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"On the Structure of Rationalizability for Arbitrary Spaces of Uncertainty",
Second Version

by

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On the Structure of Rationalizability for Arbitrary Spaces of Uncertainty.*

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Abstract

This note characterizes the set $\mathcal{A}_i^\infty$ of actions of player i that are uniquely rationalizable for some hierarchy of beliefs on an arbitrary space of uncertainty Θ. It is proved that for any rationalizable action a_i for the type t_i, if a_i belongs to $\mathcal{A}_i^\infty$ and is justified by conjectures concentrated on $\mathcal{A}_{-i}^\infty$, then there exists a sequence of types converging to t_i for which a_i is uniquely rationalizable.

Keywords: Rationalizability, incomplete information, robustness, refinement, higher order beliefs, dominance solvability, richness.

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1 Introduction

This note characterizes the set $\mathcal{A}_i^\infty$ of actions of player i that are uniquely rationalizable for *some* hierarchy of beliefs on an arbitrary space of uncertainty Θ. It is proved that for any rationalizable action a_i for the type t_i, if a_i belongs to $\mathcal{A}_i^\infty$ and is justified by conjectures concentrated on $\mathcal{A}_{-i}^\infty$, then there exists a sequence of types converging to t_i for which a_i is uniquely rationalizable.

Assuming that Θ contains regions of strict dominance for each player's strategy (the *richness condition*), Weinstein and Yildiz (2007) prove a version of the above result:

- **(Non-) Robustness (R.1):** whenever a model has multiple *rationalizable* outcomes, any of these is *uniquely* rationalizable in a model with beliefs arbitrarily close to the original ones.

*This paper benefited from countless conversations with George Mailath.

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Result (R.1) essentially corresponds to the case $\mathcal{A}_i^\infty = A_i$ for each i.

An important implication of (R.1) is that, under the richness condition, the strongest predictions that are robust to perturbations of higher order beliefs are those based on rationalizability alone. In many situations though, imposing richness may be an unnecessarily demanding robustness test: The richness condition on Θ implies that it is *not* common knowledge that any strategy is *not* dominant. However, as modelers, we may wish to assume that some features of the environment actually are common knowledge. For example: common knowledge that some strategies are not dominant. In that case, the underlying space of uncertainty does not satisfy *richness*. The main purpose of this paper is to explore the structure of rationalizability without assuming *richness*.

By guaranteeing that $\mathcal{A}_i^\infty = A_i$ for each i, the richness condition also delivers the following striking result (Weinstein and Yildiz, 2007):

- **Generic Uniqueness (R.2):** in the space of hierarchies of beliefs, the set of types with a unique *rationalizable* action is open and dense (i.e. models are generically *dominance-solvable*);

Result (R.2) generalizes an important insight from the global games literature, that multiplicity is often the consequence of the modeling assumptions of common knowledge.[1] If such assumptions are relaxed (e.g. assuming *richness*), hierarchies of beliefs "typically" have a unique rationalizable outcome. The case of multiplicity corresponds to a knife-edge situation, at the boundary of regions of uniqueness for each of the rationalizable actions.

Multiplicity is pervasive in applied models. Yet, from a theoretical point of view, there is a sense in which a *complete model* should be able to deliver a unique prediction. By way of analogy, consider the dynamics of a coin toss: If all the information about the intervening forces, the mass and shape of the coin, air pressure and so on (the initial conditions) were available, according to Newtonian mechanics we could predict the outcome of the coin toss. The "practical" unpredictability of a coin toss is rather a consequence of the "imperfection" of our model for the initial conditions: indeterminacy does not pertain to the underlying phenomenon; rather, it stems from the modeling activity.[2]

[1] Notice though that result (R.1) is in sharp contrast with that literature: In the global games' approach, the relaxation of common knowledge assumptions supports the robust selection of a unique equilibrium. In contrast, (R.1) implies that if one considers a richer class of perturbations, any selection from rationalizability is not robust. See Morris and Shin (2003) for a thorough survey of the literature.

[2] The last paragraph presumes that the underlying phenomenon, i.e. the object of the model, is *not* "intrinsically indeterminate". It is not a statement that no such indeterminate objects exist. If one believes that the object of study is intrinsically indeterminate, then the statement should be rejected. A debate in phylosophy of science is open on the issue.

Result (R.2) can be interpreted as saying that the typical indeterminacy of standard game theoretical models does not pertain to the object of study; rather, it is a consequence of the simplifying assumptions that we make on higher order beliefs.

It will be shown that very weak conditions on Θ suffice to obtain Weinstein and Yildiz's results in their full strength, without imposing the *richness condition*. For instance, it suffices to assume that there exists a state in Θ for which payoff functions are supermodular, plus dominance regions for the corresponding extreme actions only. In other words, if it is *not* common knowledge that the game is *not* supermodular, and that the corresponding extreme actions are *not* dominant, then the strongest predictions that are robust to perturbations of higher order beliefs are those based on rationalizability alone.

2 Game Theoretic Framework

I consider static games with payoff uncertainty, i.e. tuples $G = \left\langle N, \Theta, (A_i, u_i)_{i\in N}\right\rangle$ where N is the set of players; for each $i \in N$, A_i is the set of actions and $u_i : A \times \Theta \to \mathbb{R}$ is i's payoff function, where $A := \times_{i\in N} A_i$ and Θ is a parameter space representing agents' incomplete information about the payoffs of the game. Assume that the sets N, A and Θ are all finite. As standard, hierarchies of beliefs can be represented by means of type spaces: a *type space* is a tuple $\mathcal{T} = (T_i, \tau_i)_{i\in N}$ s.t. for each $i \in N$, T_i is the (compact) set of *types* of player i, and the continuous function $\tau_i : T_i \to \Delta(\Theta \times T_{-i})$ assigns to each type of player i his beliefs about Θ and the opponents' types. Let T_i^* be the set of all *coherent* hierarchies; we denote by $\mathcal{T}^* = (T_i^*, \tau_i^*)_{i\in N}$ the Θ-*based universal type space* (Mertens and Zamir, 1985). Elements of T_i^* will be referred to as *types* or *hierarchies*. A type $t_i \in T_i^*$ is finite if $\tau_i^*(t_i) \in \Delta\left(\Theta \times T_{-i}^*\right)$ has finite support; the set of finite types in the universal type space is denoted by $\hat{T}_i \subseteq T_i^*$. The function $\pi_i^{\mathcal{T}} : T_i \to T_i^*$ represents the belief morphism assigning to each type in a type space the corresponding hierarchy in the universal type space.

Attaching a type space-representation of the players' hierarchies of beliefs to the game with payoff uncertainty G, one obtains a *Bayesian model*, i.e. the Bayesian game $G^{\mathcal{T}} = \left\langle N, \Theta, (A_i, T_i, \hat{u}_i)_{i\in N}\right\rangle$, with payoff functions defined as $\hat{u}_i : A \times \Theta \times T \to \mathbb{R}$. Since players' types are payoff irrelevant, with a slight abuse of notation we write u_i and drop the dependence on T.[3]

[3] In Weinstein and Yildiz (2007) and in the present settings *types* are payoff-irrelevant, or purely epistemic (capturing beliefs). Penta (2009) instead considers the general case which allows for *payoff-types*: the space of uncertainty is $\Theta = \Theta_0 \times \Theta_1 \times ... \times \Theta_n$ and each player i observes the i-th component of the realized θ. A player's type $t_i = (\theta_i, e_i)$ is made of two parts: a payoff-relevant component (what i *knows*, θ_i) and a purely epistemic component (e_i), representing his *beliefs* about what he doesn't know:

Given a Bayesian model $G^{\mathcal{T}}$, player i's conjectures are denoted by $\psi^i \in \Delta(\Theta \times A_{-i} \times T_{-i})$.[4] For each type t_i, his *consistent conjectures* are

$$\Psi_i(t_i) = \left\{\psi^i \in \Delta(\Theta \times A_{-i} \times T_{-i}) : marg_{\Theta \times T_{-i}} \psi^i = \tau_i(t_i)\right\}.$$

For any $B_i \subseteq A_i$, let $BR_i^{B_i}(\psi^i)$ denote the set of best responses in B_i to conjecture ψ^i, and write $BR_i(\psi^i) \equiv BR_i^{A_i}(\psi^i)$. Formally:

$$BR_i^{B_i}(\psi^i) = \arg\max_{a_i \in B_i} \sum_{(\theta, a_{-i}, t_{-i})} u_i(\theta, a_i, a_{-i}) \cdot \psi^i(\theta, a_{-i}, t_{-i}).$$

If $a_i \in BR_i^{B_i}(\psi^i)$, we say that ψ^i *justifies* action a_i in B_i. Appealing again to the payoff-irrelevance of the epistemic types, with another abuse of notation we will write $BR_i(\beta^i)$ for conjectures $\beta^i \in \Delta(\Theta \times A_{-i})$.

We define next the solution concept, *Interim Correlated Rationalizability (ICR)*, introduced by Dekel et al. (2007):

Definition 1 *Fix a Bayesian model $G^{\mathcal{T}}$. For each $t_i \in T_i$ and $i \in N$, set $ICR_i^0(t_i) = A_i$. For $k = 0, 1, ...$, let ICR_i^k be such that $(a_i, t_i) \in ICR_i^k$ if and only if $a_i \in ICR_i^k(t_i)$ and $ICR_{-i}^k = \times_{j \neq i} ICR_j^k$. Then recursively, for $k = 1, 2, ...$*

$$ICR_i^{k,\mathcal{T}}(t_i) = \Big\{a_i \in A_i : \exists \psi^{a_i} \in \Psi_i(t_i) \ \ s.t.: a_i \in BR_i(\psi^{a_i}) \ \ and \\ \text{supp}\left(\text{marg}_{A_{-i} \times T_{-i}} \psi^{a_i}\right) \subseteq ICR_{-i}^{k-1,\mathcal{T}}\Big\}$$

Then, let $ICR_i^{\infty,\mathcal{T}}(t_i) = \bigcap_{k \geq 0} ICR_i^{k,\mathcal{T}}(t_i)$.

ICR is a version of rationalizability (Pearce, 1984 and Bernheim, 1984) applied to the interim normal form, with the difference that the opponents' strategies may be correlated in the eyes of a player.[5] Dekel et al. (2007) proved that whenever two types $t_i \in T_i$

his residual uncertainty about θ and the opponents' beliefs (i.e. $\Theta_0 \times \Theta_{-i} \times E_{-i}$). Penta (2009) shows how the distinction between payoff-relevant and purely epistemic types is irrelevant for the purpose of Weinstein and Yildiz's analysis of static settings. (For the same reason, payoff-types are not considered here.) The distinction instead is relevant in dynamic settings, as it affects the possibility that players have to revise their beliefs after observing unexpected moves. If Θ is sufficiently *rich*, and no payoff-types are allowed, *sequential rationality* has no bite in dynamic settings. Not so if payoff-types are considered.

[4] Throughout the paper, the convention is maintained that "beliefs" are about Θ and the opponents' beliefs about Θ. That is, "beliefs" are about *exogenous variables* only. The term "conjectures" instead refers to beliefs that also encompass the opponents' strategies.

[5] Ely and Peski (2006) studied *Interim (Independent) Rationalizability*, that is simply Pearce's solution concept applied to the interim normal form. Battigalli et al. (2008) studied the connections between these and other versions of rationalizability for incomplete information games.

and $t_i' \in T_i'$ are such that $\pi_i^{\mathcal{T}}(t_i) = \pi_i^{\mathcal{T}'}(t_i')$, $ICR_i^{\infty,\mathcal{T}'}(t_i') = ICR_i^{\infty,\mathcal{T}}(t_i)$: That is, ICR is completely determined by a type's hierarchies of beliefs, irrespective of the type space representation. Hence, we can drop the reference to the specific type space $\mathcal{T}$, and without loss of generality work with the universal type space.

2.1 Structure of Rationalizability without *Richness*.

Let $\mathcal{A}_i^0 \subseteq A_i$ be the set of actions of player i for which there exists a dominance state $\theta^{a_i} \in \Theta$. For each $k = 0, 1, ...,$ set $\mathcal{A}_{-i}^k = \times_{j \neq i} \mathcal{A}_j^k$ and $\mathcal{A}^k = \times_{i \in N} \mathcal{A}_i^k$. Recursively, for each $k = 1, 2, ...,$ define

$$\mathcal{A}_i^k = \left\{ a_i \in A_i : \exists \beta^i \in \Delta\left(\Theta \times \mathcal{A}_{-i}^{k-1}\right) \text{ s.t. } \{a_i\} = BR_i\left(\beta^i\right) \right\}$$

and let $\mathcal{A}_i^\infty = \bigcup_{k \geq 0} \mathcal{A}_i^k$.

In words, for each $k = 1, 2, ...,$ the set $\mathcal{A}_i^k$ is set of player i's actions that are *unique* best response to conjectures concentrated on $\mathcal{A}_i^{k-1}$. Actions in $\mathcal{A}_i^0$ are those for which there exists dominance states. For k then, each action in $\mathcal{A}_i^k$ can be "traced back" to such dominance regions through a finite sequence of strict best responses.

Remark 1 *It is easy to verify that, for each $k = 1, 2, ...,$ $\mathcal{A}_i^{k-1} \subseteq \mathcal{A}_i^k$. Also, since each A_i is finite, there exists $K \in \mathbb{N}$ such that for each $i \in N$, $\mathcal{A}_i^K = \mathcal{A}_i^{K+1} = \mathcal{A}_i^\infty$.*

The next lemma shows that for each k and for each action $a_i \in \mathcal{A}_i^k$, there exists a finite type for which a_i is the only action that survives after $(k+1)$ rounds of iterated deletion of dominated actions.

Lemma 1 *For each $k = 0, 1, ...,$ for each $a_i \in \mathcal{A}_i^k$ there exists a finite type $t_i' \in \hat{T}_i$ such that $ICR_i^{k+1}(t_i') = \{a_i\}$.*

Proof. The proof is by induction:

Initial Step: this is immediate, as for $a_i \in \mathcal{A}_i^0$, there exists $\theta^{a_i} \in \Theta$ that makes a_i strictly dominant, and letting t_i' denote the type corresponding to common belief of θ^{a_i}, $ICR_i^1(t_i') = \{a_i\}$.

Inductive Step: Let $a_i \in \mathcal{A}_i^k$, then there exists $\beta^i \in \Delta\left(\Theta \times \mathcal{A}_{-i}^{k-1}\right)$ such that $\{a_i\} = BR_i\left(\beta^i\right)$. From the inductive hypothesis, there exists a function $\kappa_{-i}^{k-1} : \mathcal{A}_{-i}^{k-1} \to \hat{T}_{-i}$ such that for each $a_{-i} \in \mathcal{A}_{-i}^{k-1}$, $\{a_{-i}\} = ICR_{-i}^k\left(\kappa_{-i}^{k-1}(a_{-i})\right)$. We want to show that there exists $t_i' \in \hat{T}_i$ such that $ICR_i^{k+1}(t_i') = \{a_i\}$. Let $\mu^i \in \Delta\left(\Theta \times \mathcal{A}_{-i}^{k-1} \times \hat{T}_{-i}\right)$ be defined as

$$\mu^i\left(\theta, a_{-i}, \kappa_{-i}^{k-1}(a_{-i})\right) = \beta^i(\theta, a_i)$$

and let t_i' be defined as $\tau_i^*(t_i') = marg_{\Theta\times\hat{T}_{-i}}\mu^i$. Then, by construction:

$$\left\{\mu^i\right\} = \left\{\psi_i \in \Psi_i(t_i') : supp\left(marg_{A_{-i}\times\hat{T}_{-i}}\psi^i\right) \subseteq ICR_{-i}^k\right\}$$
$$\text{and } \{a_i\} = BR_i\left(\mu^i\right).$$

Hence: $ICR_i^{k+1}(t_i') = \{a_i\}$.■

Definition 2 *From lemma 1, let* $\kappa_i^k : \mathcal{A}_i^k \to \hat{T}_i$ *be defined as a mapping such that for each* $a_i \in \mathcal{A}_i^k$, $\{a_i\} = ICR_i^k\left(\kappa_i^k(a_i)\right)$ *and* $\kappa_i : \mathcal{A}_i^\infty \to \hat{T}_i$ *as a mapping such that for each* $a_i \in \mathcal{A}_i^\infty$, $\{a_i\} = ICR_i^\infty(\kappa_i(a_i))$. *Given* $\kappa_i : \mathcal{A}_i^\infty \to \hat{T}_i$, *define the set of types* $\bar{T}_i \subseteq \hat{T}_i$ *as follows:*

$$\bar{T}_i := \left\{t_i \in \hat{T}_i : t_i = \kappa_i(a_i) \text{ for some } a_i \in \mathcal{A}_i^\infty\right\}.$$

Remark 2 *Since* $\mathcal{A}_i^\infty$ *is finite, the set* $\bar{T}_i$ *is finite.*

As already mentioned, Weinstein and Yildiz assume *richness*, that is $\mathcal{A}_i^0 = A_i$ for each i. Hence, it is immediate to construct types with a unique rationalizable action. Given such "dominance" types, they prove their main result through an "infection argument" to obtain the generic uniqueness result. Their proof is articulated in two main steps: first, a type's beliefs are perturbed so that any rationalizable action for that type, is also "strictly rationalizable" for a nearby type (lemma 6 in Weinstein and Yildiz, 2007); then, they show that by a further perturbation, each "strictly rationalizable" action can be made uniquely rationalizable for an arbitrarily close type, perturbing higher order beliefs only (lemma 7, ibid.).

With arbitrary spaces of uncertainty (without *richness*), the argument requires two main modifications: first, the set of types $\bar{T}_i$ which will be used to start the "infection argument" had to be constructed (definition 2); then, a result analogous to Weinstein and Yildiz's lemma 6 is proved (lemma 3 below), but with a different solution concept than "strict rationalizability", which will be presented shortly (def 3). The difference with respect to "strict rationalizability", parallels the difference between Weinstein and Yildiz's "dominance types" and types $\bar{T}_i$ constructed above, so to be able to "trace back" a type's hierarchies to the dominance regions for actions in $\mathcal{A}^0$. Given these preliminary steps, the further perturbations of higher order beliefs needed to obtain the result is completely analogous to Weinstein and Yildiz's: lemma 4 below entails minor modifications of Weinstein and Yildiz's equivalent (lemma 7).

The proof of the main result is based on the following solution concept:

Definition 3 *For each* $i \in N$ *and* $t_i \in T_i^*$, *set* $\mathcal{W}_i^0(t_i) = \mathcal{A}_i^0$. *For* $k = 0, 1, ...,$ *let* $\mathcal{W}_i^k$ *be such that* $(a_i, t_i) \in \mathcal{W}_i^k$ *if and only if* $a_i \in \mathcal{W}_i^k(t_i)$ *and* $\mathcal{W}_{-i}^k = \times_{j \neq i} \mathcal{W}_j^k$. *For* $k = 1, 2, ...$*define recursively*

$$\mathcal{W}_i^k(t_i) = \Big\{a_i \in \mathcal{A}_i^k : \exists \psi^i \in \Delta\left(\Theta \times \mathcal{W}_{-i}^{k-1}\right) \; s.t. \; \mathrm{marg}_{\Theta \times T_{-i}} \psi^i = \tau_i(t_i) \\ \text{and } \{a_i\} = BR_i\left(\psi^i\right)\Big\}$$

Let $K \in \mathbb{N}$ *be such that for each* $k \geq K$, $\mathcal{W}_i^{k+1}(t_i) \subseteq \mathcal{W}_i^k(t_i)$ *for all* t_i *and* i *(such* K *exists because of remark 1 above). Finally, define* $\mathcal{W}_i^\infty(t_i) := \bigcap_{k \geq K} \mathcal{W}_i^k(t_i)$.

Notice that for $k < K$, $\mathcal{W}_i^k(t_i)$ may be non-monotonic in k, as for $k < K$ the sets $\mathcal{A}_i^k$ are increasing. Hence, up to K, $\mathcal{W}_i^k(t_i)$ may increase. When $k \geq K$ though, $\mathcal{A}_i^k = \mathcal{A}_i^\infty$ is constant, and the condition "$\exists \psi^i \in \Delta\left(\Theta \times \mathcal{W}_{-i}^{k-1}\right)$" becomes (weakly) more and more stringent, making the sequence $\left\{\mathcal{W}_i^k(t_i)\right\}_{k>K}$ monotonically (weakly) decreasing. Being always non-empty, $\mathcal{W}_i^\infty(t_i) := \bigcap_{k>K} \mathcal{W}_i^k(t_i)$ is also non-empty and well-defined (as long as $\mathcal{A}_i^0 \neq \emptyset$).

The following lemma states a standard fixed point property, and it is an immediate implication of lemma 5 in Weinstein and Yildiz (2007).[6]

Lemma 2 *For any family of sets* $\{V_i(t_i)\}_{t_i \in T_i, i \in N}$ *such that* $V_i(t_i) \subseteq \mathcal{A}_i^\infty$ *for all* $i \in N$ *and* $t_i \in T_i$. *If for each* $a_i \in V_i(t_i)$, *there exists* $\psi^i \in \Delta(\Theta \times A_{-i} \times T_{-i})$ *such that* $\{a_i\} = BR_i\left(\psi^i\right)$, $\mathrm{marg}_{\Theta \times T_{-i}} \psi^i = \tau_i(t_i)$ *and* $\psi^i(\theta, a_{-i}, t_{-i}) > 0 \Rightarrow a_{-i} \in V_{-i}(t_{-i})$, *then* $V_i(t_i) \subseteq \mathcal{W}_i^\infty(t_i)$ *for each* t_i.

We turn next to the analysis of higher order beliefs: the next lemma shows how for each t_i and each action $a_i \in ICR_i^\infty(t_i) \cap \mathcal{A}_i^\infty$ that is justified by conjectures concentrated on $\mathcal{A}_{-i}^\infty$, we can construct a sequence of types converging to t_i for which a_i survives to the iterated deletion procedure introduced in definition 3.

Lemma 3 *Let* $a_i \in ICR_i^\infty(t_i) \cap \mathcal{A}_i^\infty$ *be such that there exists a justifying conjecture* $\psi^{a_i} \in \Psi_i(t_i)$ *such that* $\mathrm{supp}\left(\mathrm{marg}_{A_{-i}} \psi^{a_i}\right) \subseteq \mathcal{A}_{-i}^\infty$. *Then there exists* $t_i(\varepsilon) \to t_i$ *as* $\varepsilon \to 0$ *such that for each* $\varepsilon > 0$, $a_i \in \mathcal{W}_i^\infty(t_i(\varepsilon))$ *and* $t_i(\varepsilon) \in \hat{T}_i$ *(hence* $a_i \in \mathcal{W}_i^k(t_i(\varepsilon))$ *for all* $k \geq K$*)*

Proof. Since $a_i \in ICR_i^\infty(t_i) \cap \mathcal{A}_i^\infty$, $\exists \psi^{a_i} \in \Delta\left(\Theta \times ICR_{-i}^\infty\right)$ such that $a_i \in BR_i\left(\psi^i\right)$ and $marg_{\Theta \times T_{-i}} \psi^i = \tau_i(t_i)$ and there exists $\beta^i \in \Delta\left(\Theta \times \mathcal{A}_{-i}^\infty\right)$ such that $\{a_i\} = BR_i\left(\beta^i\right)$.

[6] This is because $\mathcal{W}^\infty$ coincides with Weinstein and Yildiz's W^∞ applied to the game with actions $\mathcal{A}^\infty$.

Let κ_{-i} be as in definition 2, and $v_i^{(t_i,a_i)} \in \Delta\left(\Theta \times \bar{T}_{-i}\right)$ such that for each $(\theta, a_{-i}) \in \Theta \times \mathcal{A}_{-i}^{\infty}$, $v_i^{(t_i,a_i)}(\theta, \kappa_{-i}(a_{-i})) = \beta^i(\theta, a_{-i})$. For each $\varepsilon \in [0,1]$, consider the set of types $T_i^{\varepsilon} \subseteq T_i^*$ such that each $T_i^{\varepsilon} = \bar{T}_i \cup T_i^{\iota}$. That is, T_i^{ε} consists of two kinds of types: types $\bar{t}_i^{a_i} \in \bar{T}_i$ (see definition 2), which have a unique rationalizable action, and for each t_i, a_i and ε types $\bar{\iota}_i(t_i, a_i, \varepsilon) \in T_i^{\iota}$ with hierarchies of beliefs are implicitly defined as follows:

$$\tau_i^*(\bar{\iota}_i(t_i, s_i, \varepsilon)) = \varepsilon \cdot v_i^{(t_i,a_i)} + (1-\varepsilon)\left[\psi^{a_i} \circ \hat{\iota}_{-i,\varepsilon}^{-1}\right],$$

where $\hat{\iota}_{-i,\varepsilon} : \Theta \times \mathcal{A}_{-i}^{\infty} \times T_{-i} \to \Theta \times T_{-i}^{\varepsilon}$ is the mapping given by $\hat{\iota}_{-i,\varepsilon}(\theta, a_{-i}, t_{-i}) = (\theta, \bar{\iota}_{-i}(t_{-i}, a_{-i}, \varepsilon))$, and $\left[\psi^{a_i} \circ \hat{\iota}_{-i,\varepsilon}^{-1}\right]$ denotes pushforward of ψ^{a_i} given by $\hat{\iota}_{-i,\varepsilon}^{-1}$.

Define $\gamma : \Theta \times T_{-i}^{\varepsilon} \to \Theta \times \mathcal{A}_{-i}^{\infty} \times T_{-i}^{\varepsilon}$ such that:

$$\begin{aligned} &\forall \bar{\iota}_{-i}(t_{-i}, a_{-i}, \varepsilon) \in T_{-i}^{\iota} : \\ &\gamma(\theta, \bar{\iota}_{-i}(t_{-i}, a_{-i}, \varepsilon)) = (\theta, a_{-i}, \bar{\iota}_{-i}(t_{-i}, a_{-i}, \varepsilon)) \\ &\text{and for every } \bar{t}_{-i}^{a_{-i}} \in \bar{T}_{-i} \subseteq T_{-i}^{\varepsilon}, \\ &\gamma\left(\theta, \bar{t}_{-i}^{a_{-i}}\right) = \left(\theta, a_{-i}, \bar{t}_{-i}^{a_{-i}},\right). \end{aligned}$$

Consider the conjectures $\psi^i \in \Delta\left(\Theta \times \mathcal{A}_{-i}^{\infty} \times T_{-i}^{\varepsilon}\right)$ defined as $\psi^i = \left(\tau_{\bar{\iota}_i(t_i,a_i,\varepsilon)}^{\varepsilon} \circ \gamma^{-1}\right)$. By construction, they are consistent with type $\tau_{\bar{\iota}_i(t_i,a_i,\varepsilon)}^{\varepsilon}$. Being a mixture of the beliefs ψ^{a_i} (which made a_i best reply) and of β^i (which makes a_i strict best reply), we have that $\{a_i\} = BR_i\left(\psi^i\right)$. Hence, setting $V_i(\bar{\iota}_i(t_i, a_i, \varepsilon)) = \{a_i\}$ and $V_i(\bar{t}_i^{a_i}) = \{a_i\}$ as in lemma 2, we have that $\{a_i\} \in \mathcal{W}_i^{\infty}(t_i)$ for all $t_i \in T_i^{\varepsilon}$. Finally, $\bar{\iota}_i(t_i, a_i, \varepsilon) \to t_i$ as $\varepsilon \to 0$.■

The next lemma shows that for any type t_i and for any $a_i \in \mathcal{W}_i^k(t_i)$, $k = 0, 1, ...$, there exists a type that differs from t_i only for beliefs of order higher than k, for which a_i is the unique action which survives $(k+1)$ rounds of the ICR-procedure.

For any type $t_i \in T_i^*$, let t_i^m denote the *m-th order beliefs* of type t_i. (By definition of T_i^*, any $t_i \in T_i^*$ can be written as $t_i = (t_i^m)_{m=1}^{\infty}$.)

Lemma 4 *For each $k = 0, 1, ...$, and for each $a_i \in \mathcal{W}_i^k(t_i)$, there exists $\tilde{t}_i : \tilde{t}_i^m = t_i^m$ for all $m \leq k$ and such that $\{a_i\} = ICR_i^{k+1}\left(\tilde{t}_i\right)$*

Proof. The proof is by induction. For $k = 0$, $a_i \in \mathcal{W}_i^0(t_i) = \mathcal{A}_i^0$, so there exists a dominance state for action a_i, θ^{a_i}. Let $\tilde{t}_i$ denote common belief of θ^{a_i}, so that $\{a_i\} = ICR_i^1\left(\tilde{t}_i\right)$ (condition $\tilde{t}_i^0 = t_i^0$ holds vacuously). For the inductive step, write each t_{-i} as $t_{-i} = (l, h)$ where

$$\begin{aligned} l &= \left(t_{-i}^1, ..., t_{-i}^k\right) \text{ and} \\ h &= \left(t_{-i}^{k+1}, t_{-i}^{k+2}, ...\right). \\ \text{Let } L &= \left\{l : \exists h \text{ s.t. } (l, h) \in T_{-i}^*\right\}. \end{aligned}$$

Let $a_i \in \mathcal{W}_i^k(t_i)$, and $\psi^{a_i} \in \Delta\left(\Theta \times \mathcal{W}_{-i}^{k-1}\right)$ the corresponding conjecture s.t. $marg_{\Theta\times T_{-i}}\psi^{a_i} = \tau_i(t_i)$ and $\{a_i\} = BR_i(\psi^{a_i})$. Under the inductive hypothesis, for each $(a_{-i}, t_{-i}) \in supp\big(marg_{A_{-i}\times T_{-i}}\psi^{a_i}\big)$, $\exists \tilde{t}_{-i}(a_{-i}) = \left(l, \tilde{h}(a_{-i})\right)$ s.t. $ICR_{-i}^k\left(\tilde{t}_{-i}(a_{-i})\right) = \{a_{-i}\}$. Define the mapping

$$\varphi : supp\left(marg_{\Theta\times A_{-i}\times L}\psi^{a_i}\right) \to \Theta \times T_{-i}^*$$

by $\varphi(\theta, a_{-i}, l) = \left(\theta, \tilde{t}_{-i}(a_{-i})\right)$. Define $\tilde{t}_i$ by

$$\tau_i^*\left(\tilde{t}_i\right) = \left(marg_{\Theta\times A_{-i}\times L}\psi^{a_i}\right) \circ \varphi^{-1}$$

By construction,

$$\begin{aligned} marg_{\Theta\times A_{-i}\times L}\tau_i^*\left(\tilde{t}_i\right) &= \psi^{a_i} \circ proj^{-1}_{\Theta\times A_{-i}\times L} \circ \varphi^{-1} \circ proj^{-1}_{\Theta\times L} \\ &= \psi^{a_i} \circ proj^{-1}_{\Theta\times L} \\ &= \psi^{a_i} \circ proj^{-1}_{\Theta^*\times A_{-i}\times T_{-i}^*} \circ proj^{-1}_{\Theta\times L} \\ &= marg_{\Theta\times A_{-i}\times L}\tau_i(t_i) \end{aligned}$$

where the first equality exploits the definition of lower order beliefs and the construction of type $\tilde{t}_i$, the second follows from the definition of φ, for which

$$proj^{-1}_{\Theta\times L\times A_{-i}} \circ \varphi^{-1} \circ proj^{-1}_{\Theta\times L} = proj^{-1}_{\Theta\times L}$$

The third is simply notational, and the last one by definition. Hence, by construction, we have $ICR_i^{k+1}\left(\tilde{t}_i\right) = \{a_i\}$, which completes the inductive step. ■

We are now in the position to present the main result:

Proposition 1 *For each $t_i \in \hat{T}_i$ and for each $a_i \in ICR_i^\infty(t_i) \cap \mathcal{A}_i^\infty$ such that* $\mathrm{supp}\big(\mathrm{marg}_{A_{-i}}\psi^{a_i}\big) \subseteq \mathcal{A}_{-i}^\infty$*, there exists a sequence $\{t_i^\nu\} \subseteq \hat{T}_i$ s.t. $t_i^\nu \to t_i$ and for each $\nu \in \mathbb{N}$, $\{a_i\} = ICR_i^\infty(t_i^\nu)$.*

Proof: Take any $t_i \in \hat{T}$ and any $a_i \in ICR_i^\infty(t_i) \cap \mathcal{A}_i^\infty$ such that $supp\big(marg_{A_{-i}}\psi^{a_i}\big) \subseteq \mathcal{A}_{-i}^\infty$: from lemma 3, there exists a sequence of finite types $t_i(\varepsilon) \to t_i$ (as $\varepsilon \to 0$) such that $a_i \in \mathcal{W}_i^\infty(t_i(\varepsilon))$ for each $\varepsilon > 0$, hence, there exists a sequence $\{t_i(n)\}_{n\in\mathbb{N}}$ converging to t_i such that $a_i \in \mathcal{W}_i^k(t_i(n))$ for all $k \geq K$. Then we can apply lemma 4 to the types $t(n)$: for each n, for each $k \geq K$ and for each $a_i \in \mathcal{W}^k(t_i(n))$, there exists $\tilde{t}_i(k,n)$ such that $\tilde{t}_i^k(k,n) = t_i^k(n)$ and $\{a_i\} = ICR_i^{k+1}\left(\tilde{t}_i(k,n)\right)$. Hence, for each n, the sequence $\left\{\tilde{t}_i(k,n)\right\}_{k\in\mathbb{N}}$ converges to $t_i(n)$ as $k \to \infty$. Because the universal type-space T^* is metrizable, there exists a sequence $t_i(n,k_n) \to t_i$ such that $ICR_i^\infty(t_i(n,k_n)) = \{a_i\}$. Set $t_i^\nu = t_i(n,k_n)$: $t_i^\nu \to t_i$ as $\nu \to \infty$ and $ICR_i^\infty(t_i^\nu) = \{a_i\}$ for each ν.■

3 Discussion

If it is common knowledge that no action is dominant ($\mathcal{A}_i^0 = \emptyset$), proposition 1 is vacuous. Weinstein and Yildiz's *richness condition* amounts to assuming that Θ is such that $\mathcal{A}_i^0 = A_i$ for each i: In this case, proposition 1 coincides with proposition 1 in Weinstein and Yildiz (2007).

Of more interest is the observation that all results in Weinstein and Yildiz (2007) (including the generic uniqueness result) hold true, *without richness,* whenever $\mathcal{A}^\infty = A$.

Moreover, suppose that there exists a payoff state $\theta^* \in \Theta$ for which payoff functions are supermodular, with player i's higher and lower actions a_i^h and a_i^l respectively; and for each i, $\mathcal{A}_i^0 = \{a_i^l, a_i^h\}$. Then under these conditions $\mathcal{A}^\infty = A$, and Weinstein and Yildiz's full results are again obtained. This corresponds to the case considered by the global games literature, in which the underlying game has strategic complementarities and dominance regions are assumed for the extreme actions only. The difference is that in that literature supermodularity is assumed at all states (so that it is commonly known).[7] In contrast, here it may be assumed for only one state, which only entails relaxing common knowledge that payoffs are *not* supermodular. This observation clarifies that, on the one hand, the equilibrium selection results obtained in the global games literature, which contrast with the non-robustness result (R.2), are exclusively determined by the particular class of perturbations that are considered. On the other hand, the generic uniqueness result can be obtained without assuming common knowledge of supermodularity or imposing richness: as argued, relaxing common knowledge that payoffs are *not* supermodular and that the corresponding extreme actions are *not* dominant would suffice to obtain the full results of Weinstein and Yildiz.

With minor changes, the proof above can be used to obtain a slightly stronger result (although not directly in terms of the primitives) just setting $\mathcal{A}_i^0$ as the set of actions of player i that are *uniquely ICR* for *some* type (clearly, this would always include the set of actions that are dominant in some state): Then, $\mathcal{A}_i^\infty$ characterizes the set of actions that are uniquely rationalizable (hence robust) for some type.[8]

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[7] See Morris and Shin (2003) and references therein.

[8] Since the *ICR*-correspondence is upper hemicontinuous in T^*, the uniqueness regions are open and locally constant: the corresponding predictions are thus robust.

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