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"Indivisibility, Lotteries, and Monetary Exchange"

by

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Indivisibilities, Lotteries, and Monetary Exchange*

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Abstract

We introduce lotteries – that is, randomized trading – into search-theoretic models of monetary exchange. In the model with indivisible goods and fiat money, we show that in any monetary equilibrium goods change hands with probability 1 and money changes hands with probability τ where $\tau < 1$ iff the buyer has sufficient bargaining power. In the model with divisible goods, a nonrandom quantity of goods q changes hands with probability 1 and, again, money changes hands with probability τ where $\tau < 1$ iff the buyer has sufficient bargaining power. Hence, the implicit assumption made in the previous literature that lotteries are ruled out is restrictive. Moreover, q may be less than but can never exceed the efficient quantity (a result that cannot be shown without lotteries). We also consider the implications of lotteries for models with direct barter or commodity money. If commodity money has sufficient intrinsic value, we show the equilibium quantity q is necessarily efficient (another result that cannot be shown without lotteries).

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1 Introduction

In this paper we introduce *lotteries* – that is, randomized trading – into search-theoretic models of monetary exchange. There are several reasons for doing so. First, consider the most basic version of the model, with indivisible goods and indivisible fiat money and agents with single unit storage capacity (as in Kiyotaki and Wright [1991, 1993]). Although this model is simplistic, we think that it still has virtues; in particular, since every trade is a one-for-one swap, one can relatively easily study certain aspects of the exchange process and illustrate certain essential features of money without having to determine exchange rates or the distribution of inventories. However, it is well known from the study of various economic environments with indivisibilities or other nonconvexities that, in such an environment, agents can often do better using randomized rather than deterministic trading mechanisms.[1] So agents in this model may want to use lotteries, and we see no compelling reason to constrain them not to.

In the model with indivisible goods and money, bargaining over lotteries means bargaining over the joint probability distribution of (q, m), where $q \in \{0, 1\}$ is the amount of the good and $m \in \{0, 1\}$ the amount of money to be exchanged. We show first that if buyers (i.e., agents with money) have bargaining power θ below some threshold a monetary equilibrium will not exist, and for θ above this threshold a unique monetary equilibrium does exist. If θ is above the threshold but not too large, then when a buyer meets a seller with a good he desires he gives the seller the money with probability 1 and the seller gives him the good with probability 1. If θ is larger, however, then the money changes hands with probability $\tau < 1$ and the good changes hands with probability 1 independent of whether

[1] See, for example, Prescott and Townsend (1984a, 1984b), Rogerson (1988), Shell and Wright (1993), or Diamond (1990).

the seller gets the money. Hence, at least for some parameter values, the implicit assumption made in the previous literature that lotteries are not allowed is restrictive. Moreover, allowing lotteries means that we can discuss a notion of prices even though goods and money are both indivisible, since τ is the average amount of currency that changes hands for a unit of consumption goods.[2]

Now consider the search model of money with divisible consumption goods, where even if (in the interests of tractability) we continue to assume that money is indivisible and agents have a unit storage capacity, one can determine prices by letting agents bargain over the quantity of goods that a buyer gets for a unit of currency (as in Shi [1995] and Trejos and Wright [1995]). In this model bargaining over lotteries again means bargaining over the joint probability distribution of (q, m), but now $q \in [0, \infty)$. We prove that there exists a unique monetary equilibrium for all parameter values, and when a buyer meets a seller with a good he desires, he gives the seller the money with probability τ, where again τ is strictly less than 1 if and only if the buyer has sufficient bargaining power, and the seller gives him q units of the good with probability 1, where q is deterministic and independent of whether or not the money changes hands. Thus, at least for some parameter values, assuming no lotteries is also restrictive with divisible goods.

Further, we find in the divisible goods model that the equilibrium quantity q may be

[2]We emphasize here that lotteries are different from mixed strategy equilibria. In particular, in the basic model with indivisible goods and indivisible money, without lotteries, if there exists a pure strategy equilibrium where money is universally accepted then generically there also exists a mixed strategy equilibrium where money is accepted with probability less than 1. One thing to note is that sellers are indifferent between having and not having money in this mixed strategy equilibrium, while in the model with lotteries sellers strictly prefer having money – but so do buyers, and the bargaining solution determines the probability that each gets it. Moreover, once we allow agents to bargain over lotteries, mixed strategy equilibria of the above variety no longer exist. Thus, introducing lotteries serves to eliminate the somewhat unnatural equilibria where agents randomize by sometimes accepting and sometimes rejecting money.

less than but can never exceed the efficient quantity q^*, to be defined precisely below. In particular, if the bargaining power of the buyer θ is below a critical value $\tilde{\theta}$, we have $\tau = 1$ and $q < q^*$; as θ increases towards $\tilde{\theta}$, τ stays at 1 and q increases towards q^*; and for $\theta > \tilde{\theta}$, q stays at q^* and τ decreases below 1. We think that this is interesting because it has been argued in Trejos and Wright (1995) that having q less then q^* is a natural result for a monetary model. But one can only rule out $q > q^*$ for one of the two versions of the bargaining game considered in that paper, and even in that version one can not rule $q > q^*$ for all bargaining power parameters. Once we allow agents to use lotteries, it turns out that $q \leq q^*$ holds for both versions of the bargaining game and for all parameter values. Moreover, we will show that welfare is strictly higher for some parameter values, and never lower, if lotteries are allowed than if they are ruled out in this model.

A feature of both the indivisible and divisible goods models is the asymmetry between the way in which goods and money are traded: goods always change hands with probability 1 while for some parameters money changes hands with probability $\tau < 1$. We claim this is due to the *fiat* nature of the monetary object (i.e., it has no intrinsic value and has use only as a medium of exchange). To demonstrate this we do two things. First, we consider a nonmonetary model where agents barter consumption goods directly, and show that any good that is indivisible may be traded with probability less than one. Second, we consider a model with commodity rather than fiat money by assuming the monetary object has a flow utility yield γ. With commodity money, low θ does not rule out the existence of monetary equilibria the way it did in the model with fiat money and indivisible goods. Moreover, with commodity money it is possible for indivisible consumption goods to change hands with probability less than 1. Also, with commodity money and divisible goods, we show that if γ is sufficiently large then monetary equilibria are necessarily efficient, in the sense that $q = q^*$. Finally, note that in a commodity money model without lotteries, if γ is too big then agents will hoard the money and all trade will cease. With lotteries, we show that if γ becomes large money will change hands with a low probability, but still with a positive probability, and goods will still change hands with probability 1.

The rest of the paper is organized as follows. In Section 2 we present the basic assumptions underlying the model. In Section 3 we analyze the version with indivisible goods. In Section 4 we analyze the version with divisible goods. Section 5 considers the barter and commodity money models. Section 6 concludes.[3]

2 The General Model

The economy is populated by a $[0, 1]$ continuum of infinitely-lived agents who specialize in consumption and production. Assume consumption goods are non-storable (to rule out commodity money, so that we may concentrate on fiat money, for now). Let X_i be the set of goods that agent i consumes. No agent i produces a good in X_i. Moreover, for a pair of agents i and j selected at random, the probability that i produces a good in X_j and also j produces a good in X_i is 0 (there are no double coincidences of wants), while the probability that i produces a good in X_j but j does not produce a good in X_i is $x \in (0, 1)$. For example, if there are N goods and N types, $N > 2$, and each type i agent consumes only good i and produces only good $i + 1 \pmod N$, then $x = 1/N$. Let Q denote the set of feasible quantities that agents can produce. We will consider two cases: the indivisible goods model where $Q = \{0, 1\}$, and the divisible goods model where $Q = \Re_+$.

For every agent i, preferences are described as follows. He derives utility $u(q)$ from

[3] We do not consider models where both money and goods are divisible, or where money is indivisible but agents can hold more than a single unit in inventory, such as such as Molico (1996), Green and Zhou (1997), Zhou (1998), Camera and Corbae (1998), Taber and Wallace (1998) or Berentsen (1998).

q units of any good in X_i and incurs in disutility $c(q)$ from q units of a good that he produces. We always assume $u(0) = c(0) = 0$. For the divisible goods model, we assume that both u and c are C^2 real-valued functions, with $u'(q) > 0$, $c'(q) > 0$ for all $q > 0$, and $u''(q) \leq 0$ and $c''(q) \geq 0$, with at least one strict inequality, for all $q > 0$. We also assume $u'(0) > c'(0) = 0$, and that there exists a $\overline{q} > 0$ such that $u(\overline{q}) = c(\overline{q})$. For the indivisible goods model, let $u(1) = U$ and $c(1) = C$ and assume $U > C > 0$. The rate of time preference is $r > 0$.

In addition to the consumption goods described above, there is also an object that cannot be produced or consumed by anyone called *fiat money.* We assume that money is indivisible and that individuals have a single unit storage capacity, so that if a fraction $M \in (0, 1)$ of the population are each initially endowed with one unit of money then (at least as long as no one disposes of the stuff) there will always be M agents with and $1 - M$ agents without money. We call agents with money *buyers* and agents without *sellers.* Agents meet randomly according to a Poisson process with arrival rate α. Thus, the probability per unit time that buyer i meets a seller j such that j produces a good in X_i is $\alpha(1 - M)x$, and the probability that seller j meets a buyer i such that j produces a good in X_i is αMx. Without lost of generality, we can normalize $\alpha x = 1$ by choosing the units of time appropriately.

We want to consider exchanges where the amounts of goods and money that are traded may be random. To this end, define an event to be a pair (q, m), where $q \in Q$ denotes the quantity of the good and $m \in \{0, 1\}$ the amount of money that is traded. Let $E \equiv Q \times \{0, 1\}$ denote the space of such events and $\mathcal{E}$ denote the Borel σ-algebra. Define a *lottery* to be a probability measure λ on the measurable space $(E, \mathcal{E})$. One can always write $\lambda(q, m) = \lambda_m(m)\lambda_{q|m}(q)$, where λ_m is the marginal probability measure of m and

$\lambda_{q|m}$ is the conditional probability measure of q given m. Then to reduce notation let $\lambda_m(m=1)=\tau$ and $\lambda_m(m=0)=1-\tau$, where $\tau\in[0,1]$ is the probability that the unit of money changes hands. A lottery can be completely described by the probability τ and the two probability measures $\lambda_{q|0}$ and $\lambda_{q|1}$.[4]

Let V_m denote the value function for an agent with $m\in\{0,1\}$ units of money in inventory. Then the expected payoffs from a lottery for a buyer and a seller are given by

$$\begin{aligned}\Pi_1 &= \tau\left[\int u(q)\lambda_{q|1}(dq)+V_0\right]+(1-\tau)\left[\int u(q)\lambda_{q|0}(dq)+V_1\right]\\ \Pi_0 &= \tau\left[-\int c(q)\lambda_{q|1}(dq)+V_1\right]+(1-\tau)\left[-\int c(q)\lambda_{q|0}(dq)+V_0\right].\end{aligned}$$

We focus on symmetric equilibria, where in any meeting between a buyer i and a seller where j produces a good in X_i, the agents agree to the same lottery. Then we can write Bellman's equations as follows:

$$\begin{aligned} rV_1 &= (1-M)(\Pi_1-V_1)\\ rV_0 &= M(\Pi_0-V_0).\end{aligned}\tag{1}$$

For example, the first of these equations sets the flow value to being a buyer, rV_1, equal to the rate at which he meets sellers who produce a good in X_i, which is simply $1-M$ given the normalization $\alpha x=1$, times his net gain from playing the lottery.

We now discuss bargaining. In this paper we employ the generalized Nash solution.

[4] One may question how agents can commit to the outcome of the lottery. For example, suppose that we agree to randomize so that you give me the good for sure and we flip a coin to see whether I give you the money. If the coin comes up so that I keep the money, will you still give me the good? Of course, in any exchange some notion of commitment is required, but perhaps it is more delicate when objects are not exchanged simultaneously. We are simply assuming that agents can commit to the outcome of the lottery;. To the extent that one might worry about this, however, there are devices that get around the problem. For example, if I am supposed to give you the money with probability n/m, I can put it in one of m boxes and shuffle them, and then we can simultaneously swap n of the boxes for the good.

That is, we determine τ, $\lambda_{q|0}$ and $\lambda_{q|1}$ by solving

$$\max \left(\Pi_1 - T_1\right)^{\theta} \left(\Pi_0 - T_0\right)^{1-\theta} \tag{2}$$

where T_1 and T_0 are the threat points of the buyer and the seller, respectively, and $\theta \in [0, 1]$ is the bargaining power of the buyer. It is well known that using the generalized Nash solution is equivalent to using an explicit strategic bargaining model of the sort developed by Rubinstein (1982) when the time between rejected offers and counteroffers is small, where θ and T_j depend on details of the strategic environment.[5] In what follows, we allow θ to take on any value in $[0, 1]$ and consider two cases for the threat points: $T_j = V_j$, which follows from the strategic model if we assume that individuals continue to meet other potential trading partners between bargaining rounds; and $T_j = 0$, which follows from the strategic model if we assume that they cannot meet other trading partners between bargaining rounds. Additionally, we impose incentive compatibility conditions to guarantee that agents voluntarily agree to bargain:

$$\Pi_1 \geq V_1 \text{ and } \Pi_0 \geq V_0. \tag{3}$$

A steady state equilibrium for this economy is a list $\left(V_1, V_0, \tau, \lambda_{q|0}, \lambda_{q|1}\right)$ such that: the value functions satisfy the Bellman equations in (1) taking the lottery as given; and the lottery solves the maximization problem in (2) subject to the constraints in (3) taking the value functions as given. If $\lambda_{q|0}(0) = \lambda_{q|1}(0) = 1$ or $\tau = 0$ the equilibrium is called nonmonetary, and otherwise it is called monetary. It is clear that a nonmonetary equilibrium always exists. From now on we focus on monetary equilibria. In any monetary equilibria,

[5] See Binmore, Rubinstein and Wolinsky (1986) or Osborne and Rubinstein (1990); see Coles and Wright (1998) for an exposition in the context of monetary search models.

the second constraint in (3) can be rearranged to yield

$$V_1 - V_0 \geq \int c(q)\lambda_{q|1}(dq) + \frac{1-\tau}{\tau} \int c(q)\lambda_{q|0}(dq) > 0. \tag{4}$$

Hence, $V_1 > V_0$. In the next two sections, we analyze in turn the two models, where $Q = \{0, 1\}$ and where $Q = \Re_+$.

3 The Indivisible Goods Model

When $q \in \{0, 1\}$, a lottery is completely described by τ, plus two numbers, $\lambda_1 \equiv \lambda_{q|1}(q = 1)$ and $\lambda_0 \equiv \lambda_{q|0}(q = 1)$, which give the probabilities that the good changes hands conditional on money changing hands and conditional on money not changing hands, respectively (of course, λ_0 is irrelevant if $\tau = 1$ and vice-versa). Given any lottery, one can solve (1) for the value functions, substitute into (3), and verify that the first constraint holds for all parameters, while the second holds if and only if

$$rC \leq \tau(1 - M)(U - C). \tag{5}$$

Notice that λ_1 and λ_0 do not appear in this expression. Also notice that we require $C \leq (1 - M)U/(r + 1 - M)$ for a monetary equilibrium to exist, since otherwise (5) could not be satisfied for any $\tau \leq 1$. To facilitate the presentation, we ignore the non-generic case and assume this holds with strict inequality in what follows:

$$C < \left(\frac{1 - M}{r + 1 - M}\right) U \tag{6}$$

We begin by briefly reviewing the standard model, where lotteries are ruled out. To allow for mixed strategy equilibria, let Ω denote the probability that money is accepted by a seller. Then we have

$$rV_1 = (1 - M)\,\Omega\,(U + V_0 - V_1)$$

$$rV_0 = M\Omega\,(V_1 - V_0 - C)\,.$$

In this model (which is basically Kiyotaki and Wright [1993]), there is nothing to bargain over, and an equilibrium is simply a list (V_1, V_0, Ω) such that either: $V_1 - V_0 - C \geq 0$ and $\Omega = 1$; $V_1 - V_0 - C \leq 0$ and $\Omega = 0$; or $0 < \Omega < 1$ and $V_1 - V_0 - C = 0$. There is always an equilibrium with $\Omega = 0$. Setting $\Omega = 1$ implies $V_1 - V_0 - C = (1-M)(U-C) - rC$, and so an equilibrium with $\Omega = 1$ exists as long as (6) holds. Also, as long as (6) holds, there exists an equilibrium where $\Omega = rC/(1-M)(U-C) \in (0,1)$ and $V_1 - V_0 - C = 0$.

We claim that the equilibrium with $\Omega \in (0,1)$ is an artifact of ruling out lotteries in this model. To see this, notice that in such an equilibrium the seller is indifferent between trading and not trading, $V_1 - V_0 - C = 0$, while the buyer strictly prefers to trade, $U + V_0 - V_1 > 0$. This means that trading with probability less than 1 is inconsistent with efficient bargaining. To see this, think about the strategic game of alternating offers that underlies the Nash solution, and suppose that buyer i makes seller j the following offer: i gives j the money with probability 1 and j gives i the good with probability λ_1. Then there will be $\lambda_1 < 1$ such that both i and j strictly prefer to trade. Consequently, there can be no equilibria where $\Omega \in (0,1)$ and $V_1 - V_0 - C = 0$ if we allow lotteries. This is why we implicitly assumed that $\Omega = 1$ in the previous section, and will continue to do so in what follows.[6]

As stated earlier, we will analyze separately the two cases, $T_j = V_j$ and $T_j = 0$. The following proposition characterizes the set of equilibria for the former case.

Proposition 1 *Assume* $T_j = V_j$. *Then there are critical values* $\underline{\theta}_1$ *and* $\bar{\theta}_1$ *constructed in the proof, with* $0 < \underline{\theta}_1 < \bar{\theta}_1 < 1$, *such that the following is true: if* $\theta < \underline{\theta}_1$ *there is*

[6]This is essentially the same argument that rules out mixed strategy monetary equilibria in the divisible goods model, except that there the buyer offers to take a slightly smaller quantity while here he offers to take the indivisible quantity with a slightly lower probability. Note that we will actually show below that in any monetary equilibrium the good changes hands with probability $\lambda_1 = 1$ in this model; the argument that set $\lambda_1 < 1$ was only used to show that $\Omega < 1$ is *not* an equilibrium.

no monetary equilibrium; if $\theta \in [\underline{\theta}_1, \bar{\theta}_1]$ *there exists a unique monetary equilibrium and it entails* $\tau = 1$ *and* $\lambda_1 = 1$*; and if* $\theta > \bar{\theta}_1$ *there exists a unique monetary equilibrium and it entails* $\lambda_1 = \lambda_0 = 1$ *and* $\tau = \tau_1 \in (0, 1)$*, where*

$$\tau_1 = \frac{r\,[\theta C + (1-\theta)U]}{(\theta - M)(U - C)}.$$

Proof: In this model, (2) reduces to choosing $(\tau, \lambda_0, \lambda_1) \in [0,1] \times [0,1] \times [0,1]$ to solve

$$\max\,(\Pi_1 - V_1)^{\theta}\,(\Pi_0 - V_0)^{1-\theta}$$

where $\Pi_1 = \tau(\lambda_1 U + V_0) + (1-\tau)(\lambda_0 U + V_1)$ and $\Pi_0 = \tau(-\lambda_1 C + V_1) + (1-\tau)(-\lambda_0 C + V_0)$, taking V_1 and V_0 as given. Necessary and sufficient conditions for a solution are

$$\begin{aligned}
&\theta\,[V_0 - V_1 + (\lambda_1 - \lambda_0)\,U]\,(\Pi_0 - V_0) \\
&\qquad + (1-\theta)\,[V_1 - V_0 - (\lambda_1 - \lambda_0)\,C]\,(\Pi_1 - V_1) - \eta_\tau \le 0, \quad = \text{ if } \tau > 0 \\
&\qquad\quad \theta\tau U\,(\Pi_0 - V_0) - (1-\theta)\,\tau C\,(\Pi_1 - V_1) - \eta_1 \le 0, \quad = \text{ if } \lambda_1 > 0 \\
&\theta\,(1-\tau)\,U\,(\Pi_0 - V_0) - (1-\theta)\,(1-\tau)\,C\,(\Pi_1 - V_1) - \eta_0 \le 0, \quad = \text{ if } \lambda_0 > 0.
\end{aligned} \tag{7}$$

where the η_j's are nonnegative multipliers for the constraints that the choice variables cannot exceed 1.

We are looking for monetary equilibria, which means that $\tau > 0$ and the first condition in (7) holds with equality. First consider the case $\tau < 1$, which implies $\eta_\tau = 0$. If $\lambda_1 \in [0, 1)$ then $\eta_1 = 0$ and $\theta\tau U\,(\Pi_0 - V_0) \le (1-\theta)\,\tau C\,(\Pi_1 - V_1)$, and combining this with the first condition in (7) yields $U \le C$, which is a contradiction. A similar contradiction results if $\lambda_0 \in [0, 1)$. Hence, $\tau < 1$ implies $\lambda_1 = \lambda_0 = 1$. Given this, we can solve (1) for the V_j's, substitute them into first condition in (7) at equality, and solve for $\tau = \tau_1$, where τ_1 is defined in the statement of the proposition. Notice that $\tau_1 \in (0, 1)$ if and only if $\theta > \bar{\theta}_1$, where

$$\bar{\theta}_1 = \frac{(r + M)U - MC}{(1 + r)(U - C)}.$$

One can easily check that the incentive condition (5) is satisfied at $\tau = \tau_1$. We conclude that there exists an equilibrium with $\lambda_1 = \lambda_0 = 1$ and $\tau = \tau_1 \in (0,1)$ if and only if $\theta > \bar{\theta}_1$.

Now consider the case where $\tau = 1$. This means that λ_0 is irrelevant and $\lambda_1 > 0$ in any monetary equilibrium. Inserting the V_j's into the second equation in (7) at equality and rearranging, we arrive at:

$$\lambda_1 \{\theta U [(1-M)U - (r+1-M)C] - (1-\theta)C[(r+M)U - MC]\} = (1+r)\eta_1. \qquad (8)$$

Suppose $\lambda_1 < 1$; then $\eta_1 = 0$, and (8) can be satisfied only for the nongeneric parameter value $\theta = \underline{\theta}_1$ where

$$\underline{\theta}_1 = \frac{C(1+r)}{(1-M)U + MC}\bar{\theta}_1.$$

Hence, except for the nongeneric case $\theta = \underline{\theta}_1$, the only solution to (8) with $\lambda_1 < 1$ is $\lambda_1 = 0$. Therefore, in any monetary equilibrium we have $\lambda_1 = 1$. But this means that (8) holds if and only if the left hand side is non-negative, which is true if and only if $\theta \geq \underline{\theta}_1$. So monetary equilibria are only possible if $\theta \geq \underline{\theta}_1$ and $\lambda_1 = 1$. Given this, $\tau = 1$ satisfies the first condition in (7) if and only if $\theta \leq \bar{\theta}_1$. One can easily check that (5) is satisfied at $\tau = 1$. Hence, we conclude that there exists an equilibrium with $\lambda_1 = 1$ and $\tau = 1$ if and only if $\underline{\theta}_1 \leq \theta \leq \bar{\theta}_1$.

Summarizing, an equilibrium with $\tau \in (0,1)$ exists if and only if $\theta > \bar{\theta}_1$ and an equilibrium with $\tau = 1$ exists if and only if $\underline{\theta}_1 \leq \theta \leq \bar{\theta}_1$. Finally, one can verify that $0 < \underline{\theta}_1 < \bar{\theta}_1 < 1$ using (6). This completes the proof. ■

In terms of existence results, the version of the model with $T_j = 0$ has exactly the same qualitative properties, although τ, $\underline{\theta}$ and $\bar{\theta}$ change quantitatively. Since the argument is basically the same as the proof of Proposition 1, we simply state the results here and relegate the proof to the Appendix.

Proposition 2 *Assume* $T_j = 0$. *Then there are critical values* $\underline{\theta}_0$ *and* $\bar{\theta}_0$ *constructed in the proof, with* $0 < \underline{\theta}_0 < \bar{\theta}_0 < 1$, *such that the following is true: if* $\theta < \underline{\theta}_0$ *there exists no monetary equilibrium; if* $\theta \in [\underline{\theta}_0, \bar{\theta}_0]$ *there exists a unique monetary equilibrium and it entails* $\tau = 1$ *and* $\lambda_1 = 1$; *and if* $\theta > \bar{\theta}_0$ *there exists a unique monetary equilibrium and it entails* $\lambda_1 = \lambda_0 = 1$ *and* $\tau = \tau_0 \in (0, 1)$, *where*

$$\tau_0 = \frac{r\left[\theta(r+M)C + (1-\theta)(r+1-M)U\right]}{\left[r(\theta - M) + M(1-M)(2\theta - 1)\right](U - C)}.$$

Several comments are in order concerning these results. First, since $\bar{\theta} < 1$, we have $\tau \in (0, 1)$ in a region of parameter space with positive measure. Hence, the implicit restriction made in the previous literature, that lotteries are not allowed, is indeed restrictive. Second, we want to emphasize the strong asymmetry in the model: money may change hands randomly, but goods either change hands with probability 1 or not at all. This is depicted in Figure 1, which plots τ and λ as functions of θ (for either the model with $T_j = V_j$ or $T_j = 0$, since the results are qualitatively the same). As is clear, for $\theta > \bar{\theta}$ goods trade with probability 1 and money trades randomly, for intermediate $\theta \in [\underline{\theta}, \bar{\theta}]$ both objects trade with probability 1, and for $\theta < \underline{\theta}$ monetary equilibria do not exist. We will discuss this asymmetry further in Section 5.

Also, there is a clear sense in which τ measures the price level: it is the average number of units of money that it takes to buy a consumption good. Hence, one can discuss prices even though all objects are indivisible here. One can show that both τ_1 and τ_0 are decreasing in θ, increasing in r, increasing in C, and decreasing in U (the last result is perhaps counterintuitive, but can be explained by nooting that when U increases money becomes more desirable, and so sellers are willing to settle for less). The effects of changes in M depend on which version of the model we use: one can show $\partial\tau_1/\partial M > 0$, but, perhaps surprisingly, $\partial\tau_0/\partial M > 0$ if and only if r and M are not too small. Also, as

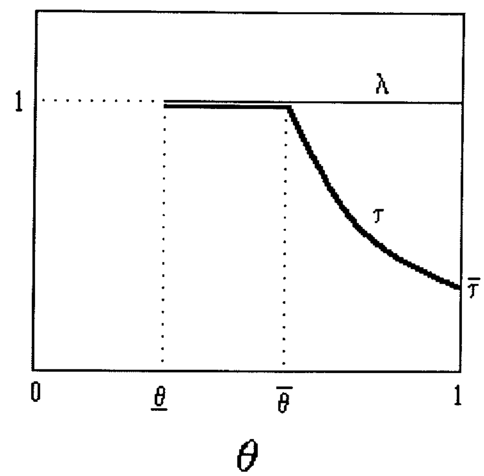

Figure 1: Monetary Equilibrium as a Function of θ.

$r \to 0$, we have $\tau \to 0$ for all $\theta > \bar{\theta}$ (the τ curve in Figure 1 approaches a vertical line at $\theta = \bar{\theta}$), and so if $\theta > \bar{\theta}$ and agents are very patient, one can get the good virtually for free.[7] When r is near 0, a seller is willing to produce when τ near 0 because, even though he gets paid with a very low probability, if he does get paid he expects to be able to buy a large number of goods with the money.

Finally, to close this section, we mention welfare. For low θ there is no monetary equilibrium, and so the only equilibrium is the nonmonetary equilibrium which implies $V_1 = V_0 = 0$. By way of contrast, if lotteries are ruled out there is an equilibrium where money is accepted with probability 1 and $V_1 > V_0 > 0$ for all parameters. Hence, allowing lotteries can actually reduce welfare. This is not too surprising, since in the monetary equilibrium without lotteries buyers get to consume every time they meet an appropriate seller, and

[7] Note that $\bar{\theta}$ also changes: as $r \to 0$, $\bar{\theta}_1 \to M$ and $\underline{\theta}_1 \to CM/\left[(1-M)U + MC\right]$ in the model with $T_j = V_j$, and $\bar{\theta}_0 \to 1/2$ and $\underline{\theta}_0 \to C/\left(U+C\right)$ in the model with $T_j = 0$. For completeness, we report some other differences between the models with different threat points. When $T_j = 0$, $\bar{\theta}_0 > 1/2$ for all $r > 0$, and so lotteries are not used when buyers and sellers have equal bargaining power; but when $T_j = V_j$, it is possible to have $\tau < 1$ when $\theta = 1/2$. Also, one can show that as long as τ_1 and τ_0 are in $(0,1)$, the difference $\tau_0 - \tau_1$ is proportional to $1 - 2M$, and so $\tau_0 < \tau_1$ if and only if $M > 1/2$.

there is no concern about *how much* they consume because the good is indivisible. All the result says is that agents may be better off ex ante if they can commit to $\tau = \lambda = 1$, rather than bargaining over these variables in each bilateral meeting. We will see in the model with divisible goods in Section 4, and also in a model with indivisible goods but commodity rather than fiat money in Section 5, that lotteries may well increase welfare.

4 The Divisible Goods Model

When $q \in \Re_+$, a lottery is generally described by τ and two conditional probability distributions, $\lambda_{q|0}$ and $\lambda_{q|1}$. However, we claim that the amount of goods that changes hand is degenerate and independent of whether money changes hands.

Proposition 3 *There is a q (that depends on parameter values) such that $\lambda_{q|0}(q) = \lambda_{q|1}(q) = 1$.*

Proof: The Nash bargaining problem is to choose $\tau \in [0,1]$ and probability measures $\lambda_{q|0}$ and $\lambda_{q|1}$ to solve

$$\max \left\{ \tau \left[\int u(q)\lambda_{q|1}(dq) + V_0 \right] + (1-\tau) \left[\int u(q)\lambda_{q|0}(dq) + V_1 \right] - T_1 \right\}^{\theta} \times \left\{ \tau \left[-\int c(q)\lambda_{q|1}(dq) + V_1 \right] + (1-\tau) \left[-\int c(q)\lambda_{q|0}(dq) + V_0 \right] - T_0 \right\}^{1-\theta}$$

subject to the incentive constraints in (3), taking V_0 and V_1 as given. Suppose that the solution implies that $\lambda_{q|0}$ and $\lambda_{q|1}$ are nondegenerate, and let $q_0 = \int q\lambda_{q|0}(dq)$ and $q_1 = \int q\lambda_{q|1}(dq)$. Since $u(q)$ is concave and $c(q)$ is convex, by Jensen's inequality, the incentive constraints are still satisfied and the Nash product is higher when $\lambda_{q|0}(q_0) = \lambda_{q|1}(q_1) = 1$, which is a contradiction. Hence, $\lambda_{q|0}$ and $\lambda_{q|1}$ are degenerate at q_0 and q_1, respectively. Now suppose $q_0 \neq q_1$, and let $Eq = \tau q_1 + (1-\tau)q_0$. Again, since $u(q)$ is concave and $c(q)$

is convex, the incentive constraints are still satisfied and the Nash product is higher at Eq, which is a contradiction. This completes the proof. ■

The above result makes the analysis simpler because we can now restrict attention to lotteries that are completely characterized by two numbers, τ and q. Given any such lottery, one can, as in the previous section, solve (1) for the value functions, substitute in (3), and verify that the first constraint is never binding and the second is satisfied if and only if

$$rc(q) - \tau(1-M)[u(q) - c(q)] \leq 0. \tag{9}$$

In particular, if $\tau = 1$, then (9) holds if and only if

$$\varphi(q) \equiv rc(q) - (1-M)[u(q) - c(q)] \leq 0. \tag{10}$$

It is easy to see that $\varphi(0) = 0$, $\varphi'(0) < 0$, $\varphi''(q) \geq 0$ for all q, and $\varphi(q) > 0$ for large q; hence, if $\tau = 1$ the constraints are satisfied if and only if q is below some critical value $\widehat{q}$. Also, let q^* be the *efficient* quantity, defined by $u'(q^*) = c'(q^*)$. It is easy to verify that q^* is the quantity that maximizes welfare, $W = MV_1 + (1-M)V_0$. If $q = q^*$ then (9) holds if and only if

$$\tau \geq \widehat{\tau} \equiv \frac{rc(q^*)}{(1-M)[u(q^*) - c(q^*)]}, \tag{11}$$

which can only hold if r is not too big.

We consider in turn the case with and without threat points. The following proposition characterizes the set of equilibria for the former case.

Proposition 4 *Assume $T_j = V_j$. If $\theta = 0$, there does not exist a monetary equilibrium. If $\theta > 0$, then there is a critical value $\tilde{\theta}_1$ constructed in the proof, where $\widetilde{\theta}_1 > 0$ for all* *parameter values and* $\widetilde{\theta}_1 < 1$ *if and only if* $r < (1-M)[u(q^*) - c(q^*)]/c(q^*)$, *such that the following is true: if* $\theta < \tilde{\theta}_1$ *there exists a unique monetary equilibrium and it entails* $\tau = 1$ *and* $q < q^*$, *with* $\partial q/\partial\theta > 0$ *and* $\lim_{\theta \to \tilde{\theta}_1} q = q^*$; *and if* $\theta > \widetilde{\theta}_1$ *there exists a unique monetary equilibrium and it entails* $q = q^*$ *and* $\tau = \widetilde{\tau}_1 \in (0,1)$, *where*

$$\widetilde{\tau}_1 = \frac{r\left[\theta c\left(q^*\right) + (1-\theta)\, u\left(q^*\right)\right]}{(\theta - M)\left[u\left(q^*\right) - c\left(q^*\right)\right]}.$$

Proof: If $\theta = 0$ then the bargaining solution is equivalent to take-it-or-leave-it offers by the seller, which implies $u(q) = \tau(V_1 - V_0)$. Inserting this into (1), we find $V_1 = 0$, and therefore $V_0 < 0$ by (4). But a seller can always achieve $V_0 = 0$ by not trading. Hence, there cannot exist a monetary equilibrium when $\theta = 0$.

Now assume $\theta > 0$. Then (2) reduces to choosing $(\tau, q) \in [0,1] \times \Re_+$ to solve

$$\max\left(\Pi_1 - V_1\right)^{\theta}\left(\Pi_0 - V_0\right)^{1-\theta},$$

where $\Pi_1 = u(q) + \tau V_0 + (1-\tau)V_1$ and $\Pi_0 = -c(q) + \tau V_1 + (1-\tau)V_0$, taking V_1 and V_0 as fixed. Necessary and sufficient conditions for a solution are

$$\begin{aligned} \theta u'(q)\left(\Pi_0 - V_0\right) - (1-\theta)\, c'(q)\left(\Pi_1 - V_1\right) &\leq 0, \quad = \text{ if } q > 0 \\ \theta\left(V_0 - V_1\right)\left(\Pi_0 - V_0\right) + (1-\theta)\left(V_1 - V_0\right)\left(\Pi_1 - V_1\right) - \eta_\tau &\leq 0, \quad = \text{ if } \tau > 0 \end{aligned} \tag{12}$$

where η_τ is the nonnegative multiplier on the constraint $\tau \leq 1$. We are looking for monetary equilibria, which implies that both conditions hold with equality.

First consider the case where $\tau < 1$, which implies that $\eta_\tau = 0$. Then combining the two first order conditions yields $u'(q) = c'(q)$, and so $q = q^*$. Solving (1) for the V_j's and inserting the solutions, as well as $q = q^*$, into the second condition in (12), we can solve for $\tau = \widetilde{\tau}_1$ where $\widetilde{\tau}_1$ is defined in the statement of the proposition. Notice that $\widetilde{\tau}_1 \in (0,1)$

if and only if $\theta > \widetilde{\theta}_1$ where

$$\widetilde{\theta}_1 = \frac{(r+M)u\left(q^*\right) - Mc\left(q^*\right)}{(1+r)\left[u\left(q^*\right) - c\left(q^*\right)\right]}.$$

One can check that $\widetilde{\tau}_1 \geq \hat{\tau}$, where $\hat{\tau}$ is defined in (11), and therefore the incentive condition (9) holds at $\tau = \tilde{\tau}_1$ and $q = q^*$. Hence, we conclude that there exists an equilibrium with $\tau = \tilde{\tau}_1$ and $q = q^*$ if and only if $\theta > \widetilde{\theta}_1$.

Now consider the case where $\tau = 1$, which implies $\eta_\tau \geq 0$. By combining the two conditions in (12), we get $u'(q) \geq c'(q)$, and this implies $q \leq q^*$ in any equilibrium with $\tau = 1$, with strict inequality as long as $\eta_\tau > 0$. Inserting the V_j's and $\tau = 1$, we can rewrite the first order condition for q as

$$\frac{(1-\theta)\,c'(q)}{\theta u'(q)} = \frac{1 - M - (r+1-M)c(q)/u(q)}{r + M - Mc(q)/u(q)}. \tag{13}$$

The left hand side of (13) is zero at $q = 0$ and it is strictly increasing. As $q \to 0$, the right hand side approaches $(1-M)/(r+M) > 0$, because $c(q)/u(q) \to 0$ by l'Hopital's rule, and it is strictly decreasing and equals 0 when $q = \hat{q}$, where recall that $\hat{q}$ is the solution to (10) at equality. Hence, there exists an unique solution to (13), call it $\chi = \chi(\theta)$, in $(0, \hat{q})$. Moreover, it is easy to check that $\chi'(\theta) > 0$ and that $\chi(\tilde{\theta}_1) = q^*$. Since we need $\chi(\theta) \leq q^*$ for an equilibrium with $\tau = 1$, an equilibrium of this type cannot exist if $\theta > \tilde{\theta}_1$. If $\theta < \tilde{\theta}_1$ then $\chi(\theta) < q^*$, and we now show that this also implies the first order condition for τ is satisfied at $\tau = 1$. To see this, rearrange the first order condition for τ as

$$\theta \leq \frac{(r+M)u(q) - Mc(q)}{(1+r)[u(q) - c(q)]}. \tag{14}$$

The right hand side of (14) is decreasing in θ and approaches $(r+M)/(1+r) > 0$ as $q \to 0$. Also, (14) is satisfied at equality when $\theta = \tilde{\theta}_1$. Hence, (14) is satisfied if and only if $\theta \leq \tilde{\theta}_1$. We conclude that $\tau = 1$ and $q = \chi(\theta)$ satisfy the first order conditions if and only if $\theta \leq \tilde{\theta}_1$.

Moreover, since $\chi(\theta) < \hat{q}$, it satisfies the incentive condition (10), and hence satisfies all of the conditions for an equilibrium.

Finally, it is obvious that $\tilde{\theta}_1 > 0$, and that $\tilde{\theta}_1 < 1$ if and only if $r < (1 - M)[u(q^*) - c(q^*)]/c(q^*)$. This completes the proof. ■

As in the previous section, the version of the model with $T_j = 0$ has the same qualitative properties, although τ and $\tilde{\theta}$ change quantitatively. Thus, we simply state the results here and again relegate the proof to the Appendix.

Proposition 5 *Assume $T_j = 0$. If $\theta = 0$ there does not exist a monetary equilibrium. If $\theta > 0$ then there is a critical value $\tilde{\theta}_0$, where $\widetilde{\theta}_0 > 0$ for all parameter values and $\widetilde{\theta}_0 < 1$ if and only if $r < (1 - M)[u(q^*) - c(q^*)]/c(q^*)$, such that the following is true: if $\theta < \tilde{\theta}_0$ there exists a unique monetary equilibrium and it entails $\tau = 1$ and $q < q^*$, with $\partial q/\partial \theta > 0$ and $\lim_{\theta \to \tilde{\theta}_0} q = q^*$; and if $\theta > \widetilde{\theta}_0$ there exists a unique monetary equilibrium and it entails $q = q^*$ and $\tau = \widetilde{\tau}_0 \in (0, 1)$, where*

$$\widetilde{\tau}_0 = \frac{r\left[(1-\theta)\left(1-M+r\right)u\left(q^*\right)+\theta\left(M+r\right)c\left(q^*\right)\right]}{\left[r\left(\theta-M\right)+M\left(1-M\right)\left(2\theta-1\right)\right]\left[u\left(q^*\right)-c\left(q^*\right)\right]}.$$

Several comments are in order. First, as in the previous section, we emphasize that the two objects are traded asymmetrically: randomization may be used for trading money never for trading goods. Figure 2 shows τ and q as functions of θ (for either $T_j = V_j$ or $T_j = 0$, since the outcome is qualitatively the same). Note, however, that $\tilde{\theta} < 1$ if and only if r is not too big. If agents are extremely impatient, then $\tilde{\theta} > 1$, and we have $\tau = 1$ for all θ. If individuals are not extremely impatient, then $\tilde{\theta} < 1$, and as long as the buyer has sufficient bargaining power we have $\tau < 1$. Hence, as in the model with indivisible goods, there are parameters values for which the implicit restriction made in the previous literature, that lotteries are not allowed, is restrictive.

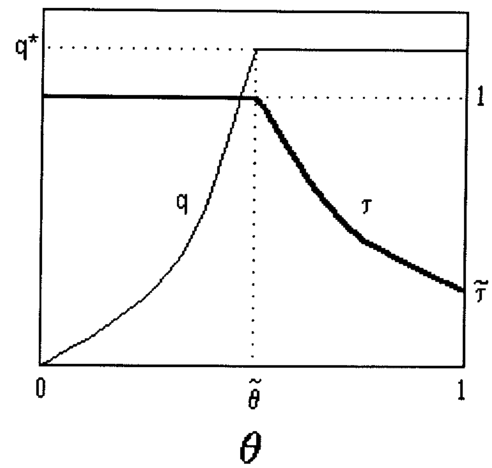

Figure 2: Monetary Equilibrium as a Function of θ.

Next, notice that $q \leq q^*$ for all θ, with strict inequality if and only if $\theta < \tilde{\theta}$, where q^* is the efficient quantity. It is argued in Trejos and Wright (1995) that it is natural to have q below q^* in a monetary economy, although in the model in that paper, without lotteries, the result does not hold very generally: it holds when $\theta = 1/2$ in the model where $T_j = 0$, but it may not hold for other values of θ, and it may not hold in the model where $T_j = V_j$ even if $\theta = 1/2$. With lotteries, we find that q can never exceed q^*, irrespective of the threat points or bargaining power parameter.[8]

Some results in this section are similar to those in the indivisible goods model. For example, we have the asymmetry that money may change hands randomly but goods

[8] One way to interpret this heuristically is as follows. Agents generally would like first to maximize the size of the surplus and then bargain over how to distribute that surplus. With lotteries, they therefore would like first to set $q = q^*$ and then hold a lottery to see who gets to keep the money in order to distribute the surplus according to the bargaining solution. This can be done if the buyer is due sufficient gains from trade (i.e., if θ is big). Otherwise, even if the seller is getting the money with probability 1 he may not be getting his share, and so q will have to be reduced below q^*. In the model without lotteries, the only way to distribute the surplus across agents is to vary q, which simultaneously affect the size of the surplus. Without lotteries, then, q^* will almost certainly not satisfy the bargaining solution for given parameters.

never change hands randomly, and the behavior of τ with respect to the parameters θ, r and M is qualitatively the same as in the previous section.[9] There are also differences between the indivisible and divisible goods models. For one thing, in the indivisible goods model we have $\bar{\theta} < 1$, and hence we definitely have $\tau < 1$ for high θ; but in the divisible goods model $\tilde{\theta} < 1$ if and only if r is not too big, and hence we can guarantee $\tau < 1$ for high θ if and only if r is not too big. Also, in the indivisible goods model we showed $\underline{\theta} > 0$, and so for low θ monetary equilibria do not exist; but in the divisible goods model a monetary equilibrium exists for all $\theta > 0$. Finally, recall that lotteries could reduce welfare in the indivisible goods model, but it is easy to see that lotteries can only improve welfare in this model: for $\theta \leq \tilde{\theta}$, q is the same with or without lotteries; and for $\theta > \tilde{\theta}$, $q = q^*$ with lotteries and $q > q^*$ without lotteries.

5 Discussion

We have seen that although agents may agree to a lottery where money changes hands with probability less than 1, they will never agree to a lottery where goods change hands with probability less than 1. What lies behind this asymmetry? It is not due to the assumption that money is indivisible while goods are divisible, because the same asymmetry arises when goods and money are both indivisible. Rather, the asymmetry seems to be due more to the *fiat* nature of the monetary object (i.e., it has no intrinsic worth, and derives its value solely from its role as a medium of exchange). To develop this further, we now present some variations and extensions of the basic framework.

First, consider a model with direct *barter* instead of monetary exchange. There are N_1

[9]One can show that τ is increasing and q is decreasing in r in either version of the model, and that τ is increasing and q is decreasing in M in the version with $T_j = V_j$, but not necessarily in the version with $T_j = 0$. Also, as $r \to 0$, $\tau \to 0$ for all $\theta > \tilde{\theta}$ (the τ curve in Figure 2 approaches a vertical line at $\theta = \tilde{\theta}$).

agents who consume good 1 and produce good 2, and $N_2 = 1 - N_1$ agents who consume good 2 and produce good 1. For now, both goods are indivisible. Production of good j costs $C_j > 0$ and its consumption yields utility $U_j > C_j$. Agents meet at random, and when two agents of the opposite type meet they bargain over lotteries. If two agents agree to a lottery, the random trade is executed, after which they return to the market to search for new trading partners. This model is meant to be as close as possible to the monetary search model, except that both objects are treated symmetrically in the sense that both are real commodities.[10]

Let τ_j be the probability that an agent of type j trades his production good to the other agent, and let V_j be the value function of type j.[11] The Bellman equations are

$$\begin{aligned} rV_1 &= N_2 \left(\tau_2 U_1 - \tau_1 C_2\right) \\ rV_2 &= N_1 \left(\tau_1 U_2 - \tau_2 C_1\right). \end{aligned} \tag{15}$$

The bargaining solution chooses (τ_1, τ_2) to solve

$$\max \left(\tau_2 U_1 - \tau_1 C_2 + V_1 - T_1\right)^{\theta} \left(\tau_1 U_2 - \tau_2 C_1 + V_2 - T_2\right)^{1-\theta}$$

[10]This model can also be thought of as a version of Rubinstein and Wolinsky (1985) where both goods are indivisible. The only difference is that agents in their setup exit the model after trading, while here they reenter the market to search for new partners. This is not at all important, however, and we have derived qualitatively similar results for versions where agents trade and exit, and also where agents trade and switch from desiring good 1 to desiring good 2. Details of these and other extensions, including different choices for the threat points for the various models in this section, are available on the Web at http://www-vwi.unibe.ch/staff/berentsen/aleks.htm.

[11]As in Secion 3, with two indivisible objects, the most general lottery needs to be defined in terms of τ, the probability the first object changes hands, plus the *conditional* probabilities λ_0 and λ_1 (i.e., the probability the second object changes hands conditional on the first changing hands and the probability the second changes conditional on the first not changing hands). However, one really only needs a single λ, because we showed $\lambda_0 = \lambda_1$ (i.e., whether the good changes hands is *independent* of whether the money changes hands). A similar independence result can be used here to restrict attention to lotteries defined in terms of the pair (τ_1, τ_2), where τ_j is the probability good j changes hands, and so in order to reduce notation, this is what we do in this section.

where T_j is the threat point of type j and θ is the bargaining power of type 1 agents. For brevity, we only present results for $T_j = V_j$ (the case $T_j = 0$ is similar), and relegate the proof to the Appendix.

Proposition 6 *Assume $U_1 > C_2$ and $U_2 > C_1$. Then there are critical values $\underline{\theta}$ and $\bar{\theta}$ constructed in the proof, with $0 < \underline{\theta} < \bar{\theta} < 1$, such that the following is true: if $\theta < \underline{\theta}$ there exists a unique equilibrium and it entails $\tau_1 = 1$ and $\tau_2 = \widetilde{\tau}_2 \in (0,1)$, where*

$$\widetilde{\tau}_2 = \frac{\theta U_1 U_2 + (1-\theta) C_1 C_2}{C_1 U_1};$$

if $\underline{\theta} \leq \theta \leq \bar{\theta}$ there exists a unique equilibrium and it entails $\tau_1 = \tau_2 = 1$; and if $\theta > \bar{\theta}$ there exists a unique equilibrium and it entails $\tau_2 = 1$ and $\tau_1 = \widetilde{\tau}_1 \in (0,1)$, where

$$\widetilde{\tau}_1 = \frac{(1-\theta) U_1 U_2 + \theta C_1 C_2}{C_2 U_2}.$$

The results are depicted in Figure 3. First notice that if one good is changes hands with probability less that 1 then the other good changes hands with probability 1. Also, either agent can get his consumption good with probability less than 1 if he has sufficiently low bargaining power θ, while for intermediate θ both agents get their good with probability 1. The main point is that this model is symmetric, and thereby differs from the model where one of the objects is fiat money. In the monetary model, as the bargaining power of the money holder declines, the probability he gets his good jumps discretely from 1 to 0 as θ crosses $\underline{\theta}$ (recall Figure 1); in the present model, as his bargaining power declines the probability gets his good declines smoothly from 1 to C_2/U_1.[12]

[12]The Proposition assumes $U_1 > C_2$ and $U_2 > C_1$, but this is not necessary to get trade. In fact, if $U_1 \leq C_2$ and $U_2 > C_1$, then for all θ there exists a unique equilibrium and it entails $\tau_2 = 1$ and $\tau_1 = \widetilde{\tau}_1$; and if $U_2 \leq C_1$ and $U_1 > C_2$, then for all θ there exists a unique equilibrium and it entails $\tau_1 = 1$ and $\tau_2 = \widetilde{\tau}_2 \in (0,1)$. Thus, even if the cost of production exceeds the benefit of consumption, if we allow lotteries, an agent is still willing to produce with some probability in order to consume.

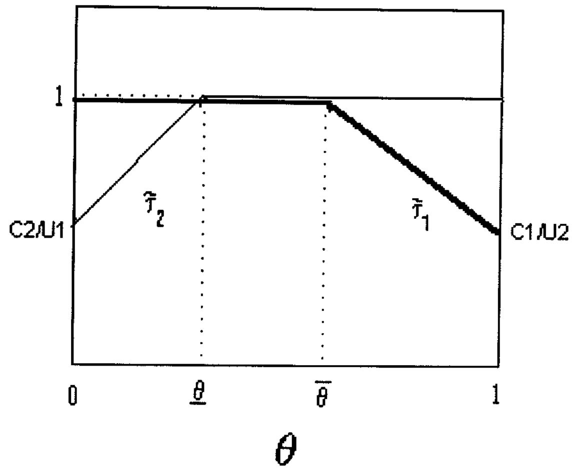

Figure 3: Equilibrium with Two Indivisible Real Commodities

Next, consider a model with one indivisible good and money, as in Section 3, except that now we assume that the money is a *commodity* money in the sense that it yields a direct utility flow $\gamma > 0$ to someone holding it.[13] Letting τ and λ be the probabilities that money and goods change hands, the value functions now satisfy

$$\begin{aligned} rV_0 &= M\left[\tau\left(V_1 - V_0\right) - \lambda C\right] \\ rV_1 &= (1-M)\left[\tau\left(V_0 - V_1\right) + \lambda U\right] + \gamma. \end{aligned} \tag{16}$$

The bargaining solution is the same as in Section 3. Again, for brevity we only present the case $T_j = V_j$ (the other case is similar), and relegate the proof to the Appendix.

Proposition 7 *Let* $\bar{\gamma} = (r+M)U - MC$. *If* $\gamma \in (0, \bar{\gamma})$ *then there are critical values* $\underline{\theta}$ *and* $\bar{\theta}$ *constructed in the proof, with* $0 < \underline{\theta} < \bar{\theta} < 1$, *such that the following is true: if* $\theta < \underline{\theta}$

[13]This is one notion of commodity money – it yields a real rate of return, like gold jewelry, say; another notion is that it yields utility only if consumed, like cigarettes, say.

there exists a unique monetary equilibrium and it entails $\tau = 1$ *and* $\lambda = \widetilde{\lambda} \in (0,1)$, *where*

$$\widetilde{\lambda} = \frac{\gamma[\theta U + (1-\theta)C]}{(U-C)[M(1-\theta)C - \theta(1-M)U] + rCU};$$

if $\theta \in [\underline{\theta}, \bar{\theta}]$ *there exists a unique monetary equilibrium and it entails* $\tau = 1$ *and* $\lambda = 1$; *and if* $\theta > \bar{\theta}$ *there exists a unique monetary equilibrium and it entails* $\lambda = 1$ *and* $\tau = \widetilde{\tau} \in (0,1)$, *where*

$$\widetilde{\tau} = \frac{r[\theta C + (1-\theta)U]}{\gamma + (\theta - M)(U-C)}.$$

If $\gamma > \bar{\gamma}$ *then for all* θ *there exists a unique monetary equilibrium and it entails* $\lambda = 1$ *and* $\tau = \widetilde{\tau} \in (0,1)$.

There are three parts of Proposition 7 that we want to emphasize. First, when $\gamma > 0$ there exists a monetary equilibrium for all $\theta \geq 0$, while with fiat money there did not exist a monetary equilibrium for small θ. Second, when $\gamma > 0$ we can have $\lambda \in (0,1)$, while with fiat money we either have $\lambda = 1$ or $\lambda = 0$. Third, for large γ we must have $\lambda = 1$ and $\tau \in (0,1)$.

To develop some further intuition for this, we now present an alternative but equivalent way of analyzing the model. Given economy-wide probabilities λ and τ, the value functions satisfy (16). Then, when a particular buyer-seller pair meet, they bargain over the probabilities $\check{\lambda}$ and $\check{\tau}$ that they will use to trade, taking λ and τ as given. This generates something akin to a best response function mapping (λ, τ) into $(\check{\lambda}, \check{\tau})$, of which an equilibrium is a fixed point. Let us look for equilibria with $\tau = \check{\tau} = 1$. The best response function mapping λ into $\check{\lambda}$ is given by $\check{\lambda} = \min\{\Lambda(\lambda), 1\}$, where $\Lambda(\lambda)$ is the linear function

$$\Lambda(\lambda) = \frac{\theta U + (1-\theta)C}{CU(1+r)}\{\gamma + [(1-M)U + MC]\lambda\}.$$

On the one hand, assume $\gamma = 0$ so that we are back to the fiat money model. Then the intercept of $\Lambda(\lambda)$ is zero. When $\theta < \underline{\theta}$, the slope of $\Lambda(\lambda)$ is less than 1; in this case, given any economy-wide $\lambda \in [0, 1]$ a particular pair of agents will bargain to $\check{\lambda} < \lambda$, and so the only fixed point is $\lambda = 0$. When $\theta > \underline{\theta}$, the slope is greater than 1; in this case, given any economy wide $\lambda > 0$ the bargaining solution implies $\check{\lambda} > \lambda$, and so $\lambda = 0$ and $\lambda = 1$ are both fixed points. On the other hand, in the commodity money model with $\gamma > 0$, the intercept of $\Lambda(\lambda)$ is strictly positive, and so the unique fixed point is strictly positive. Intuitively, even if other agents are giving goods with probability $\lambda = 0$ in exchange for money, $\gamma > 0$ implies that you would be willing to trade your good with some positive probability to get a unit of money. In particular, if $\gamma > 0$ and $\theta < \underline{\theta}$, then in equilibrium goods change hands with probability strictly between 0 and 1.

Note that $\tau \to 0$ as $\gamma \to \infty$, but for any given γ we have $\tau > 0$, and the agent with the money can still get his consumption good. This is *not* generally true in a model without lotteries. One can show that when lotteries are not allowed, if $\gamma > rU$, then money is hoarded and never used to purchase goods (see Velde, Weber and Wright [1998] for an application of such a model). Intuitively, if γ is very big agents do not want to spend all of their money to get their consumption good, and since the money is indivisible they cannot spend part of it. With lotteries, however, agents trade regardless of γ because they can adjust the probability τ that the money changes hands. Hence, for large γ the velocity of money may be low, but buyers still get their consumption good in every opportunity. This means that for large γ welfare is higher if lotteries are allowed than if they are ruled out, because every trade generates a positive surplus $U - C$, and it does not matter from an ex ante welfare point of view who gets the money after the trade.

To close this section we briefly consider the model with divisible consumption goods and indivisible money, where the money is now commodity money with flow utility $\gamma > 0$.[14] One can derive the following results for the divisible goods model, analogous to those discussed in the previous paragraph for the indivisible goods model. If lotteries are ruled out, one can show that money will be hoarded and trade will cease if $\gamma > ru(q')$, where q' is given by $u(q') = c(q')$. With lotteries, however, $\tau \to 0$ as $\gamma \to \infty$, but for any given γ we have $\tau > 0$ and the agent with the money can still get his consumption good. Furthermore, recalling that $\tilde{\theta}$ is the threshold such that $q = q^*$ for all $\theta > \tilde{\theta}$, one can show that $\tilde{\theta}$ is decreasing in γ and that there is a γ^* such that $\tilde{\theta} = 0$. Therefore, for all $\gamma > \gamma^*$ the equilibrium is efficient in the sense that buyers get their consumption good in every opportunity, and the amount that they get is q^*.

6 Conclusion

We think that introducing lotteries has been interesting for the following reasons. First, in general, individuals may want to use lotteries in this environment, and we see no compelling reason to constrain them not to. Second, the use of lotteries in the models analyzed here affects aggregate welfare. Third, in the model with indivisible goods and money, lotteries give agents something to bargain over and thereby give us a way to discuss prices. Fourth, introducing lotteries eliminates the somewhat unnatural mixed strategy equilibria (where agents are indifferent between accepting and rejecting money) that appear in models in the

[14]One can also consider a model with no money but with two consumption goods, one divisible and one indivisible. In this model, a nonrandom quantity q of the divisible good always changes hands with probability 1, while the indivisible good changes hands with probability τ, where $\tau < 1$ iff the agent with the indivisible good has sufficient bargaining power. Note that this is essentially the model in Rubinstein and Wolinsky (1985), except that here the instantaneous payoffs are $u(q) - C$ and $U - c(q)$ for agents who consume and produce q units of the divisible good, while Rubinstein and Wolinsky assume the payoffs are q and $1 - q$. Agents never need to use lotteries in their model only because of the assumption that the utility and disutilty functions are linear.

literature without lotteries. Fifth, it is not difficult to get a rather complete characterization of the outcome: with either indivisible or divisible goods, in any equilibrium with valued fiat money, good changes hands with probability 1 and money changes hands with probability τ, where $\tau < 1$ if and only if the buyer has sufficient bargaining power. Sixth, when goods are divisible one can prove $q \leq q^*$ in the model with lotteries, where q^* is the efficient quantity, but one cannot prove this in general without lotteries. Seventh, if we allow lotteries, commodity money will never drop out of circulation entirely, and even if it is traded with low probability money holders still acquire consumption goods in every opportunity, while without lotteries a sufficiently valuable commodity money will drop out of circulation and trade will cease. And, finally, with divisible goods one can show that a sufficiently valuable commodity money not only can be used to acquire consumption goods in every opportunity, it always commands the efficient quantity q^*.

Based on these findings it seems that allowing lotteries can be important in at least some versions of this framework for at least some applications. One may be able to derive additional insights by introducing lotteries into other related models. For example, in would seem possible to include lotteries in models where the medium of exchange is derived endogenously, such as the model in Kiyotaki and Wright (1989). This could give us a way to discuss prices in that framework, among other things, hopefully without complicating the analysis too much. We leave this to future research.

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Appendix

Proof of Proposition 2

The argument mimics that of Proposition 1. Now the bargaining solution is $\max \Pi_1^\theta \Pi_0^{1-\theta}$, where $\Pi_1 = \tau(\lambda_1 U + V_0) + (1-\tau)(\lambda_0 U + V_1)$ and $\Pi_0 = \tau(-\lambda_1 C + V_1) + (1-\tau)(-\lambda_0 C + V_0)$. Necessary and sufficient conditions for a solution are:

$$\begin{aligned} &\theta\left[V_0 - V_1 + (\lambda_1 - \lambda_0)U\right]\Pi_0 \\ &+ (1-\theta)\left[V_1 - V_0 - (\lambda_1 - \lambda_0)C\right]\Pi_1 - \eta_\tau \le 0, \quad = \text{ if } \tau > 0 \\ &\theta\tau U\Pi_0 - (1-\theta)\tau C\Pi_1 - \eta_1 \le 0, \quad = \text{ if } \lambda_1 > 0 \\ &\theta(1-\tau)U\Pi_0 - (1-\theta)(1-\tau)C\Pi_1 - \eta_0 \le 0, \quad = \text{ if } \lambda_0 > 0 \end{aligned} \tag{17}$$

Since $\tau > 0$, the first condition in (17) holds with equality. First consider the case $\tau < 1$, which implies $\lambda_1 = \lambda_0 = 1$. Then we can solve for the V_j's, substitute them into first condition in (17), and solve for $\tau = \tau_0$ where τ_0 is defined in the statement of the Proposition. Notice $\tau_0 \in (0,1)$ if and only if $\theta > \bar{\theta}_0$, where

$$\bar{\theta}_0 = \frac{(r+1-M)\left[(r+M)U - MC\right]}{(r+1-M)\left[(r+M)U - MC\right] + (r+M)\left[(1-M)U - (r+1-M)C\right]}.$$

One can check (5) is satisfied at $\tau = \tau_0$. Hence, there exists an equilibrium with $\lambda_1 = \lambda_0 = 1$ and $\tau = \tau_0 < 1$ if and only if $\theta > \bar{\theta}_0$.

Now consider the case where $\tau = 1$ and $\lambda_1 > 0$. Inserting the V_j's into the second equation in (17) at equality and rearranging, we have

$$\lambda_1\left\{\theta U(r+M)\left[(1-M)U - (r+1-M)C\right] - (1-\theta)C(r+1-M)\left[(r+M)U - MC\right]\right\} = r(1+r)\eta_1. \tag{18}$$

Suppose $\lambda_1 < 1$; then $\eta_1 = 0$, and (18) can be satisfied only for the nongeneric parameter value $\theta = \underline{\theta}_0$ where

$$\underline{\theta}_0 = \frac{C(r+1-M)\left[(r+M)U - MC\right]}{(1-M)(r+M)U^2 - M(r+1-M)C^2}.$$

Hence, except for $\theta = \underline{\theta}_0$, the only solution to (18) with $\lambda_1 < 1$ is $\lambda_1 = 0$. Therefore, we must have $\lambda_1 = 1$, which means that (18) holds if and only if the left hand side is non-negative, which is true if and only if $\theta \geq \underline{\theta}_0$. So monetary equilibria are only possible if $\theta \geq \underline{\theta}_0$ and $\lambda_1 = 1$. Given this, $\tau = 1$ satisfies the first condition in (17) if and only if $\theta \leq \bar{\theta}_0$. One can easily check that (5) is satisfied at $\tau = 1$. Hence, there exists an equilibrium with $\lambda_1 = 1$ and $\tau = 1$ if and only if $\underline{\theta}_0 \leq \theta \leq \bar{\theta}_0$. Finally, one can verify $0 < \underline{\theta}_0 < \bar{\theta}_0 < 1$ using (6). ■

Proof of Proposition 5

The argument mimics that of Proposition 4. As there, if $\theta = 0$ there is no monetary equilibrium. For $\theta > 0$, the bargaining problem reduces $\max \Pi_1^{\theta}\Pi_0^{1-\theta}$ where $\Pi_1 = u(q) + \tau V_0 + (1-\tau)V_1$ and $\Pi_0 = -c(q) + \tau V_1 + (1-\tau)V_0$. Necessary and sufficient conditions:

$$\begin{aligned} \theta u'(q)\,\Pi_0 - (1-\theta)\,c'(q)\,\Pi_1 \leq 0, &\quad = \text{ if } q > 0 \\ \theta\,(V_0 - V_1)\,\Pi_0 + (1-\theta)\,(V_1 - V_0)\,\Pi_1 - \eta_\tau \leq 0, &\quad = \text{ if } \tau > 0 \end{aligned} \tag{19}$$

In monetary equilibria both conditions hold with equality.

First consider the case $\tau < 1$, which implies that $\eta_\tau = 0$. Then combining the first order conditions yields $u'(q) = c'(q)$, and so $q = q^*$. Insering the V_j's into the second condition in (19), we can solve for $\tau = \widetilde{\tau}_0$ where $\widetilde{\tau}_0$ is defined in the statement of the Proposition. Notice $\widetilde{\tau}_0 \in (0,1)$ if and only if $\theta > \widetilde{\theta}_0$ where

$$\widetilde{\theta}_0 = \frac{(r+1-M)\left[(r+M)\,u(q^*) - Mc(q^*)\right]}{(r+1-M)\left[(r+M)\,u(q^*) - Mc(q^*)\right] + (r+M)\left[(1-M)\,u(q^*) - (r+1-M)\,c(q^*)\right]}.$$

One can check that (9) holds at $\tau = \tilde{\tau}_0$ and $q = q^*$. Hence, there exists an equilibrium with $\tau = \tilde{\tau}_0$ and $q = q^*$ if and only if $\theta > \widetilde{\theta}_0$.

Now consider the case $\tau = 1$, which implies $\eta_\tau \geq 0$. Combining the conditions in (19), we get $u'(q) \geq c'(q)$, or $q \leq q^*$, with strict inequality as long as $\eta_\tau > 0$. Inserting the V_j's we can rewrite the first order condition for q as

$$\frac{(1-\theta)\,c'(q)}{\theta u'(q)}=\frac{(r+M)\left[1-M-(r+1-M)c(q)/u(q)\right]}{(r+1-M)\left[r+M-Mc(q)/u(q)\right]}. \tag{20}$$

As in Proposition 4, there exists an unique solution to (20), call it $\chi=\chi(\theta)$, in $(0,\hat{q})$. Moreover, it is easy to check that $\chi'(\theta)>0$ and that $\chi(\tilde{\theta}_0)=q^*$. Since we need $\chi(\theta)\le q^*$ for an equilibrium with $\tau=1$, an equilibrium of this type cannot exist if $\theta>\tilde{\theta}_0$. If $\theta<\tilde{\theta}_0$ then $\chi(\theta)<q^*$, and this also implies the first order condition for τ is satisfied at $\tau=1$. We conclude that $\tau=1$ and $q=\chi(\theta)$ satisfy the first order conditions if and only if $\theta<\tilde{\theta}_0$. Moreover, $\chi(\theta)$ satisfies the incentive condition, and hence all of the conditions for an equilibrium. This completes the proof. ■

Proof of Proposition 6

Necessary and sufficient conditions for a solution to the maximization problem are:

$$\begin{aligned} -\theta C_2\left(U_2\tau_1-C_1\tau_2\right)+(1-\theta)\,U_2\left(U_1\tau_2-C_2\tau_1\right)-\eta_1\le 0, &\quad = \text{ if } \tau_1>0 \\ \theta U_1\left(U_2\tau_1-C_1\tau_2\right)-(1-\theta)\,C_1\left(U_1\tau_2-C_2\tau_1\right)-\eta_2\le 0, &\quad = \text{ if } \tau_2>0 \end{aligned} \tag{21}$$

The incentive compatibility constraints are $U_1\tau_2-C_2\tau_1\ge 0$ and $U_2\tau_1-C_1\tau_2\ge 0$. We are interested in equilibria where τ_1 and $\tau_2>0$, and (21) holds with equality. First consider the case $\tau_1<1$ and $\tau_2<1$, which implies $\eta_1=\eta_2=0$. Manipulating (21) yields $U_1U_2=C_1C_2$, a contradiction.

Consider now the case $\tau_1=1$ and $\tau_2<1$, implying $\eta_2=0$. Solving the second equation at equality for τ_2 yields:

$$\tilde{\tau}_2=\frac{\theta U_1U_2+(1-\theta)\,C_1C_2}{U_1C_1},$$

which is less than 1 if and only if $\theta < \underline{\theta}$ where

$$\underline{\theta} = \frac{C_1 (U_1 - C_2)}{U_1 U_2 - C_1 C_2}.$$

Note that $U_1 \leq C_2$ implies $\underline{\theta} \leq 0$, and so there is no equilibrium of this type. Thus, we must have $U_1 > C_2$. In fact, $0 < \underline{\theta} < 1$ if and only if $U_1 > C_2$ and $U_2 > C_1$. If $U_2 < C_1$ then $\underline{\theta} > 1$ and thus $\tilde{\tau}_2 < 1$ for any θ. One can also verify that $\tau_1 = 1$ and $\tau_2 = \tilde{\tau}_2$ satisfy the incentive constraints.

The case $\tau_1 < 1$ and $\tau_2 = 1$ is completely symmetric, and we simply report:

$$\bar{\theta} = \frac{U_2 (U_1 - C_2)}{U_1 U_2 - C_1 C_2}.$$

Finally, consider the case $\tau_1 = \tau_2 = 1$, which implies $\eta_j \geq 0$. For the first order conditions to hold the following two inequalities must be satisfied: $\theta C_2 (U_2 - C_1) \geq (1 - \theta) U_2 (U_1 - C_2)$ and $\theta U_1 (U_2 - C_1) \leq (1 - \theta) C_1 (U_1 - C_2)$. Also, the incentive compatibility constraints are satisfied if and only if $U_1 \geq C_2$ and $U_2 \geq C_1$. Under this condition the two inequalities are satisfied if and only if $0 \leq \underline{\theta} < \theta < \bar{\theta} \leq 1$. ■

Proof of Proposition 7

Necessary and sufficient conditions for a solution are:

$$\begin{aligned} -\theta \left[\tau (V_1 - V_0) - \lambda C\right] (V_1 - V_0) \\ + (1 - \theta) \left[\tau (V_0 - V_1) + \lambda U\right] (V_1 - V_0) - \eta_\tau \leq 0, \quad = \text{ if } \tau > 0 \end{aligned} \tag{22}$$

$$\theta \left[\tau (V_1 - V_0) - \lambda C\right] U - (1 - \theta) \left[\tau (V_0 - V_1) + \lambda U\right] C - \eta_\lambda \leq 0, \quad = \text{ if } \lambda > 0$$

Since $\tau > 0$ the first condition in (22) holds with equality. Consider the case $\tau < 1$, which implies $\lambda = 1$. Then we can substitute the V_j's into first condition in (22) at equality and solve for $\tau = \widetilde{\tau}$, where $\widetilde{\tau}$ is defined in the statement of the Proposition. Notice $\widetilde{\tau} \in (0,1)$ if and only if $\theta > \bar{\theta}$, where

$$\bar{\theta} = \frac{(r+M)U - MC - \gamma}{(1+r)(U-C)}.$$

Also, $\bar{\theta} > 0$ if and only if $(r+M)U - MC > \gamma$. The incentive conditions are satisfied at $\tau = \widetilde{\tau}$ and $\lambda = 1$. Hence, there exists an equilibrium with $\lambda = 1$ and $\tau = \widetilde{\tau} \in (0,1)$ if and only if $\theta > \bar{\theta}$.

Now consider the case where $\tau = 1$. Inserting the V_j's into the second equation in (22) at equality and rearranging, we get

$$\lambda \left\{ \theta U \left[(1-M)U - (r+1-M)C + \gamma \right] - (1-\theta)C \left[(r+M)U - MC - \gamma \right] \right\} = (1+r)\eta_\lambda.$$

Consider the case $\lambda < 1$, which implies $\eta_\lambda = 0$ and $\tau = 1$. Given this, we can substitute the V_j's into second condition in (22) at equality and solve for $\lambda = \widetilde{\lambda}$, where $\widetilde{\lambda}$ is defined in the statement of the Proposition. Notice $\widetilde{\lambda} \in (0,1)$ if and only if $\theta < \underline{\theta}$, where

$$\underline{\theta} = \frac{C\left[(r+M)U - MC - \gamma \right]}{(U-C)\left(U(1-M) + CM + \gamma \right)}.$$

The incentive conditions are satisfied at $\lambda = \widetilde{\lambda}$. Hence, there exists an equilibrium with $\tau = 1$ and $\lambda = \widetilde{\lambda} < 1$ if and only if $\theta < \underline{\theta}$.

Consider the case $\lambda = \tau = 1$. Note that $\tau = 1$ satisfies the first condition in (22) if and only if $\theta \leq \bar{\theta}$ and $\lambda = 1$ satisfies the second condition if and only if $\theta \geq \underline{\theta}$. Also, the incentive compatibility constraints are satisfied at $\tau = \lambda = 1$. Hence, there exists an equilibrium with $\lambda = 1$ and $\tau = 1$ if and only if $\underline{\theta} \leq \theta \leq \bar{\theta}$. Finally, one can verify that $\gamma < \bar{\gamma}$ implies $0 < \underline{\theta} < \bar{\theta} < 1$. ■