Penn Institute for Economic Research
Department of Economics
University of Pennsylvania
3718 Locust Walk
Philadelphia, PA 19103-6297

pier@ssc.upenn.edu
http://www.econ.upenn.edu/pier

PIER Working Paper 98-008

"Efficient Non-Contractible Investment"

by

Harold L. Cole, George J. Mailath and Andrew Postlewaite

Efficient Non-Contractible Investments*

Harold L. Cole
Research Department
Federal Reserve Bank
90 Hennepin Avenue
Minneapolis, MN 55480-0291
hlc@res.mpls.frb.fed.us

George J. Mailath[†]
Department of Economics
3718 Locust Walk
University of Pennsylvania
Philadelphia, PA 19104-6297
gmailath@econ.sas.upenn.edu

Andrew Postlewaite[†]
Department of Economics
3718 Locust Walk
University of Pennsylvania
Philadelphia, PA 19104-6297
apostlew@econ.sas.upenn.edu

August 17, 1998

*This paper is a substantial revision of an earlier paper "Bargaining and Investment in a Two-Sided Matching Model". We thank Daron Acemoglu, Jere Behrman, John Moore, Mark Rosenzweig, Randall Wright and the seminar audiences at numerous universities for helpful discussion and comments. The views expressed herein are those of the authors and not necessarily those of the Federal Reserve Bank of Minneapolis or the Federal Reserve System.

[†]Mailath and Postlewaite thank the Economics Department at the University of Bonn and the Minneapolis FRB for their hospitality. They also acknowledge financial support from the NSF, the Deutsche Forschungsgemeinschaft (SFB 303), and the Minneapolis FRB.

Efficient Non-Contractible Investments

by

Harold L. Cole, George J. Mailath, and Andrew Postlewaite

Abstract

This paper addresses the question of whether agents will invest efficiently in attributes that will increase their productivity in subsequent matches with other individuals. We present a two-sided matching model in which buyers and sellers make investment decisions prior to a matching stage. Once matched, the buyer and seller bargain over the transfer price. In contrast to most matching models, preferences over possible matches are affected by decisions taken before the matching process. We show that if bargaining respects the existence of outside options (in the sense that the resulting allocation is in the core of the assignment game), then efficient decisions can always be sustained in equilibrium. However, there may also be inefficient equilibria. Our analysis identifies a potential source of inefficiency not present in most matching models.

Efficient Non-Contractible Investments

by

Harold L. Cole, George J. Mailath, and Andrew Postlewaite

1. Introduction

Complementary investments are often made by different individuals; for example, a worker may invest in human capital while a firm invests in machinery that utilizes that human capital. Do investors making complementary investments face the correct incentives, especially when they cannot contract with each other prior to their decisions? The traditional answer is no (Williamson [17] and Grossman and Hart [8]). An agent's investment is a sunk cost by the time the agents bargain over the split of the surplus that results from the investment. Since bargaining typically allocates part of the surplus generated by an agent's investment to the other party, the failure of that agent to capture the full benefit of his investment leads to underinvestment.

In the analysis of this holdup problem, the degree to which the benefits of an agent's investment cannot be captured by that agent is related to asset specificity. The share of the surplus that an agent gets in any plausible bargaining process will be constrained by his outside options. A worker whose skills are nearly as valuable on a machine other than that owned by the person he is currently bargaining with can play the two owners off against each other. In many circumstances, competition between potential partners provides protection against the holdup problem, and agents capture the bulk of the benefits of their investments, and, consequently, have incentives to invest efficiently. The polar extreme to this case is that an agent's investment is of value to a single individual, for example, a worker who becomes expert on a unique machine. The value of the asset he invests in is specific to the match with the owner of that machine. Intuitively, the lack of outside options for such an agent should lead him to expect a smaller share of the surplus generated by his investment than when there is potential competition for his services.

While there is a large literature that analyzes the effect of asset specificity on investment, the degree to which investments are specific is typically taken to be exogenous. That analysis considers a single pair in isolation, taking as given other agents' investments, and the outside options inherent in those investments. The difficulty with analyzing investments of a single pair is that those investments determine (at least in part) the outside options of other pairs. Consider a matching problem in which there are a number of people on each side who might make investments in hopes of subsequently pairing with someone who has made a complementary investment. The return any individual can expect from investing will be the outcome of the bargaining with his future partner, which will depend on the outside options of both individuals. These outside options, of course, are determined precisely by the investment decisions of the agents involved.

Our aim is to analyze the investment decisions of agents who, subsequent to investing, pair off, produce a surplus, and share that surplus through some bargaining process. We treat the agents' investment decisions as a noncooperative game, each agent's decision depending on the (equilibrium) investment choices of other agents like him and of the agents with whom he can potentially match. In this way the asset specificity of agents' investments is endogenously determined, rather than exogenously assumed. We are particularly interested in comparing the investments agents make when they can contract prior to investing and those they make when they cannot. If agents can contract over the investment levels they make, investments will be efficient. We take those investments as a benchmark to which we compare investments when ex ante contracting is impossible. When ex ante contracting is impossible, there will always be an equilibrium in which agents invest efficiently, but there may be additional equilibria characterized by inefficient investments. The analysis also suggests that, in many situations, the efficient investment equilibrium is implausible. We further show that for some problems, regardless of the bargaining process, underinvestment may occur. On the other hand, we conjecture that there are bargaining processes for which overinvestment cannot occur.

In order to focus on the efficiency of investment choices and bargaining over the resulting surplus, we label the two sides in the relationship "buyers" and "sellers." There is of course nothing important about this, and we could have used the terms "workers" and "firms."

The outline of the paper is as follows. In the next section, we present a simple example that illustrates the investment and matching process. We then present the formal model for the case of a finite number of agents (Section 3) and characterize the bargaining outcomes (Section 4). Section 5 provides sufficient conditions for agents to receive the social value of their investments, and Section 6 compares the cases in which agents can and cannot contract prior to investing. The simplest version of the sufficient conditions for agents to fully appropriate the value of their decisions (to use the language of Makowski and Ostroy [15]) involve binding outside options, so that all agents' payoffs are completely determined by the payoffs that any single agent receives. With a finite population, this requires that many agents are choosing the attributes that are also chosen by other agents. If each agent is idiosyncratic (for example, has different costs of acquiring attributes), then efficient attribute choices will not imply binding outside options. Efficiency then results only if the bargaining between agents results in a particular outcome (see in particular the discussion after Proposition 5). On the other hand, the outside options do limit the agreements that agents can come to, and the richer the set of chosen attributes, the closer to binding the outside options become. A plausible (but incorrect) conjecture is that as the number of agents becomes large, outside options become binding and so in large economies, we have full appropriation. The reason the conjecture fails is that even when the set of agents is rich (in the sense that each agent has a close competitor in exogenous characteristics), since attributes are endogenous, agents may not have a close competitor in attributes. Moreover, even if all agents have close competitors, the outside options that need to bind to ensure full appropriation may not.

A second reason for exploring a model with a large number of agents is that with a finite population, a change in a single agent's investment decision can affect the matching and payoffs to all other agents. While this may be plausible for some problems, it is not for others, particularly when there are many agents. In Section 7, we present a model with a continuum of agents, which allows us to assume that a single agent's decisions do not affect the other agents. We obtain analogs of the results for the finite case: when agents cannot contract prior to investing, there is always an equilibrium in which they invest efficiently, but there may be inefficient equilibria as well. As in the finite case, the inefficient equilibria may well be more plausible than the efficient equilibria (Section 8). We further discuss our model and results in Section 9, while Section 10 closes with a discussion of related literature.

2. A simple example

As discussed in the introduction, we are interested in the interaction between the way in which the surplus is divided in matched pairs and the incentives individuals have to invest in attributes. We begin by illustrating several issues with a simple example. There are two buyers, $\{1,2\}$, and two sellers, $\{1,2\}$. For now, we fix the attributes of the buyers and sellers as in the following table. The surplus generated by a pair (b,s) is given by the product of their attributes, $b \cdot s$. Figure 2.1 displays one particular outcome for this environment with each of the two columns representing a matched pair and the split of the surplus for that pair. Total surplus is maximized by the indicated matching and the split of the surplus for the pairs is unique if the sharing rule is symmetric with respect to buyers and sellers.

Suppose now that attributes are not fixed, but chosen from the set $\{2,3\}$.

buyer's share (x_i)	2	$4\frac{1}{2}$
buyer's attribute (b_i)	2	3
buyer (i)	1	2
seller (j)	1	2
seller's attribute (s_j)	2	3
seller's share (p_j)	2	$4\frac{1}{2}$

Figure 2.1: An example with two buyers and sellers.

x_i	3	$4\frac{1}{2}$
b_i	2	3
i	1	2
j	1	2
s_j	3	3
p_j	3	$4\frac{1}{2}$

Figure 2.2: Seller 1 with attribute $s = 3$.

We focus on the behavior of seller 1, with the attributes of the other agents unchanging.[1] If the surplus is always divided equally and seller 1 chose instead $s = 3$, then the matching and surplus division are as in Figure 2.2.

In this example, equal division violates equal treatment: The two sellers have the same attribute but receive different payoffs. As we will see in the next section, such a specification of payoffs is not "stable" since seller 1 could make buyer 2 a marginally better offer than he gets when matched with seller 2.

Equal division may also prevent efficient attribute choices. If, for example the cost of attribute 2 to seller 1 is 0, while the cost of attribute 3 is $\frac{3}{2}$, then the increase in surplus when seller 1 chooses attribute 3 rather than attribute 2 is 2 while the increased cost to seller 1 of choosing the higher attribute is only $\frac{3}{2}$. This is, of course, a simple consequence of having a sharing rule that gives part of the increase in output that results from seller 1's investment to the buyer that is matched with seller 1.

There are sharing rules that satisfy equal treatment (and so are stable); Figure 2.3 gives one such.

[1] We can choose the cost functions for the two buyers and for seller 2 to ensure (assuming the bargaining is monotonic) that their optimal choice of attributes is as in the table above. Specifically, denote by $\psi(b, i)$ the cost to buyer i of acquiring attribute b and by $c(s, j)$ the cost of attribute s to seller j, and suppose $\psi(2,1) = \psi(b,2) = c(s,2) = 0, \psi(3,1) = 10$.

x_i	$1\frac{1}{2}$	$4\frac{1}{2}$
b_i	2	3
i	1	2
j	1	2
s_j	3	3
p_j	$4\frac{1}{2}$	$4\frac{1}{2}$

Figure 2.3: Equal treatment and inefficiency.

x_i	2	5
b_i	2	3
i	1	2
j	1	2
s_j	3	3
p_j	4	4

Figure 2.4: Equal treatment and efficiency.

While we obtain equal treatment here, there are still incentives for inefficient choices. For example, if the cost of attribute 3 to seller 1 is $2\frac{1}{4}$, then seller 1 chooses $s = 3$, even though it is inefficient to do so. The problem now is that the payoff to the buyer who is matched with seller 1 falls in response to the higher attribute of the seller.

There is, however, a specification of payoffs for this vector of buyers' and sellers' attribute choices that satisfies equal treatment, is stable, and implies efficient choices by seller 2 (given in Figure 2.4).

In this case, the surplus division between buyer and seller 2 has changed, even though the characteristics of the match did not change.

This simple example illustrates the issues we address in this paper and the approach we take. If agents must make complementary investments before contracting over outcomes, they may choose inefficiently. Further, the manner in which bargaining takes place after matching (that is, the sharing rule) will affect the efficiency of investments, even if the sharing rules are constrained by stability.

3. Description of the investment problem

An *investment problem* Γ is the collection $\{I, J, B, S, \psi, c, v\}$, where

- I and J are disjoint finite sets of buyers and sellers;
- B and S are respectively the set of possible attributes (income, wealth, or willingness to pay) buyers can choose from, and the set of possible attributes (quality of good) for sellers;
- $\psi : B \times I \to \Re_+$, where $\psi(b, i)$ is the cost to buyer i of attribute b;
- $c : S \times J \to \Re_+$, where $c(s, j)$ is the cost to seller j of attribute s; and
- $v : B \times S \to \Re_+$, where $v(b, s)$ is the surplus generated by a buyer with attribute b matching with a seller with attribute s.

We assume B and S are compact subsets of $\Re_+$. We assume (without loss of generality) that there are equal populations of buyers and sellers.[2] We assume that $v(b, s)$ displays complementarities in attributes (v is supermodular): for $b < b'$ and $s < s'$, $v(b', s) + v(b, s') \leq v(b, s) + v(b', s')$. Equivalently, if v is C^2, $\partial^2 v/\partial b \partial s \geq 0$. We will sometimes assume that the surplus function is *strictly* supermodular, i.e., $v(b', s) + v(b, s') < v(b, s) + v(b', s')$ for all $b < b'$ and $s < s'$. We also assume v is strictly increasing in b and in s.

We model the bargaining and matching process that follows the attribute choices as a cooperative game. Given a fixed distribution of attributes of buyers and sellers, the resulting cooperative game is an *assignment game*: there are two populations of agents (here, buyers and sellers), with each pair of agents (one from each population) generating some value. To distinguish this assignment game from the assignment game we describe in Section 6, we call this assignment game *the ex post assignment game* (indicating that attribute choices are taken as fixed). An outcome in the assignment game is a *matching* (each buyer matching with no more than one seller and each seller matching with no more than one buyer) and a *bargaining outcome* or *payoff* (a division of the value generated by each matched pair between members of that pair). We denote the buyer's share of the surplus by $x \geq 0$ and the seller's share by $p \geq 0$, with $x + p \leq v(b, s)$.[3]

Definition 1. *A* ***matching*** m *is a function* $m : I \to J \cup \{\emptyset\}$, *where* m *is one-to-one on* $m^{-1}(J)$, *and* $\emptyset$ *is interpreted as no match.*

Definition 2. *A bargaining outcome* $(\mathbf{x}, \mathbf{p}) \in \Re^I_+ \times \Re^J_+$ *is* ***feasible for the matching*** m *if* $x_i + p_{m(i)} \leq v(b_i, s_{m(i)})$ *whenever* $m(i) \neq \emptyset$, $x_i = 0$ *whenever*

[2] The case of more buyers than sellers, for example, is handled by adding additional sellers with attribute 0 and setting $v(b, 0) = 0$ for all b.

[3] Note that shares are amounts, not fractions.

$m(i) = \emptyset$, and $p_j = 0$ whenever $j \notin m(I)$. *A bargaining outcome is* **feasible** *if it is feasible for some matching.*

Definition 3. *A bargaining outcome* $(\mathbf{x}, \mathbf{p})$ *is* **stable** *if it is feasible and for all* $i \in I$ *and* $j \in J$,

$$x_i + p_j \geq v(b_i, s_j). \tag{3.1}$$

A matching associated with a stable bargaining outcome vector is a ***stable matching****.*

It is clear that there are no transfers across matched pairs in a stable bargaining outcome. As usual in assignment games, stable bargaining outcomes are core allocations of the assignment game and conversely, where the characteristic function of the assignment game has value $V(A)$ at a coalition $A \subset I \cup J$ given by the maximum of the sum of surpluses of matched pairs (the maximum is taken over all matchings of buyers and sellers in A). Since buyer attributes are described by the vector $\mathbf{b}$ and seller attributes are described by the vector $\mathbf{s}$, we sometimes write $V(\mathbf{b}, \mathbf{s})$ for $V(I \cup J)$.[4]

We are thus modelling the game facing buyers and sellers as one of simultaneously choosing attributes, and subsequent to the choice of attributes, matching and sharing the surplus generated by the matches. We restrict attention to matches and payoffs that are stable given the choice of attributes. Since v is supermordular, it is straightforward to show that there always exists a stable payoff for any vector of attribute choices.

There is, however, one important issue in considering the attribute investment decisions as a noncooperative game. Typically there is not a unique stable outcome associated with a vector of attributes; in fact, as we will see, there is usually a continuum of stable outcomes. In order to treat attribute choices as a noncooperative game, each agent must be able to compare the payoffs from two different attribute choices, given other agents' choices. This requires a well-defined (stable) payoff associated with every possible set of attribute investments. That is, there must be a *bargaining outcome function* $g : B^I \times S^J \to \Re^I_+ \times \Re^J_+$, with $g(\mathbf{b}, \mathbf{s}) = (\mathbf{x}, \mathbf{p})$ a stable outcome for each vector of attribute choices $(\mathbf{b}, \mathbf{s})$. We denote by $x_i(\mathbf{b}, \mathbf{s})$ buyer i's share when the vector of attributes is $(\mathbf{b}, \mathbf{s})$ and $p_j(\mathbf{b}, \mathbf{s})$ the j-th seller's share. Observe that given g, buyers and sellers are simultaneously choosing attributes, with payoffs $x_i(\mathbf{b}, \mathbf{s}) - \psi(b_i, i)$ to buyer i and

[4] Assignment games have received considerable attention in the literature. The core of any assignment game is nonempty, and coincides with the set of Walrasian allocations (Kaneko [12] and Quinzii [16] for the finite population case and Gretsky, Ostroy, and Zame [7] for the continuum population case). Our case is particularly simple, since v is supermodular.

$p_j(\mathbf{b}, \mathbf{s}) - c(s_j, j)$ to seller j. This is a standard strategic form game. The notion of *weak ex post contracting equilibrium* combines the requirement that every vector of attribute choices lead to a stable payoff of the induced ex post assignment game with the requirement that attribute choices are a Nash equilibrium of the strategic form game.

Definition 4. *Given an investment problem $\Gamma = \{I, J, B, S, \psi, c, v\}$, a* ***weak ex post contracting equilibrium*** *is a pair $\{g^*, (\mathbf{b}^*, \mathbf{s}^*)\}$ such that:*

1. *$g^* : B^I \times S^J \to \Re^I_+ \times \Re^J_+$, where for any choice of characteristics $(\mathbf{b}, \mathbf{s})$, $g^*(\mathbf{b}, \mathbf{s}) = (\mathbf{x}^*(\mathbf{b}, \mathbf{s}), \mathbf{p}^*(\mathbf{b}, \mathbf{s}))$ is a stable payoff for $(\mathbf{b}, \mathbf{s})$, and*

2. *for each $i \in I$ and $b'_i \in B$, $x^*_i(\mathbf{b}^*_{-i}, b^*_i, \mathbf{s}^*) - \psi(b^*_i, i) \geq x^*_i(\mathbf{b}^*_{-i}, b'_i, \mathbf{s}^*) - \psi(b'_i, i)$ and for each $j \in J$ and $s'_j \in S$, $p^*_j(\mathbf{b}^*, \mathbf{s}^*_{-j}, s^*_j) - c(s^*_j, j) \geq p^*_j(\mathbf{b}^*, \mathbf{s}^*_{-j}, s'_j) - c(s'_j, j)$.*

This equilibrium notion is a combination of a cooperative notion (stability) with two non-cooperative ones (Nash and perfection). Each individual is best replying to the actions of every one else, the future consequences of any attribute choice are correctly foreseen, and any attribute choice must lead to a stable payoff.

We think of the bargaining outcome function, g, as capturing the way bargaining transpires in an investment problem. Restricting the sharing of the surpluses arising from a given vector of attribute choices $(\mathbf{b}, \mathbf{s})$ to stable payoffs still leaves considerable indeterminacy since there is typically a multitude of stable allocations for a given vector of attributes choices. For some investment problems, that indeterminacy might be resolved through bargaining that favors the buyers as much as possible, given the constraints imposed by stability. For other problems, bargaining might resolve the indeterminacy in favor of the sellers, while in still others, bargaining might result in as equal a division as is consistent with stability.

An alternative to including the bargaining outcome function in the definition of the equilibrium is to include it in the description of the investment problem. For example, if bargaining favors buyers, the bargaining outcome function capturing this could be included in the specification of the investment problem, leading to a "buyer-friendly" bargaining problem. There are two difficulties with this approach. First, the bargaining outcome function is endogenous. Second, for some bargaining outcome functions, there may be no pure strategy equilibrium. This nonexistence reflects an inconsistency between an exogenously specified bargaining outcome function and the given data of an investment problem, I, J, B,

S, ψ, c, and v. The way in which bargaining resolves indeterminacy must be endogenously determined in concert with agents' investment choices.

We impose further restrictions on weak ex post contracting equilibria, in an equilibrium selection spirit. As defined, for a given set of attribute investments, the outcome selected by the bargaining outcome function can depend on the identity of the individuals who have chosen particular attributes. We focus on the case in which bargaining is anonymous in the sense that it depends only on attributes, independent of the identities of the agents choosing those attributes.

There is also nothing in the definition of ex post contracting equilibrium that prevents the bargaining outcome function from selecting the stable outcome that is most favorable to buyers as long as no buyers deviate, and selecting the stable outcome that is most favorable to sellers otherwise.[5] This trigger specification of payoffs breaks any link between the marginal social return from an investment and its private return. We need some way of reducing the arbitrariness of the specification of stable payoffs. One simple way that is always consistent with stability is to pin the split at the bottom pair.

Definition 5. *An **ex post contracting equilibrium (EPCE)** is a weak ex post contracting equilibrium, $\{g^*, (\mathbf{b}^*, \mathbf{s}^*)\}$ that is anonymous and, if for any two attribute vectors $(\mathbf{b}', \mathbf{s}')$ and $(\mathbf{b}'', \mathbf{s}'')$, there exists $i \in I$ such that $b'_i = \min_{\ell \in I}\{b'_\ell\} = b''_i = \min_{\ell \in I}\{b''_\ell\}$ and there exists $j \in J$ such that $s'_j = \min_{\ell \in J}\{s'_\ell\} = s''_j = \min_{\ell \in J}\{s''_\ell\}$, then*

$$\begin{aligned} x_i^*(\mathbf{b}', \mathbf{s}') &= x_i^*(\mathbf{b}'', \mathbf{s}'') \text{ and} \\ p_j^*(\mathbf{b}', \mathbf{s}') &= p_j^*(\mathbf{b}'', \mathbf{s}''). \end{aligned}$$

If there is an imbalance between the number of buyers and sellers, then we (along the lines of footnote 2) add enough dummy agents to equalize the number of buyers and sellers. In this case, the bottom pair necessarily receives a payoff of zero, and so the stable outcome necessarily favors agents on the short side of the market.

4. Characterization of stable allocations for a finite population

We now characterize the stable allocations. The simple proposition below summarizes several characteristics: For any attribute vector $(\mathbf{b}, \mathbf{s})$, any stable outcome

[5] In our context, all buyers agree as to the ranking of stable payoff vectors (and all sellers have the reverse ranking). Moreover, with a finite set of buyers, even in an anonymous equilibrium any deviation is detected, since any deviation results in a different empirical distribution over attributes.

matches attributes positively assortatively; all buyers with equal attributes receive equal payoffs, and similarly for sellers; and finally, in checking stability, one need not examine all unmatched pairs, but only those unmatched pairs for which the partners have attributes which are "close" to those of their matches. Before stating the proposition we make the following definition:

Definition 6. *A matching m is* **positively assortative** *if $m(I) = J$ and for any $i, j \in I, b_i > b_j \Rightarrow s_{m(i)} \geq s_{m(j)}$.*

Proposition 1. *Consider a vector of attributes $(\mathbf{b}, \mathbf{s})$, and a labeling of agents so that $I, J = \{1, ..., n\}$, and attributes are weakly increasing in index. Then*

1. *every stable matching is positively assortative on attributes;*
2. *every stable payoff exhibits equal treatment: $b_i = b_{i'} = b \Rightarrow x_i = x_{i'} \equiv x_b$ and $s_j = s_{j'} = s \Rightarrow p_j = p_{j'} \equiv p_s$; and*
3. *a payoff $(\mathbf{x}, \mathbf{p})$ is stable if and only if for all i,*

$$\begin{aligned} x_i + p_i &= v(b_i, s_i), \\ x_i + p_{i+1} &\geq v(b_i, s_{i+1}), \text{ and} \\ x_{i+1} + p_i &\geq v(b_{i+1}, s_i). \end{aligned}$$

Proof. The first two statements are straightforward. Without loss of generality, the stable matching can be taken to be by index, yielding $x_i + p_i = v(b_i, s_i)$. The two inequalities are immediate implications of stability.

In order to show sufficiency, we argue to a contradiction. Suppose there exists a $k > 1$ such that $x_i + p_{i+k} < v(b_i, s_{i+k})$. Then

$$\begin{aligned} x_{i+1} + p_{i+k} &< x_{i+1} + v(b_i, s_{i+k}) - x_i \\ &\leq x_{i+1} + v(b_i, s_{i+k}) - v(b_i, s_{i+1}) + p_{i+1} \\ &= v(b_{i+1}, s_{i+1}) + v(b_i, s_{i+k}) - v(b_i, s_{i+1}) < v(b_{i+1}, s_{i+k}), \end{aligned}$$

where the last inequality holds because v is strictly supermodular. Induction then yields $x_{i+k-1} + p_{i+k} < v(b_{i+k-1}, s_{i+k})$, a contradiction. ∎

The third part of this proposition is useful since it implies that in order to check the stability of a payoff vector, we need only check adjacent pairs in a positively assortative matching. If no buyer (or seller) can block when matched with a partner adjacent to his or her current partner, the payoff vector is stable.

x	2	2	4	4	4	7	7	7	15
b	2	2	3	3	3	4	4	4	6
s	2	2	2	3	3	3	4	4	4
p	2	2	2	5	5	5	9	9	9

Figure 4.1: Equal treatment can imply stability.

Since stable payoffs exhibit equal treatment, we sometimes refer to the payoffs to an attribute rather than the payoffs to an individual, and we often will not distinguish between the two.

Proposition 1 states that equal treatment is necessary for stability; in some cases, it is sufficient for stability as well. Consider the allocation in Figure 4.1 with our usual $v(b, s) = b \cdot s$. As before, matches should be read by columns.

Once the bottom (left-most) pair's shares have been determined in this example, all other agents' payoffs are uniquely determined by equal treatment because of the "overlap" in the players' attributes. Moreover, when the attribute vectors have this overlap property, any division of the surplus for a specified pair is consistent with a stable payoff.

The next proposition and corollary formalize the intuition illustrated by this example. If we order the *values* of chosen attributes of the buyers from low to high, we denote by $b_{(\kappa)}$ the κ-th value, and similarly by $s_{(\kappa)}$ the κ-th value for the seller.[6]

Definition 7. *The pair of attribute vectors* $(\mathbf{b}, \mathbf{s})$ *is* ***overlapping*** *if, for a positively assortative matching* m, *and any* κ, *there exists* i, i' *such that* $b_i = b_{(\kappa)}$, $b_{i'} = b_{(\kappa+1)}$, $s_{m(i)} = s_{m(i')}$.

Overlapping attribute vectors have the following more transparent formulation. Suppose we index the buyers and sellers by the integers 1 through n so that attributes are weakly increasing in index. Matching by index (i.e., $i = m(i)$) is then positively assortative on attributes. The attribute vectors are overlapping if $b_{j-1} \neq b_j \Rightarrow s_{j-1} = s_j$. Note that the notion is symmetric, since $b_{i-1} \neq b_i \Rightarrow s_{i-1} = s_i$ implies $s_{i-1} \neq s_i \Rightarrow b_{i-1} = b_i$.

Proposition 2. *Suppose the attribute vectors are overlapping and* $(\mathbf{x}, \mathbf{p})$ *is a payoff vector for a positively assortative matching that satisfies:*

[6] Note that we are not ordering the chosen attributes, so $b_{(\kappa-1)} < b_{(\kappa)} < b_{(\kappa+1)}$ even if two buyers have attribute $b_{(\kappa)}$.

1. *equal treatment;*

2. *and no waste:* $x_i + p_i = v(b_i, s_i)$.

Then $(\mathbf{x}, \mathbf{p})$ *is stable.*

Proof. Since we have assumed no waste, we need only check to see that for adjacent pairs, if the matching is switched, neither of the new pairs can block. But since the vectors of attributes is overlapping, either both buyers have the same attribute or both sellers have the same attribute, and the assumption that $x_i + p_i = v(b_i, s_i)$ ensures that neither of the new pairs can block. ■

Corollary 1. *Suppose* $(\mathbf{b}, \mathbf{s})$ *is overlapping and the pair* $(x_{b_{(1)}}, p_{s_{(1)}})$ *satisfies* $x_{b_{(1)}} \geq 0$, $p_{s_{(1)}} \geq 0$, *and* $x_{b_{(1)}} + p_{s_{(1)}} = v(b_{(1)}, s_{(1)})$. *Define* (x, p) *recursively as follows:*

$$x_{b_{(\kappa+1)}} = x_{b_{(\kappa)}} + [v(b_{(\kappa+1)}, s) - v(b_{(\kappa)}, s)],$$

where $s = s_{m(i)} = s_{m(i')}$ *and* $b_i = b_{(\kappa)}$, $b_{i'} = b_{(\kappa+1)}$ *for some positive assortative matching* m *and* $i, i' \in I$, *and similarly for the sellers.*[7] *Then the payoffs* (x, p) *are stable, and every stable payoff can be constructed in this way.*

Proof. Since there is a unique positive assortative matching of attributes, there is a unique seller attribute that satisfies, for any positively assortative matching of agents, $s = s_{m(i)} = s_{m(i')}$ and $b_i = b_{(\kappa)}$, $b_{i'} = b_{(\kappa+1)}$ for some $i, i' \in I$. Moreover, the hypothesis of overlapping attribute vectors ensures s exists and that for all matched attributes (b, s), $x_b + p_s = v(b, s)$. Hence, we have equal treatment and no waste, and Proposition 2 applies.

Equal treatment in stable payoffs guarantees that every stable payoff has this property. ■

Corollary 1 provides a complete characterization of stable outcomes when attribute vectors are overlapping. When attribute vectors don't overlap, there is a degree of indeterminacy in stable payoffs, even fixing the division of the value for the bottom pair. One can, however, construct stable payoffs for a vector of positively assortative, nonoverlapping attributes in a straightforward way: Fix the share for the bottom pair. If there is an overlapping subset of attributes containing this bottom pair of attributes, equal treatment determines the payoffs to those attributes. Where there is a gap between the attributes for this subset of agents and those higher, Proposition 1 puts constraints on how the surplus

[7] This defines the payoffs to attributes. Every agent with the same attribute receives that payoff.

for the pair above the gap can be divided. Choose an arbitrary distribution of surplus for that pair, subject to those constraints. Allocate the surplus for the adjoining pairs so long as there is overlap, and each time a gap is encountered, proceed as above.

We now formalize this idea and provide bounds on the indeterminacy of stable payoffs. Given an attribute vector $(\mathbf{b}, \mathbf{s})$, and a stable matching m, relabel the agents so that I, $J = \{1, \ldots, n\}$, attributes are weakly increasing in index and buyer i is matched with seller $m(i) = i$. (This is always possible, because every stable matching is positively assortative in attributes.) Let $(\mathbf{b}^\dagger, \mathbf{s}^\dagger)$ denote the vector of attributes for a population of agents $(I^\dagger, J^\dagger)$, $I \subset I^\dagger$ and $J \subset J^\dagger$, with overlap constructed as follows: if there exists i such that $b_i \neq b_{i+1}$ and $s_i \neq s_{i+1}$, then in the extended population, there is an additional buyer (with index $i + \frac{1}{2}$) with attribute b_i and an additional seller (also with index $i + \frac{1}{2}$) with attribute s_{i+1}. We refer to $(\mathbf{b}^\dagger, \mathbf{s}^\dagger)$ as the *buyer-first extension* of $(\mathbf{b}, \mathbf{s})$. Note that a stable matching for $(\mathbf{b}^\dagger, \mathbf{s}^\dagger)$ is given by $m^\dagger(i) = i$ for all i. This maintains the original matching on I, and extends it to the new agents by matching any new buyer $i + \frac{1}{2}$ with the new seller $i + \frac{1}{2}$. Similarly, let $(\mathbf{b}^\ddagger, \mathbf{s}^\ddagger)$ denote the vector of attributes for the population $(I^\ddagger, J^\ddagger)$ obtained from $(\mathbf{b}, \mathbf{s})$ by giving attribute b_{i+1} to buyer $i + \frac{1}{2}$ and attribute s_i to seller $i + \frac{1}{2}$. We refer to $(\mathbf{b}^\ddagger, \mathbf{s}^\ddagger)$ as the *seller-first extension* of $(\mathbf{b}, \mathbf{s})$. Note that $(\mathbf{b}^\ddagger, \mathbf{s}^\ddagger)$ also has no gaps, and $I^\ddagger = I^\dagger$ and $J^\ddagger = J^\dagger$. Note also that for any stable payoff for either extension, the restriction of the payoff to the original agents, $I \cup J$, is stable.

The attribute vectors $(\mathbf{b}^\dagger, \mathbf{s}^\dagger)$ and $(\mathbf{b}^\ddagger, \mathbf{s}^\ddagger)$ are minimal extensions of $(\mathbf{b}, \mathbf{s})$ that yield overlapping attribute vectors by adding just enough of the "right" attributes. Note that the bottom pair of matched attributes is unaffected by the extension, so that $b_1^\dagger = b_1^\ddagger = \min b_i \equiv \underline{b}$ and $s_1^\dagger = s_1^\ddagger = \min s_j \equiv \underline{s}$. Since $(\mathbf{b}^\dagger, \mathbf{s}^\dagger)$ is overlapping, by Corollary 1, there is a unique stable payoff corresponding to each value of $x_{\underline{b}}^\dagger$, and similarly for $(\mathbf{b}^\ddagger, \mathbf{s}^\ddagger)$.

The following proposition shows that the vector $(\mathbf{b}^\dagger, \mathbf{s}^\dagger)$ uniformly favors buyers in the sense that it gives the maximal payoff to buyers over stable payoffs given $x_{\underline{b}}^\dagger$. Suppose that buyers and sellers are positively assortatively matched and there are adjacent pairs for which both the buyers' and sellers' attributes differ. The buyer-first extension maximizes the buyer's payoff by having a seller with the same attribute as his partner match with a buyer with a lower attribute, which minimizes the payoff to that attribute (by Proposition 1). The buyer then receives the remainder. Analogously, the seller-first extension $(\mathbf{b}^\ddagger, \mathbf{s}^\ddagger)$ gives the maximum payoff to the seller subject to the bound.

Proposition 3. *Suppose* $(\mathbf{b}, \mathbf{s})$ *is a vector of attributes and* $(\mathbf{x}, \mathbf{p})$ *is a stable* *payoff. Let* $(\mathbf{x}^\dagger, \mathbf{p}^\dagger)$ *be the unique stable payoff for the buyer-first extension of* $(\mathbf{b}, \mathbf{s})$ *satisfying* $x^\dagger_{\underline{b}}=x_{\underline{b}}$*, and* $(\mathbf{x}^\ddagger, \mathbf{p}^\ddagger)$ *the unique stable payoff for the seller-first extension of* $(\mathbf{b}, \mathbf{s})$ *satisfying* $x^\ddagger_{\underline{b}} = x_{\underline{b}}$*. Then,*

$$x^\ddagger_b \leq x_b \leq x^\dagger_b, \;\; \forall b, \tag{4.1}$$

and

$$p^\dagger_s \leq p_s \leq p^\ddagger_s, \;\; \forall s. \tag{4.2}$$

Moreover, for any attribute in $(\mathbf{b}, \mathbf{s})$*, any share* x_b *satisfying (4.1) or* p_s *satisfying (4.2), there is a stable payoff for* $(\mathbf{b}, \mathbf{s})$ *giving shares* x_b *to* b *or* p_s *to* s*.*

Proof. Since no new attributes are introduced in $(\mathbf{b}^\dagger, \mathbf{s}^\dagger)$ or $(\mathbf{b}^\ddagger, \mathbf{s}^\ddagger)$, and every pair of attributes in $(\mathbf{b}, \mathbf{s})$ matched in a stable matching remains matched when the attribute vector is $(\mathbf{b}^\dagger, \mathbf{s}^\dagger)$ or $(\mathbf{b}^\ddagger, \mathbf{s}^\ddagger)$, it is enough to show that $x_b \leq x^\dagger_b \; \forall b$ to verify (4.1) and (4.2).

Let $b_{(\kappa)}$ be the first buyer attribute at which there is no overlap, and note that $b_{(\kappa)} = b^\dagger_{(\kappa)}$. The attribute $b_{(\kappa)}$'s stable payoff is at a maximum when the stable payoff of the sellers with attribute s^κ is at a minimum, where s^κ is the smallest seller attribute matched with the buyer attribute $b_{(\kappa)}$. This occurs when $x_{b_{(\kappa-1)}} + p_{s^\kappa} = v(b_{(\kappa-1)}, s^\kappa)$. Thus,

$$x_{b_{(\kappa)}} \leq x_{b_{(\kappa-1)}} + v(b_{(\kappa)}, s^\kappa) - v(b_{(\kappa-1)}, s^\kappa) \equiv x^\dagger_{(\kappa)}(x_{b_{(\kappa-1)}}),$$

with equality yielding a payoff that is consistent with stability. Moreover, $x^\dagger_{(\kappa)}(x_{b_{(\kappa-1)}})$ is the payoff of attribute $b_{(\kappa)}$ when the population attribute vector is $(\mathbf{b}^\dagger, \mathbf{s}^\dagger)$, since attribute $b_{(\kappa-1)}$ receives a payoff of $x_{b_{(\kappa-1)}}$. Note also that $x^\dagger_{(\kappa)}(x_{b_{(\kappa-1)}})$ is increasing in $x_{b_{(\kappa-1)}}$. Proceeding recursively up buyer and seller attributes shows that buyer attribute $b_{(\kappa)}$'s maximum stable payoff is calculated as if there is the pattern of overlap of $(\mathbf{b}^\dagger, \mathbf{s}^\dagger)$.

Now consider the sufficiency of (4.1) for a single buyer attribute's share to be stable. Fix some share satisfying (4.1) for an attribute b. We now proceed inductively to fill in shares to the other attributes above and below. For attributes above b, apply the procedure described just after Corollary 1. The same procedure can also be applied for attributes below b, starting at b and working down. The bounds (4.1) guarantee that each step will be feasible and result in the bottom pair receiving the split $(x_{\underline{b}}, p_{\underline{s}})$. ■

Note that the proposition does not assert that any vector of shares that satisfies (4.1) for all attributes can be achieved in a single stable payoff. Propositions

x	2	4	4	4	7	7	7	11	15
b	2	3	3	3	4	4	4	5	6
s	2	2*	2	3	3*	3	4	4*	4
p	2	2	2	5	5	5	9	9	9

Figure 5.1: The result of a buyer's change in attribute.

2 and 3 characterize the stable outcomes associated with any attribute vector $(\mathbf{b}, \mathbf{s})$. These propositions provide the tools with which we analyze the incentives agents have in making investment decisions in the next section.

5. Incentives for efficient investment

We can use the example in Figure 4.1 to compare the private and social returns to an individual who changes his or her attribute. Suppose, for example, a buyer with attribute $b = 2$ changed his attribute to $b = 5$. If we leave unchanged the bottom pair's division, the unique payoffs consistent with equal treatment are as in Figure 5.1 (an asterisk indicates a seller for whom the matched buyer has a different attribute level as a result of the change).

The share to the buyer whose attribute changed increased by 9. In principle, this need not be the change in the social value. The change in the buyer's attribute from 2 to 3 alters the matching of buyers and sellers; a buyer who increases his attribute will "leapfrog" other buyers, and match with a higher attribute seller. This will result in some of the other buyers being matched with lower attribute sellers than they had originally been matched with and some of the sellers being matched with higher attribute buyers than before. In other words, when this buyer (or other buyers or sellers) chooses an attribute, he imposes an externality on other players simply because the matching is changed. While an increase in a buyer's attribute causes some of the other players to be in matches with higher total surplus and others to be in matches with lower total surpluses, it is unambiguous that the aggregate surplus is increased. When a buyer increases his or her attribute, a number of the sellers are matched with higher attribute buyers following the increase while none is matched with a lower attribute buyer. Hence, the increase in the social value is the sum of the increases in the total surplus of those pairs with sellers matched with higher attribute buyers after the increase.

These externalities may lead individuals to either overinvest or underinvest from a social perspective. While it is true that in general changes in attribute can result in changes to the individual's payoff that differ from the change in social value, it is not the case in this example. The particular pattern of overlapping attributes for the vectors of attributes results in each of the players whose attribute is unchanged getting the same payoff after the specified player's change as before. Since no other agent's payoff is changed by the buyer increasing his attribute, it follows trivially that this buyer captures the full social value of the attribute change. The qualitative characteristics of this example are quite general as shown by the next proposition (which is proved in the appendix).

Proposition 4. *Let* $(\mathbf{b}, \mathbf{s})$ *and* $(\mathbf{b}', \mathbf{s})$ *denote two vectors of attributes satisfying* $b_i = b_i'$, $\forall i \neq \ell$. *Let* $\{(\mathbf{x}, \mathbf{p}), m\}$ *denote a stable payoff and matching for* $(\mathbf{b}, \mathbf{s})$, *and* $\{(\mathbf{x}', \mathbf{p}'), m'\}$ *a stable payoff and matching for the attributes* $(\mathbf{b}', \mathbf{s})$. *If* $p_{m(\ell)} = p'_{m(\ell)}$ *and* $p_{m'(\ell)} = p'_{m'(\ell)}$, *then*

$$x'_\ell = x_\ell + V(\mathbf{b}', \mathbf{s}) - V(\mathbf{b}, \mathbf{s}).$$

Definition 8. *The pair of attribute vectors* $(\mathbf{b}, \mathbf{s})$ *is* ***doubly overlapping*** *if* $(\mathbf{b}, \mathbf{s})$ *is overlapping and each matched pair of attributes appears at least twice.*

Corollary 2. *Let* $(\mathbf{b}, \mathbf{s})$ *and* $(\mathbf{b}', \mathbf{s})$ *denote two vectors of attributes satisfying* $b_i = b_i'$, $\forall i \neq \ell$. *Let* $\{(\mathbf{x}, \mathbf{p}), m\}$ *denote a stable payoff and matching for* $(\mathbf{b}, \mathbf{s})$, *and* $\{(\mathbf{x}', \mathbf{p}'), m'\}$ *a stable payoff and matching for the attributes* $(\mathbf{b}', \mathbf{s})$ *satisfying* $x'_{\underline{b}} = x_{\underline{b}}$. *If* $(\mathbf{b}, \mathbf{s})$ *is doubly overlapping, then*

$$x'_\ell = x_\ell + V(\mathbf{b}', \mathbf{s}) - V(\mathbf{b}, \mathbf{s}).$$

Proof. If $(\mathbf{b}, \mathbf{s})$ is doubly overlapping, then the vector of attributes following any single agent's change of attribute is overlapping. It is straightforward to see that if $x'_{\underline{b}} = x_{\underline{b}}$, the construction in corollary1 results in $p_{m(\ell)} = p'_{m(\ell)}$ and $p_{m'(\ell)} = p'_{m'(\ell)}$, and hence the proposition applies. ■

The proposition and corollary provide sufficient conditions that rule out one source of inefficiency in investments. If the attribute choice vector is doubly overlapping, each agent captures exactly the incremental aggregate surplus that results from his attribute choice. Competition among future potential partners eliminates any "holdup problem" that might arise due to the investment choice being made prior to matching and bargaining over the surplus.

Double overlap is not necessary for agents to receive the correct incentives for efficient attribute choice; there are trivial examples for which double overlap may fail, yet Proposition 4 still holds. There are, however, trivial examples for which there are equilibria for which an agent will not capture the change in surplus that results from a change in his attribute when the conditions for Proposition 4 fail.

It is important to note that Proposition 4 does *not* say that when the hypotheses of the proposition hold, the outcome is efficient. The proposition only guarantees that any inefficiency in the investment choices does not stem from a single person's decision. There remains the possibility of inefficiencies due to coordination failures resulting from the choices of multiple agents. For example, if we consider the surplus function that we have used in the examples above, $v(b, s) = b \cdot s$, it is clearly an equilibrium for all buyers and sellers to choose attribute 0 if the cost of choosing this attribute is 0, regardless of the cost of higher investment levels. The problem, of course, is that unilateral deviations from no investment have no value. We will show in the next section, however, that for any investment problem, there is always one equilibrium for which each agent will capture precisely the change in surplus that results from a change in attribute, and further, that no pair of agents can *jointly* change their attributes in a way that increases their surplus, net of investment cost (or other set of agents for that matter).

6. Ex ante contracting equilibrium

We now compare the investments taken in an ex post contracting equilibrium with the investments agents would make if buyers and sellers could contract with each other over matches, the investments to be undertaken, and the sharing of the resulting surplus. If a buyer i and seller j agree to match and make investments b and s respectively, then the total surplus so generated is $v(b,s)-\psi(b,i)-c(s,j)$. In a world of ex ante contracting, investments maximize this total surplus. Thus, if buyer i and seller j are considering matching, they are bargaining over the surplus $\varphi(i,j) = \max_{b,s} v(b,s) - \psi(b,i) - c(s,j)$. The *ex ante assignment game* is the assignment game with the population I of buyers, J of sellers, and value function φ. Just as we considered stable outcomes for the ex post assignment, we impose stability on outcomes of the ex ante assignment game. A stable outcome, together with the implied attribute investments, is an *ex ante contracting equilibrium*:

Definition 9. *The outcome of the ex ante assignment game* $\{m^*, (\mathbf{b}^*, \mathbf{s}^*), (\mathbf{x}^*, \mathbf{p}^*)\}$ *is an* ***ex ante contracting equilibrium (EACE)*** *if*

1. $(b_i^*, s_{m^*(i)}^*)$ *maximizes* $v(b,s) - \psi(b,i) - c(s, m^*(i))$ *if* $m^*(i) \in J$;
2. $(\mathbf{x}^*, \mathbf{p}^*)$ *is feasible for* m^*; *and*
3. *for all* $i \in I$ *and* $j \in J$,
$$x_i^* - \psi(b_i^*, i) + p_j^* - c(s_j^*, j) \geq \varphi(i,j).$$

Since the ex ante assignment game is a finite assignment game, ex ante contracting equilibria exist (see footnote 4). It is immediate that $(\mathbf{x}^*, \mathbf{p}^*)$ is a stable payoff of the ex post assignment game associated with the attribute vector $(\mathbf{b}^*, \mathbf{s}^*)$.

We pointed out in the previous section that investments could be inefficient. Given the bargaining outcome function in the equilibrium, some agents might not be able to capture the incremental surpluses that would result from altering their investments in attributes. Further, regardless of the bargaining outcome function, there may be coordination failures in which Pareto improvements are possible, but only if pairs of agents jointly change their attributes.

We should not be surprised that an inability to contract over investment choices in the presence of complementarities can lead to inefficiency. Indeed, one might expect that in such an environment inefficiency is inevitable, but this is not the case. The following proposition states that any outcome achievable under ex ante contracting is part of an ex post contracting equilibrium.

Proposition 5. *Given an ex ante contracting equilibrium $\{m^*, (\mathbf{b}^*, \mathbf{s}^*), (\mathbf{x}^*, \mathbf{p}^*)\}$, there exists g^* such that $(g^*, (\mathbf{b}^*, \mathbf{s}^*))$ is an ex post contracting equilibrium.*

Proof. If necessary, relabel buyers and sellers so $I = J = \{1, \ldots, n\}$ and $m^*(i) = i$. Define $g^*(\mathbf{b}^*, \mathbf{s}^*) = (\mathbf{x}^*, \mathbf{p}^*)$. Since ex post contracting equilibria are Nash equilibria, we need only be concerned with unilateral deviations (any specification of g^* for multilateral deviations consistent with the definition of an ex post contracting equilibrium will work). Consider then an attribute vector $(\mathbf{b}^*_{-\ell}, b_\ell, \mathbf{s}^*)$ for some $b_\ell \in B$ and $\ell \in I$ (the extension of g^* to a deviation by a seller is identical). Denote the stable payoff we are defining by $(\mathbf{x}, \mathbf{p})$.

Suppose $b_\ell < b^*_\ell$, and let i' satisfy $b^*_{i'-1} < b_\ell \leq b^*_{i'}$ (where $b^*_0 \equiv -1$); clearly $i' \leq \ell$. Since stable matchings are positively assortative in attributes, $m(i) = i$ for $i < i'$, $m(i) = i + 1$ for $i \geq i'$, $i \neq \ell$, and $m(\ell) = i'$ is a stable matching for $(\mathbf{b}^*_{-\ell}, b_\ell, \mathbf{s}^*)$. Since $m(i) = m^*(i)$, we can set $(x_i, p_i) = (x^*_i, p^*_i)$ for $i < i'$. Set

$$p_{i'} = v(b_\ell, s^*_{i'}) - v(b_\ell, s^*_{i'-1}) + p^*_{i'-1},$$

(this is the most that seller i' can receive consistent with stability and $p_{i'-1}$—Proposition 1),[8] and then complete g^* as described above. Before considering $b_\ell > b^*_\ell$, we show that $b_\ell < b^*_\ell$ is not a profitable choice with this specification. The difference in payoffs is

$$x^*_\ell - \psi(b^*_\ell, \ell) - \{v(b_\ell, s^*_{i'}) - p_{i'} - \psi(b_\ell, \ell)\}$$

[8] If $i' = \ell$, then there is no rematching as a result of the lower attribute choice, and p^*_i may be feasible in a stable outcome. If it is, then setting $p_i = p^*_i$ also works.

$$
\begin{aligned}
&= x_\ell^* - \psi(b_\ell^*, \ell) - \{v(b_\ell, s_{i'-1}^*) - p_{i'-1}^* - \psi(b_\ell, \ell)\} \\
&= x_\ell^* - \psi(b_\ell^*, \ell) + p_{i'-1}^* - \{v(b_\ell, s_{i'-1}^*) - \psi(b_\ell, \ell)\} \\
&\geq \varphi(\ell, i'-1) - \{v(b_\ell, s_{i'-1}^*) - \psi(b_\ell, \ell) - c(s_{i'-1}^*, i'-1)\} \geq 0,
\end{aligned}
$$

where the first inequality comes from the stability of ex ante contracting outcomes in the ex ante assignment game and the second from the definition of φ.

Now, suppose $b_\ell > b_\ell^*$, and now let i' satisfy $b_{i'}^* < b_\ell \leq b_{i'+1}^*$ (where $b_{n+1}^* \equiv \infty$); clearly $i' \geq \ell$. Set $m(i) = i$ for $i < \ell$, $m(i) = i - 1$ for $\ell < i \leq i'$, $m(\ell) = i'$, and $m(i) = i$ for $i \geq i' + 1$. As before, for $i < \ell$, we set $(x_i, p_i) = (x_i^*, p_i^*)$. Potentially all the matches between seller ℓ and seller i' (inclusive) involve the seller being matched with a new buyer attribute than under m^*. Moreover, all these sellers are matching, under m, with buyers whose attributes are at least as large as under m^*. Then it is still feasible (and stable) to set $p_\ell = p_\ell^*$ and $x_{\ell+1} = v(b_{\ell+1}^*, s_\ell^*) - p_\ell$ (note that $x_{\ell+1} \leq x_{\ell+1}^*$). We now proceed inductively, setting $p_{i+1} = v(b_{i+2}^*, s_{i+1}^*) - (v(b_{i+2}^*, s_i^*) - p_i)$ and $x_{i+2} = v(b_{i+2}^*, s_i^*) - p_i$ for $i \geq \ell$. By Proposition 1, we have described a stable outcome of the ex post assignment game associated with $(\mathbf{b}_{-\ell}^*, b_\ell, \mathbf{s}^*)$. Moreover, $p_{i'} \geq p_{i'}^*$. [The proof is by induction. Note that $p_\ell \geq p_\ell^*$ and suppose $p_i \geq p_i^*$. Then, $p_{i+1} = v(b_{i+2}^*, s_{i+1}^*) - x_{i+2} = v(b_{i+2}^*, s_{i+1}^*) - v(b_{i+2}^*, s_i^*) + p_i \geq v(b_{i+2}^*, s_{i+1}^*) - v(b_{i+2}^*, s_i^*) + p_i^* \geq v(b_{i+1}^*, s_{i+1}^*) - v(b_{i+1}^*, s_i^*) + p_i^* \geq v(b_{i+1}^*, s_{i+1}^*) - x_{i+1}^* = p_{i+1}^*$ (where the first inequality follows from $p_i \geq p_i^*$, the second from the supermodularity of v, and the third from stability).] The difference in payoffs for buyer ℓ for the deviation to b_ℓ is then

$$
\begin{aligned}
& x_\ell^* - \psi(b_\ell^*, \ell) - \{v(b_\ell, s_{i'}^*) - p_{i'} - \psi(b_\ell, \ell)\} \\
\geq\; & x_\ell^* - \psi(b_\ell^*, \ell) - \{v(b_\ell, s_{i'}^*) - p_{i'}^* - \psi(b_\ell, \ell)\} \\
=\; & x_\ell^* - \psi(b_\ell^*, \ell) + p_{i'}^* - \{v(b_\ell, s_{i'}^*) - \psi(b_\ell, \ell)\} \\
\geq\; & \varphi(\ell, i') - \{v(b_\ell, s_{i'}^*) - \psi(b_\ell, \ell) - c(s_{i'}^*, i')\} \geq 0,
\end{aligned}
$$

where the second inequality comes from the stability of ex ante contracting outcomes in the ex ante assignment game and the third from the definition of φ. ■

If the ex ante contracting attribute choices are doubly overlapping, then the result is trivial. The nontriviality comes from the possibility that there may be gaps in the attribute matchings (after a deviation), so that stability and the bottom pair do not uniquely determine attribute payoffs. This indeterminacy is important; it is because of this indeterminacy that we do not interpret Proposition 5 as a strong positive result. It is true that for any outcome that is supportable as part of an EACE, there is an EPCE yielding the same investments and payoffs. But the EPCE that does this depends crucially on the bargaining outcome function. The issue is the interpretation of bargaining outcome function. We suggested above that we could think of it as generally determining how surplus is shared subject to the constraints of competition implicit in stability. For some problems sellers might capture most of this and in others, it may be the buyers. But in Proposition 5, the bargaining outcome function responds to changes in the underlying investment problem (e.g., changes in the costs of investment ψ or c), since it depends upon $(\mathbf{b}^*, \mathbf{s}^*)$.

While the indeterminacy is eliminated if the ex ante contracting attribute choices are doubly overlapping, there is good reason not to expect double overlap. Typically, if each agent has different costs of acquiring attributes and attributes are continuous variables, then the efficient attribute choices $(\mathbf{b}^*, \mathbf{s}^*)$ will not be doubly overlapping. Proposition 3, on the other hand, provides bounds that suggest that as the set of chosen attributes become sufficiently rich (in the sense that the set of attributes looks like an interval), the indeterminacy in stable payoffs disappears (in the limit, Lemma 1 would apply). However, attributes are endogenous and even if there are many agents, the set of chosen attributes may *not* be rich. The complementarity of attributes means that in general (in particular, when the complementarity is strong), in the limit the set of efficient attributes forms a disconnected set (footnote 9 in Section 7 contains an example). Consider an increasing sequence of finite populations of agents, with the space of their exogenous characteristics becoming increasingly rich (so that in the limit, every agent has close competitors, in the sense that the limit space of characteristics is an interval). The efficient attributes along the sequence must then eventually fail to be doubly overlapping, and so the failure of double overlapping is not a "small numbers" problem.

6.1. Inefficient investment: underinvestment

We mentioned at the end of Section 5 that ex post contracting equilibrium outcomes might easily be inefficient (the example of $v(b,s) = b \cdot s$ and all buyers and sellers choosing attribute 0). While having all agents choose attribute 0 is a particularly simple way to illustrate the possibility of inefficiency, it isn't difficult to construct examples in which all agents are choosing positive attributes. In fact, we can modify any investment problem to generate inefficiency; moreover, this inefficiency cannot eliminated by *any* restrictions on the bargaining outcome function.

Fix an investment problem $\Gamma = \{I, J, B, S, \psi, c, v\}$ and define $B' \equiv B \cup \{b'\}$ and $S' \equiv S \cup \{s'\}$, where $b' > \bar{b} \equiv \max B$ and $s' > \bar{s} \equiv \max S$. Extend the definition of v to $B' \times S'$ by setting $v(b, s') = v(b, \bar{s})$ for all $b \in B$ and $v(b', s) =$

$v(\bar{b}, s)$ for all $s \in S$, and $v(b', s') = v(\bar{b}, \bar{s}) + \max_i \psi(\bar{b}, i) + \max_j c(\bar{s}, j) + 2a + 1$, where $a > v(\bar{b}, \bar{s})$. Extend the cost functions by setting $\psi(b', i) = \psi(\bar{b}, i) + a$ for all $i \in I$ and $c(s', j) = c(\bar{s}, j) + a$ for all $j \in J$. Note that, unless both the buyer and the seller in a pair choose the added elements b' and s', the new attributes are simply expensive substitutes for $\bar{b}$ and $\bar{s}$.

The only efficient outcome in the investment problem $\Gamma' = \{I, J, B', S', \psi, c, v\}$ is for every buyer to choose b' and every seller s' (since $v(b', s') - \psi(b', i) - c(s', j) = v(\bar{b}, \bar{s}) + \max_i \psi(\bar{b}, i) + \max_j c(\bar{s}, j) + 2a + 1 - \psi(\bar{b}, i) - a - c(s', j) = c(\bar{s}, j) - a \geq v(\bar{b}, \bar{s}) + 1)$.

Fix an ex post contracting equilibrium of Γ' and denote its bargaining outcome function by g. We claim that there is another ex post contracting equilibrium of Γ' with the *same* bargaining outcome function g that involves inefficient attribute choices. Consider the strategic form game implied by g on the attribute sets B and S. This has an equilibrium (perhaps in mixed strategies). Moreover, this will be an ex post contracting equilibrium of Γ': If all other agents are choosing attributes in B and S, then no matter how the bargaining outcome function divides the surplus, since $a > v(\bar{b}, \bar{s})$, there is insufficient total surplus to justify choosing the added attribute.

6.2. Inefficient investment: overinvestment

The previous subsection illustrated an ex post contracting equilibrium outcome with agents making inefficiently low investment in attributes. There is a similar possibility of overinvestment, but with an important difference. We first give a simple example with overinvestment

There are two buyers, $\{1, 2\}$ and two sellers, $\{1, 2\}$; the possible characteristics for buyers and sellers are $B = S = \{4, 6\}$. The surplus function is $v(b, s) = b \cdot s$. The cost functions are $\psi(4, i) = c(4, j) = 5$, $i, j = 1, 2$; $\psi(6, i) = c(6, j) = 16$, $i, j = 1, 2$. The efficient attribute choices are for all buyers and sellers to choose attribute level 4. These efficient choices can be part of an EPCE. Suppose that when all agents choose attribute 4, the surpluses are shared as in the left of Figure 6.1, and as in the right of Figure 6.1 if a single agent (here, a buyer) deviates and chooses attribute 6.

Since a single agent switching from attribute 4 to attribute 6 decreases his net payoff from 3 to 0, the efficient choice of attribute level 4 for all agents is an EPCE. However, there may be another EPCE in which all agents overinvest, that is, all agents choose the high attribute level 6. Suppose that the payoffs resulting from all agents choosing attribute level 6 and those following a single agent deviating and choosing level 4 are as given in Figure 6.2.

$x_i - \psi$	3	3
x_i	8	8
b_i	4	4
i	1	2
j	1	2
s_j	4	4
p_j	8	8
$p_j - c$	3	3

$x_i - \psi$	3	0
x_i	8	16
b_i	4	6
i	1	2
j	1	2
s_j	4	4
p_j	8	8
$p_j - c$	3	3

Figure 6.1: The efficient equilibrium.

$x_i - \psi$	2	2
x_i	18	18
b_i	6	6
i	1	2
j	1	2
s_j	6	6
p_j	18	18
$p_j - c$	2	2

$x_i - \psi$	1	2
x_i	6	18
b_i	4	6
i	1	2
j	1	2
s_j	6	6
p_j	18	18
$p_j - c$	2	2

Figure 6.2: The overinvestment equilibrium.

These make clear that it is an EPCE for all agents to choose the inefficient attribute level 6. Note that there is a common bargaining outcome function g that supports (that is, is part of) both equilibria.

This illustrates that we can get inefficient overinvestment as well as inefficient underinvestment, but as we stated above, there is a difference between the two cases. For the example in the previous section illustrating an equilibrium with underinvestment, we pointed out that the inefficiency would arise regardless of the bargaining rule g (that is, there was no g for which the underinvestment outcomes would not be an equilibrium.

We conjecture that there are bargaining outcome functions g that will preclude overinvestment for many investment problems. For example, consider a finite symmetric investment problem. Suppose, moreover, that the net surplus function $v(b,s)-\psi(b,i)-c(s,j)$ is concave in attributes. It can be shown that every EPCE with the following g cannot involve overinvestment. If the vector of attributes $(\mathbf{b},\mathbf{s})$ is such that $b=s$, g divides the surplus equally for each pair. For a vector $(\mathbf{b},\mathbf{s})$ in which $b \neq s$, let i be the first pair (under assortative matching) for which $b_i \neq s_i$. The agent with the smaller attribute receives under g half the surplus that would have resulted had he been matched with an agent with the same attribute as his own, and his partner receives the residual. Define g for all buyers and sellers with smaller index to be equal division and for buyers and sellers with higher index, let g give the highest share to that side for which i has the smaller attribute consistent with this, and with stability. While we believe a similar bargaining outcome function will also work in the absence of concavity of the net surplus function, an investigation of this would take us too far afield.

7. Characterization of stable allocations for a continuum of agents

We now describe a model with a continuum of agents, analogous to that in the earlier sections with a finite set of agents, that allows us to analyze individual agents' behavior when they are negligible with respect to the aggregate, that is, when individual deviations leave other agents' payoffs unchanged.

The populations of buyers and sellers are each described by Lebesgue measure on the unit interval, so that $I = J = [0,1]$. We denote buyers' behavior by the function $\beta : [0,1] \rightarrow \Re_+$ and sellers' by the function $\sigma : [0,1] \rightarrow \Re_+$.

Stability is as before: (x,p) is *stable* if it is feasible and $x(i) + p(j) \geq v(\beta(i), \sigma(j))$ for all i and j (note that we have not defined feasibility for the continuum model as yet). Intuitively, stability should again require that matching be positively assortative in attributes. If β and σ are strictly increasing, we could then specify that i matches with $j = i$.

If the attribute functions, β and σ, are continuous and strictly increasing, and the matching is positively assortative in index, feasibility is adequately captured by the finite population pairwise feasibility requirement: $x(i)+p(i) \leq v(\beta(i), \sigma(i))$ for all i. However, since the surplus function is supermodular, there is no reason to believe that endogenous attribute choices will be continuous functions of agent characteristics. In particular, efficient attribute choices need not be continuous.[9] We therefore need to describe feasible payoffs when the attribute functions are increasing, but not necessarily continuous.

We begin by illustrating the issues through an example: Suppose first that $v(b,s) = b \cdot s$, $\beta(i) = 1+i$ for all i, $\sigma(j) = 1+j$ for all j, and matching is positively assortative by index (equivalently, by attribute). Then the bottom pair generates a surplus of 1 and equal division of the payoff for each pair is feasible under the pairwise feasibility requirement and stable. Suppose now the bottom buyer's attribute is 0 rather than 1 (i.e., $\beta(0) = 0$). The pairwise feasibility requirement forces $p(0) = 0$. However, the point of modelling the set of agents as a continuum is to eliminate the possibility that a single agent's actions affect the feasible payoffs available to other agents.

Consider the sequence of matchings $\{m_n\}_{n=2}^{\infty}$ where i matches with $j = i$, except that buyers 0 and $\frac{1}{n}$ exchange partners.[10] If payoffs under m_n are determined by equal division of the induced surpluses, then the payoffs for all agents, *except* buyer 0, converge to the payoffs they received under equal division when $\beta(0) = 1$. This *includes* seller 0. Thus, there is a sequence of matchings that yield payoffs that satisfy the pairwise feasibility requirement, and yet their limit does not. Note, moreover, that in the case $\beta(0) = 0$, the pairwise feasibility re-

[9] For example, suppose the cost functions are given by $\psi(b,i) = b^3/i$, and $c(s,j) = s^3/(8j)$. The surplus function is

$$v(b,s) = \begin{cases} b \cdot s, & \text{if } b \cdot s < \frac{1}{2}, \\ (3 - 4 \cdot b \cdot s) \cdot b \cdot s + (4 \cdot b \cdot s - 2)\left(2 \cdot b \cdot s - \frac{14}{27}\right), & \text{if } \frac{1}{2} \leq b \cdot s \leq \frac{3}{4}, \\ 2 \cdot b \cdot s - \frac{14}{27}, & \text{if } b \cdot s > \frac{3}{4}. \end{cases}$$

Aggregate net surplus is maximized by matching buyer i with seller $j = i$. The net surplus maximizing choices of attribute are (β^*, σ^*) where

$$\beta^*(i) = \begin{cases} \frac{2i}{3}, & i < \frac{1}{2}, \\ \frac{4i}{3}, & i \geq \frac{1}{2}, \end{cases}$$

and

$$\sigma^*(j) = \begin{cases} \frac{4j}{3}, & j < \frac{1}{2}, \\ \frac{8j}{3}, & j \geq \frac{1}{2}. \end{cases}$$

[10] That is, $m_n : I \to J$ is given by $m_n(0) = \frac{1}{n}$, $m_n(\frac{1}{n}) = 0$, and $m_n(i) = i$ for all $i \neq 0, \frac{1}{n}$. Note that m_n is one-to-one and preserves measure.

quirement with stability forces $p(j) \to 0$ as $j \to 0$. At an intuitive level, we would like the bargaining outcome payoff (x^*, p^*), where $x^*(0) = 0$, $x^*(i) = (1+i)^2/2$ for $i > 0$, and $p^*(j) = (1+j)^2/2$ for all j, to be feasible and stable.[11]

As mentioned above, our goal in moving to a continuum of agents is to eliminate the effects that a single agent might have on the possible stable payoffs to other agents. We can accomplish this by altering the definition of feasible payoffs for a continuum of agents' attribute choices. Rather than giving a complete treatment of feasibility in assignment games with a continuum of agents and arbitrary attribute choice functions, we define feasibility in the simple case in which the attribute choice functions are strictly increasing, and positively assortative matching on index is effectively imposed.[12] Almost everywhere positive assortative matching by attribute can be deduced from stability and the notion of feasibility used by Gretsky, Ostroy, and Zame [7] or used by Kamecke [11]. Our notion of feasibility is:[13]

Definition 10. *Suppose β and σ are strictly increasing. A bargaining outcome (x, p) is* ***feasible*** *if*

$$x(i') \leq \max\left\{ \limsup_{j \to i'} \left[v(\beta(i'), \sigma(j)) - p(j) \right], 0 \right\},$$

[11] It is not critical to this example that the bottom buyer has chosen an isolated attribute. The same issue arises whenever there is a discontinuity in the attribute choice functions. Suppose for example that the buyer attribute choice function is discontinuous. We would like the set of sellers' feasible payoffs to be the same when the buyer attribute choice function only differs in whether it is continuous from the left or from the right.

[12] We will need to extend the notion of feasibility in the next section to cover the case where, due to a *single* agent's choice of attribute, the attribute function is not strictly increasing. This extension is obvious and trivial.

[13] This notion of feasibility differs from that in Gretsky, Ostroy, and Zame [7] and in Kamecke [11]. As we indicated above, our definiton only applies to positively assortative matchings, so we have not described feasibility for "most" matchings. Our definition has the important advantage that when combined with stability, it uniquely determines a single agent's payoff as a function of the other agents' payoffs. This is necessary if an agent is to compare payoffs from different attribute choices. The measure-theoretic notion of feasibility in Gretsky, Ostroy, and Zame [7], when combined with feasibility, does not force isolated attributes to have unique payoffs (when other agents' payoffs are fixed). The notion of feasibility in Kamecke [11] effectively requires that the attribute functions be continuous. Kamecke defines a bargaining outcome to be feasible if it can be approximated, in the sense of uniform convergence, by payoffs that are pairwise feasible. In our example, (x^*, p^*) would not be feasible under this notion. Simply requiring pointwise convergence, on the other hand, is too weak, since under this notion of feasibility, there are feasible and stable payoffs that violate equal treatment: Consider again the example, but with $\beta(i) = \sigma(j) = 1$ for all i and j. Let m_n be the matching described in footnote 10. The payoff (x_n, p_n) given by $x_n(0) = \frac{3}{4}$, $x_n(i) = \frac{1}{2}$, $p_n(\frac{1}{n}) = \frac{1}{4}$, and $p_n(j) = \frac{1}{2}$ is feasible for m_n. Moreover, it converges pointwise to the stable payoffs $(\tilde{x}, \tilde{p})$, where $\tilde{x}(0) = \frac{3}{4}$, $\tilde{x}(i) = \frac{1}{2}$, and $\tilde{p}(j) = \frac{1}{2}$.

and

$$p(j') \le \max\left\{ \limsup_{i \to j'} \left[v(\beta(i), \sigma(j')) - x(i) \right], 0 \right\}.$$

To motivate this definition, first note that if all the relevant functions (β, σ, x, and p) are continuous and the nonnegativity constraints are not binding, this reduces to the pairwise feasibility definition for positively assortative matching by index. Second, the role of the nonnegativity constraint (which we show below cannot bind almost everywhere) is to describe agents like buyer 0 in the example above. Finally, as in the example, with a continuum of agents, an agent i may not be matching with precisely $j = i$. Rather, he may be matching with agents arbitrarily close to $j = i$. Moreover, these matches may yield higher possible payoffs. Taking the lim sup captures these possibilities.[14]

It is immediate that the definition of stability implies that the inequalities in the definition of feasibility hold as equalities for stable payoffs. In the finite case, equal treatment implies that if stable payoffs have been fixed for all but one buyer (similar statements hold for sellers), and if that buyer has the same attribute as a second buyer, then that buyer's payoff is determined by the second buyer's payoff. There is a similar result for the continuum agent case. Suppose that stable payoffs have been fixed for all but one buyer. Then that buyer's payoff is determined by that of any other buyers whose attributes are arbitrarily close:

Lemma 1. *Suppose v is strictly supermodular and C^1. Suppose β and σ are strictly increasing. For any stable payoffs (x, p), x and p are strictly increasing (and so their left hand and right hand limits exist). Moreover, x and p inherit the continuity properties of β and σ, respectively (i.e., if β is continuous from the left at i', then x is continuous from the left at i', etc.).*

Proof. See appendix.

Let $C(\beta, \sigma)$ be the set of common continuity points of β and σ. By Lemma 1, for $i' \in C(\beta, \sigma)$, stable x and p are both continuous at i', and so $x(i') \le \max\{v(\beta(i'), \sigma(i')) - p(i'), 0\}$. Hence, $p(i') \le \max\{v(\beta(i'), \sigma(i')) - x(i'), 0\}$, implying $x(i')$, $p(i') \le v(\beta(i'), \sigma(i'))$ and so $x(i') + p(i') = v(\beta(i'), \sigma(i'))$. We can thus assume that buyer i with attribute $b = \beta(i)$ is matching with precisely seller $j = i$ with attribute $s = \sigma(i)$. This allows us to define the function $\tilde{s} : \beta(C(\beta, \sigma)) \to S$ given by $\tilde{s}(b) = \sigma(\beta^{-1}(b))$ and the function $\tilde{b} : \sigma(C(\beta, \sigma)) \to B$ given by $\tilde{b}(s) = \beta(\sigma^{-1}(s))$.

[14] We need to take the lim sup, rather than simply taking limits, because the limit does not exist when the attribute functions are discontinuous.

For $b \in \beta(C(\beta,\sigma))$, $\tilde{s}(b)$ is the attribute of the seller that the buyer with attribute b matches with (and similarly for the buyer attribute $\tilde{b}(s)$). It is also helpful to have specific notation for the share of the surplus that a particular attribute receives in a stable payoff (x,p). Suppose β and σ are strictly increasing. Define $\hat{x}(b) \equiv x(\beta^{-1}(b))$ and $\hat{p}(s) \equiv p(\sigma^{-1}(s))$. Equivalently, $(x,p) = (\hat{x}\circ\beta, \hat{p}\circ\sigma)$. The payoffs $(\hat{x},\hat{p})$ is *stable* if $(\hat{x}(\beta),\hat{p}(\sigma))$ is stable.

To simplify notation and eliminate a special case that, while straightforward to analyze, does not add anything substantive, we rule out isolated attribute choices in the statement of the characterization result (isolated attribute choices are addressed in footnote 17). We first make the following definition:

Definition 11. *A function is* **well-behaved** *if it is strictly increasing, discontinuous at only a finite number of points, differentiable where continuous, and at every point, either continuous from the left or from the right.*

We now characterize the stable payoffs of the assignment game for a particular class of attribute choice functions. Kamecke [11] has previously shown that stability implies (7.2) and (7.3) a.e. for general v when β and σ are continuous. As usual, $f(x+)$ denotes the right hand limit ($f(x+) = \lim_{\varepsilon\downarrow 0} f(x+\varepsilon)$) and $f(x-)$ denotes the left hand limit ($f(x-) = \lim_{\varepsilon\downarrow 0} f(x-\varepsilon)$).

Lemma 2. *Suppose v is strictly supermodular and C^1, and that β and σ are both well behaved. Stable payoffs (x,p) exist. The payoffs (x,p) are stable if and only if the following hold:*

1. *No waste:*
$$x(i) + p(i) = v(\beta(i),\sigma(i))\ \forall i \in C(\beta,\sigma), \tag{7.1}$$

2. *x and p are continuous at all $i \in C(\beta,\sigma)$,*

3. *$\hat{x}$ and $\hat{p}$ are differentiable on $\beta(C(\beta,\sigma))$ and $\sigma(C(\beta,\sigma))$ respectively, with derivatives*
$$\hat{x}'(b) = \frac{\partial v(b,\tilde{s}(b))}{\partial b} \text{ for all } b \in \beta(C(\beta,\sigma)), \text{ and} \tag{7.2}$$
$$\hat{p}'(s) = \frac{\partial v(\tilde{b}(s),s)}{\partial s} \text{ for all } s \in \sigma(C(\beta,\sigma)), \tag{7.3}$$

and

4. *at any point of discontinuity i,*

$$\begin{aligned} x(i+) + p(i+) &= v(\beta(i+), \sigma(i+)), \\ x(i+) - x(i-) &\geq v(\beta(i+), \sigma(i-)) - v(\beta(i-), \sigma(i-)), \text{ and} \\ p(i+) - p(i-) &\geq v(\beta(i-), \sigma(i+)) - v(\beta(i-), \sigma(i-)). \end{aligned} \tag{7.4}$$

Proof. See appendix.

8. Ex post contracting equilibrium in the continuum

In this section, we analyze ex ante and ex post contracting equilibria of the continuum agent model. As in the finite player case, in the ex post contracting game agents make simultaneous choices of attribute prior to matching, followed by a stable payoff in the assignment game resulting from those choices. The ex ante contracting game is the game in which buyers and sellers contract with each other over matches, the investments to be undertaken, and the sharing of the resulting surplus.

As in the finite case, we need to determine the change in payoff to an agent who unilaterally changes his attribute. After such a deviation by buyer i', say, the attribute choice function β is no longer strictly increasing. However, it fails to be strictly increasing only because of a single agent's choice of attribute. Accordingly, we *assume* that all the agents' payoffs, except for buyer i', are determined as if β and σ are strictly increasing. That is, we consider the attribute choice functions $(\tilde{\beta}, \sigma)$, where $\tilde{\beta}(i) = \beta(i)$ for all $i \neq i'$ and $\tilde{\beta}(i')$ is any attribute b satisfying $\lim_{i \uparrow i'} \beta(i) \leq b \leq \lim_{i \downarrow i'} \beta(i)$. Note that if $\lim_{i \uparrow i'} \beta(i) < \lim_{i \downarrow i'} \beta(i)$, the indeterminacy of $\tilde{\beta}(i')$ can only be reflected in an indeterminacy in the payoff of seller i', and then only if σ is both discontinuous from the right and from the left at i'. Finally, the payoff of buyer i' is determined from:

$$x(i') = \max\left\{v(\beta(i'), \sigma(i'-) - p(i'-),\, v(\beta(i'), \sigma(i')) - p(i'),\, v(\beta(i'), \sigma(i'+) - p(i'+)\right\}. \tag{8.1}$$

We make some standard assumptions on the surplus and cost functions.

Assumption 8.1. *The surplus function $v : \Re^2_+ \to \Re_+$ is C^2 with $\partial v(b,s)/\partial b > 0$, $\partial v(b,s)/\partial s > 0$, $\partial^2 v(b,s)/\partial b \partial s > 0$, $\partial^2 v(b,s)/\partial b^2 < 0$, and $\partial^2 v(b,s)/\partial s^2 < 0$ for all $(b,s) \in \Re^2_+$.*

There exists $\overline{B} : [0,1] \to \Re_{++}$ such that the buyers' cost function $\psi : \{(b,i) \mid b \in [0, \overline{B}(i)), i \in [0,1]\} \to \Re_+$ is C^2 and satisfies $\psi(0,i) = 0$, $\partial \psi(0,i)/\partial b = 0$, for all $i \in [0,1]$, and $\partial \psi(b,i)/\partial b > 0$, $\partial^2 \psi(b,i)/\partial b^2 > 0$ and $\partial^2 \psi/\partial b \partial i < 0$ for $b > 0$. Moreover, $\lim_{b \to \overline{B}(i)} \psi(b,i) = \infty$.

There exists $\overline{S} : [0,1] \to \Re_{++}$ such that the sellers' cost function $c : \{(s,j) \mid s \in [0, \overline{S}(j)), j \in [0,1]\} \to \Re_+$ satisfies the same properties as ψ.

Assumption 8.1 implies the problem

$$\max_{b,s} v(b,s) - \psi(b,i) - c(s,i) \tag{8.2}$$

has an interior solution for all $i \in [0,1]$, and that any solution is strictly increasing in index. For the analysis that follows, it is convenient to make the following assumption:

Assumption 8.2. *There is a well-behaved pair of attribute choice functions, (β^*, σ^*), such that $(\beta^*(i), \sigma^*(i))$ solves (8.2) for all i.*

While this is a direct assumption on the efficient attribute choice functions, it is one that will be typically satisfied. Our first result is the counterpart of Proposition 5.

Proposition 6. *Under assumptions 8.1 and 8.2, there exists a bargaining outcome function g^* such that $(g^*, (\beta^*, \sigma^*))$ is an ex post contracting equilibrium.*

Proof. See appendix.

There is an important difference between Propositions 5 and 6. First consider the case of doubly overlapping ex ante efficient attribute vectors, $(\mathbf{b}^*, \mathbf{s}^*)$, for the finite population. Stable payoffs are determined completely by the division for the bottom pair of attributes and equal treatment. It is immediate that there is a bargaining function g^* such that $(g^*, (\mathbf{b}^*, \mathbf{s}^*))$ is an ex post contracting equilibrium: Since a single agent (i say) changing attribute in the doubly overlapping attribute vector $(\mathbf{b}^*, \mathbf{s}^*)$ does not remove any matched pair of attribute *values*,[15] the resulting attribute vector is still overlapping and so the original stable payoff is still stable (with, if necessary, the new attribute's payoff determined in the obvious way). Since no other agents's payoff is changed as a result of i's play, i captures the full value of any attribute change and so $(g^*, (\mathbf{b}^*, \mathbf{s}^*))$ is an ex post contracting equilibrium. The difficulty in the finite case arises in dealing with the possibility that $(\mathbf{b}^*, \mathbf{s}^*)$ may not be doubly overlapping. If a single agent's change in attribute results in a completely new attribute matching, the stable payoff to most attributes necessarily changes. Finally, the bargaining outcome function used in Proposition 5 only depends on the cost functions through $(\mathbf{b}^*, \mathbf{s}^*)$.

[15]While the attributes of some the agents' partners will be different as a result of the rematching, any pair of attribute *values* that was matched in the absence of the deviation will be matched when there is a deviation.

While this is somewhat misleading, since if the attributes are continuous variables, changes in ψ and c will necessarily affect $(\mathbf{b}^*, \mathbf{s}^*)$, there is a sense in which, given $(\mathbf{b}^*, \mathbf{s}^*)$, the cost functions do not affect the required bargaining outcome function.

The continuum case is quite different. By construction, any change of attribute by a single agent leaves all other payoffs unchanged, and a single agent's attribute choice has no impact on social value. None the less, there is a similarity between the case of doubly overlapping attributes in a finite population and the case of continuous attribute choice functions, β^* and σ^*, in a continuum population. For the continuum population, stable payoffs are determined completely by the division for the bottom pair of attributes and (7.2) and (7.3). The two marginal conditions, (7.2) and (7.3), essentially assert that each attribute is paid its marginal social value, and so it is not surprising that Proposition 6 holds in this case. Moreover, the definition of g^* is trivial, since it is given by the division for the bottom pair of attributes and (7.2) and (7.3), and (8.1) for deviating attributes outside the range of β^* and σ^*.

The case of discontinuous attribute choice functions in a continuum population is more interesting. As we noted at the beginning of the previous paragraph, any change of attribute by a single agent leaves all other payoffs unchanged, and so there is no problem in determining stable payoffs for the other agents. Suppose β^* (and so σ^*) are discontinuous at i. From (7.4), at i, there is a range of possible divisions that is consistent with stability. However, *only one* division is consistent with (β^*, σ^*) being an ex post contracting equilibrium, namely the division that makes the buyer indifferent between the choices $\beta^*(i-)$ and $\beta^*(i+)$ and, at the same time, makes the seller indifferent between $\sigma^*(i-)$ and $\sigma^*(i+)$:

$$x(i+) - \psi(\beta^*(i+), i) = x(i-) - \psi(\beta^*(i-), i)$$

and

$$p(i+) - c(\sigma^*(i+), i) = p(i-) - c(\sigma^*(i-), i).$$

(This division is feasible because the total net surplus at $i-$ equals that at $i+$). There is thus a sense in which the appropriate g^* is "special." Moreover, unlike the bargaining outcome function for the finite population case, given (β^*, σ^*), the bargaining outcome function does depend on the cost functions directly, as well as through their determination of (β^*, σ^*).

The following lemma captures the idea that the coordination failure exhibited by inefficient ex post contracting equilibria is not a failure of matching, but rather a failure of attribute choice.

Lemma 3. *Suppose (β^*, σ^*) are the attribute choice functions for an ex ante contracting equilibrium and $(\bar{\beta}, \bar{\sigma})$ are well-behaved attribute choice functions for* *an ex post contracting equilibrium. Suppose for buyer i, $\bar{\beta}(i) \neq \beta^*(i)$; then there does not exist j such that $\bar{\sigma}(j) = \sigma^*(i)$; similarly, if for seller j, $\bar{\sigma}(j) \neq \sigma^*(j)$, then there does not exist i such that $\bar{\beta}(i) = \beta^*(j)$.*

Proof: Since marginal costs of attributes are decreasing in index, $\bar{\beta}$ and $\bar{\sigma}$ are increasing in index. Since any stable matching is positively assortative in attributes, in any ex post contracting equilibrium, buyer i is matched with seller i. First note that for all i,

$$\bar{\beta}(i) \in \operatorname*{argmax}_b v(b, \bar{\sigma}(i)) - \psi(b, i) - \bar{p}(i) \tag{8.3}$$

and

$$\bar{\sigma}(i) \in \operatorname*{argmax}_s v(\bar{\beta}(i), s) - c(s, i) - \bar{x}(i). \tag{8.4}$$

[We prove the first. If it did not hold, then there exists $b \neq \bar{\beta}(i)$ such that

$$v(b, \bar{\sigma}(i)) - \psi(b, i) - \bar{p}(i) > v(\bar{\beta}(i), \bar{\sigma}(i)) - \psi(\bar{\beta}(i), i) - \bar{p}(i).$$

Since $(\bar{\beta}, \bar{\sigma})$ is an ex post contracting equilibrium,

$$v(\bar{\beta}(i), \bar{\sigma}(i)) - \psi(\bar{\beta}(i), i) - \bar{p}(i) \geq \sup_j v(b, \bar{\sigma}(j)) - \psi(b, i) - \bar{p}(j),$$

which implies

$$v(b, \bar{\sigma}(i)) - \bar{p}(i) > v(b, \bar{\sigma}(i)) - \bar{p}(i),$$

a contradiction.]

Suppose that for some buyer i', $\bar{\beta}(i') \neq \beta^*(i')$, while for some seller j', $\bar{\sigma}(j') = \sigma^*(i')$. Then $j' \neq i'$, since if $j' = i'$, (8.3) and the efficiency of (β^*, σ^*) imply that $\bar{\beta}(i')$ and $\beta^*(i')$ are maximizing the same strictly concave function, $v(b, \sigma^*(i')) - \psi(b, i')$, which is inconsistent with $\bar{\beta}(i') \neq \beta^*(i')$.

Suppose $j' > i'$ (the other case is handled similarly). Since $\partial^2\psi/\partial b\partial i < 0$, the solution to the problem $\max_b v(b, \sigma^*(i')) - \psi(b, i)$ is increasing in i, and hence, $\bar{\beta}(j') > \beta^*(i')$. Since $\partial^2 v/\partial b\partial s > 0$, the solution to $\max_s v(b, s) - c(s, j')$ is increasing in b, and consequently, $\bar{\sigma}(j') > \operatorname{argmax}_s v(\beta^*(i'), s) - c(s, j')$. Finally, $\partial^2 c/\partial s\partial j < 0$ implies that $\operatorname{argmax}_s v(\beta^*(i'), s) - c(s, j)$ is increasing in j, so that $\bar{\sigma}(j') > \sigma^*(i')$, a contradiction. Hence, there cannot be a seller with index than $j' > i'$ for which $\bar{\sigma}(j') = \sigma^*(i')$. ■

In addition, ex post contracting equilibria are "constrained" efficient for the continuum case. This is in contrast to the finite case in which ex post contracting equilibrium outcomes need not be "constrained" efficient, as is illustrated by the example in section 2.

Lemma 4. *Suppose $(\hat{\beta}, \hat{\sigma})$ are attribute choice functions for an ex post contracting equilibrium that are not part of an ex ante contracting equilibrium. Then for any blocking coalition $(\tilde{\imath}, \tilde{\jmath})$ with attribute choices $(\tilde{b}, \tilde{s})$, there does not exist i'such that $\tilde{b} = \hat{\beta}(i')$, nor does there exist j' such that $\tilde{s} = \hat{\sigma}(j')$.*

Proof: Suppose $(\tilde{\imath}, \tilde{\jmath})$ is a blocking coalition with attribute choices $(\tilde{b}, \tilde{s})$ and shares $(\tilde{x}, \tilde{p})$. Then,

$$\begin{aligned}
\tilde{x} + \tilde{p} &= v(\tilde{b}, \tilde{s}), \\
\tilde{x} - \psi(\tilde{b}, \tilde{\imath}) &> \hat{x}(\hat{\beta}(\tilde{\imath})) - \psi(\hat{\beta}(\tilde{\imath}), \tilde{\imath}), \text{ and} \\
\tilde{p} - c(\tilde{s}, \tilde{\jmath}) &> \hat{p}(\hat{\sigma}(\tilde{\jmath})) - c(\hat{\sigma}(\tilde{\jmath}), \tilde{\jmath}).
\end{aligned}$$

The proof is by contradiction. Suppose there exists j' such that $\tilde{s} = \hat{\sigma}(j')$. Since $\tilde{p} - c(\tilde{s}, \tilde{\jmath}) > \hat{p}(\hat{\sigma}(\tilde{\jmath})) - c(\hat{\sigma}(\tilde{\jmath}), \tilde{\jmath}) \geq \hat{p}(\tilde{s}) - c(\tilde{s}, \tilde{\jmath})$, we have $\tilde{p} > \hat{p}(\tilde{s})$, and so $\tilde{x} - \psi(\tilde{b}, \tilde{\imath}) = v(\tilde{b}, \tilde{s}) - \tilde{p} - \psi(\tilde{b}, \tilde{\imath}) < v(\tilde{b}, \tilde{s}) - \hat{p}(\tilde{s}) - \psi(\tilde{b}, \tilde{\imath})$. But stability, the hypothesis that stable payoffs to nondeviating players are unchanged, and the fact that $(\hat{\beta}, \hat{\sigma})$ is are part of an ex post contracting equilibrium implies that $v(\tilde{b}, \tilde{s}) - \hat{p}(\tilde{s}) - \psi(\tilde{b}, \tilde{\imath})$ is a lower bound on buyer $\tilde{\imath}$'s payoff in equilibrium, and so we have a contradiction. An identical argument, mutatis mutandis, shows that there cannot exist an i' such that $\tilde{b} = \hat{\beta}(i')$. ■

9. Discussion

We saw in the continuum case that when agents are choosing attribute choices efficiently, there may be a jump in the investments at some point. When the cost and surplus functions are well-behaved and there is a discontinuous increase in attribute for one side, there must also be a discontinuous increase in attribute for the other side as well. The gross payoff to agents must be discontinuous at the point of discontinuity as well: agents arbitrarily close will receive boundedly different payoffs in any efficient equilibrium. There is not, however, any discontinuity in utility; the increase in gross payoff is exactly offset by the increase in cost in acquiring the attribute necessary to attain that payoff. This is, of course, not surprising; when cost functions are continuous, if there was a discontinuous increase in payoff net of the cost of investment, agents just below the point of discontinuity would have an incentive to make the higher investment. This characteristic has the flavor of the argument that rents will be dissipated by agents' expending resources in competition for those rents.

It is important to note that this is not inefficient here, but rather may be a property of efficient choice of attributes. The variance of income (that is, gross payoffs) may be greater or less than the variance of utility (payoffs net of investment costs), depending on the cost functions. In the extreme case that all agents on one side of the market have nearly the same cost of acquiring any given attribute, there will be hardly any variance in utility while there may be large variance in income. Variance in utility is a direct consequence of differences across agents in acquiring given attributes. The degree to which such differences in cost translate into different *gross* payoffs depends on the shape of the surplus function, v.

We treat in this paper the case in which the relevant groups for production are pairs. We could easily have extended the analysis to cover the case in which production necessitated a group of people, one of each of a number of different types. With analogous assumptions on the surplus and cost functions, we would have had similar results regarding positive assortative matching, equal treatment, etc. An interesting extension that is not so direct is to treat the case in which groups may or may not have one of each type of agent with the surplus they generate depending on the composition of the group.

In our model matching is frictionless, that is, there is no cost in agents' searching out appropriate partners. It is clear that frictionless matching drives some of the qualitative results; for example, we would not expect to see perfectly assortative matching if matching is accomplished through costly search.[16] For given vectors of attributes for the agents, the matching and sharing that we focus

For many of the problems the model is meant to address—such as matching workers to firms—the process of matching and production is ongoing. That is, there is a sequence of periods in which matching may take place, and once matched, the pair may stay matched for several periods. A natural way to model such a problem would be with a new cohort of individuals on each side of the market entering each period, making investments in the first period of their lives and entering the matching market the next period. If the cost functions vary stochastically across cohorts, individuals who are looking for partners might find it profitable to defer matching until later periods in the hope of finding a better match. The static nature of our model clearly precludes an analysis of such behavior; extending it to such an environment would be difficult, but potentially quite interesting.

[16]See Burdett and Coles ([2]) for an analysis of such a model, although one in which attributes are given exogenously.

10. Related literature

Our focus is on whether agents have the right incentives in terms of their investment decisions, given that a core allocation of the induced assignment game will result. Since the core in this case coincides with the set of Walrasian allocations, a question related to ours is whether in a competitive equilibrium, agents have incentives to make efficient ex ante investments. This question has been addressed by Hart [9, 10], Makowski [14], and Makowski and Ostroy [15]. In these papers, there is a set of exogenously specified possible commodities that can be produced (or in [15], occupations that can be chosen). Firms first decide which goods to produce, and then, given these decisions, a price-taking equilibrium results. Hart and Makowski ask whether firms choose to produce an efficient mix of commodities. They both conclude that if firms are perfectly competitive, then the resulting equilibrium commodity choices are constrained efficient. Makowski and Ostroy [15] are interested in the role of full appropriation and noncomplementarities in leading to full efficiency.

Our work is distinguished from this work in two ways: First, the production technology available to our agents is very different, due to the matching and complementarities in attributes. Second, the matching process allows to us to focus on the bargaining between the agents and the role of outside options in affecting efficiency. On the other hand, the qualitative properties exhibited by the equilibria in their models are similar to those in our model.

Acemoglu [1] analyzes a model in which workers and firms are matched in which there may be inefficient underinvestment in human capital. The inefficiency in his model stems from the fact that while a matched firm and worker can contract over how the incremental surplus that would result from additional investment would be shared, there is an exogenously specified probability of a negative productivity shock to the pair that would necessitate rematching. It is assumed that it is impossible for workers to contract with future employers over the sharing of surplus, leading to inefficient underinvestment. The inefficiency in that paper is due to assumed labor market imperfections, namely that following the dissolution of a match, there is a costly search process to rematching. The frictions in the matching process prevent a worker from capturing the entire social value of his investment, leading to underinvestment. The model presented here differs in that we assume frictionless matching. We demonstrate that for both the case in which there is a finite number of agents and the case in which there is a continuum of agents, inefficient investment can occur in equilibrium.

Besides these papers, there are several other papers that are related, but less closely. Cole and Prescott [3] and Ellickson, Grodal, Scotchmer, and Zame [5],

[4] analyze general equilibrium models in which agents, in addition to exchanging goods, can belong to clubs, and care about the characteristics of the other members of the clubs. Their models allow more general possible groupings of agents than we do, but take the agents' characteristics as given. Farrell and Scotchmer [6] study formation of coalitions when output is divided equally and show existence and (generic) uniqueness of the core. When agents differ in ability, coalitions are inefficiently small. The inefficiency in their model arises from the heterogeneity of agents, and would not arise if the there were sufficiently many agents of each ability. MacLeod and Malcomson [13] study the hold-up problem associated with investment decisions taken prior to contracting and provide, in a specific model, the idea that ex ante investments will be efficient, as long as the investments are general and there are outside options. That investments in their model are general leads to competition for the individual making the investment, assuring him of the incremental surplus that results from the investment; this is similar to the effect of "local competition" in our overlap case above. Their model, however, doesn't give rise to the coordination inefficiencies in our model.

References

[1] D. Acemoglu. Training and innovation in an imperfect labour market. *Review of Economic Studies*, 64:445–464, 1997.

[2] K. Burdett and M. G. Coles. Marriage and class. *Quarterly Journal of Economics*, 112:141–168, 1997.

[3] H. L. Cole and E. C. Prescott. Valuation equilibrium with clubs. *Journal of Economic Theory*, 74:19–39, 1997.

[4] B. Ellickson, B. Grodal, S. Scotchmer, and W. Zame. Clubs and the market: Continuum economies. mimeo, University of California, Berkeley, 1997.

[5] B. Ellickson, B. Grodal, S. Scotchmer, and W. Zame. Clubs and the market: Large finite economies. mimeo, University of California, Berkeley, 1997.

[6] J. Farrell and S. Scotchmer. Partnerships. *Quarterly Journal of Economics*, 103:279–297, 1988.

[7] N. E. Gretsky, J. M. Ostroy, and W. R. Zame. The nonatomic assignment model. *Economic Theory*, 2:103–127, 1992.

[8] S. J. Grossman and O. Hart. The cost and benefits of ownership: A theory of vertical and lateral integration. *Journal of Political Economy*, 94:691–719, 1986.

[9] O. D. Hart. Monopolistic competition in a large economy with differentiated commodities. *Review of Economic Studies*, 46:1–30, 1979.

[10] O. D. Hart. Perfect competition and optimal product differentiation. *Journal of Economic Theory*, 22:279–312, 1980.

[11] U. Kamecke. On the uniqueness of the solution to a large linear assignment problem. *Journal of Mathematical Economics*, 21:509–521, 1992.

[12] M. Kaneko. The central assignment game and the assignment markets. *Journal of Mathematical Economics*, 10:205–232, 1982.

[13] W. B. MacLeod and J. M. Malcomson. Investments, holdup, and the form of market contracts. *American Economic Review*, 83:811–837, 1993.

[14] L. Makowski. Perfect competition, the profit criterion, and the organization of economic activity. *Journal of Economic Theory*, 22:222–242, 1980.

[15] L. Makowski and J. M. Ostroy. Appropriation and efficiency: A revision of the first theorem of welfare economics. *American Economic Review*, 85:808–827, 1995.

[16] M. Quinzii. Core and competitive equilibria with indivisibilities. *International Journal of Game Theory*, 13:41–60, 1984.

[17] O. E. Williamson. *The Economic Institutions of Capitalism.* Free Press, New York, 1985.

A. Appendix

Proof of Proposition 4. This will follow from the following 2 lemmata.

Lemma A. *Let $(\mathbf{b}, \mathbf{s})$ and $(\mathbf{b}', \mathbf{s})$ denote two vectors of attributes satisfying $b_i = b_i'$, $\forall i \neq \ell$. Let $\{(x,p),m\}$ denote a stable payoff and matching for $(\mathbf{b}, \mathbf{s})$, and $\{(x',p'),m'\}$ a stable payoff and matching for the attributes $(\mathbf{b}', \mathbf{s})$. If $(\mathbf{b}', \mathbf{s})$ are overlapping and $p_{m(\ell)} = p'_{m(\ell)}$, then*

$$x'_\ell = x_\ell + V(\mathbf{b}', \mathbf{s}) - V(\mathbf{b}, \mathbf{s}).$$

(A similar result holds for the sellers.)

Proof. Suppose $b'_\ell > b_\ell$ (the same argument applies, mutatis mutandis, to the case $b'_\ell < b_\ell$). Let κ' denote the rank order of b'_ℓ in $\mathbf{b}'$, i.e., $b'_\ell = b'_{(\kappa')}$, and let κ'' denote the rank order of $\min\{b_i : b_i > b_\ell\}$ in $\mathbf{b}'$. Since $(\mathbf{b}', \mathbf{s})$ has no gaps,

$$x'_\ell = x'_{b'_{(\kappa')}} = x'_{b'_{(\kappa'')}} + \sum_{k=\kappa''}^{\kappa'} [v(b'_{(k)}, s^k) - v(b'_{(k-1)}, s^k)], \tag{A.1}$$

where, for each k, $s^k = s_{m'(i)} = s_{m'(i')}$ and $b'_i = b'_{(k)}$, $b'_{i'} = b'_{(k-1)}$ for some positive assortative matching m' and $i, i' \in I$.

Since the only difference between the attribute vector $(\mathbf{b}, \mathbf{s})$ and $(\mathbf{b}', \mathbf{s})$ is that one worker has a higher attribute, the only *attribute* matchings that are different involve exactly one matching for each of the attributes $\{s^k : k = \kappa'', \ldots, \kappa'\}$. For each $k = \kappa'' + 1, \ldots, \kappa'$, one seller of attribute s^k matches with a worker with attribute $b'_{(k-1)}$ under $(\mathbf{b}, \mathbf{s})$, and matches with a worker with the next higher attribute $b'_{(k)}$ under $(\mathbf{b}', \mathbf{s})$. For $k = \kappa''$, one seller of attribute $s^{\kappa''}$ matches with a worker who has the same attribute (b_ℓ) as worker ℓ under $(\mathbf{b}, \mathbf{s})$, and matches with a worker with attribute $b'_{(\kappa'')}$ under $(\mathbf{b}', \mathbf{s})$. Thus,

$$V(\mathbf{b}', \mathbf{s}) - V(\mathbf{b}, \mathbf{s}) = v(b'_{(\kappa'')}, s^{\kappa''}) - v(b_\ell, s^{\kappa''}) + \sum_{k=\kappa''+1}^{\kappa'} [v(b'_{(k)}, s^k) - v(b'_{(k-1)}, s^k)].$$

Now, using $x'_{b'_{(\kappa'')}} + p'_{m(\ell)} = v(b'_{(\kappa'')}, s^{\kappa''})$ and $p'_{m(\ell)} = p_{m(\ell)}$, equation (A.1) can be rewritten as

$$x'_\ell \;=\; v(b'_{(\kappa'')}, s^{\kappa''}) - p_{m(\ell)} + \sum_{k=\kappa''}^{\kappa'} \left[v(b'_{(k)}, s^k) - v(b'_{(k-1)}, s^k)\right]$$

$$= \quad v(b'_{(\kappa'')}, s^{\kappa''}) - \left[v(b_\ell, s^{\kappa''}) - x_\ell\right] + \sum_{k=\kappa''}^{\kappa'} \left[v(b'_{(k)}, s^k) - v(b'_{(k-1)}, s^k)\right]$$

$$= \quad x_\ell + V(\mathbf{b}', \mathbf{s}) - V(\mathbf{b}, \mathbf{s}).$$

■

Lemma B. *Let* $(\mathbf{b}, \mathbf{s})$ *and* $(\mathbf{b}', \mathbf{s})$ *denote two vectors of attributes satisfying* $b_i = b'_i$, $\forall i \neq \ell$. *Let* $\{(x, p), m\}$ *denote a stable payoff and matching for* $(\mathbf{b}, \mathbf{s})$, *and* $\{(x', p'), m'\}$ *and* $\{(x'', p''), m'\}$ *be two stable payoffs and matchings for the attributes* $(\mathbf{b}', \mathbf{s})$. *If* $p_{m(\ell)} = p'_{m(\ell)}$ *and* $p_{m'(\ell)} = p''_{m'(\ell)}$, *then*

$$x'_\ell - x_\ell \leq V(\mathbf{b}', \mathbf{s}) - V(\mathbf{b}, \mathbf{s}) \leq x''_\ell - x_\ell.$$

(A similar result holds for the sellers.)

Proof. The bound on x''_ℓ is immediate, given the bound on x'_ℓ (reverse $\mathbf{b}$ and $\mathbf{b}'$). If $(\mathbf{b}', \mathbf{s})$ has no gaps, the value of x'_ℓ is determined uniquely once $p'_{m(\ell)}$ is fixed, and by Lemma A, the bound holds with equality.

Suppose now that $(\mathbf{b}', \mathbf{s})$ has gaps and $b'_\ell > b_\ell$ (the same argument applies, mutatis mutandis, to the case $b'_\ell < b_\ell$).

Consider the impact of buyer ℓ's attribute change in a related collection of buyers and sellers that is a combination of the buyer and seller attributes that are rematched. Let $I' = \{i : b_\ell \leq b_i \leq b'_\ell\}$, $J' = m(I')$ and $J'' = m'(I')$. Consider an economy $(\tilde{I}, \tilde{J}, (\tilde{\mathbf{b}}, \tilde{\mathbf{s}}))$ with $|\tilde{I}| = |\tilde{J}| = 2 \cdot |I'|$ buyers and sellers, and $\tilde{\mathbf{b}} = ((b_i)_{i \in I'}, (b_i)_{i \in I'})$ and $\tilde{\mathbf{s}} = ((s_j)_{j \in J'}, (s_j)_{j \in J''})$. (Note that $\{s : s = s_j,\ j \in J'\} = \{s : s = s_j,\ j \in J''\}$.) The attribute vector of buyers after buyer ℓ changes attribute is $\tilde{\mathbf{b}}' = ((b_i)_{i \in I'}, (b'_i)_{i \in I'})$. Observe that $(\tilde{\mathbf{b}}', \tilde{\mathbf{s}})$ has no gaps, and that $(\tilde{\mathbf{b}}', \tilde{\mathbf{s}})$ is the buyer-first extension of $(\mathbf{b}', \mathbf{s})$, apart from the bottom matched pair (but the seller's attribute in that pair is the same as in $(\mathbf{b}', \mathbf{s})$) and some repeated matched pairs. By Lemmas A and ,

$$x'_\ell \leq x_\ell + V(\tilde{\mathbf{b}}', \tilde{\mathbf{s}}) - V(\tilde{\mathbf{b}}, \tilde{\mathbf{s}}),$$

which yields the desired upper bound, because $V(\tilde{\mathbf{b}}', \tilde{\mathbf{s}}) - V(\tilde{\mathbf{b}}, \tilde{\mathbf{s}}) = V(\mathbf{b}', \mathbf{s}) - V(\mathbf{b}, \mathbf{s})$. ■

Proof of Lemma 1: We first argue that x and p are strictly increasing. Suppose there exists $i' < i$ such that $x(i') \geq x(i)$. For $\eta > 0$ small, let $\varepsilon = \frac{1}{2}\{v(\beta(i), \sigma(i' - \eta)) - v(\beta(i'), \sigma(i' - \eta))\}$. Since β is strictly increasing, $\varepsilon > 0$.

Moreover, since σ is also strictly increasing and v is strictly supermodular, $v(\beta(i),\sigma(j)) - v(\beta(i'),\sigma(j)) > 2\varepsilon$ for all $j > i' - \eta$. Feasibility implies that there exists $j \in (i' - \eta, i' + \eta)$ such that

$$x(i') \leq v(\beta(i'),\sigma(j)) - p(j) + \varepsilon,$$

and so

$$\begin{aligned} x(i) + p(j) &\leq x(i') + p(j) \leq v(\beta(i'),\sigma(j)) + \varepsilon \\ &\leq v(\beta(i),\sigma(j)) - \varepsilon, \end{aligned}$$

contradicting the stability of (x, p), and so x is strictly increasing. A similar argument applies to p.

Consider now the case of β continuous from the left at i'. Suppose $x(i') > \liminf_{i\uparrow i'} x(i)$. Let $\varepsilon = [x(i') - \liminf_{i\uparrow i'} x(i)]/4$. Suppose $\limsup_{j\to i'}[v(\beta(i'),\sigma(j)) - p(j)] > 0$. (If the reverse weak inequality held, $x(i') = 0$, contradicting the assumption that x jumps up at i'.) There exists j close to i' such that $x(i') + p(j) < v(\beta(i'),\sigma(j)) + \varepsilon$. Moreover, for i close to (but less than) i', $v(\beta(i'),\sigma(j)) \leq v(\beta(i),\sigma(j)) + \varepsilon$ and $x(i) + 3\varepsilon \leq x(i')$. Thus,

$$\begin{aligned} x(i) + p(j) &\leq x(i') + p(j) - 3\varepsilon \\ &< v(\beta(i'),\sigma(j)) - 2\varepsilon \\ &< v(\beta(i),\sigma(j)) - \varepsilon < v(\beta(i),\sigma(j)). \end{aligned}$$

But this contradicts stability, and so $x(i') \leq \liminf_{i\uparrow i'} x(i)$.

Now suppose $x(i') < \limsup_{i\uparrow i'} x(i)$. Note that this implies that $\limsup_{i\uparrow i'} x(i) > 0$. Let $\varepsilon = \left[\limsup_{i\uparrow i'} x(i) - x(i')\right]/4$. Since v is uniformly continuous, there exists i close to (but less than) i', such that $|v(\beta(i), s) - v(\beta(i'), s)| < \varepsilon$ for all $s \in S$. Moreover, i can be chosen so that $x(i') \leq x(i) - 3\varepsilon$. There also exists j close to i such that $x(i) + p(j) < v(\beta(i),\sigma(j)) + \varepsilon$. Thus,

$$\begin{aligned} x(i') + p(j) &\leq x(i) + p(j) - 3\varepsilon \\ &< v(\beta(i),\sigma(j)) - 2\varepsilon \\ &< v(\beta(i'),\sigma(j)) - \varepsilon < v(\beta(i'),\sigma(j)). \end{aligned}$$

But this also contradicts stability, and hence, $\liminf_{i\uparrow i'} x(i) \geq x(i') \geq \limsup_{i\uparrow i'} x(i)$; i.e., x is continuous from the right at $\beta(i')$.

The other possibilities are covered similarly. ■

Proof of Lemma 2: Let $\{i_1, i_2, \ldots, i_T\}$ be the discontinuity points of β and σ, and define $I_t = (i_t, i_{t+1})$ for $t = 1, \ldots, T-1$, $I_0 = [0, i_1)$, and $I_T = (i_T, 1]$. Then, $C(\beta,\sigma) = \cup_{t=0}^{T} I_t$.

Existence of stable payoffs is addressed after the characterization. We have already argued that the no waste and continuity conditions must hold for any stable payoffs. These in turn imply at any point of discontinuity i_t, $x(i_t-) + p(i_t-) = v(\beta(i_t-), \sigma(i_t-))$ and $x(i_t+) + p(i_t+) = v(\beta(i_t+), \sigma(i_t+))$. The two inequalities in (7.4) are then equivalent to the local stability conditions:

$$\begin{aligned} x(i_t+) + p(i_t-) &\geq v(\beta(i_t+), \sigma(i_t-)), \text{ and} \\ x(i_t-) + p(i_t+) &\geq v(\beta(i_t-), \sigma(i_t+)), \end{aligned}$$

which (from continuity) are clearly necessary. The local condition (7.3) follows from the observation that since the payoffs are stable, for $b' \in \beta(C(\beta, \sigma))$ and all $s \in \sigma(C(\beta, \sigma))$,

$$v(b', \tilde{s}(b')) - \hat{p}(\tilde{s}(b')) = \hat{x}(b') \geq v(b', s) - \hat{p}(s), \tag{A.2}$$

while (7.2) follows from fixing $s' \in \sigma(C(\beta, \sigma))$ in the same inequality and considering the value to the seller of matching with different buyers.

Now we turn to sufficiency. Fix a pair of nonnegative payoffs $(x(0), p(0))$ that satisfy

$$x(0) + p(0) = v(\beta(0), \sigma(0)).$$

Since any stable payoff must satisfy (7.2) and (7.3), we have

$$x(i) = x(0) + \int_{\beta(0)}^{\beta(i)} \frac{\partial v(b, \tilde{s}(b))}{\partial b} db, \text{ for } i \in I_0, \tag{A.3}$$

and

$$p(j) = p(0) + \int_{\sigma(0)}^{\sigma(j)} \frac{\partial v(\tilde{b}(s), s)}{\partial s} ds, \text{ for } j \in I_0. \tag{A.4}$$

Note that these equations determine $x(i_1-)$ and $p(i_1-)$. (We show below that (7.1), (A.3), and (A.4) are consistent.) It remains to extend x and p to the rest of $[0,1]$. As on I_0, (7.2) and (7.3) determine x and p on I_t once the initial values, $x(i_t+)$ and $p(i_t+)$, have been determined. Let $(x(i_t+), p(i_t+))$ be any pair of payoffs satisfying (7.4). If, for example, β is continuous at i_t, then $x(i_t+) = x(i_t-)$, and there is only one choice for $(x(i_t+), p(i_t+))$. The payoff for buyer i_t is then determined by the continuity property of β: if β is continuous from the left, then $x(i_t) = \beta(i_t-)$, while if β is continuous from the right, $x(i_t) = \beta(i_t+)$ (the same considerations apply for seller i_t)[17]

[17] If both β and σ are discontinuous from the left and the right at i_t, then any choice $(x(i_t+), p(i_t+))$ satisfying

$$x(i_t+) = \max\{v(\beta(i_t), \sigma(i_t-) - p(i_t-),\ v(\beta(i_t), \sigma(i_t)) - p(i_t),\ v(\beta(i_t), \sigma(i_t+) - p(i_t+)\} \tag{A.5}$$

We next verify feasibility for $i \in C(\beta,\sigma)$. Suppose $i \in I_t$. By assumption, $x(i_t+) + p(i_t+) = v(\beta(i_t+), \sigma(i_t+))$, and for $i \in I_t$,

$$\begin{aligned} x(i) + p(i) &= x(i_t+) + \int_{\beta(i_t+)}^{\beta(i)} \frac{\partial v(b, \tilde{s}(b))}{\partial b} db \\ & \quad + p(i_t+) + \int_{\sigma(i_t+)}^{\sigma(i)} \frac{\partial v(\tilde{b}(s), s)}{\partial s} ds \end{aligned}$$

$$= v(\beta(i_t+), \sigma(i_t+)) + \int_{i_t}^{i} \frac{dv(\beta(i), \sigma(i))}{di} di = v(\beta(i), \sigma(i)),$$

so each pair efficiently shares the surplus.

We now verify stability. Lemma 3 implies

$$x(i_t-) + p(i_{t+k}+) \geq v(\beta(i_t-), \sigma(i_{t+k}+)) \text{ for all } k.$$

If (x,p) is not stable, then there is a pair i and j satisfying $x(i) + p(j) < v(\beta(i), \sigma(j))$. Suppose $i \in I_t$ and $j \in I_{t+k}$, $k \geq 1$ (the case of i and j in the same continuity interval is an obvious modification of the following, as is the case in which i and j are reversed). Then,

$$\begin{aligned} x(i_{t+1}-) + p(j) &< x(i_{t+1}-) + v(\beta(i), \sigma(j)) - x(i) \\ &= v(\beta(i), \sigma(j)) + \int_{\beta(i)}^{\beta(i_{t+1}-)} \frac{\partial v(b, \tilde{s}(b))}{\partial b} db \\ &< v(\beta(i), \sigma(j)) + \int_{\beta(i)}^{\beta(i_{t+1}-)} \frac{\partial v(b, \sigma(j))}{\partial b} db \\ &= v(\beta(i_{t+1}-), \sigma(j)), \end{aligned}$$

where the second inequality comes from the strict supermodularity of v. But then,

$$\begin{aligned} x(i_{t+1}-) + p(i_{t+k}+) &< v(\beta(i_{t+1}-), \sigma(j)) - p(j) + p(i_{t+k}+) \\ &= v(\beta(i_{t+1}-), \sigma(j)) - \int_{\sigma(i_{t+k}+)}^{\sigma(j)} \frac{\partial v(\tilde{b}(s), s)}{\partial s} ds \\ &< v(\beta(i_{t+1}-), \sigma(j)) - \int_{\sigma(i_{t+k}+)}^{\sigma(j)} \frac{\partial v(\beta(i_{t+1}-), s)}{\partial s} ds \\ &= v(\beta(i_{t+1}-), \sigma(i_{t+k}+)), \end{aligned}$$

---

and

$$p(i_t+) = \max\left\{v(\beta(i_t-), \sigma(i_t)) - x(i_t-),\ v(\beta(i_t), \sigma(i_t)) - x(i_t),\ v(\beta(i_t+), \sigma(i_t)) - x(i_t+)\right\}, \tag{A.6}$$

is stable and feasible.

a contraction. Thus, (x,p) is stable. ■

Proof of Proposition 6: As in the proof of Lemma 2, i_t denotes the tth discontinuity point of β^* and σ^*. Then

$$\begin{aligned} v(\beta^*(i_t-),\sigma^*(i_t-)) \quad & -\psi(\beta^*(i_t-),i_t) - c(\sigma^*(i_t-),i_t) = \\ & v(\beta^*(i_t+),\sigma^*(i_t+) - \psi(\beta^*(i_t+),i_t) - c(\sigma^*(i_t+),i_t). \end{aligned} \tag{A.7}$$

Equilibrium requires

$$x(i_t+) - \psi(\beta^*(i_t+),i_t) = x(i_t-) - \psi(\beta^*(i_t-),i_t) \tag{A.8}$$

and

$$p(i_t+) - c(\sigma^*(i_t+),i_t) = p(i_t-) - c(\sigma^*(i_t-),i_t), \tag{A.9}$$

where $x(i_t+)$ $(x(i_t-))$ is the share of a buyer with attribute $\beta^*(i_t+)$ $(\beta^*(i_t-))$, and $p(i_t+)$ $(p(i_t-))$ is the share of a seller with attribute $\sigma^*(i_t+)$ $(\sigma^*(i_t-))$. If the stable payoffs do not satisfy these equalities, then clearly buyers and sellers close to i_t (either just above or just below) have an incentive to deviate.

It remains to show that the payoffs implied by (A.8) and (A.9) are consistent with stability. Now,

$$\begin{aligned} x(i_t+) + p(i_t-) &= x(i_t-) + p(i_t-) + \psi(\beta^*(i_t+),i_t) - \psi(\beta^*(i_t-),i_t) \\ &= v(\beta^*(i_t-),\sigma^*(i_t-)) + \psi(\beta^*(i_t+),i_t) - \psi(\beta^*(i_t-),i_t) \\ &\geq v(\beta^*(i_t+),\sigma^*(i_t-)), \end{aligned}$$

since $v(\beta^*(i_t-),\sigma^*(i_t-)) - \psi(\beta^*(i_t-),i_t) - c(\sigma^*(i_t-),i_t) \geq v(\beta^*(i_t+),\sigma^*(i_t-)) - \psi(\beta^*(i_t+),i_t) - c(\sigma^*(i_t-),i_t)$.

We need to show that (A.8) and (A.9) are sufficient for equilibrium. Fix $(x^*(0),p^*(0))$ such that $x^*(0)+p^*(0) = v(\beta^*(0),\sigma^*(0))$. The payoffs (x^*,p^*) are now obtained from (A.3), (A.4), (A.8), and (A.9). From Lemma 2, these payoffs are stable. These determine the payoffs to a buyer (seller) choosing any attribute in the range of β^* (σ^*). Attributes outside the range are dealt with directly through feasibility: Let b_t^* solve $v(b,\sigma^*(i_t+)) - p(i_t+) = v(b,\sigma^*(i_t-)) - p(i_t-)$ and set $b_0^* = 0$ and $b_{T+1}^* = \bar{B}(i)$ (and similarly for s_t^*). Then, $\beta^*(i_t-) < b_t^* < \beta^*(i_t+)$, $v(\beta^*(i_t-),\sigma^*(i_t+)) - p(i_t+) < v(\beta^*(i_t-),\sigma^*(i_t-)) - p(i_t-)$, and $v(\beta^*(i_t+),\sigma^*(i_t+)) - p(i_t+) > v(\beta^*(i_t+),\sigma^*(i_t-)) - p(i_t-)$. Then, for $b \in [\beta^*(i_t-),b_t^*]$, $\hat{x}(b) = v(b,\sigma^*(i_t-)) - p(i_t-)$, and for $b \in [b_t^*,\beta^*(i_t+)]$, $\hat{x}(b) = v(b,\sigma^*(i_t+)) - p(i_t+)$. Similar statements hold for the seller.

Consider now the buyer's problem (the argument for the seller is symmetric). We first argue that $\beta^*(i)$ is a maximizing attribute choice for buyer $i \in [i_t, i_{t+1}]$ from the attribute set $[b_t^*, b_{t+1}^*]$.[18] The problem for buyer $\hat{\imath}$ is to choose $b \in [b_t^*, b_{t+1}^*]$ to maximize

$$\hat{x}(b) - \psi(b, \hat{\imath}).$$

Consider first choices of $b \in (\beta^*(i_t+), \beta^*(i_{t+1}-))$. Since buyer $\hat{\imath}$'s payoff function is differentiable over that domain (by Lemma 2), any maximizing choice of $b \in (\beta^*(i_t+), \beta^*(i_{t+1}-))$ must satisfy the first order condition

$$\hat{x}'(b) = \frac{\partial \psi(b, \hat{\imath})}{\partial b}.$$

By construction,

$$\hat{x}'(\beta^*(i)) = \frac{\partial v(\beta^*(i), \sigma^*(i))}{\partial b} = \frac{\partial \psi(\beta^*(i), i)}{\partial b} \quad \forall i \in (i_t, i_{t+1}).$$

Suppose that $\hat{x}'(b) = \partial\psi(b,i)/\partial b$ for some $b \neq \beta^*(i)$, $b \in (\beta^*(i_t+), \beta^*(i_{t+1}-))$. Since $b \in (\beta^*(i_t+), \beta^*(i_{t+1}-))$, there exists $\tilde{\imath}$ with $\beta^*(\tilde{\imath}) = b$ and so

$$\frac{\partial \psi(b, i)}{\partial b} = \hat{x}'(b) = \frac{\partial \psi(b, \tilde{\imath})}{\partial b},$$

which is impossible, since $\partial\psi/\partial b$ is a strictly decreasing function of i. Thus, the first order condition has a unique solution in $b \in (\beta^*(i_t+), \beta^*(i_{t+1}-))$.

We now argue that $\beta^*(\hat{\imath})$ a local maximizer for $\hat{\imath}$. In what follows, partial derivatives are indicated by subscripts. It is enough to show that the second derivative of buyer $\hat{\imath}$'s payoff function is strictly negative. The second derivative is

$$v_{bb}(\beta^*(\hat{\imath}), \sigma^*(\hat{\imath})) + v_{bs}(\beta^*(\hat{\imath}), \sigma^*(\hat{\imath})) \left.\frac{d\tilde{s}}{db}\right|_{b=\beta^*(\hat{\imath})} - \psi_{bb}(\beta^*(\hat{\imath}), \hat{\imath}). \tag{A.10}$$

Now, $\left.\frac{d\tilde{s}}{db}\right|_{b=\beta^*(\hat{\imath})} = (d\sigma^*(\hat{\imath})/di)\,(d\beta^*(\hat{\imath})/di)^{-1}$ and $d\beta^*/di > 0$, so that (A.10) can be rewritten as

$$(d\beta^*(\hat{\imath})/di)^{-1}\left\{(v_{bb} - \psi_{bb})\left(\frac{d\beta^*}{di}\right) + v_{bs}\left(\frac{d\sigma^*}{di}\right)\right\} = (d\beta^*(\hat{\imath})/di)^{-1}\,\psi_{bi} < 0.$$

Thus, $\beta^*(i)$ is the unique optimal choice from $(\beta^*(i_t+), \beta^*(i_{t+1}-))$. By continuity, $\beta^*(i)$ is an optimal choice for $i = i_t$ and i_{t+1} from $[\beta^*(i_t+), \beta^*(i_{t+1}-)]$.

[18] The same argument shows that for buyers in the bottom interval $[0, i_1)$, $\beta^*(i)$ is optimal in the set $[0, b_1^*]$, and that for buyers in the top interval $(i_T, 1]$, $\beta^*(i)$ is optimal in the set $[b_T^*, \overline{B}(i)]$.

We now turn to choices of $b \notin (\beta^*(i_t+), \beta^*(i_{t+1}-))$. Since stable matchings require positive assortative matching in attributes, if buyer $\hat{\imath}$ chooses $b \in [b_t^*, \beta^*(i_t+))$, then he is effectively matched with the seller with attribute $\sigma^*(i_t+)$, while a choice of $b \geq \beta^*(i_{t+1}-)$ leads to a match with $\sigma^*(i_{t+1}-)$. Consider the first possibility. In the first case, $\hat{x}(b) = v(b, \sigma^*(i_t+)) - \hat{p}(\sigma^*(i_t+))$, while in the second, $\hat{x}(b) = v(b, \sigma^*(i_{t+1}-)) - p(\sigma^*(i_{t+1}-))$.

We first consider $b \leq \beta^*(i_t+)$ and argue to a contradiction. Suppose there exists $b \leq \beta^*(i_t+)$ such that

$$\hat{x}(\beta^*(\hat{\imath})) - \psi(\beta^*(\hat{\imath}), \hat{\imath}) < v(b, \sigma^*(i_t+)) - \hat{p}(\sigma^*(i_t+)) - \psi(b, \hat{\imath}).$$

Let $\epsilon \equiv v(b, \sigma^*(i_t+)) - \hat{p}(\sigma^*(i_t+)) - \psi(b, \hat{\imath}) - [\hat{x}(\beta^*(\hat{\imath})) - \psi(\beta^*(\hat{\imath}), \hat{\imath})] > 0$. Since $\hat{p}$ is continuous, there exists an $i < \hat{\imath}$ (and close to i_t) such that $|\hat{p}(\sigma^*(i)) - \hat{p}(\sigma^*(i_t+))| < \epsilon/2$. For this i,

$$v(\beta^*(i), \sigma^*(i)) - v(b, \sigma^*(i)) \geq \psi(\beta^*(i), i) - \psi(b, i) > \psi(\beta^*(i), \hat{\imath}) - \psi(b, \hat{\imath}),$$

where the first inequality follows from the optimality of (β^*, σ^*) for i, and the second from $\partial\psi/\partial b \partial i < 0$. Then,

$$\begin{aligned}
\hat{x}(\beta^*(\hat{\imath})) - \psi(\beta^*(\hat{\imath}), \hat{\imath}) &\geq \hat{x}(\beta^*(i)) - \psi(\beta^*(i), \hat{\imath}) \\
&= v(\beta^*(i), \sigma^*(i)) - \psi(\beta^*(i), \hat{\imath}) - \hat{p}(\sigma^*(i)) \\
&> v(b, \sigma^*(i)) - \psi(b, \hat{\imath}) - \hat{p}(\sigma^*(i)) \\
&> v(b, \sigma^*(i_t+)) - \psi(b, \hat{\imath}) - \hat{p}(\sigma^*(i_t+)) - \epsilon/2 \\
&= \hat{x}(\beta^*(\hat{\imath})) - \psi(\beta^*(\hat{\imath}), \hat{\imath}) + \epsilon - \epsilon/2,
\end{aligned}$$

which implies $0 \geq \epsilon$, a contradiction.

We now consider $b \geq \beta^*(i_{t+1}-)$. Note first that it is obviously a best reply for buyer i_{t+1} to choose $\beta^*(i_{t+1}-)$. Consider the difference between buyer i's payoff from following β^* and choosing b:

$$\Delta(i; b) \equiv \hat{x}(\beta^*(i)) - \psi(\beta^*(i), i) - [v(b, \sigma^*(i_{t+1}-)) - \hat{p}(\sigma^*(i_{t+1}-)) - \psi(b, i)].$$

Differentiating with respect to i yields:

$$\begin{aligned}
\frac{\partial \Delta(i; b)}{\partial i} &= (\hat{x}'(\beta^*(i)) - \psi_b(\beta^*(i), i)) \frac{d\beta^*}{di} - \psi_i(\beta^*(i), i) + \psi_i(b, i) \\
&= \left(\frac{\partial v(\beta^*(i), \sigma^*(i))}{\partial b} - \psi_b(\beta^*(i), i) \right) \frac{d\beta^*}{di} - \psi_i(\beta^*(i), i) + \psi_i(b, i) \\
&= \psi_i(b, i) - \psi_i(\beta^*(i), i) = \int_{\beta^*(i)}^{b} \psi_{bi} < 0,
\end{aligned}$$

so that if $\Delta(\hat{\imath}; b) < 0$ for some $b > \beta^*(i_{t+1}-)$, then $\Delta(i_{t+1}, b) < 0$, contradicting the optimality of $\beta^*(i_{t+1}-)$ for buyer i_{t+1}.

We now argue that $\beta^*(i)$ is a maximizing attribute choice for buyer $i \in [i_t, i_{t+1}]$ from the full attribute set $[0, \bar{B}(i)]$. Fix $i \in [i_t, i_{t+1}]$, $t \geq 1$, and consider an attribute $b \in [b^*_{t-1}, b^*_t)$. Then

$$x(i) - \psi(\beta^*(i), i) \geq \hat{x}(\beta^*(i_t+)) - \psi(\beta^*(i_t+), i),$$

and

$$x(i_t-) - \psi(\beta^*(i_t-), i_t) \geq \hat{x}(b) - \psi(b, i_t).$$

Combining these two inequalities with

$$x(i_t+) - \psi(\beta^*(i_t+), i_t) = x(i_t-) - \psi(\beta^*(i_t-), i_t)$$

gives

$$x(i) - \psi(\beta^*(i), i) \geq \hat{x}(b) - \psi(b, i_t),$$

and so

$$x(i) - \psi(\beta^*(i), i) \geq \hat{x}(b) - \psi(b, i),$$

that is, $\beta^*(i)$ is a maximizing choice for i from $[b^*_{t-1}, b^*_{t+1}]$. An obvious induction completes the argument. ∎