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"Identification and Estimation of Auction Model with Two-Dimensional Unobserved Heterogeneity"

by

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Identification and Estimation of Auction Model with Two-Dimensional Unobserved Heterogeneity

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Abstract

This paper investigates theempirical importance of allowing for multi-dimensional sources of unobserved heterogeneity in auction models with private information. It in turn develops the estimation procedure that recovers the distribution of private information in the presence of two distinct sources of unobserved heterogeneity. It is shown that this estimation procedure identifies components of the model and produces uniformly consistent estimators of these components. The estimation procedure is applied to the data from highway procurement. The results of the estimation indicate that allowing for two-dimensional unobserved heterogeneity may significantly affect the results of estimation as well as policy-relevant instruments derived from the estimated distributions of bidders' costs.

Keywords: unobserved auction heterogeneity, procurement auctions, reserve price.

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1 Introduction

Auctions are extensively used by governments and private organizations as a price-setting mechanism in markets with private information. However, the performance of a specific auction mechanism as well as the choice of the optimal policy instruments (such as reserve price) depend on the exact distribution of private information in a given auction environment. Thus, it is important in empirical auction analysis to be able to non-parametrically identify the distribution of bidders' private information from the available data.

A large literature on non-parametric identification of auction models has emerged to provide a theoretical foundation for empirical analysis. In a seminal contribution, Guerre, Perrigne and Vuong (2000) established that the first-order condition of bidder optimization problem can be used to recover the distribution of private information from the distribution of bids under independent symmetric private values. Subsequent literature extended this result to settings with affiliated private values, asymmetric bidders and settings with risk-averse bidders. An important assumption underlying this literature is that the researcher has access to all the common information available to bidders.

When a researcher may not have access to all the common information incorporated in bidding decisions, the environment is said to feature unobserved auction heterogeneity. More recently, it has been shown that models with independent private values are identified in the presence of unobserved auction heterogeneity. Krasnokutskaya (2009) shows identification and proposes an estimation procedure for the model with an unobserved heterogeneity factor that multiplicatively affects bidders' costs. She shows that accounting for unobserved heterogeneity has important implications for policy analysis. Hu, McAdams and Shum (2008) obtain more general identification result that allows for a flexible relationship between the distribution of bidders' costs and the unobserved heterogeneity factor. These papers, however, assume that the unobserved heterogeneity factor is one-dimensional and, therefore, affects the moments of the distribution of bidders' costs in a coordinated way. The restriction of unobserved heterogeneity to be one-dimensional is potentially an important one. However, the literature provides neither the identification results nor estimation procedure in case of multi-dimensional unobserved heterogeneity. Consequently, little is known about its empirical relevance. This paper attempts to fill this gap in the literature.

In particular, this paper extends the framework in Krasnokutskaya (2009) to allow for two-dimensional unobserved heterogeneity so that independent factors may affect the mean and the variance of the distribution of bidders' costs. I prove that such a model is identified from bid data and show how the identification argument can be translated into an estimation procedure that produces uniformly consistent estimators. The latter step involves significant modification of the argument developed in the one-dimensional case. In the one-dimensional case the consistency argument relies in part on the results from the classical measurement error literature developed by Li and Vuong (1998). However, these results require that all the distributions should have bounded support. In the two-dimensional case the intermediate steps of estimation procedure require working with the distributions that violate this assumption. Therefore, an independent proof of consistency has to be developed which exploits restrictions on the tail behavior of the distributions in question.

I apply the proposed estimation procedure to the data from California highway procurement auctions to investigate the empirical importance of allowing for multi-dimensional unobserved heterogeneity. The results show that allowing for two-dimensional unobserved heterogeneity may significantly affect the results of estimation as well as the choice of policy relevant instruments derived from the estimated distributions of bidders' costs. In particular, I study the data on auctions for (a) bituminous resurfacing and (b) small construction projects. I recover the distributions of the private information and the unobserved heterogeneity under the two alternative assumptions on the structure of unobserved heterogeneity, i.e. one- or two-dimensional. In the latter case two non-trivial components of unobserved heterogeneity are recovered for both sets of projects. However, in the case of bituminous resurfacing, the distribution of private information remains virtually the same under the two specifications. In the case of small construction auctions, the variance of the private cost component almost doubles when going from the model that allows for one-dimensional unobserved heterogeneity to the model that allows for two-dimensional heterogeneity. Similarly, I find only small differences in the mark-ups over the bidders' costs and in the optimal reserve price computed for the two specifications in the set of resurfacing projects. In contrast, for the set of small construction projects, the model that allows for two-dimensional heterogeneity recovers mark-ups which are 30% higher then those recovered in the model that allows for only one-dimensional heterogeneity. Similarly, the optimal reserve price derived from the estimates obtained in the model with two-dimensional heterogeneity results in a cost of procurement which is 15% lower relative to the costs that arise when the reserve price is computed on the basis of the estimates from the model with one-dimensional heterogeneity. These finding indicate that allowing for a flexible relationship between the distribution of bidders' costs and unobserved heterogeneity may have important implications for policy variables and have a sizable economic impact.

Hu, McAdams and Shum (2009) provide a very general identification result allowing for a flexible relationship between the the distribution of bidders' costs and the unobserved heterogeneity factor in the setting with one-dimensional unobserved heterogeneity. They introduce an unobserved project heterogeneity as a factor conditional on which bidders' valuations are independent. The authors require that a functional should exist that extracts the realization of unobserved heterogeneity in a given auction from the auction-specific distribution of bids. They show that if such functional exists then the distribution of valuations conditional on unobserved heterogeneity and the distribution of unobserved heterogeneity are identified. It seems that their argument may be extended to allow for multi-dimensional unobserved heterogeneity. However, the estimation strategy based on this identification result has not yet been developed and, therefore, cannot be used to empirically assess the importance of multi-dimensional unobserved heterogeneity in the data.

The rest of the paper is organized as follows: The remainder of this section discusses the prior literature. Section 2 describes the model with two-dimensional unobserved heterogeneity. Section 3 outlines and proves the identification result. Section 4 describes an estimation algorithm and analyzes statistical properties of the estimation procedure. Section 5 describes the market for highway procurement projects and presents results of the estimation and policy analysis. Section 6 concludes.

1.1 Literature

This paper relates to several strands of the empirical auction literature. The first strand concerns estimation of auction models with private information. These are some of the most influencial papers in this literature. Donald and Paarsch (1993, 1996) and Laffont, Ossard and Vuong (1995) develop parametric methods to recover the distribution of costs from the observed distribution of bids. Guerre, Perrigne and Vuong (2000) study identification of the first price auction model with symmetric bidders and propose a uniformly consistent estimation procedure. Li, Perrigne and Vuong (2000, 2002) extend the result to the affiliated private values and the conditionally independent private values models. Campo, Perrigne and Vuong (2003) prove identification and develop a uniformly consistent estimation procedure for first price auctions with asymmetric bidders and affiliated private values.

The second strand concerns the literature that studies unobserved auction heterogeneity. Campo, Perrigne and Vuong (2003) as well as Bajari and Ye (2003) rely on the assumption that the number of bidders can serve as a sufficient statistic for unobserved auction heterogeneity. Haile, Hong and Shum (2003) appeal to the instrumental variables approach to control for the variation generated by unobserved factors. Hong and Shum (2002) account for unobserved auction heterogeneity by modeling the median of the bid distribution as a normal random variable with a mean that depends on the number of bidders. Athey and Haile (2001) study identification of auction models with unobserved auction heterogeneity in the context of second price and English auctions. Chakraborty and Deltas (1998) assume that the distribution of bidders' valuations belongs to a two-parameter distribution family. They use this assumption to derive small sample estimates for the corresponding parameters of the auction-specific valuation distributions. The estimates are later regressed on observable auction characteristics to determine the percentage of values variation that is due to unobserved auction heterogeneity. Hu, McAdams and Shum (2009), Krasnokutskaya (2009), Guerre, Perrigne, Vuong (2009), Roberts (2008) propose alternative methods to identify auction model with one-dimensional unobserved heterogeneity.

Highway procurement auctions have been extensively studied in the literature. Porter and Zona (1993) find evidence of collusion in Long Island highway procurement auctions. Hong and Shum (2002) find some evidence of common values in bidders' costs in the case of New Jersey highway construction auctions. Bajari and Ye (2003) reject the hypothesis of collusive behavior in procurement auctions conducted in Minnesota, North Dakota and South Dakota. Jofre-Bonet and Pesendorfer (2003) find evidence of capacity constraints in California highway procurement auctions. Bajari and Tadelis (2001) and Bajari, Houghton and Tadelis (2004) study the implications of the incompleteness of procurement contracts. Decarolis (2008) studies Italian highway procurement auctions where the average bid is used to determine the winner.

2 Model

This section describes the first-price procurement auction model under unobserved auction heterogeneity and summarizes properties of the equilibrium bidding strategies.

The seller offers a single project for sale to m bidders. Bidder i's cost is equal to

$$C_i = Y_1 + Y_2 X_{ij} \tag{1}$$

where Y_1 and Y_2 represent common cost components known to all bidders; X_i is an individual cost component and private information of bidder i. I use capital letters to denote random variables summarizing the common and individual cost components. The small letters y_1, y_2 and x denote realizations of common components and the vector of individual components.

The random variables (Y_1, Y_2, X) are distributed on their respective supports $S(Y_1) = [\underline{y}_1, \overline{y}_1]$, $S(Y_2) = [\underline{y}_2, \overline{y}_2]$, $S(X) = [\underline{x}, \overline{x}]^m$, $\underline{y}_2 > 0$, $\underline{x} > 0$, according to the probability distribution functions F_{Y_1}, F_{Y_2}, F_X.

Asymmetries between bidders: I assume that there are two groups of bidders; m_1 bidders are from group 1, and m_2 bidders, $m_2 = (m - m_1)$, are from group 2. Thus, the vector of independent cost components is given by $X = (X_{11}, .., X_{1m_1}, X_{2(m_1+1)}, .., X_{2m})$. The model and all the results can easily be extended to the case of m groups. I focus on the case of two groups for the sake of expositional clarity. Groups are defined from the observable characteristics of bidders.

Assumptions $(D_1) - (D_4)$ are maintained throughout the paper.

(D_1) Y_1, Y_2 and X_j's are mutually independent.

(D_2) The probability density functions of the individual cost components, f_{X_1} and f_{X_2}, are continuously differentiable and strictly positive on the interior of $[\underline{x}, \overline{x}]$.

(D_3) $EY_1 = 0$ and $EX_{1j} = 1$.

(D_4) (a) The number of bidders is common knowledge;[1]

(b) There is no binding reservation price.

The assumption (D_2) ensures the existence and uniqueness of the equilibrium in the auction game; (D_1) and (D_3) provide a basis for the identification argument; assumption (D_3) is used to fix the locations of the common components; and (D_4) summarizes miscellaneous assumptions about the auction environment.

The auction environment can be described as a collection of auction games indexed by the different values of common components. An auction game corresponding to the common components values $y_1 \in [\underline{y}_1, \overline{y}_1]$, $y_2 \in [\underline{y}_2, \overline{y}_2]$ is analyzed below.

In this game, the cost realizations of bidder i are given by $y_1 + y_2 x_i$, for the realization of the individual cost component x_i. The bidding strategy of bidder i is a real-valued function defined on $[\underline{x}, \overline{x}]$

$$\beta_i(.|y_1, y_2) : \ [\underline{x}, \overline{x}] \to [0, \infty].$$

Small Greek letter β with subscript i is used to denote the strategy of bidder i as a function of the individual cost components and a small Roman letter b to denote the value of this function at a particular realization x.

Expected profit. The profit realization of bidder i, $\pi_i(b_i, b_{-i}, x_i|y_1, y_2)$, equals $(b_i - y_1 - y_2 x_i)$ if bidder i wins the project and zero if he loses. The symbol b_i denotes the bid submitted by bidder i, and the symbol b_{-i} denotes the vector of bids submitted by bidders

[1]Note that the model does not assume that the number of bidders is exogenous. All the results in this paper are valid if the number of bidders is endogenous and depends on the realization of unobserved heterogeneity. For the details of the model with endogenous participation see Krasnokutskaya and Seim (2009).

other than i. At the time of bidding, bidder i knows (y_1, y_2) and x_i but not b_{-i}. The bidder who submits the lowest bid wins the project. The interim expected profit of bidder i is given by

$$E[\pi_i|X_i = x_i, Y_l = y_l] = (b_i - y_1 - y_2 x_i) \Pr(b_i \leq b_j, \forall j \neq i|X_i = x_i, Y_l = y_l).$$

A Bayesian Nash equilibrium is then characterized by a vector of functions

$$\beta(.|y_1, y_2) = \{\beta_1(.|y_1, y_2), ..., \beta_m(.|y_1, y_2)\} \text{ such that } b_{y_1,y_2;i} = \beta_i(x_i|y_1, y_2) \text{ maximizes}$$
$$E[\pi_i|X = x_i, Y_l = y_l], \text{ when } b_{y_1,y_2;j} = \beta_j(x_j|y_1, y_2), \ j \neq i, \ j = 1, .., m;$$
$$\text{for every } i = 1, .., m \text{ and for every realization of } X_i.$$

McAdams (2003) and others establish that, under assumptions $(D_1)-(D_2)$, a vector of equilibrium bidding strategies $\beta(.|y_1, y_2) = \{\beta_1(.|y_1, y_2), ..., \beta_m(.|y_1, y_2)\}$ exists and is unique. The strategies are strictly monotone and differentiable.

Next, I characterize a simple property of the equilibrium bidding strategies.

Proposition 1

If $(\alpha_1(.), ..., \alpha_m(.))$ *is a vector of equilibrium bidding strategies in the game with* $y_1 = 0$ *and* $y_2 = 1$*, then the vector of equilibrium bidding strategies in the game with* (y_1, y_2)*,* $y_l \in [\underline{y}_l, \overline{y}_l]$*, is given by* $\beta_i(.|y_1, y_2) = \{\beta_1(.|y_1, y_2), ..., \beta_m(.|y_1, y_2)\}$*, such that* $\beta_i(x_i|y_1, y_2) = y_1 + y_2\alpha_i(x_i)$*,* $i = 1, ..., m$*.*

The proposition shows that the bid function has a factor structure similar to costs with the individual bid component given by $\alpha_i(.)$. The proof of this proposition is based on the comparison of two sets of first-order conditions and follows immediately from the assumption that the factor structure of bidders' costs and the common components are known to all bidders.

The equilibrium inverse individual bid function for a group "k" bidder is denoted by ξ_k. Since the function $\alpha_k(.)$ is strictly monotone and differentiable, the function $\xi_k(.)$ is well-defined and differentiable. The necessary first-order conditions for the set of equilibrium strategies when $y_1 = 0, y_2 = 1$ are then given by

$$\frac{1}{a - \xi_{k(i)}(a)} = (m_{k(i)} - 1)\frac{f_{X_{k(i)}}(\xi_{k(i)}(a))\xi'_{k(i)}(a)}{1 - F_{X_{k(i)}}(\xi_{k(i)}(a))} + m_{-k(i)}\frac{f_{X_{-k(i)}}(\xi_{-k(i)}(a))\xi'_{-k(i)}(a)}{1 - F_{X_{-k(i)}}(\xi_{-k(i)}(a))}, \quad (2)$$

where $\xi'_k(.)$ denotes the derivative of $\xi_k(.)$.

Equation (2) characterizes the equilibrium inverse individual bid function when $y_1 = 0$ and $y_2 = 1$. It describes a trade-off the bidder faces when choosing a bid: an increase in the mark-up over the cost may lead to a higher ex-post profit if bidder i wins, but it reduces the probability of winning. The bid a is chosen in such a way that the marginal effects of an infinitesimal change in a bid on the winner's profit and the probability of winning sum to zero.

3 Identification

I assume that the econometrician has access to bid data, based on n independent draws from the joint distribution of (Y_1, Y_2, X). The observable data are in the form $\{b_{ij}\}$, where i denotes the identity of the bidder, $i = 1, .., m$; and j denotes project, $j = 1, ..., n$. If data represent equilibrium outcomes of the model with two-dimensional unobserved auction heterogeneity, then

$$b_{ij} = \beta_{k(i)}(x_{ij}|y_{1j}, y_{2j}) \tag{3}$$

(i.e., b_{ij} is a value of bidder i's equilibrium bidding strategy corresponding to (y_{1j}, y_{2j}) evaluated at the point x_{ij}).

I use B_i to denote the random variable that describes the bid of bidder i of group $k(i)$ with distribution function $G_{B_{k(i)}}$ and the associated probability density function $g_{B_{k(i)}}$; b_{ij} denotes the realization of this variable in auction j. The econometrician observes the joint distribution function of $(B_{i_1}, .., B_{i_l})$ for all subsets $(i_1, ..., i_l)$ of $(1, ..., m)$[2].

As was shown in the previous section, b_{ij} depends on the realizations of the common and individual cost components as well as on the distributions of the individual cost components. This section examines under what conditions on available data there exists a unique tuple $\{\{x_{ij}\}, F_{Y_1}, F_{Y_2}, F_X\}$ that satisfies (3), i.e., under what conditions the model from a previous section is identified.

Proposition 1 establishes that

$$b_{ij} = y_{1j} + y_{2j} a_{ij},$$

where a_{ij} is a hypothetical bid that would have been submitted by bidder i if y_1 were equal to zero and y_2 were equal to one. I use A_i to denote the random variable with realizations equal to a_{ij}. The associated distribution function is denoted by $G_{A_{k(i)}}$ with the probability density function $g_{A_{k(i)}}$. Notice that the econometrician does not observe (y_{1j}, y_{2j}) and

[2] In fact, it is not necessary to observe joint distribution for all subsets. For details, see the formulation of Theorem 1.

neither therefore a_{ij}. The distribution of A_i is latent.

The following theorem is the main result of this section. It formulates sufficient identification conditions for the model with two dimensional unobserved heterogeneity.

Theorem 1

If conditions $(D_1)-(D_4)$ are satisfied, then the probability density functions f_{Y_1}, f_{Y_2} are uniquely identified from the joint distribution of four arbitrary bids $(B_{i_1}, B_{i_2}, B_{i_3}, B_{i_4})$. The probability density functions f_{X_j}, $j = 1, 2$, are also uniquely identified from the joint distribution of four arbitrary bids $(B_{i_1}, B_{i_2}, B_{i_3}, B_{i_4})$ if $k(i_l) = j$ for some $l = 1, ..., 4$. [3].

The proof of Theorem 1 relies on a statistical result by Kotlarski (1966),[4] which establishes that the marginal distributions of mutually independent random variables (Z_1, Z_2, Z_3) are identified from the joint distribution of random variables (W_1, W_2) such that

$$W_1 = Z_1 + Z_3, \;\; W_2 = Z_1 + Z_3.$$

This result requires that the characteristic functions of Z_1, Z_2, Z_3 should be non-vanishing. Under these conditions it is possible to solve for the characteristic functions of $Z_i's$ from the joint characteristic function of (W_1, W_2). More specifically, let $\Psi(.,.)$ and $\Psi_1(.,.)$ denote the joint characteristic function of (W_1, W_2) and the partial derivative of this characteristic function with respect to the first component respectively. Also, let $\Phi_{Z_i}(.)$ denote characteristic functions of $Z_i's$. Then,

$$\begin{aligned}
\Phi_{Z_3}(t) &= \exp\Big(\int_0^t \frac{\Psi_1(0,u_2)}{\Psi(0,u_2)} du_2 - itE[Z_1]\Big), \qquad (4)\\
\Phi_{Z_1}(t) &= \frac{\Psi(t,0)}{\Phi_{Z_3}(t)},\\
\Phi_{Z_2}(t) &= \frac{\Psi(0,t)}{\Phi_{Z_3}(t)},
\end{aligned}$$

Once characteristic functions of Z_1, Z_2, Z_3 are known the probability density functions of $Z_i's$ can be recovered using inverse Fourier transformation. In fact, since there is a one-to-one distribution between characteristic and density functions, the distribution of random variable is identified if the characteristic function of this distribution can be recovered.

[3] Therefore, at least four bids are needed to identify the model with two-dimensional unobserved heterogeneity.

[4] See Rao (1992).

Proof

Lemma 1 (see Appendix) establishes that all the random variables considered in this proof have non-vanishing characteristic functions. The rest of the proof is organized in 3 steps.

Step 1

First, I form the pair-wise bid differences for two pairs of distinct bids: $W_{i_1,i_2} = B_{i_1} - B_{i_2}$ and $W_{i_3,i_4} = B_{i_3} - B_{i_4}$. The identification of the probability density function f_{Y_2} is established by applying Kotlarski's argument to the joint distribution of $(\log W_{i_1,i_2},\ \log W_{i_3,i_4})$ conditional on $(W_{i_1,i_2} > 0, W_{i_3,i_4} > 0)$. The later condition is equivalent to $(A_{i_1} - A_{i_2} > 0, A_{i_3} - A_{i_4} > 0)$. Since there is no special rule according to which indexes (i_1, i_2, i_3, i_4) are fixed, then $\log(A_{i_1} - A_{i_2})$ and $\log(A_{i_3} - A_{i_4})$ conditional on $(A_{i_1} - A_{i_2} > 0, A_{i_3} - A_{i_4} > 0)$ are independent of each other and of $\log(Y_2)$. Therefore, conditions of Kotlarski's theorem are satisfied. At this point I impose normalization $E[\log(Y_2)] = 0$. I will re-adjust recovered distributions later so as to satisfy condition (D_3).

Step 2

(a) The joint characteristic function of W_{i_1,i_3} and W_{i_2,i_3} conditional on $W_{i_1,i_3} > 0$, $W_{i_2,i_3} > 0$ together with the characteristic function of Y_2 (identified in (a)) identifies the joint characteristic functions and therefore joint distributions of $(A_{i_1} - A_{i_3},\ A_{i_2} - A_{i_3})$ conditional on $(A_{i_1} - A_{i_3} > 0,\ A_{i_2} - A_{i_3} > 0)$. The joint distributions of $(A_{i_1} - A_{i_3},\ A_{i_2} - A_{i_3})$ conditional on $(A_{i_1} - A_{i_3} > 0,\ A_{i_2} - A_{i_3} < 0)$, $(A_{i_1} - A_{i_3} < 0,\ A_{i_2} - A_{i_3} > 0)$, $(A_{i_1} - A_{i_3} < 0,\ A_{i_2} - A_{i_3} < 0)$ are identified in a similar way. The probabilities of observing $(A_{i_1} - A_{i_3} > 0,\ A_{i_2} - A_{i_3} > 0)$, $(A_{i_1} - A_{i_3} > 0,\ A_{i_2} - A_{i_3} < 0)$, $(A_{i_1} - A_{i_3} < 0,\ A_{i_2} - A_{i_3} > 0)$ or $(A_{i_1} - A_{i_3} < 0,\ A_{i_2} - A_{i_3} < 0)$ are identified from the data. Therefore, the joint distribution of $(A_{i_1} - A_{i_3},\ A_{i_2} - A_{i_3})$ is also identified.

(b)The Kotlarski argument, then, is applied to the joint distribution of $(A_{i_1} - A_{i_3},\ A_{i_2} - A_{i_3})$ to identify the probability density functions of A_{i_1}, A_{i_2} and A_{i_3} under normalization that $E[A_{i_1}] = 0$.

(c) The argument developed in Laffont and Vuong (1996) and used in Krasnokutskaya (2009) establishes identification of the probability density functions of $X_{i_1}, X_{i_2}, X_{i_3}$ from the probability distributions of A_{i_1}, A_{i_2} and A_{i_3}.

(d) Let e_{Y_2} and e_{X_1} denote the expectations of Y_2 and X_1 under above normalization, then the random variables $\tilde{Y}_2 = \frac{Y_2}{e_Y}$, $\tilde{X}_1 = e_Y X_1 - e_Y e_{X_1} + 1$ and $\tilde{X}_2 = e_Y X_2 - e_Y e_{X_1} + 1$ represent components of the model that corresponds to the normalization postulated in (D_3).

Step 3

The probability density functions $g_{A_{i_1}}$, f_{Y_2} uniquely determine the probability distribution and thus the characteristic function of $Y_2 \cdot A_{i_1}$, which allows unique identification of the probability distribution of Y_1 from the characteristic function of B_{i_1}. **End of proof.**

Thus, f_{Y_1}, f_{Y_2}, f_{X_1}, f_{X_2} are identified from the joint distribution of four arbitrary bids. Similar to the one-dimensional case, the exact realizations of y_{1j}, y_{2j} and $\{x_{ij}\}$ are not uniquely identified.

4 Estimation

The econometrician has data for n auctions. For each auction j, $(m_j, \{b_{ij}\}_{i=1}^{i=m_j}, z_j)$ are observed, where m_j is the number of bidders in the auction j, with m_{j1} bidders of group 1 and m_{j2} bidders of group 2; $\{b_{ij}\}_{i=1}^{i=m_j}$ is a vector of bids submitted in the auction j; and z_j is a vector of auction characteristics.

In the estimation procedure which follows the observable covariates could be handled in two ways. An index assumption could be made, i.e. $c_{ij} = \mu_j + \sigma_j(y_{1j} + y_{2j}x_{ij})$ where $\mu_j = z_j\alpha$ and $\sigma_j = z_j\gamma$. From Proposition 1 it follows that $b_{ij} = \mu_j + \sigma_j(y_{1j} + y_{2j}b^0_{ij})$. Then, in the first step the indices μ_j and σ_j are estimated conditional on the number of bidders and normalized bids are formed: $b^0_{ij} = (b_{ij} - \mu_j)/\sigma_j$. The remaining steps of estimation procedure are applied to the normalized bids. I follow this procedure in the empirical part of this paper. Alternatively, the estimation steps below could be implemented conditional on the observable project characteristics. More specifically, the researcher should condition on discrete attributes and use kernel smoothing over the continuous attributes.

The steps of the estimation procedure closely follow the steps of identification argument. I assume that at least four bids, $(B_{i1}, B_{i2}, B_{i3}, B_{i4})$ are available per project. For the convenience of exposition it is assumed that index i_1 corresponds to the bids submitted by the bidders from the group 1 whereas all other bids are submitted by the bidders from the group 2. It is straightforward to adjust the steps of estimation procedure if the configuration of bidder set is different. Finally, I use $\Delta_{k,l}X$ to denote the difference between the observations of variable X subscripted i_k and i_l, i.e. $\Delta_{k,l}X = X_{i_k} - X_{i_l}$; $L\Delta_{k,l}X$ denotes logarithm of $\Delta_{kl}X$.

Step 1

1. First, the researcher selects a subsample such that $(B_{i_1} - B_{i_2}) > 0$, $(B_{i_3} - B_{i_4}) > 0$. Let us denote the number of projects in this subsample by n_{01}. This subsample is

used to estimate the joint characteristic function of $(\log(B_{i_1} - B_{i_2}),\ \log(B_{i_3} - B_{i_4}))$ as

$$\hat{\Psi}_{(L(\Delta_{1,2}B),L(\Delta_{3,4}B))}(t_1, t_2) = \frac{1}{n}\sum_{j=1}^{n_{01}} \exp(it_1 \log(B_{i_1} - B_{i_2}) + it_2 \log(B_{i_3} - B_{i_4}))$$

and the derivative of $\Psi(.,.)$ with respect to the first argument, $\Psi_1(.,.)$, by

$$\hat{\Psi}_{1,(L(\Delta_{1,2}B),L(\Delta_{3,4}B))}(t_1, t_2) = \frac{1}{n}\sum_{j=1}^{n_{01}} i\log(B_{i_1}-B_{i_2})\exp(it_1 \log(B_{i_1}-B_{i_2})+it_2 \log(B_{i_3}-B_{i_4})).$$

The researcher should average over all possible quadruples to enhance efficiency. If bidders are symmetric, the efficiency could be further improved by using $(-(B_{i_1} - B_{i_2}),\ -(B_{i_3} - B_{i_4}))$ for $B_{i_1} - B_{i_2} < 0,\ B_{i_3} - B_{i_4} < 0$.

2. The characteristic function of $\log(Y_2)$ is estimated as

$$\hat{\varphi}_{LY_2}(t) = \exp(\int_0^t \frac{\hat{\Psi}_{1,(L(\Delta_{1,2}B),L(\Delta_{3,4}B))}(0, u_2)}{\hat{\Psi}_{(L(\Delta_{1,2}B),L(\Delta_{3,4}B))}(0, u_2)} du_2 - itE[\log(B_{i_1} - B_{i_2})]).$$

Here I adopt normalization $E[\log(Y_2)] = 0$. As in the identification argument the researcher would re-normalize all the variables in the later steps.

3. Next, I use inversion formula to estimate $\tilde{f}_{LY_2}(.)$.

$$\tilde{f}_{LY_2}(y) = \frac{1}{2\pi}\int_{-T}^{T} \exp(-ity)\hat{\Phi}_{LY_2}(t)dt$$

for $y \in S(\log Y_2)$, where T is a smoothing parameter.

4. Finally, I obtain $\tilde{f}_{Y_2}(.)$ as

$$\tilde{f}_{Y_2}(y) = \frac{\tilde{f}_{LY_2}(\log(y))}{y}$$

for $y \in S(Y_2)$.

Step 2

1. I use $\hat{\varphi}_{LY_2}(t)$ to estimate the joint characteristic function of $(\log(A_{i_1} - A_{i_3}),\ \log(A_{i_2} - A_{i_3}))$ from the subsample with $(B_{i_1} - B_{i_3} > 0,\ B_{i_2} - B_{i_3} > 0)$

and, therefore, $(A_{i_1} - A_{i_3} > 0,\ A_{i_1} - A_{i_3} > 0)$

$$\hat{\varphi}_{L(\Delta_{1,3}A),L(\Delta_{2,3}A)}(t_1,t_2) = \frac{\widehat{\Psi}_{(L(\Delta_{1,2}B),L(\Delta_{3,4}B))}(t_1,t_2)}{\widehat{\varphi}_{LY_2}(t_1+t_2)}.$$

Similarly, I obtain

$$\begin{gathered}
\text{From subsample with } (\Delta_{1,3}B < 0,\ \Delta_{2,3}B < 0): \\
\hat{\varphi}_{L(-\Delta_{1,3}A),L(-\Delta_{2,3}A)}(t_1,t_2) = \frac{\hat{\Psi}_{(L(-\Delta_{1,2}B),L(-\Delta_{3,4}B))}(t_1,t_2)}{\hat{\varphi}_{LY_2}(t_1+t_2)}; \\
\text{from subsample with } (\Delta_{1,3}B < 0,\ \Delta_{2,3}B > 0): \\
\hat{\varphi}_{L(-\Delta_{1,3}A),L(\Delta_{2,3}A)}(t_1,t_2) = \frac{\hat{\Psi}_{(L(-\Delta_{1,2}B),L(\Delta_{3,4}B))}(t_1,t_2)}{\hat{\varphi}_{LY_2}(t_1+t_2)}; \\
\text{from subsample with } (\Delta_{1,3}B > 0,\ \Delta_{2,3}B < 0): \\
\hat{\varphi}_{L(\Delta_{1,3}A),L(-\Delta_{2,3}A)}(t_1,t_2) = \frac{\hat{\Psi}_{(L(\Delta_{1,2}B),L(-\Delta_{3,4}B))}(t_1,t_2)}{\hat{\varphi}_{LY_2}(t_1+t_2)}.
\end{gathered}$$

2. I use the inversion formula to obtain

$$\begin{gathered}
\tilde{f}^{(1)}_{L(\Delta_{1,3}A),L(\Delta_{2,3}A)}(u_1,u_2) = \frac{1}{(2\pi)^2}\int_{-T}^{T}\int_{-T}^{T}\exp(-it_1u_1 - it_2u_2)\hat{\varphi}_{L(\Delta_{1,3}A),L(\Delta_{2,3}A)}(t_1,t_2)dt \\
\text{conditional on } (\Delta_{1,3}A > 0, \Delta_{2,3}A > 0), \text{ for } u_1 \in S(L\Delta_{1,3}A|\Delta_{1,3}A > 0), \\
u_2 \in S(L\Delta_{2,3}A|\Delta_{2,3}A > 0); \\
\tilde{f}^{(2)}_{L(-\Delta_{1,3}A),L(-\Delta_{2,3}A)}(u_1,u_2) = \frac{1}{(2\pi)^2}\int_{-T}^{T}\int_{-T}^{T}\exp(-it_1u_1 - it_2u_2)\hat{\varphi}_{L(-\Delta_{1,3}A),L(-\Delta_{2,3}A)}(t_1,t_2)dt \\
\text{conditional on } (\Delta_{1,3}A < 0, \Delta_{2,3}A < 0), \text{ for } u_1 \in S(L(-\Delta_{1,3}A)|\Delta_{1,3}A < 0), \\
u_2 \in S(L(-\Delta_{2,3}A)|\Delta_{2,3}A < 0); \\
\tilde{f}^{(3)}_{L(-\Delta_{1,3}A),L(\Delta_{2,3}A)}(u_1,u_2) = \frac{1}{(2\pi)^2}\int_{-T}^{T}\int_{-T}^{T}\exp(-it_1u_1 - it_2u_2)\hat{\varphi}_{L(-\Delta_{1,3}A),L(\Delta_{2,3}A)}(t_1,t_2)dt \\
\text{conditional on } (\Delta_{1,3}A < 0, \Delta_{2,3}A > 0), \text{ for } u_1 \in S(L(-\Delta_{1,3}A)|\Delta_{1,3}A < 0), \\
u_2 \in S(L\Delta_{2,3}A|\Delta_{2,3}A > 0); \\
\tilde{f}^{(4)}_{L(\Delta_{1,3}A),L(-\Delta_{2,3}A)}(u_1,u_2) = \frac{1}{(2\pi)^2}\int_{-T}^{T}\int_{-T}^{T}\exp(-it_1u_1 - it_2u_2)\hat{\varphi}_{L(\Delta_{1,3}A),L(-\Delta_{2,3}A)}(t_1,t_2)dt \\
\text{conditional on } (\Delta_{1,3}A > 0, \Delta_{2,3}A < 0), \text{ for } u_1 \in S(L\Delta_{1,3}A|\Delta_{1,3}A > 0), \\
u_2 \in S(L(-\Delta_{2,3}A)|\Delta_{2,3}A < 0).
\end{gathered}$$

3. Next, I derive

$$\tilde{f}^{(1)}_{\Delta_{1,3}A,\Delta_{2,3}A}(u_1,u_2) = \frac{\tilde{f}^{(1)}_{L(\Delta_{1,3}A),L(\Delta_{2,3}A)}(\log(u_1),\log(u_2))}{u_1,u_2}$$

conditional on $(\Delta_{1,3}A > 0, \Delta_{2,3}A > 0)$, for $u_1 \in S(L\Delta_{1,3}A|\Delta_{1,3}A > 0)$, $u_2 \in S(L\Delta_{2,3}A|\Delta_{2,3}A > 0)$;

$$\tilde{f}^{(2)}_{-\Delta_{1,3}A,-\Delta_{2,3}A}(u_1,u_2) = \frac{\tilde{f}^{(2)}_{L(-\Delta_{1,3}A),L(-\Delta_{2,3}A)}(\log(u_1),\log(u_2))}{u_1,u_2}$$

conditional on $(\Delta_{1,3}A < 0, \Delta_{2,3}A < 0)$, for $u_1 \in S(L(-\Delta_{1,3}A)|\Delta_{1,3}A < 0)$, $u_2 \in S(L(-\Delta_{2,3}A)|\Delta_{2,3}A < 0)$;

$$\tilde{f}^{(3)}_{-\Delta_{1,3}A,\Delta_{2,3}A}(u_1,u_2) = \frac{\tilde{f}^{(3)}_{L(-\Delta_{1,3}A),L(\Delta_{2,3}A)}(\log(u_1),\log(u_2))}{u_1,u_2}$$

conditional on $(\Delta_{1,3}A < 0, \Delta_{2,3}A > 0)$, for $u_1 \in S(L(-\Delta_{1,3}A)|\Delta_{1,3}A < 0)$, $u_2 \in S(L\Delta_{2,3}A|\Delta_{2,3}A > 0)$;

$$\tilde{f}^{(4)}_{\Delta_{1,3}A,-\Delta_{2,3}A}(u_1,u_2) = \frac{\tilde{f}^{(4)}_{L(\Delta_{1,3}A),L(-\Delta_{2,3}A)}(\log(u_1),\log(u_2))}{u_1,u_2}$$

conditional on $(\Delta_{1,3}A > 0, \Delta_{2,3}A < 0)$, for $u_1 \in S(L\Delta_{1,3}A|\Delta_{1,3}A > 0)$, $u_2 \in S(L(-\Delta_{2,3}A)|\Delta_{2,3}A < 0)$.

4. I use frequency estimators[5]

$$\hat{Prob}(\Delta_{1,3}B > 0,\ \Delta_{2,3}B > 0) = \frac{1}{n_{01}}\sum_{j=1}^{n_{01}} I(\Delta_{1,3}B > 0,\ \Delta_{2,3}B > 0);$$

$$\hat{Prob}(\Delta_{1,3}B < 0,\ \Delta_{2,3}B < 0) = \frac{1}{n_{02}}\sum_{j=1}^{n_{02}} I(\Delta_{1,3}B < 0,\ \Delta_{2,3}B < 0);$$

$$\hat{Prob}(\Delta_{1,3}B < 0,\ \Delta_{2,3}B > 0) = \frac{1}{n_{03}}\sum_{j=1}^{n_{03}} I(\Delta_{1,3}B < 0,\ \Delta_{2,3}B > 0);$$

$$\hat{Prob}(\Delta_{1,3}B > 0,\ \Delta_{2,3}B < 0) = \frac{1}{n_{04}}\sum_{j=1}^{n_{04}} I(\Delta_{1,3}B > 0,\ \Delta_{2,3}B < 0);$$

to obtain the probability density function of the unconditional distribution of

[5] Here n_{02} is the number of projects with $(\Delta_{1,3}B < 0,\ \Delta_{2,3}B < 0)$, n_{03} is the number of projects with $(\Delta_{1,3}B < 0,\ \Delta_{2,3}B > 0)$, n_{04} is the number of projects with $(\Delta_{1,3}B > 0,\ \Delta_{2,3}B < 0)$.

$(A_{i_1} - A_{i_3},\ A_{i_2} - A_{i_3})$:

$$
\begin{aligned}
\tilde{f}_{\Delta_{1,3}A,\Delta_{2,3}A}(u_1,u_2) = \tilde{f}^{(1)}_{\Delta_{1,3}A,\Delta_{2,3}A}(u_1,u_2)\hat{Prob}(\Delta_{1,3}B > 0,\ \Delta_{2,3}B > 0) + \\
\tilde{f}^{(2)}_{-\Delta_{1,3}A,-\Delta_{2,3}A}(-u_1,-u_2)\hat{Prob}(\Delta_{1,3}B < 0,\ \Delta_{2,3}B < 0) + \\
\tilde{f}^{(3)}_{-\Delta_{1,3}A,\Delta_{2,3}A}(-u_1,u_2)\hat{Prob}(\Delta_{1,3}B < 0,\ \Delta_{2,3}B > 0) + \\
\tilde{f}^{(4)}_{\Delta_{1,3}A,-\Delta_{2,3}A}(u_1,-u_2)\hat{Prob}(\Delta_{1,3}B > 0,\ \Delta_{2,3}B < 0).
\end{aligned}
$$

5. This allows us to construct

$$
\begin{aligned}
\hat{\Phi}_{\Delta_{1,3}A,\Delta_{2,3}A}(t_1,t_2) &= \int\int \exp(it_1u_1 + it_2u_2)\tilde{f}_{\Delta_{1,3}A,\Delta_{2,3}A}(u_1,u_2)du_1\,du_2 \\
\hat{\Phi}_{1,\Delta_{1,3}A,\Delta_{2,3}A}(t_1,t_2) &= \int\int iu_1 \exp(it_1u_1 + it_2u_2)\tilde{f}_{\Delta_{1,3}A,\Delta_{2,3}A}(u_1,u_2)du_1\,du_2.
\end{aligned}
$$

6. The characteristic functions of the individual bid components A_{i_k}, $k = 1,3$, are estimated as

$$
\begin{aligned}
\hat{\varphi}_{A_{i_3}}(t) &= \exp\Big(\int_0^t \frac{\hat{\Psi}_{1,\Delta_{1,3}A,\Delta_{2,3}A}(0,u_2)}{\hat{\Psi}_{\Delta_{1,3}A,\Delta_{2,3}A}(0,u_2)}du_2 - itE[A_{i_1}]\Big), \\
\hat{\varphi}_{A_{i_k}}(t) &= \frac{\hat{\Psi}_{\Delta_{k,3}A,\Delta_{-k,3}A}(t,0)}{\hat{\Phi}_{A_{i_3}}(t)} \text{ for } k = 1,2.
\end{aligned}
$$

Here I use normalization that $E[A_{i_3}] = 0$. I re-normalize all the variables in the later steps.

7. The inversion formula is used to estimate densities $\widetilde{g}_{A_{i_k}}$, $k = 1,3$,

$$
\tilde{f}_{A_{i_k}}(u) = \frac{1}{2\pi}\int_{-T}^{T} \exp(-itu)\hat{\Phi}_{A_{i_k},n}(t)dt.
$$

8. The individual inverse bid function at a point $a \in S(A_k)$ is estimated as

$$
\hat{\xi}_{k,n}(a) = a - \frac{(1-\tilde{F}_{A_1,n}(a))\cdot(1-\tilde{F}_{A_2,n}(a))}{(m_k-1)\cdot\tilde{f}_{A_k,n}(a)\cdot(1-\tilde{F}_{A_{-k},n}(a)) + m_{-k}\cdot\tilde{f}_{A_{-k},n}(a)\cdot(1-\tilde{F}_{A_k,n}(a))}
$$

where

$$\tilde{F}_{A_k,n}(a) = \int_{\underline{\hat{a}}_n}^{a} \tilde{f}_{A_k,n}(z)dz$$

and $\underline{\hat{a}}_n$ is an estimate of the lower bound of the support of $f_{A_k}(.)$, which corresponds to the normalizations $E[\log Y_2] = 0$ and $E[A_{i_3}] = 0$ (see the Appendix for the discussion of the support estimation).

9. Here the re-normalization should be performed as described in step 2 of the identification argument.

Step 3

1. I estimate $f_{Y_2 A_{i_1}}(.)$ as

$$\hat{f}_{Y_2 A_{i_1}}(z) = \int_{S(Y_2)} \frac{1}{y} \tilde{f}_{A_{i_1}}(\frac{z}{y}) \hat{f}_{Y_2}(y)\, dy.$$

2. I then estimate $\varphi_{Y_2 A_{i_1}}(t)$ and $\varphi_{Y_1}(t)$ as

$$\hat{\varphi}_{Y_2 A_{i_1}}(t) = \int_{S(Y_2 A_{i_1})} \exp(itu) \hat{f}_{Y_2 A_{i_1}}(u) du$$

$$\hat{\varphi}_{Y_1}(t) = \frac{\hat{\Phi}_{B_{i_1}}(t)}{\hat{\varphi}_{Y_2 A_{i_1}}(t)}, \text{ where}$$

$$\hat{\Phi}_{B_{i_1}}(t) = \frac{1}{n} \sum_{j=1}^{n} \exp(itB_{i_1}).$$

3. The inversion formula is used to estimate the density $\hat{f}_{Y_1}$

$$\hat{f}_{Y_1}(y) = \frac{1}{2\pi} \int_{-T}^{T} \exp(-ity) \hat{\varphi}_{Y_1}(t) dt.$$

5 Properties of the Estimators

The estimation procedure yields uniformly consistent estimators of the relevant distributions. This result is derived under the following restrictions on the tail behavior of characteristic functions.

(D_5) The characteristic functions φ_{LY_2}, φ_{Y_1}, φ_{LA_k}, φ_{A_k} and $\varphi_{Y_2A_k}$ are ordinary-smooth.[6]

This property holds, for example, when cumulative probability functions of cost components admit up to R, $R > 1$ continuous derivatives on the support interior such that M of them, $1 \leq M \leq R$, can be continuously extended to the real line.

Theorem 2 summarizes properties of the estimator.

Theorem 2

If conditions (D_1)−(D_5) are satisfied, then $\hat{f}_{Y_1}$, $\hat{f}_{Y_2}$ and $\hat{f}_{X_k}$ are uniformly consistent estimators of f_{Y_1}, f_{Y_2} and f_{X_k}, $k = 1, 2$, respectively.

Notice that in this setting I cannot directly apply results obtained in Li and Vuong (1998) on the uniform consistency of the estimators derived from the Kotlarski's theorem. This is because their results require that all the random variables involved have bounded support. This property does not hold in this setting. The random variables A_{i_k}, $k = 1, .., 3$, have the same support. As a result the support of $(A_{i_k} - A_{i_l})$ contains zero and the support of $\log(A_{i_k} - A_{i_l})$ conditional on $A_{i_k} - A_{i_l} > 0$ is given by $(-\infty, M]$ for some $M > 0$. In order to derive the uniform convergence of estimators in the case with unbounded support I will exploit the tail behavior of $\log(A_{i_k} - A_{i_l})$ which is established in Lemma 3 (in the Appendix).

6 Application

I apply the methodology presented in Section 4 to data from highway procurement auctions. I use data provided by the California Department of Transportation (CalTrans), which is responsible for construction and maintenance of roads and highways within California. CalTrans allocates the work which needs to be done to companies in the form of projects through a first price sealed bid auction. The project usually involves a small number of tasks, such as resurfacing or replacing the base or filling in cracks.

Projects are advertised four weeks prior to the letting date. Companies interested in the project can obtain a detailed description from CalTrans. CalTrans constructs a cost estimate for every project. This estimate is based on the engineer's assessment of the work required to perform each task and prices derived from the winning bids for similar projects let in the past. The costs are then adjusted through a price deflator. The reserve price,

[6]Following Fan (1991): The distribution of random variable Z is ordinary-smooth of order $\varkappa$ if its characteristic function $\Phi_z(t)$ satisfies $d_0|t|^{-\varkappa} \leq |\Phi_z(t)| \leq d_1|t|^{-\varkappa}$ as t$\to \infty$ for some positive constants d_0,d_1,$\varkappa$.with $\varkappa > 1$.

while formally present, is not enforced.

It is unclear if the auction participants have a good idea about the number of their competitors. The existing literature on highway procurement auctions tends to argue that this is a small market where participants are well informed about each other and can accurately predict the identities of auction participants.[7] I follow this tradition and assume that the number of actual bidders is known to auction participants.

I allow for cost asymmetries between bidders. In particular, I distinguish between two types of bidders: regular (large) bidders and fringe bidders. The set of regular bidders is defined to include companies that consistently won at least $10 million in projects during each year in my data set and have at least 100 employees.

The analysis focuses on two types of projects: (1) bituminous resurfacing and (2) small construction projects. The projects in the first set involve stripping the old surface off, correcting the road base and laying out new surface. These projects are quite similar and well defined. After I control for the size of the project, time allocated, location and type of road, the remaining variation (not observed in the data) is associated with possible curvature, incline or elevation of the road, ground conditions, etc. In comparison, small construction projects usually involve building small parking lots, culverts and small bridges. The projects in this set are less homogeneous and may have substantial amount of project-specific variation which is difficult to summarize in the data. Such projects usually involve excavation, levering the ground, laying the base, building a stand alone structure, etc. They are much simpler than projects in the construction category because they involve building simple objects according to known and well-defined blueprints. The completion of such project does not require a lot of time and therefore is not associated with long-run risks, planning and commitments.

Table 3 provides summary statistics for the two sets of projects. I focus on the medium-size projects in both categories so that engineer's estimates are similar across the two sets. The small construction projects are allowed longer duration (on average 25% longer than the duration of resurfacing projects) and tend to have a higher number of tasks.

Table 2 reports the estimates from the OLS regression of the logs of the bids on the project characteristics for the two sets of projects used in the estimation. The results indicate that observable characteristics explain a higher portion of variation in log-bids in the case of bituminous resurfacing. In additional, the engineer's estimate plays a more important role in the case of bituminous resurfacing. This indicates that this measurement

[7] See, for example, Bajari and Ye (2003).

Table 1: Summary Statistics

Variables	**Bituminous Resurfacing**	**Small Construction**
Engineer's estimate	6.05	6.0
(hundreds of thousands)	(1.6)	(1.2)
Duration (days)	69.5	44.1
	(15.1)	(25.3)
Tasks	4.2	9.4
	(1.32)	(4.46)
$[n_{regular},\ n_{fringe}]$	[4, 0]	[4, 0]
Number of projects	252	270

Note: The standard deviations are shown in the parenthesis.

is more precise for resurfacing projects.

Table 2: Log-Bid Regression

Variables	**Bituminous Resurfacing**	**Small Construction**
Constant	0.273	-0.0061
	(0.087)	(0.002)
Engineer's estimate	0.903	1.013
	(0.024)	(0.012)
Duration	0.0011	0.0003
	(0.0002)	(0.0001)
Tasks	0.0008	0.0006
	(0.0002)	(0.0001)
Other controls: year, month and district dummy variables.		
R^2	0.91	0.82

Note: The standard errors are shown in the parenthesis.

To account for the observable project characteristics I assume that

$$\log(b_{ij}) = x_j\beta + \log y + \log \tilde{b}_{ij}$$

for the specification with one-dimensional unobserved heterogeneity and

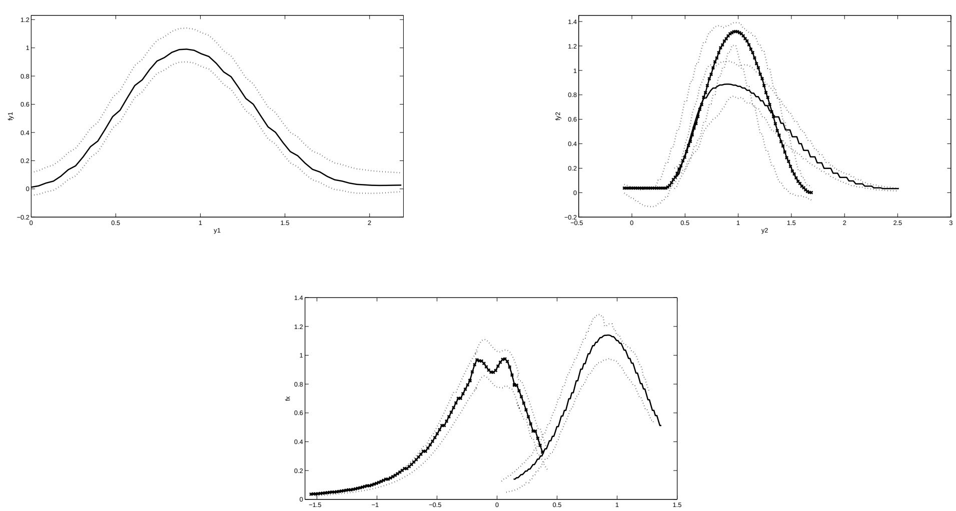
$$\log(b_{ij}) = x_j\beta + \log(y_1 + y_2\tilde{b}_{ij})$$

for the specification with two-dimensional unobserved heterogeneity. That is, I extract observable project variation by using OLS projection of bids on observable project characteristics and use residuals from this regression in further estimation.

Figures 1 and 2 depict the estimated densities of the costs components under one- and two-dimensional unobserved heterogeneity, and for the two sets of projects. Table 2 summarizes the results of the estimation. For both groups of projects the estimation under the assumption of two-dimensional unobserved heterogeneity recovers three non-trivial cost components. In both cases, the variance of the scaling component (Y_2) is smaller under two-dimensional specification relative to one-dimensional specification. The variance of the distribution of the individual cost component is very similar across specifications in the case of bituminous resurfacing and increases substantially in the case of small construction projects. Similarly, the estimated markups over the bidders' costs differ very little across specifications in the case of resurfacing projects whereas they increase from 7% (under one-dimensional specification) to 9.3% (under two-dimensional specification) in the case of small construction projects.

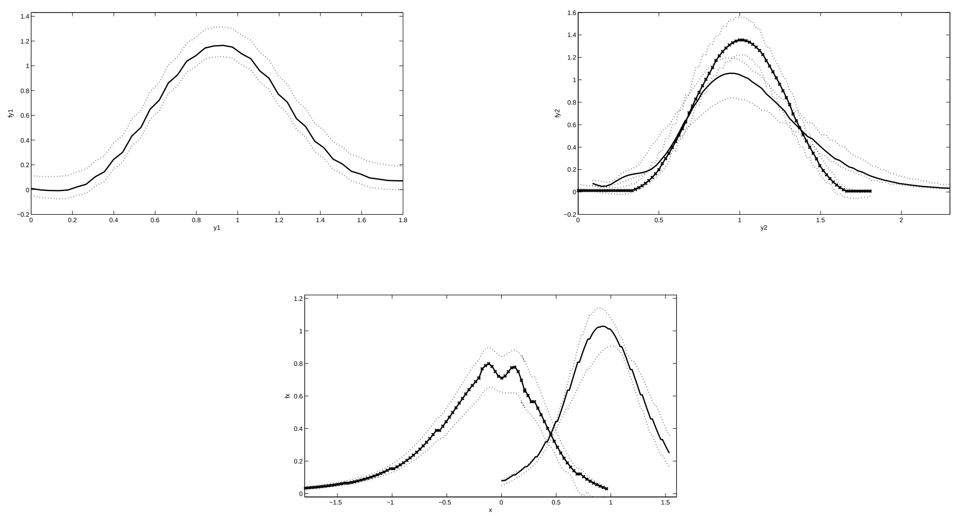
The results of estimation, thus, underscore the potential for misspecification bias. The model with two-dimensional heterogeneity mitigates the bias by allowing for greater flexibility in estimation.

Figure 1: Bituminous Resurfacing



Note: The top panel shows the estimated densities of the unobserved auction heterogeneity components. The lower panel reports the estimated density of bidder private information. The solid line corresponds to the case of one-dimensional unobserved heterogeneity while the line with a cross-marker depicts the density estimated under two-dimensional unobserved heterogeneity. Dotted lines represent 5% - 95% quantiles of pointwise density estimators.

Figure 2: Small Construction Projects



Note: The top panel shows the estimated densities of the unobserved auction heterogeneity components. The lower panel reports the estimated density of bidder private information. The solid line corresponds to the case of one-dimensional unobserved heterogeneity while the line with a cross-marker depicts the density estimated under two-dimensional unobserved heterogeneity. Dotted lines represent 5% - 95% quantiles of pointwise density estimators.

Further, I study the importance of allowing for greater flexibility in the specification of the model with unobserved heterogeneity by comparing the optimal reserve price derived from the estimates obtained under the assumption of (a) one-dimensional and (b) two-dimensional unobserved heterogeneity.

The government chooses a reserve price to minimize the expected cost of procurement, which consists of two parts: the expected cost of not allocating the job today and the expected cost of completing the work today given the reserve price r. Let us denote the first component c_0. It represents the sum of the cost of waiting another period and the expected cost at which the project can be completed in the future. Then the objective function of the government is therefore given by

$$C = c_0 \Pr(b_{ij} > r,\ i = 1, ..., n) + \int_{\underline{b}}^{r} bn(1 - F_B(b))^{n-1} f_B(b) db.$$

I do not have data on the magnitude of c_0. Therefore, I use a plausible value for c_0 and derive an optimal reserve price for this value.

Table 3: **Estimation Results**

	One Factor Model	Two Factor Model
Bituminous resurfacing		
$\sigma^2_{Y_2}$	0.12	0.062
	[0.11, 0.128]	[0.053, 0.068]
σ^2_X	0.11	0.124
	[0.102, 0.125]	[0.11,0.132]
$\sigma^2_{Y_1}$	-	0.11
		[0.106, 0.118]
avrg. mark-up	6.6%	7%
	[6.3,6.9]	[6.5, 7.2]
Small Structures		
$\sigma^2_{Y_2}$	0.16	0.07
	[0.153,0.172]	[0.064,0.8]
σ^2_X	0.08	0.13
	[0.07,0.085]	[0.12,0.14]
$\sigma^2_{Y_1}$	-	0.13
		[0.12,0.126]
avrg. mark-up	5.7%	7.8%
	[5.3,6.4]	[7.2,8.3]

Note: The 5% - 95% quantiles of the estimators are shown in the parenthesis.

The results of the analysis are summarized in the Table 4. The table records for every case (1) the reserve price, (2) the probability with which a bid is submitted and (3) the cost of procurement as a percent of c_0.

For each specification I consider two cases: (a) realization of unobserved heterogeneity is known to the government with the cost to the government given by

$$C(y) = c_0 \Pr(b_{ij} > r,\ i = 1, ..., n|y) + \int_{\underline{b}}^{r} bn(1 - F_B(b|y))^{n-1} f_B(b|y) db;$$

(b) realization of unobserved heterogeneity is unknown to the government and the reserve price is derived to minimize the average cost of procurement, where the average is taken with respect to the distribution of unobserved auction heterogeneity, i.e.

$$C = \int (c_0 \Pr(b_{ij} > r,\ i = 1, ..., n|y) + \int_{\underline{b}}^{r} bn(1 - F_B(b|y))^{n-1} f_B(b|y) db)\ f_Y(y) dy.$$

For the case in (a) Table 4 reports (1) the average reserve price, (2) the average probability with which a bid is submitted and (3) the average cost of procurement as a percent of c_0. The average is taken with respect to the distribution of unobserved heterogeneity.

I consider both (a) and (b) cases because the case (a) may not be implementable in practice if the government does not know the realization of unobserved auction heterogeneity. In this case the reserve price derived in (b) can be used.

The table shows that in the set of small construction projects the reserve price based on the distributions estimated under the assumption of two-dimensional unobserved heterogeneity is higher than the reserve price based on the distributions estimated under the assumption of one-dimensional unobserved heterogeneity. It also results in higher participation and lower cost of procurement. The table does not record significant differences between one- and two-dimensional cases in the case of bituminous resurfacing.

7 Conclusion

This paper analyzes the first price auction model with two-dimensional unobserved auction heterogeneity. I show that such a model is identified from the bid data, and develop an estimation methodology to recover the distribution of bidders' private information and the distributions of two-dimensional unobserved auction heterogeneity. I show that this methodology produces uniformly consistent estimators of the distributions in question.

Table 4: Reserve Price, $c_0 = 7$

		One-dimensional		Two-dimensional	
		(a)	**(b)**	**(a)**	**(b)**
Bituminous Resurfacing					
1.	Reserve Price	5.2	5.4	5.5	5.6
		[5.11, 5.29]	[5.28, 5.51]	[5.4,5.61]	[5.48,5.71]
2.	Probability of Submitting a Bid	0.56	0.57	0.55	0.57
		[0.55, 0.57]	[0.56, 0.58]	[0.54, 0.56]	[0.56, 0.58]
3.	Expected Cost of Procurement	70%	68%	72%	70%
	(as % of c_0)	[68.5%, 71.3%]	[66.6%, 69.4%]	[70.3%, 73.2%]	[68.6%, 71.8%]
Small Construction					
1.	Reserve Price	5.5	5.7	6.4	6.5
		[5.4, 5.6]	[5.56, 5.83]	[6.3, 6.5]	[6.33, 6.64]
2.	Probability of Submitting a Bid	0.65	0.68	0.53	0.56
		[0.64, 0.66]	[0.664, 0.693]	[0.515, 0.554]	[0.544, 0.573]
3.	Expected Cost of Procurement	61%	58%	75%	72%
	(as % of c_0)	[60.2%, 62.1%]	[57.0%,59.2%]	[73.4%,76.7%]	[70.6%,73.7%]

Note: The 5% - 95% quantiles of the estimators are shown in the parenthesis.

I apply this methodology to the sets of projects associated with bituminous resurfacing and small structures. I find that while in the case of bituminous resurfacing projects the estimated distribution of the private information differs little across specifications, in the case of small structure projects, allowing for two-dimensional unobserved heterogeneity results in significantly different estimates.

I also show that accounting for the two-dimensional nature of unobserved heterogeneity has important implications for the computation of optimal reserve prices. In particular, I find that in the set of small construction projects where two distinct dimensions of unobserved heterogeneity are present, the optimal reserve price is higher and calls for higher participation compared to the reserve price derived from the estimates obtained under the assumption of one-dimensional unobserved heterogeneity. I also find that the reserve price based on estimates from the misspecified model results in procurement costs which are 15% higher than the procurement costs under optimal reserve price.

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8 Appendix

Definition

The characteristic function of the variable X is non-vanishing if for every $T > 0$ there exists t such that $|t| > T$ and $\varphi_X(t) \neq 0$.

Lemma 1

Let Y and A denote random variables with bounded supports $[\underline{y}, \overline{y}]$ and $(\underline{a}, \overline{a}]$ such that $\underline{y} > 0$, $\underline{a} = 0$. Then, the characteristic functions of (a) Y and (b) $\log A$ are non-vanishing.

Proof

(a) The non-vanishing property of the characteristic function of Y is established as in Krasnokutskaya (2009). The proof introduces a function which is an extension of the characteristic function to the complex plane. It is shown that such a function is infinitely differentiable everywhere in the complex plane. It, therefore, is an entire function. Thus, the number of points where $\varphi_Y(t)$ is equal to zero cannot be more than countable, which means that $\varphi_Y(t)$ is non-vanishing.

(b) I follow a similar strategy to show that the characteristic function of $\log A$ is non-vanishing. Notice that the density function of $\log A$ is given by $f_{\log A}(x) = f_A(e^x)e^x$. Then, the characteristic function of $\log A$ is given by $\varphi_{\log A}(t) = \int_{-\infty}^{\log \overline{a}} e^{ita} f_{\log A}(a)da = \int_{-\infty}^{\log \overline{a}} e^{ita} f_A(e^a)e^a da$. It is easy to see that the characteristic function can be extended to the complex plane. The k-th derivative of the characteristic function, $\varphi^{(k)}_{\log A}(t) = \int_{-\infty}^{\log \overline{a}} (ia)^k e^{ita} f_A(e^a)e^a da$, is well defined and finite everywhere on the complex plane. Therefore, $\varphi_{\log A}$ is an entire function. As before this implies that $\varphi_Y(t)$ is non-vanishing.

Lemma 2

Let $X = (X_1, X_2)$ denote a vector of random variables such that

1. *The support of X, S_X, is unbounded, i.e. $S_X = [-\infty, M]^2$ for some $M > 0$;*
2. *$Prob(|X| \geq x) \leq L_0 e^{-x}$ for some $L_0 > 0$.*

Then, provided that $T_n = O((\frac{n}{\log n})^{\alpha})$ *for some* $\alpha > 0$*:*

$$(a)\ sup_{[-T_n,\, T_n]}|\int e^{itX} d(\hat{F}_{n;X} - F_X)| = O((\frac{\log n}{n})^{0.5})\ a.s.$$

Further, if the following conditions are satisfied

3. $Prob(|X_2| \geq x) \leq L_1 e^{-x}$ *for some* $L_1 > 0$*;*

4. $E[X_1^k|X_2] \leq L_k < \infty$ *for some* $L_k > 0,\ k = 1, 2$*;*

5. $E[X_1^k|X_2] \leq L_2 * L_3^{k-2} k! < \infty$ *for some* $L_3 > 0,\ k > 2$*;*

then, provided that $T_n = O((\frac{n}{\log n})^{\alpha})$ *for some* $\alpha > 0$*:*

$$(b)\ sup_{[-T_n,\, T_n]}|\int iX_1 e^{itX_2} d(\hat{F}_{n;X} - F_X)| = O((\frac{\log n}{n})^{0.5})\ a.s.$$

Proof

The (a) statement of Lemma 2 follows from Theorem 1 in Csorgo (1980). The latter result establishes that

$$\Delta_n(T_n) = sup_{[-T_n,\, T_n]}|\int e^{itX} d(\hat{F}_{n;X} - F_X)| = O(R_n) \text{ a.s.}$$

$$\text{if } \sum_{n=n_0}^{\infty} e^{-M_1 R_n^2 n} + \sum_{n=n_0}^{\infty} (K_n T_n / R_n)^2 e^{-M_2 R_n^2 n} < \infty$$

for some $M_1,\, M_2 > 0$ such that $n_0 = n_0(M_1,\, M_2) = \inf\{n:\ R_n \leq 1/4\sqrt{\max(M_1,\, M_2)}\}$ and $K_n = \inf\{x > 0:\ Prob(|X| > x) \leq R_n\}$.

It is straightforward to verify that condition (5) is satisfied for $R_n = (\frac{\log n}{n})^{0.5}$ and $T_n = O((\frac{n}{\log n})^{\alpha})$ with $\alpha > 0$ when $Prob(|X| \geq x) \leq L_0 e^{-x}$ for some $L_0 > 0$. The latter implies that $K_n = -0.5(\log\log n - \log n) - \log(L_0)$. Substituting all the appropriate values into

(5) obtains:

$$\sum_{n=n_0}^{\infty} e^{-M_1 R_n^2 n} + \sum_{n=n_0}^{\infty} (K_n T_n/R_n)^2 e^{-M_2 R_n^2 n}$$
$$\sum_{n=n_0}^{\infty} n^{-M_1} + \sum_{n=n_0}^{\infty} ((-0.5(\log\log n - \log n) - log(L_0))(\frac{n}{\log n})^{\alpha+0.5})^2 n^{-M_2} \leq$$
$$\sum_{n=n_0}^{\infty} n^{-M_1} + \sum_{n=n_0}^{\infty} (\log n)^2 (\frac{n}{\log n})^{2\alpha+1} n^{-M_2}$$
$$\sum_{n=n_0}^{\infty} n^{-M_1} + \sum_{n=n_0}^{\infty} \frac{n^{2\alpha+1-M_2}}{(\log n)^{2\alpha-1}} < \infty \text{ for } M_1 > 1 \text{ and } M_2 > 2(\alpha+1).$$

(b) The result in Csorgo (1980) can be extended to the case of

$$\Delta_n(T_n) = sup_{[-T_n, T_n]} | \int iX_1 e^{it_2 X_2} d(\hat{F}_{n;X} - F_X)|$$

when random vector X satisfies conditions (1-5). The statement exactly identical to the one in Theorem 1 of Csorgo (1980) obtains with the only modification that $n_0 = n_0(M_1, M_2) = \inf\{n : R_n \leq L_2/4\sqrt{\max(M_1, M_2)}\}$.

Lemma 3

Let $X_1 = \log(B_{i_1} - B_{i_2})|B_{i_1} - B_{i_2} > 0$ and $X_2 = \log(B_{i_3} - B_{i_4})|B_{i_3} - B_{i_4} > 0$ for some $i_1, ..., i_4$ such that $i_1 \neq i_2$ and $i_3 \neq i_4$. Then the the following properties hold:

1. *The support of X, S_X, is unbounded, i.e. $S_X = [-\infty, M_0]^2$ for some $M_0 > 0$;*
2. *$Prob(|X_2| \geq z) \leq L_0 e^{-z}$ as $z \to -\infty$ and for some $L_0 > 0$.;*
3. *$Prob(|X| \geq z) \leq L_{01} e^{z}$ as $z \to -\infty$ and for some $L_{01} > 0$.*
4. *$E[X_1^k|X_2] \leq L_k < \infty$ for some $L_k > 0$, $k = 1, 2$;*
5. *$E[X_1^k|X_2] \leq L_2 * L_3^{k-2} k! < \infty$ for some $L_3 > 0$, $k > 2$.*

Proof

1. According to the assumptions of the model $S(B_{i_k}) = [\underline{b}, \overline{b}]$. Then, $S(\Delta_{kl}|\Delta_{kl} > 0) = (0, \overline{b} - \underline{b}]$. Finally, $S(L\Delta_{kl}|\Delta_{kl} > 0) = (-\infty, \log(\overline{b} - \underline{b})]$. Denoting $M_0 = \log(\overline{b} - \underline{b})$ obtains the result.

2. Here I use that $\log(B_{i_k}) = \log(A_{i_k}) + \log(Y_2)$. Then,

$$\Pr(\log(A_{i_k} - A_{i_l}) + \log Y_2 \leq z | A_{i_k} - A_{i_l} > 0) = \frac{\Pr(\log(A_{i_k} - A_{i_l}) + \log Y_2 \leq z,\ A_{i_k} - A_{i_l} > 0)}{\Pr(A_{i_k} - A_{i_l} > 0)}.$$

Further,

$$Pr(\log(A_{i_k} - A_{i_l}) + \log Y_2 \leq z,\ A_{i_k} - A_{i_l} > 0) = \tag{5}$$
$$\int_{\underline{y}}^{\overline{y}} \int_{\underline{a}}^{\overline{a}} \int_{a_1}^{a_1 + e^{z-y}} f_{A_k}(a_2) f_{A_l}(a_1)\, da_2\, da_1\, f_{LY_2}(y)\, dy =$$
$$\int_{\underline{y}}^{\overline{y}} \int_{\underline{a}}^{\overline{a}} (F_{A_k}(a_1 + e^{z-y}) - F_{A_k}(a_1)) f_{A_l}(a_1)\, da_1\, f_{LY_2}(y)\, dy =$$

$$\int_{\underline{y}}^{\overline{y}} \int_{\underline{a}}^{\overline{a}} (f_{A_k}(a_1) e^z + o(e^z)) f_{A_l}(a_1)\ da_1\, f_{LY_2}(y)\, dy =$$
$$e^z \int_{\underline{y}}^{\overline{y}} \int_{\underline{a}}^{\overline{a}} f_{A_k}(a_1) f_{A_l}(a_1)\ da_1\, f_{LY_2}(y)\, dy + o(e^z) \int_{\underline{y}}^{\overline{y}} \int_{\underline{a}}^{\overline{a}} f_{A_k}(a_1)\ da_1\, f_{LY_2}(y)\, dy \leq W_1 e^z + o(e^z)$$
as $z \to -\infty$.

The last inequality holds because A_k, A_l, LY_2 have finite support and continuous density functions. Therefore, f_{A_k}, f_{A_l}, f_{LY_2} as well as $\int_{\underline{y}}^{\overline{y}} \int_{\underline{a}}^{\overline{a}} f_{A_k}(a_1) f_{A_l}(a_1)\ da_1\, f_{LY_2}(y)\, dy$ and $\int_{\underline{y}}^{\overline{y}} \int_{\underline{a}}^{\overline{a}} f_{A_k}(a_1)\ da_1\, f_{LY_2}(y)\, dy$ are bounded by some constant.

$$\text{In addition, } \Pr(A_{i_k} - A_{i_l} > 0) = \int_{\underline{a}}^{\overline{a}} \int_{a_1}^{\overline{a}} f_{A_k}(a_2)\, da_2\, f_{A_l}(a_1)\, da_1 =$$
$$\int_{\underline{a}}^{\overline{a}} (1 - F_{A_k}(a_1)) f_{A_l}(a_1)\ da_1 \geq W_2 \text{ for some } W_2 > 0. \tag{6}$$

Combining (5) and (6) proves the result of the lemma.

3. Similarly,

$$\Pr(\log(A_{i_1}-A_{i_2})+\log Y_2\le z_1, \log(A_{i_3}-A_{i_4})+\log Y_2\le z_2 | A_{i_1}-A_{i_2}>0,\ A_{i_3}-A_{i_4}>0)=$$
$$\frac{\int_{\underline{y}}^{\overline{y}}\int_{\underline{a}}^{\overline{a}}(F_{A_1}(a_1+e^{z_1-y})-F_{A_1}(a_1))f_{A_2}(a_1)\,da_1\int_{\underline{a}}^{\overline{a}}(F_{A_3}(a_2+e^{z_2-y})-F_{A_3}(a_2))f_{A_4}(a_2)\,da_2\,f_{LY_2}(y)\,dy}{\int_{\underline{a}}^{\overline{a}}(1-F_{A_1}(a_1+y))f_{A_2}(a_1)\,da_1\int_{\underline{a}}^{\overline{a}}(1-F_{A_3}(a_2+y))f_{A_4}(a_2)\,da_2}.$$

As above,

$$\int_{\underline{a}}^{\overline{a}}(1-F_{A_1}(a_1))f_{A_2}(a_1)\,da_1\int_{\underline{a}}^{\overline{a}}(1-F_{A_3}(a_2))f_{A_4}(a_2)\,da_2>W_3>0.$$

Further,

$$\int_{\underline{y}}^{\overline{y}}\int_{\underline{a}}^{\overline{a}}(F_{A_1}(a_1+e^{z_1-y})-F_{A_1}(a_1))f_{A_2}(a_1)\,da_1\int_{\underline{a}}^{\overline{a}}(F_{A_3}(a_2+e^{z_2-y})-F_{A_3}(a_2))f_{A_4}(a_2)\,da_2\,f_{LY_2}(y)\,dy=$$
$$\int_{\underline{y}}^{\overline{y}}\int_{\underline{a}}^{\overline{a}}(e^{z_1-y}f_{A_1}(a_1)+o(e^{z_1}))f_{A_2}(a_1)\,da_1\int_{\underline{a}}^{\overline{a}}(e^{z_2-y}f_{A_3}(a_2)+o(e^{z_2}))f_{A_4}(a_2)\,da_2\,f_{LY_2}(y)\,dy=$$

$$e^{z_1}e^{z_2}\int_{\underline{y}}^{\overline{y}}\int_{\underline{a}}^{\overline{a}}(e^{-y}f_{A_1}(a_1)+o(1))f_{A_2}(a_1)\,da_1\int_{\underline{a}}^{\overline{a}}(e^{-y}f_{A_3}(a_2)+o(1))f_{A_4}(a_2)\,da_2\,f_{LY_2}(y)\,dy\le$$
$$e^{z_1}e^{z_2}W_4 \text{ for some } W_4>0 \text{ as } z_1,\ z_2\to-\infty.$$

In addition,

$$\Pr(\log(A_{i_1}-A_{i_3})+\log Y_2\le z_1, \log(A_{i_2}-A_{i_3})+\log Y_2\le z_2 | A_{i_1}-A_{i_3}>0,\ A_{i_2}-A_{i_3}>0)=$$
$$\frac{\int_{\underline{y}}^{\overline{y}}\int_{\underline{a}}^{\overline{a}}(F_{A_1}(a+e^{z_1-y})-F_{A_1}(a))(F_{A_2}(a+e^{z_2-y})-F_{A_2}(a))f_{A_3}(a)\,da\,f_{LY_2}(y)\,dy}{\int_{\underline{a}}^{\overline{a}}(1-F_{A_1}(a))(1-F_{A_2}(a))f_{A_3}(a)\,da},$$

and,

$$\int_{\underline{y}}^{\overline{y}}\int_{\underline{a}}^{\overline{a}}(F_{A_1}(a+e^{z_1-y})-F_{A_1}(a))(F_{A_2}(a+e^{z_2-y})-F_{A_2}(a))f_{A_3}(a)\,da\,f_{LY_2}(y)\,dy=$$

$$\int_{\underline{y}}^{\overline{y}}\int_{\underline{a}}^{\overline{a}}(e^{z_1-y}f_{A_1}(a)+o(e^{z_1}))(e^{z_2-y}f_{A_2}(a)+o(e^{z_2}))f_{A_3}(a)\,da\,f_{LY_2}(y)\,dy=$$

$$e^{z_1}e^{z_2}\int_{\underline{y}}^{\overline{y}}\int_{\underline{a}}^{\overline{a}}(e^{-y}f_{A_1}(a)+o(1))(e^{-y}f_{A_2}(a)+o(1))f_{A_3}(a)\,da\,f_{LY_2}(y)\,dy\leq$$

$$e^{z_1}e^{z_2}W_5 \text{ for some } W_5>0 \text{ as } z_1,\, z_2\to-\infty$$

with $\int_{\underline{a}}^{\overline{a}}(1-F_{A_1}(a+e^y))(1-F_{A_2}(a+e^y))f_{A_3}(a)\,da>W_6>0$.

4. The probability density function of the conditional distribution of $\log(B_{i_1}-B_{i_2})$ conditional on $\log(B_{i_3}-B_{i_4})$, $\Delta_{1,2}B>0$, $\Delta_{3,4}B>0$ is given by

$$f_{\Delta_{1,2}B|\Delta_{3,4}B}(b_1|b_2)=$$

$$\frac{\int_{\underline{y}}^{\overline{y}}\int_{\underline{a}}^{\overline{a}}e^{b_1-y}f_{A_1}(e^{b_1-y}+a_1)f_{A_2}(a_1)\,da_1\int_{\underline{a}}^{\overline{a}}e^{b_2-y}f_{A_3}(e^{b_2-y}+a_2)f_{A_4}(a_2)\,da_2\,f_{LY_2}(y)\,dy}{\int_{\underline{y}}^{\overline{y}}\int_{\underline{a}}^{\overline{a}}e^{b_2-y}f_{A_3}(e^{b_2-y}+a_2)f_{A_4}(a_2)\,da_2\,f_{LY_2}(y)\,dy}=$$

$$\frac{e^{b_1}\int_{\underline{y}}^{\overline{y}}e^{-2y}\int_{\underline{a}}^{\overline{a}}f_{A_1}(e^{b_1-y}+a_1)f_{A_2}(a_1)\,da_1\int_{\underline{a}}^{\overline{a}}f_{A_3}(e^{b_2-y}+a_2)f_{A_4}(a_2)\,da_2\,f_{LY_2}(y)\,dy}{\int_{\underline{y}}^{\overline{y}}e^{-y}\int_{\underline{a}}^{\overline{a}}f_{A_3}(e^{b_2-y}+a_2)f_{A_4}(a_2)\,da_2\,f_{LY_2}(y)\,dy}.$$

Then,

$$E[|B_1||B_2]=\int|b_1|dF_{B_1|B_2}=$$

$$\int_{-\infty}^{M}|b_1|\frac{e^{b_1}\int_{\underline{y}}^{\overline{y}}e^{-2y}\int_{\underline{a}}^{\overline{a}}f_{A_1}(e^{b_1-y}+a_1)f_{A_2}(a_1)\,da_1\int_{\underline{a}}^{\overline{a}}f_{A_3}(e^{b_2-y}+a_2)f_{A_4}(a_2)\,da_2\,f_{LY_2}(y)\,dy}{\int_{\underline{y}}^{\overline{y}}e^{-y}\int_{\underline{a}}^{\overline{a}}f_{A_3}(e^{b_2-y}+a_2)f_{A_4}(a_2)\,da_2\,f_{LY_2}(y)\,dy}db_1\leq$$

$$\int_{-\infty}^{M} |b_1| \frac{e^{b_1} \int_{\underline{y}}^{\overline{y}} e^{-2y} \int_{\underline{a}}^{\overline{a}} f_{A_1}(e^{b_1-y}+a_1) f_{A_2}(a_1)\, da_1 \int_{\underline{a}}^{\overline{a}} f_{A_3}(e^{b_2-y}+a_2) f_{A_4}(a_2)\, da_2\, f_{LY_2}(y)\, dy}{W_7} db_1 \leq$$

$$\int_{-\infty}^{M} |b_1| e^{b_1} \frac{\frac{-e^{-\overline{y}}+e^{-\underline{y}}}{2} M_{LY_2} M_A^2 (\overline{a}-\underline{a})^2}{W_7} db_1 \leq W_8 \int_{-\infty}^{M} |b_1| e^{b_1} db_1 = W_8 (\int_0^M b_1 e^{b_1} db_1 +$$

$$+ \int_{-\infty}^{0} |b_1| e^{b_1} db_1) = W_8 (e^M (M-1) + 1 - \int_0^{\infty} b_1 e^{-b_1} db_1) = W_8 e^M (M-1).$$

The first inequality holds for every $b_2 \neq \underline{b}, \overline{b}$ since due to absolute continuity of f_A and f_{LY_2} there exists non-empty sets of y's and a_2's such that the integrand is positive over these sets. The second inequality also arises due to the continuity of f_A and f_{LY_2} and compactness of $[\underline{y}, \overline{y}]$ and $[\underline{a}, \overline{a}]$. All the equalities are derived by direct computation. Similarly,

$$E[B_1^2|B_2] = \int b_1^2 dF_{B_1|B_2} \leq$$

$$W_8 \int_{-\infty}^{M} b_1^2 e^{b_1} db_1 = W_8 (\int_0^M b_1^2 e^{b_1} db_1 +$$

$$+ \int_{-\infty}^{0} b_1^2 e^{b_1} db_1) = W_8 (e^M (M^2 - 2(M-1)) - 2 - \int_0^{\infty} b_1^2 e^{-b_1} db_1) = W_8 (e^M (M^2 - 2(M-1)).$$

5. Finally, for the k-order moment I have

$$E[|B_1|^k|B_2] = \int |b_1|^k dF_{B_1|B_2} \leq$$

$$W_8 \int_{-\infty}^{M} |b_1|^k e^{b_1} db_1 = W_8 (\int_0^M b_1^k e^{b_1} db_1 +$$

$$+ \int_{-\infty}^{0} |b_1|^k e^{b_1} db_1) = W_8 (\int_0^M b_1^k e^{b_1} db_1 - \int_0^{\infty} b_1^k e^{-b_1} db_1).$$

Denote $M_k^0 = \int_0^M b_1^k e^{b_1} db_1$ and $M_k^1 = \int_0^{\infty} b_1^k e^{-b_1} db_1$. Then,

$$M_k^0 = e^M M^k - k M_{k-1}^0$$
$$M_k^1 = k M_{k-1}^1.$$

Using the recursive formulas above obtains:

$$M_k^1 = k!$$
$$M_k^0 = e^M(M^k + \sum_{l=1}^{l=k-1} \frac{k!}{(k-l)!} M^{k-l}(-1)^l) - k!$$

This gives us

$$E[|B_1|^k|B_2] \leq W_8 e^M (M^k + \sum_{l=1}^{l=k-1} (M^k + \frac{k!}{(k-l)!} M^{k-l}(-1)^l) =$$
$$W_8 e^M M^k k!(1 + \sum_{l=1}^{l=k-1} \frac{1}{(k-l)!} M^{-l}(-1)^l) = W_8 e^M M^k k! W_9 = L_2(M W_9^{\frac{1}{k-2}})^{k-2} k!$$

for some $W_9 > 0$.

Lemma 4

Let X be a random variable with the probability density function $f_X(.)$ and such that

1. *The characteristic function of X, $\varphi_X(t)$ is ordinarily smooth, i.e. $|\varphi_X(t)| \geq d_0|t|^{-\beta_x}$ for some $d_0 > 0$ and $\beta_x > 1$;*
2. *The estimator of $\varphi_X(t)$, $\hat{\varphi}_{X;n}(t)$ is such that $sup_{t\in[-T_n,T_n]}|\hat{\varphi}_{X;n}(t) - \varphi_X(t)| = C_{\varphi;n}$;*
3. *The estimator of $f_X(.)$, $\hat{f}_{n;X}(x)$ is given by $\hat{f}_{n;X}(x) = \int_{-T_n}^{T_n} e^{-itx}\hat{\varphi}_{X;n}(t)dt$.*

   Then

$$\sup_{x\in S(X)} |\hat{f}_{n;X}(x) - f_X(x)| \leq 2T_n C_{n;\varphi} + T_n^{1-\beta_x} \ a.s.$$

Proof

$$|\hat{f}_{n;X}(x) - f_X(x)| \leq \frac{1}{2\pi} \int_{-T_n}^{T_n} |\hat{\varphi}_{X;n}(t) - \varphi_X(t)|dt +$$
$$\frac{1}{2\pi} \int_{-\infty}^{-T_n} |\varphi_X(t)|dt + \frac{1}{2\pi} \int_{T_n}^{\infty} |\varphi_X(t)|dt \leq$$

$$2T_n \sup_{t\in[-T_n,T_n]} |\hat{\varphi}_{X;n}(t) - \varphi_X(t)| + \frac{1}{\pi}\int_{T_n}^{\infty} |d_0 t^{-\beta_x}|dt =$$

$$2T_n C_{\varphi;n} + \frac{1}{\pi} d_0 T_n^{1-\beta_x}.$$

Proof of Theorem 2

Step 1

First, I begin by establishing that $\sup_{[-T_n,T_n]} |\hat{\varphi}_{n;LY_2}(t) - \varphi_{LY_2}(t)| = O((\frac{\log n}{n})^{0.5})$. In Step 1 I always condition on $\Delta_{1,2}B > 0$, $\Delta_{3,4}B > 0$. I suppress conditioning in the notations for the ease of exposition.

Applying Taylor approximation to

$$\hat{\varphi}_{LY_2}(t) = \exp(\int_0^t \frac{\widehat{\Psi}_{1;L\Delta_{1,2}B,L\Delta_{3,4}B}(0,u_2)}{\widehat{\Psi}_{L\Delta_{1,2}B,L\Delta_{3,4}B}(0,u_2)} du_2$$

obtains

$$|\hat{\varphi}_{LY_2}(t) - \varphi_{LY_2}(t)| = \sum_{l=1}^{\infty} \frac{\varphi_{LY_2}(t)}{l!} (\int_0^t \frac{\widehat{\Psi}_{1;L\Delta_{1,2}B,L\Delta_{3,4}B}(0,u_2)}{\widehat{\Psi}_{L\Delta_{1,2}B,L\Delta_{3,4}B}(0,u_2)} du_2 - \int_0^t \frac{\Psi_{1;L\Delta_{1,2}B,L\Delta_{3,4}B}(0,u_2)}{\Psi_{L\Delta_{1,2}B,L\Delta_{3,4}B}(0,u_2)} du_2)^l$$

Denote $|\Delta_n| = |\int_0^t \frac{\widehat{\Psi}_{1;L\Delta_{1,2}B,L\Delta_{3,4}B}(0,u_2)}{\widehat{\Psi}_{L\Delta_{1,2}B,L\Delta_{3,4}B}(0,u_2)} du_2 - \int_0^t \frac{\Psi_{1;L\Delta_{1,2}B,L\Delta_{3,4}B}(0,u_2)}{\Psi_{L\Delta_{1,2}B,L\Delta_{3,4}B}(0,u_2)} du_2)|$. Then

$$|\hat{\varphi}_{LY_2}(t) - \varphi_{LY_2}(t)| \le \sum_{l=1}^{\infty} |\Delta_n|^l.$$

Using von Mises differentials I have

$$\Delta_n = \sum_{k=1}^{\infty} \frac{1}{k!} d_k T(F_{L\Delta_{1,2}B,L\Delta_{3,4}B}; \hat{F}_{n;L\Delta_{1,2}B,L\Delta_{3,4}B} - F_{L\Delta_{1,2}B,L\Delta_{3,4}B}),$$

where $d_k T(F_{L\Delta_{1,2}B,L\Delta_{3,4}B}; \hat{F}_{n;L\Delta_{1,2}B,L\Delta_{3,4}B} - F_{L\Delta_{1,2}B,L\Delta_{3,4}B}) =$

$$\frac{d^k}{d\lambda^k} \int_0^t \frac{\int ib_1 e^{iu_2 b_2} d(F_{L\Delta_{1,2}B,L\Delta_{3,4}B} + \lambda(\hat{F}_{n;L\Delta_{1,2}B,L\Delta_{3,4}B} - F_{L\Delta_{1,2}B,L\Delta_{3,4}B}))}{\int e^{iu_2 b_2} d(F_{L\Delta_{1,2}B,L\Delta_{3,4}B} + \lambda(\hat{F}_{n;L\Delta_{1,2}B,L\Delta_{3,4}B} - F_{L\Delta_{1,2}B,L\Delta_{3,4}B}))} du_2 |_{\lambda=0}.$$

By direct differentiation I establish that

$$d_k T(F_{L\Delta_{1,2}B,L\Delta_{3,4}B}; \hat{F}_{n;L\Delta_{1,2}B,L\Delta_{3,4}B} - F_{L\Delta_{1,2}B,L\Delta_{3,4}B}) = (-1)^k k! \int_0^t \frac{A(u_2)B(u_2)^{k-1}}{\Psi_{L\Delta_{1,2}B,L\Delta_{3,4}B}(0,u_2)^{k+1}}\ du_2$$

where

$$A(u_2) = \int ib_1 e^{iu_2b_2} d(\hat{F}_{n;L\Delta_{1,2}B,L\Delta_{3,4}B} - F_{L\Delta_{1,2}B,L\Delta_{3,4}B}) \int e^{iu_2b_2} dF_{L\Delta_{1,2}B,L\Delta_{3,4}B}$$
$$- \int ib_1 e^{iu_2b_2} dF_{L\Delta_{1,2}B,L\Delta_{3,4}B} \int e^{iu_2b_2} d(\hat{F}_{n;L\Delta_{1,2}B,L\Delta_{3,4}B} - F_{L\Delta_{1,2}B,L\Delta_{3,4}B}).$$
$$B(u_2) = \int e^{iu_2b_2} d(\hat{F}_{n;L\Delta_{1,2}B,L\Delta_{3,4}B} - F_{L\Delta_{1,2}B,L\Delta_{3,4}B}).$$

Lemma 2 and Lemma 3 imply that

$$\sup_{t\in[-T_n,T_n]} |B(u_2)| = \sup_{t\in[-T_n,T_n]} |\int e^{iu_2b_2} d(\hat{F}_{n;L\Delta_{1,2}B,L\Delta_{3,4}B} - F_{L\Delta_{1,2}B,L\Delta_{3,4}B})| = O(R_n)$$

where $R_n = (\frac{\log n}{n})^{0.5}$ *and* $T_n = (\frac{\log n}{n})^{\alpha}$ for some $\alpha > 0$;

$$\sup_{t\in[-T_n,T_n]} |\int ib_1 e^{iu_2b_2} d(\hat{F}_{n;L\Delta_{1,2}B,L\Delta_{3,4}B} - F_{L\Delta_{1,2}B,L\Delta_{3,4}B})| = O(R_n)$$

where $R_n = (\frac{\log n}{n})^{0.5}$ *and* $T_n = (\frac{\log n}{n})^{\alpha}$ for some $\alpha > 0$.

$$|\int ib_1 e^{iu_2b_2} dF_{L\Delta_{1,2}B,L\Delta_{3,4}B}| \leq L_{1;L\Delta B}.$$

Therefore,

$$|A(u_2)| \leq |\int ib_1 e^{iu_2b_2} d(\hat{F}_{n;L\Delta_{1,2}B,L\Delta_{3,4}B} - F_{L\Delta_{1,2}B,L\Delta_{3,4}B})||\int e^{iu_2b_2} dF_{L\Delta_{1,2}B,L\Delta_{3,4}B}| +$$
$$|\int ib_1 e^{iu_2b_2} dF_{L\Delta_{1,2}B,L\Delta_{3,4}B}||\int e^{iu_2b_2} d(\hat{F}_{n;L\Delta_{1,2}B,L\Delta_{3,4}B} - F_{L\Delta_{1,2}B,L\Delta_{3,4}B})| \leq$$
$$|\int ib_1 e^{iu_2b_2} d(\hat{F}_{n;L\Delta_{1,2}B,L\Delta_{3,4}B} - F_{L\Delta_{1,2}B,L\Delta_{3,4}B})| +$$

$$|\int ib_1 e^{iu_2b_2} dF_{L\Delta_{1,2}B,L\Delta_{3,4}B}||\int e^{iu_2b_2} d(\hat{F}_{n;L\Delta_{1,2}B,L\Delta_{3,4}B} - F_{L\Delta_{1,2}B,L\Delta_{3,4}B})| = O(R_n).$$

Next,

$$d_k T(F_{L\Delta_{1,2}B,L\Delta_{3,4}B}; \hat{F}_{n;L\Delta_{1,2}B,L\Delta_{3,4}B} - F_{L\Delta_{1,2}B,L\Delta_{3,4}B}) =$$

$$(-1)^k k! \int_0^t \frac{A(u_2)B(u_2)^{k-1}}{\Psi_{L\Delta_{1,2}B,L\Delta_{3,4}B}(0,u_2)^{k+1}}\, du_2 = \int_0^t \frac{Q_1 R_n R_n^{k-1}}{\Psi_{L\Delta_{1,2}B,L\Delta_{3,4}B}(0,u_2)^{k+1}}\, du_2 \leq$$

$$Q_1 R_n^k \int_0^{T_n} \frac{1}{d_{L\Delta B} T_n^{-\beta_{L\Delta B}(k+1)}}\, du_2 = Q_2 R_n^k T_n^{1+\beta_{L\Delta B}(1+k)} \text{ for some } Q_1,\, Q_2 > 0.$$

The following reasoning justifies the inequality: $\varphi_{LY_2}(t)$ and $\varphi_{L\Delta_{k,l}A}(t)$ are ordinarily smooth with parameters β_{LY_2} and $\beta_{L\Delta A}$ respectively. Since $L\Delta_{k,l}B = LY_2 + L\Delta_{k,l}A$ then $\Psi_{L\Delta_{1,2}B,L\Delta_{3,4}B}(0,u_2)$ is ordinarily smooth with parameter $\beta_{L\Delta B} = \beta_{LY_2} + \beta_{L\Delta A}$. Further, it can be shown (see Li and Voung (1998)) that if T_n is large enough then $|\Psi_{L\Delta_{1,2}B,L\Delta_{3,4}B}(0,u_2)| > d_{L\Delta B}|T_n|^{-\beta_{L\Delta B}}$ for an appropriate $d_{L\Delta B} > 0$.

Then,

$$|\Delta_n| = \sum_{k=1}^{\infty} Q_2 (\frac{\log n}{n})^{k/2} T_n^{1+\beta_{L\Delta B}(1+k)} = Q_2 \frac{T_n^{1+2\beta_{L\Delta B}} (\frac{\log n}{n})^{0.5}}{1 - T_n^{\beta_{L\Delta B}} (\frac{\log n}{n})^{0.5}}.$$

Therefore,

$$|\Delta_n| \leq O(T_n^{1+2\beta_{L\Delta B}} (\frac{\log n}{n})^{0.5}) = O((\frac{\log n}{n})^{0.5-\alpha(1+2\beta_{L\Delta B})})$$

and

$$|\hat{\varphi}_{LY_2}(t) - \varphi_{LY_2}(t)| \leq \frac{|\Delta_n|}{1-|\Delta_n|} = O((\frac{\log n}{n})^{0.5-\alpha(1+2\beta_{L\Delta B})})$$

if $\alpha \leq \frac{1}{2(1+2\beta_{L\Delta B})}$.

Next, Lemma 4 implies that

$$\sup_{y \in S_{LY_2}} |\hat{f}_{LY_2}(y) - f_{LY_2}(y)| \leq O(T_n (\frac{\log n}{n})^{0.5-\alpha(1+2\beta_{L\Delta B})}) + T_n^{1-\beta_{LY_2}}$$

$$\text{or } \sup_{y \in S_{LY_2}} |\hat{f}_{LY_2}(y) - f_{LY_2}(y)| \leq O((\frac{\log n}{n})^{0.5-2\alpha(1+\beta_{L\Delta B})}) + O((\frac{\log n}{n})^{\alpha(1-\beta_{LY_2})}).$$

Since $\underline{y}_2 > 0$ and $\overline{y}_2 < \infty$:

$$\sup_{y \in S_{Y_2}} |\hat{f}_{Y_2}(y) - f_{Y_2}(y)| = \sup_{y \in S_{Y_2}} |\frac{\hat{f}_{LY_2}(\log y) - f_{LY_2}(\log y)}{y}| \leq$$
$$O((\frac{\log n}{n})^{0.5-2\alpha(1+\beta_{L\Delta B})}) + O((\frac{\log n}{n})^{\alpha(1-\beta_{LY_2})}).$$

I use $C_{f_{Y_2}}$ to denote $\sup_{y \in S_{Y_2}} |\hat{f}_{Y_2}(y) - f_{Y_2}(y)|$ in the rest of the proof.

Step 2

All the analysis below is performed conditional on $\Delta_{1,3}A$, $\Delta_{2,3}A$ unless otherwise noted. The conditioning is suppressed for the ease of exposition. I begin by deriving

$$\sup_{[-T_n, T_n]} |\hat{\varphi}_{L\Delta_{1,3}A, L\Delta_{2,3}A}(t_1, t_2) - \varphi_{L\Delta_{1,3}A, L\Delta_{2,3}A}(t_1, t_2)|.$$

Taylor expansion gives:

$$\hat{\varphi}_{L\Delta_{1,3}A, L\Delta_{2,3}A}(t_1, t_2) - \varphi_{L\Delta_{1,3}A, L\Delta_{2,3}A}(t_1, t_2) =$$
$$\sum_{k=1}^{k=\infty} \frac{\varphi_{L\Delta_{1,3}A, L\Delta_{2,3}A}(t_1, t_2)}{k!} (\log(\hat{\varphi}_{L\Delta_{1,3}A, L\Delta_{2,3}A}(t_1, t_2)) - \log(\varphi_{L\Delta_{1,3}A, L\Delta_{2,3}A}(t_1, t_2)))^k.$$

Further,

$$|\hat{\varphi}_{L\Delta_{1,3}A,\, L\Delta_{2,3}A}(t_1, t_2) - \varphi_{L\Delta_{1,3}A,\, L\Delta_{2,3}A}(t_1, t_2)| \leq$$
$$\sum_{k=1}^{k=\infty} |(\log(\hat{\varphi}_{L\Delta_{1,3}A,\, L\Delta_{2,3}A}(t_1, t_2)) - \log(\varphi_{L\Delta_{1,3}A,\, L\Delta_{2,3}A}(t_1, t_2))|^k =$$
$$O(\log(\hat{\varphi}_{L\Delta_{1,3}A,\, L\Delta_{2,3}A}(t_1, t_2)) - \log(\varphi_{L\Delta_{1,3}A,\, L\Delta_{2,3}A}(t_1, t_2)))$$

when $|\log(\hat{\varphi}_{L\Delta_{1,3}A,\, L\Delta_{2,3}A}(t_1, t_2)) - \log(\varphi_{L\Delta_{1,3}A,\, L\Delta_{2,3}A}(t_1, t_2))| < 1$.

Then,

$$|\log(\hat{\varphi}_{L\Delta_{1,3}A,\, L\Delta_{2,3}A}(t_1, t_2)) - \log(\varphi_{L\Delta_{1,3}A,\, L\Delta_{2,3}A}(t_1, t_2))| =$$
$$|\log(\hat{\Psi}_{L\Delta_{1,3}A,\, L\Delta_{2,3}A}(t_1, t_2)) - \log(\Psi_{L\Delta_{1,3}A,\, L\Delta_{2,3}A}(t_1, t_2)) +$$
$$\log(\hat{\varphi}_{LY_2}(t_1 + t_2)) - \log(\varphi_{LY_2}(t_1 + t_2))| \leq$$
$$O(|\frac{\hat{\Psi}_{L\Delta_{1,3}B, L\Delta_{2,3}B}(t_1, t_2) - \Psi_{L\Delta_{1,3}B, L\Delta_{2,3}B}(t_1, t_2)}{\Psi_{L\Delta_{1,3}B, L\Delta_{2,3}B}(t_1, t_2)}|) +$$
$$O(\frac{|\hat{\varphi}_{LY_2}(t_1 + t_2) - \varphi_{LY_2}(t_1 + t_2)|}{\varphi_{LY_2}(t_1 + t_2)}).$$

Similar to Step 1, ordinary smoothness of $\varphi_{L\Delta_{1,3}A,L\Delta_{2,3}A}$ and of φ_{LY_2} implies that

$$|\varphi_{L\Delta_{1,3}A,L\Delta_{2,3}A}(t_1,\,t_2)| > d_{L\Delta A,L\Delta A}|t_1|^{-\beta_{L\Delta_{1,3}A}}|t_2|^{-\beta_{L\Delta_{2,3}A}}$$
$$|\varphi_{LY_2}(t)| > d_{LY_2}|t|^{-\beta_{LY_2}}.$$

Applying Lemma 2 and Lemma 3, I obtain

$$\sup_{t_1,t_2\in[-T_n,T_n]^2} |(\log(\hat{\varphi}_{L\Delta_{1,3}A,\,L\Delta_{2,3}A}(t_1,t_2)) - \log(\varphi_{L\Delta_{1,3}A,L\Delta_{2,3}A}(t_1,t_2))| =$$
$$O(\frac{\log n}{n})^{0.5-\alpha(\beta_{L\Delta_{1,3}B}+\beta_{L\Delta_{2,3}B})} + O((\frac{\log n}{n})^{0.5-\alpha(1+2\beta_{L\Delta B_{3,4}}+\beta_{LY_2})}) = O((\frac{\log n}{n})^{0.5-\alpha\beta^*})$$
$$\text{if } T_n = O(\frac{n}{\log n})^{\alpha},\ \text{with } \alpha \leq \frac{1}{2\beta^*} \text{ where}$$
$$\beta^* = \min\{(\beta_{L\Delta_{1,3}B} + \beta_{L\Delta_{2,3}B}),\,(1 + 2\beta_{L\Delta B_{3,4}} + \beta_{LY_2})\}.$$

Notice that in above $(t_1 + t_2) \in [-2T_n, 2T_n]$ where as $2T_n$ is still $O((\frac{n}{\log n})^{\alpha})$.

Using Lemma 4 and the fact that $\varphi_{L\Delta_{1,3}A,L\Delta_{2,3}A}$ is ordinarily smooth with parameters $\beta_{L\Delta A_{1,3}}$, $\beta_{L\Delta A_{2,3}}$ obtains:

$$\sup_{a_1,\,a_2\in S(L\Delta_{1,3}A,L\Delta_{2,3}A)} |\hat{f}_{L\Delta_{1,3}A,L\Delta_{2,3}A}(a_1,a_2) - f_{L\Delta_{1,3}A,L\Delta_{2,3}A}(a_1,a_2)| \leq$$
$$O(T_n\frac{\log n}{n}^{0.5-\alpha\beta^*}) + T_n^{2-\beta_{L\Delta_{1,3}A}-\beta_{L\Delta_{1,3}A}}$$
$$\text{or } |\hat{f}_{L\Delta_{1,3}A,\,L\Delta_{2,3}A}(a_1,a_2) - f_{L\Delta_{1,3}A,\,L\Delta_{2,3}A}(a_1,a_2)| =$$
$$O((\frac{\log n}{n})^{0.5-\alpha(1+\beta^*)}) + O((\frac{\log n}{n})^{\alpha(\beta_{L\Delta_{1,3}A}+\beta_{L\Delta_{2,3}A}-2)}) = O((\frac{\log n}{n})^{0.5-\alpha(1+\beta^*)})$$
$$\text{if } \alpha < \frac{1}{2(\beta^* + \beta_{L\Delta_{1,3}A} + \beta_{L\Delta_{2,3}A} - 1)}.$$

Also, for every subset, $SC = [\varepsilon_n, M_A]^2$, of $S(\Delta_{1,3}A, \Delta_{2,3}A) = (0, M_A]^2$

$$\sup_{[\varepsilon_n,M_A]^2} |\hat{f}_{\Delta_{1,3}A,\Delta_{2,3}A}(a_1,a_2) - f_{\Delta_{1,3}A,\Delta_{2,3}A}(a_1,a_2)| =$$
$$\sup_{[\varepsilon_n,M_A]^2} \frac{|\hat{f}_{L\Delta_{1,3}A,L\Delta_{2,3}A}(a_1,a_2) - f_{L\Delta_{1,3}A,L\Delta_{2,3}A}(\log a_1, \log a_2)|}{a_1a_2} \leq$$
$$= O(\varepsilon_n^{-2}(\frac{\log n}{n})^{0.5-\alpha(1+\beta^*)}).$$

Then,

$$|\hat{f}_{\Delta_{1,3}A,\Delta_{2,3}A}(a_1,a_2) - f_{\Delta_{1,3}A,\Delta_{2,3}A}(a_1,a_2)| = O((\frac{\log n}{n})^{0.5-\alpha(1+\beta^*)-2\gamma}).$$

if $\varepsilon_n = (\frac{\log n}{n})^{\gamma}$ for some $\gamma > 0$.

Next, I investigate the convergence of the estimator for the density of the unconditional distribution of $\Delta_{1,2}A$, $\Delta_{3,4}A$.

$$\sup_{a_1,\, a_2 \in S(\Delta_{1,3}A,\Delta_{2,3}A)} |\hat{f}_{\Delta_{1,3}A,\, \Delta_{2,3}A}(a_1,a_2) - f_{\Delta_{1,3}A,\, \Delta_{2,3}A}(a_1,a_2)| = O((\frac{\log n}{n})^{0.5-\alpha(1+\beta^*)-2\gamma}).$$

It is so because

$$|\hat{\Pr}(\Delta_{1,3}B >_< 0,\ \Delta_{2,3}B >_< 0) - \Pr(\Delta_{1,3}B >_< 0,\ \Delta_{2,3}B >_< 0)| = O(n^{\frac{1}{2}}).$$

Next, I use the uniform convergence of $\hat{f}_{\Delta_{1,3}A,\Delta_{2,3}A}(a_1,a_2)$ to show the uniform convergence for $\hat{\varphi}_{A_{i_3}}$, $\hat{\varphi}_{A_{i_k}}$ as well as $\hat{f}_{A_{i_3}}$ and $\hat{f}_{A_{i_k}}$ for $k = 1, 2$. I begin as in Step 1 by using a Taylor approximation to obtain that

$$|\hat{\varphi}_{A_{i_3}}(t) - \varphi_{A_{i_3}}(t)| \le \sum_{l=1}^{\infty} |\Delta_n|^l.$$

where $|\Delta_n| = |\int_0^t \frac{\widehat{\Psi}_{1;\Delta_{1,3}A,\Delta_{2,3}A}(0,u_2)}{\widehat{\Psi}_{\Delta_{1,3}A,\Delta_{2,3}A}(0,u_2)} du_2 - \int_0^t \frac{\Psi_{1;\Delta_{1,3}A,\Delta_{2,3}A}(0,u_2)}{\Psi_{\Delta_{1,3}A,\Delta_{2,3}A}(0,u_2)} du_2)|$, and

$$\widehat{\Psi}_{\Delta_{1,3}A,\Delta_{2,3}A}(u_1,u_2) = \int e^{i(u_1a_1+u_2a_2)} \hat{f}_{\Delta_{1,3}A,\Delta_{2,3}A}(a_1,a_2)\, da_1 da_2$$

$$\widehat{\Psi}_{1;\Delta_{1,3}A,\Delta_{2,3}A}(u_1,u_2) = \int ia_1 e^{i(u_1a_1+u_2a_2)} \hat{f}_{\Delta_{1,3}A,\Delta_{2,3}A}(a_1,a_2)\, da_1 da_2.$$

Using von Mises differentials I have

$$\Delta_n = \sum_{k=1}^{\infty} \frac{1}{k!} d_k T(f_{\Delta_{1,3}A,\Delta_{2,3}A}; \hat{f}_{n;\Delta_{1,3}A,\Delta_{2,3}A} - f_{\Delta_{1,3}A,\Delta_{2,3}A}),$$

where

$$d_k T(f_{\Delta_{1,3}A,\Delta_{2,3}A}; \hat{f}_{n;\Delta_{1,3}A,\Delta_{2,3}A} - f_{\Delta_{1,3}A,\Delta_{2,3}A}) =$$
$$\frac{d^k}{d\lambda^k}\int\limits_0^t \frac{\int a_1 e^{iu_2a_2}(f_{\Delta_{1,3}A,\Delta_{2,3}A} + \lambda(\hat{f}_{n;\Delta_{1,3}A,\Delta_{2,3}A} - f_{\Delta_{1,3}A,\Delta_{2,3}A}))da_1da_2}{\int e^{iu_2a_2}(f_{\Delta_{1,3}A,\Delta_{2,3}A} + \lambda(\hat{f}_{n;\Delta_{1,3}A,\Delta_{2,3}A} - f_{\Delta_{1,3}A,\Delta_{2,3}A}))da_1da_2} du_2|_{\lambda=0} =$$
$$(-1)^k k! \int\limits_0^t \frac{A(u_2)B(u_2)^{k-1}}{\Psi_{\Delta_{1,3}A,\Delta_{2,3}A}(0,u_2)^{k+1}}\ du_2$$

with

$$A(u_2) = \int ia_1 e^{iu_2a_2}(\hat{f}_{n;\Delta_{1,3}A,\Delta_{2,3}A} - f_{\Delta_{1,3}A,\Delta_{2,3}A})da_1da_2 \int e^{iu_2a_2} f_{\Delta_{1,3}A,\Delta_{2,3}A}da_1da_2 -$$
$$\int ia_1 e^{iu_2a_2} f_{\Delta_{1,3}A,\Delta_{2,3}A}da_1da_2 \int e^{iu_2a_2} d(\hat{f}_{n;\Delta_{1,3}A,\Delta_{2,3}A} - f_{\Delta_{1,3}A,\Delta_{2,3}A})da_1da_2.$$
$$B(u_2) = \int e^{iu_2a_2}(\hat{f}_{n;\Delta_{1,3}A,\Delta_{2,3}A} - f_{\Delta_{1,3}A,\Delta_{2,3}A})da_1da_2.$$

In contrast to Step 1 all the random variables in the expression above have bounded support. Therefore,

$$|B(u_2)| \leq (\Delta a)^2 C_{f_{\Delta A}}$$
$$|A(u_2)| \leq (\overline{a}^2 - \underline{a}^2)(\Delta a + \Delta a^2) O(C_{f_{\Delta A}}).$$

As in Step 1 I use the fact that $\Delta_{k,l}A$ is ordinarily smooth with parameter $\beta_{\Delta A}$:

$$|\Delta_n| \leq \sum_{k=1}^{\infty} Q_3 C^k_{f_{\Delta A}} \tilde{T}_n^{\beta_{\Delta A}(1+k)} = \frac{\tilde{T}_n^{2\beta_{\Delta A}} C_{f_{\Delta A}}}{1 - \tilde{T}_n^{\beta_{\Delta A}} C_{F_{\Delta A}}} = O(\tilde{T}_n^{2\beta_{\Delta A}} C_{f_{\Delta A}}),$$
where $\tilde{T}_n = O((\frac{n}{\log n})^{\alpha_1})$ for some $\alpha_1 > 0$.

Therefore,

$$|\hat{\varphi}_{A_{i_3}}(t) - \varphi_{A_{i_3}}(t)| \leq O(|\Delta_n|) = O((\frac{\log n}{n})^{0.5-\alpha(1+\beta^*)-2\gamma-2\alpha_1\beta_{\Delta A}})$$
with $\alpha_1 < \dfrac{0.5 - \alpha(1+\beta^*) - 2\gamma}{2\beta_{\Delta A}}$.

The rate of convergence for A_{i_1} and A_{i_2} is obtained as at the beginning of Step 2.

$$
\begin{aligned}
&|\hat{\varphi}_{A_{i_k}}(t) - \varphi_{A_{i_k}}(t)| = O(C_{\Psi_{\Delta_{1,3}A,\Delta_{2,3}A}})) + O(\tilde{T}_n^{\beta_{A_{i_3}}} C_{\varphi_{A_{i_3}}}) \\
&O(C_{f_{\Delta A}}) + O(\tilde{T}_n^{\beta_{A_{i_3}}} C_{\varphi_{A_{i_3}}}) = \\
&O((\frac{\log n}{n})^{0.5-\alpha(1+\beta^*)-2\gamma}) + O((\frac{\log n}{n})^{0.5-\alpha(1+\beta^*)-2\gamma-\alpha 1(2\beta_{\Delta A}+\beta_{A_{i_3}})}) \\
&O((\frac{\log n}{n})^{0.5-\alpha(1+\beta^*)-2\gamma-\alpha 1(2\beta_{\Delta A}+\beta_{A_{i_3}})}) \text{ for } k = 1, 2.
\end{aligned}
$$

Finally, the rate of convergence for densities follows from Lemma 4 and is given by

$$
\begin{aligned}
&|\hat{f}_{A_{i_k}}(a) - f_{A_{i_k}}(a)| \leq 2\tilde{T}_n C_{\varphi_{A_{i_k}}} + \frac{1}{\pi} d_{A_{i_k}} \tilde{T}_n^{1-\beta_{A_{i_k}}} \text{ for } k = 1, 2, 3 \\
&|\hat{f}_{A_{i_3}}(a) - f_{A_{i_3}}(a)| \leq O((\frac{\log n}{n})^{0.5-\alpha(1+\beta^*)-2\gamma-\alpha_1(2\beta_{\Delta A}+1)}) + O((\frac{\log n}{n})^{\alpha_1(\beta_{A_{i_3}}-1)}) = \\
&O((\frac{\log n}{n})^{0.5-\alpha(1+\beta^*)-2\gamma-\alpha_1(2\beta_{\Delta A}+1)}) \text{ for } \frac{0.5-\alpha(1+\beta^*)-2\gamma}{\beta_{\Delta A}+2\beta_{A_{i_3}}} < \alpha_1 < \frac{0.5-\alpha(1+\beta^*)-2\gamma}{2\beta_{\Delta A}} \\
&|\hat{f}_{A_{i_k}}(a) - f_{A_{i_k}}(a)| \leq O((\frac{\log n}{n})^{0.5-\alpha(1+\beta^*)-2\gamma-\alpha_1(2\beta_{\Delta A}+\beta_{A_{i_3}})}) + O((\frac{\log n}{n})^{\alpha_1(\beta_{A_{i_k}}-1)}) = \\
&O((\frac{\log n}{n})^{0.5-\alpha(1+\beta^*)-2\gamma-\alpha_1(2\beta_{\Delta A}+\beta_{A_{i_3}})}) \text{ for } \frac{0.5-\alpha(1+\beta^*)-2\gamma}{2\beta_{\Delta A}+\beta_{A_{i_3}}+\beta_{A_{i_k}}-1} < \alpha_1 < \frac{0.5-\alpha(1+\beta^*)-2\gamma}{2\beta_{\Delta A}}.
\end{aligned}
$$

The uniform consistency of the estimator for the density of cost distribution is shown exactly like in Krasnokutskaya (2009). The only modification needed concerns the derivation of the estimators for the support bounds. More specifically, if $[\underline{\Delta}a, \overline{\Delta}a]$ denotes the support of $\Delta_{k,l}A$ variables, then under normalization $E[\log Y_2] = 1$ the following restrictions hold:

$$
\begin{aligned}
&\overline{\Delta}B = \overline{y}_2 \overline{\Delta}a \\
&\underline{\Delta}B = \underline{y}_2 \underline{\Delta}a \\
&\int_{\log \underline{y}_2}^{\log \overline{y}_2} f_{LY_2}(y)\, dy = 1 \\
&\int_{\log \underline{y}_2}^{\log \underline{y}_2} y f_{LY_2}(y)\, dy = 0.
\end{aligned}
$$

Alternatively, I could have used restriction that $\overline{\Delta}a = -\underline{\Delta}a$ since it holds even under the normalization above. The last two equations uniquely identify $\underline{y}_2$ and $\overline{y}_2$ whereas the first two equation will then identify $\underline{\Delta}a$ and $\overline{\Delta}a$ consistent with $E[\log Y_2] = 1$ normalization. The latter set of values can be used to identify $\underline{a}$ and $\overline{a}$ from the following restrictions:

$$\overline{a} - \underline{a} = \overline{\Delta}a$$
$$\int_{\underline{a}}^{\overline{a}} f_{A_{i_1}}(a)\,dy = 1.$$

This set of restrictions is used to derive estimators for the support bounds.

Step 3

I first derive

$$C_{f_{Y_2A_1}} = \sup_{z\in S(Y_2A_1)} |\hat{f}_{Y_2A_{i_1}}(z) - f_{Y_2A_{i_1}}(z)| \leq$$
$$\int \frac{1}{y}|\hat{f}_{Y_2}(y)\hat{f}_{A_{i_1}}(\frac{z}{y}) - \hat{f}_{Y_2}(y)f_{A_{i_1}}(\frac{z}{y}) + \hat{f}_{Y_2}(y)f_{A_{i_1}}(\frac{z}{y}) - f_{Y_2}(y)f_{A_{i_1}}(\frac{z}{y})|dy$$
$$\leq \int \frac{1}{y}(|\hat{f}_{Y_2}(y)||\hat{f}_{A_{i_1}}(\frac{z}{y}) - f_{A_{i_1}}(\frac{z}{y})| + |f_{A_{i_1}}(\frac{z}{y})||\hat{f}_{Y_2}(y) - f_{Y_2}(y)|)dy =$$
$$\int \frac{1}{y}((M_{1,Y_2} + C_{f_{Y_2}})C_{f_{A_{i_1}}} + M_{1,A}C_{f_{Y_2}})dy =$$
$$M_{0,Y_2}M_{1,Y_2}C_{f_{A_{i_1}}} + M_{0,Y_2}M_{1,A_1}C_{f_{Y_2}} + M_{0,Y_2}C_{f_{Y_2}}C_{f_{A_{i_1}}} \leq Q_3C_{f_{A_{i_1}}} =$$

$$= O((\frac{\log n}{n})^{0.5-\alpha(1+\beta^*)-2\gamma-\alpha_1(2\beta_{\Delta A}+\beta_{A_{i_3}})}) \text{ for some } Q_3 > 0.$$

Here $\int \frac{1}{y}dy \leq M_{0,Y_2}$, $|\hat{f}_{Y_2}(y)| \leq M_{1,Y_2}$, $|f_{A_{i_1}}(\frac{z}{y})| \leq M_{1,A_1}$ with $M_{0,Y_2} > 0$, $M_{1,Y_2} > 0$ $M_{1,A_1} > 0$.

This, then, implies that

$$|\hat{\varphi}_{Y_2A_{i_1}}(t) - \varphi_{Y_2A_{i_1}}(t)| \leq$$
$$\int |\hat{f}_{Y_2A_{i_1}}(y) - f_{Y_2A_{i_1}}(y)|\,dy = M_{0,Y_2A_{i_1}}Q_3C_{f_{Y_2A_{i_1}}} = O((\frac{\log n}{n})^{0.5-\alpha(1+\beta^*)-2\gamma-\alpha_1(2\beta_{\Delta A}+\beta_{A_{i_3}})}),$$

where $\int\limits_{S(Y_2A_{i_1})} dy \leq M_{0,Y_2A_{i_1}}$.

Similar to Step 2:

$$|\hat{\varphi}_{Y_1}(t) - \varphi_{Y_1}(t)| = O(|\log(\hat{\varphi}_{Y_1}(t)) - \log(\varphi_{Y_1}(t))|$$

and

$$
\begin{aligned}
&C_{\varphi_{Y_1}} = \sup_{t\in[-T_n,T_n]} |\log(\hat{\varphi}_{Y_1}(t)) - \log(\varphi_{Y_1}(t))| \leq O(|\frac{\hat{\Psi}_{B_{i_1}}(t) - \Psi_{B_{i_1}}(t)}{\Psi_{B_{i_1}}(t)}|) + \\
&O(|\frac{\hat{\varphi}_{Y_2A_{i_1}}(t) - \varphi_{Y_2A_{i_1}}(t)}{\varphi_{Y_2A_{i_1}}(t)}| = C_{F_B}\hat{T}_n^{1+\beta_B} + C_{\varphi_{Y_2A_{i_1}}}\hat{T}_n^{\beta_{Y_2A_1}} = \\
&O((\frac{\log\log n}{n})^{0.5-\alpha_2(1+\beta_B)}) + O((\frac{\log n}{n})^{0.5-\alpha(1+\beta^*)-2\gamma-\alpha_1(2\beta_{\Delta A}+\beta_{A_{i_3}})-\alpha_2\beta_{Y_2A_1}}) = \\
&O((\frac{\log n}{n})^{0.5-\alpha(1+\beta^*)-2\gamma-\alpha_1(2\beta_{\Delta A}+\beta_{A_{i_3}})-\alpha_2\beta_{Y_2A_1}}) \\
&\text{if } \beta_{Y_2A_1} < 1+\beta_B, \text{ or } \alpha_2 \leq \frac{\alpha(1+\beta^*)+2\gamma+\alpha_1(2\beta_{\Delta A}+\beta_{A_{i_3}})}{1+\beta_B+\beta_{Y_2A_1}}.
\end{aligned}
$$

Here I use the ordinary smoothness of $\Psi_{B_{i_1}}(t)$ and $\varphi_{Y_2A_{i_1}}(t)$.
The value for $\sup_{t\in[-T_n,T_n]}|\hat{\Psi}_{B_{i_1}}(t) - \Psi_{B_{i_1}}(t)|$ is obtained from integration by parts

$$
\begin{aligned}
&\sup_{t\in[-\hat{T}_n,\hat{T}_n]} |\hat{\Psi}_{B_{i_1}}(t) - \Psi_{B_{i_1}}(t)| = |\int (\hat{F}_{n;B}(b) - F_{n;B}(b)) ite^{itb} db| = \\
&= C_{F_B}\hat{T}_n = O((\frac{\log\log n}{n})^{0.05-\alpha_2}) \\
&\text{for } \hat{T}_n = O((\frac{n}{\log n})^{\alpha_2}) \text{ with } \alpha_2 > 0.
\end{aligned}
$$

The value for C_{F_B} obtains by the log-log law (see Chung, 1949; Serfing, 1980). Finally, from Lemma 4 I have

$$
\begin{aligned}
&|\hat{f}_{Y_1}(t) - f_{Y_1}(t)| \leq 2C_{\varphi_{Y_1}}\hat{T}_n + d_{Y_1}\hat{T}_n^{\,1-\beta_{Y_1}} = \\
&O((\frac{\log n}{n})^{0.5-\alpha(1+\beta^*)-2\gamma-\alpha_1(2\beta_{\Delta A}+\beta_{A_{i_3}})-\alpha_2(1+\beta_{Y_2A_1})}) + O((\frac{\log n}{n})^{\alpha_2(\beta_{Y_1}-1)}).
\end{aligned}
$$