Penn Institute for Economic Research
Department of Economics
University of Pennsylvania
3718 Locust Walk
Philadelphia, PA 19103-6297

pier@ssc.upenn.edu
http://www.econ.upenn.edu/pier

PIER Working Paper 97-034

"Continuous Approximations of Stochastic Evolutionary Game Dynamics"

by

Valentina Corradi and Rajiv Sarin

Continuous Approximations of Stochastic Evolutionary Game Dynamics[1]

Valentina Corradi
Department of Economics
University of Pennsylvania
Philadelphia, PA 19104
U.S.A.

Rajiv Sarin
Department of Economics
Texas A&M University
College Station, TX 77843
U.S.A.

First Version: December 1995
This Version: June 1996

[1] We are grateful to Joel Sobel and Peyton Young for helpful comments.

Abstract

We derive continuous approximations of stochastic evolutionary dynamics in games. Depending on how we construct the continuous limit, we obtain a continuous approximation that is either an ordinary differential equation (ODE) or a stochastic differential equation (SDE). Our SDE approximation result provides the first derivation of a SDE from an underlying discrete stochastic evolutionary game model. In deriving both an ODE and a SDE limit from the same model, our results provide information regarding the conditions under which the different limits arise.

1 Introduction

In this paper we construct and study continuous approximations of a discrete stochastic model of evolutionary game dynamics. The state variable in the model is the proportion of players of different types in a large but finite population. The model describes the state variable evolving in discrete random steps over time. In the continuous limit the state variable evolves continuously over time.

In some of the continuous limits we construct the state variable evolves according to an ordinary differential equation (ODE).[1] In others the state variable evolves according to a stochastic differential equation (SDE).[2] The objective of this paper is to study the conditions under which the different limits obtain, and also manner and time horizon over which the limiting process approximates the underlying discrete stochastic model.

The manner in which we take the continuous limit plays an important role in determining which of the continuous limiting processes obtain. We show the importance of the *rate* at which the continuous limit is taken. Specifically, if the rate at which the continuous limit is taken is "fast", then in some situations the SDE limit obtains. When we obtain the SDE limit, the (interpolated) underlying discrete process weakly converges to a diffusion solution of a SDE over the whole real line. This contrasts with the manner in which the ODE approximates the interpolated process, in which case the (interpolated) process converges uniformly in probability to the ODE up to an arbitrary, given, finite time.

The ODE we obtain is the continuous time replicator equation of evolutionary game theory. The drift term in our SDE is given by the replicator expression. The variance of our diffusion term vanishes as we approach the boundaries of the state space. The vertices of the state space are absorbing. Hence, the drift term in our SDE is like that in Foster and Young (1990), and distinct from Fudenberg and Harris (1992), even while our diffusion term is distinct from either. We study the properties of the SDE we obtain in repeated normal form games, and compare these properties with others that have

[1] Continuous limits that are described by an ODE have recently been constructed by Binmore, Samuelson and Vaughn (1995), Börgers and Sarin (1995) and Boylan (1995).

[2] Binmore, Samuelson and Vaughn (1995), Cabrales (1993), Foster and Young (1990), and Fudenberg and Harris (1992) have studied related SDEs in evolutionary games. However, none of these authors have derived the SDE from an underlying discrete evolutionary game model.

been studied in evolutionary game theory.

In our model there is a large but finite population of individuals. The individuals may be of a finite number of *types.* We think of the type of an individual as describing the pure strategy the individual plays. The initial population composition is assumed to be given. There are two distinct sources of randomness in the stochastic path followed by the state variable of the discrete model.

First, there is the *background* or *payoff independent* noise. This arises as the population next period is composed of a proportion of the present population and a randomly selected proportion of other individuals. The manner in which some individuals are randomly selected to be a part of next periods population is *selection with replacement.* In selection with replacement a pair of individuals is selected, one after the other, at random from the population. The selected pair is replaced into the population before the next pair is selected in the same manner. It is clearly possible that an individual is selected more than once in any period. We think of this as reflecting that some individuals may be more "active" than others in the period.[3]

The second source of randomness is *payoff dependent.* The payoff dependent randomness arises as each randomly selected pair of individuals is *matched* to play a finite normal form game. Payoffs in the game are offspring. After the agents have played the game, a number of them, equal to the number of newborns and excluding the newborns born in the period, are randomly selected and killed. The death protocol we consider is *death without replacement*, which involves randomly selecting an individual to die, before selecting the next to die, until the total number to die have been killed.

The *discrete* change in the state in any period depends both on the proportion of population that is selected in any period to be part of next periods population and also on the payoffs the selected individuals receive from playing the game. This enables us to construct alternative continuous limits, each of which involves "shrinking" the size of the change, or step size, of the state variable as we reduce the "real" time between

[3]An alternative matching protocol in which individuals cannot be selected to play more than once in any period, which we call *selection without replacement*, has recently been considered by Boylan (1995). In selection without replacement any pair selected in a period is not replaced into the population before the next pair is selected. In Proposition 2, in Section 3, we shows that a continuous limit when selection is without replacement is identical to the continuous limit when selection is with replacement.

consecutive periods. For our results, it is essential that we have both sources of change in the discrete model, and, hence, alternative ways of taking continuous limits.

The first continuous limit we consider involves reducing the proportion of pairs of individuals selected to play the game in every time interval, and simultaneously reducing the "real" time that elapses between consecutive rounds of the selection-birth-death process.[4] The continuous limit we obtain is an ODE. This equation is the usual continuous time replicator equation.[5] The intuition behind this result is that as a smaller proportion of individuals are selected and play in every interval of time and as the population grows larger, a law of large numbers applies in every interval of time. Thus, the selection-birth-death noise gets eliminated in the limiting process. We show that the same limit obtains regardless of whether selection is with replacement or is without replacement. This is true in either matching protocol because the two protocols "approach" one another as the proportion of individuals selected to play is reduced and the population grows large.

The second continuous limit we consider involves reducing the payoffs the individuals obtain with any particular play of the game and simultaneously reducing the "real" time that elapses between consecutive rounds of the selection-birth-death process. We reduce these at a fast enough rate. The proportion of individuals selected is as in the original model. The continuous limit we obtain is a SDE. The intuition for this result is that in any interval of time the same proportion of individuals is selected as in the original discrete process and, hence, the payoff independent randomness doesn't get eliminated even in the limit. Furthermore, the rate at which we take the continuous limit is fast enough to preserve the noise in the limiting process.

The third continuous limit we consider involves reducing the payoffs the individuals obtain with any particular play of the game and simultaneously reducing the "real" time that elapses between consecutive rounds of the matching-birth-death process *but* at a slower rate than we do in the second limit that we consider. The proportion of individuals matched is as in the original model. The continuous limit we obtain

[4]The precise manner in which this is done, in this and other continuous limits, is explained in Section 3.1 of the paper.

[5]In the discrete model, the *expected* change in the proportion of people of any particular type, conditional on the state of the population, is given by the replicator expression.

is an ODE. Hence, even while the limiting process preserves the payoff independent randomness, the continuous limit is deterministic. The slower rate at which we take the continuous limit explains the different result we get in this case.

Note that, in the first continuous limit, the "step size" or change in the state is reduced (as the "real" time between successive periods is reduced) by reducing the number of individuals selected to play. This leads to a limiting process where the selections are more evenly distributed along the real line than in the original discrete, finite population process. In contrast with the first limit, in the second and third limits, the "step size" or change in the state is reduced (as the "real" time between successive periods is reduced) by reducing the payoffs the individuals receive with any play of the game. This leads to a limiting process where selections are very frequent and payoffs are very small in any particular selection. We feel both kinds of limits may be appropriate according to the situation being modeled.[6]

In each case the limiting process is an ODE, our result regarding how it approximates the discrete (stochastic) model strengthen those in the literature. Specifically, the approximation results of Börgers and Sarin (1995, Proposition 4), and Boylan (1995, Proposition 1) are *pointwise* results. Roughly speaking, these results say that the "distance" between the state of the discrete model at any particular finite point in time and the solution of the ODE at that same point in time converges to zero, in probability. These results, in particular, say nothing about the supremum of the "distance" between the two processes at any intermediate time. Our results are *uniform*. They state that the supremum of the "distance" between the two processes goes to zero for all points in time upto a certain finite point in time, in probability. We directly prove the uniform convergence rather than show pointwise convergence and stochastic equicontinuity. It is known (e.g. Newey (1991)) that on a compact set pointwise convergence and stochastic equicontinuity imply (and are implied by) uniform convergence.

In contrast to the time horizon over which the ODE approximates the underlying discrete model, the SDE approximates the underlying model over the whole real line. Furthermore, the SDE convergence result is a distributional result. Specifically, when

[6]Boylan (1995) contructs a continuous limit by reducing the proportion of individuals who are matched in any particular round of play. Börgers and Sarin (1995) construct a continuous limit where the payoffs the agents obtain in any round are reduced.

we obtain the SDE limit, the (interpolated) underlying discrete model weakly converges to a diffusion solution of a SDE over the whole real line.

We do not consider the asymptotic properties of the ODE we obtain as they are well known (e.g. Hofbauer and Sigmund (1988), Weibull (1995)). However, we analyze the asymptotic properties of the SDE we obtain for the class of generic symmetric 2x2 games and compare these properties with the SDE studied by Fudenberg and Harris (1992). We show that our SDE asymptotically gets absorbed in the boundary of the state space, and that the probability with which it converges depends both on the initial state and on the payoffs in the game.

The paper is structured as follows. The next Section introduces the discrete model, and provides a Lemma regarding the moments of the conditional change in the state. In the first subsection of Section 3, we construct the continuous limits we consider. The next subsection contains our main results. Section 4 provides information on the asymptotics. Two Appendices contain proofs.

2 Discrete Model

There is a population of $2N$ individuals. The individuals can be one of $j \in \{1, ..., J\}$ *types*. We think of the type of an individual as describing the pure strategy the individual plays. In each period $n = 0, 1, 2, ...$ there are $N_j(n)$ individuals of type j in the population, where $\sum_{j=1}^{J} N_j(n) = 2N$. The state in the model in period n is given by $P^N(n) = \left(P_1^N(n), ..., P_J^N(n)\right)$, where $P_j^N(n) \equiv \frac{N_j(n)}{2N}$. $P^N(0)$ is assumed given.

In every period $n = 1, 2, ...$, αN *pairs*, $\alpha \in (0,1)$ of individuals are selected at random from the population.[7] These individuals are selected in pairs according to the selection protocol of *selection with replacement.* In selection with replacement a pair of individuals is selected, one after the other, at random from the population. The selected pair is replaced into the population before the next pair is selected in the same manner until $2(\alpha N)$ individuals have been selected.

[7] Here, and generally, we disregard the requirement the there can be only be a integer valued number of births, deaths, etc. This explains why we assume the population to be large. All our results stand if we replace non-integer amounts by their closest integer approximation.

In every period, the randomly selected pairs are matched to play a finite normal form game, the payoffs in which are offspring. Individuals breed asexually and adopt the type of their parent. When an agent of type j plays with an agent of type k the former receives a payoff of U_{jk} and the latter receives a payoff of U_{kj}. We denote the (symmetric) payoff matrix by U. We assume that U_{jk} is a (positive) integer, for all $j,\ k \in J$. Offspring born in any period cannot be selected to play in that period.

In every period, after the selected individuals have played the game, individuals *excluding the newborns*, are randomly, and one at a time, selected to die. We call this death procedure *death without replacement.* The number of individuals selected to die in a period is equal to the number of individuals born in the period. This ensures that the population remains constant over time. For the death process to be well defined we assume that the total number of births in any period is less than the existing population in that period. Hence, we assume that the payoff matrix satisfies $\alpha \cdot \max_{j,\,k} U_{jk} \leq 1-\alpha$.

We now state, more formally, how the population composition in period $(n+1)$ is obtained given the population composition in period n. Denote the random number of matchings in period n between individuals of type j with individuals of type k by $M_{jk}(n),\ j,k \in J$. Denote the random number of deaths of individuals of type j at time n, and that occur after the births have taken place, by $N_j^D(n),\ j \in J$. Then, the number of j types in the population at time $(n+1)$, for all $j \in J$, is given by,

$$N_j(n+1) = (1-\alpha)\,N_j(n) + \sum_{k=1}^{J} M_{jk}(n)\,(1+U_{jk}) + \sum_{k=1}^{J} M_{kj}(n)\,(1+U_{jk}) - N_j^D(n)$$

The individuals selected and matched to play in period n are selected from the population in which there are $N_j(n)$ individuals who are of type j. The individuals selected to die in period n are selected from a population where there $(1-\alpha)\,N_j(n)$ individuals of type j. This specification ensures that $N_j(n) \geq 0$, for all j and for all n.

To obtain intuition about the above equation it is useful to consider the case when all payoffs are identically equal to zero, and so the number of deaths is also zero. In this case, a proportion $(1-\alpha)$ of the population stays as it is, i.e. it's type does not change. There is still randomness in the population composition because a proportion α of the population is newly selected from the existing population according to the selection protocol of selection with replacement. This payoff independent randomness reflects how the population composition continually changes from period to period in a way unrelated to the payoffs the selected individuals obtain upon playing the game.

If payoffs are not all equal to zero then there is a second source of randomness, which is *payoff dependent.* When payoffs are not all zero, there are newborns. The number of newborns is random because it depends on the randomly selected individuals who play the game. Further, there is another source of randomness, which is due to the manner in which individuals die in every period.

Observe that, for any finite population size the state variable evolves via random discrete jumps that form a Markov chain. The complex nature of the stochastic dynamics make the discrete model very difficult to analyze.

The probability that an individual of type j is matched with an individual of type k in period n is given by $\frac{N_j(n)}{2N} \cdot \frac{N_k(n)-\delta_{jk}}{2N-1}$, where $\delta_{jk} = 1$ if $j = k$ and zero otherwise. As matching is with replacement, this probability remains the same for all matches within any period. Hence, the distribution of M_{jk} can be approximated by a multinomial distribution. It is also straightforward to see that the probability that any individual is killed, conditional on the total number of individuals who will be killed, follows a hypergeometric distribution. Knowledge of the probability distributions implied by the randomness in the model facilitates in the proof of the following Lemma. The Lemma states the first two moments of the change in the state $\Delta P_j^N(n) \equiv P_j^N(n+1) - P_j^N(n)$ for all $j \in J$, conditional on the state at time n being p, which we denote by $E\left[\Delta P_j^N(n)/p\right]$ and $Cov\left[\Delta P_j^N(n),\, \Delta P_k^N(n)/p\right]$, respectively. A detailed proof of the Lemma is contained in Appendix 2.

To state the Lemma we introduce the following notation. Let the expected payoff of type j against a population state p be denoted $U(j,\, p)$, and the expected payoff of the population p against itself be denoted $U(p,\, p)$. That is, $U(j,\, p) = \sum_{k=1}^{J} p_k U_{jk}$ and $U(p,\, p) = \sum_{j=1}^{J}\sum_{k=1}^{J} p_j p_k U_{jk}$.

Lemma 1 *For all $j, k \in J$, and for all n,*

$$\begin{aligned} E\left[\Delta P_j^N(n)/p\right] &= \alpha p_j \left(U(j,\, p) - U(p,\, p)\right) + O\left(\frac{1}{N}\right) \\ Cov\left[\Delta P_j^N(n),\, \Delta P_k^N(n)/p\right] &= \frac{\alpha p_j\left(\delta_{jk} - p_k\right)}{N}\left(1 + \frac{U(p,\, p)}{2}\left(1 + \frac{\alpha U(p,\, p)}{(1-\alpha)}\right)\right) + O\left(\frac{1}{N}\right) \end{aligned}$$

Lemma 1 says that conditional on the state, the (discrete) expected movement in the proportion of individuals playing the different pure strategies places greater weight on strategies that do better than average in the current period. The Lemma also states the conditional second moment of the change in the state of the population. As we shall see in the next Section, these two conditional moments of the discrete model play a central role in the continuous limits we obtain in the next Section.

3 Continuous Limits

3.1 Construction

We say that we get a continuous limit of the discrete process $P^N(n)$ if the process we get in continuous time units is itself a continuous function of time. We shall see that the limiting process associated with $P^N(n)$ may be the solution to a ordinary differential equation[8] for given initial conditions, or it may the solution of a stochastic differential equation[9] for given initial conditions.

As a first step in the construction of the continuous limit, we connect the random sequence of points $P^N(n)$, $n = 0, 1, \ldots$, defined only for discrete time units n, to obtain $P^N(t)$, defined for continuous time units $t \in \Re_+$. This is achieved by connecting the points $P^N(n)$ by a step function. Specifically,

$$P^N(t) = P^N(n) \qquad n \leq t < n+1$$

Observe that the process $P^N(t)$ which is defined for continuous time units is not a continuous function of time.[10] We shall refer to both $P^N(n)$ and $P^N(t)$ as the discrete process.

In the discrete model, given the population size, two factors determine how much the state changes in any period. That is, two factors determine the *step size*[11] of the state variable: (i) the proportion of the people selected in any time period, and (ii) the payoffs the selected individuals receive in any time period. Taking a continuous limit

[8] In which case the limiting process is a continuous *and* differentiable function of time.

[9] In which case the limiting process is a continuous, but nowhere differentiable, function of time.

[10] It is, however, a right continuous function of time with left limits.

[11] The step size may be defined as $\max_t \left\| P^N(t+1) - P^N(t) \right\|$.

involves "shrinking" the step size as the time interval between two consecutive periods (of the selection-birth-death process) is reduced.

We shall consider two ways of doing this. In the first we "shrink" the proportion of people selected as the "real" time that elapses between consecutive periods θ_N is reduced. In the second, we "shrink" the payoffs that people obtain with any particular matching as the "real" time between consecutive periods θ_N is reduced.[12] In this latter case, we distinguish two variants which differ only in the *rate* at which the continuous limit is taken. That is, the rate at which the step size and "real" time between consecutive periods is reduced. In either way of controlling the step-size (or change) of the process we suppose that $\theta_N \to 0$ as $N \to \infty$.

Specifically, we construct three different continuous limits of the discrete process $P^N(t)$. Each arises as $N \to \infty$. These are:

1. In every period θ_N, $2\theta_N$, ... the selection-birth-death process occurs, with $\theta_N(\alpha N)$ pairs of individuals being randomly matched in every period. The length of the time period is $\theta_N = KN^{-\delta}$, $0 < K < \infty$, $\delta \in (0,1]$. We say δ is the *rate* at which the continuous limit is taken. Clearly, this rate is faster the larger is δ. Hence, as $N \to \infty$, the interval length θ_N approaches zero, the proportion of people matched goes to zero, and the number of matchings $\theta_N(\alpha N)$ goes to infinity when $\delta \in (0, 1)$. If $\delta = 1$, then the proportion matched goes to zero, but the number of matchings in any period is equal to αK. Payoffs are given by U.

2. We leave the proportion of individuals selected as in the discrete model, i.e. α, and let the payoffs agents obtain with each play of the game go to zero as the time interval θ_N goes to zero. In this case we consider two limits:

    (a) In every period θ_N, $2\theta_N$, ... the selection-birth-death process occurs, with αN pairs of individuals being randomly matched in every period. Payoffs that agents obtain with each play are given by $\tilde{U} = \theta_N U$, where $\theta_N = KN^{-\delta}$, $\delta = 1$, for some $0 < K < \infty$. Hence, as $N \to \infty$, the interval length θ_N

[12] A continuous limit where *only* the proportion of the population that is selected to play in every time interval shrinks, as the time interval between consecutive selections shrinks, has been constructed by Boylan (1995). Our construction in this case is similar to his. In Börgers-Sarin (1995) the continuous limit is taken by *only* shrinking the size of the payoffs (with every matching) as the time interval shrinks.

approaches zero and the payoffs the agent obtain with each play of the game go to zero.

(b) In every period θ_N, $2\theta_N$, ... the selection-birth-death process occurs, with αN pairs of individuals being randomly matched in every period. Payoffs that agents obtain with each play are given by $\tilde{U} = \theta_N U$, where $\theta_N = KN^{-\delta}$, $\delta \in (0, 1)$. As $N \to \infty$, the interval length θ_N approaches zero and the payoffs agents obtain with each play of the game goes to zero. The difference with the previous continuous limit lies in the different speed at which the continuous limit is taken. In particular, now the continuous limit is taken at a slower rate than before.

3.2 Results

Our first result concerns the continuous limit described in (**3.1**.1). That is, we consider the discrete model and take the limit as $N \to \infty$ in the manner described in (**3.1**.1). To state the result, we define the value of the state variable at time t, $p(t) = (p_1(t), ..., p_J(t))$, in the continuous time replicator equation. This is obtained by solving the J differential equations

$$\frac{dp_j(t)}{dt} = \alpha p_j(t)\{U(j, p(t)) - U(p(t), p(t))\}$$

$j \in \{1, ..., J\}$, for given $p(0)$.

Proposition 1 *Suppose that at times θ_N, $2\theta_N$, ... there are $\theta_N(\alpha N)$ matchings with replacement, and $\theta_N = KN^{-\delta}$, for some $\delta \in (0,1]$. Assume $\left\|P^N(0) - p(0)\right\| \to 0$ as $N \to \infty$. Then, for any $T < \infty$, and for all $\varepsilon > 0$*

$$\lim_{N\to\infty} \Pr\left[\sup_{t\in[0,T]} \left\|P^N(t) - p(t)\right\| > \varepsilon\right] = 0$$

Observe the manner in which the interpolated model converges to the replicator equation is uniform convergence in probability upto an arbitrarily finite time T. That is, the distance between the interpolated model and the continuous process goes to zero as $N \to \infty$, for all times upto the arbitrarily chosen time T. This result strengthens the pointwise results of Binmore, Samuelson and Vaughn (1995), Börgers and Sarin (1995) and Boylan (1995). The proof proceeds along the lines of Boylan, but we invoke Doob's inequality[13] to obtain uniform convergence in probability.

The intuition behind this result is that as a smaller proportion of individuals are selected in every interval of time and as the population grows larger a law of large numbers applies in every interval of time. The reduction in the proportion of people selected in any period eliminates both the payoff independent and the payoff dependent noise in the limiting process. The rate at which the continuous limit is taken does not play any role in the result. That is, we can choose any $\delta \in (0, 1]$.

Our next results says that the conclusion of Proposition 1 continues to be true if matching is without replacement.

Proposition 2 *Suppose that at times θ_N, $2\theta_N$, ... there are $\theta_N (\alpha N)$ matchings without replacement and $\theta_N = KN^{-\delta}$, for some $\delta \in (0,1]$. Assume $\left\| P^N(0) - p(0) \right\| \to 0$, as $N \to \infty$. Then, for any $T < \infty$, and for all $\varepsilon > 0$*

$$\lim_{N \to \infty} \Pr \left[\sup_{t \in [0,T]} \left\| P^N(t) - p(t) \right\| > \varepsilon \right] = 0$$

The equivalence of the continuous limiting processes resulting from selection with replacement and selection without replacement is due to the fact that the two selection protocols approach one another when a decreasing proportion of an increasing population is matched to play the game. Specifically, in either matching protocol, the proportion of people selected to play is of order θ_N. As $N \to \infty$ and $\theta_N \to 0$, the probability of a match between an individual of type j with an individual of type k approaches the same limit in either matching protocol.

The continuous approximation resulting from the limit described in (**3.1**.2.(a)) is a SDE. The interpolated model weakly converges to this SDE over the whole real line. Denote weak convergence by $\Longrightarrow$, and let D_Γ denote the set of right continuous functions with left limits defined on the set Γ.

[13] See, e.g., Revuz and Yor (1991, p. 52)

Proposition 3 *Suppose that at times $\theta_N, 2\theta_N, \ldots,$ there are αN matchings with replacement. Assume that payoffs are $\tilde{U}_{jk} = \theta_N U_{jk}$, where $\theta_N = KN^{-\delta}$, $\delta = 1$, with $0 < K < \infty$, and $\left\| P^N(0) - \tilde{P}(0) \right\| \to 0$ as $N \to \infty$. Then,*

$$P^N \Longrightarrow \tilde{P}$$

in $D_S[0, \infty)$, where S is defined as

$$S = \left\{ (p_1, \ldots p_J) \in [0,1]^J, \sum_i^J p_i = 1 \right\}$$

and $\tilde{P}$ is a diffusion whose generator is given by

$$A(p) = \frac{1}{2} \sum_{j=1}^J \sum_{k=1}^J a_{jk}(p) + \sum_{j=1}^J b_j(p)$$

where

$$b_j(p) = \alpha p_j K \left(U(j, p) - U(p, p) \right)$$

and

$$a_{jk}(p) = \alpha p_j \left(\delta_{jk} - p_k \right)$$

Observe that the randomness in the discrete model does not get eliminated in this continuous limit. Observe also the "longer" time horizon over which the SDE approximation result is valid, as also the different kind of convergence which arises in this case.

The limiting result in the above Proposition is driven by the fact that in any interval of time the same proportion of individuals is selected as in the original process. Hence the randomness due to the payoff independent noise does not get eliminated in the limiting process even while the payoff dependent randomness gets eliminated in the limiting process. A critical factor in obtaining the stochastic continuous limit is the "fast enough" rate, specifically $\delta = 1$, at which the continuous limit is taken. The role played by the rate at which the step size decreases becomes apparent with Proposition 4. We shall discuss the role of δ in the continuous limits after stating Proposition 4.

The following Corollary is an immediate consequence of Proposition 3.

Corollary 1 *When there are only two types 1 and 2 in the population, the diffusion process generated by A is the solution to the following SDE*

$$d\tilde{P}_1(t) = \alpha K \tilde{P}_1(t)\tilde{P}_2(t)(U(1, p) - U(2, p))\,dt + \sqrt{\alpha \tilde{P}_1(t)\tilde{P}_2(t)}dW(t)$$

where $W(t)$ is a one-dimensional Weiner process.

In Section 4 we shall study the properties of this SDE as $t \to \infty$. We shall then also compare these properties with other SDEs studied in the evolutionary game theory.

The continuous approximation resulting from (**3**.**1**.2.(b)) is an ODE. Again the kind of convergence is as in the former ODE approximation results. This ODE is the usual continuous time replicator equation.

Proposition 4 *Suppose that at times θ_N, $2\theta_N$, ... there are αN pairs selected with replacement. Assume that payoffs are $\tilde{U}_{jk} = \theta_N U_{jk}$, where $\theta_N = KN^{-\delta}$, $0 < K < \infty$, $\delta \in (0, 1)$. If $\left\|P^N(0) - p(0)\right\| \to 0$ as $N \to \infty$, then, for any $T < \infty$, and for all $\varepsilon > 0$*

$$\lim_{N\to\infty} \Pr\left[\sup_{t\in[0,T]} \left\|P^N(t) - p(t)\right\| > \varepsilon\right] = 0$$

The intuition behind why the limiting process in Proposition 4 is different from that obtained in the previous result is a little more intricate. The main difference with Proposition 3, in which we obtained a SDE limit, is the rate at which the step size goes to zero. In Proposition 3 we had $\delta = 1$, whereas in Proposition 4 we have $\delta \in (0, 1)$. The faster rate at which the step size is reduced in the former result yields a continuous limits which is an SDE, whereas the slower rate in the latter result yields a continuous limit that is an ODE.

As will become clear from the proof of the Proposition 4, what drives this result is the fact that the conditional first moment is of order $N^{-\delta}$, $\delta \in (0, 1)$, while the conditional variance is of order N^{-1}. Hence, as N increases, the conditional variance converges to zero quicker than the conditional mean. As a consequence, the conditional variance gets eliminated in the limit as $N \to \infty$. In contrast, in Proposition 3, both the conditional first moment and the conditional variance were of order N^{-1}. Consequently, the conditional variance does not get eliminated in the limit as $N \to \infty$.

4 Asymptotic Analysis

The asymptotic properties as $t \to \infty$ of the ODE we obtained in the previous Section, the continuous time replicator equation, are well known (e.g. Hofbauer and Sigmund (1988), Weibull (1995)).[14] In this Section we study the asymptotic properties of the SDE limit we derived. We compare these properties with the SDE studied by Fudenberg and Harris (1992).[15]

We shall consider the class of symmetric 2x2 games. This class of games can be represented by the following payoff matrix,

	1	2
1	U_{11}, U_{11}	U_{12}, U_{21}
2	U_{21}, U_{12}	U_{22}, U_{22}

Following Fudenberg and Harris we distinguish three cases:

(i) The case of a strictly dominant strategy. The game has a dominant strategy equilibrium when $U_{11} > U_{21}$ and $U_{12} > U_{22}$, in which case strategy 1 is the dominant strategy. Analogously, if $U_{11} < U_{21}$ and $U_{12} < U_{22}$, then strategy 2 is the dominant strategy. In the former case $P_1 = 1$ is the unique (pure strategy) Nash equilibrium, and in the latter case, $P_1 = 0$ is the unique equilibrium.

(ii) The coordination case. If $U_{11} > U_{21}$ and $U_{12} < U_{22}$ then the game has three Nash equilibria. The two pure strategy equilibria are when $P_1 = 1$, and $P_1 = 0$. The mixed strategy equilibrium is at $P_1^* = (U_{22} - U_{12}) \,/\, ((U_{22} - U_{12}) + (U_{11} - U_{21}))$.

(iii) The case of a mixed strategy equilibrium. The game has a unique mixed strategy equilibrium if $U_{11} < U_{21}$ and $U_{12} > U_{22}$. The unique mixed equilibrium is then at $P_1^* = (U_{22} - U_{12}) \,/\, ((U_{22} - U_{12}) + (U_{11} - U_{21}))$.

The following Proposition provides information on the asymptotic properties of the SDE we obtained in the previous Section in symmetric 2x2 games.

[14] It is known that the ODE approximation results we obtained in the previous Section are, in general, not true in the limit as $T \to \infty$ (see, e.g., Boylan (1995)).

[15] We do not provide a comparative analysis with the SDE studied by Foster and Young (1990). The SDE they studied was defined only for a compact subset in the interior of the state space.

Proposition 5 *As $t \to \infty$, (i) $\tilde{P}_1(t)$ converges to 1 with probability $\frac{I_1}{I_1+I_2}$ and converges to 0 with probability $\frac{I_2}{I_1+I_2}$, where*

$$I_1 = \int_0^{\tilde{P}_1(0)} \exp\left[K\left((U_{12}-U_{22})-(U_{11}-U_{21})\right)\left(x^2-z^2\right)-2\left(K\left(U_{12}-U_{22}\right)\right)(x-z)\right]dx$$

z is an arbitrary number in $(0, 1)$, and

$$I_2 = \int_{\tilde{P}_1(0)}^{1} \exp\left[K\left((U_{12}-U_{22})-(U_{11}-U_{21})\right)\left(x^2-z^2\right)-2\left(K\left(U_{12}-U_{22}\right)\right)(x-z)\right]dx$$

Proof: Following Fudenberg and Harris (1992, p.429), we have

$$I_1 = \int_0^{\tilde{P}_1(0)} \exp\left[-2\int_z^x \frac{b(y)}{a(y)}dy\right]dx$$

and

$$I_2 = \int_{\tilde{P}_1(0)}^{1} \exp\left[-2\int_z^x \frac{b(y)}{a(y)}dy\right]dx$$

The expression for I_1 and I_2 given in the statement of the Proposition are obtained by plugging in the expression for $b(y)$ and for $a(y)$, as defined in Section 3.2, and solving the integral inside the brackets. As $\max_{j,k} U_{jk} < \infty$ both I_1 and I_2 are finite. Hence, by Theorem 16.1 in Gihman and Skorohod (1972), it follows that $\tilde{P}_1(t)$ converges to 1 with probability $\frac{I_1}{I_2+I_1}$ and to zero with probability $\frac{I_2}{I_1+I_2}$. ■

The above result tells us that the SDE gets absorbed in either of the vertices of the state space. That is, the boundaries of the state space are absorbing in our model. The probability with which it gets absorbed in either vertex depends both on the initial state of the process and the payoffs in the game. The following Corollary of Proposition 5 provides further information in a special instance of the strictly dominant strategy case when the difference between the payoffs from the two strategies are the same, and strategy 1 is the dominant strategy.

Corollary 2 *Suppose $U_{11} - U_{21} = U_{12} - U_{22} > 0$. Then $\tilde{P}_1(t)$ converges to 1 with probability equal to,*

$$\frac{1-\exp\left(-2K\left(U_{12}-U_{22}\right)\tilde{P}_1(0)\right)}{1-\exp\left(-2K\left(U_{12}-U_{22}\right)\right)}$$

and converges to 0 with probability,

$$\frac{\exp\left(-2K\left(U_{12}-U_{22}\right)\tilde{P}_1(0)\right)-\exp(-2K(U_{12}-U_{22}))}{1-\exp\left(-2K\left(U_{12}-U_{22}\right)\right)}$$

Proof: If $U_{12} - U_{22} = U_{11} - U_{21} > 0$, then

$$\begin{aligned} I_1 &= \int_0^{\tilde{P}_1(0)} \exp\left(-2K\left(U_{12} - U_{22}\right)(x - z)\right) dx \\ &= -\frac{\exp\left(-2K\left(U_{12} - U_{22}\right)\tilde{P}_1(0)\right)}{\exp\left(-2K\left(U_{12} - U_{22}\right)\right)} + \frac{1}{\exp\left(-2K\left(U_{12} - U_{22}\right)\right)} \end{aligned}$$

and,

$$I_2 = \frac{\exp\left(-2K\left(U_{12} - U_{22}\right)\right)\tilde{P}_1(0)}{\exp\left(-2K\left(U_{12} - U_{22}\right)\right)} - 1$$

The statement of the result follows. ■

Corollary 2 tells us the probabilities with which the SDE gets absorbed in either vertex in a particular variant of the strictly dominant strategy case. From the Corollary it is readily seen that as strategy 1 becomes more "attractive", the probability of converging to that strategy increases.

We now proceed to compare our SDE with that studied by Fudenberg and Harris (1992). Fudenberg and Harris studied the SDE

$$dP_1(t) = P_1(t)\,P_2(t)\left\{\left[U(1, P(t)) - U(2, P(t))\right]dt + \left[\sigma_2^2 P_2(t) - \sigma_1^2 P_1(t)\right]dt + \sigma_1 dW_1 - \sigma_2 dW_2\right\}$$

where σ_i^2 is the variance term.

Notice first how their drift term differs from ours, and from the replicator expression. Specifically, their drift term includes a second term reflecting the weighted difference of the variances. This arises due to the particular way in which they construct the SDE. Specifically, they add continuous (aggregate payoff) shocks to the continuous version of the equation that governs the change in the sizes of the different infinite populations. They then derive their stochastic replicator equation by applying Ito's Lemma. Their diffusion term is different for a similar reason. We now turn to compare the asymptotic properties of their SDE with the one we obtained.

In the dominant strategy case, the Fudenberg and Harris SDE converges to the dominant strategy *provided* the variances of the shocks are *sufficiently small.* In our SDE the variances of the shocks are endogenously determined and their magnitude ensures that either pure strategy outcome will be asymptotically realized.

In the coordination case, the Fudenberg and Harris SDE converges to either of the pure strategy equilibria *provided* that the variances of the shocks are *sufficiently small.*

Each pure strategy equilibria is reached with positive probability starting from any interior point in the state space. This is true also for our SDE.

In the mixed strategy case, the Fudenberg and Harris SDE possesses an ergodic distribution, which collapses to P_1^* as the variance of the shocks converges to zero. In contrast our SDE converges to a pure strategy outcome even in this case. This difference arises due to the different nature of the noise that is present in the two models.

The differences between the properties of the two SDEs arise largely because of the different ways the SDEs are motivated and derived. We start with a discrete stochastic finite population model with randomness due to the selection-birth-death process. We, then, take an (appropriate) continuous limit as the population size goes to infinity to obtain the SDE. Fudenberg and Harris derive their SDE from the consideration of how aggregate payoff shocks affect the population sizes of different (infinite) populations. They add continuous shocks to a continuous equation (and apply Ito's Lemma) to obtain their SDE.

APPENDIX 1

Proof of Proposition 1: Let $\Delta X(n\theta)$ denote $X((n+1)\theta) - X(n\theta) = X(n\theta+\theta) - X(n\theta)$ and let $[x]$ denote the integer part of x. Assuming there are $\theta_N N$ matchings (or pairs selected) per period each of length θ_N and following the proof of Lemma 1 (see Appendix 2), we can compute the first two moments of the change in the proportion of individuals of the different types to be

$$\begin{aligned}
E\left[\Delta P_j(n\theta_N)/p\right] &= \theta_N \alpha p_j \left(U(j,\, p) - U(p,\, p)\right) + O\left(\frac{\theta_N}{N}\right) \\
&\equiv \theta_N H_j(p) + O\left(\frac{\theta_N}{N}\right) \\
Cov\left[\Delta P_j(n\theta_N),\, \Delta P_k(n\theta_N)/p\right] &= \frac{\theta_N \alpha}{N} p_j\left(\delta_{jk} - p_k\right)\left(1 + \frac{U(p,p)}{2}\left(1 + \frac{\theta_N \alpha}{1-\alpha} U(p,p)\right)\right) \\
&\quad + O\left(\frac{\theta_N}{N^2}\right)
\end{aligned}$$

Let $H(p) = (H_1(p),\, ...,\, H_J(p))$. Note that for all $p,\ p' \in S$,

$$\left\|H(p) - H(p')\right\| \leq L\left\|p - p'\right\|$$

for some constant $L < \infty$.

By construction,

$$\begin{aligned}
P^N(t) &= P^N(0) + \sum_{n=0}^{[\theta_N^{-1} t - 1]} \left[P^N((n+1)\theta_N) - P^N(n\theta_N)\right] \\
&\equiv P^N(0) + \sum_{n=0}^{[\theta_N^{-1} t - 1]} \Delta P^N(n\theta_N)
\end{aligned}$$

Define the right continuous martingale,

$$M^N(t) \equiv P^N(t) - P^N(0) - \sum_{n=0}^{[\theta_N^{-1} t - 1]} E\left[\Delta P^N(n\theta_N)/P^N(n\theta_N)\right]$$

Clearly,

$$P^N(t) = M^N(t) + P^N(0) + \sum_{n=0}^{[\theta_N^{-1}t-1]} E\left[\Delta P^N(n\theta_N)/P^N(n\theta_N)\right]$$

By definition,

$$p(t) = p(0) + \int_0^t H(p(s))\,ds$$

So,

$$\begin{aligned} P^N(t) - p(t) &= M^N(t) + \left(P^N(0) - p(0)\right) \\ &\quad + \left\{ \sum_{n=0}^{[\theta_N^{-1}t-1]} E\left[\Delta P^N(n\theta_N)/P^N(n\theta_N))\right] - \int_0^t H(p(s))\,ds \right\} \end{aligned}$$

But,

$$\begin{aligned} \sum_{n=0}^{[\theta_N^{-1}t-1]} E\left[\Delta P^N(n\theta_N)/P^N(n\theta_N)\right] &= \theta_N \sum_{n=0}^{[\theta_N^{-1}t-1]} H\left(P^N(n\theta_N)\right) \\ &= \sum_{n=0}^{[\theta_N^{-1}t-1]} \int_{n\theta_N}^{(n+1)\theta_N} H(P^N(s))ds - \int_{[\theta_N^{-1}t]\theta_N}^{t} H\left(P^N(s)\right) ds \\ &= \int_0^t H\left(P^N(s)\right) ds - \int_{[\theta_N^{-1}t]\theta_N}^{t} H\left(P^N(s)\right) ds \end{aligned}$$

Thus,

$$\begin{aligned} P^N(t) - p(t) &= M^N(t) + \left(P^N(0) - p(0)\right) \\ &\quad + \int_0^t \left[H\left(P^N(s)\right) - H(p(s))\right] ds - \int_{[\theta_N^{-1}t]\theta_N}^{t} H\left(P^N(s)\right) ds \end{aligned}$$

and so,

$$\begin{aligned} \left\|P^N(t) - p(t)\right\| &\leq \left\|M^N(t)\right\| + \left\|P^N(0) - p(0)\right\| \\ &\quad + L\int_0^t \left\|P^N(s) - p(s)\right\| ds + \int_{[\theta_N^{-1}t]\theta_N}^{t} \left\|H(P^N(s))\right\| ds \end{aligned}$$

By Gronwall's lemma (Revuz and Yor, 1990, p. 499), given a positive locally bounded Borel function on $\Re_{+}$, say ϕ, such that $\phi(t) \leq a + b\int_0^t \phi(s)ds$, it follows that $\phi(t) \leq ae^{bt}$. Let $\phi(t) = \left\|P^N(t) - p(t)\right\|$, we have

$$\left\|P^N(t) - p(t)\right\| \leq \left(\left\|M^N(t)\right\| + \left\|P^N(0) - p(0)\right\| + \int_{[\theta_N^{-1}t]\theta_N}^{t} \left\|H(P^N(s)\right\| ds\right) e^{Lt}$$

Recalling that the sup of the sum is less than equal the sum of sups, we have

$$\sup_{t\in[0,T]} \left\|P^N(t) - p(t)\right\| \leq \left(\sup_{t\in[0,T]} \left\|M^N(t)\right\| + \left\|P^N(0) - p(0)\right\| + \sup_{t\in[0,T]} \left\|H(P^N)(t)\right\| \int_{[\theta_N^{-1}t]\theta_N}^{t} ds\right) e^{LT}$$

Now $\left\|P^N(0) - p(0)\right\| \to 0$ by assumption, and because of the boundedness of H,

$$\sup_{t\in[0,T]} \left\|H(P^N(t))\right\| \int_{[\theta_N^{-1}t]\theta_N}^{t} ds \leq L\left(t - \left[\theta_N^{-1}t\right]\theta_N\right)$$

Thus in order to show that

$$\lim_{N\to\infty} \Pr\left(\sup_{t\in[0,T]} \left\|P^N(t) - p(t)\right\| > \varepsilon\right) = 0$$

we only need to show,

$$\lim_{N\to\infty} \Pr\left(\sup_{t\in[0,T]} \left\|M^N(t)\right\| > \varepsilon\right) = 0$$

By the Doob inequality for right continuous martingales (Revuz and Yor (1991, p. 52, for $p = 2$)), for $j = 1, \ldots, J$,

$$\begin{aligned}
\Pr\left(\sup_{t\in[0,T]} \left\|M_j^N(t)\right\| > \varepsilon\right) &\leq \frac{1}{\varepsilon^2} \sup_{t\in[0,T]} E\left[M_j^N(t)\right]^2 \\
&= \frac{1}{\varepsilon^2} E\left[M_j^N(T)\right]^2 \\
&\leq C\frac{1}{\varepsilon^2}\frac{\left[\theta_N^{-1}T\right]\theta_N}{N}
\end{aligned}$$

where the equality and the last inequality on the right hand side above come from the fact that

$$\sup_{t\in[0,T]} E\left[M_j^N(t)\right]^2 = \sup_{t\in[0,T]} \sum_{n=0}^{[\theta_N^{-1}t]-1} Var\left[\Delta P_j^N(n\theta_N) / P^N(n\theta_N)\right]$$

$$
\begin{aligned}
&= \sum_{n=0}^{[\theta_N^{-1}T]-1} Var\left[\Delta P_j^N(n\theta_N)/P^N(n\theta_N)\right] \\
&= \alpha\frac{\theta_N}{N}\sum_{n=0}^{[\theta_N^{-1}T]-1} P_j^N(n\theta_N)\left(1-P_j^N(n\theta_N)\right) \\
&\quad \times\left(1+\frac{U\left(P^N(n\theta_N),P^N(n\theta_N)\right)}{2}\right)\left(1+\frac{\alpha}{1-\alpha}U\left(P^N(n\theta_N),P^N(n\theta_N)\right)\right) \\
&\le C\frac{\theta_N[\theta_N^{-1}T-1]}{N}
\end{aligned}
$$

for some $0<C<\infty$, because of the boundedness of U. As

$$
\sup_{t\in[0,T]}\left\|M^N(t)\right\| \le \sup_{t\in[0,T]}\left\|M_1^N(t)\right\| + \ldots + \sup_{t\in[0,T]}\left\|M_J^N(t)\right\|
$$

the desired result follows. ■

Proof of Proposition 2: Suppose at period n, we have already selected $\eta\theta_N\alpha N$ pairs, with $\eta\in(0,1)$. Suppose that $\gamma_k\eta\theta_N 2\alpha N$ were of type k and $\gamma_j\eta\theta_N 2\alpha N$ were of type j, with $\sum_i\gamma_i=1$. We want to show that, as N gets large, the probability of a matching between individuals of type j with individuals of type k is the same within a given period with either matching protocol. At any period, the probability of a jk match at the $\eta\theta_N\alpha 2N$-th draw is given by,

$$
\begin{aligned}
\frac{N_j(n)-\gamma_j\eta\alpha\theta_N 2N}{2(N-\eta\alpha\theta_N N)}\frac{N_k(n)-\gamma_k\eta\alpha\theta_N 2N-\delta_{jk}}{2(N-\eta\alpha\theta_N N)-1} &= \frac{N_j(n)}{2(N-\eta\alpha\theta_N N)}\frac{N_k(n)}{2(N-\eta\alpha\theta_N N)-1} \\
&\quad -\frac{\gamma_j\eta\alpha\theta_N N N_k(n)}{(N-\eta\alpha\theta_N N)(2(N-\eta\alpha\theta_N)N-1)} \\
&\quad -\frac{N_j(n)(\gamma_k\eta\alpha\theta_N N-\delta_{jk})}{(N-\eta\alpha\theta_N N)(2(N-\eta\alpha\theta_N N)-1)} \\
&\quad +\frac{\gamma_j\eta\alpha\theta_N N(\gamma_k\eta\alpha\theta_N N-\delta_{jk})}{(N-\eta\alpha\theta_N N)(N-\eta\alpha\theta_N N)-1} \\
&= \frac{N_j(n)}{(2N-\eta\alpha\theta_N N)}\frac{N_k(n)}{2(N-\eta\alpha\theta_N N)-1}+O(\theta_N) \\
&= P_j(n)P_k(n)+O(\theta_N)
\end{aligned}
$$

where $P_j(n)=\frac{N_j(n)}{2N}$ and the last term is $O(\theta_N)$ uniformly in η,γ_j and γ_k.

Thus as N gets large, the probability of a jk match is the same within a given period, regardless the selection being with or without replacement. For this reason we can approximate the moments of the matching distribution with the moments of the multinomial distribution in that, as $N \to \infty$ the two set of moments converge to the same limit. Thus by recalling that we have $\theta_N N$ matchings per period, by the same argument as in Lemma 1, we have,

$$\begin{aligned} E\left[M_{jk}/p\right] &= \theta_N \alpha N p_j p_k + O\left(\theta_N\right) \\ Var\left[M_{jk}/p\right] &= \theta_N \alpha N p_j p_k \left(1 - p_j p_k\right) + O\left(\theta_N\right) \\ Cov\left[M_{jk}, M_{j'k'}/p\right] &= \theta_N \alpha N p_j p_k p_{j'} p_{k'} + O\left(\theta_N\right) \end{aligned}$$

where the $O\left(\theta_N\right)$ term comes from the approximation we have taken.

Recalling that the both the matchings and the deaths occur at period $\theta_N, 2\theta_N, \ldots$, by the same argument as Lemma 1, we obtain for all j, $k \in J$, and for all n,

$$\begin{aligned} E\left[\Delta P_j\left(n\right)/p\right] &= \theta_N \alpha p_j \left(U\left(j,\, p\right) - U\left(p,\, p\right)\right) + O\left(\frac{\theta_N}{N}\right) \\ Cov\left[\Delta P_j\left(n\right),\, \Delta P_k\left(n\right)/p\right] &= \frac{\theta_N \alpha p_j \left(\delta_{jk} - p_k\right)}{N}\left(1 + \frac{U\left(p,\, p\right)}{2}\left(1 + \frac{\theta_N \alpha U\left(p,\, p\right)}{1-\alpha}\right)\right) + O\left(\frac{\theta_N}{N^2}\right) \end{aligned}$$

The result then follows by the argument used in the proof of Proposition 1. ■

Proof of Proposition 3: We can compute the moments under the current hypotheses by following Lemma 1. We get,

$$E\left[\Delta P_j^N\left(n\theta_N\right)/p\right] = \theta_N \alpha p_j \left(U\left(j,\, p\right) - U\left(p,\, p\right)\right) + O\left(\frac{\theta_N}{N}\right) \tag{1}$$

$$\begin{aligned} Cov\left[\Delta P_j^N\left(n\theta_N\right),\, \Delta P_k^N\left(n\theta_N\right)/p\right] &= \theta_N \alpha p_j \left(\delta_{jk} - p_k\right)\left(1 + \frac{\theta_N U\left(p,\, p\right)}{2}\left(1 + \frac{\theta_N \alpha U\left(p,\, p\right)}{1-\alpha}\right)\right) \\ &\quad + O\left(\frac{\theta_N}{N}\right) \\ &= \theta_N \alpha p_j \left(\delta_{jk} - p_k\right) + O\left(\frac{\theta_N}{N}\right) \end{aligned} \tag{2}$$

In order to show weak convergence to a diffusion process, we first need to show that, $\forall j$,

$$E\left[\left(\Delta P_j^N\left(n\theta_N\right)\right)^4/p\right] = O\left(\frac{1}{N^2}\right)$$

This is done in Lemma 2 (the Proof of which is contained in Appendix 2).

Lemma 2 *For all* j,

$$E\left[\left(\Delta P_j^N\left(n\theta_N\right)\right)^4/p\right] = O\left(\frac{1}{N^2}\right)$$

Given Lemma 2, a straightforward application of Chebychev's inequality yields,

$$Pr\left[\left|\Delta P_j^N\left(n\theta_N\right)/p\right| > \epsilon\right] \leq \frac{1}{\epsilon^4} E\left|\Delta P_j^N\left(n\theta_N\right)/p\right|^4 = O\left(\frac{1}{N^2}\right) \quad (3)$$

uniformly in $p \in S_N$, where

$$S_N = \left[(2N)^{-1}\beta, \beta \in \left(Z_+\right)^J, \sum_i^J \beta_i = 2N\right]$$

and Z_+ is the set of positive integers. Note that S_N is the state space of the Markov chain P^N.

We want to show that (1),(2) and (3) imply condition (1.22) in Ethier and Kurtz (1996, p.415), that is

$$\lim_{N\to\infty} \sup_{p\in S_N} \left|2N\left(T_N - I\right) f(p) - Af(p)\right| = 0 \quad (4)$$

with $f \in C^2(S)$, A is the generator of the diffusion in S as defined in Section 3.2 in the text and

$$T_N f(p) = \int f(y)\, \mu_N(p, dy)$$

where μ_N is the transition function of the discrete chain; all integrals are defined over S_N, unless otherwise specified.

Given (4), the result follows from Theorem 1.1 in Ethier and Kurtz (1986, p.415).

Define

$$NL_N f(p) = \int \left(f(y) - f(p)\right) \mu_N(p, dy)$$

By noting that

$$\lim_{N\to\infty} \sup_{p\in S_N} \left|\left(T_N - I\right) f(p) - NL_N f(p)\right| = 0,$$

it suffices to show that

$$\lim_{N\to\infty} \sup_{p\in S_N} \left|NL_N f(p) - Af(p)\right| = 0$$

Let $A_N f$ be the operator with coefficient $b_j^N = NE\left[\Delta P_j^N\left(n\theta_N\right)/p\right]$ and $a_{ij}^N = NCov\left[\Delta P_i^N\left(n\theta_N\right), \Delta P_j^N\left(n\theta_N\right)/p\right]$. From (1) and (2), we have that

$$\lim_{N\to\infty} \sup_{p\in S_N} \left|A_N f(p) - Af(p)\right| = 0.$$

Thus it suffices to show that

$$\lim_{N\to\infty}\sup_{p\in S_N}|NL_Nf(p)-A_Nf(p)|=0$$

Following Stroock and Varadhan (1979, p.268-269), for any $f\in C^2(S)$, we can define

$$H(p,y)=\sum_i^J(p_i-y_i)\frac{\partial f}{\partial p_i}f(p)+\frac{1}{2}\sum_i^J\sum_j^J(p_i-y_i)(p_j-y_j)\frac{\partial^2 f}{\partial p_i\partial p_j}(p)$$

By the Taylor theorem, there exist a constant C_f, $0<C_f<\infty$, such that $\forall p,y$,

$$|f(y)-f(p)-H(p,y)|\le C_f|y-p|^3$$

Thus we also have that

$$|NL_Nf(p)-A_Nf(p)|\le C_f\int|y-p|^3\mu_N(p,dy)$$

Now, as $N\to\infty$ the right hand side converges to zero uniformly on S_N by (3) and by Cauchy-Schwartz inequality. This concludes the proof. ■

Proof of Corollary 1: Straightforward from Proposition 3; as in this case

$$a_{11}=a_{22}=\alpha p_1(1-p_1)$$

and

$$a_{12}=a_{21}=-\alpha p_1(1-p_1)$$

also

$$b(p)=p_1^2(1-p_1)(U_{11}-U_{21})+(p_1(1-p_1))(1-p_1)(U_{12}-U_{22})$$

■

Proof Proposition 4: Under the current hypotheses, following the proof of Lemma 1, we have

$$\begin{aligned}E[\Delta P_j(n\theta_N)/p] &= \theta_N\alpha p_j(U(j,p)-U(p,p))+O\left(\frac{\theta_N}{N}\right)\\ Cov[\Delta P_j(n\theta_N),\Delta P_k(n\theta_N)/p] &= \frac{\alpha}{N}p_j(\delta_{jk}-p_k)\left(\left(1+\frac{\theta_NU(p,p)}{2}\right)\left(1+\frac{\theta_N\alpha U(p,p)}{1-\alpha}\right)\right)\\ &\quad +O\left(\frac{\theta_N}{N}\right)\end{aligned}$$

The result then follows by the argument used in the proof of Proposition 1. ■

APPENDIX 2

Proof of Lemma 1: Given that,

$$N_j(n+1) = (1-\alpha) N_j(n) + \sum_k^J M_{jk}(n)(1+U_{jk}) + \sum_k^J M_{kj}(n)(1+U_{jk}) - N_j^D(n)$$

$$E[N_j(n+1)/p] = (1-\alpha) 2Np_j + \sum_k^J \{E[M_{jk}(n)/p](1+U_{jk}) + E[M_{kj}(n)/p](1+U_{jk})\} \\ -E\left[N_j^D(n)/p\right]$$

As the probability of a M_{jk} match is equal to the probability of a M_{kj} match, we have (by, e.g., Johnson and Kotz (1977, p.90)),

$$2\sum_k^J E[M_{jk}(n)/p](1+U_{jk}) = \alpha 2Np_j \sum_k^J p_k(1+U_{jk}) + O(1)$$

where the $O(1)$ term arises as we approximate $\frac{N_j(n)}{2N}\frac{N_k(n)-\delta_{jk}}{2N-1}$ by $\frac{N_j}{2N}\frac{N_k}{2N}$, and the consequent order $O(1/N)$ error becomes an $O(1)$ term when multiplied by N.

To calculate $E\left[N_j^D(n)/p\right]$, we first observe that the total number of deaths in any period n is equal to the total number of newborns in that period. Thus, $\sum_j^J N_j^D(n) \equiv N^D(n) \equiv N^B(n) \equiv \sum_j^J N_j^B(n) = \sum_j^J \sum_k^J M_{jk}(n) U_{jk} + \sum_j^J \sum_k^J M_{kj} U_{jk}$. This is a random variable, as M_{jk} is random $\forall\ j, k$. However, after the births are realized, we observe the realization of $N^B(n)$. Conditional on a given realization n_i of the total number of newborns, n_i individuals, excluding the newborns, are randomly selected to die from a population with $(1-\alpha)N_j$ type j individuals. The distribution of deaths follows a multivariate hypergeometric distribution. Thus, (e.g. Johnson and Kotz, 1977, p.92),

$$E\left[N_j^D(n)/n_i,\, p\right] = n_i p_j$$

Now,

$$
\begin{aligned}
E\left[N_j^D(n)/p\right] &= \sum_i E\left[N_j^D(n)/n_i, p\right] \times \Pr\left[N^D(n) = n_i/p\right] \\
&= p_j \sum_i n_i \Pr\left[N^D(n) = n_i/p\right] \\
&= p_j E\left[N^B(n)/p\right] \\
&= \alpha 2N p_j \sum_l^J \sum_k^J p_l p_k U_{lk} + O(1)
\end{aligned}
$$

as $N^B(n) = \sum_{l=1}^J N_l^B(n)$. Hence,

$$
\begin{aligned}
E\left[\Delta P_j(n)/p\right] &= E\left[\frac{N_j(n+1)}{2N}/p\right] - p_j \\
&= \alpha p_j \left(\sum_k^J p_k U_{jk} - \sum_l^J \sum_k^J p_l p_k U_{lk}\right) + O\left(\frac{1}{N}\right) \\
&= \alpha p_j \left(U(j, p) - U(p, p)\right) + O\left(\frac{1}{N}\right)
\end{aligned}
$$

We now compute the variance. To simplify notation we denote $M_{jk}(n)$ by M_{jk}. As the probability of a M_{jk} match is equal to the probability of a M_{kj} match, $Var(M_{jk}/p) = Var(M_{kj}/p)$. Hence,

$$
\begin{aligned}
Var\left[\Delta N_j(n)/p\right] &= Var\left[N_j(n+1)/p\right] \\
&= 4Var\left[\sum_k^J M_{jk}(1+U_{jk})/p\right] + Var\left[N_j^D(n)/p\right]
\end{aligned}
$$

The covariance term is zero given the independence of the matching and the death process, which in turn follows from the fact that the population composition for the matching process and the "initial" population composition for the death process are the same. So,

$$
Var\left[\Delta P_j(n)/p\right] = \frac{1}{N^2} Var\left[\sum_k M_{jk}(1+U_{jk})/p\right] + \frac{1}{N^2} Var\left[N_j^D(n)/p\right].
$$

Now,

$$
\frac{1}{N^2} Var\left[\sum_k^J M_{jk}(1+U_{jk})/p\right] = \frac{1}{N^2} \sum_{i=1}^J Var\left[M_{ji}/p\right](1+U_{ji})^2
$$

$$
\begin{aligned}
& +\frac{2}{N^2}\sum_{i=1}^{J}\sum_{k>i}^{J} Cov\left[M_{ji},\, M_{jk}/p\right](1+U_{ji})(1+U_{jk}) \\
= \; & \alpha\frac{1}{N}\left\{\sum_{i=1}^{J} p_j p_i (1-p_j p_i)(1+U_{ji})^2\right\} \\
& -\alpha\frac{2}{N}\left\{\sum_{i=1}^{J}\sum_{k>i}^{J} p_j^2 p_i p_k (1+U_{ji})(1+U_{jk})\right\} + O\left(\frac{1}{N^2}\right) \\
= \; & \alpha\frac{\pi_j(1-\pi_j)}{N} + O\left(\frac{1}{N}\right) + O\left(\frac{1}{N^2}\right) \\
= \; & \alpha\frac{p_j(1-p_j)}{N} + O\left(\frac{1}{N}\right) + O\left(\frac{1}{N^2}\right)
\end{aligned}
$$

where $\pi_j = p_j \sum_k^J p_k(1+U_{jk})$, and the $O(1/N^2)$ terms captures the fact we have approximated $\frac{N_j}{2N}\frac{N_k-\delta_{jk}}{2N-1}$ with $\frac{N_j}{2N}\frac{N_k}{2N}$ and the $O(1/N)$ terms in the last two equations come from the fact that we are neglecting terms bounded by $\sum_{k=1}^J U_{jk}/N + \sum_h^J \sum_l^J U_{jl}U_{kh}/N$. As for the off diagonal terms, by the same argument used for the diagonal term, we have,

$$
\begin{aligned}
& \frac{2}{N^2} Cov\left[\sum_l^J M_{jl}(1+U_{jl}),\, \sum_h^J M_{kh}(1+U_{kh})/p\right] \\
= \; & \frac{2}{N^2}\sum_{l=1}^{J}\sum_{h=1}^{J} Cov\left[M_{jl}(1+U_{jl}),\, M_{kh}(1+U_{kh})\right] \\
= \; & -\alpha\frac{\pi_j\pi_k}{N} + O\left(\frac{1}{N^2}\right) + O\left(\frac{1}{N}\right) \\
= \; & -\alpha\frac{p_j p_k}{N} + O\left(\frac{1}{N^2}\right) + O\left(\frac{1}{N}\right)
\end{aligned}
$$

where the $O(1/N)$ terms takes into account terms bounded $\sum_k^J U_{jk}/N$.

Recalling that conditional on the total number of newborns deaths have a hypergeometric distribution, and the individuals selected to die come from a population with $(1-\alpha)N_j$ type j individuals, we have

$$
\begin{aligned}
Var\left[N_j^D(n)/n_i,\, p\right] &= \left(\frac{(1-\alpha)2Nn_i}{(1-\alpha)2N-1}\frac{n_i^2}{(1-\alpha)2N-1}\right) p_j(1-p_j) \\
&= \left(n_i - \frac{n_i^2}{(1-\alpha)2N}\right) p_j(1-p_j) + O\left(\frac{1}{N}\right)
\end{aligned}
$$

Thus,

$$
Var\left[N_j^D(n)/p\right] = p_j(1-p_j)\sum_i \left(n_i - \frac{n_i^2}{(1-\alpha)2N}\right)\Pr\left[N^B(n) = n_i/p\right] + O\left(\frac{1}{N}\right)
$$

$$
\begin{aligned}
&= p_j(1-p_j)\left\{E\left[N^B(n)/p\right] - \frac{1}{(1-\alpha)2N}E\left[N^B(n)^2/p\right]\right\} + O\left(\frac{1}{N}\right) \\
&= p_j(1-p_j)\left(E\left[N^B(n)/p\right]\right) \\
&\quad -p_j(1-p_j)\frac{1}{(1-\alpha)2N}\left\{Var\left[N^B(n)/p\right] - \left\{E\left[N^B(n)/p\right]\right\}^2\right\} \\
&\quad +O\left(\frac{1}{N}\right)
\end{aligned}
$$

As, $N^B(n) = \sum_j^J \sum_k^J M_{jk}U_{jk} + \sum_j^J \sum_k^J M_{kj}U_{jk}$,and recalling that the probability of a M_{jk} match is equal to the probability of a M_{kj} match, it follows that,

$$
\begin{aligned}
Var\left[N^B(n)/p\right] &= 4\alpha N \sum_j^J \sum_k^J p_j p_k (1 - p_j p_k) U_{jk}^2 \\
&\quad -8\alpha N \sum_j^J \sum_k^J \sum_{i>k}^J p_j^2 p_k p_i U_{jk} U_{ji} + O\left(\frac{1}{N}\right)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
Var\left[\frac{N_j^D}{2N}(n)/p\right] &= \alpha\frac{p_j(1-p_j)}{2N}\left[\sum_j^J \sum_k^J p_k p_j U_{jk}\left(1 + \frac{\alpha}{1-\alpha}\sum_j^J \sum_k^J p_j p_k U_{jk}\right)\right] \\
&\quad -\alpha\frac{p_j(1-p_j)}{(1-\alpha)N^2}\left(1/2\sum_k^J p_j p_k (1-p_j p_k) U_{jk}^2 - \sum_k^J \sum_{i>k} p_j^2 p_k p_i U_{jk} U_{ji}\right) \\
&\quad +O\left(\frac{1}{N}\right) \\
&= \alpha\frac{p_j(1-p_j)}{N}\left(1/2\sum_j^J \sum_k^J p_j p_k U_{jk}\left(1 + \frac{\alpha}{1-\alpha}\sum_j^J \sum_k^J p_j p_k U_{jk}\right)\right) \\
&\quad +O\left(\frac{1}{N}\right) + O\left(\frac{1}{N^2}\right)
\end{aligned}
$$

Thus,

$$
Var\left[\Delta P_j(n)/p\right] = \alpha\frac{p_j(1-p_j)}{N}\left(1 + \frac{U(p,p)}{2}\left(1 + \frac{\alpha}{1-\alpha}U(p,p)\right)\right) + O\left(\frac{1}{N}\right)
$$

Recalling the formula for the covariance term of the multivariate hypergeometric distribution (e.g. Johnson and Kotz 1977, p.92), by the same argument as above, it follows that

$$
Cov\left[\Delta P_j(n), \Delta P_k(n)/p\right] = -\alpha\frac{p_j p_k}{N}\left(1 + \frac{U(p,p)}{2}\left(1 + \frac{\alpha}{1-\alpha}U(p,p)\right)\right) + O\left(\frac{1}{N}\right)
$$

where the $O(1/N)$ terms takes account of terms bounded by $\alpha \sum_k^J U_{jk}/N$. ■

Proof of Lemma 2: Recalling that

$$\begin{aligned} P_j^N((n+1)\theta_N) &= \frac{N_j((n+1)\theta_N)}{2N} \\ &= \frac{(1-\alpha)N_j(n\theta_N)}{2N} + \frac{\sum_k^J M_{jk}(n\theta_N)\left(1+\tilde{U}_{jk}\right)}{2N} \\ &\quad + \frac{\sum_k^J M_{kj}(n\theta_N)\left(1+\tilde{U}_{jk}\right)}{2N} - \frac{N_j^D(n\theta_N)}{2N} \end{aligned}$$

where $\tilde{U}_{jk} = \theta_N U_{jk}$, we have that,

$$\begin{aligned} E\left[\left(\Delta P_j^N(n\theta_N)\right)^4/p\right] &= \frac{1}{16N^4}E\left[\left(\sum_{k=1}^J 2M_{jk}(n\theta_N)\left(1+\tilde{U}_{jk}\right) - \alpha 2Np_j\right)^4/p\right] \\ &\quad + \frac{1}{16N^4}E\left[\left(N_j^D(n\theta_N)\right)^4/p\right] \\ &\quad + \frac{6}{16N^4}E\left[\left(\sum_{k=1}^J 2M_{jk}(n\theta_N)\left(1+\tilde{U}_{jk}\right) - \alpha 2Np_j\right)^2\left(N_j^D(n\theta_N)\right)^2/p\right] \\ &\quad - \frac{4}{16N^4}E\left[\left(\sum_{k=1}^J 2M_{jk}(n\theta_N)\left(1+\tilde{U}_{jk}\right) - \alpha 2Np_j\right)^3 N_j^D(n\theta_N)/p\right] \\ &\quad - \frac{4}{16N^4}E\left[\left(\sum_{k=1}^J 2M_{jk}(n\theta_N)\left(1+\tilde{U}_{jk}\right) - \alpha 2Np_j\right)\left(N_j^D(n\theta_N)\right)^3/p\right] \end{aligned} \quad (5)$$

As for the first term on the right hand side of (5) we know that $M_{jk}(n\theta_N)$ is distributed according to a multinomial, then by Karlin and Taylor (1982, p. 179) and from Ethier and Kurtz (1986, p. 415), we have,

$$\frac{1}{N^4}E\left[\left(\sum_{k=1}^J M_{jk}(n\theta_N)\left(1+\tilde{U}_{jk}\right) - \alpha 2Np_j\right)^4/p\right] = O\left(\frac{1}{N^2}\right)$$

We now consider the second term on the RHS of (5). We know that $\frac{\alpha}{1-\alpha}\max_{j,k} U_{jk} \leq 1$ and $\tilde{U}_{jk} = \theta_n U_{jk} = \frac{K}{N}U_{jk}$. So at most we can have $(1-\alpha)2K$ newborn per period and consequently no more than $(1-\alpha)2K$ individuals will be selected to die. It follows that

$$E\left[\left(\frac{N_j^D(n\theta_N)}{2N}\right)^4/p\right] = O\left(\frac{1}{N^4}\right)$$

The third, fourth and fifth terms are $O\left(1/N^2\right)$ because of Holder inequality, and because

$$E\left[\left(\frac{N_j^D\left(n\theta_N\right)}{2N}\right)^k/p\right]=O\left(\frac{1}{N^k}\right)$$

We have thus shown that the conditional fourth moment is $O(1/N^2)$. ■

REFERENCES

Binmore, K., L. Samuelson and R. Vaughn (1995): "Musical Chairs: Modelling Noisy Evolution", *Games and Economic Behavior*, 11, 1-35.

Börgers, T. and R. Sarin (1995): "Learning Through Reinforcement and Replicator Dynamics", mimeo, University College London and Texas A&M University.

Boylan, R. (1995): "Continuous Approximation of Dynamical Systems with Randomly Matched Individuals", *Journal of Economic Theory*, 66, 615-625.

Cabrales, A. (1993): "Stochastic Replicator Dynamics", mimeo, University of California, San Diego.

Ethier, S. and T. Kurtz (1986): *Markov Processes: Characterization and Convergence*, Wiley.

Foster, D. and P. Young (1990): "Stochastic Evolutionary Game Dynamics", *Theoretical Population Biology,* 38, 219-232.

Fudenberg, D. and C. Harris (1992): "Evolutionary Dynamics with Aggregate Shocks", *Journal of Economic Theory*, 57, 420-441.

Gihman, I. and A.V. Skorohod (1972): *Stochastic Differential Equations*, Springer-Verlag..

Hofbauer, J. and K. Sigmund (1988): *The Theory of Evolution and Dynamical Systems*, Cambridge.

Johnson, N. and S. Kotz (1977): *Urn Models and Their Applications*, Wiley.

Karlin, S. and H. Taylor (1982): *A Second Course in Stochastic Processes*, Academic Press.

Newey, W.K. (1991): "Uniform Convergence in Probability and Stochastic Equicontinuity", *Econometrica*, 59, 1161-1167.

Revuz, D. and M. Yor (1990): *Continuous Martingales and Brownian Motion*, Springer-Verlag.

Strook, D.W. and S.R.S. Varadhan (1979): *Multidimensional Diffusion Processes*, Springer-Verlag.

Weibull, J. (1995): *Evolutionary Game Theory*, MIT Press.