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"Risk Aversion and Optimal Reserve Prices in First and Second-Price Auctions"

by

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Risk Aversion and Optimal Reserve Prices in First and Second-Price Auctions[1]

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Risk Aversion and Optimal Reserve Prices in First and Second-Price Auctions

Abstract

This paper analyzes the effects of buyer and seller risk aversion in first and second-price auctions. The setting is the classic one of symmetric and independent private values, with ex ante homogeneous bidders. However, the seller is able to optimally set the reserve price. In both auctions the seller's optimal reserve price is shown to decrease in his own risk aversion, and more so in the first-price auction. Thus, greater seller risk aversion increases the ex post efficiency of both auctions, and especially that of the first-price auction. The seller's optimal reserve price in the first-price, but not in the second-price, auction decreases in the buyers' risk aversion. Thus, greater buyer risk aversion also increases the ex post efficiency of the first but not the second-price auction. At the interim stage, the first-price auction is preferred by all buyer types in a lower interval, as well as by the seller.

Keywords: first-price auction, second-price auction, risk aversion, reserve price

JEL classification: D44

1 Introduction

Much of the literature that compares the effects of risk aversion across auctions assumes each auction has the same, exogenously determined reserve price. The predominant example is the comparison of a first-price auction (FPA) to a second-price auction (SPA) with the same reserve price, in a symmetric independent private values setting.[1] The well-known result in this case is that risk averse bidders bid more in the FPA than they do in the SPA.[2]

However, the reserve price in most real auctions is set by the seller. To the extent that it influences bidding behavior and depends on the type of auction, the endogeneity of the reserve price should be taken into account. In particular, the comparative statics of the optimal reserve price are of direct interest because they bear on ex post efficiency. Lowering the reserve price decreases the probability of the inefficient event in which no sale occurs because the maximum value of the bidders exceeds the seller's value but not the reserve price.

This paper focuses on the effects of buyer and seller risk aversion on the seller's optimal reserve price in standard first and second-price auctions. Sharp results are obtained by restricting attention to the otherwise simplest setting, that of symmetric and independent private values. Our three main results are Theorems $1-3$.

Theorem 1 establishes that if the seller and/or the buyers are risk averse, then the seller sets a lower reserve price in the FPA than in the SPA. This is in contrast to when all parties are risk neutral, in which case the revenue equivalence theorem implies that the seller's optimal reserve price is the same in both auctions.

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[1] We use the term FPA for both the first-price sealed-bid auction and the strategically equivalent Dutch (descending) auction. We use the term SPA for both the second-price sealed-bid (Vickrey) auction and the "button" model of the English ascending-bid auction, as they have the same dominant strategy equilibria in our private values setting. (Milgrom and Weber, 1982).

[2] This and related results are established, e.g., by Holt (1980), Riley and Samuelson (1981), Harris and Raviv (1981), Milgrom and Weber (1982), Matthews (1983, 1987), Maskin and Riley (1984), Cox et al. (1982, 1988), Smith and Levin (1996), and Eso and White (2004).

Risk aversion thus makes the FPA more ex post efficient than the SPA. The result hinges on how the FPA equilibrium bid function is affected by a marginal increase in the reserve price. Risk averse bidders increase their bids less than do risk neutral bidders, and a risk averse seller values the increase in the bids of the high bidders relatively less than does a risk neutral seller because of diminishing marginal utility. Both forces lower the seller's marginal incentive to raise the reserve price.

Theorem 2 establishes that in either auction, a more risk averse seller sets a lower reserve price. Thus, the more risk averse the seller, the more ex post efficient are both auctions. The intuition is straightforward: a more risk averse seller values more (on the margin) a decrease in the risk of not selling the object. The proof, however, is surprisingly intricate.[3]

Theorem 3 establishes that in the FPA, the seller sets a lower reserve price if the bidders are more risk averse. (Bidder risk aversion does not affect the SPA equilibrium.) This theorem requires more assumptions: either (a) the reverse hazard rate function of the buyers' values is decreasing, and the buyers and/or the seller exhibit nonincreasing absolute risk aversion, or (b) the buyers exhibit constant absolute risk aversion. Under these conditions the rate at which the FPA bid function increases in the reserve price is smaller when the bidders are more risk averse, and so the seller has less incentive to raise the reserve price. This effect is stronger if the seller is also risk averse, as then the fact that more risk averse bidders bid higher than less risk averse bidders implies that the seller has a lower marginal utility valuation for the increase in their bids caused by an increase in the reserve price.

The remainder of the paper begins with the model in Section 2. Useful technical results are in Section 3. The FPA equilibrium is studied in Section 4. The seller's preferences over auctions with the same reserve price are determined in Section 5, and his optimal reserve prices are examined in Section 6. Section 7 concludes.

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[3] Theorem 3 in Waehrer et al. (1998) is our Theorem 2 for the case of risk neutral bidders (and a more general information structure). Their proof relies on a point-by-point renormalization of the utility functions that we do not understand. Our proof takes a different approach.

2 Model

An indivisible object is to be possibly sold to one of $n \geq 2$ potential buyers through either a FPA or a SPA with a reserve price. Each buyer $i \in N = \{1, ..., n\}$ has a private value for the object, v_i, which is unknown to the others. Ex ante, these values are independently distributed on an interval $[L, H]$ according to the same distribution function F, which has a density function $f = F'$ that is strictly positive on $[L, H]$ and continuously differentiable on (L, H). Some of our results are obtained under the assumption of a decreasing reverse hazard rate function:

$$\text{(DRH)} \quad \frac{f(v)}{F(v)} \text{ strictly decreases on } (L, H).$$

Each participant maximizes expected utility. Each buyer has the same utility function, $u_B : \mathbb{R} \to \mathbb{R}$. If a buyer with value v wins and pays a price b, his utility is $u_B(v - b)$; his utility is $u_B(0)$ if he loses.[4] We assume u_B is twice continuously differentiable, with $u_B' > 0$ and $u_B'' \leq 0$, and normalized so that $u_B(0) = 0$.

The seller has a value $v_0 \in [L, H)$ for the object, and a twice continuously differentiable utility function, $u_S : \mathbb{R} \to \mathbb{R}$, satisfying $u_S' > 0$ and $u_S'' \leq 0$. The seller's utility is $u_S(b)$ if a sale occurs at price b, and it is $u_S(v_0)$ otherwise.

We consider first and second-price auctions with a reserve price $r \in (L, H)$.[5] In either auction a buyer with a value $v < r$ abstains from bidding. In a SPA, the dominant strategy of a buyer with $v \geq r$ is to submit a bid equal to v. We restrict attention to this equilibrium of the SPA.

Turning to the FPA, it is useful to define $G \equiv F^{n-1}$. If a buyer has value v, then $G(v)$ is the probability that every other buyer has a lower value. Let $g \equiv G'$ be the associated density, and let $\ell(v) = g(v)/G(v)$. Lastly, for $i = B, S$, define

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[4] A more general formulation would have $u_B = u(v, -b)$ as the winning bidder's payoff. Under appropriate assumptions, as in Maskin and Riley (1984) or Matthews (1987), our main results would extend to this generalization.

[5] This is without loss of generality. As we shall see, $v_0 \in [L, H)$ implies that the seller's optimal reserve price is in (L, H).

the function $\gamma = u_B/u_B'$. Then $b(v,r)$, the unique symmetric equilibrium bidding function of the FPA, is defined for $v \geq r$ as the solution to the differential equation,

$$b_1(v,r) = \frac{g(v)u_B(v-b)}{G(v)u_B'(v-b)} = \ell(v)\gamma(v-b), \tag{1}$$

that satisfies the initial condition $b(r,r) = r$ (e.g., Maskin and Riley, 1984). We restrict attention to this equilibrium of the FPA.[6] Observe that $b_1(r,r) = 0$, and $b_2(r,r) = 1 - b_1(r,r) = 1$.

If u_B satisfies CARA, so that $R_B \equiv a$ for some $a \geq 0$, we let b^a denote the FPA equilibrium bid function. In this case the differential equation (1) can be solved to yield

$$b^a(v,r) = \frac{1}{a}\ln\left(e^{av} - a\int_r^v \left(\frac{G(y)}{G(v)}\right)e^{ay}dy\right) \quad \text{for } v \geq r. \tag{2}$$

From (2) we see that for CARA buyers, the equilibrium bid function strictly increases in the reserve price:

$$b_2^a(v,r) = \frac{G(r)}{G(v)}e^{-a(b-r)} > 0, \tag{3}$$

and that increasing the risk aversion of the bidders lowers the rate at which the bid function increases in the reserve price: for all $v > r$, $\partial b_2^a/\partial a = (r-b)b_2^a < 0$.

3 Technical Preliminaries

For $i = B, S$, we let $R_i = -u_i''/u_i'$ denote the Arrow-Pratt measure of absolute risk aversion. The function $\gamma = u_B/u_B'$ is related to R_B by the equation $\gamma' = 1 + R_B\gamma$. Because of the normalization $u_B(0) = 0$, for $t \geq 0$ we have $\gamma(t) \geq 0$, and hence $\gamma'(t) > 0$. It is well known that if $\hat{u}_B$ is another utility function such that $\hat{u}_B(0) = 0$ and $\widehat{R}_B > R_B$, then $\hat{\gamma}(t) > \gamma(t)$ for $t > 0$ (Pratt, 1964, Theorem 1).

We make repeated use of the following two lemmas, which are proved in the Appendix. The first is a variation of the "Ranking Lemma" of Milgrom (2004).

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[6] It is the unique equilibrium if R_B is nonincreasing (Maskin and Riley, 2003).

Lemma 1 *For* $c < d \leq \infty$ *and* $h : [c, d] \to \mathbb{R}$ *differentiable, if* $h(c) \geq 0$ *then*

(i) $h \geq 0$ *on* $(c, d]$ *if* $[\forall t \geq c,\ h(t) \leq 0 \Rightarrow h'(t) \geq 0]$,

(ii) $h > 0$ *on* $(c, d]$ *if* $[\forall t \geq c,\ h(t) = 0 \Rightarrow h'(t) > 0]$,

(iii) $h > 0$ *on* $(c, d]$ *if* $[\forall t > c,\ h(t) \leq 0 \Rightarrow h'(t) > 0]$.

Lemma 2 *For* $c < d \leq \infty$ *and* $i = 1, 2$, *let the functions* $h_i : [c, d] \to \mathbb{R}$ *be differentiable and satisfy* $h_1' < h_2'$ *on* (c, d). *Let* t_i *maximize* h_i *on* $[c, d]$. *Then, if* $t_i \in (c, d)$ *for* $i = 1$ *or* $i = 2$, *we have* $t_1 < t_2$.

4 Properties of the FPA Equilibrium

The following are well known and easily proved properties of the FPA equilibrium: $b(v, r) < v$, $b_1(v, r) > 0$, and $b_2(v, r) \geq 0$ for any $L < r < v$.[7] As we are concerned with the seller's incentives for setting the reserve price, we need to further determine how the equilibrium varies with the reserve price. Our first proposition establishes an upper bound on $b_2(v, r)$. This derivative is equal to $G(r)/G(v)$ if the bidders are risk neutral, as can be seen by setting $a = 0$ in (3). This is in fact the upper bound.

Proposition 1 *For* $v > r$, $b_2(v, r) \leq G(r)/G(v)$. *This inequality is strict if* $R_B > 0$.[8]

Proof. For a fixed $r \in (L, H)$, let $h(v) = G(r) - b_2(v, r)G(v)$. Then $h'(v) = -g(v)b_2 - G(v)b_{12}$. Differentiating (1) with respect to r yields $b_{12} = -(1 + R_B\gamma)\ell b_2$, where R_B and γ are evaluated at $v - b(v, r) > 0$. This and $\ell = g(v)/G(v)$ yield

$$h'(v) = R_B \gamma g(v) b_2 \geq 0. \tag{4}$$

Hence, since $h(r) = 0$, we have $h(v) \geq 0$, and so $b_2 \leq G(r)/G(v)$, for $v > r$. Now assume $R_B > 0$. We apply Lemma 2(iii) to h on $[r, H]$. Suppose $h(v) \leq 0$ for some

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[7] Applying parts (ii) and (i) of Lemma 1, respectively, to $b_1(\cdot, r)$ and $b_2(\cdot, r)$ yields $b_1 > 0$ and $b_2 \geq 0$ for $v > r$, using (1) to obtain b_{11} and b_{12}. Then $b < v$ for $v > r$ follows from (1) and $b_1 > 0$.

[8] Here and below, $R_B > 0$ is a functional inequality, meaning that $R_B(y) > 0$ for all y in the relevant interval, which is $[0, H - L]$.

$v > r$. At this (v, r), we have $b_2 \geq G(r)/G(v) > 0$. It follows that $h'(v) > 0$, since each term in (4) (evaluated at this v) is positive. Now Lemma 2(iii) implies that in fact, $h > 0$, and so $b_2 < G(r)/G(v)$, for $v > r$. ■

Our second proposition concerns the implications of a decreasing reverse hazard rate. Given (DRH), part (i) of Proposition 2 shows that the bid function increases in the reserve price (as we have already seen is true if u_B is CARA). Part (ii) establishes the intuitive property that a bidder's profit conditional on winning, $v - b(v, r)$, increases in v. The proof of Proposition 2 is at the end of this section.

Proposition 2 *If* (DRH) *holds, then the* FPA *equilibrium satisfies, for* $v > r$,

(i) $b_2(v, r) > 0$, *and*

(ii) $b_1(v, r) < 1$.

Our third proposition determines the effects of the bidders becoming more risk averse. Part (i) shows that the bid function increases in their risk aversion, generalizing the well-known result that bids are higher when the bidders are risk averse than when they are risk neutral. The remainder of the proposition establishes more surprising results, assuming that (DRH) holds, and the seller and/or the buyers exhibit nonincreasing absolute risk aversion. Parts (ii) and (iii), respectively, show that then, the more risk averse are the bidders, the more rapidly the bid function increases in a bidder's value, but the more slowly it increases in the reserve price. The latter property is largely why the seller's optimal reserve price decreases in the risk aversion of the bidders, as we shall see.

Proposition 3 *Let* $\hat{u}_B$ *be another function satisfying the same assumptions as* u_B, *with an absolute risk aversion measure satisfying* $\widehat{R}_B > R_B$ *on* $[0, H - L]$. *Then*

(i) $\hat{b}(v, r) > b(v, r)$ *for* $v > r$.

If (DRH) *holds, and* R_B *and/or* $\widehat{R}_B$ *is nonincreasing, then*

(ii) $\hat{b}_1(v, r) > b_1(v, r)$ *for* $v > r$, *and*

(iii) $\hat{b}_2(v, r) < b_2(v, r)$ *for* $v > r$.

Proofs of Propositions 2 and 3. It is efficient to prove the parts in a certain order. Let P2*i* refer to Proposition 2(*i*), etc.

P2*ii*. We apply Lemma 1(*ii*) to $1 - b_1(\cdot, r)$. Recall $1 - b_1(r, r) > 0$. Suppose $1 - b_1 = 0$ at some $v \geq r$. Then $v > r$. Differentiating (1) with respect to v, and evaluating the result at this (v, r), yields

$$b_{11} = \ell'\gamma + \ell\gamma'(1 - b_1) = \ell'\gamma.$$

We have $\gamma(v - b) > 0$ because $b < v$, and $\ell'(v) < 0$ by (DRH). Hence, at this (v, r), $\partial \left[1 - b_1\right] / \partial v = -b_{11} > 0$. Lemma 1(*ii*) now implies that for any $v \geq r$, $1 - b_1 > 0$.

P3*i*. We apply Lemma 1(*iii*) to $\hat{b}(\cdot, r) - b(\cdot, r)$. We have $\hat{b}(r, r) = b(r, r)$. Suppose $\hat{b} \leq b$ for some $v > r$. Then $\hat{\gamma}(v - \hat{b}) \geq \hat{\gamma}(v - b)$, since $\hat{\gamma}$ is increasing on $\mathbb{R}_+$. Since $\widehat{R}_B > R_B$ on $[0, H - L]$, we have $\hat{\gamma}(v - b) > \gamma(v - b)$. Hence, $\hat{\gamma}(v - \hat{b}) > \gamma(v - b)$. This and (1) yields

$$\hat{b}_1 - b_1 = \left[\hat{\gamma}(v - \hat{b}) - \gamma(v - b)\right] \ell(v) > 0.$$

Lemma 1(*iii*) now implies $\hat{b} > b$ for all $v > r$.

P3*ii*. We apply Lemma 1(*ii*) to $\hat{b}_1(\cdot, r) - b_1(\cdot, r)$ on intervals of the form $[\xi_k, H]$, where $\xi_k \downarrow r$ as $k \to \infty$. We will show that $\hat{b}_1(\cdot, r) > b_1(\cdot, r)$ on each interval, and hence on $(r, H]$. To obtain ξ_k, let $\{v_k\}$ be a sequence such that $v_k \downarrow r$. Since $\hat{b}(r, r) = b(r, r)$ and $\hat{b}(v_k, r) > b(v_k, r)$, the mean value theorem implies $\xi_k \in (r, v_k)$ exists such that $\hat{b}_1(\xi_k, r) > b_1(\xi_k, r)$. Note that $\xi_k \downarrow r$. Now, suppose $\hat{b}_1(v, r) = b_1(v, r)$ for some $v \geq \xi_k$. Since R_B or $\widehat{R}_B$ is nonincreasing and $\hat{b} > b$ at (v, r), we have

$$\widehat{R}_B(v - \hat{b}) > R_B(v - b). \tag{5}$$

Because $\hat{b}_1 = b_1$ at (v, r), from (1) we obtain $\hat{\gamma}(v - \hat{b}) = \gamma(v - b)$. Hence, using (1) to differentiate $\hat{b}_1$ and b_1 yields

$$\begin{aligned} \hat{b}_{11} - b_{11} &= \left[\ell'\hat{\gamma} + \ell\left(1 + \widehat{R}_B\hat{\gamma}\right)(1 - \hat{b}_1)\right] \\ &\qquad - \left[\ell'\gamma + \ell\left(1 + R_B\gamma\right)(1 - b_1)\right] \\ &= \left[\widehat{R}_B(v - \hat{b}) - R_B(v - b)\right] \hat{b}_1(1 - \hat{b}_1) > 0, \end{aligned}$$

where the inequality follows from (5), $\hat{b}_1 > 0$, and $\hat{b}_1 < 1$ (by Proposition 2(*ii*), since we have (DRH) here). Lemma 1(*ii*) now implies $\hat{b}_1(\cdot, r) > b_1(\cdot, r)$ on each $(\xi_k, H]$.

P2i. The continuity of R_B implies the existence of $a < \infty$ such that $a > R_B$ on $[0, H - L]$. Let $\hat{b}$ be the equilibrium for a CARA $\hat{u}_B$ with $\widehat{R}_B \equiv a$. Then $\hat{b}$ is given by (2) and satisfies $\hat{b}_2 > 0$ for any $v \geq r$. We apply Lemma 1(*iii*) to $b_2(\cdot, r) - \hat{b}_2(\cdot, r)$. We have $b_2(r, r) = \hat{b}_2(r, r)$. Suppose $b_2 \leq \hat{b}_2$ for some $v > r$. As (1) holds for both b_1 and $\hat{b}_1$, differentiating $b_1 - \hat{b}_1$ with respect to r yields

$$\begin{aligned} b_{12} - \hat{b}_{12} &= -(1 + R_B\gamma)\ell b_2 + (1 + \widehat{R}_B\hat{\gamma})\ell\hat{b}_2 \\ &= -(\ell + R_B b_1)b_2 + (\ell + \widehat{R}_B\hat{b}_1)\hat{b}_2 \\ &= \left(\hat{b}_2 - b_2\right)\ell + \widehat{R}_B\hat{b}_1\hat{b}_2 - R_B b_1 b_2. \end{aligned}$$

Thus, because $b_2 \leq \hat{b}_2$ and $\ell > 0$,

$$b_{12} - \hat{b}_{12} \geq \left(\widehat{R}_B\hat{b}_1 - R_B b_1\right)\hat{b}_2. \tag{6}$$

Since (DRH) holds and $\widehat{R}_B$ is a constant function, Proposition 3(*ii*) implies $\hat{b}_1 > b_1$. Our choice of $\widehat{R}_B = a$ implies $\widehat{R}_B > R_B$. Thus, since $\hat{b}_2 > 0$, from (6) we obtain $b_{12} - \hat{b}_{12} > 0$. Lemma 1(*iii*) now implies $b_2 > \hat{b}_2$, and hence $b_2 > 0$, for $v > r$.

P3iii. This proof follows that of (P2*i*), applying Lemma 1(*iii*) to $b_2(\cdot, r) - \hat{b}_2(\cdot, r)$. Again $b_2(r, r) = \hat{b}_2(r, r)$, and if $b_2 \leq \hat{b}_2$ for some $v > r$, then (6) holds at (v, r). Since (DRH) holds and R_B or $\widehat{R}_B$ is nonincreasing, Proposition 3(*ii*) implies $\hat{b}_1 > b_1$. Furthermore, as (5) holds, $\widehat{R}_B > R_B$. Lastly, we now have $\hat{b}_2 > 0$ by Proposition 2(*i*). Thus, we again obtain $b_{12} - \hat{b}_{12} > 0$ from (6). Lemma 1(*iii*) now implies $b_2 > \hat{b}_2$ for all $v > r$. ■

5 Seller Preferences over Auctions with the Same Reserve Price

Let $i = I, II$ denote, respectively, the FPA and the SPA, and let $V_i(r)$ be the seller's equilibrium expected utility in auction i with reserve price r. The revenue equivalence theorem establishes that $V_I(r) = V_{II}(r)$ if all participants are risk neutral.

As shown by Maskin and Riley (1984), risk aversion on the part of the seller and/or the buyers causes the seller to prefer the FPA to the SPA if both have the same reserve price.[9] This is due to two effects. The first is a direct "revenue effect": buyer risk aversion causes them to bid more in the FPA. The second is a "risk effect": the high bid in a FPA is a less risky random variable than it is in a SPA, and so preferred by a risk averse seller.

For future reference we record this result as part (i) of the following proposition. Part (ii) records the result that in a FPA, the seller prefers the buyers to be more risk averse, a consequence of the fact that they then bid more.

Proposition 4 *(i) If R_B and/or R_S is positive, then $V_I(r) > V_{II}(r)$ for $r < H$.*

(ii) If $\hat{u}_B$ satisfies the same assumptions as u_B, with $\widehat{R}_B > R_B$, and $\widehat{V}_I(r)$ is the corresponding FPA equilibrium seller payoff, then $\widehat{V}_I(r) > V_I(r)$ for $r < H$.

Proof. Part (i) follows from Theorem 5 in Maskin and Riley (1984). To prove (ii), fix $r < H$ and $(v_1, \ldots, v_n) \in [L, H]^n$. Let $v^m = \max_i v_i$. In either case, u_B or $\hat{u}_B$, a sale occurs if and only if $v^m \geq r$. The price is then $b(v^m, r)$ or $\hat{b}(v^m, r)$, since b_1 and $\hat{b}_1$ are positive. By Proposition 3(i), $\hat{b}(v, r) > b(v, r)$ for $v > r$. Thus, for almost all value vectors resulting in a sale, the sale price is higher given $\hat{u}_B$ than u_B. Since a sale occurs with positive probability because $r < H$, we have $\widehat{V}_I(r) > V_I(r)$. ■

The seller's preferences over auctions with the same fixed reserve price extend immediately to the setting in which the seller sets reserve prices. For example, if R_B and/or R_S is positive, and r_I (r_{II}) is an optimal reserve price for the seller in the FPA (SPA), then Proposition 4(i) immediately implies $V_I(r_I) > V_{II}(r_{II})$.

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[9] Related results are in Vickrey (1961), Matthews (1980), Riley and Samuelson (1981), Waehrer et al. (1998), and Milgrom (2004, Theorem 4.10).

6 Optimal Reserve Prices

We now derive expressions for $V_i(r)$ and $V_i'(r)$ in order to study the seller's optimal reserve prices. The rules of the auctions and the nature of their equilibria imply

$$V_I(r) = n\int_r^H u_S(b(v,r))G(v)dF(v) + F(r)^n u_S(v_0), \tag{7}$$

$$V_{II}(r) = nG(r)(1-F(r))u_S(r) + n\int_r^H u_S(y)(1-F(y))dG(y) + F(r)^n u_S(v_0). \tag{8}$$

Differentiating (7) yields

$$V_I'(r) = n\int_r^H u_S'(b(v,r))b_2(v,r)G(v)dF(v) - nG(r)f(r)\left[u_S(r) - u_S(v_0)\right]. \tag{9}$$

The first term in (9) is the seller's marginal benefit from raising the reserve price in the FPA, due to the resulting increase in the bid function on $[r, H]$. The second term is the marginal cost, due to the lost sales at price $b(r,r) = r$ caused by a marginal increase in the reserve price.

Differentiating (8) yields

$$V_{II}'(r) = nG(r)\left(1-F(r)\right)u_S'(r) - nG(r)f(r)\left[u_S(r) - u_S(v_0)\right]. \tag{10}$$

Again, the first and second terms are the seller's marginal benefit and marginal cost of raising the reserve price. Comparing (9) to (10) shows that the marginal cost is the same in the SPA as in the FPA. The marginal benefit of raising the reserve price in the SPA differs, as it is due to the resulting increase in the price received in the event that precisely one bidder has a value greater than r.

Let $\mathcal{R}_i$ denote the set of reserve prices that maximize $V_i(r)$. The next proposition establishes that in both auctions, optimal reserve prices exist and are in the interval (v_0, H). Furthermore, the optimal reserve price in the SPA is unique and invariant to the number of bidders under the regularity assumption of Myerson (1981), that a bidder's virtual valuation increases in his value.

The proposition refers to the function

$$\Phi(r) \equiv \frac{u_S(v_0) - u_S(r)}{u_S'(r)} + \frac{1-F(r)}{f(r)}. \tag{11}$$

From (10) we obtain $V'_{II}(r) = nG(r)f(r)u'_S(r)\Phi(r)$.

Proposition 5 *Both $\mathcal{R}_I$ and $\mathcal{R}_{II}$ are nonempty subsets of (v_0, H). Any $r_{II} \in \mathcal{R}_{II}$ satisfies $\Phi(r_{II}) = 0$, and $\mathcal{R}_{II}$ is a singleton and independent of the number of bidders if $v - \frac{1-F(v)}{f(v)}$ is strictly increasing on (L, H).*

Proof. The first term in (9) is nonnegative for $r \leq H$, since $b_2 \geq 0$. Hence, from this and (10) we see that V_i is constant on $(-\infty, L]$, and $V'_i > 0$ on (L, v_0). For $r \geq H$, $V_i(r) = u_S(v_0)$, and $V'_i(H) < 0$. Thus, any $r_i \in \mathcal{R}_i$ is in (v_0, H). We know $\mathcal{R}_i \neq \emptyset$ because $[v_0, H]$ is compact and V_i is continuous. Any $r_{II} \in \mathcal{R}_{II}$ satisfies $V'_{II}(r_{II}) = 0$ and $r_{II} > L$, and hence $\Phi(r_{II}) = 0$. Differentiating (11) yields

$$\Phi'(r) = \left(\frac{u_S(v_0) - u_S(r)}{u'_S(r)}\right) R_s(r) - \left(r - \frac{1 - F(r)}{f(r)}\right)'.$$

The first term is nonpositive for $r \geq v_0$. Hence, if $v - \frac{1-F(v)}{f(v)}$ is strictly increasing, then $\Phi' < 0$ on $[v_0, H]$. This interval then contains a unique r_{II} satisfying $\Phi(r_{II}) = 0$, and so $\mathcal{R}_{II} = \{r_{II}\}$. Since Φ does not depend on n, neither does this r_{II}. ■

We now show that the seller sets a lower reserve price in the FPA than in the SPA if he and/or the bidders are risk averse. The proof is based on the observation that because the seller's marginal cost of raising the reserve price is the same in both auctions, the difference in his incentives is the difference in the marginal benefits: (9) and (10) yield

$$V'_I(r) - V'_{II}(r) = \underbrace{n\int_r^H u'_S(b(v,r))b_2(v,r)G(v)dF(v)}_{MB_I} - \underbrace{nG(r)\left(1 - F(r)\right)u'_S(r)}_{MB_{II}}. \quad (12)$$

It is easy to see that this difference is negative if the bidders and/or the seller is risk averse. By the revenue equivalence theorem, $V'_I(r) = V'_{II}(r)$ if they are all risk neutral, and so then $MB_I = MB_{II}$. As the seller becomes risk averse, the ratio $u'_S(b(v,r))/u'_S(r)$ falls because $b(v,r) > r$, and hence MB_I falls relative to MB_{II}. As the bidders become risk averse, b_2 falls by Proposition 1, which lowers MB_I and leaves MB_{II} unchanged. The proof of our first theorem makes this logic precise.

Theorem 1 *Suppose R_B and/or R_S is positive. Then, for any $r_I \in \mathcal{R}_I$ and $r_{II} \in \mathcal{R}_{II}$, we have $r_I < r_{II}$.*

Proof. Write (12) as

$$V_I'(r) - V_{II}'(r) = nG(r)u_S'(r) \int_r^H \left[\left(\frac{u_S'(b(v,r))}{u_S'(r)} \right) \left(\frac{G(v)}{G(r)} b_2(v,r) \right) - 1 \right] dF(v).$$

Since $r > v_0 \geq L$, this expression is positive if and only if the integral is positive. Fix $v > r$. Since $b(v,r) > r$ and u_S is concave we have $u_S'(b(v,r))/u_S'(r) \leq 1$, and this inequality is strict if $R_S > 0$. From Proposition 1 we have $G(v)b_2(v,r)/G(r) \leq 1$, and this inequality is strict if $R_B > 0$. Hence, as at least one of R_S and R_B is positive, the integrand in the above expression is negative at all $v \in (r, H]$. This proves $V_I' < V_{II}'$ on (L, H). Since $r_{II} \in (v_0, H)$ by Proposition 5, Lemma 2 now implies $r_I < r_{II}$. ■

Our second theorem shows that in either auction, a more risk averse seller sets a lower reserve price. The intuition is that the more risk averse the seller is, the more he wishes to avoid the risk of not selling the object for a profitable price.

Theorem 2 *Let $\hat{u}_S$ satisfy the same assumptions as u_S, with $\widehat{R}_S > R_S$. Let $\widehat{\mathcal{R}}_i$ and $\mathcal{R}_i$ be the sets of optimal reserve prices given $\hat{u}_S$ and u_S, for $i = I, II$. Then, for any $\hat{r}_i \in \widehat{\mathcal{R}}_i$ and $r_i \in \mathcal{R}_i$, we have $\hat{r}_I < r_I$ and $\hat{r}_{II} < r_{II}$.*

Proof. We first prove $\hat{r}_{II} < r_{II}$. W.l.o.g., we may assume $\hat{r}_{II} = \max \widehat{\mathcal{R}}_{II}$. It is convenient to normalize $\hat{u}_S$ so that $\hat{u}_S(v_0) = u_S(v_0)$ and $\hat{u}_S(\hat{r}_{II}) = u_S(\hat{r}_{II})$. Then, since $\widehat{R}_S > R_S$, by Pratt (1964, Theorem 1) we have

$$u_S(y) < \hat{u}_S(y) \text{ for } y \in (v_0, \hat{r}_{II}). \tag{13}$$

From (8) we obtain $V_{II}(r) - \widehat{V}_{II}(r) = T_1(r) + T_2(r)$, where

$$T_1(r) \equiv n\, G(r)(1 - F(r)) \left[u_S(r) - \hat{u}_S(r) \right],$$
$$T_2(r) \equiv n \int_r^H \left[u_S(y) - \hat{u}_S(y) \right] (1 - F(y)) dG(y).$$

Let $r \in (v_0, \hat{r}_{II})$. Then (13) implies $T_1(r) < 0 = T_1(\hat{r}_{II})$ and $T_2(r) < T_2(\hat{r}_{II})$. Hence, $V_{II}(r) - \widehat{V}_{II}(r) < V_{II}(\hat{r}_{II}) - \widehat{V}_{II}(\hat{r}_{II})$. We conclude that for all $r \in (v_0, \hat{r}_{II})$,

$$V_{II}(\hat{r}_{II}) - V_{II}(r) > \widehat{V}_{II}(\hat{r}_{II}) - \widehat{V}_{II}(r) \geq 0,$$

where the second inequality follows from $\hat{r}_{II} \in \widehat{\mathcal{R}}_{II}$. This proves $\hat{r}_{II} \leq r_{II}$. Now, again by Pratt (1964, Theorem 1), for any $y > v_0$ we have

$$\frac{\hat{u}_S(v_0) - \hat{u}_S(y)}{\hat{u}'_S(y)} < \frac{u_S(v_0) - u_S(y)}{u'_S(y)}. \tag{14}$$

Hence, using $\hat{u}_S$ in (11) to define $\widehat{\Phi}$, $\widehat{\Phi}(r) < \Phi(r)$ for $r > v_0$. This and Proposition 5 imply $\Phi(\hat{r}_{II}) > \widehat{\Phi}(\hat{r}_{II}) = 0 = \Phi(r_{II})$, and so $\hat{r}_{II} \neq r_{II}$. Thus, $\hat{r}_{II} < r_{II}$.

We now use a similiar approach to prove $\hat{r}_I < r_I$. W.l.o.g., we may assume $\hat{r}_I = \max \widehat{\mathcal{R}}_I$. Normalize $\hat{u}_S$ so that $\hat{u}_S(v_0) = u_S(v_0)$ and $\hat{u}_S(\hat{r}_I) = u_S(\hat{r}_I)$. Then, from $\widehat{R}_S > R_S$ we have $u_S(y) < \hat{u}_S(y)$ for $y \in (v_0, \hat{r}_I)$, and $u_S(y) > \hat{u}_S(y)$ for $y \notin [v_0, \hat{r}_I]$. For any $y > v_0$ we still have (14), and hence

$$\frac{u'_S(y)}{\hat{u}'_S(y)} > \frac{u_S(y) - u_S(v_0)}{\hat{u}_S(y) - \hat{u}_S(v_0)}.$$

We thus have $u'_S(y) > \hat{u}'_S(y)$ for $y > \hat{r}_I$, using $\hat{u}_S(v_0) = u_S(v_0)$ and $u_S(y) > \hat{u}_S(y)$. This implies that for $v_0 < r \leq \hat{r}_I < v$,

$$u_S(b(v, \hat{r}_I)) - \hat{u}_S(b(v, \hat{r}_I)) > u_S(b(v, r)) - \hat{u}_S(b(v, r))$$

holds if $b(v, r) > \hat{r}_I$ (since $b_2 \geq 0$). This inequality holds also if $b(v, r) \leq \hat{r}_I$, since then its right side is nonpositive, but its left side is positive because $b(v, \hat{r}_I) > \hat{r}_I$. From (7) we therefore obtain, for $r \in (v_0, \hat{r}_I)$,

$$\begin{aligned} V_I(\hat{r}_I) - \widehat{V}_I(\hat{r}_I) &= n \int_{\hat{r}_I}^{H} \left[u_S(b(v, \hat{r}_I)) - \hat{u}_S(b(v, \hat{r}_I))\right] G(v) dF(v) \\ &> n \int_{\hat{r}_I}^{H} \left[u_S(b(v, r)) - \hat{u}_S(b(v, r))\right] G(v) dF(v). \end{aligned}$$

Observe that for $v_0 < r < v < \hat{r}_I$, we have $b(v, r) < b(\hat{r}_I, r) < \hat{r}_I$, and so $u_S(b(v, r)) < \hat{u}_S(b(v, r))$. This implies that for $r \in (v_0, \hat{r}_I)$,

$$\int_{r}^{\hat{r}_I} \left[u_S(b(v, r)) - \hat{u}_S(b(v, r))\right] G(v) dF(v) < 0.$$

The two previous displays yield, for any $r \in (v_0, \hat{r}_I)$,

$$\begin{aligned} V_I(\hat{r}_I) - \widehat{V}_I(\hat{r}_I) &> n \int_r^H \left[u_S(b(v,r)) - \hat{u}_S(b(v,r))\right] G(v) dF(v) \\ &= V_I(r) - \widehat{V}_I(r). \end{aligned}$$

This implies $V_I(\hat{r}_I) - V_I(r) > \widehat{V}_I(\hat{r}_I) - \widehat{V}_I(r) \geq 0$, where the second inequality follows from $\hat{r}_I \in \widehat{\mathcal{R}}_I$. Hence, $\hat{r}_I \leq r_I$. To rule out equality, observe from (9) that $V_I'(r) = nG(r)f(r)u_S'(r)\Psi(r)$, where

$$\Psi(r) \equiv \frac{u_S(v_0) - u_S(r)}{u_S'(r)} + \int_r^H \frac{u_S'(b(v,r))}{u_S'(r)} \frac{b_2(v,r)G(v)}{G(r)f(r)} dF(v).$$

Define $\widehat{\Psi}(r)$ similarly from $\hat{u}_S$. Since $b(v,r) > r$ for $v > r$, from Pratt (1964, (20)) we have

$$\frac{u_S'(b(v,r))}{u_S'(r)} > \frac{\hat{u}_S'(b(v,r))}{\hat{u}_S'(r)} \text{ for } v > r.$$

This implies $\Psi(\hat{r}_I) > \widehat{\Psi}(\hat{r}_I)$ (using $b_2 \geq 0$, $\hat{r}_I > v_0$, and (14)). Hence, $V_I'(\hat{r}_I) > \widehat{V}_I'(\hat{r}_I) = 0 = V_I'(r_I)$. This proves $\hat{r}_I \neq r_I$, and hence $\hat{r}_I < r_I$. ■

Our third and final theorem establishes that in the FPA, the seller sets a lower reserve price if the bidders are more risk averse. The theorem requires that either the bidders satisfy CARA, or that (DRH) holds and at least one of the two bidder utility functions being compared satisfies nonincreasing absolute risk aversion. The logic of the result is twofold. First, under these conditions the FPA bid function increases in the reserve price at a slower rate when the bidders are more risk averse. This lowers the incentive of the seller to raise the reserve price. Second, because more risk averse bidders bid more, the increase in their bids in response to an increase in the reserve price generates a lower marginal utility increase for the (weakly) risk averse seller. The proof reflects these two forces.

Theorem 3 *Let* $\hat{u}_B$ *satisfy the same assumptions as* u_B*, with* $\widehat{R}_B > R_B$*. Let* $\mathcal{R}_I$ *(*$\widehat{\mathcal{R}}_I$*) be the set of optimal reserve prices for the seller given* u_B *(*$\hat{u}_B$*). Then, for any* $r_I \in \mathcal{R}_I$ *and* $\hat{r}_I \in \widehat{\mathcal{R}}_I$*, we have* $\hat{r}_I < r_I$ *if either*

(a) (DRH) *holds and* R_B *and/or* $\widehat{R}_B$ *is nonincreasing, or*

(b) *both* u_B *and* $\hat{u}_B$ *satisfy CARA.*

Proof. Letting $\widehat{V}(r)$ be the seller's payoff given $\hat{u}_B$ and reserve r, (9) implies

$$\widehat{V}_I'(r) - V_I'(r) = n \int_r^H \left[u_S'(\hat{b}(v,r))\hat{b}_2(v,r) - u_S'(b(v,r))b_2(v,r) \right] G(v)dF(v).$$

The concavity of u_S, together with $\hat{b}(v,r) > b(v,r)$ (by Proposition 3(i)), yield $u_S'(\hat{b}) \leq u_S'(b)$ for $v > r$. Both (a) and (b) imply $\hat{b}_2 < b_2$ for $v > r$, by either Proposition 3(iii) or equation (3) (which implies $\partial b_2^a / \partial a < 0$). Hence, $\widehat{V}_I'(r) < V_I'(r)$ for any $r > L$. Since $r_I \in (v_0, H)$ by Proposition 5, Lemma 2 now yields $\hat{r}_I < r_I$. ■

7 Concluding Discussion

We have shown that when the seller sets the reserve price, he sets it lower the more risk averse he is and, in a first-price auction, the more risk averse the buyers are. The seller's optimal reserve price is lower in the first-price than in the second-price auction, unless all parties are risk neutral. Risk aversion thus reduces the probability of not selling the object when a buyer's value for it exceeds that of the seller, especially in the first-price auction.

The buyers may agree, ex ante, with the seller's preference for the first-price auction. Indeed, if they exhibit constant (or increasing) absolute risk aversion, every type of buyer weakly (strictly) prefers at the interim stage the first-price to a second-price auction that has the same reserve price (Matthews, 1987). *Ipso facto,* in these cases the buyers prefer the first-price auction if it has the lower reserve price, as it does when the seller sets the reserve price and he or the buyers are risk averse. By continuity, the buyers must also prefer the first-price auction if their absolute risk aversion measure is approximately constant, so long as they and/or the seller are risk averse.[10] More generally, buyers with values in the interval $(r_I, r_{II}]$ strictly

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[10] Formally, if $|R_B(y) - a| < \varepsilon$ for all y, and if $a > 0$ and/or $R_S > 0$, then $\bar{\varepsilon} > 0$ exists such

prefer the FPA, and hence so must the buyers with values in some interval $(r_I, \hat{v})$, where $\hat{v} > r_{II}$.

We have focused tightly in this paper on the effects of risk aversion on optimal reserve prices in two standard auctions, holding fixed their other features. Endogenizing these other features and determining the effects of risk aversion on their levels is a topic for future research. For example, if the seller is able to charge bidders an entry fee, he may wish to do so if the bidders are risk averse (Maskin and Riley, 1983), but not if he is risk averse and can also set the reserve price (Waehrer et al., 1998). The nature of optimal combinations of entry fees and reserves when the seller or buyers are risk averse is unknown. Another example is entry: if each of a large number of potential bidders must pay a cost to learn his value, the number of bidders becomes endogenous. In this case the seller may want to lower the reserve price in order to increase entry.[11] Our results suggest that risk aversion on the part of the seller or buyers should strengthen this effect, especially in the first-price auction.[12].

Future work may also generalize the setting of our results. It may be fruitful, for example, to consider asymmetric bidders with different value distributions, which give rise to a different ex post inefficiency (sale to the wrong bidder) than the one (no sale) that we have considered. Settings with ex post risk or interdependent values are naturally of interest as well.

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that when $\varepsilon < \bar{\varepsilon}$, every type of buyer interim prefers the FPA to the SPA when the seller sets the reserve prices.

[11] The effects of endogenous entry on optimal reserve prices are studied, in risk neutral settings, by McAfee and McMillan (1987), Engelbrecht-Wiggans (1993), and Levin and Smith (1994).

[12] Endogenous entry can reverse the seller's preference for the FPA, since the SPA may induce more entry if the buyers have DARA risk preferences, as is shown in Smith and Levin (1996). This reversal should occur less often, however, when the seller sets the reserve price, since he sets it lower in the FPA.

Appendix

Proof of Lemma 1. (*i*) Proved in Milgrom (2004).[13]

(*ii*) Assume $h(t) \leq 0$ for some $t \in (c, d]$. The hypothesis and the continuity of h imply the existence of $\hat{t} \in [c, t)$ such that $h(\hat{t}) < 0$. Let $\bar{s} = \sup\{s \in [c, \hat{t}) : h(s) \geq 0\}$. As h is continuous, $\bar{s} < \hat{t}$ and $h(\bar{s}) = 0$. The hypothesis now implies the existence of $s \in (\bar{s}, \hat{t})$ such that $h(s) > 0$. This contradicts the definition of $\bar{s}$.

(*iii*) Assume $h(t) \leq 0$ for some $t \in (c, d]$. Let m be the largest minimizer of h on $[c, t]$. Since $h(c) \geq h(t)$, $m > c$. We thus have $h'(m) \leq 0$, as well as $h(m) \leq h(t) \leq 0$. This contradicts the hypothesis. ■

Proof of Lemma 2. Let $i \in \{1, 2\}$ be such that $t_i \in (c, d)$, and let $j \neq i$ be the other index. Then $h_j'(t_i) \neq h_i'(t_i) = 0$. This proves $h_j(t_j) > h_j(t_i)$, and hence $t_1 \neq t_2$. Defining $h = h_2 - h_1$, we now have $h(t_2) > h(t_1)$. By the mean value theorem, there exists t strictly between t_1 and t_2 such that

$$(t_2 - t_1)h'(t) = h(t_2) - h(t_1) > 0.$$

This proves $t_1 < t_2$, since $h'(t) > 0$. ■

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[13] The Ranking Lemma in Milgrom (2004, p. 124) is (*i*) and (*ii*), stated for $c = \infty$ and a continuous h'. Only (*i*) is proved, however, and its proof does not use these extra assumptions.

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