Penn Institute for Economic Research
Department of Economics
University of Pennsylvania
3718 Locust Walk
Philadelphia, PA 19104-6297
pier@econ.upenn.edu
http://economics.sas.upenn.edu/pier

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by

David Cass, Abhinash Borah, Kyungmin Kim,
Maxim Kryshko, Antonio Penta, and Jonathan Pogach

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Robustness of the Uniqueness of Walrasian Equilibrium with Cobb-Douglas Utilities.*

David Cass, Abhinash Borah, Kyungmin Kim,
Maxim Kryshko, Antonio Penta, Jonathan Pogach
University of Pennsylvania, Dept. of Economics

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Abstract

The majority of results in the literature on general equilibrium are not for *an* economy (i.e. given an endowment and preferences), but rather, for a *set* of economies (i.e. a set of endowments given preferences). Therefore, we argue that the most appropriate robustness result requires perturbing economies uniformly over the space of endowments for which the result is obtained. In this paper, we examine the robustness of the uniqueness of Walrasian endowment economies with Cobb-Douglas utility functions under this interpretation of robustness. Namely, we prove that for economies described by Cobb-Douglas utilities and all endowments in a fixed set, uniqueness of equilibrium is robust to perturbations of the utility functions.

JEL Codes: D50, D51.

1 Introduction

There is a long literature aimed at proving various robustness results of Walrasian equilibrium. In an endowment economy, defined by preferences and endowments, these results establish that small perturbations to our economy in both preferences and endowments do not produce large changes in the equilibrium prices or allocations. The spirit of such

*This paper was presented at the "IV Caress-Cowles Conference on General Equilibrium and its Applications" (April 2008), a few days after Dave's death. Unfortunately, Dave could not take part to the final draft of the working paper. The rest of us thus take full responsibility for all remaining errors. This paper is the result of a class project in the course on Advanced General Equilibrium that David held at Penn in Spring 2006. As unique and original as his contribution to theoretical economics is, Dave's dedication to teaching is probably even harder to parallel. We are honored to have been his students, and have had the opportunity to move our first steps in research inspired by his passion.

exercises is to ensure that other results we might obtain regarding Walrasian economies are not, in general, particular to the parameters of the economy we have chosen.

However, the majority of results in the literature are not for *an* economy (i.e. given an endowment and preferences), but rather, for a *set* of economies (i.e. a set of endowments given preferences). Therefore, for results obtained over a set of endowments the more appropriate robustness result would not be to look at how prices and allocations respond to perturbations of individual economies, but rather to perturb economies uniformly over the space of endowments for which the result is obtained. In this paper, we examine the robustness of the uniqueness of Walrasian endowment economies with Cobb-Douglas utility functions under the latter interpretation of robustness. Namely, we prove that for economies described by Cobb-Douglas utilities and all endowments in a fixed set, uniqueness of equilibrium is robust to perturbations of the utility functions.

To highlight the distinction between our work and previous work, consider Smale (1974). Smale shows that the set of regular economies, where an economy is an endowment-utility pair, is open and dense in the endowment-utility space. Given uniqueness of equilibrium and regularity of the economy with Cobb-Douglas utility functions, Smale's result would immediately imply that given any endowment, there is an open neighborhood in the endowment-utility space such that equilibrium is unique. However, this neighborhood depends on the particular endowment chosen. In contrast, our result is that for a fixed set of endowments and a uniform perturbation of the Cobb-Douglas utility functions, equilibrium is unique for all endowments in that set. We believe that our result is more in spirit with the robustness exercise.

In the following section we establish notation and represent some well known results in the literature that are necessary for our proof. We then formally state our result and proof.

2 Notation

We will work in the Walrasian endowment economy with H households and G commodities, with generic household $h \in \mathcal{H} = \{1, ..., H\}$ and commodity $g \in \{1, ..., G\}$, respectively. An economy is described by an endowment $e_{-1} \in E_{-1}$ and utility functions $u \in U$ where $E_{-1} = \left\{ e_{-1} \in \mathbb{R}_{++}^{G(H-1)} : \sum_{h \neq 1} e_{-1} < r \right\}$ with $r \in \mathbb{R}_{++}^{G}$ the total amount of resources of the economy and $U = \times_h U_h$, where U_h is the set of $\mathcal{C}^2$ differentiably strictly increasing and differentiably strictly quasi concave functions from $\mathbb{R}_{++}^{G}$ to $\mathbb{R}$. Note that parameters of the economy r and e_{-1} immediately pin down e_1, the endowment of household 1, which we treat as "endogenous".

To fix notation further, we will denote the set of Cobb-Douglas utility functions (where each good is assigned a strictly positive value) as $U^A \subset U$, with generic element denoted u^α. Formally,

$$U^A = \left\{ u \in U | \forall h \in \mathcal{H}, \forall x_h \in X_h,\, u_h^\alpha(x_h) = \sum_{g=1}^{G} \alpha_h^g \log x_h^g \right\}$$

where $\alpha \in A \equiv \left\{ \alpha \in \mathbb{R}_{++}^{G\times H} | \sum_{g=1}^{G} \alpha_h^g = 1,\, \forall h \right\}$. $X_h \subseteq \mathbb{R}_{++}^G$ is household's h consumption space.

Endow U with the topology of $\mathcal{C}^2$ uniform convergence on compact sets of $\mathbb{R}_{++}^G$. For the reader's benefit, we reproduce the definition of our topology from Munkres (1975, p.282):

Definition 1 *A sequence $f_n : X \to Y$ of functions converges to the function f in the topology of compact convergence if and only if for each compact subset $C \subseteq X$, the sequence $f_n|_C$ converges uniformly to $f|_C$.*

U is made into a metric space in the following way (cf. Allen, 1981): Let $\{K_n\}_{n=1}^\infty$ be an arbitrary sequence of compact sets in $\mathbb{R}_{++}^G$ such that $\cup_{n=1}^\infty K_n = \mathbb{R}_{++}^G$. Then define the metric d by

$$d(u,v) = \sum_{n=1}^{\infty} \frac{1}{2^n} \min\{(||u-v||_{2,K_n}), 1\}$$

for $u, v \in U$, where $||\cdot||_{2,K_n}$ is the $\mathcal{C}^2$ uniform norm of $\mathcal{C}^2(K_n, \mathbb{R})$, namely:

$$\begin{aligned} ||u-v||_{2,K_n} &= ||\,(u|_{K_n} - v|_{K_n})\,||_2 \\ &= \sup_{x\in K_n} |u(x)-v(x)| + \sup_{x\in K_n} ||Du(x)-Dv(x)|| + \sup_{x\in K_n} ||D^2u(x)-D^2v(x)||. \end{aligned}$$

We will denote the set of parameters as $\Theta = E_{-1} \times U$ and the set of endogenous variables as $\Xi = X \times \Delta \times \Lambda \times E_1$ where Δ is the unit simplex $\{p | p^g \geq 0, \sum_g p^g = 1\}$, $\Lambda = \mathbb{R}_+^H$, and $E_1 = \{e_1 | 0 < e_1^g < r^g, \forall g\}$. Also, we will denote by $\pi : \Theta \times \Xi \to \Theta$ the natural projection of $\Theta \times \Xi$ onto Θ.

Definition 2 $(p^*, x^*, \lambda^*, e_1) \in \Xi$ *is an equilibrium if*

$$x_h^* \in \arg\max_{x_h \in B_h(p^*)} u_h(x_h), \forall h$$

$$\sum_h e_h^g - x_h^g = 0, \forall g$$

$$r - \sum_{h\neq 1} e_h = e_1$$

where $B_h(p^*) = \{x_h \in X_h | p^*(e_h - x_h) = 0\}$

Equations characterizing equilibrium will be written as $\Phi(\theta, \xi) = 0$ where $\theta \in \Theta$ and $\xi \in \Xi$.

Before the statement of our theorem, we make note of a few well known results necessary for our proof. In Lemmas 1 and 2 we restate the uniqueness and regularity results with Cobb-Douglas utility functions, proofs of which appear in the Appendix. Proof of Theorem 1 can be found in Balasko (1975).

Lemma 1 (Uniqueness) *Suppose that $u \in U^A$ and $e_{-1} \in E_{-1}$, then the equilibrium is unique.*

Lemma 2 (Regularity) *Suppose that $u \in U^A$ and $e_{-1} \in E_{-1}$, then*

$$\det D_\xi \Phi(e_{-1}, \xi) \neq 0.$$

Theorem 1 (Balasko, 1975) *For all $u \in U$, there exists an open set, V around the Pareto set such that for all $e_{-1} \in V$, $|\pi^{-1}(e_{-1})| = 1$ and endowments on the Pareto set are regular.*

3 The Problem

Given these well established results, we introduce one final piece of notation before the formal statement of our theorem. Define:

$$\bar{E}(\delta, u) = \{x \in X \bigcap E | u_h(x_h) \geq u_h(\delta \mathbf{1}_G)\}$$

where $\mathbf{1}_G$ is a $1 \times G$ vector of ones. This set represents the intersection of upper contour sets in the Edgeworth box of each individual receiving δ units of each good. The necessity of the use of this restriction will be made clear in the proof.

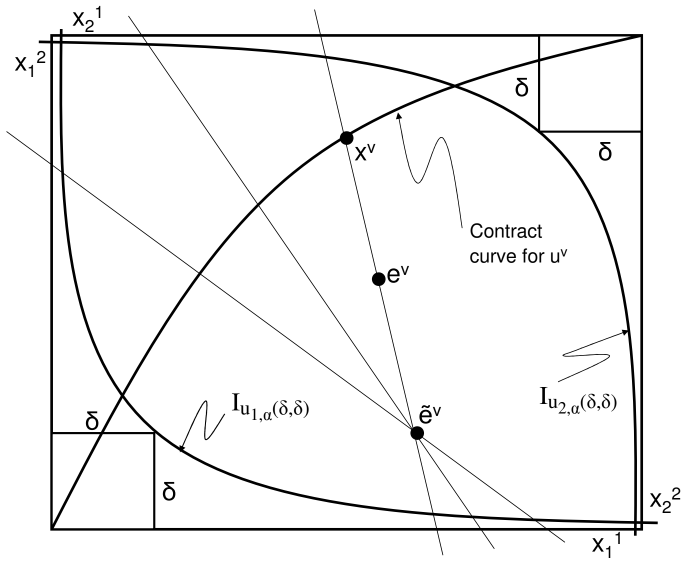
Theorem 2 $\forall u^\alpha \in U^A$, $\forall \delta > 0$ *there exists a neighborhood of u^α, $N(u^\alpha) \subseteq U$, such that $\forall e_{-1} \in \bar{E}_{-1}$, $u \in N(u^\alpha) \Rightarrow |\pi^{-1}(e_{-1}, u)| = 1$.*

The proof can be easily understood graphically using a 2×2 Edgeworth box and we include such a representation in the proof.

Proof. The proof is by contraposition. Suppose that the statement of the theorem is false. Then, there exists some $\alpha \in A$, $\delta > 0$ and a sequence $\{u^\nu\}_{\nu \in \mathbb{N}} \subseteq U$ such that $u^\nu \to u^\alpha$ and a corresponding sequence of endowments, $\{\tilde{e}^\nu_{-1}\}_{\nu \in \mathbb{N}} \to e_{-1} \in \bar{E}_{-1}(u, \delta)$ such that $\forall \nu \in \mathbb{N}$, $\left|\pi^{-1}\left(\tilde{e}^\nu_{-1}, u^\nu\right)\right| > 1$.

Let x^ν be an associated sequence of equilibrium allocations for $(\tilde{e}^\nu_{-1}, u^\nu)$. By Balasko's theorem, we know that: $\forall \nu$, $\exists e^\nu_{-1} \in E_{-1}$ such that $e^\nu_{-1} = \eta^\nu \tilde{x}^\nu_{-1} + (1 - \eta^\nu)\, \tilde{e}^\nu_{-1}$, where

Figure 1: Graphical Representation of the Proof.



$\eta^\nu \in [0, 1)$, such that $\xi^\nu \in \pi^{-1}(e_1^\nu, u^\nu)$ so that $\tilde{x}^\nu$ is an equilibrium allocation given e^ν and e_{-i}^ν is a critical value, so that det $\left(D_\xi \Phi\left(\xi^\nu, e_{-1}^\nu, u^\nu\right)\right) = 0$. This follows from the fact that we have uniqueness and regularity around the Pareto set and multiplicity at $\tilde{e}_{-1}^\nu$. Thus, there must be a critical value between, as Balasko (1975) shows that path connected regular values have the same number of equilibria.

Continuity of D^2u implies that $D_\xi\Phi$ is continuous in u (and it is continuous in its other arguments as well).

Because $\det(D_\xi\Phi(\theta^\nu, \xi^\nu)) = 0$ for all ν, the continuity in θ, and continuity of the determinant it must be that $\det(D_\xi\Phi(\theta, \xi)) = 0$ where $(\theta^\nu, \xi^\nu) \to (\theta, \xi)$. To conclude the proof we must establish that $x^\nu \to x \in \bar{E}(\delta, u^\alpha)$, so that we may conclude that there exists a critical value $e_{-1} \in E_{-1}$. This follows from the upper hemi-continuity of the upper contour set in u (see Lemma 3 in the Appendix). This contradicts Lemma 2, completing the proof.

The idea of the proof is illustrated in Figure 1.[1] Note first that the lens is fixed by our selection of u^α. Our sequence $\tilde{e}^\nu$ must lie in the lens by assumption. Balasko guarantees that there exists a corresponding sequence of critical points e^ν that are convex

[1]In the graph $I_{u_h^\alpha(\delta,\delta)}$ represents household h's indifference curve through consumption bundle (δ, δ) given Cobb-Douglas utility functions described by u^α. The contract curve is for those preferences u^ν in the sequence used to construct the contradiction. Other notation in the graph is as described in the proof.

combinations of our endowment sequence and associated equilibrium allocations. For the proof to work, we need the critical points, e^ν to converge to a point in our lens to arrive at a contradiction. We use the lens to guarantee that our sequence of critical points is converging in the interior of the Edgeworth box. This is resolved by the upper hemi-continuity of the upper contour sets. ■

This statement, in contrast to previous results, establishes robustness of the uniqueness of equilibrium across endowments to uniform perturbations of the utility functions. Given that results are often established over a fixed set of endowments, we believe that ours is the appropriate robustness exercise rather than one established point by point in the parameter space. In addition, notice that for any set of endowments whose closure is in the interior of the Edgeworth Box, we can find a sufficiently small $\delta > 0$ so that our lens, $\bar{E}(\delta, u^\alpha)$, strictly contains that set of endowments.

Only the uniqueness and regularity of Cobb-Douglas utility functions were necessary for our proof. Therefore, so long as we can find some lens on which a set of utility function exhibits uniqueness and regularity, we would be able to obtain the same robustness result for the set of endowments in that lens.

4 Appendix

Proof of Lemma 1. Equilibrium is described by the following system of equations:

$$\Phi(\theta, \xi) = \begin{pmatrix} \frac{\alpha_h^g}{x_h^g} - p^g \lambda_h, \forall h, g \\ \sum_g p^g (x_h^g - e_h^g), h = 1, ..., H \\ \sum_h (x_h^g - e_h^g), g = 1, ..., G-1 \\ r^g - \sum_h e_h^g, g = 1, ..., G \end{pmatrix} = 0$$

We follow the proof strategy by Gale (1960). First, we can reduce the extended form equations to yield that any equilibrium price vector must satisfy $p(I_{g\times g} - AE') = 0$ where A is a $G \times H$ matrix with elements $A_{gh} = \alpha_h^g$ and E is a $G \times H$ matrix with normalized endowment elements $E_{gh} = e_h^g / r^g = \hat{e}_h^g$. Let $\Gamma = AE'$ with generic element $\gamma_{ij} = \sum_g \alpha_i^g \hat{e}_j^g$.[2] First, note that equilibrium prices are strictly positive due to u_h strictly increasing.

To show uniqueness, consider two equilibrium price vectors p and p'. Let $\zeta = \min_g p^g / p'^g$, w.l.o.g. let $\zeta = p^1 / p'^1$. By positive prices, $\zeta > 0$ and $p^g - \zeta p'^g \geq 0$. Define $p'' = p - \zeta p' \geq 0$

[2] First order conditions on x yield $\frac{\alpha_h^g}{x_h^g} - \lambda_h p^g = 0$. Summing over g and letting $w_h = \sum_g p^g e_h^g = 1/\lambda_h$ we get that $\frac{\alpha_h^g w_h}{p^g} = x_h^g$. Summing over h and applying the market clearing condition (where we have normalized resources to 1), we obtain $p^g = \sum_h \alpha_h^g w_h$, which can be expressed as above as $p(I_{g\times g} - AE') = 0$.

so that $\Gamma p'' = p''$. However, $p''^1 = 0$ by construction. Therefore, $p'' = 0$ so $p = \zeta p'$, which is possible in the simplex only if $\zeta = 1$.■

Proof of Lemma 2. It is enough to show that for any $u^\alpha \in U^A$, $D_\xi \Phi(e_{-1}, \xi) \cdot \Delta\xi = 0$ implies that $\Delta\xi = 0$ (where $\Delta\xi = (\Delta x, \Delta p, \Delta\lambda, \Delta e_1)$ denotes the vector of variations of the endogenous variables). For convenience, we renormalize the price vector taking good G as the numeraire (i.e. $p^G = 1$), so that $\Delta p \in \mathbb{R}^{G-1}$, and we obtain:

$$D_\xi \Phi(e_{-1}, \xi) \cdot \Delta\xi = \begin{pmatrix} -\frac{\alpha_h^g}{(x_h^g)^2} \Delta x_h^g - \lambda_h \Delta p^g - p^g \Delta\lambda_h, & \forall h, \forall g \neq G & (1) \\ -\frac{\alpha_h^G}{(x_h^G)^2} \Delta x_h^G - \Delta\lambda_h, & \forall h & (2) \\ \sum_{g=1}^{G} p^g \Delta x_h^g - \sum_{g=1}^{G-1} \Delta p^g x_h^g, & \forall h & (3) \\ \sum_{h=1}^{H} \Delta x_h^g, & \forall g & (4) \\ \Delta e_1^g, & \forall g & (5) \end{pmatrix}.$$

Clearly, if the $D_\xi \Phi(e_{-1}, \xi) \cdot \Delta\xi = 0$, then $\Delta e_1 = 0$ from (5). Looking at (2) and combining it with (1) for each h, and the equilibrium relation $p^g = \alpha_h^g / (\lambda_h x_h^g)$ we obtain

$$\gamma_h^g p^g \Delta x_h^G = p^g \Delta x_h^g - \Delta p^g x_h^g$$

where $\gamma_h^g = \frac{\alpha_h^G x_h^g}{\lambda_h (x_h^G)^2} > 0$. Then, summing over the the first $G-1$ goods and using (3) we arrive to

$$\Delta x_h^G \sum_{g=1}^{G-1} \gamma_h^g p^g = \sum_{g=1}^{G-1} (p^g \Delta x_h^g - \Delta p^g x_h^g) = \Delta x_h^G.$$

Given that γ and p are strictly positive, this implies that $\Delta x_h^G = 0$ for all h. By (2), $\Delta\lambda = 0$. From (1) this implies that for any g, Δx_h^g and Δp^g have the opposite sign for all h, so that for any g, Δx_h^g and $\Delta x_{h'}^g$ have the same sign for all h, h'. However, given (4), this is possible only when $\Delta p^g = \Delta x_h^g = 0$, concluding the proof.■

Lemma 3 (UHC of Upper Contour Sets) *$\forall h \in H, \forall x_h \in X_h$, define $B_h : U \rightrightarrows E$ such that $\forall u \in U$, $B_h(u) = \bar{X}_h(u, x_h) \cap E$. The correspondence B_h is upper hemi-continuous.*

Proof: We need to show:

$$\begin{aligned} &\forall \{u_h^\nu\} : u_h^\nu \to u_h, \forall \{x_h^\nu\} \subseteq X_h \text{ such that} \\ &\quad \forall \nu \in \mathbb{N}, x_h^\nu \in B_h(u^\nu) : \\ &\quad x_h^\nu \to \bar{x}_h \in B_h(u). \end{aligned}$$

Since $\{x_h^\nu\} \subseteq E$, which is compact, $x_h^\nu \to \bar{x}_h$ for some $\bar{x}_h \in E$. By contradiction, suppose that $\bar{x}_h \notin B_h(u_h)$. This means $\bar{x}_h \notin \bar{X}_h(u, x_h)$, that is $u_h(x_h) - u_h(\bar{x}_h) = \varepsilon > 0$. Then we have:

$$\begin{aligned}
\varepsilon &= u_h(x_h) - u_h(\bar{x}_h) \\
&= u_h(x_h) - u_h^\nu(x_h) + u_h^\nu(x_h) - u_h(\bar{x}_h) \\
&\leq u_h(x_h) - u_h^\nu(x_h) + u_h^\nu(x_h^\nu) - u_h(\bar{x}_h) \\
&\leq |u_h(x_h) - u_h^\nu(x_h)| + |u_h^\nu(x_h^\nu) - u_h(\bar{x}_h)| \\
&\leq |u_h(x_h) - u_h^\nu(x_h)| + |u_h^\nu(x_h^\nu) - u_h(x_h^\nu)| + |u_h(x_h^\nu) - u_h(\bar{x}_h)| \\
&< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon \quad \text{(a contradiction).}
\end{aligned}$$

where the first inequality derives from the fact that by definition, $\forall \nu \in \mathbb{N}, u_h^\nu(x_h^\nu) \geq u_h^\nu(x_h)$. The third inequality is for the triangular inequality, and the last one is implied by:

Convergence $u_h^\nu \to u_h$ and continuity of u_h. Convergence of $u_h^\nu \to u_h$ implies that $\forall\varepsilon$, $\exists N : \forall \nu > N, d(u_h^\nu, u_h) < \varepsilon/3$ which in our case implies that $\forall \nu > N$:

$$\begin{aligned}
\varepsilon/3 &> sup_{x_h' \in X_h} |u_h^\nu(x_h') - u_h(x_h')| \geq |u_h^\nu(x_h) - u_h(x_h)| \text{ and} \\
\varepsilon/3 &> sup_{x_h' \in X_h} |u_h^\nu(x_h') - u_h(x_h')| \geq |u_h^\nu(x_h^\nu) - u_h(x_h^\nu)|
\end{aligned}$$

Continuity of u_h implies that $\forall\varepsilon/3$,$\exists N$:$\forall \nu > N$, $|u_h(x_h^\nu) - u_h(\bar{x}_h)| < \varepsilon/3$.■

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