( \frac{1}{2} \) is optimal if a player knows that someone has switched to the permanent defect and is small.

Lemma 10 There exists a \( b > 0 \) such that \( V^{1/2} ; 1 \text{ i} : p ; 1 \) is maximized by \( \frac{1}{2} \) for any \( p_0 \) if \( 1 \text{ i} \) \((\mu) = 1 \) for any \( \mu \neq 0 \).

Proof. Take \( \frac{1}{2} \) and any strategy which starts with \( C \). The least deviation gain is \( (1 \text{ i} + b \mu) \). The largest loss caused by the difference in continuation payoffs with \( \frac{1}{2} \) and the latter strategy is \( b \mu \). Setting \( b \) small enough guarantees that \( (1 \text{ i} + b \mu > b \mu V \) for any \( 2 \) \((0, b) \). Then, \( D \) must be the optimal action for any such \( \mu \). Since players are using \( \frac{1}{2} \), \( 1 \text{ i} \) \((\mu) = 1 \) for some \( \mu \neq 0 \) in the next period. This implies that \( D \) is the unique optimal action in all the following periods.

Using \( p(\mu) \) and given the fact that all players are playing either \( \frac{1}{2} \) or \( \frac{1}{2} \), we define a transition probability of the number of players who have switched to \( \frac{1}{2} \). Let \( q(l|jm) \) be a probability that \( l \) players will play \( \frac{1}{2} \) from the next period when \( m \) players are playing \( \frac{1}{2} \) now. In other words, this \( q(l|jm) \) is a probability that \( l \text{ i} m \) players playing \( C \) receive the signal \( d \) when \( n \text{ i} m \) players play \( C \) and \( m \) players play \( D \). Of course, \( q(l|jm) > 0 \) if \( l = m \) and \( q(l|jm) = 0 \) if \( l < m \).

The following lemma provides various informative and useful bounds on the variations of discounted average payoffs caused by introducing small imperfectness in private monitoring.

Lemma 11

\footnote{Also note that convergence of \( V^{1/2} ; 1 \text{ i} : p ; 1 \) to \( V^{1/2} ; 1 \text{ i} : p_0 ; 1 \) is independent of the choice of the associated sequence \( p \) because of the definition of \( P \).}
is using " which discounted average payoffs functions are continuous. Of course, this if she knew the true continuation strategies of her opponents at each possible state. To see that this additional information is valuable, suppose that the continuation strategy of $V$ optimality of $V$.

Proof. (1): For any $2 (0;1)$ and $p-2 p$

$$V (\frac{3}{2};0:p;\pm) = (1_i \pm \pm (1_i q(0)0)) V (\frac{3}{2};0:p;\pm) + \pm (1_i q(0)0) V$$

(39)

So,

$$V (\frac{3}{2};0:p;\pm) = (1_i \pm \pm (1_i q(0)0)) V (\frac{3}{2};0:p;\pm) + \pm (1_i q(0)0) V$$

(40)

(2): Given $2 \pm 2 q(0)0;1$; it is easy to check that $V (\frac{3}{2};0:p;\pm) > V (\frac{3}{2};0:p;\pm)$: Pick $\pm$ small enough such that (i) $V (\frac{3}{2};0:p;\pm) > V (\frac{3}{2};0:p;\pm)$ for any $p$ and (ii) $\pm < b$. Let $\frac{3}{2}$ be the optimal strategy given that everyone is using $\frac{3}{2}$: Suppose that $\frac{3}{2}$ assigns $D$ for the first period. Then for any $2 (0;1)$

$$V (\frac{3}{2};0:p;\pm) \pm q(1)1 V (\frac{3}{2};0:p;\pm) + q(1)1 V (\frac{3}{2};0:p;\pm)$$

(41)

In this inequality, the second component represents what player $i$ could get if she knew the true continuation strategies of her opponents at each possible state. To see that this additional information is valuable, suppose that the continuation strategy of $\frac{3}{2}$ leads to a higher expected payoff than $V (\frac{3}{2};0:p;\pm)$ or $V (\frac{3}{2};0:p;\pm)$ at the corresponding states, then this contradicts the optimality of $\frac{3}{2}$ or $\frac{3}{2}$ by Lemma 10. Hence this inequality should hold.

Then, for any $2 (0;1)$

$$V (\frac{3}{2};0:p;\pm) = \frac{1_i \pm \pm \pm \pm (1_i q(1)1)}{1_i \pm \pm (1_i q(1)1)} V (\frac{3}{2};0:p;\pm)$$

(42)

$$V (\frac{3}{2};0:p;\pm) \pm q(1)1 V (\frac{3}{2};0:p;\pm) + q(1)1 V (\frac{3}{2};0:p;\pm)$$

Such $\frac{3}{2}$ exists because the strategy space is a compact space in product topology, on which discounted average payoff functions are continuous. Of course, this $\frac{3}{2}$ depends on the choice of $p$.
Since this contradicts the optimality of \( \frac{3}{6} \); \( \frac{3}{6} \) has to assign C for the first period.

Now,

\[
V \left( \frac{3}{6}; 0 : p : \pm 5 \right) (1 \pm 5 + \pm q(0)0) V \left( \frac{3}{6}; 0 : p : \pm + \pm (1 \pm q(0)0) \right) \nabla
\] (43)

So,

\[
V \left( \frac{3}{6}; 0 : p : \pm 5 \right) \frac{(1 \pm 5 + \pm (1 \pm q(0)0)) \nabla}{1 \pm q(0)0} \frac{(1 \pm 5 + \pm \nabla)}{1 \pm (1 \pm \nabla)}
\] (44)

This implies that \( \sup_{\frac{3}{6}, \pm \nabla} V_i \left( \frac{3}{6}; 0 : p : \pm 5 \right) \frac{(1 \pm 5 + \pm \nabla)}{1 \pm (1 \pm \nabla)} \) for any \( \pm 2 [0, \nabla] \).

(1) means that a small departure from the perfect monitoring does not reduce the payoff of \( \frac{3}{6} \) much when all the other players are using \( \frac{3}{6} \). (2) means that there is not much to be exploited by using other strategies than \( \frac{3}{6} \) with a small imperfection in the private signal as long as all the other players are using a \( \frac{3}{6} \).

The following result is an almost complete characterization of the optimal action as a function of \( ^{11}_{11} \).

Proposition 12 Given \( \pm ; \) for any \( \nu > 0 \); there exists a \( \mu > 0 \) such that for any \( \mu \),

\[
^1_2 \text{ it is not optimal to play } C \text{ for player } i \text{ if } \mu \text{ satis}..es \quad \hat{\lambda} \left( \frac{1}{i} \right) \geq \frac{1}{i} g \left( \frac{1}{i} \right) \left( \frac{0}{i} + \frac{\nabla}{i} \right) + \frac{1}{i} \hat{\lambda} \left( \frac{1}{i} \right) \left( \frac{0}{i} + \frac{\nabla}{i} \right)
\] (45)

\[
^2 \text{ it is not optimal to play } D \text{ for player } i \text{ if } \mu \text{ satis}..es \quad \hat{\lambda} \left( \frac{1}{i} \right) \geq \frac{1}{i} g \left( \frac{1}{i} \right) \left( \frac{0}{i} + \frac{\nabla}{i} \right) + \frac{1}{i} \hat{\lambda} \left( \frac{1}{i} \right) \left( \frac{0}{i} + \frac{\nabla}{i} \right)
\]

Proof: (1): It is not optimal to play C if

\[
(1 \pm 5 + \pm g \left( \frac{1}{i} \right) \left( \frac{0}{i} + \frac{\nabla}{i} \right)) \sup \left( \frac{3}{6}; 0 : p : \pm + \pm \nabla + \nabla \right) \nabla
\] (46)

is satis..ed because then any strategy which plays C now is dominated by \( \frac{3}{6} \). By Lemma 11.2., this inequality is satis..ed for any \( \pm 2 [0, \nabla] \) and any \( \mu \) if

\[
\left( \frac{1}{i} \right) \geq \frac{1}{i} g \left( \frac{1}{i} \right) \left( \frac{0}{i} + \frac{\nabla}{i} \right) \nabla
\] (46)
LHS converges to \((1_i \neq g^{i_1}_{1, i}; p - \xi)\) and RHS converges to \(\pm A^{i_1}_{1, i} \xi\) as \(\varepsilon \to 0\). So, if \(1_i \xi\) satisfies \(A^{i_1}_{1, i} \neq 1_i \neq g^{i_1}_{1, i}; p - \xi\) for any \(\xi > 0\), then there exists a \(a_i \neq \xi; 1_i \neq 2(0; \varepsilon)\) and a neighborhood \(B^{i_1}_{1, i} \xi\) of \(1_i \xi\) such that \(C\) is not optimal for any \(p_{0}(\varepsilon; 1_i)\) and any \(1_i \neq 2 B^{i_1}_{1, i} \xi\). This \(a_i \neq \xi; 1_i \neq 2(0; \varepsilon)\) can be set independent of \(1_i \xi\) by standard arguments because \(1_i \xi|A^{i_1}_{1, i} \xi = 0\) is a compact subset in a \(n_i\) 1 dimensional simplex.

(2): It is not optimal to play \(D\) if

\[
(1_i \neq g^{i_1}_{1, i}; p - \xi) < \pm A^{i_1}_{1, i} \xi f(1_i \varepsilon) V(1/\varepsilon; 0; p - \xi) + "Vg + i_1 \xi A^{i_1}_{1, i} \xi "V_i "g^\varepsilon\]
\]

This inequality converges to \(A^{i_1}_{1, i} \xi = 1_i \neq g^{i_1}_{1, i}; p - \xi\) as \(\varepsilon \to 0\). So, if \(1_i \xi\) satisfies \(A^{i_1}_{1, i} \xi = 1_i \neq g^{i_1}_{1, i}; p - \xi\) for any \(\xi > 0\), there exists a \(a_i \neq \xi; 1_i \xi\) such that \(D\) is not optimal for any \(p_{0} 2 P_{0}(\varepsilon; 1_i)\) and any \(1_i \neq 2 B^{i_1}_{1, i} \xi\). Again, \(a_i \neq \xi; 1_i \xi\) can be set independent of \(1_i \xi\).

Finally, setting \(\varepsilon = \min f^{\varepsilon(\pm \varepsilon)}\) completes the proof.

This proposition implies that the optimal action can be completely characterized except for an arbitrary small area around the manifold satisfying \(A^{i_1}_{1, i} \xi = 1_i \neq g^{i_1}_{1, i}; p - \xi\) in a \(n_i\) 1 dimensional simplex; where player \(i\) is indifferent between \(\varepsilon/\xi\) and \(\varepsilon/\xi\) with perfect monitoring, but we also characterize the optimal action for this region later.

Although this argument is essentially the path dominance argument used in Sekiguchi [13] for \(n = 2\), it extends to any \(n_i\) player case and provides a sharper characterization even for \(n = 2\).

An immediate corollary of this proposition is that \(C\) is the unique optimal action given that \(A\) is sufficiently close to \(1, \pm > 1/\varepsilon; \xi\); and \(\varepsilon\) is small:

**Corollary 13** Given \(\pm > 1/\varepsilon; \xi\); there exists \(A > 0\) and \(\varepsilon > 0\) such that for any \(p_{0}\); it is not optimal for player \(i\) to play \(D\) if \(\varepsilon > A\).

Since the optimal action is almost characterized as a function of \(i\); now we need to know the dynamics of \(1_i \xi\) associated with \(\varepsilon/\xi\) and \(\varepsilon/\xi\). Given the
optimal action shown above, what we need for consistency is that \( i, j \) stays in the "C area" described by Proposition 12 as long as player \( i \) has observed fully cooperative signals from the beginning and \( i, j \) stays in the "D area" once player \( i \) received a bad signal and started playing defection for herself:

Let \( W_k = i, j A^1_{i, j} = \frac{1}{i, j} + g_{i, j}^1 ; p_0 i, k \); where \( k \geq 0 \); \( g_{i, j} \). When \( k \) and " is small, \( W_k \) just covers the region where \( C \) is the unique optimal action and the unique optimal action is indeterminate. We show that \( A^1_{t+n} \) is always above \( \bar{A} \) if \( t(0; 1) \) if a player plays \( C \), observes \( c; \bar{A} \) \( W_k \), and monitoring is almost perfect.

Lemma 14 For any \( \bar{A} > 0 \); \( k \geq 0 \); \( i, j A^1_{i, j} \); there exists \( \delta \) such that for any \( " \geq 2 \) \( (0; \delta \) \( i, j A^1_{i, j} ; p \in \bar{A} \) for \( h_t^{n+1} = (h_t^1; (C; c) ; \ldots ; (C; c)) \) when \( i, j (h_t^1) \) \( W_k \).

Proof. Let \( h_t^{n+1} = (h_t^1; (C; c)) \) and \( \bar{A}^t = i, j A^1_{i, j} (h_t^1; p) \).

Applying Bayes' rule,

\[
A^{t+1}_i = i, j A^1_{i, j} h_t^{n+1} ; p \in \bar{A}^t
\]

\[
= \frac{\bar{A}^t p(\text{c|C})}{\bar{A}^t p(\text{c|C}) + \int \text{p}}
\]

This function is increasing in \( \bar{A}^t \) and crosses 45° line once. Note that this function is bounded below by \( i, j A^1_{i, j} \) \( \text{c|c} \) \( \bar{A}^t \). Let \( \hat{A} \) be the unique \( \text{c|c} \) point of this mapping. Given that \( \hat{A}^t = i, j A^1_{i, j} (h_t^1; p) \) \( W_k \); it is easy to see that \( i, j A^1_{i, j} \) and \( \hat{A} \) can be made larger than \( \bar{A} > 0 \) by choosing " small enough.

If \( \hat{A}^t < \hat{A} \); then, as long as players continue to observe \( c; i, j A^1_{i, j} \) is going to increase monotonically to \( \hat{A} \). On the other hand, since \( \hat{A}^t \geq i, j A^1_{i, j} \) for \( n = 1, 2, \ldots \) and, \( i, j A^1_{i, j} \) is monotonically increasing, \( \hat{A}^t \) is larger than \( i, j A^1_{i, j} \); hence larger than \( i, j A^1_{i, j} \) for any \( p = 1, 2, \ldots \) On the other hand, if \( \hat{A}^t = \hat{A} \); then it is clear that \( \hat{A}^t \geq i, j A^1_{i, j} = \hat{A} > \bar{A} \). These imply that \( \hat{A}^t \) \( n=1 \) is always above \( \bar{A} \).

The above lemma guarantees that players are cooperating after they observed a stream of cooperative signals and played \( C \) all the time. The next lemma is used to show that player \( i \) plays \( \frac{1}{p} \) once a bad signal is observed or \( D \) has been played in the previous period. Let us define \( p \geq 2 (0; 1) \) as the smallest number such that
\[ p_{\theta} = \frac{p(c|c)P(!i = !j|c)}{p(c!i; j|c)P(!i = !j|c; f_i(!i))} \text{ for any } !i \theta c \] (50)

and

\[ p_{\theta} = \frac{p(!i; c|d; a_i = c)}{p(!i = !j|d; a_i = c)} \text{ for any } !i \theta 0 \]

where \( f_i : !i \theta !j \theta f_j(!i, !j) \theta 2 \) \( !i \theta A!i \) is a mapping such that \( f_j(!j) = D \) if and only if \( !j \theta c \). The first condition implies that \( A!i \) will be below \( p_\theta \) even if it is the first time for a player to observe anything other than \( c \) while \( C \) has been played. The second condition means that \( A!i \) will be below \( p_\theta \) independent of the signal received when \( D \) is played. We impose the following regularity condition on the information structure we focus.

Assumption B: For some \( \theta < 1; \theta = \theta_\theta \).

Note that this assumption is not satisfied in the two player case in the previous sections. For example, \( \hat{A}_C(1) \) or \( \hat{A}_D(1) \) can be arbitrary close to 1 even if monitoring is almost perfect. However, we don't need this assumption when \( n = 2 \). This assumption helps us to establish our result for \( n = 3 \). The following is an example of information structure which satisfies this assumption when \( n = 3 \) independent of \( \theta \): 9

Example: Totally Decomposable Case

\[ p(!ja) = \prod_{i,j} f(!i, j|a_j) \text{ for all } a \in A \text{ and } ! \theta 2 - \]

where \( f(!ja) \) is a distribution function on \( f_c, d \theta \) such that \( ! = a \) with very high probability. Given the action by player \( j \); the probability that player \( i \theta j \) receives the right signal or the wrong signal about player \( j \)'s action is the same across \( i \theta j \). Also note that players' signals are conditionally independent over players.

The next lemma is an easy consequence of this assumption.

Lemma 15 A \( \prod_{i} \hat{1} \prod_{i} (h^t_i; p^t) \) \( \leq \theta \) after histories such as \( h^t_i = \prod_{i} (C; c) ; \hat{1} \hat{C} ; 1^t \hat{C}; 1^t \hat{C} \) for \( t = 3 \) where \( !i \theta 1 \theta c \) or \( h^t_i = \prod_{i} (D; d) ; 1^t \hat{C} \).

Proof. See appendix.

\(^9\)An example of a more general class of \( p^t \) which satisfies this assumption independent of the level of \( \theta \) can be found in Obara [10].
4.2 Construction of Sequential Equilibrium

Let us introduce the following notations:

\[ U^C = \mathbb{G}_{i,j} V^{i_3 \epsilon}; i_i : p_p, p_c \succ V^{i_3 \beta}; i_i : p_p, p_c \] (51)

\[ U^L = \mathbb{G}_{i,j} V^{i_3 \epsilon}; i_i : p_p, p_c \succ V^{i_3 \beta}; i_i : p_p, p_c \]

\[ U^D = \mathbb{G}_{i,j} V^{i_3 \epsilon}; i_i : p_p, p_c \prec V^{i_3 \beta}; i_i : p_p, p_c \]

These are subsets of the belief space \( U \). \( U^L \) is a manifold where player \( i \) is indifferent between \( \epsilon \) and \( \beta \) in particular, \( \mathbb{G}_{i,j} (\pm p_p) \) 2 \( U^L \) by definition. Note that \( U^C \) converges to \( U^C = \mathbb{G}_{i,j} i'_{A^1} i_i \epsilon > \frac{1}{2} \frac{g_1}{i_i} i_i^p \) and \( U^D \) converges to \( U^D = \mathbb{G}_{i,j} i'_{A^1} i_i \epsilon < \frac{1}{2} \frac{g_1}{i_i} i_i^p \) as \( \epsilon \) 0.

Now define \( \bar{\epsilon} \) as a mapping from \( i_i 2 U \) to \( 4 \frac{C}{D} \frac{g_1}{p_0} \) in the following way:

\[ \frac{8}{9} i_i \epsilon = \begin{cases} \frac{C}{D} & \text{if } i_i 2 U^C, \frac{1}{2} (\pm p_p) \text{ if } i_i 2 U^L, \frac{1}{2} (\pm p_p) \text{ if } i_i 2 U^D \end{cases} \] (52)

We know from Proposition 12 that this function assigns the best response action almost everywhere except for a neighborhood of \( U^L \) when \( \epsilon \) is small.

Now we use \( \bar{\epsilon} \) to construct a Nash equilibrium approximating the efficient outcome, for which there exists a realization equivalent sequential equilibrium. All we have to do is to make sure that players are actually playing \( \epsilon \) or \( \beta \) on the equilibrium path after they initially randomize between \( C \) and \( D \).

Proposition 16 Suppose that Assumption B is satisfied. Then there exists a \( \Xi > \frac{g_1}{4 + g_1} \) such that for any \( \Xi > \frac{g_1}{4 + g_1} \) there is a \( (\pm) > 0 \) where, for any \( p_p \), there exists a symmetric sequential equilibrium which is realization equivalent to \( \bar{\epsilon} \) \( i_i \epsilon \), hence realization equivalent to \( \bar{\epsilon} \) \( 1/4 (\pm p_p) \) \( \epsilon \) \( (1/4 (\pm p_p)) \) \( \beta \).

Proof. Pick any \( \Xi > \frac{g_1}{4 + g_1} \) such that if \( \pm 2 \frac{g_1}{4 + g_1} ; \Xi \); then \( \epsilon < \frac{1}{2} (p_0; \Xi) \). First we prove that if \( A \) \( 5 \) then \( D \) is the unique optimal action, hence the optimal continuation strategy is \( \beta \) when \( \epsilon \) is small enough.

Note that player \( i \) is indifferent between \( \epsilon \) and \( \beta \) with belief \( \bar{\epsilon} = \frac{1}{i_i} (p_0; \Xi) \) with no noise, hence the following equality holds:

\[ \forall \mu, 1 \frac{1}{i_i} (p_0; \Xi) \mu \succ V (\mu; p_0; \Xi) \] (53)

Suppose that \( \bar{A} = \frac{1}{i_i} (p_0; \Xi) \succ 1 \frac{1}{i_i} (p_0; \Xi) \) \( 5 \) \( \epsilon < 1 \frac{1}{i_i} (p_0; \Xi) \): We show that \( i\beta \frac{1}{i_i} (p_0; \Xi) \mu \succ V (\mu; p_0; \Xi) < 0 \) by comparing it to the above equality. If
player $i$ plays $C$ here, then the positive payoff player $i$ can get when $\mu = 0$ decreases by at least $1^{\frac{1}{2}} \sum_i (p; 0) 0^{(i)} 2 V(\mu; 0) \frac{1}{\mu}$ compared to the case when $A_i = 1^{\frac{1}{2}} (p; 0):$ On the other hand, the additional gain from playing $C$ when $A_i 0^{(i)} \mu \neq 0$ is bounded above by $1^{\frac{1}{2}} \sum_i (p; 0) 0^{(i)} 2 g$ compared to the case when $A_i = 1^{\frac{1}{2}} (p; 0):$ Since $1^{\frac{1}{2}} (p; 0) 0^{(i)} 1 as $g^{(i)} 0^{(i)}$ by Lemma 9, playing $C$ is strictly dominated when $\mu$ is chosen to be close to $1^{\frac{1}{2}} g^{(i)}.$

Any strategy playing $C$ now continues to be dominated by $\frac{3}{2} \alpha$ even monitoring is almost perfect by the same argument as in Lemma 11$^{10}$.

Now, we check players' incentive on the equilibrium path to prove that $1^{\frac{1}{2}} i^{(i)}$ is a symmetric Nash equilibrium. Players randomize between $C$ and $D$ with probability $\frac{1}{2}(\pm p); \frac{1}{2}(\pm p)$ respectively in the $1^{st}$ period. First of all, $A_i$ is strictly above $A$ as long as Assumption B has been observed by Lemma 14. $C$ is the unique optimal action for such $A_i$ by Lemma 13. Next, when a player $1^{st}$ observes $(C; * (C))$ after the second period, $A_i$ gets below $0$ by Lemma 15. Hence, the unique optimal action is $D$ by the above argument: If $(C; * (C))$ is observed in the $1^{st}$ period; then $A_i$ is again clearly below $0$ for small $\mu$ because $\mu$ is interpreted as a signal of $\frac{3}{2} \alpha$ being chosen in the $1^{st}$ period rather than an error. So, $D$ is always the unique optimal action after this kind of history which ends with $(C; * (C))$: Finally, when $D$ is played, it is always the case that $A_i 0^{(i)}$ in the next period; hence the unique optimal action is again $D$: These imply that $1^{\frac{1}{2}} i^{(i)} 2 U^{(i)}$ after $(C; c)$ has been observed and $1^{\frac{1}{2}} i^{(i)} 2 U^{(i)}$ after $(C; * (C))$ is observed or $D$ is played in the previous period by definition of $U^{(i)}$ and $U^{(i)}$. So, $1^{\frac{1}{2}} i^{(i)}$ is a symmetric Nash equilibrium, which is clearly realization equivalent to $\frac{1}{2}(\pm p) \frac{3}{2} \alpha + (1^{\frac{1}{2}} \frac{1}{2}(\pm p)) \frac{3}{2} \alpha$: Existence of a sequential equilibrium which is realization equivalent to $1^{\frac{1}{2}} i^{(i)} \frac{1}{2}$ follows from the fact that this game is in a class of games with non-observable deviation. See Sekiguchi [13] for detail.$^{9}$

Since the probability that everyone chooses $\frac{3}{2} \alpha$ in this sequential equilibrium; $\frac{1}{2}(\pm p)^{n1}$ gets closer to 1 as $\pm$ gets closer to $\frac{g(0)}{1 + g(0)}$ by Lemma 9, an outcome arbitrary close to the efficient outcome can be achieved for arbitrary close to $\frac{g(0)}{1 + g(0)}$. For high $\pm$ we can use a public randomization device again to reduce $\pm$ exactly or use Ellison's trick as in Ellison [4] to achieve an almost efficient outcome although the strategy is more complex and no longer a grim trigger. Hence, the following result is obtained.

Proposition 17 Suppose that Assumption B is satisfied. Fix $2 \frac{g(0)}{1 + g(0)} i $ Then for any $\xi > 0$; there is a $> 0$ such that for any $p$; there is a sequential equilibrium whose symmetric equilibrium payoff is more than $1^{\frac{1}{2}} \xi$.

Finally we show that $1^{\frac{1}{2}} i^{(i)}$ itself is actually sequential equilibrium with one more assumption:

Assumption C: If $1^{\frac{1}{2}} i^{(i)} 2 U^{(i)}$; then $A_i 0^{(i)}$ after $(C; * (C))$ is observed.

$^{10}$ This argument is not necessary when $n = 2$ because of the property: $A_{0;1} (1) < 1$: 28
Proposition 18: Suppose that Assumption B and C is satisfied. For any $\pm 2^{\frac{g(0)}{2g(0)}} > 0$ if $\alpha$ is small enough, then

1. $C$ is the unique optimal action if and only if $1_{1i} 2 U^C$

2. $D$ is the unique optimal action if and only if $1_{1i} 2 U^D$

Hence, $\frac{1}{2} 1_{1i} 1_{0}$ itself is a sequential equilibrium.

Proof: Fix $\alpha(\alpha) > 0$ in Proposition 12 and set $k(\alpha) > 0$ slightly larger than $\alpha(\alpha) > 0$ for each $\pm 2^{\frac{g(0)}{2g(0)}} > 0$. Take any $1_{0}$ such that $A^{1}_{1i} 1_{0} > \frac{1}{2} g 1_{1i} 1_{0}; p_{0} 1_{k}$ and $1_{1i} 2 U^{D}$. We prove that $D$ is the unique optimal action in this region if $\alpha$ is sufficiently small. If a player plays $C$, then Lemma 14 and Assumption C imply that the continuation strategy is $\frac{1}{2} 1_{k}$ for small $\alpha$. This is because $A_{k} > A(\alpha)$ as long as $(C; c)$ realizes and $A_{5} > 0$ holds otherwise. Since $\frac{1}{2} 1_{k}$ is dominated by $\frac{1}{2} 1_{0}$ in this region, the unique optimal action should be $D$.

Similarly, take any $1_{1i} 1_{0}$ such that $\frac{1}{2} g 1_{1i} 1_{0}; p_{0} 1_{k}$ and $1_{1i} 2 U^{C}$. If $D$ is played, then again the continuation strategy is $\frac{1}{2} 0$ by Lemma 15. Since $\frac{1}{2} 0$ is dominated by $\frac{1}{2} 1_{k}$ in this region, the unique optimal action should be $C$.

Since any other $1_{1i} 2, U^{D}$ satisfies $A^{1}_{1i} 1_{0} = \frac{1}{2} g 1_{1i} 1_{0}; p_{0} 1_{k}$ and any other $1_{1i} 2 U^{C}$ satisfies $A^{1}_{1i} 1_{0} = \frac{1}{2} g 1_{1i} 1_{0}; p_{0} 1_{k}$ for small $\alpha > 0$, the proof is complete.

5 Concluding Comments

The main point of this paper has been to develop “belief-based” strategies as a way of constructing sequential equilibria in repeated games with private monitoring. This affords a major simplification as compared to the traditional method of analysis. While our construction has been restricted to the prisoners’ dilemma, and to a strategy profile which consists only of two continuation strategies, the idea underlying this simplification is generalizable. If player $i$ starts with a finitely complex (mixed) strategy which induces $k$ possible continuation strategies, then the state space or the set of possible beliefs for player $j$ for the entire repeated game is a $k \times 1$ dimensional simplex.

The approach of the present paper is based on generalizing “trigger strategy” equilibria to the private monitoring context. Under perfect or imperfect public monitoring, such trigger strategy can be constructed so as to provide strict incentives for players to continue with their equilibrium actions at each information set. Mailath and Morris [8] show that one can construct equilibria which provide similar strict incentives under private monitoring which is “almost-public”. However, if private signals are not sufficiently correlated, pure trigger strategy profiles fail to be equilibria. The approach in the present paper, as in previous works such as Bhaskar and van Damme [3] and Sekiguchi [13],
relies on approximating the grim trigger strategy with a mixed strategy. In the basic construction, a player is indifferent between cooperating and defecting in the initial period, but has strict incentives to play the equilibrium action at every subsequent information set. In particular, player i’s strategy is measurable with respect to her beliefs about player j’s continuation strategy. As Bhaskar [2] shows, such mixed strategies are robust to a small amount of incomplete payoff information as in Harsanyi’s [6]. In particular, Bhaskar [2] shows in the context of the repeated prisoners’ dilemma, where stage game payoffs are random and private, there exists a strict equilibrium with behavior corresponding to that of the mixed equilibrium of the present paper.\footnote{This result is relevant since Bhaskar [1] considers a overlapping generations game with private monitoring and shows that incomplete payoff information as in Harsanyi implies an anti-folk theorem — players must play Nash equilibrium of the stage game in every period.}

In the initial period, a player plays C for some realizations of his private payoff information, and D for other realizations, and continues with a trigger strategy in subsequent periods, independent of their payoff information.

The alternative approach to constructing non-trivial repeated game equilibria with private monitoring is due to Piccione [12] and Ely and Välimäki [5].\footnote{See also the work of Kandori [7] in the context of a nitely repeated game.} This approach relies on using player j’s mixed strategy to make a player i indifferent between playing C and D at every information set. Since player i is so indifferent, she is likewise willing to randomize so as to make j also indifferent between his actions at each information set. In this approach, beliefs are irrelevant, since a player’s continuation payoff function does not depend upon her beliefs. Such equilibria seem to be less likely to survive if there is private payoff information, and indeed this question is the subject of current research.

Appendix.

Proof of Lemma 9.

When $\pm = \frac{q(0)}{1+q(0)}; \frac{1}{4}(\pm + p_0) = 1$ is the solution of the equation in $^1$:

$$V^{i}_{\frac{1}{2} \epsilon};^{1}_{i i}: p_0; \pm \epsilon, V^{i}_{\frac{1}{2} \epsilon};^{1}_{i i}: p_0; \pm \epsilon = 0$$

(54)

where $^{1}_{i i}(\mu) = (1 \ i \ 1)^{\mu_1 n_1} i^{1};^{\mu_1 n_1} \epsilon$ for $\mu = 0; \ldots; n \ i \ 1$:

We just need to show that $\frac{\partial V(\pm, p_0)}{\partial \pm} \bigg|_{\pm = \frac{q(0)}{1+q(0)}} < 0$ using the implicit function theorem. Since,

$$V^{i}_{\frac{1}{2} \epsilon};^{1}_{i i}: p_0; \pm \epsilon, V^{i}_{\frac{1}{2} \epsilon};^{1}_{i i}: p_0; \pm \epsilon = (1 \ i \ \pm) (1 \ i \ 1)^{\mu_1 n_1} i^{1};^{\mu_1 n_1} \epsilon$$

(55)

$$= \ i (1 \ i \ \pm) (1 \ i \ 1)^{\mu_1 n_1} i^{1};^{\mu_1 n_1} \epsilon \ g(\mu) + \epsilon^{n_1}$$

$$\mu = 0$$
a straightforward calculation gives the desired result as follows.

\[
\frac{\mathbb{E}(\hat{A}^t_\pm \mathbf{r}_0)}{\mathbb{E}} \overset{q(0)}{=} \mathbb{E}
\]

\[
\mathbb{E}
\]

\[
= i \left( 1 + g(0) \right) \frac{1}{n} \left( 1 + g(0) \right)^2 < 0
\]

\[

\]

***Proof of Lemma 15***

Suppose that \( h_t = \mathbb{I}, \ldots, (C; c); i C; i \frac{\hat{A}^t}{\hat{A}^t} \) with \( \hat{A}^t = \hat{A}^t \) the \& c and \( t = 3: \hat{A}_t^3 \) is bounded above by \( \hat{A}_t^3 \) which is obtained by Bayes' rule after such an observation when \( \hat{A}_t^{i_1^t} = 1 \): That \( \hat{A}_t^0 \) is given by

\[
\frac{p(i_1^t i_2^t i_3^t)}{p(i_1^t i_2^t i_3^t)} = \frac{(c; i_1^t)}{p(id)} I_i^t 2 = 0 \]

\[
= i \left( 1 + g(0) \right) \frac{1}{n} \left( 1 + g(0) \right)^2 < 0
\]

\[

\]

Similarly, when \( h_t = \mathbb{I}, \ldots, (D; c); i i^t \frac{\hat{A}^t}{\hat{A}^t} \) with \( \hat{A}^t = \hat{A}^t \) \& \( \hat{A}_t^3 \) is bounded by

\[
\hat{A}_t^3 = \hat{A}_t^3 + \hat{A}_t^3 \frac{p(i)}{p(i)} \]

\[
= i \left( 1 + g(0) \right) \frac{1}{n} \left( 1 + g(0) \right)^2 < 0
\]

\[

\]
References


