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"Bounds on Revenue Distributions in Counterfactual Auctions with Reserve Prices"

by

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Abstract

In first-price auctions with interdependent bidder values, the distributions of private signals and values cannot be uniquely recovered from bids in Bayesian Nash equilibria. Non-identification invalidates structural analyses that rely on the exact knowledge of model primitives. In this paper I introduce tight, informative bounds on the distribution of revenues in counterfactual first-price and second-price auctions with binding reserve prices. These robust bounds are identified from distributions of equilibrium bids in first-price auctions under minimal restrictions where I allow for affiliated signals and both private- and common-value paradigms. The bounds can be used to compare auction formats and to select optimal reserve prices. I propose consistent nonparametric estimators of the bounds. I extend the approach to account for observed heterogeneity across auctions, as well as binding reserve prices in the data. I use a recent data of 6,721 first-price auctions of U.S. municipal bonds to estimate bounds on counterfactual revenue distributions. I then bound optimal reserve prices for sellers with various risk attitudes.

KEYWORDS: Structural auction models, interdependent values, affiliated signals, partial identification, counterfactual revenue distributions, U.S. municipal bond auctions

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1 Introduction

In a structural auction model, a potential bidder does not know his own valuation of the auctioned object, but has some noisy private signal about its value. Bidders make their decisions conditional on these signals and their knowledge of the distribution of their competitors' private signals and values. A structural approach for empirical studies of auctions posits the distribution of bids observed can be rationalized by a joint distribution of bidder values and signals in Bayesian Nash equilibria, and defines this joint distribution as the model primitive. The objective is to extract information about this primitive from the distribution of bids, and to use it to answer policy questions such as the choice of optimal reserve prices or auction formats. (See Hendricks and Porter (2007) for a survey.) Depending on whether bidders would find rivals' signals informative about their own values conditional on their own signals, an auction belongs to one of the two mutually exclusive types : *private values* (PV), and *common values* (CV).[1] These two types have distinct implications for revenue distributions under a given auction format.

In this paper I propose tight, informative bounds on counterfactual revenue distributions that can be constructed from the distribution of bids in a general class of first-price auctions with interdependent values and affiliated signals. The counterfactual formats considered in this paper include both first- and second-price auctions with reserve prices.[2] Thus I introduce a unified approach of policy analyses for both PV and CV auctions that does not require exact identification of model primitives. My method is motivated by several empirical challenges related to structural CV models. First, several policy questions have not been addressed outside the restrictive case of PV auctions due to difficulties resulting from non-identification of signal and value distributions.[3] For a fixed reserve price, theory ranks expected revenue for general interdependent value auctions with affiliated signals, but the magnitude of expected revenue differences remains an empirical question.[4] Another open issue is the choice of optimal reserve prices in general interdependent value auctions with affiliated signals and finite number of bidders.[5] Since model primitives cannot be recovered

[1] I use the term "interdependent values" for a larger class of auctions that encompass both PV and CV auctions. The formal definition of a PV auction is one in which bidders' values are mean independent from rival signals conditional on their own signals.

[2] In this paper, I use the term "second-price auctions" exclusively for the sealed-bid format. This does not include the open formats, or "English auctions".

[3] For a proof of non-identification, see Laffont and Vuong (1996).

[4] The only exception is the case with i.i.d. signals, where expected revenue from first-price, second-price and English auctions are the same regardless of value interdependence.

[5] An exception is symmetric, independent private value auctions, where the optimal reserve price is

from equilibrium bids in CV auctions, these questions cannot be addressed as in PV auctions, where point identification of signal distributions helps exactly recover revenue distributions in counterfactual formats.[6] Second, it is difficult to distinguish PV and CV auctions from the distribution of bids alone under a given auction format, even though the two have distinct implications in counterfactual revenue analyses. Laffont and Vuong (1996) proved for a given number of potential bidders, distributions of equilibrium bids in CV auctions can always be rationalized by certain PV structures. Empirical methods that have been proposed for distinguishing between the two types often have practical limitations. They either rely on assumptions that may not be valid in some applications (such as exogenous variations of number of bidders, as in Haile, Hong and Shum (2003)), or they may entail strong data requirements (an ex post measure of bidder values as in Hendricks, Pinkse and Porter (2003), or many bids near a binding reserve price as in Hendricks and Porter (2007)).[7] Third, the empirical auction literature has not considered the magnitude of the bias if a CV environment is analyzed with a PV model in counterfactual revenue analyses.

I propose a structural estimation method through partial identification of revenue distributions to address the questions above. First, the bounds on revenue distributions are constructed directly from the bids, and do not rely on pinpointing the underlying signal and value distributions. Second, the bounds only require a minimum set of general restrictions on value and signal distributions that encompass both PV and CV paradigms. Third, the bounds are tight and sharp within the general class of first-price auctions. The lower bound is the true counterfactual revenue distribution under a PV structure, while the upper bound can be close to the truth for certain types of CV models. Hence the distance between the bounds can be interpreted as a measure of maximum error possible when a CV structure is analyzed as PV in counterfactual analyses. The bounds can be nonparametrically estimated consistently. Although I do not provide point estimates of revenue distributions, the bounds are informative for answering policy questions, as they can be used to compare auction formats, or to bound revenue maximizing reserve prices. The analysis can be extended to risk-averse sellers immediately given the sellers' utility functions.

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identified from the distribution of equilibrium bids. Levin and Smith (1994) also showed in symmetric first-price auctions, where signals are affiliated and values are interdependent through a common unobserved component, the optimal reserve price converges to the seller's true value as the number of potential bidders n goes to infinity. Yet the theory is otherwise silent about identifying optimal reserve prices with a finite n.

[6] See Guerre et.al (2000), Li, Perrigne and Vuong (2002) and Li, Perrigne and Vuong (2003) for details. Also note in PV auctions, the distribution of signals $\{X_i\}_{i=1}^n$ are equivalent to the distribution of values $\{V_i\}_{i=1}^n$ under the normalization $E(V_i|X_i = x) = x$.

[7] A binding reserve price is one that is high enough to have a positive probability of screening out some bidders.

My paper is related to the literature on robust inference in auction models. Haile and Tamer (2003) use incomplete econometric models to bound the optimal reserve price in independent PV English auctions, where the equilibrium bidding assumption is replaced with two intuitive behavior assumptions. In contrast, my paper focuses on first-price CV auctions. Incompleteness here arises from the range of possible rationalizing signal and value distributions, instead of a flexible interpretation of bids. Hendricks, Pinkse and Porter (2003) introduce nonparametric structural analyses to CV auctions. They use an ex post measure of bidder values to test the assumption of equilibrium bidding. They also provide evidence that the winner's curse effect dominated the competition effect, leading to less aggressive bidding in equilibrium as the number of bidders increase. Shneyerov (2006) introduces an approach for counterfactual revenue analyses in common-value auctions without the need to identify model primitives. In particular, he shows that for any given reserve price, equilibrium bids from first-price auctions can be used to identify the expected revenues in second-price auctions with the same reserve price. He also shows how to bound the expected gains in revenues from English auctions under the general restriction of monotone value functions and affiliated signals.

My paper makes three novel contributions. First, the focus on revenue distributions, as opposed to distributional parameters such as expectations, allows more general revenue analyses. Auction theory usually uses expected revenue as a criterion to compare auction designs, but central tendency may not be justifiable in practice, say if the seller is not risk-neutral. Knowledge of distributions is necessary for other criteria, such as maximizing expected seller utility. (A seller may also choose a design to maximize the probability that revenue falls in a certain range.) Second, bounds on revenue distributions can be constructed for hypothetical reserve prices. One can then compare reserve prices within first-price or second-price formats. In CV auctions, a counterfactual, binding reserve price r creates serious challenges in policy analyses. The probability that no one bids higher than r in equilibrium cannot be pinpointed from bids in the data, since the screening level can not be identified without further restrictions.[8] Moreover, the mapping from equilibrium bids in the data to those under the counterfactual r cannot be uniquely recovered. I address this issue by bounding the bid that a marginal bidder under a counterfactual binding r actually places in equilibrium under the data-generating auction format.[9] These bounds in turn lead to bounds on the revenue distribution under r. Finally, the bounds on revenue distributions are robust and independent from exact forms of signal affiliations and value interdependence,

[8] A screening level under r is the value of signal such that only bidders with signals higher than the screening level will choose to submit bids above r in equilibrium. See Section 2 below for a formal definition.

[9] A marginal bidder under r is the one whose signal is exactly equal to the screening level.

and are identified from the distribution of equilibrium bids alone. This robustness comes with the price of partial identification of revenue distributions. Nonetheless, one can obtain informative answers for some policy questions.

The remainder of the paper proceeds as follows. Section 2 introduces bounds on counterfactual revenue distributions in a benchmark model where data is collected from homogenous auctions with exogenous participation. Section 3 defines nonparametric estimators for bounds and proves their pointwise consistency. Section 4 provides Monte Carlo evidence about the performance of the bound estimators. Section 5 extends the benchmark model to allow for observable auction heterogeneity and endogenous participation under binding reserve prices in the data. Section 6 applies the proposed method to U.S. municipal bond auctions on the primary market. Section 7 concludes.

2 Bounds on Counterfactual Revenue Distributions in the Benchmark Model

This section focuses on a benchmark case where bids are observed from increasing, symmetric pure-strategy Bayesian Nash Equilibria ($PSBNE$) in homogenous single-unit first-price auctions with a fixed number of bidders and no binding reserve prices. I show how to use joint distributions of these bids to construct tight bounds on revenue distributions in counterfactual first-price and second-price auctions with a binding reserve price $r > 0$. Extensions to cases where bids are observed from heterogenous auctions or auctions with endogenous participation due to binding reserve prices are discussed in Section 5.

2.1 Model specifications

Consider a single-unit, first-price auction with N potential risk-neutral bidders and no reserve price. Each bidder receives a private signal X_i but cannot observe his own valuation V_i. The distribution of all bids submitted in equilibrium (denoted $\mathbf{B}_N^0 \equiv \{B_i^0\}_{i=1,..,N}$) is observed from a random sample of independent, identical auctions, but neither X_i nor V_i can be observed. For simplicity, X_i and V_i are both scalars. Throughout the paper I use upper case letters to denote random variables, lower case letters for realized values, and bold letters for vectors. The following assumptions are maintained for the rest of the paper.

A1 (Symmetric, Affiliated Signals) Private signals $\mathbf{X}_N \equiv \{X_i\}_{i=1,..,N}$ are affiliated with support $S_X^N \equiv [x_L, x_U]^N$, and the joint distribution $F_{\mathbf{X}_N}$ is exchangeable in all arguments.[10]

A2 (Interdependent Values) A bidder's value satisfies $V_i = \theta_N(X_i, \mathbf{X}_{-i})$, where $\theta_N(.)$ is a nonnegative, bounded, continuous function exchangeable in $\mathbf{X}_{N-i} \equiv \{X_1, ., X_{i-1}, X_{i+1}, ., X_N\}$, non-decreasing in all signals, and increasing in his own signal X_i over $S_X \equiv [x_L, x_U]$.

Note *A2* implies private signals are drawn from identical marginal distributions on S_X, and *A1* includes private values (PV) as a special case, where $\theta_N(x_i, \mathbf{x}_{-i})$ does not depend on $\mathbf{x}_{-i}$ for all $(x_i, \mathbf{x}_{-i}) \in S_X^N$. Common values ($CV$) correspond to value functions that are non-degenerate in rival signals $\mathbf{X}_{-i}$. A pure strategy for a bidder under a given auction structure $(N, \theta_N, F_{\mathbf{X}_N})$ is a function $b_{i,N}^0(.; \theta_N, F_{\mathbf{X}_N}) : X_i \to \mathbb{R}_+^1$. A pure-strategy Bayesian Nash equilibrium is a portfolio $\{b_{i,N}^0(.)\}_{i=1,..,N}$ such that for all i, $b_{i,N}^0(.)$ is the best response to $\{b_{j,N}^0(.)\}_{j\in\{1,..,N\}\setminus\{i\}}$. (The superscript 0 signifies that there is no reserve price.) That is, for all i and $x \in S_X$,

$$b_{i,N}^0(x) = \arg\max_b E(V_i - b | \max_{j\in\{1,.,N\}\setminus\{i\}} b_{j,N}^0(X_j) \leq b, X_i = x) \Pr(\max_{j\in\{1,.,N\}\setminus\{i\}} b_{j,N}^0(X_j) \leq b | X_i = x)$$

The regularity conditions for existence of such a $PSBNE$ is collected in *A3* below. McAdams (2006) proved *A1,2,3* are sufficient for the existence of unique symmetric, increasing $PSBNE$ in first-price auctions. The restrictions in *A3* are otherwise inessential for the main result of partial identification in this paper.

A3 (Regularity Conditions) (i) $\theta_N(.)$ is twice continuously differentiable; *(ii)* The joint density of $\{X_i\}_{i=1,..,N}$ exists on S_X^N, is continuously differentiable, and $\exists f_{low}, f_{high} > 0$ such that $f(\mathbf{x}) \in [f_{low}, f_{high}]$ $\forall \mathbf{x} \in S_X^N$.

Definition 1 *A joint distribution of bids $\{b_{i,N}^0\}_{i=\{1,..,N\}}$ in first-price auctions with no reserve price (denoted $G_{\mathbf{B}_N^0}$) is **rationalized** by an auction structure $\{\theta_N, F_{\mathbf{X}_N}\}$ if $G_{\mathbf{B}_N^0}$ is the distribution of bids in a symmetric, increasing PSBNE in the auction. Two structures $\{\theta_N, F_{\mathbf{X}_N}\}$ and $\{\tilde{\theta}_N, \tilde{F}_{\mathbf{X}_N}\}$ are **observationally equivalent** if they generate the same distribution $G_{\mathbf{B}_N^0}$ in symmetric, increasing PSBNE of first-price auctions. The **identified set** (relative to $G_{\mathbf{B}_N^0}$)is the set of all structures that are observationally equivalent given the bid distribution $G_{\mathbf{B}^0}$.*

[10]Let Z be a random vector in $\mathbb{R}^K$ with joint density f. Let $\vee$ and $\wedge$ denote respectively component-wise maximum and minimum of any two vectors in $\mathbb{R}^K$. Variables in Z are *affiliated* if, for all z and z' in $\mathbb{R}^K$, $f(z \vee z')f(z \wedge z') \geq f(z)f(z')$. For a more formal definition, see Milgrom and Weber (1982).

The first-order condition in *PSBNE* is characterized by:

$$b_N^{0\prime}(x) = [v_{h,N}(x,x) - b_N^0(x)]\frac{f_{Y_N|X}(x|x)}{F_{Y_N|X}(x|x)} \tag{1}$$

for all $x \in S_X$, where $Y_{i,N} \equiv \max_{j\in\{1,.,N\}\setminus\{i\}} X_j$, $F_{Y_N|X}(t|x) \equiv \Pr(Y_{i,N} \leq t|X_i = x)$, and $f_{Y_N|X}(t|x)$ denotes the corresponding conditional density. And $v_{h,N}(x,y;\theta_N, F_{\mathbf{X}_N})$ is a bidder's expected value conditional on winning with a pivotal bid, i.e. $E(V_i|X_i = x, Y_{i,N} = y)$. The equilibrium boundary condition is $b_N^0(x_L) = v_{h,N}(x_L, x_L)$. For notational ease, subscripts for bidder indices are dropped due to the symmetry in $F_{\mathbf{X}_N}$ and θ_N. In an increasing *PSBNE* where $b_N^{0\prime}(.) > 0$ on S_X, Guerre, Perrigne and Vuong (2000) established a link between the auction structure and $G_{\mathbf{B}_N^0}$ by reformulating (1) using change-of-variable :

$$v_{h,N}(x,x) = b_N^0(x) + \frac{G_{M_N^0|B_N^0}(b_N^0(x)|b_N^0(x))}{g_{M_N^0|B_N^0}(b_N^0(x)|b_N^0(x))} \equiv \xi(b_N^0(x); G_{\mathbf{B}_N^0}) \tag{2}$$

where $B_{i,N} \equiv b_N^0(X_i)$ is bidder i's bid in equilibrium, $M_{i,N} \equiv \max_{j\neq i} b_N^0(X_j)$ is the highest rival bid for bidder i, $G_{M_N^0|B_N^0}(t|b) = \Pr(M_N^0 \leq t|b_N^0 = b)$, and $g_{M_N^0|B_N^0}(t|b)$ is the corresponding conditional density.[11] Again, indices for bidders are dropped due to symmetry. *Furthermore, subscripts N will also be dropped for the rest of this section and the following section, as I focus on bid distributions from auctions with a fixed number of bidders.*

2.2 Review of literature on PV auctions

In this subsection, I review the literature on identification of signal distributions and optimal reserve prices in private value auctions. The objective is to highlight how unique identification of bidders' signal distributions and screening levels leads to exact knowledge of the optimal reserve price. This motivates my approach of partial identification when the screening level can not be exactly pinned down in more general interdependent value auctions.

Guerre, Perrigne and Vuong (2000) and Li, Perrigne and Vuong (2002) showed the joint distribution of bidder values are nonparametrically identified from distribution of equilibrium bids in first-price, PV auctions with no reserve prices. This result holds regardless of the form of dependence between private signals. The main idea is that in PV auctions, the left-hand side of (2) is independent from the second argument (the highest rival signal) and therefore

[11] Following a convention in the literature, I assume the second order conditions are always satisfied and thus first-order conditions are sufficient for characterizing the equilibrium.

can be normalized to the signal x itself. Hence the inverse bidding function can be fully recovered from $G_{\mathbf{B}^0}$, for both independent and affiliated signals.[12] Another simplification peculiar to PV auctions is that the screening level under a binding reserve price r is equal to r itself. That is, bidders choose not to bid above r in equilibrium if and only if their private signals are below r. To see this, note the screening level under r in a general interdependent value auction is defined as :

$$x^*(r) \equiv \inf\{x \in S_X : E(V_i|X_i = x, Y_i \leq x) \geq r\}$$

If $E(V_i|X_i = x, Y_i \leq x) < r$ for all $x \in S_X$, let $x^*(r) = x_U$. In PV auctions, $E(V_i|X_i = x, Y_i \leq x) = E(V_i|X_i = x)$ and the normalization $E(V_i|X_i = x) = x$ implies $x^*(r) = r$. Thus in PV auctions, both the signal distribution $F_{\mathbf{X}}$ and $x^*(r)$ are exactly recovered from $G_{\mathbf{B}^0}$.

In principle, knowledge of $F_{\mathbf{X}}$ in PV auctions is sufficient for finding counterfactual revenue distributions under a binding reserve price r. It follows that the optimal r which maximizes the expected revenue is also identified. Yet in reality it can be impractical to implement this fully nonparametric estimation due to data deficiencies, especially when the signals are affiliated. Li, Perrigne and Vuong (2003) proposed a nonparametric algorithm for estimating optimal reserve prices that is implemented with less intensive computations. The idea is to express expected seller revenue under r as a functional of the observed distribution of equilibrium bids and r. Then optimizing a sample analog of this objective function over reserve prices gives a consistent estimator of the optimal reservation price. Again the assumption of private values is indispensable for two reasons. First, it implies $x^*(r) = r$ under appropriate normalizations, which is used for defining the objective function; Second, it ensures full nonparametric identification of the distributions of counterfactual equilibrium bidding strategies.

This approach cannot be applied to CV auctions with affiliated signals immediately because of two non-identification results. First, the screening level cannot be pinned down without further restrictions on how bidders' signals and valuations are correlated. Second, inverse bidding functions can not be recovered without knowledge of θ. Hence underlying structure $\{\theta, F_{\mathbf{X}}\}$ can not be identified. These pose a major challenge for identifying counterfactual revenue distributions in CV auctions.

[12] In private value auctions, the conventional normalization of the signals is $E(V_i|X_i = x) = x$ for all x.

2.3 Observational equivalence of PV and CV

In this subsection I prove the observational equivalence of PV and CV paradigms when $G_{\mathbf{B}^0}$ is observed from first-price auctions with a fixed number of bidders. That is, $G_{\mathbf{B}^0}$ can be rationalized by a PV structure *if and only if* it can be rationalized by a CV structure.[13] This preliminary question sheds lights on the scope of bid distributions where a unified approach of counterfactual analyses for both PV and CV structures is needed. To understand this point, suppose there could exist certain $\tilde{G}_{\mathbf{B}^0}$ that would be rationalized only by a PV structure but not any CV ones. In this case, the issue of first-order importance would be to derive testable implications of all such $\tilde{G}_{\mathbf{B}^0}$, and then fully recover the underlying PV structures for those bid distributions satisfying such implications.

The rest of the subsection proves such a $\tilde{G}_{\mathbf{B}^0}$ cannot exist. Thus a robust approach of counterfactual analyses that does not count on distinguishing PV and CV structures is needed for any observed distributions of bids from a given auction structure. Let $\mathcal{F}$ denote the set of joint signal distributions that satisfy *A1*, and Θ the set of value functions that satisfy *A2*. Let Θ_{CV} denote a subset of Θ that is non-degenerate in rival signals $\mathbf{X}_{-i}$. Below I give *necessary and sufficient* conditions for $G_{\mathbf{B}^0}$ to be rationalized by some element of $\Theta_{CV} \otimes \mathcal{F}$.

Proposition 1 *A joint distribution of bids $G_{\mathbf{B}^0}$ observed in first-price auctions with non-binding reserve prices can be rationalized by some $\{\theta, F_{\mathbf{X}}\} \in \Theta_{CV} \otimes \mathcal{F}$ if and only if (i) $G_{\mathbf{B}^0}$ is affiliated and exchangeable in all arguments; and (ii) $\xi(b; G_{\mathbf{B}^0}) = b + G_{M_i^0|B_i^0}(b|b)/g_{M_i^0|B_i^0}(b|b)$ is strictly increasing on the support of individual bids $[b_L^0, b_U^0]$.*

Li, Perrigne and Vuong (2002) showed conditions *(i)* and *(ii)* are also necessary and sufficient for $G_{\mathbf{B}^0}$ to be rationalized by some PV structure. It follows that a $G_{\mathbf{B}^0}$ is rationalized by some PV structures *if and only if* it is also rationalized by some $\theta \in \Theta_{CV}$. This suggests that researchers cannot distinguish PV and CV paradigm only using bid distributions from homogenous auctions with a fixed number of bidders and no reserve prices.[14]

[13] Laffont and Vuong (1996) proved the sufficiency as they showed the non-identification of CV auctions. It remains unanswered whether the converse is true.

[14] Recent literature in empirical auctions have developed ways to distinguish two structures with augmented data containing bids from more than one auction formats. These include the use of exogenous variations in the number of bidders as in Haile, Hong and Shum (2003), ex post measures of bidder values as in Hendricks, Pinkse and Porter (2003), and bid distributions under a strictly binding reserve price as in Hendricks and Porter (2007).

2.4 Bounding revenue distributions in counterfactual 1st-price auctions

A conventional criterion for choosing optimal reserve prices is the expected revenue for the seller. The Revenue Equivalence Theorem states that in auctions with independent private values, optimal reserve prices are the same for both 2nd-price and 1st-price auctions, and are independent from the number of potential bidders. On the other hand, there is no theoretical result about the choice of optimal reserve prices in general 1st-price auctions with affiliated signals, interdependent values, and a finite number of bidders. The answer depends on the specifics of model primitives and is open for empirical analyses. Besides, expected revenue is not an appropriate criterion to use if the seller is not risk neutral. Knowledge of revenue distributions in counterfactual auction formats helps address both concerns. For a binding reserve price r, I propose informative bounds on $F_{R^I(r)}$ that are constructed from $G_{\mathbf{B}^0}$ alone.

2.4.1 The link between $G_{\mathbf{B}^0}$ and $F_{R^I(r)}$

I start by establishing links between observed bid distributions $G_{\mathbf{B}^0}$ and bidders' equilibrium bidding strategies as well as the distribution of revenues in counterfactual 1st-price auctions with a binding reserve price r. The equilibrium strategy in first-price auctions under a reserve price $r \geq 0$ has a closed form:

$$\begin{aligned} b^r(x;\theta,F_{\mathbf{X}}) &= rL(x^*(r)|x;F_{\mathbf{X}}) + \int_{x^*(r)}^{x} v_h(s,s;\theta,F_{\mathbf{X}})dL(s|x;F_{\mathbf{X}}) \quad \forall x \geq x^*(r) \\ b^r(x;\theta,F_{\mathbf{X}}) &< r \quad \forall x < x^*(r) \end{aligned}$$

where $L(s|x;F_{\mathbf{X}}) \equiv \exp\{-\int_s^x \Lambda(u;F_{\mathbf{X}})du\}$ and $\Lambda(x;F_{\mathbf{X}}) \equiv f_{Y|X}(x|x)/F_{Y|X}(x|x)$. This section focuses on a benchmark model where the bid distribution is observed from auctions with a fixed number of bidders. Hence the superscript N is suppressed for notational ease. For any given x on the closed interval S_X, $L(s|x;F_{\mathbf{X}})$ is a well-defined distribution function with support $[x_L, x]$ and is first-order stochastically dominated by the distribution of the second highest signal (i.e. $F_{Y|X}(s|x)/F_{Y|X}(x|x)$).[15] The two distributions are identical when bidders' private signals are i.i.d.. The range of r for nontrivial counterfactual analyses is $S_{RP} \equiv [\xi_L^0, \xi_U^0]$, where $\xi_k^0 \equiv \xi(b^0(x_k);G_{\mathbf{B}^0})$ for $k = L, U$. This is because for $r < \xi_L^0$, $x^*(r) = x_L$ and there is no effective screening of bidders, while for $r > \xi_U^0$, all bidders are

[15] That $L(.|x)$ is a well-defined distribution on $[x_L, x]$ is shown in Krishna (2002). Furthermore $L(s|x) \geq \exp\{-\int_s^x \frac{f_{Y|X}(u|x)}{F_{Y|X}(u|x)}du\} = \frac{F_{Y|X}(s|x)}{F_{Y|X}(x|x)}$, where the inequality follows from the fact that $F_{Y|X}(x|z)/f_{Y|X}(x|z)$ is decreasing in z for all x when signals are affiliated.

screened out with probability 1. Let v_0 denote the seller's own reserve value of the auctioned asset. For all $r \in S_{RP}$ and $r > v_0$, the distribution of revenue in counterfactual first-price auctions with a binding reserve price r (denoted $R^I(r)$) is:

$$\begin{aligned} F_{R^I(r)}(t;\psi) &= 0 \quad \forall t < v_0 \\ &= \Pr\{X^{(1)} < x^*(r;\psi)\} \quad \forall t \in [v_0, r) \\ &= \Pr\{X^{(1)} \leq \eta^r(t;\psi)\} \quad \forall t \in [r, +\infty) \end{aligned}$$

where $X^{(k)}$ denotes the k-th highest out of N signals, $\eta^r(t)$ denotes the inverse function of the equilibrium strategy $b^r(.)$ at a given bid level t, and $\psi \in \Theta \otimes \mathcal{F}$ denotes the underlying structure (where $\Theta \otimes \mathcal{F}$ is the set of primitives that satisfy *A1* and *A2*).[16] This distribution would be exactly identified from $G_{\mathbf{B}^0}$ if a mapping between the strategy under r and that with no reserve prices can be fully recovered from $G_{\mathbf{B}^0}$. For any rationalizable distribution $G_{\mathbf{B}^0}$ that satisfies conditions (i) and (ii) in *Proposition 1*, let $\Psi(G_{\mathbf{B}^0})$ denote a subset of structures in $\Theta \otimes \mathcal{F}$ that rationalizes $G_{\mathbf{B}^0}$. For notational ease, the dependence of b^0 and b^r on the structure $\psi \in \Theta \otimes \mathcal{F}$ is suppressed.

Lemma 1 *Consider a rationalizable $G_{\mathbf{B}^0}$. For all $\psi \in \Psi(G_{\mathbf{B}^0})$ in first-price auctions with any $r \in S_{RP}$ and $x \geq x^*(r;\psi)$, $b^r(x;\psi) = \delta_r(b^0(x;\psi); G_{\mathbf{B}^0})$ where δ_r solves the differential equation*

$$\delta'(b; G_{\mathbf{B}^0}) = [\xi(b; G_{\mathbf{B}^0}) - \delta(b; G_{\mathbf{B}^0})]\tilde{\Lambda}(b; G_{\mathbf{B}^0}) \tag{3}$$

for $b \geq b^0(x^*(r;\psi);\psi)$, with $\tilde{\Lambda}(u; G_{\mathbf{B}^0})$ defined as $\frac{g_{M^0|B^0}(u|u)}{G_{M^0|B^0}(u|u)}$ and the boundary condition $\delta(b^0(x^*(r;\psi);\psi); G_{\mathbf{B}^0}) = r$.

Thus δ_r can be constructed from the observed bid distribution $G_{\mathbf{B}^0}$ up to an unknown bid that a marginal bidder under r would place in an auction with no reserve prices (denoted $b^0(x^*(r))$). This is not surprising, as binding reserve prices affect bidding strategies in equilibrium only through the boundary condition $b^r(x^*(r)) = r$.[17] My construction of bounds on

[16] By definition $\Pr(R^I(r) \leq t) = 0$ for all $t < v_0$. For all $t \in [v_0, r)$, $\Pr(R^I(r) \leq t) = \Pr(R^I(r) = v_0) = \Pr(X^{(1)} < x^*(r))$. Note $b^{r\prime}(x) > 0$ for all $r \in S_{RP}$ and $x > x^*(r)$, and $b^r(x^*(r)) = r$. Hence $b^r(x)$ is invertible on $[r, +\infty)$, and for $t \geq r$, $\Pr\{R(r) \leq t\} = \Pr\{X^{(1)} < x^*(r)\} + \Pr\{X^{(1)} \in [x^*(r), (b^r)^{-1}(t))\}$ $= \Pr(X^{(1)} \leq \eta^r(t))$ for all $t \in [r, +\infty)$.

[17] To see this, note b^r and b^0 are solutions to the differential equation:

$$b'(x) = [v_h(x, x; \theta, F_{\mathbf{X}}) - b(x)]\frac{f_{Y_N|X}(x|x)}{F_{Y_N|X}(x|x)}$$

with different boundary conditions $b(x^*(r)) = r$ and $b(x_L) = v_h(x_L, x_L)$ respectively.

$F_{R^I(r)}$ follows three intuitive steps. First, derive tight bounds on $b^0(x^*(r))$ from $G_{\mathbf{B}^0}$ under the general restriction of affiliated values and signals, and then substitute the bounds for the unknown marginal bid in the boundary condition of (3) to get envelops on δ_r. Next, for any given level of counterfactual revenue t, invert these envelops to get a possible range of $b^0(\eta^r(t))$ (the hypothetical bid that a bidder who bids t under reserve price r would place when there is no reserve price). Finally, use the distribution of winning bids in auctions with no reserve prices to construct bounds on $F_{R^I(r)}(t)$.

2.4.2 Bounds on counterfactual screening levels

I start by deriving the range of possible screening levels under a given binding reserve price r. Let $v(x,y) \equiv E(V_i|X_i = x, Y_i \leq y)$, and $v_l(x,y) \equiv E(v_h(Y,Y)|X_i = x, Y_i \leq y) = \int_{x_L}^{y} v_h(s,s)\frac{f_{Y|X}(s|x)}{F_{Y|X}(y|x)}ds$. In symmetric Bayesian Nash equilibria, $v(x,x)$ is the winner's expected value if his signal is x (which is the same in both 1st-price and 2nd-price auctions), and $v_l(x,x)$ is the winner's expected payment in 2nd-price auctions with no reserve prices. For all $x \in S_X$ and structures $\psi \in \Theta \otimes \mathcal{F}$, affiliation between signals and values implies $v_h(x,x;\psi) \geq v(x,x;\psi)$, and the equilibrium condition in 2nd-price auctions guarantees $v(x,x;\psi) \geq v_l(x,x;\psi)$. Let $x_l(r;\psi) \equiv \arg\min_{x\in S_X}[v_h(x,x;\psi) - r]^2$ and $x_h(r;\psi) \equiv \arg\min_{x\in S_X}[v_l(x,x;\psi) - r]^2$.

Lemma 2 *(i) For all $\psi \in \Theta\otimes\mathcal{F}$, $x^*(r;\psi) \in [x_l(r;\psi), x_h(r;\psi)]$ for all $r \in S_{RP}$; (ii) For any rationalizable bid distribution $G_{\mathbf{B}^0}$, $\exists\psi \in \Psi(G_{\mathbf{B}^0})$ such that $x_l(r;\psi) = x^*(r;\psi)$ for all $r \in S_{RP}$. Furthermore, for all $\varepsilon > 0$, $\exists\psi \in \Psi(G_{\mathbf{B}^0})$ such that $\sup_{r\in S_{RP}} |x_h(r;\psi) - x^*(r;\psi)| \leq \varepsilon$.*

The bounds on the screening level are robust as they are constructed in a general environment where no restriction is placed on the form of value interdependence and signal affiliations. In other words, the screening level can never fall outside the bounds, provided bidders' values and private signals are affiliated. The true screening level coincides with the upper bound if the winner finds his rivals' signals uninformative about his own value. This includes PV auctions as special cases. On the other hand, it hits the lower bound if the margin between a winner's own signal and the highest competing signal reveals no additional information about his own value. For better intuition behind the bounds, consider special cases where a bidder's value function is additively separable between his own signal and the vector of rival signals. Then the lower and upper bounds on the screening level correspond to extreme cases of weights (1 and 0 respectively) that a bidder assigns to his own signals while calculating the expected value conditional on winning.

Lemma 2 illustrates how the indeterminacy of underlying model structures ψ leads to a possible range of screening levels. However, it does not bound on the marginal bid directly, as both $\{x_k(r;\psi)\}_{k=l,h}$ and the bidding strategy $b^0(.;\psi)$ depend upon the unknown structure ψ. Lemma 3 below proposes a way to bound the marginal bid by relating winners' expected payment in 2nd-price auctions, as well as the equilibrium strategies, to observable bid distribution $G_{\mathbf{B}^0}$. Let S_{B^0} denote the support of equilibrium bids in 1st-price auctions with no reserve prices. (That is, $S_{B^0} \equiv [b_L^0, b_U^0]$ where $b_k^0 \equiv b^0(x_k)$ for $k = L, U$.) Let $\xi_l(b; G_{\mathbf{B}^0}) \equiv \int_{b^0(x_L)}^{b} \xi(s; G_{\mathbf{B}^0}) \frac{g_{M^0|B^0}(s|b)}{G_{M^0|B^0}(b|b)} ds$ for $b \geq b_L^0$. For $r \in S_{RP}$, define

$$
\begin{aligned}
b_{l,r}^0(G_{\mathbf{B}^0}) &\equiv \arg\min\nolimits_{b \in S_{B^0}} [\xi(b; G_{\mathbf{B}^0}) - r]^2 \\
b_{h,r}^0(G_{\mathbf{B}^0}) &\equiv \arg\min\nolimits_{b \in S_{B^0}} [\xi_l(b; G_{\mathbf{B}^0}) - r]^2
\end{aligned}
$$

Lemma 3 *Consider any rationalizable bid distribution $G_{\mathbf{B}^0}$. Then (i) for all $r \in S_{RP}$ and all $\psi \in \Psi(G_{\mathbf{B}^0})$, $b^0(x^*(r;\psi);\psi) \in [b_{l,r}^0(G_{\mathbf{B}^0}), b_{h,r}^0(G_{\mathbf{B}^0})]$; (ii) for all $r \in S_{RP}$ and $b \in [b_{l,r}^0(G_{\mathbf{B}^0}), b_{h,r}^0(G_{\mathbf{B}^0}))$, $\exists \psi \in \Psi(G_{\mathbf{B}^0})$ such that $b^0(x^*(r;\psi);\psi) = b$.*

To understand these results, notice that ξ and ξ_l can relate observed bid distributions in equilibria to functionals of the underlying structures $v_h(.;\psi)$ and $v_l(.;\psi)$ through the first-order conditions. More importantly, by construction, they only depend upon model primitives $\{\theta, F_{\mathbf{X}}\}$ through the observable $G_{\mathbf{B}^0}$ generated in equilibria. Therefore, these links between $v_h(.;\psi)$, $v_l(.;\psi)$ and $G_{\mathbf{B}^0}$ are invariant over the identified set of structures (i.e. for all $\psi \in \Psi(G_{\mathbf{B}^0})$). Then inverting these functions at a given binding counterfactual reserve price r gives bounds on the marginal bid. The bounds are tight and sharp in the sense that each point between the bounds depicts a marginal bid corresponding to a certain structure within the identified set (i.e. the set of observationally equivalent structures). In other words, this range of possible marginal bids have exhausted all information that can be extracted from the symmetric and affiliated properties of the values and signals, for it is impossible to reduce the distance between the bounds in the absence of additional restrictions on θ and $F_{\mathbf{X}}$.

2.4.3 Bounding the mapping between strategies and $F_{R^I(r)}$

Recall δ_r solves (3) with an unidentified boundary condition $\delta_r(b^0(x^*(r;\psi);\psi); G_{\mathbf{B}^0}) = r$. By replacing the unknown marginal bid in the boundary conditions with its bounds $\{b_{k,r}^0(G_{\mathbf{B}^0})\}_{k=l,h}$, we can derive two solutions $\{\delta_{r,k}(.; G_{\mathbf{B}^0})\}_{k=l,h}$ that are envelops on the mapping between strategies b^0 and b^r (only for bidders above the screening level under r). Again this is due to the fact that reserve prices r (and therefore the marginal bid) enters bidders' strategies only through boundary conditions. For any revenue level t considered in a counterfactual 1st-price auction with a binding reserve price r, tight bounds on the hypothetical bid $b^0\left(\eta^r\left(t;\psi\right);\psi\right)$ (that a bidder who would bid t under r actually bids in auctions with no reserve prices) can be derived by inverting the envelops $\delta_{r,k}$ at t.

Lemma 4 *Consider a rationalizable distribution $G_{\mathbf{B}^0}$. For $k \in \{l,h\}$ and $r \in S_{RP}$, let $\{\delta_{r,k}(.;G_{\mathbf{B}^0})\}_{k=l,h}$ denote solutions to the differential equation (3) with boundary conditions $\delta_{r,k}(b^0_{k,r}(G_{\mathbf{B}^0});G_{\mathbf{B}^0}) = r$ for $b \geq b^0_{k,r}(G_{\mathbf{B}^0})$. Define*

$$\delta^{-1}_{r,k}(t;G_{\mathbf{B}^0}) \equiv \arg\min_{b\in[b^0_{k,r}(G_{\mathbf{B}^0}),b^0_U]} [\delta_{r,k}(b;G_{\mathbf{B}^0}) - t]^2.$$

Then: (i) for any $\psi \in \Psi(G_{\mathbf{B}^0})$, $\delta_{r,h}(b;G_{\mathbf{B}^0}) \leq \delta_r(b;\psi,G_{\mathbf{B}^0})$ for all $b \geq b^0_{h,r}(G_{\mathbf{B}^0})$ and $\delta_{r,l}(b;G_{\mathbf{B}^0}) \geq \delta_r(b;\psi,G_{\mathbf{B}^0})$ for all $b \geq b^0(x^*(r;\psi);\psi)$, and $\delta_{r,k}(.;G_{\mathbf{B}^0})$ are increasing on $[b^0_{k,r}(G_{\mathbf{B}^0}),b^0_U]$ for $k=l,h$; (ii) for any revenue $t > r$, and all $b \in [\delta^{-1}_{r,l}(t;G^0_{\mathbf{B}}),\delta^{-1}_{r,h}(t;G^0_{\mathbf{B}}))$, $\exists\psi \in \Psi(G^0_{\mathbf{B}})$ such that $b^0(\eta^r(t;\psi);\psi) = b$.

The Lemma shows how sharp bounds on the marginal bid lead to sharp bounds on the hypothetical bid $b^0(\eta^r(t;\psi);\psi)$ for all revenue level above the counterfactual reserve price. This in turn will deliver the point-wise sharpness of bounds on revenue distributions in counterfactual auctions below. A nice property of $\{\delta_{r,k}\}_{k=l,h}$ is that $\delta_{r,l} - \delta_{r,h}$ is non-increasing in b for $b \geq b^0(x_h(r))$.[18] This implies the length of $[\delta^{-1}_{r,l}(t;G_{\mathbf{B}^0}),\delta^{-1}_{r,h}(t;G_{\mathbf{B}^0})]$ is decreasing in the revenue level t provided both $\delta_{r,l}$ and $\delta_{r,h}$ increase at a moderate rate.

Proposition 2 *Consider a rationalizable $G_{\mathbf{B}^0}$ and any $r \in S_{RP}$ with $r > v_0$. Then for all $\psi \in \Psi(G_{\mathbf{B}^0})$, $F^l_{R^I(r)}(G_{\mathbf{B}^0}) \succeq_{F.S.D.} F_{R^I(r)}(\psi) \succeq_{F.S.D.} F^u_{R^I(r)}(G_{\mathbf{B}^0})$, where $\succeq_{F.S.D.}$ denotes first-order stochastic dominance, and*

$$\begin{aligned} F^l_{R^I(r)}(t;G_{\mathbf{B}^0}) &= 0 \quad \forall t < v_0 \\ &= \Pr(b^0(X^{(1)};\psi) < b^0_{l,r}(G_{\mathbf{B}^0}) \quad \forall t \in [v_0,r) \\ &= \Pr(b^0(X^{(1)};\psi) \leq \delta^{-1}_{r,l}(t;G_{\mathbf{B}^0})) \quad \forall t \in [r,+\infty) \end{aligned}$$

[18]Proof: Note

$$\delta'_{r,l}(b;G^0_{\mathbf{B}}) - \delta'_{r,h}(b;G^0_{\mathbf{B}}) = \tilde{\Lambda}(b;G^0_{\mathbf{B}}) \left[\int_{b_0(x_l(r))}^{b_0(x_h(r))} r - \xi(\tilde{b};G^0_{\mathbf{B}})d\tilde{L}(\tilde{b}|b;G^0_{\mathbf{B}})\right] \leq 0$$

as $\xi(b;G^0_{\mathbf{B}}) \geq r$ for all $b \geq b_0(x_h(r))$ in equilibrium. The inequality is strict if $b_0(x_h(r)) > b_0(x_l(r))$.

and

$$\begin{aligned} F^u_{R^I(r)}(t;G_{\mathbf{B}^0}) &= 0 \quad \forall t<v_0 \\ &= \Pr(b^0(X^{(1)};\psi)<b^0_{h,r}(G_{\mathbf{B}^0}) \quad \forall t\in[v_0,r) \\ &= \Pr(b^0(X^{(1)};\psi)\le \delta^{-1}_{r,h}(t;G_{\mathbf{B}^0})) \quad \forall t\in[r,+\infty) \end{aligned}$$

The distribution of the highest bid is observed from auctions with no reserve prices. Therefore the bounds can be nonparametrically constructed from $G_{\mathbf{B}^0}$. Furthermore, it follows from Lemma 3 and Lemma 4 that for a given revenue level t, any point within the interval $[F^l_{R^I(r)}(t;G_{\mathbf{B}^0}),F^h_{R^I(r)}(t;G_{\mathbf{B}^0}))$ correspond to certain true distribution of counterfactual revenue $F_{R^I(r)}(t;\psi)$ for some $\psi\in\Psi\,(G_{\mathbf{B}^0})$. In other words, $\{F^k_{R^I(r)}(t;G_{\mathbf{B}^0})\}_{k=l,h}$ form point-wise tight, sharp bounds on the true revenue distributions in counterfactual first-price auctions with a binding reserve price r.

2.4.4 A simpler upper bound of $F_{R^I(r)}$

Below I propose a simpler upper bound on $F_{R^I(r)}$ (denoted $\tilde{F}^u_{R^I(r)}$) that can be constructed using observed revenue distributions from auctions with no binding reserve prices (denoted $F_{R^I(0)}$) as opposed to from $G_{\mathbf{B}^0}$. Define

$$\begin{aligned} \tilde{F}^u_{R^I(r)}(t;F_{R^I(0)}) &= 0,\ \forall t<v_0 \\ &= \Pr(b^0(X^{(1)};\psi)<r),\ \forall t\in[v_0,r) \\ &= \Pr(b^0(X^{(1)};\psi)\le t),\ \forall t\ge r \end{aligned}$$

This simpler upper bound is easy to construct, and is depicted in *Graph 1* in the appendix. It coincides with $F^u_{R^I(r)}$ when private signals are independent and identically distributed. However, if there is strict affiliation among the signals, the simpler upper bound will be less efficient than $F^u_{R^I(r)}$ in the sense that $\tilde{F}^u_{R^I(r)}$ fails to rule out some of the counterfactual revenue distributions that can not be rationalized by any element in the identified set.

Proposition 3 *Consider any rationalizable distribution $G_{\mathbf{B}^0}$. Then for all $\psi\in\Psi(G_{\mathbf{B}^0})$ and $r\in S_{RP}$, $F_{R^I(r)}(\psi)\succeq_{F.S.D.}F^u_{R^I(r)}(G_{\mathbf{B}^0})\succeq_{F.S.D.}\tilde{F}^u_{R^I(r)}(F_{R^I(0)}(\psi))$. Furthermore, $F^u_{R^I(r)}(G_{\mathbf{B}^0})=\tilde{F}^u_{R^I(r)}(F_{R^I(0)}(\psi))$ if private signals are independently, identically distributed.*

The proof uses the fact that bidders with signals above the screening level $x^*(r)$ would bid higher under the counterfactual binding r than in auctions with no binding reserve prices.

Intuitively, it is not surprising that $\tilde{F}^u_{R^I(r)}$ is in general less efficient than $F^u_{R^I(r)}$, since the latter uses full information from $G_{\mathbf{B}^0}$ while the former only uses $G_{\mathbf{B}^0}$ indirectly through a functional of $F_{R^I(0)}$. Nonetheless, this simpler upper bound is still interesting for two reasons. First, the i.i.d. restrictions on private signals is testable in symmetric equilibria using the bid distribution observed. Hence in practice when signals are tested to be i.i.d., the simpler upper bound are known to be efficient. Second, when signals are not i.i.d., comparing $F^u_{R^I(r)}(G_{\mathbf{B}^0})$ and $\tilde{F}^u_{R^I(r)}$ illustrates how the affiliation of private signals helps narrow down the scope of possible counterfactual revenue distributions corresponding to the identified set.

2.5 Bounding revenue distributions in counterfactual 2nd-price auctions

This subsection construct bounds on counterfactual revenue distributions in 2nd-price auctions under reserve price r (denoted $F_{R^{II}(r)}$) from $G_{\mathbf{B}^0}$. Theory predicts for any given reserve price r, the expected revenues in 2nd-price auctions are at least as high as those in 1st-price auctions provided bidder signals are affiliated. However, the size of this difference is an open empirical question. In addition, within the format of 2nd-price auctions, theory is silent about the choice of optimal reserve price r that maximizes expected revenue when signals are affiliated. Knowledge of $F_{R^{II}(r)}$ would help address these open questions.

The equilibrium strategy in 2nd-price auctions under a binding reserve price r is

$$\begin{aligned}
\beta^r(x;\psi) &= v_h(x,x;\psi)\ \forall x \geq x^*(r;\psi) \\
\beta^r(x;\psi) &< r\ \forall x < x^*(r;\psi)
\end{aligned}$$

Consider any structure $\psi \in \Theta \otimes \mathcal{F}$. For all $r \in S_{RP}$ and $r > v_0$, the distribution of revenues in a second-price auction with reserve price r is:[19] (for notational ease, dependence of v_h and $x^*(r)$ on the structure ψ is suppressed.)

$$\begin{aligned}
F_{R^{II}(r)}(t;\psi) &= 0 \quad \forall t < v_0 \\
&= \Pr(X^{(1)} < x^*(r)) \quad \forall t \in [v_0, r) \\
&= \Pr(X^{(2)} < x^*(r)) \quad \forall t \in [r, v_h(x^*(r), x^*(r))) \\
&= \Pr(v_h(X^{(2)}, X^{(2)}) \leq t) \quad \forall t \in [v_h(x^*(r), x^*(r)), +\infty)
\end{aligned}$$

The link between $G_{\mathbf{B}^0}$ and revenue distributions in counterfactual 2nd-price auctions is easier to see than in 1st-price auctions, since the distribution of $b^0(X^{(1)};\psi)$ and $b^0(X^{(2)};\psi)$

[19] See the proof of Proposition 3 below for details.

are both observed. Besides, the bids in 2nd-price auctions with a counterfactual reserve price r can be exactly recovered *for bidders that are known to be unscreened under* r. This is because in Bayesian Nash equilibrium, $v_h(X,X;\psi) = \xi\left(b^0(X;\psi); G_{\mathbf{B}^0}\right)$ for all $\psi \in \Psi\left(G_{\mathbf{B}^0}\right)$. However, the non-identification of the marginal bid $b^0(x^*(r))$, and therefore the expected value for the pivotal winner $v_h\left(x^*(r), x^*(r)\right)$, makes it impossible for researchers to fully recover $F_{R^{II}(r)}$. Fortunately, just as in the case of 1st-price auctions, replacing the marginal bid with $\{b^0_{k,r}\left(G_{\mathbf{B}^0}\right)\}_{k=l,h}$ in the definition of $F_{R^{II}(r)}$ leads to point-wise bounds on $F_{R^{II}(r)}$.

Proposition 4 *Consider a rationalizable distribution $G_{\mathbf{B}^0}$, and any $r \in S_{RP}$ with $r > v_0$. Then $F^l_{R^{II}(r)}(t; G_{\mathbf{B}^0}) \succeq_{F.S.D.} F_{R^{II}(r)}(t;\psi) \succeq_{F.S.D.} F^u_{R^{II}(r)}(t; G_{\mathbf{B}^0})$ for all $\psi \in \Psi\left(G_{\mathbf{B}^0}\right)$, where*

$$\begin{aligned}
F^l_{R^{II}(r)}(t; G_{\mathbf{B}^0}) &= 0 \quad \forall t < v_0 \\
&= \Pr(b^0(X^{(1)};\psi) < b^0_{l,r}(G_{\mathbf{B}^0})) \quad \forall t \in [v_0, r) \\
&= \Pr(\xi(b^0(X^{(2)};\psi); G_{\mathbf{B}^0}) \le t) \quad \forall t \in [r, +\infty)
\end{aligned}$$

and

$$\begin{aligned}
F^u_{R^{II}(r)}(t; G_{\mathbf{B}^0}) &= 0 \quad \forall t < v_0 \\
&= \Pr(b^0(X^{(1)};\psi) < b^0_{h,r}(G_{\mathbf{B}^0})) \quad \forall t \in [v_0, r) \\
&= \Pr(b^0(X^{(2)};\psi) < b^0_{h,r}(G_{\mathbf{B}^0})) \quad \forall t \in [r, \xi(b^0_{h,r}(G_{\mathbf{B}^0}); G_{\mathbf{B}^0})) \\
&= \Pr(\xi(b^0(X^{(2)};\psi); G_{\mathbf{B}^0}) \le t) \quad \forall t \in [\xi(b^0_{h,r}(G_{\mathbf{B}^0}); G_{\mathbf{B}^0}), +\infty)
\end{aligned}$$

The intuition of the proof is demonstrated in *Graph 2*. To understand this proposition, note by the definition of the identified set of structures, the highest and second-highest order statistics of equilibrium bids must be invariant among all $\psi \in \Psi\left(G_{\mathbf{B}^0}\right)$. Hence the bounds $\{F^k_{R^{II}(r)}\}_{k=l,u}$ are functionals of the observed bid distribution $G_{\mathbf{B}^0}$ only. In addition, just as with 1st-price auctions, it follows from the sharpness of $\{b^0_{k,r}(G_{\mathbf{B}^0})\}_{k=l,h}$ that $\{F^k_{R^I(r)}(t; G_{\mathbf{B}^0})\}_{k=l,h}$ form point-wise tight, sharp bounds on the revenue distributions in counterfactual 2nd-price auctions with a binding reserve price r.

3 Nonparametric Estimation of Bounds

In this section, I define three-step estimators $\{\hat{F}^k_{R^I(r)}\}_{k=l,u}$ for bounds on $F_{R^I(r)}$ and $F_{R^{II}(r)}$. The basic idea is to replace $G_{\mathbf{B}^0}$ with its sample analog in estimation. I consider the case where data reports all bids submitted in L_n independent, homogenous auctions, each with n potential bidders and no reserve prices.[20]

Let $S_B \equiv [b_L^0, b_U^0]$ denote the support of bids observed in 1st-price auctions with non-binding reserve prices. For all $(m,b) \in S_B^2$, define:

$$\hat{G}_{M,B}(m,b) = \frac{1}{L_n h_G} \sum_{l=1}^{L_n} \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}(m_{il} \leq m) K_G(\frac{b - b_{il}}{h_G})$$
$$\hat{g}_{M,B}(m,b) = \frac{1}{L_n h_g^2} \sum_{l=1}^{L_n} \frac{1}{n} \sum_{i=1}^{n} K_g(\frac{m - m_{il}}{h_g}, \frac{b - b_{il}}{h_g})$$

where b_{il} and m_{il} are bidder i's bid and the highest competing bid against him in auction l, L_n is the total number of auctions with n potential bidders, K_G and K_g are symmetric kernel functions with bounded hypercube supports with each side equal to 2, and h_g, h_G are the corresponding bandwidths. It is well known that density estimators are asymptotically biased near boundaries of the support for some $b \in [b_L^0, b_L^0 + h_g) \cup (b_U^0 - h_g, b_U^0]$. Let $\delta \equiv \max(h_g, h_G)$ and $S_{B,\delta} = [b_L^0 + \delta, b_U^0 - \delta]$ be an expanding subset of S_B (as sample size increases) where $\hat{G}_{M,B}$ and $\hat{g}_{M,B}$ are asymptotically unbiased. A natural estimators for $S_{B,\delta}$ is:

$$\hat{S}_{B,\delta} \equiv [\tilde{b}_L, \tilde{b}_U], \text{ where } \tilde{b}_L = \hat{b}_L + \delta, \quad \tilde{b}_U = \hat{b}_U - \delta$$

where $\hat{b}_L = \min_{i,l} b_{il}$ and $\hat{b}_U = \max_{i,l} b_{il}$ converge almost surely to b_L^0 and b_U^0 respectively. Nonparametric estimators for ξ and ξ_l are defined as:

$$\hat{\xi}(b) = b + \frac{\hat{G}_{M,B}(b,b)}{\hat{g}_{M,B}(b,b)}, \quad \tilde{G}_{M,B}(b,b) = \int_{\tilde{b}_L}^{b} \hat{g}_{M,B}(t,b)dt + \hat{G}_{M,B}(\tilde{b}_L, b)$$
$$\hat{\xi}_l(b) = \hat{\xi}(\tilde{b}_L)\frac{\hat{G}_{M,B}(\tilde{b}_L, b)}{\tilde{G}_{M,B}(b,b)} + \int_{\tilde{b}_L}^{b} \hat{\xi}(t)\frac{\hat{g}_{M,B}(t,b)}{\tilde{G}_{M,B}(b,b)}dt$$

where $\tilde{G}_{M,B}$ and $\hat{\xi}_l$ are defined over the random support $\hat{S}_{B,\delta}^2$ and $\hat{S}_{B,\delta}$ respectively. The first-step estimators for $\{b_{k,r}^0(G_{\mathbf{B}^0})\}_{k=l,h}$ are defined as:

$$\hat{b}_{l,r}^0 = \arg\min_{b \in \hat{S}_{\delta,B}} [\hat{\xi}(b) - r]^2, \quad \hat{b}_{h,r}^0 = \arg\min_{b \in \hat{S}_{\delta,B}} [\hat{\xi}_l(b) - r]^2$$

In the second step, I first construct kernel estimator for $\delta_{r,l}(b)$ and $\delta_{r,h}(b)$ on $\hat{S}_{\delta,B}$ using first-step estimates $\hat{b}_{l,r}^0$ and $\hat{b}_{h,r}^0$. For $k = \{l,h\}$, define:

$$\begin{aligned} \hat{\delta}_{r,k}(b; \hat{b}_{k,r}^0) &\equiv r\hat{L}(\hat{b}_{k,r}^0|b) + \int_{\hat{b}_{k,r}^0}^{b} \hat{\xi}(t)\hat{\Lambda}(t)\hat{L}(t|b)dt \quad \forall b \in (\hat{b}_{k,r}^0, \tilde{b}_U] \\ &\equiv r \quad \forall b \in [\tilde{b}_L, \hat{b}_{k,r}^0] \end{aligned}$$

[20] "Independence" here has both economic and statistical interpretations. First, there is no strategic interaction or learning across the auctions, so that the same first-order condition characterizes equilibria in all auctions. Second, the random vectors of bidders' private information are independent across auctions. "Homogeneity" means the auctioned object in all auctions have the same commonly observed characteristics.

where $\hat{\Lambda}(t) \equiv \hat{g}_{M,B}(t,t)/\hat{G}_{M,B}(t,t)$ and $\hat{L}(t|b) \equiv \exp(-\int_t^b \hat{\Lambda}(s)ds)$. Estimators for $\delta_{r,k}^{-1}(t; G_{\mathbf{B}^0})$ are:

$$\hat{\delta}_{r,l}^{-1}(t) = \arg\min_{b \in \hat{S}_{B,\delta}}[\hat{\delta}_{r,l}(b) - t]^2, \quad \hat{\delta}_{r,h}^{-1}(t) = \arg\min_{b \in \hat{S}_{B,\delta}}[\hat{\delta}_{r,h}(b) - t]^2$$

By definition, $\hat{S}_{\delta,B} \subseteq S_{\delta,B}$ with probability 1. In the final step, bounds on $F_{R^I(r)}$ are estimated as:

$$\hat{F}_{R^I(r)}^l(t) = \frac{1}{L_n}\sum\nolimits_{l=1}^{L_n} 1(B_l^{\max} \leq \hat{\delta}_{r,l}^{-1}(t)), \quad \hat{F}_{R^I(r)}^u(t) = \frac{1}{L_n}\sum\nolimits_{l=1}^{L_n} 1(B_l^{\max} \leq \hat{\delta}_{r,h}^{-1}(t))$$

where $B_l^{\max} = \max_{i=1,..,n} b_{il}$ is the highest bid in auction l.

The three-step estimators above converge in probability to the true bounds $F_{R(r)}^k(t; G_{\mathbf{B}^0})$ over all r and $t > r$. Below I strengthen restrictions in *A1-3* to include all regularity conditions needed for consistency.

S1 For $n \geq 2$, (i) The n-dimensional vectors of private signals $(x_{1l}, x_{2l}, ..x_{nl})_{l=1}^{L_n}$ are independent, identical draws from the joint distribution $F(x_1, .., x_n)$, which is exchangeable in all n arguments and affiliated with support $S_X \equiv [x_L, x_U]$; (ii) $F(x_1, .., x_n)$ has $R+n$, $R \geq 2$, continuous bounded partial derivatives on S_X^n, with density $f(\mathbf{x}) \geq c_f > 0$ for all $\mathbf{x} \in S_X^n$.

S2 (i) The value function $\theta_n(.) : S_X^n \to \mathbb{R}_+$ is positive, bounded, and continuous on the support; (ii) $\theta_n(.)$ is exchangeable in rival signals $\mathbf{X}_{-i}$, non-decreasing in all signals, and increasing in own signal X_i over S_X. (iii) $\theta_n(.)$ is at least R times continuously differentiable and $\theta(x_L) > 0$; (iv) $v_h(x_U, x_U) < \infty$ and $\frac{d}{dX}v_h(X, X)|_{X=x_U} < \infty$.

In addition to maintaining the identifying restrictions of affiliation and symmetry among signals and values in *A1-3*, the stronger version *S1-2* also includes additional regularity conditions on the smoothness of model primitives f and θ. This will lead to smooth properties of bid distributions in equilibrium, which in turn, determines asymptotic properties of non-parametric estimators.

S3 (i) The kernels $K_G(.)$ and $K_g(.)$ are symmetric with bounded hypercube supports of sides equal to 2, and continuous bounded first derivatives; (ii) $\int K_G(b) = 1$, and $\int K_g(\tilde{B}, b)d\tilde{B}db = 1$; (iii) $K_G(.)$ and $K_g(.)$ are both of order $R+n-2$.

These are standard assumptions on kernels necessary for proving the asymptotic properties of kernel estimators.

Proposition 5 *Let $h_G = c_G(\log L/L)^{1/(2R+2n-5)}$ and $h_g = c_g(\log L/L)^{1/(2R+2n-4)}$, where c_G*

and cg are constants. Suppose S1-3 are satisfied and $R > 2n-1$, then for all $r \in S_{RP}$ and $t \geq r$, $\hat{F}^k_{R^I(r)}(t) \xrightarrow{p} F^k_{R^I(r)}(t)$ for $k = l, u$.

The proof is included in Appendix B, and proceeds in several steps. First, I prove smoothness of bid distributions in equilibrium. Second, I show the kernel estimators $\hat{\xi}_l$ and $\hat{\xi}$ converge in probability to ξ_l and ξ uniformly over $\hat{S}_{B,\delta}$. Then I use a version of the Basic Consistency Theorem in Newey and McFadden (1994) that is generalized for extreme estimators with the objective functions being defined on random support) to show $\hat{b}^0_{k,r} \xrightarrow{p} b^0_{k,r}$ for $k = l, h$ and the relevant reserve prices. Next, I prove $\hat{\delta}_{k,r}(.; \hat{b}^0_{k,r}) \xrightarrow{p} \delta_{k,r}(.; b^0_{k,r})$ uniformly over $\hat{S}_{B,\delta}$ and again used the generalized Basic Consistency Theorem to show that $\hat{\delta}^{-1}_{r,k}(t) \xrightarrow{p} \delta^{-1}_{r,k}(t)$ for all relevant t. Finally, I use the Glivenko-Cantelli uniform law of large numbers to show empirical distributions of $B^{\max}_l$ evaluated at $\hat{\delta}^{-1}_{r,k}(t)$ for $k = l, h$ are consistent estimators for bounds on $F_{R^I(r)}(t)$.

Estimating bounds on revenue distributions in counterfactual 2nd-price auctions follows similar logic and is straightforward given the constructions above. Define:

$$\begin{aligned}
\hat{F}^l_{R^{II}(r)}(t) &= \frac{1}{L_n}\sum\nolimits_{l=1}^{L_n} 1(B^{(1:n)}_l < \hat{b}^0_{l,r}) \qquad \forall t \in [v_0, r) \\
&= \frac{1}{L_n}\sum\nolimits_{l=1}^{L_n} 1(B^{(2:n)}_l < \hat{\xi}^{-1}(t)) \qquad \forall t \in [r, +\infty)
\end{aligned}$$

and:

$$\begin{aligned}
\hat{F}^u_{R^{II}(r)}(t) &= \frac{1}{L_n}\sum\nolimits_{l=1}^{L_n} 1(B^{(1:n)}_l < \hat{b}^0_{h,r}) \qquad \forall t \in [v_0, r) \\
&= \frac{1}{L_n}\sum\nolimits_{l=1}^{L_n} 1(B^{(2:n)}_l < \hat{b}^0_{h,r}) \qquad \forall t \in [r, \hat{\xi}(\hat{b}^0_{h,r})) \\
&= \frac{1}{L_n}\sum\nolimits_{l=1}^{L_n} 1(B^{(2:n)}_l < \hat{\xi}^{-1}(t)) \qquad \forall t \in [\hat{\xi}(\hat{b}^0_{h,r}), +\infty)
\end{aligned}$$

where $\hat{b}^0_{k,r}$ is defined above and $\hat{\xi}^{-1}(t) \equiv \arg\min_{b \in \hat{S}_{B,\delta}} [\hat{\xi}(b) - t]^2$ for $t \geq r$. Pointwise consistency of $\hat{F}^k_{R^{II}(r)}(t)$ for $r \geq v(x_L, x_L)$ and $t \geq r$ follows directly from similar arguments for consistency of $\hat{F}^k_{R^I(r)}(t)$, and the fact that $\hat{\xi}(\hat{b}^0_{h,r}) \xrightarrow{p} \xi(b^0_{h,r}; G_{\mathbf{B}^0}) = v_h(x_h(r), x_h(r); \psi)$ for all $\psi \in \Psi(G_{\mathbf{B}^0})$.

4 Monte Carlo Experiments

Bounds on revenue distributions in counterfactual first-price and second-price auctions are efficient in that they have exhausted all information that can be extracted from $G_{\mathbf{B}^0}$. It is impossible to derive a tighter range of possible counterfactual revenue distributions without introducing further restrictions on how values and signals are related. Exactly how informative these bounds can be is determined by the unidentified underlying primitives. In this section, I report analytical as well as Monte Carlo evidence on the performance of our bound estimators in finite samples. The objective is to illustrate how the widths of estimated bounds vary with structural parameters such as the affiliation between private signals, the number of potential bidders n and the reserve price r.

4.1 Analytical impacts of signal affiliations

I start with an example where the impact of signal affiliations on how bounds on the all-screening probabilities (the probability that at no bidder bids above the reserve price in counterfactual formats) can be studied analytically. I use a parametric design where signal affiliations can be controlled.

Design 1 *($n = 2$ with pure common values and affiliated signals) Two potential bidders compete in an auction with $V_i = (X_1+X_2)/2$ for $i = 1, 2$. Private signals are noisy estimates of a common random variable, i.e. $X_i = X_0 + \varepsilon_i$ for $i = 1, 2$. For either bidder, his noise ε_i is independent from (X_0, ε_{-i}), and distributed uniformly on $[-c, c]$ for some $0 \le c \le 0.5$. The common random term X_0 is distributed uniformly on $[c, 1-c]$.*

The signals have well-defined marginal densities with a simple form on $[0, 1]$. For example, for $c = 0.25$, the density function is $f(x) = 4x$ for $0 \le x \le 0.5$ and $4 - 4x$ for $0.5 \le x \le 1$. (See Simon (2000) for details.) And their correlation coefficient is:

$$corr(X_1, X_2) = \frac{var(X_0)}{var(X_0) + var(\varepsilon_i)} = \frac{(1-2c)^2}{(1-2c)^2 + 4c^2}$$

By definition, $v_h(x, x) = x$, $v_l(x, x) = E[X_2|X_2 \le x, X_1 = x]$, and $v(x, x) = \frac{x+v_l(x,x)}{2}$. In this design, $v_l(x, x)$ has a closed form, and the impacts of signal correlations on the widths of bounds on the all-screening probability can be studied analytically. The derivation of the closed form of $v_l(x, x; c)$ is included in the appendix.

Figure 1(a) plots $v_l(x, x; c)$ and $v_h(x, x; c)$ for $c = [0.1\ 0.2\ 0.3\ 0.4]$. The distance between v_h and v_l is non-decreasing in private signals, as $v_l(x, x)$ is a truncated expectation and cannot increase faster than the threshold x itself. *Figure 1(b)* plots the boundwidth $x_h(r; c) - x_l(r; c)$ as a function of reserve prices for each c. For any given reserve price, bounds on screening levels are narrower as c decreases and correlation increases. Intuitively, this is due to the fact that conditional on winning with a pivotal bid, the difference between the winner's expected value and his expected payment in a 2nd-price auction decreases as the signals are increasingly positively correlated. When $c = 0.1$ and $c = 0.2$, the boundwidths are invariant for r high enough. For different signal correlations, *Figure 1(c)* plots the size of bounds on all-screening probabilities in 1st-price auctions as functions of counterfactual reserve prices considered. That is, $F_{X^{(1:2)}}(x_h(r;c);c) - F_{X^{(1:2)}}(x_l(r;c);c)$, where $X^{(1:2)}$ is the higher of two private signals. As the reserve price increases, the size of the bounds are unambiguously smaller when signals have higher positive correlations. This is explained by the pattern in *Figure 1(b)* and the distribution of $X^{(1:2)}$ as plotted in *Figure 1(e).* Note the probability mass of $X^{(1:2)}$ is more skewed to the left when signals are less positively correlated. For a low reserve price r, both $x_l(r;c)$ and $x_h(r;c)$ are small and the bounds on the screening level under r are very close in size for all c. On the other hand, as *Figure 1(e)* shows, $X^{(1:2)}$ has more probability mass close to 0 for higher positive signal correlations. Hence for r that are low enough, the size of bounds on the all-screening probabilities in 1st-price auctions is bigger for $c = 0.1$. For higher reserve prices r considered, the widths of bounds on the screening level is greater for higher c (and smaller positive correlations). Besides, the probability mass of $X^{(1:2)}$ is greater in the relevant range as signals are less positively correlated. Therefore, the size of bounds on all-screening probabilities in 1st-price auctions under a higher r are much bigger for auctions with less correlated private signals.

Figure 1(d) plots the widths of bounds on all-screening probabilities in counterfactual 2nd-price auctions as functions of r considered. That is, $F_{X^{(2:2)}}(x_h(r;c);c) - F_{X^{(2:2)}}(x_l(r;c);c)$ for various $c \in [0.1, 0.5]$. In this case, the boundwidths associated with a smaller c is almost unambiguously smaller than those with higher c (and smaller correlations). Likewise, the pattern is explained by arguments as demonstrated in *Figure 1(b)* and the distribution of $X^{(2:2)}$ plotted in *Figure 1(f).* An obvious departure from the case of 1st-price auctions is that, for auctions with less correlated private signals, the widths of bounds on all-screening probabilities increase faster as r increases, reaches their peaks around $r \in [0.3, 0.4]$ and then decreases faster than in the case of 1st-price auctions. As *Figure 1(f)* suggests, this is explained by the fact that when c is higher, there is more probability mass of $X^{(2:2)}$ around the center of the support, and the tail of the distribution diminishes faster.

4.2 Performance of $\hat{F}^k_{R^I(r)}$ under i.i.d. signals

This subsection focuses on the performance of three-step estimators $\hat{F}^k_{R^I(r)}$ when private signals are identically and independently distributed. The i.i.d. restriction can be tested in empirical analysis by checking the identity of marginal bid distributions and the independence of their joint distribution. Also the i.i.d. assumption helps simplify the estimation procedures. In this subsection, I vary n, r and distributional parameters and study their impacts on estimator performances.

Design 2 *($n \geq 3$ with PCV and i.i.d. uniform signals) Private signals* $\{X\}_{i=1,..,n}$ *are identically, independently distributed as uniform on* $[0,1]$. *The pure common value is* $V_i = \sum_{j=1}^{n} X_j/n$.

Design 3 *($n \geq 3$ with PCV and i.i.d. truncated normal signals) Private signals* $\{X_i\}_{i=1,..,n}$ *are identically, independently distributed as truncated normal on* $[0,1]$ *with underlying parameters* (μ, σ^2). *The pure common value is* $V_i = \sum_{j=1}^{n} X_j/n$.[21]

The two designs fit in the general framework of symmetric, interdependent value auctions, as independence is a special case of affiliation. Note the distribution of the average of signals depends on n, and therefore the number of bidders are not exogenous to bidders values. Hence both designs do not meet necessary restrictions for tests distinguishing PV and CV auctions in Haile, Hong and Shum (2003). Thus it is appealing to adopt our robust approach of partial identification for these two designs, which does not require distinction between the two paradigms. I experiment with different numbers of potential bidders and reserve prices for *Design 2*. For each (n,r), I calculate the nonparametric estimates of $\hat{F}^k_{R^I(r)}$ from $1,000$ simulated samples, each containing equilibrium bids submitted in 500 first-price auctions. For *Design 2*,

$$b^0_n(x) = \frac{n-1}{n}(\frac{1}{n} + \frac{1}{2})x$$

and bids are simulated as random draws from a uniform distribution on $[0, \frac{n-1}{n}(\frac{1}{n} + \frac{1}{2})]$. For *Design 3*, I vary distributional parameters μ and σ in addition to n and r. For each (n, r, μ, σ), I replicate the estimator for $1,000$ times, each based on a simulated sample of 500 auctions. For *Design 3*,

$$b^0_n(x) = \int_{x_L}^{x} \frac{2}{n}s + \frac{n-2}{n}\varphi(s) d\frac{F_X^{n-1}(s)}{F_X^{n-1}(x)}$$

where $\varphi(x) = \mu - \sigma\frac{\phi(\frac{x-\mu}{\sigma}) - \phi(\frac{x_L-\mu}{\sigma})}{\Phi(\frac{x-\mu}{\sigma}) - \Phi(\frac{x_L-\mu}{\sigma})}$ and $\frac{F_X(s)}{F_X(x)} = \frac{\Phi(\frac{s-\mu}{\sigma}) - \Phi(\frac{x_L-\mu}{\sigma})}{\Phi(\frac{x-\mu}{\sigma}) - \Phi(\frac{x_L-\mu}{\sigma})}$. Equilibrium bids are simulated by first drawing $500 * n$ signals x_{il} randomly from the truncated distribution, and calculating $b_{0,n}(x_{il})$ through numerical integrations. I use the classical approach of midpoint approximations for numerical integrations for the rest of the paper. For both designs and

[21] This form of value functions introduces a restriction (normalization) on signals, as it requires support of signals to be the same as that of values.

each r, the true counterfactual revenue distribution $F_{R^I(r)}$ can be recovered by inverting $b^r(.)$, which can be calculated with knowledge of their closed forms above. In symmetric equilibria, bids under both designs are i.i.d.. This can be tested using the distribution of bids observed, and in practice simplifies our estimation as $\xi(b; G^0_{\mathbf{B}_n}) = b + \frac{1}{n-1}\frac{G^0_{B_n}(b)}{g^0_{B_n}(b)}$ and $\xi_l(b; G^0_{\mathbf{B}_n}) = b$. The simplified estimator is $\hat{\xi}(b) \equiv b + \frac{1}{n-1}\frac{\hat{G}^0_{B_n}(b)}{\hat{g}^0_{B_n}(b)}$, where $\hat{G}_{B_n}(b) = \frac{1}{L_n}\sum_{l=1}^{L_n}\frac{1}{n}\sum_{i=1}^{n} 1(b_{il} \leq b)$, $\hat{g}^0_n(b) = \frac{1}{L_n h_g}\sum_{l=1}^{L_n}\frac{1}{n}\sum_{i=1}^{n} K(\frac{b_{il}-b}{h_g})$ and L_n is the number of auctions with n bidders. For estimation, I use the tri-weight kernel $K(u) = \frac{35}{32}(1-u^2)1(|u| \leq 1)$.[22] Bandwidths h_g is $2.98 * 1.06\hat{\sigma}_b(nL_n)^{-\frac{1}{4n-4}}$, where $\hat{\sigma}_b$ is the empirical standard deviation of bids in the data. The bandwidths are chosen in line with the consistency proposition in the appendix, while the constant factor $1.06\hat{\sigma}_b$ is chosen by the "rule of thumb". (See Li, Perrigne and Vuong (2002) for an example.) The multiplicative factor 2.98 is due to the use of tri-weight kernels. (See Hardle (1991) for details.)

Figure 2 plots the true revenue distribution $F_{R^I(r)}$ in *Design 2* and, for different n and r, reports the 5th percentile of $\hat{F}^l_{R^I(r)}$ and the 95th percentile of $\hat{F}^u_{R^I(r)}$ out of $1,000$ pairs of estimates. The two percentiles form an estimate of a conservative 90% pointwise confidence interval for the bounds $[F^l_{R^I(r)}, F^u_{R^I(r)}]$. (See Haile and Tamer (2003) for an example.) The true revenue distribution always falls within the interval. The intervals for a lower r are narrower, holding n constant. On the other hand, more potential bidders correspond to tighter confidence regions ceteris paribus. To understand the pattern, note the boundwidth of the all-screening probability is $\Pr(b^0_n(X^{(1:n)}) \leq r) - \Pr(b^0_n(X^{(1:n)}) \leq \frac{n-1}{n}r) = F_{X^{(1:n)}}(\frac{n}{n-1}\frac{2n}{n+2}r) - F_{X^{(1:n)}}(\frac{2n}{n+2}r)$, which is increasing in r for a given n. For a given r, $\frac{1}{n-1}\frac{2n}{n+2}r$ decreases in n and this offsets the impacts of a rising $\frac{2n}{n+2}r$ and a more left-skewed $F_{X^{(1:n)}}$ as competition increases. The simulations suggest variations in the width of estimated confidence intervals are mostly due to impacts of n and r on the boundwidths of $F_{R^I(r)}$.

Figure 3 reports $F_{R^I(r)}$ and estimates of conservative 90% confidence intervals for *Design 3*. Again, the true revenue distribution falls within 90% point-wise conservative confidence intervals for the parameters considered. The impacts of n and r on the estimated confidence intervals in *Design 3* are the same as those for *Design 2* in *Figure 2*. In addition, *Figure 3* also shows impacts of distributional parameters μ and σ on confidence intervals. First, holding n, r and σ fixed, the confidence intervals become narrower as μ increases. This is because for all t, $E(X|X \leq t)$ gets closer to t as the distribution of X is more skewed to the left. Consequently, $x^*(r)$ decreases for a given r, while the distance between v_h and v_l also

[22] The triweight kernel is of order 2. In principle when $n \geq 3$, kernels used in $\hat{g}^0_{Bn}$ should be of higher order. But can lead to the issue of negative density estimates. Therefore empirical literature typically ignore this requirement and use kernels with order 2.

becomes smaller. As a result, the bound on the all-screening probability is shifted to the left and becomes tighter. Second, the impact of σ on confidence intervals depends on μ, holding n and r fixed. A higher standard deviation increases the width of confidence intervals for signal distributions sufficiently skewed to the left, but reduce the width of confidence intervals for signal distributions sufficiently skewed to the right. The impacts are more obvious for distributions skewed to the right. This pattern is explained by similar reasons above. Again, simulations suggest variations in the width of estimated confidence intervals are mostly due to impacts of n and r on the size of bounds on $F_{R^I(r)}$.

4.3 Performance of $\hat{F}^k_{R^I(r)}$ with affiliated signals

When signals are not independently and identically distributed, there are no simplified forms for $\hat{\xi}$ and $\hat{\xi}_l$, and the full nonparametric estimates in Section 3 apply. In this subsection I extended *Design 1* for $n \geq 3$ so that $V_i = \sum_{j=1}^n X_j/n$, and experiment with the correlation parameter c to study its impact on the performance of estimators. With $n \geq 3$, it is impractical to derive the analytical form of the inverse hazard rate $f_{Y|X,n}(u|u)/F_{Y|X,n}(u|u)$. To find out the true revenue distribution, I replace $v_h(x, x; c)$ and $L(s|x; c)$ with their kernel estimates in a simulated sample of $5 * 10^5$ auctions, and calculate the equilibrium bidding strategies using these estimates and numerical integrations. The true counterfactual revenue distribution $F_{R^I(r)}$ is then recovered with knowledge of the distribution of the highest signal $X^{(1:n)}$. For each (c, n), I simulate 200 samples, with each containing $1,000$ simulated first-price auctions. For each r and revenue level t, *Figure 4* reports the point-wise 5-th percentile of $\hat{F}^l_{R^I(r)}(t)$ and the 95-th percentile for $\hat{F}^u_{R^I(r)}(t)$ out of 200 pairs of estimates. This forms estimates for a conservative 90% confidence interval for the bounds on $F_{R^I(r)}$. *Figure 4* shows the true $F_{R^I(r)}$ lies within the estimated confidence interval for $r = 0.2$ or 0.5, $c = 0.2$ or 0.4 and $n = 3$ or 4. Holding r and c constant, the widths of the estimated confidence intervals decrease slightly as n increases. For $r = 0.2$, higher correlation leads to slightly wider confidence intervals, whereas for $r = 0.5$ higher signal correlation leads to obviously narrower confidence intervals. Smaller correlations among signals implies the distribution of $X^{(1:n)}$ is more skewed to the left, and the distance between v_l and v_h are bigger. These explain why a higher c leads to wider confidence intervals when r is high at 0.5. On the other hand, when r is low at 0.2, the left-skewness of $F_{X^{(1:n)}}$ offsets the impact of a wider bound $[x_l(r; c), x_h(r; c)]$ due to a higher c, and may lead to a narrower confidence interval. Furthermore, the theory also states for $x \geq x^*(r; c)$ the bounds on $\delta_r(b^0(x; c))$ is tighter as $b^0(x; c)$ increases. For $t > r$, this counteracts the left skewness of $F_{X^{(1:n)}}$ due to lower correlations. This is consistent with patterns in *Figure 4* where confidence intervals on $F_{R^I(r)}$ never broaden substantially as revenue level t increases.

5 Extensions

5.1 Heterogenous auctions

In practice, bids are often collected from heterogenous auctions with varied characteristics of the objects for sale. If commonly observed by all bidders, such heterogeneity affects bidders' strategies and revenue distributions in counterfactual auctions. If researchers can completely control for heterogeneity across auctions by using the observables in the data, then logic for bounds on revenue distributions in homogenous counterfactual auctions extends immediately. Specifically, auctions are homogenous within subsets of the data if such features (denoted $\mathbf{Z}$) are controlled for, and the same algorithm in the benchmark model extends to bounds on revenue distributions conditional on such characteristics $F_{R^I(r)|\mathbf{Z}=\mathbf{z}}$, which can be constructed from conditional bid distribution $G^0_{\mathbf{B}|\mathbf{Z}=\mathbf{z}}$. However, real challenges can arise from observable auction heterogeneity is empirical implementation. Constructing bounds on conditional revenue distributions requires a large cross-sectional data of homogenous auctions with fixed features $\mathbf{z}$ and a fixed number of potential bidders n. This issue of data deficiency aggravates as the dimension of $\mathbf{z}$ increases. Below I show if signals are independent from observable heterogeneities conditional on n, and are additively separable from the latter in the value functions, then it is possible to "homogenize" bids across heterogenous auctions, thus alleviating the data deficiency problem.

A1' (Interdependent Values) $V_{i,N} = h(\mathbf{Z}'\boldsymbol{\gamma}) + \theta_N(X_i, \mathbf{X}_{-i})$, where $h(.)$ is differentiable, and θ_N is bounded, continuous, exchangeable in its last $N-1$ arguments, non-decreasing in all arguments, and increasing in X_i.

A4 (Conditional Independence of $\mathbf{X}$ *and* $\mathbf{Z}$*)* Conditional on $N = n$, $\{X_i\}_{i=1,..,n}$ is independent from $\mathbf{Z}$.

Then a *PSBNE* in the auction with no binding reserve price is a profile of strategies that solve:

$$b_{0i}(x, \mathbf{z}; n) = \arg\max_b E[(V_i - b)1\{\max_{j\neq i} b_{0j}(X_j, \mathbf{Z}) \leq b\}|X_i = x, \mathbf{Z} = \mathbf{z}, N = n].$$

Under these restrictions, common knowledge of auction features impact strategies of all bidders in the same way. As the proposition below shows, the separability and the index specification of value functions are inherited by bidding strategies in equilibria.

Proposition 6 *Under A1', A2, A3 and A4, bidders' equilibrium strategies satisfy :* $b_i^0(x, \mathbf{z}; n) = h(\mathbf{z}'\boldsymbol{\gamma}) + \lambda(x; n)$ *for all* x, z *and* i*, where* $\lambda(x; n) \equiv \int_{x_L}^{x} \phi(s; n) dL(s|x; n)$*, and* $\phi(s; n) \equiv E[\theta(\mathbf{X})|X_i = Y_i = s; N = n]$.

Fix the number of potential bidders n, the proposition implies $E(b_{0i}|\mathbf{Z} = \mathbf{z}, N = n) = h(\mathbf{z}'\boldsymbol{\gamma}) + E(\lambda(X; N)|\ N = n)$, where the second term is a constant independent from $\mathbf{Z}$. This becomes a single index model, and both Powell, Stock and Stocker (1989) and Ichimura (1991) showed $\boldsymbol{\gamma}$ can be identified up to scale, and estimated consistently using average derivative estimators or semiparametric least square estimators. In the special case where $h(.)$ is known to be the identity function, an OLS regression of bids from heterogenous auctions on the characteristics $\mathbf{z}$ for a fixed n will estimate $\boldsymbol{\gamma}$ consistently. Alternatively, including dummies for the number of potential bidders in a pooled regression will also give consistent coefficient estimators for $\boldsymbol{\gamma}$.

A corollary of the proposition is that for any pair of different features of auctions $\mathbf{z}$ and $\bar{\mathbf{z}}$, the equilibrium strategies for a given signal x are related as $b_0(x, \mathbf{z}; n) = b_0(x, \bar{\mathbf{z}}; n) - h(\bar{\mathbf{z}}'\boldsymbol{\gamma}) + h(\mathbf{z}'\boldsymbol{\gamma})$. Thus when h is known, bids across heterogenous auctions can be "homogenized" at any specific reference level $\mathbf{z}$ so that more observations are available for estimating $G^0_{\mathbf{B}(\mathbf{Z})}$. Larger sample size leads to better performance of estimators of bounds on $F_{R^I(r)|\mathbf{Z}=\mathbf{z}}$.

5.2 Binding reserve prices in data

In practice, bids are often collected from homogenous auctions under a commonly known reserve price r that is high enough to have a positive probability of screening out some of the bidders. This gives rise to new challenges relative to benchmark cases where there is no binding reserve price in the data. First, bids from potential bidders that are screened out may not be observed. Second, data may only include auctions with at least one bid above r, and exclude those where everyone is screened out (i.e. $X^{(1)} < x^*(r)$). In both cases, the algorithm in our benchmark model above can not be applied immediately.

In addition, a binding reserve price r in data also reduces the scope of counterfactual reserve prices that are interesting for counterfactual analyses. To understand this, note bids below r reveal no information about underlying signals, as the link between $G^r_{\mathbf{B}}$ and structures $\psi \in \Theta \otimes \mathcal{F}$ only holds for $b^r(x;\psi) \geq r$. The data at hand cannot help address the question how those bidders who only become unscreened under a lower counterfactual price will act. Hence the logic behind the bounds in our benchmark case only applies to revenue distributions in counterfactual auctions with $r' > r$. As a result, for all $r' < r$, $x^*(r')$ is lower than $x^*(r)$ and can not be bounded in its small neighborhoods using equilibrium conditions. Throughout this subsection, I focus on the bounds for $F_{R^I(r)}$. Extensions to bound $F_{R^{II}(r)}$ is straightforward and omitted.

5.2.1 Unobserved screened bidders

Unobserved bids from bidders who are screened out matter for bounding $F_{R^I(r')}$ (where $r' > r$) only in the sense that they may make the number of potential bidders unobservable. For now assume auctions with $X^{(1)} < x^*(r)$ are also observed in the data. If the number of potential bidders is known, as is often the case in applications, then the algorithm for bounding $F_{R^I(r)}$ can be applied even if data do not contain bids from bidders that are screened out. The following lemma generalizes the equilibrium condition (2) for any rationalizable distributions under a binding reserve price r.

Proposition 7 *Consider any distribution of bids $G^r_{\mathbf{B}}$ in first-price auctions with a binding reserve price r. Then for all $\psi \in \Psi(G_{\mathbf{B}^0})$, $\xi(b^r(x);G^r_{\mathbf{B}}) = v_h(x;\psi)$ for all $x \geq x^*(r;\psi)$.*

In the presence of binding reserve prices in data, the lower bound on $v(x,x;\psi)$ can no longer be identified from $G^r_{\mathbf{B}}$, as bids lower than r can not be linked to signals through equilibrium conditions. The solution is to bound $v(x,x;\psi)$ below by expected payment of a winner in second-price auctions with a reserve price r. For $x \geq x^*(r;\psi)$ define

$$v_{l,r}(x;\psi) \equiv r\tfrac{F_{Y|X}(x^*(r)|x)}{F_{Y|X}(x|x)} + \int_{x^*(r;\psi)}^{x} v_h(s,s;\psi)\tfrac{f_{Y|X}(s|x)}{F_{Y|X}(x|x)}ds$$

Then $v_{l,r}(x;\psi)$ is increasing in x by monotonicity of the value function and affiliations between signals, and $v(x,x;\psi) \geq v_{l,r}(x;\psi)$ for $x \geq x^*(r;\psi)$ by the equilibrium condition in second-price auctions with r. (The formal proof is similar to the benchmark case and omitted.) Hence for all $r' > r$, $x^*(r';\psi)$ is bounded by $x_{h,r}(r';\psi) \equiv \arg\min_{x\in[x^*(r;\psi),x_U]}(v_{l,r}(x;\psi) - r')^2$ and $x_{l,r}(r';\psi) \equiv \arg\min_{x\in[x^*(r;\psi),x_U]}(v_h(x,x;\psi) - r')^2$. Then $v_h(x,x;\psi)$ and $v_{l,r}(x;\psi)$ are identified from $G^r_{\mathbf{B}}$ for $x \geq x^*(r;\psi)$ respectively as $\xi(b^r(x);G^r_{\mathbf{B}})$ and

$$\xi_{l,r}(b^r(x);G^r_{\mathbf{B}}) \equiv r\tfrac{G^r_{M|B}(r|b^r(x))}{G^r_{M|B}(b^r(x)|b^r(x))} + \int_{r}^{b^r(x)} \xi(\tilde{b};G^r_{\mathbf{B}})\tfrac{g^r_{M|B}(r|b^r(x))}{G^r_{M|B}(b^r(x)|b^r(x))}d\tilde{b}$$

By similar reasoning as in the benchmark case, bounds on the $\delta^r_{r'}$-mapping (which maps $b^r(x)$ into $b^r(x)$ for $x \geq x^*(r';\psi)$) are identified as

$$\delta_{r,r',k}(b^r(x);G^r_{\mathbf{B}}) = r'\tilde{L}(b^r_{k,r'}|b;G^r_{\mathbf{B}}) + \int_{b^r_{k,r'}}^{b} \xi(\tilde{b};G^r_{\mathbf{B}})d\tilde{L}(\tilde{b}|b;G^r_{\mathbf{B}})$$

where $b^r_{k,r'} \equiv b^r(x_{k,r}(r';\psi))$ for $k = l,h$, and are identified as inverses of $\xi(.;G^r_{\mathbf{B}})$ and $\xi_{l,r}(.;G^r_{\mathbf{B}})$ over $[r,b^r(x_U)]$ respectively. It can be shown that $\delta_{r,r',k}(b^r(.);G^r_{\mathbf{B}})$ is increasing for $x \geq x_{k,r}(r')$, and inverting $\delta_{r,r',k}(.;G^r_{\mathbf{B}})$ at $t \geq r'$ gives bounds on $b^r(\eta^{r'}(t))$. Thus bounds on $F_{R^I(r')}$ can be constructed from the distribution of $b^r(X^{(1)})$.

5.2.2 Unobserved screened auctions (with $X^{(1)} < x^*(r)$)

When data exclude auctions with a reserve price r that screens out all bidders (i.e. $X^{(1)} < x^*(r)$), we observe the distribution of equilibrium bids b^r conditional on more than one bidder bids above r (denoted $G^r_{\mathbf{B}|B^{(1)}\geq r}$) as opposed to the unconditional $G^r_{\mathbf{B}}$. For $b > r$, $G^r_{M|B}(b|b)$ and $g^r_{M|B}(b|b)$ can still be identified from $G^r_{\mathbf{B}|B^{(1)}>r}$, and thus bounds on $b_r(x^*(r'))$ and the $\delta_{r,r'}$-mapping can be constructed as above. However, $G^r_{\mathbf{B}|B^{(1)}\geq r}$ can only be used to construct bounds on $F_{R^I(r)|X^{(1)}\geq r}$. That is, for any rationalizable $G^r_{\mathbf{B}}$ and $\psi \in \Psi(G^r_{\mathbf{B}})$,

$$\begin{aligned} & \Pr(\xi(B^{(1)}_r;G^r_{\mathbf{B}}) < r'|B^{(1)}_r \geq r) \\ \leq \quad & \Pr(X^{(1)} < x^*(r')|X^{(1)} \geq x^*(r)) \\ \leq \quad & \Pr(\xi_{l,r}(B^{(1)}_r;G^r_{\mathbf{B}}) < r'|B^{(1)}_r \geq r) \end{aligned}$$

and for $t \geq r'$,

$$\begin{aligned} & \Pr(B^{(1)}_r \leq \delta^{-1}_{r,r',l}(t;G^r_{\mathbf{B}})|B^{(1)}_r \geq r) \\ \leq \quad & \Pr(X^{(1)} \leq \eta^{r'}(t)|X^{(1)} \geq x^*(r)) \\ \leq \quad & \Pr(B^{(1)}_r \leq \delta^{-1}_{r,r',h}(t;G^r_{\mathbf{B}})|B^{(1)}_r \geq r) \end{aligned}$$

where $B^{(1)}_r$ is shorthand for $b^r(X^{(1)})$. The probability that r screens out all bidders $\Pr(X^{(1)} < x^*(r))$ is needed to bound the unconditional distribution $F_{R^I(r)}$. It is impossible to identify this probability solely from $G^r_{\mathbf{B}|B^{(1)}\geq r}$ without further restrictions on $F_{\mathbf{X}}$. However, the lemma below shows when bidder signals are i.i.d., $\Pr(X^{(1)} < x^*(r))$ can be recovered from $G^r_{\mathbf{B}|B^{(1)}\geq r}$ alone. [23]

[23] Under *A1*, the auction model still has interdependent values even when $\{X_i\}_{i\in N}$ are independent and identically distributed.

Proposition 8 *Suppose signals* $\{X_i\}_{i=1,..N}$ *are i.i.d. in first-price auctions with* N *potential bidders and a binding reservation price* r*. If both the number of active bidders and* N *are observed, then* $\Pr(X^{(1)} < x^*(r))$ *is identified even if auctions with* $X^{(1)} < x^*(r)$ *are not observed.*

5.2.3 About the number of potential bidders

That the number of potential bidders N is observed is crucial to our discussion of data generated under binding reserve prices so far. This is not an issue in some applications where N is directly reported in the data, or where good proxies exist. In other applications, the issue can be subtle. In some cases, neither bidders nor econometricians can observe N. Then strategic decisions can be modeled as based on bidders' subjective probability distributions of N given private signals (denoted $p(N = n|X = x)$).

Bidders integrate $v_{h,N}$, $f_{Y|X,N}$ over N with respect to this distribution and make strategic decisions based on these integrated primitives, so the actual number of potential bidders becomes irrelevant in equilibria. The new equilibrium conditions can also be manipulated through change of variables to get an analog of (2) that links bid distributions observed to model primitives. One example is the off-continental shelf (OCS) auctions of oil-drilling rights studied by Hendricks, Pinkse and Porter (2003). In OCS auctions, potential bidders' decisions to submit bids take multi-stages. The authors endogenize participations by introducing multiple signals, each corresponding to a stage in the decision-making. Then only those still active in the last-stage and their signals are relevant to decisions on strategic bids. The additional restrictions in the model is that decisions to remain active till the last stage only depends on signals from previous stages, and that conditional on last-stage signals, the signals in previous stages reveal no information about bidders' values. The logic of partial identification in benchmark models can be extended in principle to bound revenue distribution in such equilibria with unobserved potential bidders.

In other applications where bidder signals are i.i.d., the number of potential bidders can be identified even if data only report the number of *actual* bidders. This is because in equilibria, the number of actual bidders is distributed as Binomial (n, p) with p equal to the screening probability $\Pr(x \geq x^*(r))$. Provided the distribution of bids and actual bidders are rationalizable,[24] both n and p are uniquely identified.

[24] See Guerre et.al (2000) for conditions for rationalizability.

6 Application: U.S. Municipal Bond Auctions

Municipal bonds are a chief means of debt-financing for U.S. state and county governments. They are issued to finance public projects such as construction or renovation of schools and public transportation facilities. Interest income from municipal bonds are exempt from federal and local taxes, and hence municipal bonds appeal to investors in high tax brackets. In 2005, the total par amount of outstanding municipal bonds was $1.8 trillion. [25]

6.1 Institutional details

Muni-bonds are identified by issuers and basic features such as coupon rates, maturity dates, and par amounts.[26] Investors valuate muni-bonds based on this information and implied risks, including credit risks, interest rate risks, and liquidity risks.[27] On the primary market, muni-bonds are issued through first-price auctions to potential underwriters (mostly investment banks). Notices of these competitive sales are posted on major industry publications such as *The Bondbuyer*. In practice, issuers usually package a series of bonds for sale in one auction, and investment banks participate by bidding a single dollar price per $100 par value for the whole series. The bidder with the highest dollar price wins the right to underwrite the entire series, and may resell the series on secondary markets with a mark-up. To decide whether and how to bid, securities firms assess the creditworthiness of the municipalities and prospects of the bonds on secondary markets. For issues with a large par amount, investment banks usually form bidding syndicates, where members share responsibilities for reselling the bonds as well as the liability for unsold bonds. A syndicate is usually clearly defined for each issuance, as underwriters traditionally stay in the group where they bid on the last occasion that the issuer came to market. As of 2006, more than 2,100 securities firms are registered with the Municipal Securities Regulatory Board and authorized as legal underwriters. However, only a small number of these firms are active bidders in competitive sales. By 1990, 25 leading underwriters managed about 75 percent of the total volume of all

[25] Source of information : SIFMA(2005)

[26] A coupon rate is the interest rate stated on the bond and payable to the bondholder on a semi-annual basis. A maturity date is the date on which the bondholder will receive par value of the bond along with its final interest payment.

[27] *Credit risk* measures how likely the issuer is to default on its payment of interests and principals. *Interest rate risk* is due to flucuations in real interest rates that affect the market value of bonds (to both speculators and long-term investors). *Liquidity risk* refers to the situation where investors have difficulty finding buyers when they want to sell, and are forced to sell at a significant discount to market value.

new long-term issues either as lone bidders or leaders of syndicates.

6.2 Bond values: private or common ?

The bounds proposed introduce an approach of partial identification for policy analyses which is applicable regardless of underlying paradigms (PV or CV). This is highly relevant in the context of muni-bond auctions, as institutional details do not suggest conclusive evidence for either paradigm. Besides, empirical methods proposed so far for differentiating the two all have limitations in practice.

The value of bonds for firms in these auctions are resale prices on secondary markets. On most occasions bidders on the primary market cannot foresee at what price they can resell the bonds, and therefore only have noisy estimates. These estimates capture the syndicates' expectation on how investors on secondary markets interpret bond features, and depend on their beliefs about the skills of their sales and trading staff. The estimates are also built on companies' perception of how investors view relevant uncertainties such as the creditworthiness of municipalities and fluctuations of future real interest rates.

The crucial question is whether a bidding syndicate can extract additional useful information about bond values if they could access competitors' estimates. The auction is one with common values if and only if the answer is positive. On some occasions, all participating firms manage to pre-sell bonds to secondary investors prior to their actual bidding. Such auctions fit in the PV paradigm, as all bidders have perfect foresight of their values. On other occasions, pre-sales are not possible or limited in scope, and firms can have heterogenous source of information about municipalities' creditworthiness, or different interpretation of factors related to bond values. Unless all firms confidently believe their own information or interpretation dominates their competitors' in accuracy, they cannot dismiss competitors' estimates as uninformative, and auctions are closer to common values.

While the informational environment per se does not justify either PV or CV conclusively, data limitations also deter empirical efforts to discriminate between them. First, there is reason to believe the number of potential bidders is correlated with bond values. Therefore the test in Haile, Hong and Shum (2003) cannot be applied, for it requires the variation in the number of bidders to be exogenous with respect to the distribution of values. Second, our data does not have ex post measures of bond values that can be used to test whether $v_h(x,x) = E(V_i|B_i = b_0(x_i), B_{-i} = b_0(x_{-i}))$ is independent from B_{-i}. Finally muni-bond auctions proceed with no announced reserve prices and therefore the testable restrictions in

Hendricks, Pinkse and Porter (2003) are not useful.

This paper focuses on an robust approach for policy analyses by bounding counterfactual revenue distributions under general restrictions that encompass both PV and CV paradigms. The lower bound point-identifies counterfactual distributions only when values are private. On the other hand, if nothing is known about the interdependence between values, then any point between the bounds can be rationalized as the true counterfactual revenue distribution by certain structures in the identified set defined by the observed bid distribution.

6.3 Data description

The data contains all bids submitted in 6,721 auctions of municipal bonds on the primary market in the United States between 2004 and 2006. They are downloaded from auction worksheets at a website of *Thompson Financial.* The data reports the identity of issuers, the sale date, the date of the first coupon, par values of each bond in a series, coupon rates of each bond, S&P and Moody's ratings of each bond, the type of government credit support for the issuance (general obligation or revenue).[28] It also records whether the issuance is bank-qualified.[29] In addition, the data includes macroeconomic variables measuring opportunity costs of investing in bonds.

There are 97,936 bonds in 6,721 series, with an average of 14.5 for each issue. About 70% of the series have 10 to 20 bonds. The average coupon rate of all bonds is 4.06% and the average number of semiannual payments is 19.6. I use the par-weighted averages of coupon rates and numbers of coupon payments as a measure of "overall" interest rates and maturity for a series. About 90% of all issuances have a weighted average coupon rate between 3% and 5%. The weighted average maturity is approximately normally distributed with a mean of 20.8 and a standard deviation of 9.5. The total par of a series ranges from $0.1 million to $809 million, and is skewed to the right with a mean of $21.4 million and a median of $6 million. About 64.5% of the series are backed by full credit of municipalities, while the rest

[28]Bonds are categorized into two groups by the degree of credit support from municipalities. General obligation bonds are endorsed by the full faith and credit of the issuer, whereas revenue bonds promise repayment from a specified stream of future income, such as that generated by the public project financed by the issue. The latter usually bears higher interest rates due to risk premium.

[29]The Tax Reform Act of 1986 eliminated the tax benefits for commercial banks from holding municipal bonds in general. But exceptions were made for "bank-qualified" bonds, for which commercial banks can still accrue interests that are tax-exempt. Hence banks have a strong appetite for bank qualified bonds that are in limited supply, and bank qualified bonds carry a lower rate than non-bank qualified bonds.

are backed by limited municipal support, such as revenue stream from public works financed by the issuance. In practice, issuers have the option to include reserve prices in the notice of sale, but few issuers use this option. For each auction, the number of bidding coalitions, the number of companies within each coalition and their identities are all reported in the data. The number of syndicates ranges from 1 to 20, with a mean of 5.6 and a standard deviation of 2.6. Series that received more than 3 but fewer than 7 bids account for 68% of all auctions.

The dollar prices tendered are not always reported. However, total interest costs for all bids are always reported.[30] I use the following formula to calculate and impute missing dollar bids :

$$B = (1+TIC)^{-t_f} * \frac{\sum_{q=1}^{Q}\left(\sum_{t=0}^{T_q-1}\frac{C_q/2}{(1+\frac{TIC}{2})^t} + \frac{P_q}{(1+\frac{TIC}{2})^{T_q}}\right)}{\sum_{q=1}^{Q} P_q} * 100$$

where q indexes bonds in a series of Q bonds, T_q is the number of semi-annual periods from the date of first coupon until maturity, C_q and P_q are coupon and principal payments respectively, t_f is the time until first coupon payment and B is the dollar bid per \$100 of face value. *Table 1* summarizes the distribution of all 37,547 bids submitted in 6,721 auctions. The 1st percentile is \$95.32 and the 99th percentile is \$109.30. The median is \$99.40, the mean is \$99.92, and the standard deviation is \$2.76. The median winning dollar bid is \$99.66, the average is \$100.01, and the standard deviation is \$2.36.

6.4 Homogenization of bids

There is a wide variation of bond features in the data. In competitive sales, syndicates take these characteristics into account when they bid, and thus strategies across auctions are not homogenous as the benchmark model posits. In principle bounds still apply to subsets of homogenous auctions where bond features are controlled. The main empirical challenge in the implementation is that constructing nonparametric bounds on conditional revenue distributions require large samples for auctions with fixed specific features. Below I tackle this issue by homogenizing bids across heterogenous auctions. The working assumptions are: *(i)* firms' estimates of bond values are independent from publicly known bond features conditional on the number of participating syndicates; *(ii)* value functions are additively separable in private signals and commonly observed bond features. Under these assumptions, marginal

[30] Total interest cost (TIC) is the interest rate that equates dollar prices with discounted present value of future cashflows the series.

effects of bond characteristics on equilibrium bids are identified. (I discuss a specification test of these restrictions below.) Thus bids in distinct auctions can be homogenized by removing differences due to variations in bond features as in Section 5.

In competitive sales with n bidding syndicates, ex ante bond values for a potential bidder is :

$$V_{il} = \mathbf{Z}_l'\boldsymbol{\gamma} + \theta_n(X_{il}, \mathbf{X}_{-il})$$

where $i = 1, .., n$ indexes the bidding syndicates, $l = 1, 2, ..L_n$ indexes auctions with n syndicates, $\mathbf{Z}_l$ is a vector of publicly known features, and $\mathbf{X}_l = (X_{il}, \mathbf{X}_{-il})$ is a $\mathbb{R}^n$-valued random vector of idiosyncratic signals. This specification reflects the intuition that marginal effects of idiosyncratic information (signals $\mathbf{X}_l$) may not interact with those of public information (bond features $\mathbf{Z}_l$). Syndicates in an auction may differ in two aspects: the number of member firms, and local presence of firms' branch offices in the issuer's state. Recent empirical works suggest there is no conclusive evidence that they can lead to informational asymmetries.[31] Hence I maintain symmetry restrictions of θ_n and $F_{\mathbf{X}}$ as in the benchmark model in Section 2.

The equilibrium strategy is:

$$b_{il}(x_{il}, \mathbf{z}_l; n) = \mathbf{z}_l'\boldsymbol{\gamma} + \lambda_l(x_{il}, n) \tag{4}$$

where $\lambda(x, n) \equiv \int_{x_L}^{x} \phi_n(s) dL_n(s|x)$, $L_n(s|x) \equiv \exp\{-\int_s^x \frac{f_{Y|X,n}(u|u)}{F_{Y|X,n}(u|u)} du\}$ and $\phi_n(s) \equiv E[\theta_N(X_i, \mathbf{X}_{-i}) \mid X_i = \max_{j \neq i} X_j = s,\ N = n]$. Thus strategic bids can be decomposed into two additive components. The first term suggests marginal effects of bond features are invariant to potential competitions, and the second term captures effects of potential competition on strategic bids. The signals and competitions interact with each other and their effects cannot be separated. Regressing bids on bond features and a vector of dummies for the number of potential bidders will estimate $\boldsymbol{\gamma}$ consistently. That is, in the pooled regression,

$$b_{il}(x_{il}, \mathbf{z}_l) = \mathbf{d}_l'\boldsymbol{\delta} + \mathbf{z}_l'\boldsymbol{\gamma} + u_{il} \tag{5}$$

where $\mathbf{d}_l$ is a vector of dummies for n, the error term u_{il} is mean independent conditional on $\mathbf{d}_l$ and $\mathbf{z}_l$.[32]

[31] See Shneyerov (2006).

[32] To see this, fix n, then *Proposition 5* shows equilibrium bids are:

$$b_{il}(n) = \gamma_0(n) + \mathbf{z}_l'\boldsymbol{\gamma} + \varepsilon_{il}(x_{il}, n)$$

where $\gamma_0(n) \equiv E[\lambda_l(X_{il}, N_l)|N_l = n]$ and $\varepsilon_{il}(x_{il}, n) \equiv \lambda_l(x_{il}, n) - \gamma_0(n)$. It follows from the independence of $\mathbf{X}_l$ and $\mathbf{Z}_l$ conditional on number of bidders that $E[\varepsilon_{il}(X_{il}, N_l)|N_l = n, \mathbf{Z}_l = \mathbf{z}_l] = 0$ for all $(n_l, \mathbf{z}_l)$.

6.4.1 GLS estimates of index coefficients

When there is intracluster correlation among error terms within auctions, a simple ordinary least square estimator will be inefficient. This can happen when syndicates' signals $\mathbf{X}_l$ are strictly affiliated. One explanation for affiliated signals in the finance literature is the "herding" effect among research and sales staff across syndicates. For example, researchers in different syndicates tend to have similar professional backgrounds or trainings and hence are inclined to make similar decisions on the choice and weights of value-related factors in their analyses. Strict affiliation among signals could also happen when syndicates' estimates consist of idiosyncratic noisy measurements of a common, underlying random variable.

Table 2 below reports the GLS estimates and t-statistics of γ for equation (5). The dependent variable is the dollar price bid. The regressors include publicly known bond features : weighted average coupon rate (*wacr*), weighted average maturity (*wapn*), total par value of the series (*totpar*), a dummy for whether the series is supported by full municipal credit (*sectype*), a dummy for whether the series is bank-qualified (*BQ*), a dummy for whether the series is rated with investment grade (*HR*) and two interaction terms *type_ cr* and *HR_ pn* respectively.[33] Butler (2007) suggests local presence of syndicates in the geographical area of the issuer could also influence their private information about the credibility of the issuer and hence their estimates of the value of the series. Therefore I also include in the regressors some dummies for the regions, *MW* (Midwest), *NE* (New England), *SW* (Southwest), *South* and *West*, to test the impact of geographic location on bids.

The weighted average coupon rates and maturity are both highly significant at 1% level, with positive and negative marginal effects respectively. These estimates confirm the intuition that bond values increase with cashflows from coupons and decrease as maturity increases because of higher risks in the fluctuation of interest rate and inflation. Municipality support has a significant positive effect on the bids. Controlling for other features, the average dollar price is $2.47 higher for bonds supported by the full credit of municipalities. Bond ratings by *S&P* and *Moody's* have no significant impact on bids ceteris paribus. A possible explanation is that the syndicates' research forces do not consider ratings informative conditional on their own research on bond values. The dollar bid for bank-qualified series are on average about 84 cents lower than non bank-qualified ones. The effect is statistically significant at 1% level. Besides, an increase of $1 million in total par leads to a slight increase of 1.76 cents in the dollar price. This can be explained by the fact that average participation costs for a syndicate (e.g. time and effort on research) per $100 in par is lower

[33] The unit for *wapn* is 10 semin-annual coupon payments and the unit for *totpar* is $100 million.

for issuance with larger par amount. The interaction of *sectype* and *wacr* are also highly significant at 1% level, suggesting marginal effects of coupon rates are lower for series with full municipal credit supports. There is no conclusive evidence for regional effects on bids except that dollar prices for series issued in New England are higher on average than those issued in the Midwest.

6.4.2 Specification tests

Two identifying restrictions in the regression equation (5) are additive separability and conditional independence of bond features and signals in value functions. A testable implication of these two restrictions is that marginal effects are constant and invariant to the number of potential bidders. That is, for each n, the following regression equation holds:

$$b_{il}(n) = \gamma_0(n) + \mathbf{z}_l'\boldsymbol{\gamma} + \varepsilon_{il}(x_{il}, n)$$

where $\gamma_0(n) \equiv E[\lambda_l(X_{il}, N_l)|N_l = n]$ and $\varepsilon_{il}(x_{il}, n) \equiv \lambda_l(x_{il}, n) - \gamma_0(n)$ is mean-independent conditional on $\mathbf{Z}_l$ and n. On the other hand, if either restriction is not satisfied, bidding strategies are nonseparable in $\mathbf{Z}_l$, X_{il} and n. Consequently, marginal effects of bond features on bids change with the number of potential bidders. Therefore we can test the two restrictions jointly by comparing estimates for auctions with different numbers of bidding syndicates.

Table 3(a) reports GLS estimates in regressions for n between 4 and 8. The choice of regressors $\mathbf{z}$ is the same as that in (5). The estimates are consistent across n in signs and significance. For each significant characteristic of the series, *Table 3(b)* reports test statistics for the pair-wise hypotheses that coefficients are the same in the two regressions with different n. The statistics are constructed as the ratio of differences between GLS estimates and the standard error of the difference.[34] Under null hypotheses, the test statistics are asymptotically standard normal.

The results show differences between sizes of estimates are insignificant. With the exception of weighted average coupon rates for $n = 4$, all other estimates are not significantly different from their counterparts under a different n. There is no statistically significant evidence against the hypotheses that the value function is additively separable and bond

[34] Note GLS estimators for different n are independent, for (Z_l, N_l, X_l) are i.i.d. draws from the same joint distribution. Hence the standard deviation of the difference in two estimators can be consistently estimated by adding up their standard errors.

features have no bearing on the distribution of idiosyncratic signals conditional on the number of participating syndicates.

6.5 Results

6.5.1 Point and interval estimates for $\hat{F}^k_{R^I(r)}$ and $\hat{F}^k_{R^{II}(r)}$

This section reports bound estimates on counterfactual revenue distributions for a reference bond series when there are $n = 4$ bidding syndicates in the auction. The reference series is issued in the Midwest, bank-qualified, backed by full municipal credit, and has an investment grade from S&P and the Moody's. The reference series has a weighted average coupon rate of 4% and maturity of 5 years, as well as a total par of $4.84 million. These are the medians of features of the bond series among auctions with 4 bidding syndicates.

Figure 5(a) plots kernel density estimates of the ordered bids that are "homogenized" at the reference level, which are calculated using *GLS* estimates in regressions with 4 bidders. Distributions of the ordered bids are approximately normally distributed with similar standard deviations and the differences between the median of adjacent ordered bids are between $0.25 and $0.35 per $100 in par amount. I use the product of tri-weight kernels for estimating $G_{M,B}$ and $g_{M,B}$. The choice of bandwidths follows the "rule of thumb" discussed in Monte Carlo section.[35] The data is parse close to the both boundaries even after trimming bids that are within one bandwidth from the minimum and maximum bids reported. To avoid poor performances of the kernel estimates of $\hat{\xi}_l$ for lower dollar values, I trim the bids at the 0.5-th and 99.5-th percentile.[36] In the data, bids from the same auction are almost always trimmed together.

Figure 6 plots estimates $\hat{\xi}$ and $\hat{\xi}_l$ and suggests the distance between estimates of bounds on $b_0(x^*(r))$ only widens slowly as r increases. That $\hat{\xi}_l$ stays mostly above the 45-degree line is evidence for strict affiliations between private estimates within each auction. *Table 4* below summarizes estimated bounds on $b_0(x^*(r))$ and the probability that no one bids above r (hereafter referred to as the all-screening probability) for different reserve prices.

[35] The bandwidths h_G and h_g are respectively $2.98 * 1.06\hat{\sigma}_b * (4L_4)^{-\frac{1}{4n-5}} = 2.43$ and $2.98 * 1.06\hat{\sigma}_b * (4L_4)^{-\frac{1}{4n-4}} = 2.57$.

[36] The distance between the minimum bid and the 0.5-th percentile is about $5. The number is greater than the smoothing parameter $h_g = 2.57$ used in the estimation.

Table 4 : Estimated bounds on the all-screening probability

r	$\hat{b}_0(x_l(r))$	$\hat{b}_0(x_h(r))$	$b.w.\ of\ b_0(x^*(r))$	$\hat{F}^l_{R^I(r)}(r_-)$	$\hat{F}^u_{R^I(r)}(r_-)$
98	97.17	97.89	0.72	0.0540	0.1256
99	98.00	98.83	0.83	0.1488	0.3860
100	98.76	99.73	0.97	0.3609	0.6865
101	99.45	100.60	1.15	0.5935	0.8837
102	100.13	101.39	1.26	0.8074	0.9516
103	100.74	102.14	1.40	0.9042	0.9702

Table 4 suggests marginal bidders under r are estimated to bid lower than r in the scenario with no binding reserve price. It is consistent with the theoretical prediction that $F_{R^I(r)}(r)$ is less than $F_{R^I(0)}(r)$. The difference between the boundwidths of the all-screening probability for $r = 98$ and $r = 100$ is mostly due to the distribution of winning bids with no binding reserve prices. *Figure 5(b)* shows the distribution of $b_0^{(1:4)}$ (the winning bid out of 4 bids) has a larger mass in $[b_0(x_l(100))\ \ b_0(x_h(100))] = [98.75\ \ 99.73]$ than in $[b_0(x_l(98))\ \ b_0(x_h(98))] = [97.17\ \ 97.89]$. Therefore the bounds on the all-screening probability is much wider for $r = 100$ even though bounds on $b_0(x^*(100))$ is only slightly wider than those of $b_0(x^*(98))$.

For reserve prices between \$98 and \$103, the solid and dotted lines in the panels of *Figure 7* depict point estimates $\hat{F}^u_{R^I(r)}$ and $\hat{F}^l_{R^I(r)}$ respectively. In addition, I construct 100 bootstrap samples, each containing 1075 auctions drawn with replacement from the estimating data. For all levels of revenue, I record the 5-th percentile of $\hat{F}^l_{R^I(r)}$ and 95-th percentile of $\hat{F}^u_{R^I(r)}$. They form a conservative, pointwise 90% confidence interval of $[F^l_{R^I(r)}, F^u_{R^I(r)}]$, and are plotted in *Figure 7* as broken lines. In addition, the table below reports the bounds on major percentiles according to the estimates of bounds on $\hat{F}^l_{R^I(r)}$ and $\hat{F}^u_{R^I(r)}$.

Table 5 : Estimated bounds on quartiles of $F_{R^I(r)}$

r	$l.b.\ 1st$	$u.b.\ 1st$	$l.b.\ 2nd$	$u.b.\ 2nd$	$l.b.\ 3rd$	$u.b.\ 3rd$
98	98.47	98.66	99.18	99.28	99.93	99.98
99	99.08	99.30	99.29	99.54	100.01	100.14
100	v_0	v_0	v_0	100.07	100.14	100.48
101	v_0	v_0	v_0	v_0	v_0	101.09

Revenue distribution above the reserve price depends on the distribution of $b_0(x)$ and the δ_r functional mapping $b_0(x)$ and $G^0_{\mathbf{B}}$ into $b_r(x)$. The densities plotted in *Figure 5 (a)* illustrate homogenized winning bids are approximately normally distributed. Besides, our estimates of bounds on δ_r are approximately linear. Therefore bound estimates $\hat{F}^k_{R^I(r)}(t)$ for $t > r$ increase at decreasing rates, a pattern similar to normal distributions. By construction, estimates of bounds on the all-screening probabilities are monotone in reserve prices (i.e. $\hat{F}^k_{R^I(r)}(r_-)$ is increasing in r for $k = l, u$). In addition, our estimates suggest that for any pair of reserve prices $r < r'$, $\hat{F}^k_{R^I(r')}(t) < \hat{F}^k_{R^I(r)}(t)$ for $t \geq r'$. This is consistent with the theoretical prediction that for a given signal above the screening level, firms bid less aggressively when the reserve price is lowered. Likewise *Figure 8* plots point estimates for revenue distribution in second-price auctions as well as the 90% confidence intervals for $[F^l_{R^{II}(r)}(t), F^u_{R^{II}(r)}(t)]$.

6.5.2 Choice of optimal reserve prices

Knowledge of revenue distributions in counterfactual auctions makes it possible to use other distribution-based criteria for comparing auction revenues, instead of expectations alone.[37] This is especially useful when the seller is known to be risk-averse and expected utilities are used as criteria.

A natural consequence of our partial approach is that only bounds on these criteria functions can be calculated. Such bounds on criteria functions are also tight and exhaust all information possible from equilibrium bids without further restrictions on value functions and signal distributions. As a result, answers to policy questions above involves comparing bound estimates rather than point estimates. Bounds on criteria functions can also be used to bound optimal reserve prices following the logic in Haile and Tamer (2003).

A value for v_0 is needed for calculating both upper and lower bounds on $E(R^I(r))$ and $E(R^{II}(r))$. This should be measured by the amount of money that a municipality would be able to raise if it had borrowed through an alternative, next-cheapest channel (i.e. a creditor that requires the next lowest interests than syndicates in the auctions). The proxy for v_0 I use here is \$95.71, and it is calculated as the present value per \$100 in par of cash flows from the coupon and principal payments of a reference bond, with the discount rate being the 99-th percentile of total interest rates reported in the data.

Figure 9(a) plots estimated upper and lower bounds on $E(R^I(r))$ (denoted $\hat{E}_h(R^I(r))$ and $\hat{E}_l(R^I(r))$ respectively), which are calculated from $\hat{F}^l_{R^I(r)}$ and $\hat{F}^u_{R^I(r)}$ through discretization and numerical integration using midpoint approximations. The solid lines plot $\hat{E}_k(R^I(r))$

[37] Within first-price auctions, each $r > v_0$ can be justified as optimal under the criterion of maximizing $\Pr(R^I(\tilde{r}) \geq r)$. That is $r = \arg\max_{\tilde{r}>v_0} \Pr(R^I(\tilde{r}) \geq r)$ for all $r > v_0$.

and the dotted lines plot $\hat{E}_k(R^{II}(r))$. The upper bounds of expected revenue correspond to the case of PV auctions. Note estimates for $\hat{E}_h(R^{II}(r))$ are higher than $\hat{E}_h(R^I(r))$ for almost all r in the range. This is consistent with the implication of Revenue Ranking Principle: for a fixed level of r, the expected revenue is higher for second-price auctions when signals are affiliated. For first-price auctions, $\hat{E}_h(R^I(r))$ is maximized at $r = \$98.68$ to be \$99.29, and $\hat{E}_l(R^I(r))$ is maximized at $r = \$96.26$ to be \$99.16. An argument similar to Haile and Tamer (2003) suggests the optimal reserve price that maximizes $E(R^I(r))$ must be in the range $[\$96.12, \$99.21]$. For second-price auctions, $\hat{E}_l(R^I(r))$ and $\hat{E}_h(R^I(r))$ are both maximized at $r = \$96.57$ with the maximum \$99.94, thus providing a point estimate for $E(R^I(r))$-maximizing reserve price. Instead of calculating a range of r that maximizes the expected revenue, an alternative is to pick r that maximizes either the lower or upper bound on $E(R^I(r))$. In the case of risk-neutral bidders, estimates for $\hat{E}_l(R^{II}(r))$, $\hat{E}_h(R^{II}(r))$ and $\hat{E}_l(R^I(r))$ are all close to being monotone, and their maximizers are all close to the boundary \$96.

A major motivation for focusing on revenue distribution in counterfactual analyses is the risk aversion of the seller. Given any specification of the seller's utility function (denoted $u(t)$), $\{\hat{F}^k_{R^j}(r)\}^{k=l,u}_{j=I,II}$ can be used to estimate bounds on the seller's expected utility (denoted $\{U_k(F_{R^j(r)})\}^{k=l,u}_{j=I,II}$). Like the case with a risk-neutral seller, these bounds can be used to put a range on an optimal reserve price that maximizes $U(F_{R^j(r)})$, or be used as criteria themselves for choosing reserve prices.

I consider three specifications of the seller's utility function: $u^{DARA}(t) = \ln(t)$ ($DARA$) and $u^{CRRA}(t) = \frac{t^{1-\rho}}{1-\rho}$ with $\rho = 0.6$ and 0.9 ($CRRA$). *Figure 9(b), (c)* and *(d)* plot estimated bounds on the expected utilities in first- and second-price auctions (i.e. $\{U_k(F_{R^j(r)})\}^{k=l,u}_{j=I,II}$) for *DARA, CRRA*($\rho = 0.6$) and *CRRA*($\rho = 0.9$) utility functions respectively. *Table 6* below summarizes reserve prices that maximize estimated bounds of expected utilities in first-price auctions, as well as estimated bounds on optimal r^* maximizing expected utilities.

Table 6 : Optimal reserve prices for first-price auctions

	$\hat{U}_l^{DARA}$	$\hat{U}_h^{DARA}$	$\hat{U}_l^{\rho=0.6}$	$\hat{U}_h^{\rho=0.6}$	$\hat{U}_l^{\rho=0.9}$	$\hat{U}_h^{\rho=0.9}$
$r_{(maximizer)}$	96.19	98.65	96.23	98.68	96.25	98.66
maximum	4.593	4.594	15.711	15.719	15.822	15.824
bounds on r^*	$[96.24, 99.20]$		$[96.17, 99.32]$		$[96.21, 99.20]$	

In second-price auctions, estimates of bounds on expected utilities under different specifications are all maximized at \$96.26, with the maxima being 4.594, 15.743 and 15.808

respectively. As a result, we get a point estimate of the optimal reserve price r^* at \$96.26 for all three specifications. Both maximizers across different specifications of utility functions are close to each other and so are the interval estimates. This is because u^{DARA}, $u^{\rho=0.6}$ and $u^{\rho=0.9}$ are all approximately linear for the range of revenues considered in this application. As a result, estimated bounds on $\{U(F_{R^j(r)})\}_{j=I,II}$ as functions of r are close to being linear transformations of each other.

On the other hand, estimates for different $u(.)$ yield different implications regarding the choice of format between first- and second-price auctions. For $DARA$ utility functions, the point estimate for the optimal reserve price in second-price auctions is \$96.26, with a maximum $\hat{U}^{DARA}(F_{R^{II}(96.26)}) = 4.594$. This is equal to the maximized value for $\hat{U}_h^{DARA}(F_{R^I(98.65)})$. Hence estimates suggests a seller with decreasing absolute risk aversion should prefer second-price auctions in general, and may be indifferent between the two formats if the auction is known to belong to the PV paradigm. For $CRRA$ utilities with $\rho = 0.6$, the implication is the same as in the case with risk-neutral sellers. However, for $CRRA$ utilities with $\rho = 0.9$, estimates suggest first-price auctions should be preferred over second-price ones. The pattern is due to the fact that $F_{R^I(r)}$ always crosses $F_{R^{II}(r)}$ from below for any given r, and $u^{\rho=0.6}$ increases faster than $u^{\rho=0.9}$.

Finally a technical note is in order. Except for $\hat{E}_h(R^I(r))$ and $\hat{U}_h(R^I(r))$, other estimates of bounds on $\{E(R^j(r))\}_{j=I,II}$ and $\{U(R^j(r))\}_{j=I,II}$ are almost monotonically decreasing in r. In general this need not be the case in estimation. To see this, note that none of the estimates $\{\hat{F}^k(R^j(r))\}_{j=I,II}^{k=l,h}$ reported in *Figure 7* and *Figure 8* are stochastically ordered in r. In this incidence, the monotonicity is explained by the fact that our measure of v_0 is low at \$95.71 and that estimates $\hat{b}_r(x_h(r'))$ are close to r' for all (r, r').

7 Conclusion

In structural models of first-price auctions, interdependence of bidders' values leads to non-identification of model primitives. That is, distributions of equilibrium bids observed in a given auction format can be rationalized by more than one possible specifications of signal and value distributions. While this negative identification result rules out policy analyses that rely on exact knowledge of primitives, the distribution of bids observed in equilibria should still convey useful information about primitives that can be extracted for counterfactual revenue analyses. This paper derives bounds on revenue distributions in counterfactual auctions with binding reserve prices. The bounds are the tightest possible under restrictions of interdependent values and affiliated signals, and can be used to compare auction formats or bounds on optimal reserve prices. This approach also addresses the empirical difficulty of differentiating PV and CV paradigms in policy analyses. The bounds can be nonparametrically consistently estimated, and Monte Carlo evidence suggests these estimators also have reasonable finite sample performances.

Observed heterogeneity in auction characteristics can be controlled for by conditioning counterfactual analyses on these auction features. Under the restriction of additive separability of signals and auction characteristics in value functions, the marginal effects of auction features can be identified if signals are independent from auction features conditional on the number of bidders. By removing variations due to observable auction heterogeneity, the bids across various auctions can be "homogenized" to bids in auctions with given specific features. The issue of data generated under a binding reserve price also does not pose major challenges to the construction of bounds, provided the data report the number of potential bidders or good proxies of this number.

Applying this methodology to U.S. municipal bond auctions on the primary market yields informative bound estimates of revenue distributions in counterfactual auctions with binding reserve prices. These estimates are then used to bound the reserve prices that maximize expected revenues for risk-neutral sellers. For risk-averse sellers, bounds on revenue distributions are also used to bound optimal reserve prices which maximize their expected utility under different specifications of utility functions.

An interesting direction for future research include extensions of partial-identification methods for more complicated cases such as asymmetric information among bidders and unobserved auction heterogeneity. Another promising direction is inference using our three-step bound estimates. For example, suppose data reports exogenous variations in binding reserve prices. Then a novel test for private values can be constructed by comparing the actual revenue distribution under a higher reserve price and the hypothetical upper bound on the revenue distribution constructed from bids in auctions with lower reserve prices.

8 Appendix A: Proof of identification results

Proof of Proposition 1. To prove necessity, suppose $\psi \in \Theta_{CV} \otimes \mathcal{F}$ generates $G^0_{\mathbf{B}}$ in such an equilibrium. Then the support of $\mathbf{B}$ is $S^N_B \equiv [b^0_L, b^0_U]^N$, with $b^0_L \equiv v_h(x_L, x_L; \psi)$ and $b^0_U = b_0(x_U, x_U; \psi)$. Note $\forall \mathbf{b} \in [b^0_L, b^0_U]^N$,

$$
\begin{aligned}
G_{\mathbf{B}^0}(\mathbf{b}) &\equiv \Pr(b_{\mathbf{0}}(X_1;\psi) \leq b_1, .., b_{\mathbf{0}}(X_N;\psi) \leq b_N) \\
&= \Pr(X_1 \leq \eta^0(b_1;\psi), .., X_N \leq \eta^0(b_N;\psi)) \\
&\equiv F_{\mathbf{X}}(\boldsymbol{\eta}^0(\mathbf{b};\psi))
\end{aligned}
$$

where $\eta^0(.;\psi)$ denotes the inverse bidding strategy in equilibrium under the structure ψ and no binding reserve prices. Symmetric equilibrium and exchangeability of $F_{\mathbf{X}}$ implies $G_{\mathbf{B}^0}(\mathbf{b})$ must be exchangeable in $\mathbf{b}$ for all $\mathbf{b} \in S^N_B$. The affiliation of $\mathbf{B} = (b_0(X_1;\psi), .., b_0(X_n;\psi))$ follows from the monotonicity of $b_0(.)$ and the affiliation of $\mathbf{X}$ by Theorem 3 in Milgrom and Weber (1982). The first-order condition (2) implies $\xi(b; G_{\mathbf{B}^0}) = v_h(\eta^0(b;\psi), \eta^0(b;\psi);\psi)$ $\forall b \in [b^0_L, b^0_U]$, where $v_h\,(x, x;\psi)$ is increasing in x on S_X by the definition of $\Theta_{CV} \otimes \mathcal{F}$. Hence the strict monotonicity of $\eta^0(b;\psi)$ implies $\xi(b; G_{\mathbf{B}^0})$ is increasing over S_B. The proof of sufficiency uses a claim and an example below.

Claim A1 Suppose a rationalizable bid distribution $G_{\mathbf{B}^0}$ that satisfies conditions (i) and (ii) in Proposition 1. Then a structure $\psi = (\theta, F_{\mathbf{X}}) \in \Theta \otimes \mathcal{F}$ rationalizes $G_{\mathbf{B}^0}$ if and only ψ satisfies the following fixed-point equation for all $\mathbf{x} \in S^N_X$,

$$
F_{\mathbf{X}}(\mathbf{x}) = G_{\mathbf{B}^0}(\xi^{-1}(v_h(x_1, x_1;\psi); G_{\mathbf{B}^0}), .., \xi^{-1}(v_h(x_N, x_N;\psi); G_{\mathbf{B}^0})) \tag{6}
$$

Proof of Claim A1 Suppose $\psi \in \Theta \otimes \mathcal{F}$ rationalizes such a $G_{\mathbf{B}^0}$ in an increasing, symmetric equilibrium. Then $F_{\mathbf{X}}(\mathbf{x}) = G_{\mathbf{B}^0}(\mathbf{b}^0(\mathbf{x};\psi))$ for all $\mathbf{x} \in S^N_X$, where $b^0(.;\psi)$ is the strictly increasing strategy in equilibrium that solves (2) with the boundary condition $b^0(x_L;\psi) = v_h(x_L, x_L;\psi)$. Then the rationalizability of $G_{\mathbf{B}^0}$ and the monotonicity of $\xi\,(.; G_{\mathbf{B}^0})$ implies $b^0(x;\psi) = \xi^{-1}(v_h(x, x;\psi); G_{\mathbf{B}^0})$ for all $x \in S_X$. It follows that (6) must hold. To prove sufficiency, suppose $G_{\mathbf{B}^0}$ is symmetric and affiliated with the support $S^N_B \equiv [b^0_L, b^0_U]^N$ and $\xi(.; G_{\mathbf{B}^0})$ is increasing on the marginal support S_B. Consider a $\bar{\psi} = (\bar{\theta}, \bar{F}_{\mathbf{X}}) \in \Theta \otimes \mathcal{F}$ that satisfies (6). We need to show $G_{\mathbf{B}^0}(\mathbf{b}) = \bar{F}_{\mathbf{X}}(\boldsymbol{\eta}^0(\mathbf{b};\bar{\psi}))$ $\forall \mathbf{b} \in [b^0_L, b^0_U]^N$, where $\eta^0(.;\bar{\psi})$ is the inverse of the bidding strategy in a symmetric, increasing equilibrium $b^0(.;\psi)$ that solves (1) with the boundary condition $b^0(x_L;\bar{\psi}) = v_h(x_L, x_L;\bar{\psi})$.[38] The monotonicity of $v_h(x, x;\bar{\psi})$ in

[38] Existence of symmetric, increasing *PSBNE* is not an issue since by the definition of Θ and $\mathcal{F}$, they exist for all $\psi \in \Theta \otimes \mathcal{F}$.

x and $\xi(.;G_{\mathbf{B}^0})$ over S_B and the choice of $\bar{\psi}$ implies $G_{\mathbf{B}^0}(\mathbf{b}) = \bar{F}_{\mathbf{X}}(\mathbf{v}_h^{-1}(\boldsymbol{\xi}(\mathbf{b};G_{\mathbf{B}^0});\bar{\psi}))$ for all $\mathbf{b} \in [b_L, b_U]^N$.

Hence it suffices to show $\xi^{-1}(v_h(.,.;\bar{\psi});G_{\mathbf{B}^0})$ satisfies the first-order conditions in (1) with the boundary condition $\xi^{-1}(v_h(x_L,x_L;\bar{\psi});G_{\mathbf{B}^0}) = v_h(x_L,x_L;\bar{\psi})$. By the same argument as in Proposition 1 in Li, Perrigne and Vuong (2002), it can be shown that $\lim_{b\to b_L^0}\xi(b;G_{\mathbf{B}^0}) = b_L^0$ under the rationalizable conditions on $G_{\mathbf{B}^0}$. Hence the boundary condition is satisfied. Let $v_h(x)$ be a shorthand for $v_h(x,x;\bar{\psi})$ and $\xi^{-1}()$ for $\xi^{-1}(.;G_{\mathbf{B}^0})$. Furthermore, from the construction of $\bar{\psi}$,

$$\begin{aligned}
\bar{F}_{Y|X}(x|x) &= G_{M^0|B^0}[\xi^{-1}(\bar{v}_h(x))|\xi^{-1}(\bar{v}_h(x))] \\
\bar{f}_{Y|X}(x|x) &= g_{M^0|B^0}[\xi^{-1}(\bar{v}_h(x))|\xi^{-1}(\bar{v}_h(x))]\xi^{-1\prime}(\bar{v}_h(x))\bar{v}_h'(x)
\end{aligned}$$

Thus $\xi^{-1}(\bar{v}_h(.))$ solves (1) if

$$\bar{v}_h'(x)\xi^{-1\prime}(\bar{v}_h(x)) = [\bar{v}_h(x) - \xi^{-1}(\bar{v}_h(x))]\frac{\bar{f}_{Y|X}(x|x)}{\bar{F}_{Y|X}(x|x)}$$

or equivalently, if

$$\bar{v}_h(x) = \xi^{-1}(\bar{v}_h(x)) + \frac{G_{M^0|B^0}[\xi^{-1}(\bar{v}_h(x))|\xi^{-1}(\bar{v}_h(x))]}{g_{M^0|B^0}[\xi^{-1}(\bar{v}_h(x))|\xi^{-1}(\bar{v}_h(x))]}$$

But this must hold by the definition of $\xi(.;G_{\mathbf{B}^0})$. *Q.E.D.*

Suppose $\bar{\boldsymbol{\theta}}(\mathbf{x}) = (\{\tilde{\theta}(x_i,y_i)\}_{i=1}^N)$ for all $\mathbf{x} \in S_X^N$, where $y_i \equiv \max_{j\neq i} x_j$. That is, bidders' valuations only depend on his own signal and the highest rival signal. Then $\mathbf{v}_h(\mathbf{x};\bar{\theta},F_{\mathbf{X}}) = (\{\tilde{\theta}(x_i,y_i)\}_{i=1}^N)$ for all $F_{\mathbf{X}} \in \mathcal{F}$. Therefore, a distribution $G_{\mathbf{B}^0}$ that satisfies conditions (i) and (ii) is rationalized by any such $\bar{\theta} \in \Theta$ that satisfies the "$\max_{j\neq i} X_j$-sufficiency" with boundary conditions $\tilde{\theta}(x_k,x_k) = \xi(b_k^0)$ for $k \in \{L,U\}$, and a signal distribution $F_{\mathbf{X}}(\mathbf{x}) \equiv G_{\mathbf{B}^0}(\xi^{-1}(\tilde{\theta}(x_1,x_1)),..,\xi^{-1}(\tilde{\theta}(x_n,x_n)))$.

Proof of Lemma 1. Note the solution to (3) with the boundary condition has the following closed form:

$$\delta_r(b^0(x);G_{\mathbf{B}^0}) \equiv r\tilde{L}(b^0(x^*(r))|b^0(x)) + \int_{b^0(x^*(r))}^{b^0(x)} \xi(\tilde{b};G_{\mathbf{B}^0})d\tilde{L}(\tilde{b}|b^0(x))$$

where $\tilde{L}(\tilde{b}|b;G_{\mathbf{B}^0}) \equiv \exp\left(-\int_{\tilde{b}}^{b}\tilde{\Lambda}(u;G_{\mathbf{B}^0})du\right)$. The proof of the lemma uses the monotonicity and differentiability of $b_0(.)$. By change of variables, $\frac{f_{Y|X}(x|x)}{F_{Y|X}(x|x)} = b_0'(x)\tilde{\Lambda}(b_0(x);G_{\mathbf{B}^0})$ and for all $s \leq x$, $L(s|x;F_{\mathbf{X}}) = \tilde{L}(b_0(s)|b_0(x);G_{\mathbf{B}^0})$. Furthermore, in equilibria $v_h(x,x;\theta,F_{\mathbf{X}}) = \xi(b_0(x);G_{\mathbf{B}^0})$ for all $x \in S_X$. By definition, for $x \geq x^*(r)$,

$$\begin{aligned} b^r(x) &= rL(x^*(r)|x;F_{\mathbf{X}}) + \int_{x^*(r)}^{x} v_h(s,s;\theta,F_{\mathbf{X}})L(s|x;F_{\mathbf{X}})\Lambda(x;F_{\mathbf{X}})dx \\ &= r\tilde{L}(b^0(x^*(r))|b^0(x);G_{\mathbf{B}^0}) + \int_{x^*(r)}^{x} \xi(b^0(s);G_{\mathbf{B}^0})\tilde{L}(b^0(s)|b^0(x);G_{\mathbf{B}^0})\tilde{\Lambda}(b^0(s);G_{\mathbf{B}^0})b^{0\prime}(s)ds \\ &= \delta_r(b^0(x);G_{\mathbf{B}^0}) \end{aligned}$$

where the last equality follows from a change of variables in the integrand.

Proof of Lemma 2. The affiliation of signals and monotonicity of θ implies that $v_h(x,y)$ is increasing in x and non-decreasing in y. For all $x \geq y$,

$$v_h(x,y) \geq \int_{x_L}^{y} v_h(x,s)\frac{f_{Y|X}(s|x)}{F_{Y|X}(y|x)}ds \equiv v(x,y) \geq \int_{x_L}^{y} v_h(s,s)\frac{f_{Y|X}(s|x)}{F_{Y|X}(y|x)}ds \equiv v_l(x,y)$$

Therefore $v_h(x_L,x_L) = v(x_L,x_L) = v_l(x_L,x_L)$ and v is bounded between v_h and v_l for all $x \in S_X$. The proof of strict monotonicity of $v_h(x,x)$ in x is standard and omitted. For any $x < x'$ on support, the law of total probability implies

$$\begin{aligned} v_l(x',x') &= E(v_h(Y,Y)|X_i = x', Y_i \leq x') \\ &= E(v_h(Y,Y)|X_i = x', Y_i \leq x)P(Y_i \leq x|X_i = x', Y_i \leq x') + ... \\ &\quad E(v_h(Y,Y)|X_i = x', x < Y_i \leq x')P(x < Y_i \leq x'|X_i = x', Y_i \leq x') \end{aligned}$$

By monotonicity of v_h and $x' > x$, $E(v_h(Y,Y)|X_i = x', x < Y_i \leq x') > v_l(x,x)$. By affiliation of X and Y, $E(v_h(Y,Y)|X_i = x', Y_i \leq x) \geq v_l(x,x)$. Therefore $v_l(x',x') > v_l(x,x)$.

Then *(i)* follows immediately. For the first part of *(ii)*, note any rationalizable bid distribution can also be rationalized by a certain private-value structure. It follows that $\exists \psi \in \Psi(G_{\mathbf{B}^0})$, $v_h(x,x;\psi) = v(x,x;\psi)$. For the second part of *(ii)*, consider $\Theta_S \equiv \{\theta : \theta(x_i, \mathbf{x}_{-i}) = \alpha x_i + (1-\alpha)\max_{j\neq i} x_j \textit{ for some } a \in (0,1)\}$. By construction, all value functions in Θ_S are exchangeable and non-decreasing in $\mathbf{x}_{-i}$. Then $v_h(x,x) = x$ for all $x \in S_X$ regardless of the choice of signal distributions. Then define $F_{\mathbf{X}}(\mathbf{x}) \equiv G_{\mathbf{B}^0}(\xi^{-1}(x_1;G_{\mathbf{B}^0}),..,\xi^{-1}(x_n;G_{\mathbf{B}^0}))$. The auction structure $(\alpha, F_{\mathbf{X}})$ rationalizes $G_{\mathbf{B}^0}$ in a Bayesian Nash equilibrium with the same $F_{\mathbf{X}}$ regardless of the choice of $\alpha \in (0,1)$. By definition, $v(x,x;\alpha,F_{\mathbf{X}}) = \alpha x + (1-\alpha)E(Y_i|X_i = x, Y_i \leq x)$ while $v_l(x,x;\alpha,F_{\mathbf{X}}) = E(Y_i|X_i = x, Y_i \leq x)$. Therefore, the distance between v and v_l converges to 0 uniformly over the support S_X as α diminishes. It then follows that $\forall \varepsilon > 0$, $\exists \alpha$ small enough such that $\sup_{r\in S_{RP}} |x_h(r;\alpha,F_{\mathbf{X}}) - x^*(r;\alpha,F_{\mathbf{X}})| \leq \varepsilon$ with $F_{\mathbf{X}}$ defined above.

Proof of Lemma 3. By the non-negativity of θ in Θ, $x^*(0;\psi) = x_L$ for all $\psi \in \Theta \otimes \mathcal{F}$. Hence for all $x \geq x_L$, $v_h(x,x;\psi) = \xi(b^0(x;\psi);G_{\mathbf{B}^0})$ and $v_l(x,x;\psi) = \xi_l(b^0(x;\psi);G_{\mathbf{B}^0})$ for all $\psi \in \Psi(G_{\mathbf{B}^0}) \subseteq \Theta \otimes \mathcal{F}$. It follows $\xi(b;G_{\mathbf{B}^0}) \geq \xi_l(b;G_{\mathbf{B}^0})$ for all $b \in S_{B^0}$, and $\xi(b^0_L;G_{\mathbf{B}^0}) = \xi_l(b^0_L;G_{\mathbf{B}^0})$. By the monotonicity of b^0 and the definition of $x_l(r)$ and $x_h(r)$, $b_0(x^*(r;\psi);\psi) \in [b_0(x_l(r;\psi);\psi), b_0(x_h(r;\psi);\psi)]$ for all $\psi \in \Theta \otimes \mathcal{F}$. Furthermore, for all $\psi \in \Psi(G_{\mathbf{B}^0})$ and $r \in S_{RP}$,

$$\begin{aligned} \xi(b^0(x_l(r;\psi);\psi);G_{\mathbf{B}^0}) &= v_h(x_l(r;\psi), x_l(r;\psi);\psi) \\ \xi_l(b^0(x_h(r;\psi);\psi);G_{\mathbf{B}^0}) &= v_l(x_h(r;\psi), x_h(r;\psi);\psi) \end{aligned}$$

Note $\xi(,;G_{\mathbf{B}^0})$ and $\xi_l(,;G_{\mathbf{B}^0})$ are invariant for all $\psi \in \Psi(G_{\mathbf{B}^0})$, and ξ_l is increasing over S_{B^0} by the monotonicity of $v_l(x,x)$ and b^0 on S_X. Therefore, $b^0_{k,r}(G_{\mathbf{B}^0}) = b^0(x_l(r;\psi);\psi)$ for $k = l,h$ and all $\psi \in \Psi\left(G_{\mathbf{B}^0}\right)$, and Claim (i) in the lemma holds.

To prove Claim (ii), consider $\bar{\Theta} \equiv \{\theta : \theta(\mathbf{X}) = \alpha X_i + (1-\alpha)\max_{j\neq i} X_j \textit{ for some } \alpha \in (0,1]\}$. Then $v_h(x,x;\theta) = x$ for all $x \in S_X$ and $\theta \in \bar{\Theta}$. Any rationalizable bid distribution $G_{\mathbf{B}^0}$ can be rationalized by the same signal distribution $\bar{F}_{\mathbf{X}}(x_1,..,x_N) \equiv G_{\mathbf{B}^0}(\xi^{-1}(x_1;G_{\mathbf{B}^0}),.., \xi^{-1}(x_N;G_{\mathbf{B}^0}))$ for all $\alpha \in (0,1]$. Hence $\bar{\Theta} \otimes \{\bar{F}_{\mathbf{X}}(G_{\mathbf{B}^0})\} \subseteq \Psi\left(G_{\mathbf{B}^0}\right)$. By definition, $x^*(r;\alpha,\bar{F}_{\mathbf{X}}) = \arg\min_{x\in S_X}[r - v(x,x;\alpha,\bar{F}_{\mathbf{X}})]^2$, where $v(X,X;\alpha)$ is continuous in x and α, and $x^*(r;\alpha,\bar{F}_{\mathbf{X}})$ is always a single-valued function in r. Hence the Theory of Maximum implies $x^*(r;\alpha,\bar{F}_{\mathbf{X}})$ is continuous in α for all $r \in S_{RP}$ and the $\bar{F}_{\mathbf{X}}$ chosen for $G_{\mathbf{B}^0}$. Furthermore for all $\bar{\psi} \in \bar{\Theta} \otimes \{\bar{F}_{\mathbf{X}}\}$, the equilibrium strategy $b^0(x;\bar{\psi}) = b^0(x;\bar{F}_{\mathbf{X}}) = \int_{x_L}^{x} s dL(s|x;\bar{F}_{\mathbf{X}})$ is independent from α. Thus, α enters the marginal bid $b^0\left(x^*\left(r;\bar{\psi}\right);\bar{F}_{\mathbf{X}}\right)$ only through the screening level, and the marginal bid is also a continuous function in α for all $r \in S_{RP}$ and the chosen $\bar{F}_{\mathbf{X}}$. Note the image of a continuous mapping from any connected set is a connected set. Hence to prove Claim (ii), it suffices to show that the bounds are tight. Consider the case of a private structure with $\alpha = 1$. Then

$$b^0(x^*(r;1,\bar{F}_{\mathbf{X}});\bar{F}_{\mathbf{X}}) = b^0(x_l(r;1,\bar{F}_{\mathbf{X}});\bar{F}_{\mathbf{X}}) = b^0_{l,r}\left(G_{\mathbf{B}^0}\right)$$

and the lower bound is reached. On the other hand, the proof of part (ii) in Lemma 2 shows $x_h(r;\alpha,\bar{F}_{\mathbf{X}})$ can be uniformly close to $x^*(r;\alpha,\bar{F}_{\mathbf{X}})$ over $r \in S_{RP}$ for some α small enough. As the choice of $\bar{F}_{\mathbf{X}}$ only depends on $G_{\mathbf{B}^0}$ and is independent from α, this suggests

$$\sup_{r\in S_{RP}} |b^0(x^*(r;\alpha,\bar{F}_{\mathbf{X}});\bar{F}_{\mathbf{X}}) - b^0_{h,r}(G_{\mathbf{B}^0})| \to 0$$

as $\alpha \downarrow 0$. Combining these two results above shows for all $r \in S_{RP}$ and $b \in [b^0_{l,r}(G_{\mathbf{B}^0}), b^0_{h,r}(G_{\mathbf{B}^0}))$, $\exists \psi \in \Psi(G_{\mathbf{B}^0})$ such that $b^0(x^*(r;\psi);\psi) = b$.

Proof of Lemma 4. Proof of (i): The closed form of solutions with new boundary conditions is

$$\delta_{r,k}(b;G_{\mathbf{B}^0}) \equiv r\tilde{L}(b^0_{k,r}(G_{\mathbf{B}^0})|b;G_{\mathbf{B}^0}) + \int_{b^0_{k,r}(G_{\mathbf{B}^0})}^{b} \xi(\tilde{b};G_{\mathbf{B}^0})d\tilde{L}(\tilde{b}|b;G_{\mathbf{B}^0})$$

for $b \geq b^0_{k,r}(G_{\mathbf{B}^0})$. For any $\psi \in \Psi(G_{\mathbf{B}^0})$ and all $x \geq x_l(r;\psi)$, $v_h(x,x;\psi) \geq r\ \forall x \in [x_l(r;\psi), x_U]$, and it follows

$$b^r(x;\psi) \leq rL(x_l(r;\psi)|x;F_{\mathbf{X}}) + \int_{x_l(r;\psi)}^{x} v_h(s,s;\psi)dL(s|x;F_{\mathbf{X}})$$

Likewise, for all $x \geq x_h(r;\psi)$,

$$b^r(x;\psi) \geq rL(x_h(r;\psi)|x;F_{\mathbf{X}}) + \int_{x_h(r)}^{x} v_h(s,s;\psi)dL(s|x;F_{\mathbf{X}})$$

By non-negativity of θ, $x^*(0;\psi) = x_L$ and $x_h(r;\psi) \geq x^*(r;\psi) \geq x_l(r;\psi) \geq x^*(0;\psi)$ for all $r \in S_{RP}$. Hence the equation (2) holds for $x_l(r;\psi)$ and $x_h(r;\psi)$. Substitution and the change of variable show for all $x \geq x_l(r;\psi)$,

$$b^r(x;\psi) \leq r\tilde{L}(b^0_{l,r}(G_{\mathbf{B}^0})|b^0(x;\psi)) + \int_{b^0_{l,r}(G_{\mathbf{B}^0})}^{b^0(x;\psi)} \xi(\tilde{b};G_{\mathbf{B}^0})d\tilde{L}(\tilde{b}|b^0(x;\psi)) \equiv \delta_{r,l}(b^0(x;\psi);G_{\mathbf{B}^0})$$

and for all $x \geq x_h(r;\psi)$,

$$b^r(x;\psi) \geq r\tilde{L}(b^0_{h,r}(G_{\mathbf{B}^0})|b^0(x;\psi)) + \int_{b^0_{h,r}(G_{\mathbf{B}^0})}^{b^0(x;\psi)} \xi(\tilde{b};G_{\mathbf{B}^0})d\tilde{L}(\tilde{b}|b^0(x;\psi)) \equiv \delta_{r,h}(b^0(x;\psi);G^0_{\mathbf{B}})$$

For all $b \geq b^0_{k,r}(G_{\mathbf{B}^0})$ and $k = l,h$,

$$\delta'_{r,k}(b;G^0_{\mathbf{B}}) = \tilde{\Lambda}(b)\left[\xi(b) - \left(r\tilde{L}(b^0_{r,k}|b) + \int_{b^0_{r,k}}^{b} \xi(\tilde{b})d\tilde{L}(\tilde{b}|b)\right)\right] > 0$$

It then follows from the monotonicity of the envelops $\{\delta_{r,k}(b;G_{\mathbf{B}^0})\}_{k=l,h}$ that for all $\psi \in \Psi(G_{\mathbf{B}^0})$, $b^0(\eta^r(t;\psi);\psi) \in [\delta^{-1}_{r,l}(t;G^0_{\mathbf{B}}), \delta^{-1}_{r,h}(t;G^0_{\mathbf{B}})]$.

Proof of (ii): Note for any $t > r$ and $\psi \in \Psi(G_{\mathbf{B}^0})$, $b^0(\eta^r(t;\psi);\psi)$ is defined as the solution to the following minimization problem:

$$\beta_r(b^0_r(\psi);t,G_{\mathbf{B}^0}) = \arg\min_{s \in [b^0_r(\psi), b^0_U]}[t - \delta_r(s;b^0_r(\psi),G^0_{\mathbf{B}})]^2$$

where $b^0_r(\psi)$ is a shorthand for $b^0(x^*(r;\psi);\psi)$ and δ_r is defined as before. Fix the revenue level t and the rationalizable bid distribution $G^0_{\mathbf{B}}$. Then $b^0_r(\psi)$ can be treated as a parameter that enters both the constraint set and the objective function, which is continuous in s and $b_r^0(\psi)$. By construction, $\delta_{r,k}^{-1}(t;G_{\mathbf{B}}^0) = \beta_r(b_{k,r}^0(G_{\mathbf{B}^0});t,G_{\mathbf{B}^0})$. Furthermore, the Theorem of Maximum implies $\beta_r(b_r^0(\psi);t,G_{\mathbf{B}^0})$ must be continuous in $b_r^0(\psi)$ for all $t > r$. Note by part *(ii)* of *Lemma 3*, for all $b \in [b_{l,r}^0(G_{\mathbf{B}^0}), b_{h,r}^0(G_{\mathbf{B}^0}))$, $\exists \psi \in \Psi(G_{\mathbf{B}^0})$ such that $b_r^0(\psi) = b$. It then follows that for any $r > r$ and for all $b \in [\delta_{r,l}^{-1}(t;G_{\mathbf{B}}^0), \delta_{r,h}^{-1}(t;G_{\mathbf{B}}^0))$, $\exists \psi \in \Psi(G_{\mathbf{B}^0})$ such that $b^0(\eta^r(t;\psi);\psi) = b$.

Proof of Proposition 2. Recall from the lemmae above that $b^0(x^*(r;\psi);\psi) \in [b_{l,r}^0(G_{\mathbf{B}^0}), b_{h,r}^0(G_{\mathbf{B}^0})]$ for all $\psi \in \Psi(G_{\mathbf{B}^0})$. By construction, $\delta_{r,k}(b_{k,r}^0(G_{\mathbf{B}}^0);G_{\mathbf{B}^0}) = b^r(x^*(r;\psi);\psi) = r$. Hence both $\{\delta_{r,k}(b_0(.;\psi);G_{\mathbf{B}}^0)\}_{k\in\{l,h\}}$ are invertible at $t \geq r$ over the interval $[x_k(r;\psi), x_U]$ for $k \in \{l,h\}$ and $\psi \in \Psi(G_{\mathbf{B}^0})$. It follows that $\delta_{r,l}^{-1}(t;G_{\mathbf{B}}^0) \leq b_0(b_r^{-1}(t;\psi);\psi) \leq \delta_{r,h}^{-1}(t;G_{\mathbf{B}}^0)$ for $t \geq r$ and all $\psi \in \Psi(G_{\mathbf{B}^0})$. The rest of the proof follows immediately.

Proof of Proposition 3. First I prove that all $\psi \in \Theta \otimes \mathcal{F}$ and $r \in S_{RP}$, equilibrium strategies b^0 and b^r satisfy: *(i)* $b^0(x;\psi) \leq b^r(x;\psi)$ $\forall x \geq x^*(r;\psi)$ and *(ii)* the difference $b^r(x;\psi) - b^0(x;\psi)$ is decreasing in x for all $x \geq x^*(r;\psi)$. Then it follows immediately from *(i)* and *(ii)* that $F_{R^I(r)}(\psi) \succeq_{F.S.D.} \tilde{F}_{R^I(r)}^u(F_{R^I(0)}(\psi))$. To prove *(i)*, first note in equilibria $b^r(x^*(r;\psi);\psi) = r = v(x^*(r), x^*(r);\psi) \equiv E(V_i|X_i = x^*(r), Y_i \leq x^*(r)) \geq b^0(x^*(r;\psi);\psi)$, where the last inequality holds by equilibrium bidding conditions with no binding reserve prices. Besides, for all $x \geq x^*(r;\psi)$, $b^r(x;\psi) < b^0(x;\psi)$ implies $b_r'(x;\psi) > b_0'(x;\psi)$. It follows from Lemma 2 in Milgrom and Weber (1982) that $b^r(x;\psi) \geq b^0(x;\psi)$ for all $x \geq x^*(r;\psi)$. For *(ii)*, it suffices to note $sgn(b_r'(x;\psi) - b_0'(x;\psi)) = -sgn(b^r(x;\psi) - b^0(x;\psi))$ for all $x \geq x^*(r;\psi)$.

To see that $F_{R^I(r)}^u(G_{\mathbf{B}^0}) \succeq_{F.S.D.} \tilde{F}_{R^I(r)}^u(F_{R^I(0)})$ in general, note $\frac{F_{Y|X}(s|x)}{F_{Y|X}(x|x)} \succeq_{F.S.D.} L(s|x)$ when private signals are affiliated. It follows that $v_l(x,x;\psi) \geq b^0(x;\psi)$ for all x and therefore $x_h(r;\psi) \leq \eta^0(r;\psi)$ for $r \geq b_L^0$ and $\psi \in \Psi(G_{\mathbf{B}^0})$. Hence, $b^0(x_h(r);\psi) \leq v_l(x_h(r), x_h(r);\psi) = r$ for $r \in [\xi_L^0, \xi_l(b_U^0)]$. Furthermore, by a change-of-variables,

$$\delta_{r,h}(b_0(x;\psi);G_{\mathbf{B}^0}) = rL(x_h(r)|x;\psi) + \int_{x_h(r)}^{x} v_h(s,s;\psi)dL(s|x;\psi)$$

and can be written as a solution for the differential equation

$$\lambda'(x) = [v_h(x,x) - \lambda(x)]\frac{f_{Y|X}(x|x)}{F_{Y|X}(x|x)}$$

with the boundary condition $\lambda(x_h(r)) = r$ for this range of r. An application of *Lemma 2* in Milgrom and Weber (1982) shows $\delta_{r,h}(b_0(x;\psi);G_{\mathbf{B}^0}) = rL(x_h(r)|x) + \int_{x_h(r)}^{x} v_h(s,s)dL(s|x) \geq b_0(x)$ for all $x \geq x_h(r;\psi)$. Hence $\delta_{r,h}^{-1}(t;G_{\mathbf{B}^0}) \leq t$ for $t \geq r$ and it follows $F_{R^I(r)}^u(G_{\mathbf{B}^0}) \succeq_{F.S.D.}$

$\tilde{F}^u_{R^I(r)}(F_{R^I(0)}(\psi))$. For $r > \xi_l(b^0_U)$, the two upper bounds on the all-screening probability coincide trivially at 1. To show that the bounds collapse into the same one with i.i.d. signals, it suffices to note that all inequalities in this proof so far will hold with equality as the two distributions $\frac{F_{Y|X}(s|x)}{F_{Y|X}(x|x)}$ and $L(s|x)$ are the same under i.i.d. signals.

Proof of Proposition 4. Consider any $\psi \in \Theta \otimes \mathcal{F}$. For notational ease, dependence of $x^*(r)$, β^r and v_h on ψ is suppressed. By definition of v_0, $\Pr\{R^{II}(r) < v_0\} = 0$. Note $\Pr\{R^{II}(r) = v_0\} = \Pr\{X^{(1)} < x^*(r)\}$, $\Pr\{v_0 < R^{II}(r) < r\} = 0$ and $\Pr\{R^{II}(r) = r\} = \Pr\{X^{(1)} \geq x^*(r) \wedge X^{(2)} < x^*(r)\}$. Since $\beta^{r\prime}(x) > 0$ for all $x \in [x^*(r), x_U]$ and $\beta^r(x^*(r)) = v_h(x^*(r), x^*(r)) \geq r$, it follows $\Pr\{r < R^{II}(r) < v_h(x^*(r), x^*(r))\} = 0$. Hence:

$$\begin{aligned} F_{R^{II}(r)}(t) &= 0 \quad \forall t < v_0 \\ &= \Pr\{X^{(1)} < x^*(r)\} \quad \forall t \in [v_0, r) \\ &= \Pr\{X^{(2)} < x^*(r)\} \quad \forall t \in [r, v_h(x^*(r), x^*(r))) \end{aligned}$$

Next note for all $t \in [v_h(x^*(r), x^*(r)), +\infty)$, $\Pr\{R^{II}(r) \in [v_h(x^*(r), x^*(r)), t]\} = \Pr\{v_h(X^{(2)}, X^{(2)}) \in [v_h(x^*(r), x^*(r)), t]\}$. Hence for all t in this range,

$$\begin{aligned} \Pr\{R^{II}(r) \leq t\} = \quad & \Pr\{R^{II}(r) < v_h(x^*(r), x^*(r))\} + \Pr\{R^{II}(r) \in [v_h(x^*(r), x^*(r)), t]\} \\ = \quad & \Pr\{X^{(2)} < x^*(r)\} + \Pr\{v_h(X^{(2)}, X^{(2)}) \in [v_h(x^*(r), x^*(r)), t]\} \\ = \quad & \Pr\{v_h(X^{(2)}, X^{(2)}) \leq t\} \end{aligned}$$

This characterizes the counterfactual distribution of $R^{II}(r)$. For any $\psi \in \Psi(G_{\mathbf{B}^0})$ and $t < r$,

$$\begin{aligned} & \Pr\{b_0(X^{(1)}; \psi) < b^0_{l,r}(G_{\mathbf{B}^0})\} \equiv F^l_{R^{II}(r)}(t; G_{\mathbf{B}^0}) \\ \leq \quad & \Pr\{X^{(1)} < x^*(r; \psi)\} = F_{R^{II}(r)}(t; \psi) \\ \leq \quad & \Pr\{b_0(X^{(1)}; \psi) < b^0_{h,r}(G_{\mathbf{B}^0})\} \equiv F^u_{R^{II}(r)}(t; G_{\mathbf{B}^0}) \end{aligned}$$

For $r \leq t < v_h(x^*(r), x^*(r))$,

$$\begin{aligned} & \Pr\{v_h(X^{(2)}, X^{(2)}) \leq t\} \equiv F^l_{R^{II}(r)}(t; G_{\mathbf{B}^0}) \\ \leq \quad & \Pr\{v_h(X^{(2)}, X^{(2)}) < v_h(x^*(r), x^*(r))\} = F_{R^{II}(r)}(t; \psi) \\ \leq \quad & \Pr\{b_0(X^{(2)}) < b^0_{h,r}(G_{\mathbf{B}^0})\} \equiv F^u_{R^{II}(r)}(t; G_{\mathbf{B}^0}) \end{aligned}$$

since $b^{0\prime}(.) > 0$. For $t \in [v_h(x^*(r), x^*(r)), v_h(x_h(r), x_h(r))$,

$$\begin{aligned} F^l_{R^{II}(r)}(t) \quad &\equiv \quad \Pr\{v_h(X^{(2)}, X^{(2)}) \leq t\} \equiv F_{R^{II}(r)}(t; G_{\mathbf{B}^0}) \\ &\leq \quad \Pr\{v_h(X^{(2)}, X^{(2)}) < v_h(x_h(r), x_h(r))\} = F_{R^{II}(r)}(t; \psi) \\ &= \quad \Pr\{b_0(X^{(2)}) < b^0_{h,r}(G_{\mathbf{B}^0})\} \equiv F^u_{R^{II}(r)}(t; G_{\mathbf{B}^0}) \end{aligned}$$

due to the monotonicity of $b_0(x)$ and $v_h(x,x)$ in x. For $t \in [v_h(x_h(r), x_h(r)), +\infty)$,

$$F^l_{R^{II}(r)}(t) = F_{R^{II}(r)}(t) = F^u_{R^{II}(r)}(t) = \Pr\{v_h(X^{(2)}, X^{(2)}) \leq t\}$$

and this is because bids in 2nd-price auctions can be fully recovered from $G_{\mathbf{B}^0}$ for those who are not screened out under r. Finally, we complete the proof by noting $\forall \psi \in \Psi\left(G_{\mathbf{B}^0}\right)$, $v_h(x,x;\psi) = \xi(b^0(x;\psi); G_{\mathbf{B}^0})$ for all $x \in [x_L, x_U]$.

Proof of Proposition 6. Auction characteristics are common knowledge among all bidders. Hence in symmetric equilibria: (By symmetry among the bidders, bidder indices are dropped for notational ease.)

$$\tfrac{\partial}{\partial X} b(x,z;n) = [\tilde{v}_h(x,z;n) - b(x,z;n)] \tfrac{f_{Y|X,Z;N}(x|x,z;n)}{F_{Y|X,Z;N}(x|x,z;n)}$$

where $\tilde{v}_h(x,z;n) \equiv E(V_i|X_i = Y_i = x, Z = z; N = n)$, $Y_i \equiv \max_{j\neq i} X_i$, $F_{Y|X,Z;N}(t|x,z;n) \equiv \Pr(\max_{j\neq i} X_j \leq t|X_i = x, Z = z; N = n)$ and $f_{Y|X,Z;N}(t|x,z;n)$ is the corresponding conditional density. The equilibrium boundary condition for all (z,n) is $b(x_L,z;n) = \tilde{v}_h(x_L,z;n)$. For every z and n, the differential equation is known to have the following closed form solution :

$$b(x,z;n) = \int_{x_L}^{x} h(z'\gamma) + \phi(s;n) dL(s|x;n)$$

Independence of X_i and Z conditional on N implies both $\phi(x;n)$ and $L(s|x;n)$ are invariant to z for all s and x. Hence under assumptions *A1',A2 and A4*,

$$b(x_L,z;n) = \tilde{v}_h(x_L,z;n) = h(z'\gamma) + \phi(x_L;n)$$

For $x > x_L$, $b(x,z;n) = h(z'\gamma) + \int_{x_L}^{x} \phi(s;n) dL(s|x;n)$ for all (x,z,n).

Proof of Proposition 7. Differentiating $b_r(x)$ for $x \geq x^*(r)$ gives

$$\tfrac{b_r'(x;\psi)}{\Lambda(x;F_{\mathbf{X}})} + b_r(x;\psi) = v_h(x,x;\psi) \tag{7}$$

For all $r \geq 0$ and $x, y \geq x^*(r)$,

$$\begin{aligned} & F_{Y|X}(y|x) \equiv \Pr(Y \leq y|x = x) && (8)\\ = \; & \Pr(Y < x^*(r)|X = x) + \Pr(x^*(r) \leq Y \leq y|X = x) \\ = \; & \Pr(b_r(Y) < r|b_r(X) = b_r(x)) + \Pr(r \leq b_r(Y) \leq b_r(y)|b_r(X) = b_r(x)) \\ \equiv \; & G^r_{M|B}(b_r(y)|b_r(x)) \end{aligned}$$

The equality of the two terms follows from the facts that $Y < x^*(r)$ if and only if $b_r(Y) < r$ and that $b_r(x)$ is increasing for $x \geq x^*(r)$. Taking derivative of both sides w.r.t. y for $y \geq x^*(r)$ gives

$$f_{Y|X}(y|x) = b_r'(y) g_{M|B}^r(b_r(y)|b_r(x)) \tag{9}$$

for all $x, y \geq x^*(r)$. Substitute (9) and (8) into (7) proves the lemma.

Proof of Proposition 8. Let $X^{(i:n)}$ denote the ith largest signal among n potential bidders. Then $\Pr(X^{(2:n)} < x^*(r)|X^{(1:n)} \geq x^*(r))$ is observed. By the i.i.d. assumption, $\Pr(X^{(2:n)} < x^*(r)|X^{(1:n)} \geq x^*(r)) = \dfrac{nF_r^{n-1}(1-F_r)}{1-F_r^n}$, where $F_r \equiv \Pr(X_i \leq x^*(r))$. The expression is increasing in F_r. Therefore F_r is identified, and $\Pr(X^{(1)} < x^*(r)) = F_r^n$.

9 Appendix B: Consistency of the three-step estimator

The lemma below extends the Basic Consistency Theorem of extreme estimators to those defined over random, compact sets (as opposed to fixed, compact sets). The proof is an adaptation from that of Theorem 4.1.1 in Amemiya (1985) and is included in *Lemma A2* of Li, Perrigne and Vuong (2003).

Lemma B1 Let $Q(.)$ and $\hat{Q}_N(.)$ be nonstochastic and stochastic real-valued functions defined respectively on compact intervals $\Theta \equiv [\theta^l, \theta^u]$ and $\Theta_N \equiv [\theta_N^l, \theta_N^u]$, where $\Pr\{[\theta_N^l, \theta_N^u] \subseteq [\theta^l, \theta^u]\} = 1$ for all N and $\theta_N^k \to \theta^k$ almost surely for $k = l, u$. For every $N = 1, 2, ...$, let $\hat{\theta}_N \in \Theta_N$ be such that $\hat{Q}_N(\hat{\theta}_N) \leq \inf_{\theta \in \Theta_N} \hat{Q}_N(\theta) + o_p(1)$. If $Q(.)$ is continuous on Θ with a unique maximizer on Θ at $\theta_0 \in [\theta^l, \theta^u]$ and (ii) $\sup_{\theta \in \Theta_N} \left| \hat{Q}_N(\theta) - Q(\theta) \right| \xrightarrow{p} 0$ as $N \longrightarrow +\infty$, then $\hat{\theta}_N \xrightarrow{p} \theta_0$.

9.1 Regularity properties of $G_{M,B}$ and $g_{M,B}$

Let $f_{Y,X}$ and $F_{Y,X}$ denote the joint density and distribution of Y_i and X_i respectively. Let $\beta(.)$ be the bidding strategy in increasing, pure-strategy perfect Bayesian Nash equilibria. That is, $\beta(x) = \int_{x_L}^x v_h(s,s) dL(s|x)$ where $L(s|x) = \exp\{-\int_s^x \frac{f_{YX}(u,u)}{F_{YX}(u,u)} du\}$. The lemma below gives regularity results about the smoothness of the equilibrium bidding strategy.

Lemma B2 Under S1 and S2, (i) ξ has R continuous bounded derivatives on $[b_L, b_U]$ and $\xi'(.) \geq c > 0$ for some constant on $[b_L, b_U]$; (ii) $G_{M,B}$ and $g_{M,B}$ both have $R-1$ continuous bounded partial derivatives on $[b_L, b_U]^2$.

Proof. First, I show that under S1 and S2, the equilibrium bidding function $b^0(.)$ admits up to R continuous, bounded derivatives on $[x_L, x_U]$, and $b^{0\prime}(.)$ is bounded below from zero on S_X. Recall b^0 solves the differential equation $b^{0\prime}(x) = [v_h(x,x) - b^0(x)]\frac{f_{Y|X}(x|x)}{F_{Y|X}(x|x)}$ with the boundary condition $b^0(x_L) = v_h(x_L, x_L)$. Under S1, the joint density f has R continuous bounded derivatives on $[x_L, x_U]$. By symmetry of $F_{\mathbf{X}}$,

$$v_h(x,x) = \frac{\int_{x_L}^{x} ... \int_{x_L}^{x} \theta(x,x,x_3,..,x_n) f(x,x,x_3,..,x_n) dx_3...dx_n}{\int_{x_L}^{x} ... \int_{x_L}^{x} f(x,x,x_3,..,x_n) dx_3...dx_n}$$

Under S1, the denominator is 0 if and only if $x = x_L$. Since by S2, the product $\theta(.,.,.)f(.,.,.)$ also has R continuous, bounded derivatives, $v_h(x,x)$ has $R+n-2$ continuous, bounded derivatives on $(x_L, x_U]$. Furthermore, it can be shown that $\frac{f_{Y|X}(x|x)}{F_{Y|X}(x|x)}$ also has $R+n-2$ continuous, bounded derivatives on any compact subsets of $(x_L, x_U]$. Therefore, $b^0(.)$ has $R+n-1$ continuous, bounded derivatives on any compact subsets of $(x_L, x_U]$. As for the boundary point x_L, the proof proceeds by applying Taylor expansions of f around the zero vector in the definition of $\frac{f_{Y|X}}{F_{Y|X}}$, $L(s|x)$ and $v(x,x)$, and then showing β has R continuous, bounded derivatives at $x = x_L$. It is a direct extension from proof of Lemma A2 in Li, Perrigne and Vuong (2002) and excluded here for brevity. That $b^{0\prime}$ is bounded away from zero on $[x_L, x_U]$ follows from the same arguments as in Lemma A2 in Guerre, Perrigne and Vuong (2000) and not repeated here.

Let $g_{M,B}$ denote the joint density of equilibrium bids B^0 and highest rival bid M^0, and define $G_{M,B}(m,b) \equiv \int_{b_L}^{m} g_{M,B}(\tilde{b}, b) d\tilde{b}$. The relevant support is $S_B^2 = [b_L^0, b_U^0]^2$ where $b_L^0 = b^0(x_L) = v_h(x_L, x_L) = \theta(\mathbf{x}_L)$ and $b_U^0 = b^0(x_U)$. (For notational ease, below I use $v_h(x)$ as a shorthand for $v_h(x,x)$.) In equilibrium, $b^0(v_h^{-1}(\xi(b))) = b$ for all $b \in S_B$. Hence $\xi'(b) = \{b^{0\prime}[v_h^{-1}(\xi(b))] v_h^{-1\prime}(\xi(b))\}^{-1}$ where both $v_h^{-1\prime}(.)$ and $b^{0\prime}(.)$ are bounded away from zero and have $R-1$ continuous derivatives under *S2*. This proves part (i). For part (ii), note $\Pr(M \leq m, B \leq b) = \Pr(Y \leq v_h^{-1}(\xi(m)), X \leq v_h^{-1}(\xi(b)))$ by the monotonicity of $b^0(.)$. Hence $G_{M,B}(m,b) = \frac{\partial}{\partial B} \Pr(M \leq m, B \leq b) = v_h^{-1\prime}(\xi(b))\xi'(b) \Pr(Y \leq v_h^{-1}(\xi(m)), X = v_h^{-1}(\xi(b)))$, where $\Pr(Y \leq y, X = x)$ has $R+n-1$ bounded, continuous derivatives on S_X^2 and the first two terms have $R-1$ continuous bounded derivatives on $[b_L, b_U]^2$ as shown above. Besides, $g_{M,B}(m,b) = \frac{\partial}{\partial M} G_{M,B}(m,b) = v_h^{-1\prime}(\xi(b))\xi'(b) v_h^{-1\prime}(\xi(m))\xi'(m) f_{Y,X}[v_h^{-1}(\xi(m)), v_h^{-1}(\xi(b))]$, where the joint density $f_{Y,X}$ has $R+n-2$ bounded, continuous derivatives on S_B^2 and $v_h^{-1\prime}(.)$ has $R-1$ continuous derivatives on S_X. Hence $g_{M,B}(m,b)$ has $R-1$ continuous derivatives on S_B^2.

9.2 Consistency of $\hat{b}^0_{l,r}$ and $\hat{b}^0_{h,r}$

The following lemma establishes the rate of uniform convergence of kernel estimates $\hat{G}_{M,B}$ and $\hat{g}_{M,B}$ to $G_{M,B}$ and $g_{M,B}$ over $S^2_{B,\delta}$, and $\tilde{G}_{M,B}$ to $G_{M,B}$ over $\hat{S}^2_{B,\delta}$. It is a preliminary step for proving uniform convergence of $\hat{\xi}_l$, $\hat{\xi}$ and $\hat{\delta}_{l,r}$, $\hat{\delta}_{h,r}$.

Lemma B3 Let $h_G = c_G(\log L/L)^{1/(2R+2n-5)}$ *and* $h_g = c_g(\log L/L)^{1/(2R+2n-4)}$. *Under S1-3,*

$$\begin{aligned}
&\sup\nolimits_{S^2_{B,\delta}} |\hat{G}_{M,B} - G_{M,B}| = O(h_G^{R-1}), \quad \sup\nolimits_{S^2_{B,\delta}} |\hat{g}_{M,B} - g_{M,B}| = O(h_g^{R-1}) \\
&\sup\nolimits_{b \in \hat{S}_{B,\delta}} |\tilde{G}_{M,B}(b,b) - G_{M,B}(b,b)| = O_p(h_g^{R-1})
\end{aligned}$$

Furthermore, if $R > n$,

$$\sup\nolimits_{\tilde{b}_L \le b, b \in S_{B,\delta}} |\frac{\hat{G}_{M,B}(\tilde{b}_L, b)}{\hat{G}_{M,B}(b,b)} - \frac{G_{M,B}(\tilde{b}_L, b)}{G_{M,B}(b,b)}| = O_p(h_g^{R-n})$$

***Proof*.** That $\sup_{S^2_{B,\delta}} \left|\hat{G}_{M,B} - G_{M,B}\right| = O(h_G^{R-1})$ and $\sup_{S^2_{B,\delta}} |\hat{g}_{M,B} - g_{M,B}| = O(h_g^{R-1})$ follows from Lemma *A5* in Li, Perrigne and Vuong (2002). By triangular inequality, for all $b \in \hat{S}_{B,\delta}$, $\left|\tilde{G}_{M,B}(b,b) - G_{M,B}(b,b)\right| \le \int_{\tilde{b}_L}^b |\hat{g}_{M,B}(t,b) - g_{M,B}(t,b)|\,dt + \left|\hat{G}_{M,B}(\tilde{b}_L, b) - G_{M,B}(\tilde{b}_L, b)\right|$. Thus

$$\begin{aligned}
&\sup\nolimits_{b \ge \tilde{b}_L, b \in S_{B,\delta}} |\tilde{G}_{M,B}(b,b) - G_{M,B}(b,b)| \\
\le\ & |b_U - b_L| \sup\nolimits_{t \le b, (t,b) \in S^2_{B,\delta}} |\hat{g}_{M,B}(t,b) - g_{M,B}(t,b)| + O_p(h_G^{R-1}) = O_p(h_g^{R-1})
\end{aligned}$$

since by construction $\hat{S}_{B,\delta} \subseteq S_{B,\delta}$ with probability 1 and $h_G < h_g$ for L large enough. Furthermore, note:

$$\begin{aligned}
1\{\tilde{b}_L &\le b\} \left|\tfrac{\hat{G}_{M,B}(\tilde{b}_L,b)}{\hat{G}_{M,B}(b,b)} - \tfrac{G_{M,B}(\tilde{b}_L,b)}{G_{M,B}(b,b)}\right| \\
&\le \tfrac{\mathbf{1}\{\tilde{b}_L \le b\}}{\hat{G}_{M,B}(b,b)} \left|\hat{G}_{M,B}(\tilde{b}_L, b) - G_{M,B}(\tilde{b}_L, b) + \tfrac{G_{M,B}(\tilde{b}_L,b)}{G_{M,B}(b,b)} \left(G_{M,B}(b,b) - \hat{G}_{M,B}(b,b)\right)\right| \\
&\le \tfrac{\mathbf{1}\{\tilde{b}_L \le b\}}{\inf_{b \ge \tilde{b}_L} \left|\hat{G}_{M,B}(b,b)\right|} \left(\left|\hat{G}_{M,B}(\tilde{b}_L, b) - G_{M,B}(\tilde{b}_L, b)\right| + \left|G_{M,B}(b,b) - \hat{G}_{M,B}(b,b)\right|\right)
\end{aligned}$$

Thus

$$\begin{aligned}
&\sup\nolimits_{\tilde{b}_L \le b, b \in S_{B,\delta}} |\tfrac{\hat{G}_{M,B}(\tilde{b}_L,b)}{\hat{G}_{M,B}(b,b)} - \tfrac{G_{M,B}(\tilde{b}_L,b)}{G_{M,B}(b,b)}| \\
\le\ & \tfrac{\mathbf{1}}{\inf_{b \in S_{B,\delta}} \left|\hat{G}_{M,B}(b,b)\right|} \left\{ \begin{array}{c} \sup_{\tilde{b}_L \le b, b \in C^2_\delta(B)} |\hat{G}_{M,B}(\tilde{b}_L, b) - G_{M,B}(\tilde{b}_L, b)| + \ldots \\ \sup_{\tilde{b}_L \le b, b \in C_\delta(B)} |G_{M,B}(b,b) - \hat{G}_{M,B}(b,b)| \end{array} \right\}
\end{aligned}$$

where

$$\begin{aligned}\inf_{b\in S_{B,\delta}}|\hat{G}_{M,B}(b,b)| &\geq \inf_{b\in S_{B,\delta}}|G_{M,B}(b,b)| - \sup_{b\in S_{B,\delta}}|\hat{G}_{M,B}(b,b)-G_{M,B}(b,b)| \\ &= \inf_{b\in S_{B,\delta}}|G_{M,B}(b,b)| + O_p(h_G^{R-1})\end{aligned}$$

Note $G_{M,B}(b,b)=\int_{b_L}^{b}...\int_{b_L}^{b} g(b,b_2,b_3,..,b_n)\,db_2...db_n$ and $g(b_1,..,b_n)=f(b^{0,-1}(b_1),..,b^{0,-1}(b_n))$ has R continuous derivatives on $[b_L^0,b_U^0]^n$. Since $R>n$, we can apply a Taylor expansion of $g(.)$ around $(b_L^0,.,b_L^0)$ to get $G_{M,B}(b,b)=a(b-b_L)^{n-1}+o(|b-b_L|^{n-1})$ with $a\equiv g(b_L^0,..,b_L^0)>0$. It can then be shown $\inf_{b\in S_{B,\delta}}|G_{M,B}(b,b)|\geq\alpha\delta^{n-1}$ for some $\alpha>0$ and $\delta=\max(h_g,h_G)$.[39] Since $R>n$ and $\delta=h_g$ for L large enough, we have $\inf_{b\in S_{B,\delta}}\left|\hat{G}_{M,B}(b,b)\right|\geq\alpha h_g^{n-1}+o_p(h_g^{n-1})$. Since $\sup_{\tilde{b}_L\leq b,b\in S_{B,\delta}}|\hat{G}_{M,B}(\tilde{b}_L,b)-G_{M,B}(\tilde{b}_L,b)|$ and $\sup_{\tilde{b}_L\leq b,b\in S_{B,\delta}}|\hat{G}_{M,B}(\tilde{b}_L,b)-G_{M,B}(\tilde{b}_L,b)|$ are both bounded by $O_p(h_g^{R-1})$, it follows that $\sup_{\tilde{b}_L\leq b,b\in S_{B,\delta}}|\frac{\hat{G}_{M,B}(\tilde{b}_L,b)}{\hat{G}_{M,B}(b,b)}-\frac{G_{M,B}(\tilde{b}_L,b)}{G_{M,B}(b,b)}|=O_p(h_g^{R-n})$.

The next lemma proves the uniform convergence of $\hat{\xi}$ and $\hat{\xi}_l$ over the support $S_{B,\delta}^2$.

Lemma B4 Let $h_G=c_G(\log L/L)^{1/(2R+2n-5)}$ *and* $h_g=c_g(\log L/L)^{1/(2R+2n-4)}$. *Under S1-3,* $\sup_{b\in S_B}|\hat{\xi}(b)-\xi(b)|=O_p(h_g^{R-(n-1)})$ *if* $R>n$. *Furthermore if* $R>2(n-1)$, $\sup_{b\geq\tilde{b}_L,b\in S_B}|\hat{\xi}_l(b)-\xi_l(b)|=O_p(h_g^{R-2(n-1)})$.

Proof. Proposition *A2 (ii)* in Li, Perrigne and Vuong (2002) showed $\sup_{b\in C_\delta(B)}|\hat{\xi}(b)-\xi(b)|=O_p(h_g^{R-(n-1)})$. By definition, $\tilde{b}_L\in S_{B,\delta}$. Note :

$$\begin{aligned}&\sup_{\tilde{b}_L\leq b,(\tilde{b}_L,b)\in S_{B,\delta}^2}|\hat{\xi}_l(b)-\xi_l(b)| \\ \leq\ &\sup_{\tilde{b}_L\leq b,(\tilde{b}_L,b)\in S_{B,\delta}^2}\left|\int_{\tilde{b}_L}^{b}\hat{\xi}(t)\tfrac{\hat{g}_{M,B}(t,b)}{\hat{G}_{M,B}(b,b)}dt-\int_{\tilde{b}_L}^{b}\xi(t)\tfrac{g_{M,B}(t,b)}{G_{M,B}(b,b)}dt\right|+... \\ &\sup_{\tilde{b}_L\leq b,(\tilde{b}_L,b)\in S_{B,\delta}^2}\left|\hat{\xi}(\tilde{b}_L)\tfrac{\hat{G}_{M,B}(\tilde{b}_L,b)}{\hat{G}_{M,B}(b,b)}-\int_{b_L}^{\tilde{b}_L}\xi(t)\tfrac{g_{M,B}(t,b)}{G_{M,B}(b,b)}dt\right|\end{aligned}$$

Below I show the two terms converge in probability to 0. By definition, $\tilde{b}_L\geq b_L^0$, and $\int_{b_L^0}^{\tilde{b}_L}\xi(t)\frac{g_{M,B}(t,b)}{G_{M,B}(b,b)}dt$ is bounded between $\xi(b_L^0)\frac{G_{M,B}(\tilde{b}_L,b)}{G_{M,B}(b,b)}$ and $\xi(\tilde{b}_L)\frac{G_{M,B}(\tilde{b}_L,b)}{G_{M,B}(b,b)}$. With probability 1,

$$\begin{aligned}&\sup_{\tilde{b}_L\leq b,(\tilde{b}_L,b)\in S_{B,\delta}^2}\left|\hat{\xi}(\tilde{b}_L)\tfrac{\hat{G}_{M,B}(\tilde{b}_L,b)}{\hat{G}_{M,B}(b,b)}-\int_{b_L}^{\tilde{b}_L}\xi(t)\tfrac{g_{M,B}(t,b)}{G_{M,B}(b,b)}dt\right| \\ \leq\ &\max\{\sup_{\tilde{b}_L\leq b,(\tilde{b}_L,b)\in S_{B,\delta}^2}T_1(b;\tilde{b}_L),\sup_{\tilde{b}_L\leq b,(\tilde{b}_L,b)\in S_{B,\delta}^2}T_2(b;\tilde{b}_L)\}\end{aligned}$$

[39] For details, see Lemma *A6* in Li et.al 2002.

where $T_1(b;\tilde{b}_L) \equiv \left|\hat{\xi}(\tilde{b}_L)\frac{\hat{G}_{M,B}(\tilde{b}_L,b)}{\hat{G}_{M,B}(b,b)} - \xi(b_L^0)\frac{G_{M,B}(\tilde{b}_L,b)}{G_{M,B}(b,b)}\right|$ and $T_2(b;\tilde{b}_L) \equiv \left|\hat{\xi}(\tilde{b}_L)\frac{\hat{G}_{M,B}(\tilde{b}_L,b)}{\hat{G}_{M,B}(b,b)} - \xi(\tilde{b}_L)\frac{G_{M,B}(\tilde{b}_L,b)}{G_{M,B}(b,b)}\right|$. With probability 1, for all $(b,\tilde{b}_L) \in S_{B,\delta}^2$ such that $b \geq \tilde{b}_L$,

$$T_1(b;\tilde{b}_L) \leq \left|\hat{\xi}(\tilde{b}_L)\left(\frac{\hat{G}_{M,B}(\tilde{b}_L,b)}{\hat{G}_{M,B}(b,b)} - \frac{G_{M,B}(\tilde{b}_L,b)}{G_{M,B}(b,b)}\right)\right| + \left|\frac{G_{M,B}(\tilde{b}_L,b)}{G_{M,B}(b,b)}\left(\hat{\xi}(\tilde{b}_L) - \xi(b_L^0)\right)\right|$$

where $\left|\frac{G_{M,B}(\tilde{b}_L,b)}{G_{M,B}(b,b)}\right| \leq 1$ by construction. Thus

$$\sup_{\tilde{b}_L\leq b,(\tilde{b}_L,b)\in S_{B,\delta}^2} T_1(b;\tilde{b}_L) \leq \sup_{\tilde{b}_L\leq b,(\tilde{b}_L,b)\in S_{B,\delta}^2}\left\{\begin{array}{c}\left|\frac{\hat{G}_{M,B}(\tilde{b}_L,b)}{\hat{G}_{M,B}(b,b)} - \frac{G_{M,B}(\tilde{b}_L,b)}{G_{M,B}(b,b)}\right|\left|\hat{\xi}(\tilde{b}_L)\right| + ... \\ \left|\hat{\xi}(\tilde{b}_L) - \xi(b_L^0)\right|\end{array}\right\}$$

Note $\left|\hat{\xi}(\tilde{b}_L)\right| \xrightarrow{p} |\xi(b_L^0)| < \infty$ and $\left|\hat{\xi}(\tilde{b}_L) - \xi(b_L)\right| \xrightarrow{p} 0$ by the uniform convergence of $\hat{\xi}$ over $S_{B,\delta}$ and that $\tilde{b}_L \xrightarrow{p} b_L^0$. By Lemma A3, $\sup_{\tilde{b}_L\leq b,(\tilde{b}_L,b)\in S_{B,\delta}}\left|\frac{\hat{G}_{M,B}(\tilde{b}_L,b)}{\hat{G}_{M,B}(b,b)} - \frac{G_{M,B}(\tilde{b}_L,b)}{G_{M,B}(b,b)}\right| \xrightarrow{p} 0$ if $R > n$. Hence $\sup_{\tilde{b}_L\leq b,(\tilde{b}_L,b)\in S_{B,\delta}^2} T_1(b;\tilde{b}_L) \xrightarrow{p} 0$ if $R > n$. That $\sup_{\tilde{b}_L\leq b,(\tilde{b}_L,b)\in S_{B,\delta}^2} T_2(b;\tilde{b}_L) \xrightarrow{p} 0$ follows from similar arguments.

By the triangular inequality, for all $b \geq \tilde{b}_L$, and $(\tilde{b}_L, b) \in S_{B,\delta}^2$

$$\begin{aligned}&\left|\int_{\tilde{b}_L}^b \hat{\xi}(t)\frac{\hat{g}_{M,B}(t,b)}{\tilde{G}_{M,B}(b,b)}dt - \int_{\tilde{b}_L}^b \xi(t)\frac{g_{M,B}(t,b)}{G_{M,B}(b,b)}dt\right| \\ \leq\ & \frac{1}{\inf_{\tilde{b}_L\leq b,(\tilde{b}_L,b)\in S_{B,\delta}^2}\left|\tilde{G}_{M,B}(b,b)\right|}\left\{\begin{array}{c}\left|\int_{\tilde{b}_L}^b \hat{\xi}(t)\hat{g}_{M,B}(t,b)dt - \xi(t)g_{M,B}(t,b)dt\right| + ... \\ \xi(b_U^0)\left|\tilde{G}_{M,B}(b,b) - G_{M,B}(b,b)\right|\end{array}\right\}\end{aligned}$$

where I use $\int_{\tilde{b}_L}^b \xi(t)\frac{g_{M,B}(t,b)}{G_{M,B}(b,b)}dt \leq \xi(b_U^0)$. Also note $\left|\tilde{G}_{M,B}(b,b)\right| \geq |G_{M,B}(b,b)| - \left|\tilde{G}_{M,B}(b,b) - G_{M,B}(b,b)\right|$. It is shown in *Lemma B3* above that $\inf_{b\in S_{B,\delta}}|G_{M,B}(b,b)| \geq \alpha h_g^{n-1} + o_p(h_g^{n-1})$ with $R > n$ and $\sup_{\tilde{b}_L\leq b,(\tilde{b}_L,b)\in S_{B,\delta}^2}|\tilde{G}_{M,B}(b,b) - G_{M,B}(b,b)| = O_p(h_g^{R-1})$. Furthermore for all $b \geq \tilde{b}_L$, and $(\tilde{b}_L, b) \in S_{B,\delta}^2$,

$$\begin{aligned}&\left|\int_{\tilde{b}_L}^b \hat{\xi}(t)\hat{g}_{M,B}(t,b)dt - \xi(t)g_{M,B}(t,b)dt\right| \\ \leq\ & \int_{\tilde{b}_L}^b \left|\hat{\xi}(t) - \xi(t)\right||\hat{g}_{M,B}(t,b)|\,dt + \int_{\tilde{b}_L}^b |\xi(t)|\,|\hat{g}_{M,B}(t,b) - g_{M,B}(t,b)|\,dt\end{aligned}$$

The boundedness of $g_{M,B}$ and ξ implies

$$\sup_{\tilde{b}_L\leq b,(\tilde{b}_L,b)\in S_{B,\delta}^2}\left|\int_{\tilde{b}_L}^b \hat{\xi}(t)\hat{g}_{M,B}(t,b)dt - \xi(t)g_{M,B}(t,b)dt\right| = O_p(h_g^{R-(n-1)})$$

which is the rate of convergence of $\sup_{b\in S_{B,\delta}}\left|\hat{\xi} - \xi\right|$. As a result,

$$\sup_{\tilde{b}_L\leq b,(\tilde{b}_L,b)\in S_{B,\delta}^2}\left|\int_{\tilde{b}_L}^b \hat{\xi}(t)\frac{\hat{g}_{M,B}(t,b)}{\tilde{G}_{M,B}(b,b)}dt - \int_{\tilde{b}_L}^b \xi(t)\frac{g_{M,B}(t,b)}{G_{M,B}(b,b)}dt\right| = O_p(h_g^{R-2(n-1)})$$

and converges to zero when $R > 2(n-1)$.

The proposition below establishes the consistency of $\hat{b}^0_{k,r}$ using Lemma B1.

Proposition B1 Let $h_G = c_G(\log L/L)^{1/(2R+2n-5)}$ *and* $h_g = c_g(\log L/L)^{1/(2R+2n-4)}$. *Under S1-3,* $\hat{b}^0_{l,r} \xrightarrow{p} b^0_{l,r}(G_{\mathbf{B}^0})$ *if* $R > n-1$ *and* $\hat{b}^0_{h,r} \xrightarrow{p} b^0_{h,r}(G_{\mathbf{B}^0})$ *if* $R > 2(n-1)$ *for all* $r \in S_{RP}$.

Proof. First note that $\tilde{b}_k \to b^0_k$ almost surely for $k = L, U$, and that $\delta \longrightarrow 0$. Hence $\hat{b}_L \xrightarrow{a.s.} b^0_L$. It suffices to show that for all $r \in S_{RP}$, *(i)* $(\hat{\xi}(b) - r)^2$ and $(\hat{\xi}_l(b) - r)^2$ converge in probability to $(\xi(b) - r)^2$ and $(\xi_l(b) - r)^2$ uniformly over $\hat{S}^2_{B,\delta}$; and *(ii)* $(\xi(b) - r)^2$ and $(\xi_l(b) - r)^2$ are continuous on $[b^0_L, b^0_U]$ with unique minimizers $b^0_{l,r}$ and $b^0_{h,r}$ respectively on $[b^0_L, b^0_U]$. By *Lemma B4*, $\sup_{b\in S_{B,\delta}} |\hat{\xi}(b) - \xi(b)| \xrightarrow{p} 0$ and $\sup_{b\geq \tilde{b}_L, (\tilde{b}_L, b)\in S^2_{B,\delta}} |\hat{\xi}_l(b) - \xi_l(b)| \xrightarrow{p} 0$. And

$$\sup_{b\in S_{B,\delta}} \left|(\hat{\xi}(b) - r)^2 - (\xi(b) - r)^2\right| \leq \sup_{b\in S_{B,\delta}} \left|\hat{\xi}^2(b) - \xi^2(b)\right| + 2r \sup_{b\in S_{B,\delta}} \left|\hat{\xi}(b) - \xi(b)\right|$$

where both terms converge to 0 in probability since $\sup_{b\in S_{B,\delta}} \xi(b) \leq \xi(b^0_U) < \infty$. Likewise $\sup_{b\geq \tilde{b}_L, (\tilde{b}_L, b)\in S^2_{B,\delta}} \left|(\hat{\xi}_l(b) - r)^2 - (\xi_l(b) - r)^2\right| \xrightarrow{p} 0$ by similar arguments. Next, the continuity of $(\xi(b) - r)^2$ and $(\xi_l(b) - r)^2$ follows from the smoothness of ξ. Also both $\xi(.; G_{\mathbf{B}^0})$ and $\xi_l(.; G_{\mathbf{B}^0})$ are increasing on $[b^0_L, b^0_U]$ by the monotonicity of $v_h(., .; \psi)$ and $v_l(., .; \psi)$ as well as $b^0(.; \psi)$ on S_X for all $\psi \in \Psi(G_{\mathbf{B}^0})$. Thus for all $r \in S_{RP}$, the minimizers of $(\xi(b) - r)^2$ and $(\xi_l(b) - r)^2$ are unique on $[b^0_L, b^0_U]$.

9.3 Uniform convergence of $\hat{\delta}_{k,r}(.; \hat{b}^0_{k,r})$

Lemma B5 Let $h_G = c_G(\log L/L)^{1/(2R+2n-5)}$ *and* $h_g = c_g(\log L/L)^{1/(2R+2n-4)}$. *Under S1-3 and if* $R > 2n-1$,

$$\sup_{b\in S_{B,\delta}} \left|\frac{\hat{g}_{M,B}(b,b)}{\hat{G}_{M,B}(b,b)} - \frac{g_{M,B}(b,b)}{G_{M,B}(b,b)}\right| = O_p(h^{R-2n+1})$$

Proof. The proof is similar to the last part of Lemma B3. On the support $S^2_{B,\delta}$,

$$\left|\frac{\hat{g}_{M,B}}{\hat{G}_{M,B}} - \frac{g_{M,B}}{G_{M,B}}\right| \leq \frac{1}{|\hat{G}_{M,B}||G_{M,B}|}\left(|G_{M,B}|\,|\hat{g}_{M,B} - g_{M,B}| + |g_{M,B}|\left|\hat{G}_{M,B} - G_{M,B}\right|\right)$$

By Lemma B3, $\sup_{S^2_{B,\delta}} |\hat{G}_{M,B} - G_{M,B}| = O_p(h_G^{R-1})$ and $\sup_{S^2_{B,\delta}} |\hat{g}_{M,B} - g_{M,B}| = O_p(h_g^{R-1})$. Besides, $\sup_{S^2_{B,\delta}} |G_{M,B}| < \infty$ and $\sup_{S^2_{B,\delta}} |g_{M,B}| < \infty$ implies the supremum of the term in the bracket is $O_p(h_g^{R-1})$ as $h_g > h_G$ for L large enough. Hence $\sup_{b\in S_{B,\delta}} \left|\frac{\hat{g}_{M,B}(b,b)}{\hat{G}_{M,B}(b,b)} - \frac{g_{M,B}(b,b)}{G_{M,B}(b,b)}\right| \leq$

$\frac{O_p(h_g^{R-1})}{\inf_{b\in S_{B,\delta}}\left|\hat{G}_{M,B}\right|\inf_{b\in S_{B,\delta}}\left|G_{M,B}\right|}$, where the two terms in the denominator are bounded below by $\alpha h_g^{n-1}+o(h_g^{n-1})$ and $\beta h_g^{n-1}+o(h_g^{n-1})$ respectively by some constant α and β. It follows the denominator is bounded below by $\gamma h_g^{2n-2}+o(h_g^{2n-2})$. Hence $\sup_{b\in S_{B,\delta}}|\frac{\hat{g}_{M,B}(b,b)}{\hat{G}_{M,B}(b,b)}-\frac{g_{M,B}(b,b)}{G_{M,B}(b,b)}|=O_p(h^{R-2n+1})$ and converges in probability to 0 if $R>2n-1$.

Lemma B6 Let $h_G=c_G(\log L/L)^{1/(2R+2n-5)}$ *and* $h_g=c_g(\log L/L)^{1/(2R+2n-4)}$. *Under S1-3 and suppose* $R>2n-1$, *then* $\sup_{b\in S_{B,\delta}}\left|\hat{\delta}_{k,r}(b;\hat{b}_{k,r}^0)-\delta_{k,r}(b;b_{k,r}^0)\right|\xrightarrow{p}0$ *for all* $r\in S_{RP}$.

Proof. Consider the case where r is in the interior of S_{RP} (or equivalently, $b_{k,r}^0\in(b_L^0,b_U^0)$). By definition, $\hat{b}_{k,r}^0\in S_{B,\delta}$, and for sample sizes large enough and δ small enough, $b_{k,r}^0$ is in the interior of $S_{B,\delta}$. By the triangular inequality,

$$\begin{aligned}
&\sup_{b\in S_{B,\delta}}\left|\hat{\delta}_{k,r}(b;\hat{b}_{k,r}^0)-\delta_{k,r}(b;b_{k,r}^0)\right|\\
\leq\;&\sup_{b\in S_{B,\delta}}1(\hat{b}_{k,r}^0\leq b_{k,r}^0)1(b>b_{k,r}^0)\left|\hat{\delta}_{k,r}(b)-\delta_{k,r}(b)\right|+...\\
&\sup_{b\in S_{B,\delta}}1(\hat{b}_{k,r}^0>b_{k,r}^0)1(b>\hat{b}_{k,r}^0)\left|\hat{\delta}_{k,r}(b)-\delta_{k,r}(b)\right|+...\\
&\sup_{b\in S_{B,\delta}}1(\hat{b}_{k,r}^0\leq b_{k,r}^0)1(b\in(\hat{b}_{k,r}^0,b_{k,r}^0])\left|\hat{\delta}_{k,r}(b)-r\right|+...\\
&\sup_{b\in S_{B,\delta}}1(\hat{b}_{k,r}^0>b_{k,r}^0)1(b\in(b_{k,r}^0,\hat{b}_{k,r}^0])\left|\delta_{k,r}(b)-r\right|
\end{aligned}$$

It suffices to show all four terms (denoted A_1, A_2, A_3 and A_4 respectively) converge in probability to 0 uniformly over $b\in S_{B,\delta}$ as sample size increases. For A_1,

$$\begin{aligned}
&\sup_{b\in S_{B,\delta}}1(\hat{b}_{k,r}^0\leq b_{k,r}^0)1(b>b_{k,r}^0)\left|\hat{\delta}_{k,r}(b;\hat{b}_{k,r}^0)-\delta_{k,r}(b;b_{k,r}^0)\right|\\
\leq\;&\sup_{b_{k,r}^0\leq b\leq b_U^0-\delta}1(\hat{b}_{k,r}^0\leq b_{k,r}^0)\left\{\begin{array}{c}\left|\int_{b_{k,r}^0}^{b}\hat{\xi}(t)\hat{\Lambda}(t)\hat{L}(t|b)-\xi(t)\Lambda(t)L(t|b)dt\right|+..\\ \left|\int_{\hat{b}_{k,r}^0}^{b_{k,r}^0}\hat{\xi}(t)\hat{\Lambda}(t)\hat{L}(t|b)dt\right|+r\left|\hat{L}(\hat{b}_{k,r}^0|b)-L(b_{k,r}^0|b)\right|\end{array}\right\}
\end{aligned}$$

It can be shown $\sup_{t\leq b,(t,b)\in S_{B,\delta}^2}\left|\hat{L}(t|b)-L(t|b)\right|\xrightarrow{p}0$ using convergence results from previous lemmae.[40] Note for r in the interior of S_{RP},

$$\begin{aligned}
&\sup_{b_{k,r}^0<b\leq b_U-\delta}1(\hat{b}_{k,r}^0\leq b_{k,r}^0)r\left|\hat{L}(\hat{b}_{k,r}^0|b)-L(b_{k,r}^0|b)\right|\\
\leq\;&\sup_{b_{k,r}^0<b\leq b_U-\delta}1(\hat{b}_{k,r}^0\leq b_{k,r}^0)r\left|\hat{L}(\hat{b}_{k,r}^0|b)-L(\hat{b}_{k,r}^0|b)\right|+...\\
&\sup_{b_{k,r}^0<b\leq b_U-\delta}1(\hat{b}_{k,r}^0\leq b_{k,r}^0)r\left|L(\hat{b}_{k,r}^0|b)-L(b_{k,r}^0|b)\right|
\end{aligned}$$

[40]For details of the proof, see Li, Perrigne and Vuong (2003).

For sufficiently small δ, $b^0_{k,r} > b^0_L + \delta$. By construction $\hat{b}^0_{k,r} \in S_{B,\delta}$, and therefore $\sup_{b^0_{k,r} < b \le b_U - \delta} 1(\hat{b}^0_{k,r} \le b^0_{k,r}) r \left| \hat{L}(\hat{b}^0_{k,r}|b) - L(\hat{b}^0_{k,r}|b) \right| \xrightarrow{p} 0$. Also by the mean value theorem,

$$\begin{aligned} \left| L(\hat{b}^0_{k,r}|b) - L(b^0_{k,r}|b) \right| &= \left| \frac{\partial}{\partial t} L(t|b)|_{t=\tilde{b}^0_{k,r}} (\hat{b}^0_{k,r} - b^0_{k,r}) \right| \\ &= \left| \Lambda(\tilde{b}^0_{k,r}) L(\tilde{b}^0_{k,r}|b) \right| \left| \hat{b}^0_{k,r} - b^0_{k,r} \right| \end{aligned}$$

for some $\tilde{b}^0_{k,r}$ between $\hat{b}^0_{k,r}$ and $b^0_{k,r}$. The consistency of $\hat{b}^0_{k,r}$ for $b^0_{k,r}$ suggests $\tilde{b}^0_{k,r}$ is bounded away from b^0_L as sample size increases. Thus both $\Lambda(\tilde{b}^0_{k,r})$ and $L(\tilde{b}^0_{k,r}|b)$ converge in probability to some finite constant since $\sup_{S_{B,\delta}} |g| < \infty$ and $\inf_{S_{B,\delta}} |g'| > c > 0$ and hence $\sup_{b^0_{k,r} \le b \le b_U - \delta} 1(\hat{b}^0_{k,r} \le b^0_{k,r}) \left| L(\hat{b}^0_{k,r}|b) - L(b^0_{k,r}|b) \right|$ is $o_p(1)$. Next

$$\begin{aligned} & \sup_{b^0_{k,r} < b \le b_U - \delta} \left| \int_{b^0_{k,r}}^{b} \hat{\xi}(t) \hat{\Lambda}(t) \hat{L}(t|b) - \xi(t) \Lambda(t) L(t|b) dt \right| \\ \le \; & \left| b^0_U - b^0_{k,r} \right| \sup_{b^0_{k,r} \le t \le b \le b^0_U - \delta} \left| \hat{\xi}(t) \hat{\Lambda}(t) \hat{L}(t|b) - \xi(t) \Lambda(t) L(t|b) \right| \xrightarrow{p} 0 \end{aligned}$$

where the right hand side is $o_p(1)$ by the uniform convergence of $\hat{\xi}$, $\hat{\Lambda}$, and $\hat{L}$ over $S_{B,\delta}$ and $S^2_{B,\delta}$ under *S1-3* and the boundedness of ξ, Λ and L on the closed interval $[b^0_{k,r}, b^0_U - \delta]$. Finally

$$\begin{aligned} & \sup_{b^0_{k,r} < b \le b_U - \delta} 1(\hat{b}^0_{k,r} \le b^0_{k,r}) \left| \int_{\hat{b}^0_{k,r}}^{b^0_{k,r}} \hat{\xi}(t) \hat{\Lambda}(t) \hat{L}(t|b) dt \right| \\ \le \; & \sup_{t \le b, (t,b) \in S^2_{B,\delta}} \left| \hat{\xi}(t) \hat{\Lambda}(t) \hat{L}(t|b) \right| 1(\hat{b}^0_{k,r} \le b^0_{k,r}) \left| \hat{b}^0_{k,r} - b^0_{k,r} \right| \end{aligned}$$

Note $\sup_{t \le b, (t,b) \in S^2_{B,\delta}} \left| \hat{\xi}(t) \hat{\Lambda}(t) \hat{L}(t|b) - \xi(t) \Lambda(t) L(t|b) \right| \xrightarrow{p} 0$, and $\sup_{b^0_{k,r} < b \le b^0_U - \delta, t \le b, (t,b) \in S^2_{B,\delta}} 1(\hat{b}^0_{k,r} \le t \le b^0_{k,r}) \left| \xi(t) \Lambda(t) L(t|b) \right|$ is bounded with probability approaching 1 as $\hat{b}^0_{k,r} \xrightarrow{p} b^0_{k,r}$. Therefore $\sup_{b^0_{k,r} < b \le b_U - \delta} \left| \int_{\hat{b}^0_{k,r}}^{b^0_{k,r}} \hat{\xi}(t) \hat{\Lambda}(t) \hat{L}(t|b) dt \right| = o_p(1)$ and it follows $A_1 = o_p(1)$. For A_2,

$$\begin{aligned} & \sup_{b \in S_{B,\delta}} 1(\hat{b}^0_{k,r} > b^0_{k,r}) 1(b > \hat{b}^0_{k,r}) \left| \hat{\delta}_{k,r}(b) - \delta_{k,r}(b) \right| \\ \le \; & \sup_{b^0_{k,r} < b \le b^0_U - \delta} 1(b > \hat{b}^0_{k,r} > b^0_{k,r}) \left\{ \begin{array}{c} \left| \int_{\hat{b}^0_{k,r}}^{b} \hat{\xi}(t) \hat{\Lambda}(t) \hat{L}(t|b) - \xi(t) \Lambda(t) L(t|b) dt \right| + \ldots \\ \left| \int_{b^0_{k,r}}^{\hat{b}^0_{k,r}} \xi(t) \Lambda(t) L(t|b) dt \right| + r \left| \hat{L}(\hat{b}^0_{k,r}|b) - L(b^0_{k,r}|b) \right| \end{array} \right\} \end{aligned}$$

where the supremum of the first and last term over $b^0_{k,r} < b \le b^0_U - \delta$ are $o_p(1)$ by the same argument as above, and

$$\begin{aligned}
& \sup_{b^0_{k,r} \le b \le b_U - \delta} 1(b > \hat{b}^0_{k,r} > b^0_{k,r}) \left| \int_{\hat{b}^0_{k,r}}^{b} \hat{\xi}(t)\hat{\Lambda}(t)\hat{L}(t|b) - \xi(t)\Lambda(t)L(t|b)dt \right| \\
\le \ & \sup_{b^0_{k,r} \le b \le b_U - \delta, (t,b) \in S^2_{B,\delta}} \left| \hat{\xi}(t)\hat{\Lambda}(t)\hat{L}(t|b) - \xi(t)\Lambda(t)L(t|b) \right| \left| b - \hat{b}^0_{k,r} \right| = o_p(1)
\end{aligned}$$

For A_3,

$$\begin{aligned}
& \sup_{b \in S_{B,\delta}} 1(\hat{b}^0_{k,r} \le b^0_{k,r}) 1(b \in (\hat{b}^0_{k,r}, b^0_{k,r}]) \left| \hat{\delta}_{k,r}(b) - r \right| \\
= \ & \sup_{b \in S_{B,\delta}} 1(\hat{b}^0_{k,r} \le b^0_{k,r}) 1(b \in (\hat{b}^0_{k,r}, b^0_{k,r}]) \left| r\hat{L}(\hat{b}^0_{k,r}|b) - r + \int_{\hat{b}^0_{k,r}}^{b} \hat{\xi}(t)\hat{\Lambda}(t)\hat{L}(t|b)dt \right|
\end{aligned}$$

By construction, $\hat{b}^0_{k,r} \in S_{B,\delta}$. The uniform convergence of $\hat{L}(t|b)$ for all $t \le b$ on $S^2_{B,\delta}$, the continuity of $L(t|b)$ in both arguments, and $\hat{b}^0_{k,r} \xrightarrow{p} b^0_{k,r}$ suggest $\sup_{b \in S_{B,\delta}} 1(\hat{b}^0_{k,r} \le b^0_{k,r} \cap b \in (\hat{b}^0_{k,r}, b^0_{k,r}]) \left| r\hat{L}(\hat{b}^0_{k,r}|b) - r \right| = o_p(1)$. Also for large samples, $\hat{b}^0_{k,r}$ is bounded away from b^0_L with probability approaching to 1 and $\sup_{b \in S_{B,\delta}} 1(\hat{b}^0_{k,r} \le b^0_{k,r} \cap b \in (\hat{b}^0_{k,r}, b^0_{k,r}]) \left| \int_{\hat{b}^0_{k,r}}^{b} \hat{\xi}(t)\hat{\Lambda}(t)\hat{L}(t|b)dt \right|$ $= o_p(1)$. For A_4, note $\delta_{k,r}$ is continuous at $b^0_{k,r}$ with $\delta_{k,r}(b^0_{k,r}) = r$ and is increasing beyond $b^0_{k,r}$. Hence the consistency of $\hat{b}^0_{k,r}$ is sufficient for $A_4 \xrightarrow{p} 0$. In the boundary case where $b^0_{k,r} = b^0_L$, it suffices to show the convergence of terms $A2$ and $A4$. The same argument above applies.

9.4 Final step of the proof

Lemma B7 Let $h_G = c_G(\log L/L)^{1/(2R+2n-5)}$ and $h_g = c_g(\log L/L)^{1/(2R+2n-4)}$. Under S1-3 and suppose $R > 2n-1$, for any $r \in S_r$, $\hat{\delta}^{-1}_{k,r}(t; \hat{b}^0_{k,r}) \xrightarrow{p} \delta^{-1}_{k,r}(t; b^0_{k,r})$ for $k = \{l, h\}$ and all $t > r$.

Proof. For the range of $r \in S_{RP}$ and $t > r$, $\delta^{-1}_{k,r}(t; b^0_{k,r})$ are unique minimizers of $[\delta_{k,r}(b) - t]^2$ over S_B. *Lemma B6* showed that $\sup_{b \in S_{B,\delta}} \left| \hat{\delta}_{k,r}(b; \hat{b}^0_{k,r}) - \delta_{k,r}(b; b^0_{k,r}) \right| \xrightarrow{p} 0$ and $\delta_{k,r}(.; b^0_{k,r})$ is also continuous on S_B. Also $\tilde{b}_k \to b^0_k$ almost surely for $k = L, U$ as sample size increases. All conditions for *Lemma B1* are satisfied and claim is proven.

Lemma B8 Let $\hat{F}_n(t) = \frac{1}{n}\sum_{i=1}^{n} 1(Z_i \le t)$ where $\{Z_i\}_{i=1}^n$ is an i.i.d. sample from a population distributed as F_Z. Then $\sup_{t \in \mathbb{R}} |\hat{F}_n(t) - F_Z(t)| \xrightarrow{a.s} 0$. If $F_Z(t_0)$ is continuous at t_0 and a sequence of random variable $\hat{t}_n \xrightarrow{p} t_0$ and t_0 is a continuity point of F, then $\hat{F}_n(\hat{t}_n) \xrightarrow{p} F_Z(t_0)$.

Proof. The first claim follows from Glivenko-Cantelli Lemma and the proof of the second claim is standard (e.g. see Theorem 4.1.5 Amemiya 1985).

The proof of *Proposition 4* follows directly from results of the lemmae above.

Proof of Proposition 5. By the first part of *Lemma B8*, $\frac{1}{L_n}\sum_{l=1}^{L_n} 1(B_l^{\max} \leq b)$converge in probability to $\Pr(B_l^{\max} \leq b)$ uniformly over S_B. By *Lemma B7*, $\hat{\delta}_{r,k}^{-1}(t) \xrightarrow{p} \delta_{r,k}^{-1}(t)$ for all r and t in the stated range of interests. The second part of *Lemma B8* proves $\hat{F}_{R(r)}^{l}(t) \xrightarrow{p} F_{R(r)}^{l}(t)$ and $\hat{F}_{R(r)}^{u}(t) \xrightarrow{p} F_{R(r)}^{u}(t)$ for given r and t.

10 Appendix D: Derivations for Monte Carlo designs

A. Closed forms of $v_l(x, x; c)$ and $b^0(x; c)$ in Design 1

Let $X_i = X_0 + \varepsilon_i$ for $i = 1, 2$, where ε_i are statistically independent from (X_0, ε_{-i}). Let ε_i be uniform on $[-c, c]$ and X_0 be uniform on $[a, b]$. For the closed form of v_l and b^0, we need to calculate the conditional expectation $v_l(t, t) = E(X_2|X_2 \leq t, X_1 = t)$ and the inverse hazard ratio $\frac{f_{X_2|X_1=t}(t)}{F_{X_2|X_1=t}(t)}$. For any t_1, t_2 such that $t_2 \in [t_1 - 2c, t_1 + 2c]$,

$$
\begin{aligned}
f_{X_2|X_1=t_1}(t_2) &= \int_{S(t_1)} f_{X_2|X_0=s,X_1=t_1}(t_2) f_{X_0|X_1=t_1}(s) ds \qquad (10) \\
&= \int_{S(t_1)} f_{\varepsilon_2}(t_2 - s) f_{X_0|X_1=t_1}(s) ds
\end{aligned}
$$

where $S(t_1) = [\max(a, t_1 - c), \min(b, t_1 + c)]$ is the support of X_0 given $X_1 = t_1$. The second equality follows from the independence of ε_2 from (X_0, ε_1). The conditional density of X_0 given $X_1 = t_1$ depends on values of a, b and c. In the case $b - a \geq 2c$,

$$
\begin{array}{lll}
f_{X_0|X_1=t_1} \tilde{} & Unif[a, t_1 + c] & \forall\ t_1 \in [a - c, a + c] \\
f_{X_0|X_1=t_1} \tilde{} & Unif[t_1 - c, t_1 + c] & \forall\ t_1 \in [a + c, b - c] \\
f_{X_0|X_1=t_1} \tilde{} & Unif[t_1 - c, b] & \forall\ t_1 \in [b - c, b + c]
\end{array}
$$

and for the case $b - a < 2c$,

$$
\begin{array}{lll}
f_{X_0|X_1=t_1} \tilde{} & Unif[a, t_1 + c] & \forall\ t_1 \in [a - c, b - c] \\
f_{X_0|X_1=t_1} \tilde{} & Unif[a, b] & \forall\ t_1 \in [b - c, a + c] \\
f_{X_0|X_1=t_1} \tilde{} & Unif[t_1 - c, b] & \forall\ t_1 \in [a + c, b + c]
\end{array}
$$

Therefore equation (10) suggests conditional on $X_1 = t_1$, X_2 is distributed as the sum of two independent uniform variables with support on $S(t_1)$ and $[-c, c]$ respectively.

First consider the case $b - a \geq 2c$. If $t_1 \in [a - c, a + c)$,

$$\begin{aligned} f_{X_2|X_1=t_1}(t_2) &= \frac{1}{2c}\left(\frac{t_2 - a + c}{t_1 - a + c}\right) && \forall t_2 \in [a - c, t_1] \\ &= \frac{1}{2c} && \forall t_2 \in [t_1, a + c] \\ &= \frac{1}{2c}\left(\frac{t_1 + 2c - t_2}{t_1 + c - a}\right) && \forall t_2 \in [a + c, t_1 + 2c] \end{aligned}$$

If $t_1 \in [a + c, b - c]$,

$$\begin{aligned} f_{X_2|X_1=t_1}(t_2) &= \frac{1}{4c^2}(t_2 - t_1 + 2c) && \forall t_2 \in [t_1 - 2c, t_1] \\ &= \frac{1}{4c^2}(t_1 + 2c - t_2) && \forall t_2 \in [t_1, t_1 + 2c] \end{aligned}$$

If $t_1 \in (b - c, b + c]$,

$$\begin{aligned} f_{X_2|X_1=t_1}(t_2) &= \frac{1}{2c}\left(\frac{t_2 - t_1 + 2c}{b + c - t_1}\right) && \forall t_2 \in [t_1 - 2c, b - c] \\ &= \frac{1}{2c} && \forall t_2 \in [b - c, t_1] \\ &= \frac{1}{2c}\left(\frac{b + c - t_2}{b + c - t_1}\right) && \forall t_2 \in [t_1, b + c] \end{aligned}$$

Next consider the case $b - a < 2c$. If $t_1 \in [a - c, b - c]$,

$$\begin{aligned} f_{X_2|X_1=t_1}(t_2) &= \frac{1}{2c}\left(\frac{t_2 - a + c}{t_1 - a + c}\right) && \forall t_2 \in [a - c, t_1] \\ &= \frac{1}{2c} && \forall t_2 \in [t_1, a + c] \\ &= \frac{1}{2c}\left(\frac{t_1 + 2c - t_2}{t_1 + c - a}\right) && \forall t_2 \in [a + c, t_1 + 2c] \end{aligned}$$

If $t_1 \in [b - c, a + c]$,

$$\begin{aligned} f_{X_2|X_1=t_1}(t_2) &= \frac{1}{2c}\left(\frac{t_2 - a + c}{b - a}\right) && \forall t_2 \in [a - c, b - c] \\ &= \frac{1}{2c} && \forall t_2 \in [b - c, a + c] \\ &= \frac{1}{2c}\left(\frac{b + c - t_2}{b - a}\right) && \forall t_2 \in [a + c, b + c] \end{aligned}$$

If $t_1 \in [a+c, b+c]$,

$$\begin{aligned} f_{X_2|X_1=t_1}(t_2) &= \frac{1}{2c}\left(\frac{t_2-t_1+2c}{b+c-t_1}\right) && \forall t_2 \in [t_1-2c, b-c] \\ &= \frac{1}{2c} && \forall t_2 \in [b-c, t_1] \\ &= \frac{1}{2c}\left(\frac{b+c-t_2}{b+c-t_1}\right) && \forall t_2 \in [t_1, b+c] \end{aligned}$$

To sum up, $f_{X_2|X_1=t_1}(t_1) = \frac{1}{2c}$ $\forall t_1 \in [a-c, b+c]$ for both cases. Furthermore $F_{X_2|X_1=t_1}(t_1)$ can be calculated by integrating the corresponding densities over proper ranges. That is, $F_{X_2|X_1=t_1}(t_1) = \int_0^{t_1} f_{X_2|X_1=t_1}(s)ds$. Thus both screening level $x^*(r;c)$ can be calculated and equilibrium bids $b_0(x^*(r;c);c)$ can be calculated using numerical approximations.

B. Closed form of $v_l(x,x)$ in Design 2

Distribution and density of truncated normal distributions (with the underlying normal distribution being $N(\mu,\sigma)$ are respectively:

$$\tilde{\Phi}(x) = \frac{\Phi(\frac{x-\mu}{\sigma}) - \Phi(\frac{x_L-\mu}{\sigma})}{\Phi(\frac{x_U-\mu}{\sigma}) - \Phi(\frac{x_L-\mu}{\sigma})}; \quad \tilde{\phi}(x) = \frac{\frac{1}{\sigma}\phi(\frac{x-\mu}{\sigma})}{\Phi(\frac{x_U-\mu}{\sigma}) - \Phi(\frac{x_L-\mu}{\sigma})}$$

where ϕ and Φ denote the density and distribution of the parental (untruncated) normal distribution with mean μ and standard deviation σ, and (x_L, x_U) denotes the pair of truncation points. Suppose X is distributed as truncated normal on (x_L, x_U), then for all $a \in (x_L, x_U)$,

$$E(X|X \le a) \equiv \frac{\int_{x_L}^a x\tilde{\phi}(x)dx}{\tilde{\Phi}(a)} = \frac{\int_{x_L}^a \frac{x}{\sigma}\phi(\frac{x-\mu}{\sigma})dx}{\Phi(\frac{a-\mu}{\sigma}) - \Phi(\frac{x_L-\mu}{\sigma})}$$

Note

$$\begin{aligned} \frac{\partial}{\partial X}\phi(\frac{x-\mu}{\sigma}) &= \phi(\frac{x-\mu}{\sigma})(-\frac{x-\mu}{\sigma})\frac{1}{\sigma} \\ &= -\phi(\frac{x-\mu}{\sigma})\frac{x}{\sigma}\frac{1}{\sigma} + \phi(\frac{x-\mu}{\sigma})\frac{\mu}{\sigma}\frac{1}{\sigma} \\ \Longrightarrow \phi(\frac{x-\mu}{\sigma})\frac{x}{\sigma} &= \phi(\frac{x-\mu}{\sigma})\frac{\mu}{\sigma} - \sigma\frac{\partial}{\partial X}\phi(\frac{x-\mu}{\sigma}) \end{aligned}$$

Then the numerator is

$$\begin{aligned} & \int_{x_L}^a \phi(\frac{x-\mu}{\sigma})\frac{\mu}{\sigma}dx - \int_{x_L}^a \sigma\frac{\partial}{\partial X}\phi(\frac{x-\mu}{\sigma})dx \\ = \; & \mu(\Phi(\frac{a-\mu}{\sigma}) - \Phi(\frac{x_L-\mu}{\sigma})) - \sigma(\phi(\frac{a-\mu}{\sigma}) - \phi(\frac{x_L-\mu}{\sigma})) \end{aligned}$$

Therefore

$$\begin{aligned} E(X|X \leq a) &\equiv \frac{\mu(\Phi(\frac{a-\mu}{\sigma}) - \Phi(\frac{x_L-\mu}{\sigma})) - \sigma(\phi(\frac{a-\mu}{\sigma}) - \phi(\frac{x_L-\mu}{\sigma}))}{\Phi(\frac{a-\mu}{\sigma}) - \Phi(\frac{x_L-\mu}{\sigma})} \\ &= \mu - \sigma\frac{\phi(\frac{a-\mu}{\sigma}) - \phi(\frac{x_L-\mu}{\sigma})}{\Phi(\frac{a-\mu}{\sigma}) - \Phi(\frac{x_L-\mu}{\sigma})} \end{aligned}$$

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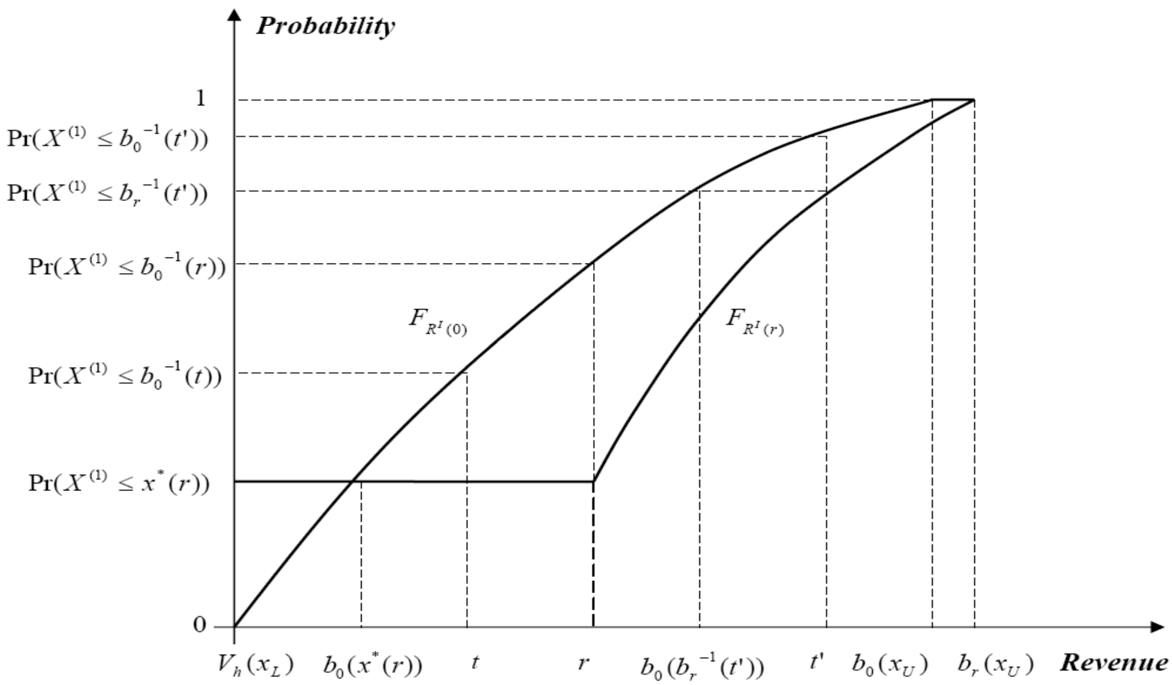
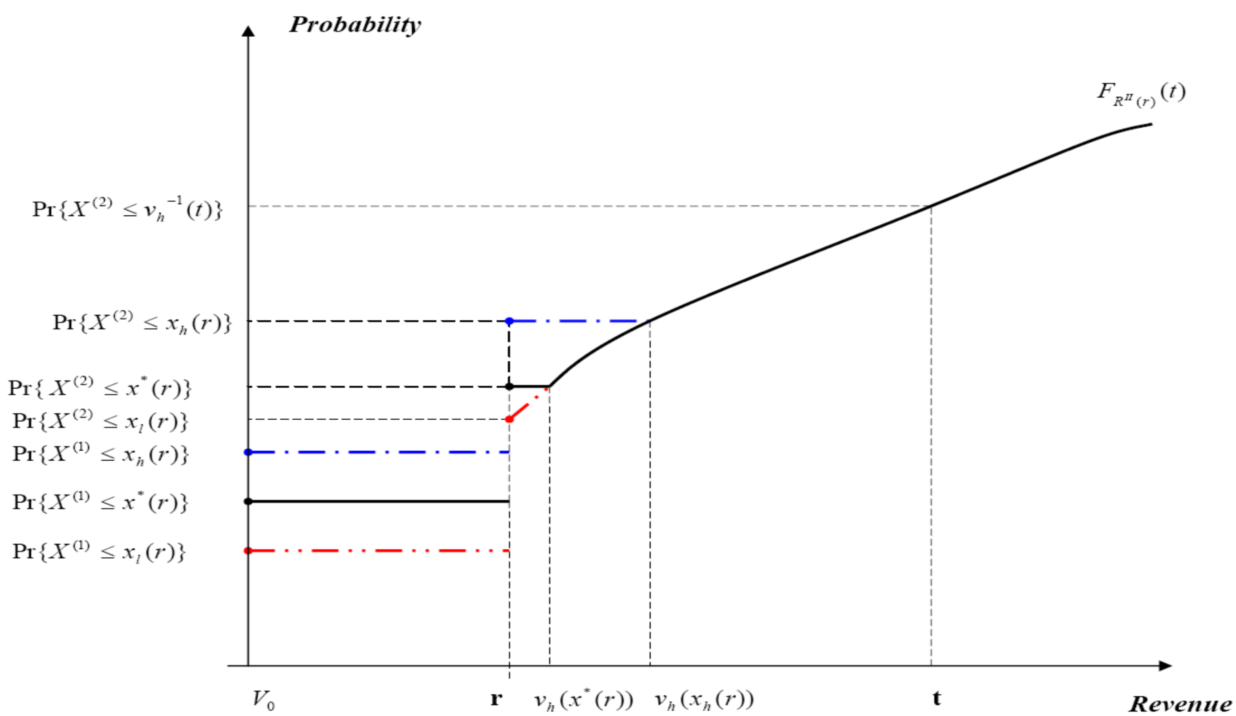
Appendix C : Graphs, Figures and Tables



Graph 1



Graph 2

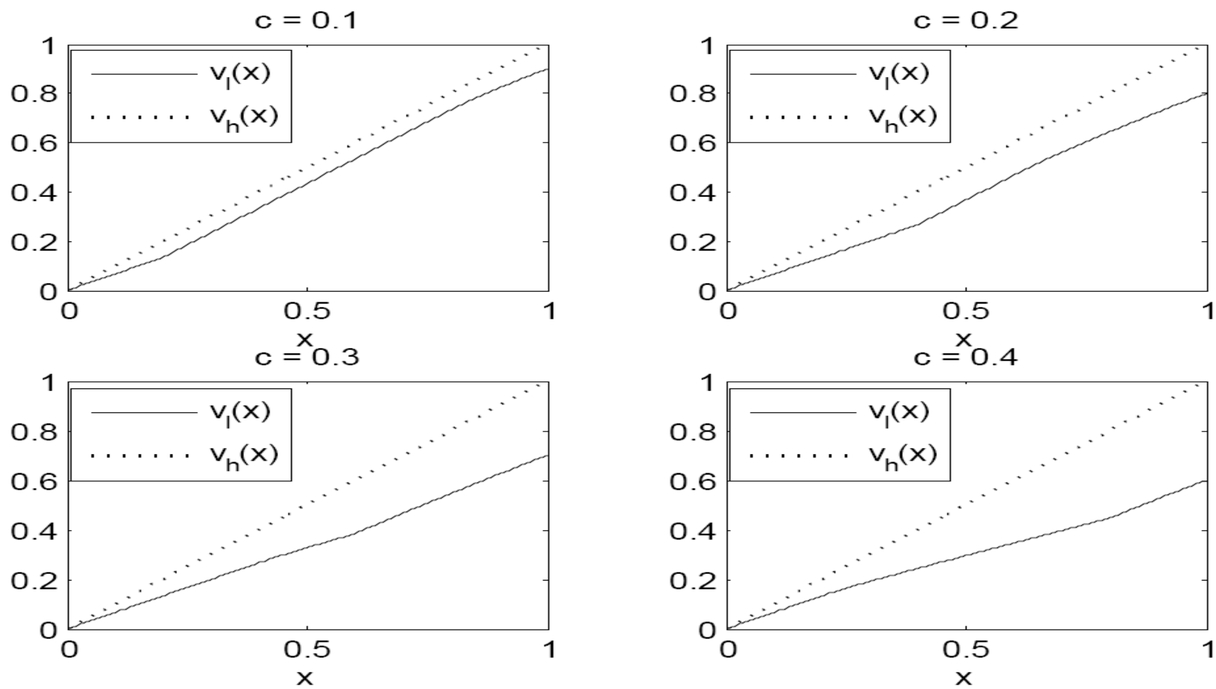


Figure 1 (a)

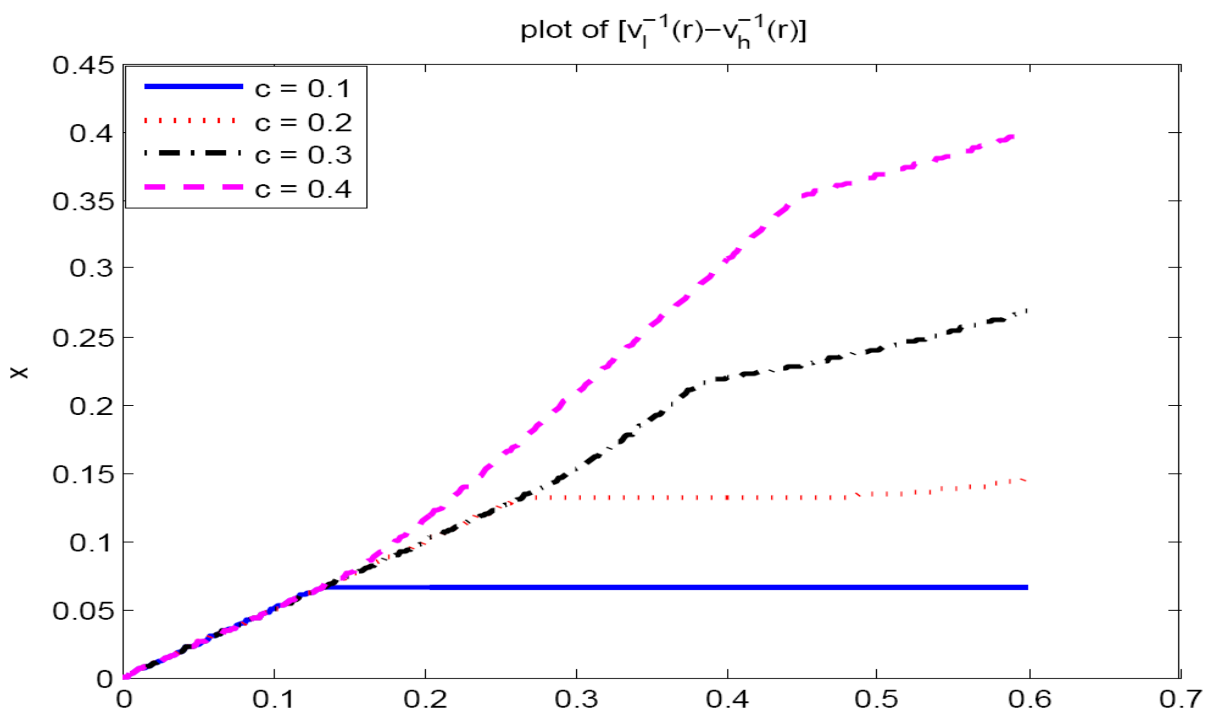


Figure 1 (b)

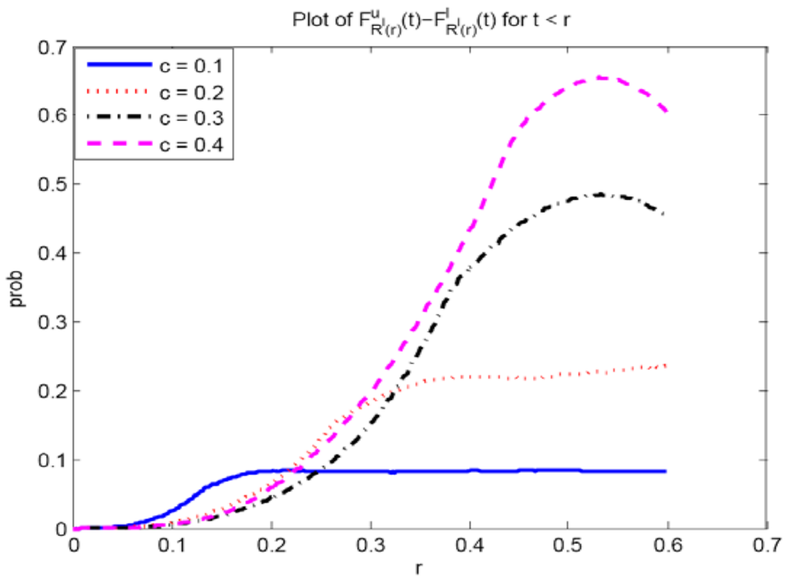


Figure 1 (c)

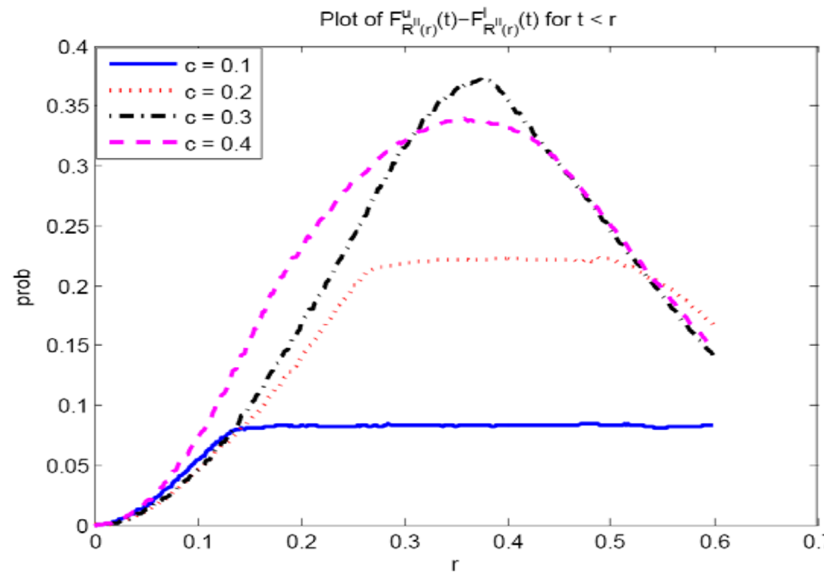


Figure 1 (d)

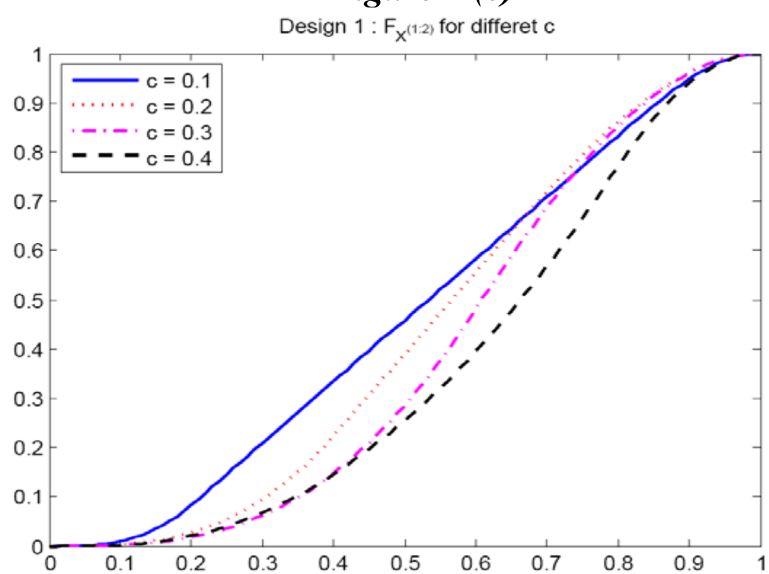


Figure 1 (e)

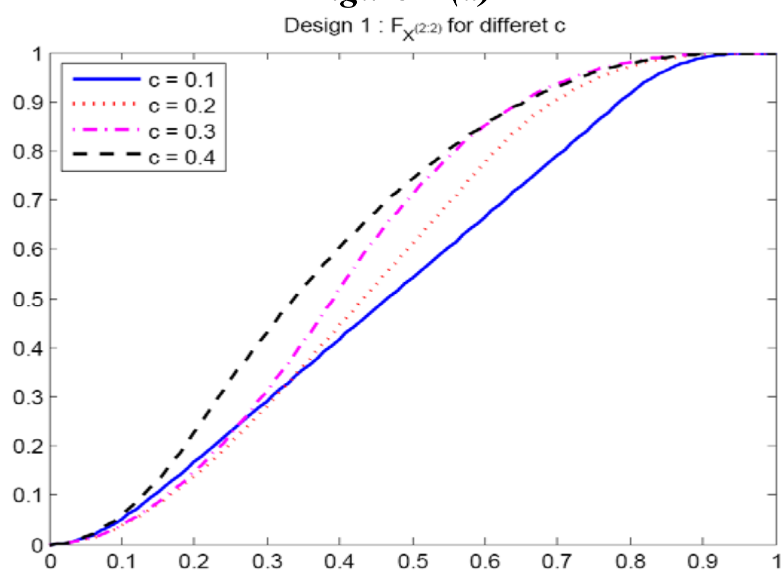


Figure 1 (f)

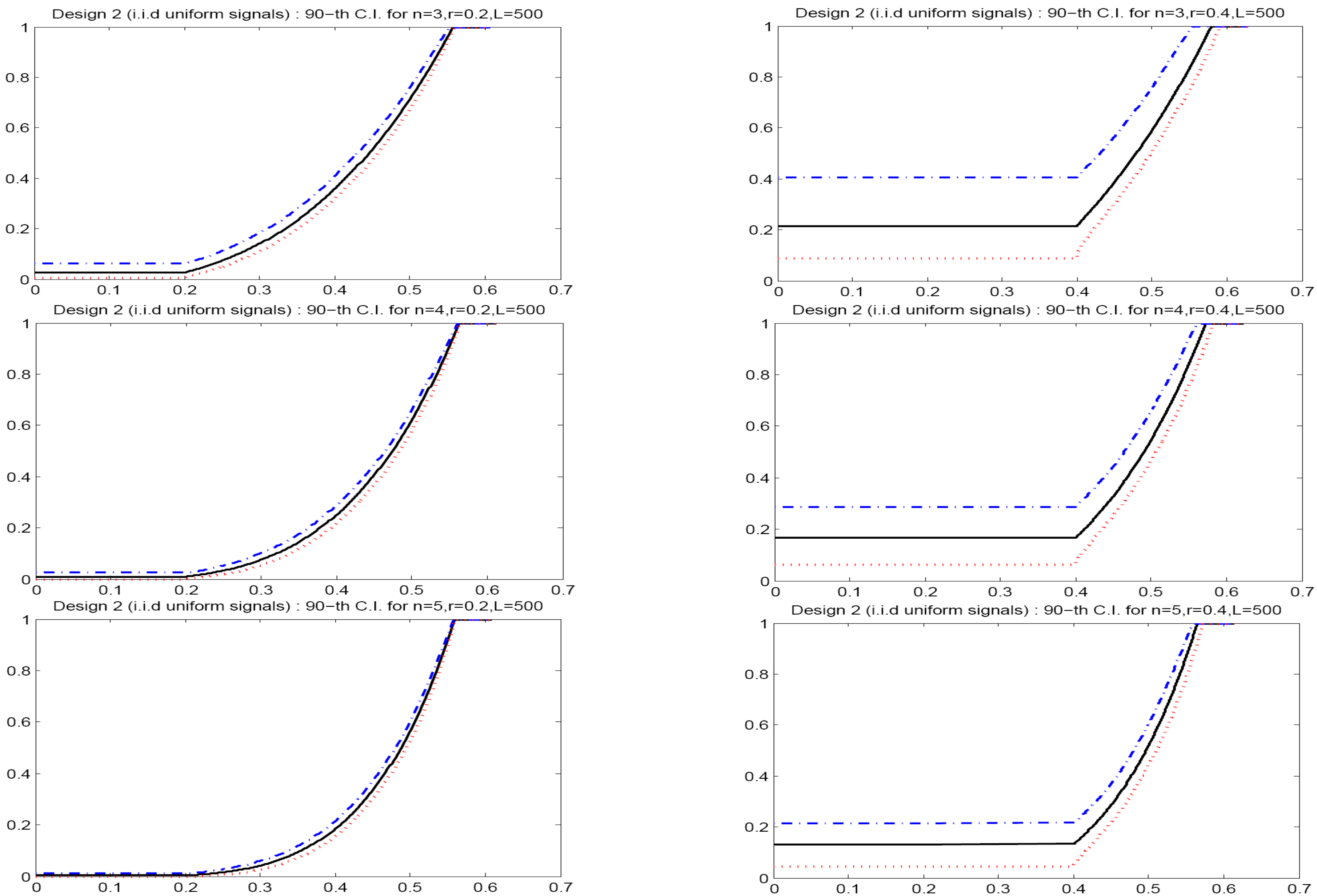

Design 2 (i.i.d uniform signals) : 90−th C.I. for n=3,r=0.2,L=500
Design 2 (i.i.d uniform signals) : 90−th C.I. for n=3,r=0.4,L=500
Design 2 (i.i.d uniform signals) : 90−th C.I. for n=4,r=0.2,L=500
Design 2 (i.i.d uniform signals) : 90−th C.I. for n=4,r=0.4,L=500
Design 2 (i.i.d uniform signals) : 90−th C.I. for n=5,r=0.2,L=500
Design 2 (i.i.d uniform signals) : 90−th C.I. for n=5,r=0.4,L=500
0
0.2
0.4
0.6
0.8
1
0.1
0.3
0.5
0.7

Figure 2

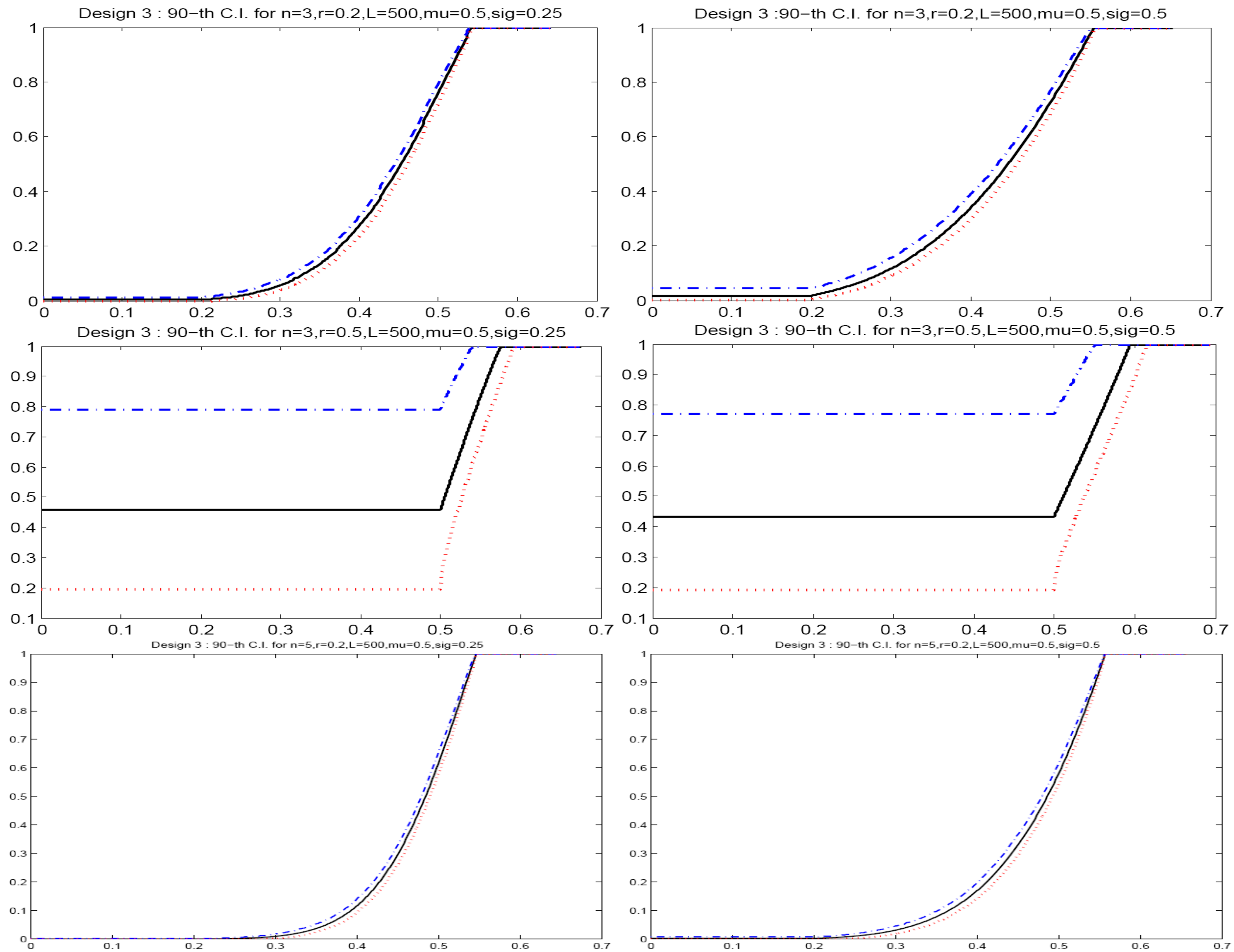
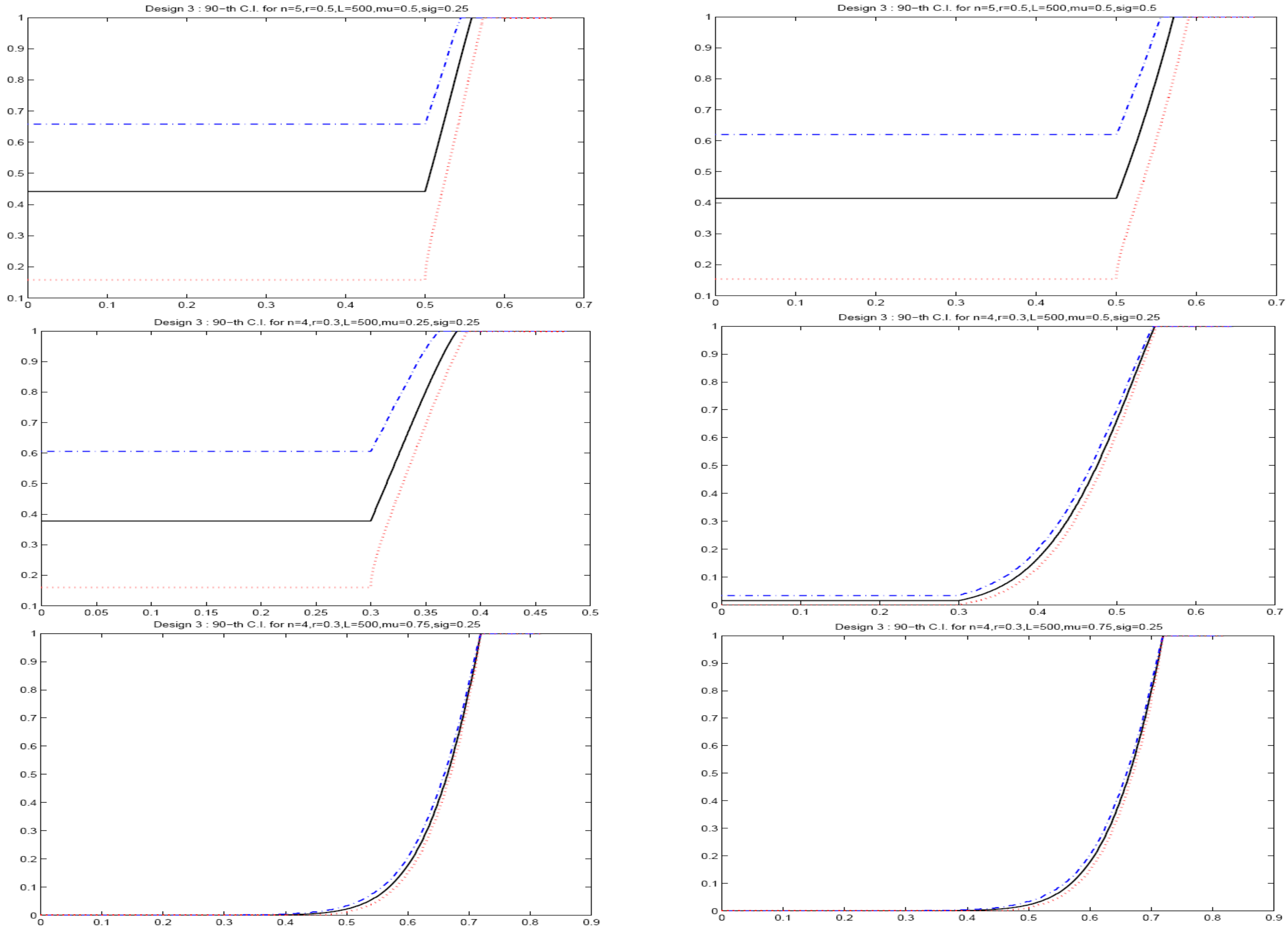

Design 3 : 90−th C.I. for n=3,r=0.2,L=500,mu=0.5,sig=0.25
Design 3 :90−th C.I. for n=3,r=0.2,L=500,mu=0.5,sig=0.5
Design 3 : 90−th C.I. for n=3,r=0.5,L=500,mu=0.5,sig=0.25
Design 3 : 90−th C.I. for n=3,r=0.5,L=500,mu=0.5,sig=0.5
Design 3 : 90−th C.I. for n=5,r=0.2,L=500,mu=0.5,sig=0.25
Design 3 : 90−th C.I. for n=5,r=0.2,L=500,mu=0.5,sig=0.5

Figure 3(a)



Design 3 : 90−th C.I. for n=5,r=0.5,L=500,mu=0.5,sig=0.25
Design 3 : 90−th C.I. for n=5,r=0.5,L=500,mu=0.5,sig=0.5
Design 3 : 90−th C.I. for n=4,r=0.3,L=500,mu=0.25,sig=0.25
Design 3 : 90−th C.I. for n=4,r=0.3,L=500,mu=0.5,sig=0.25
Design 3 : 90−th C.I. for n=4,r=0.3,L=500,mu=0.75,sig=0.25
Design 3 : 90−th C.I. for n=4,r=0.3,L=500,mu=0.75,sig=0.25

Figure 3(b)

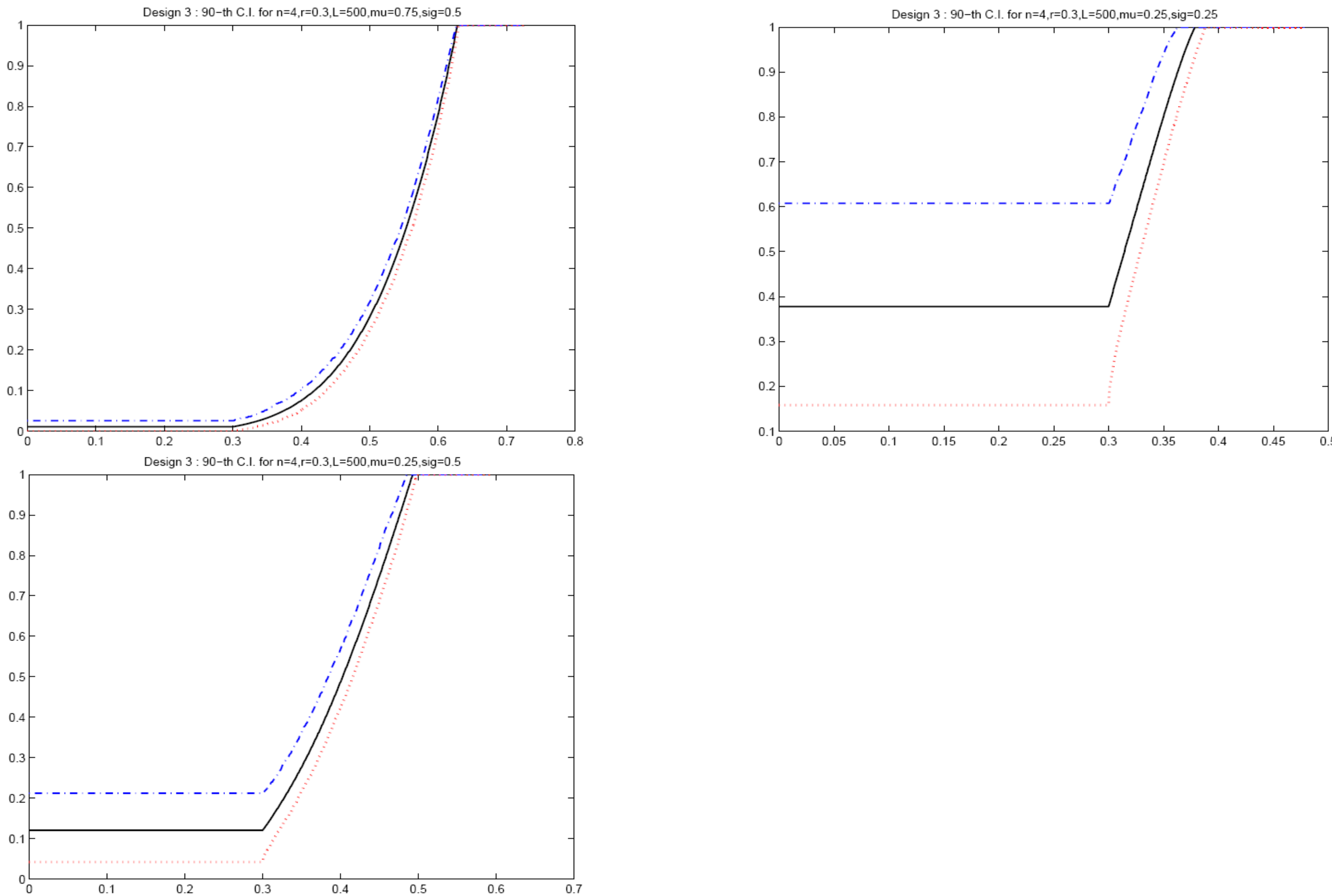

Design 3 : 90−th C.I. for n=4,r=0.3,L=500,mu=0.75,sig=0.5
Design 3 : 90−th C.I. for n=4,r=0.3,L=500,mu=0.25,sig=0.25
Design 3 : 90−th C.I. for n=4,r=0.3,L=500,mu=0.25,sig=0.5

Figure 3(c)

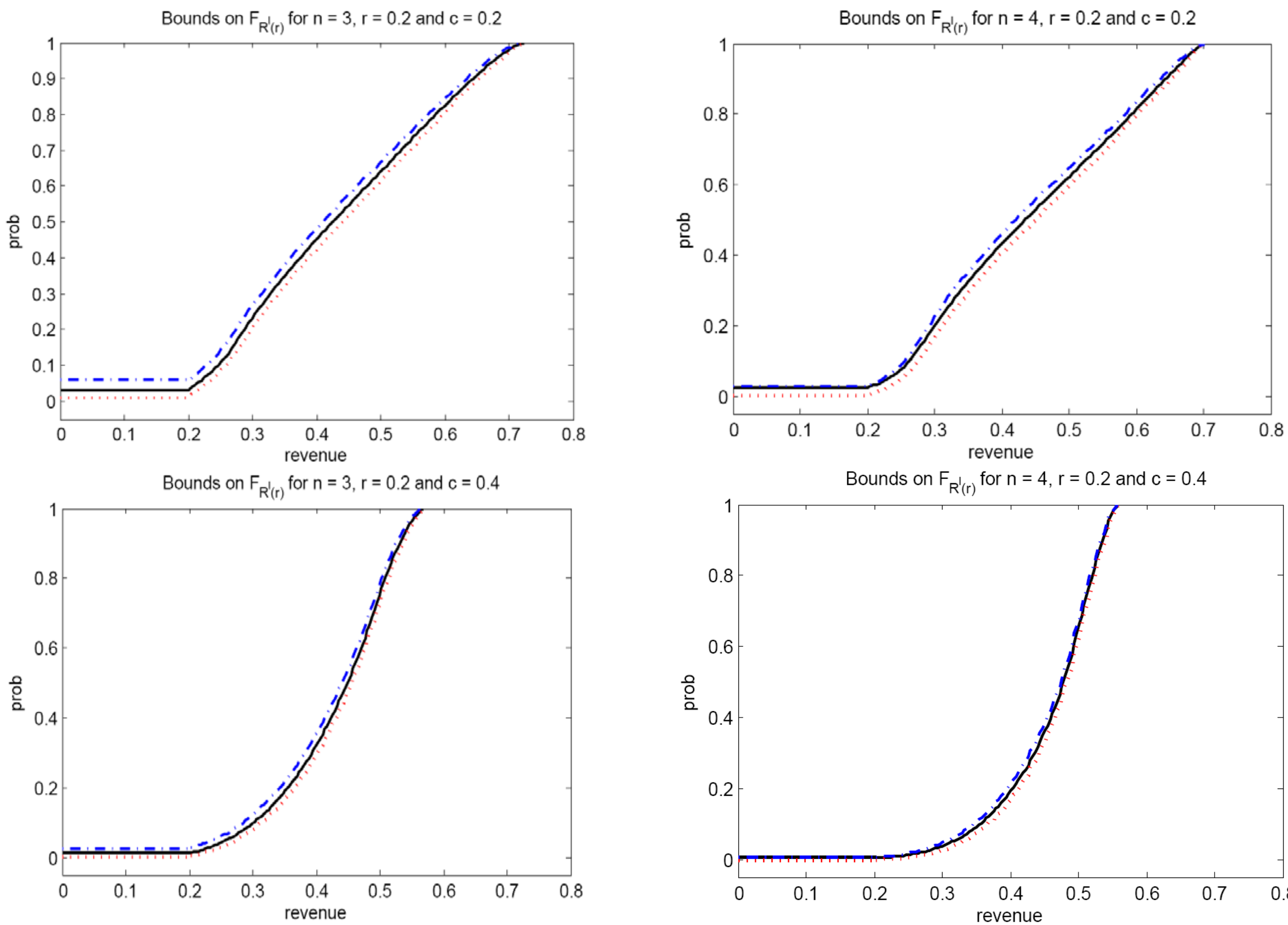

Bounds on $F_{R^{I}(r)}$ for n = 3, r = 0.2 and c = 0.2
prob
revenue
Bounds on $F_{R^{I}(r)}$ for n = 4, r = 0.2 and c = 0.2
prob
revenue
Bounds on $F_{R^{I}(r)}$ for n = 3, r = 0.2 and c = 0.4
prob
revenue
Bounds on $F_{R^{I}(r)}$ for n = 4, r = 0.2 and c = 0.4
prob
revenue

Figure 4(a)

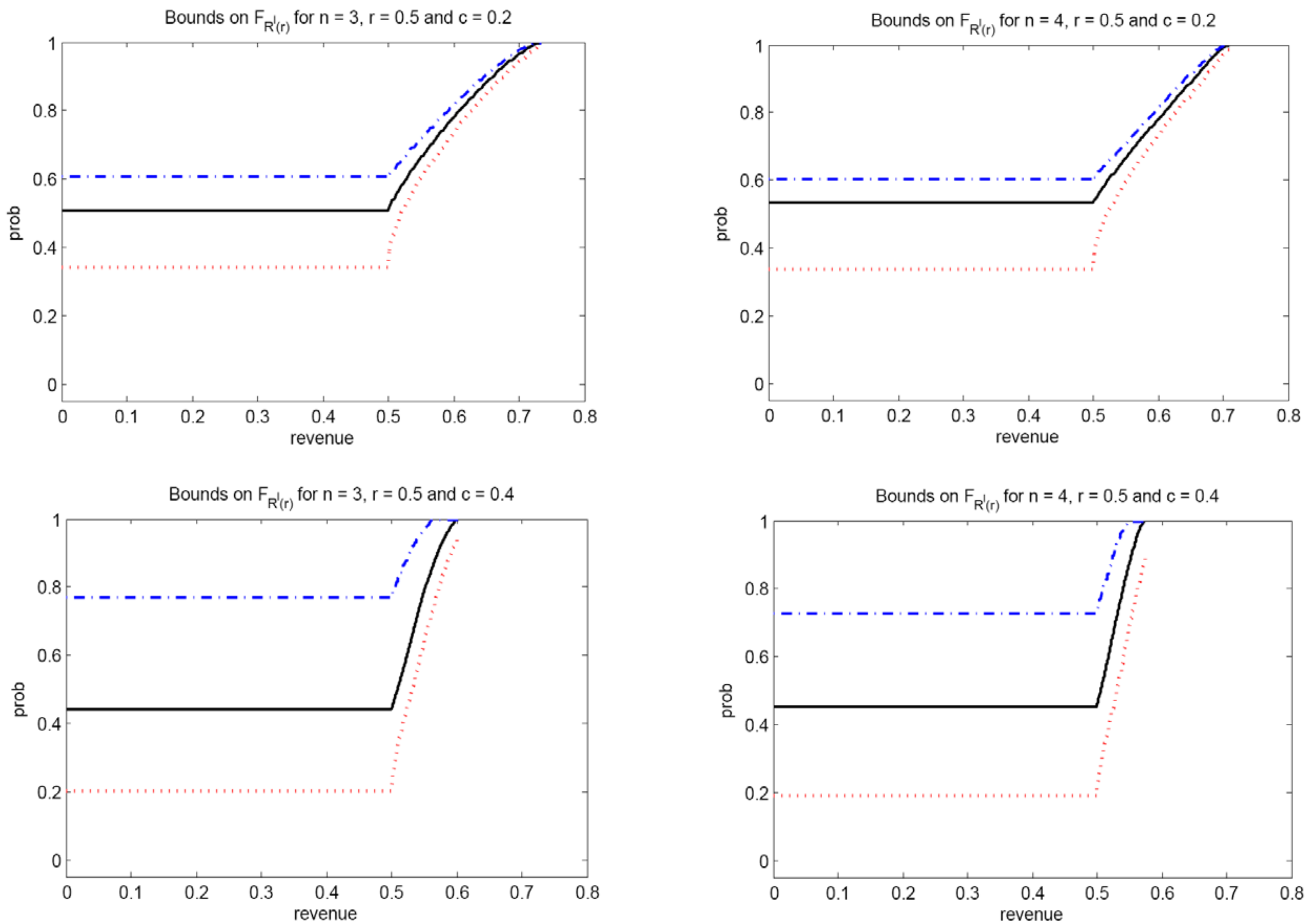

Bounds on $F_{R^l(r)}$ for n = 3, r = 0.5 and c = 0.2
Bounds on $F_{R^l(r)}$ for n = 4, r = 0.5 and c = 0.2
Bounds on $F_{R^l(r)}$ for n = 3, r = 0.5 and c = 0.4
Bounds on $F_{R^l(r)}$ for n = 4, r = 0.5 and c = 0.4
prob
revenue
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
1

Figure 4(b)

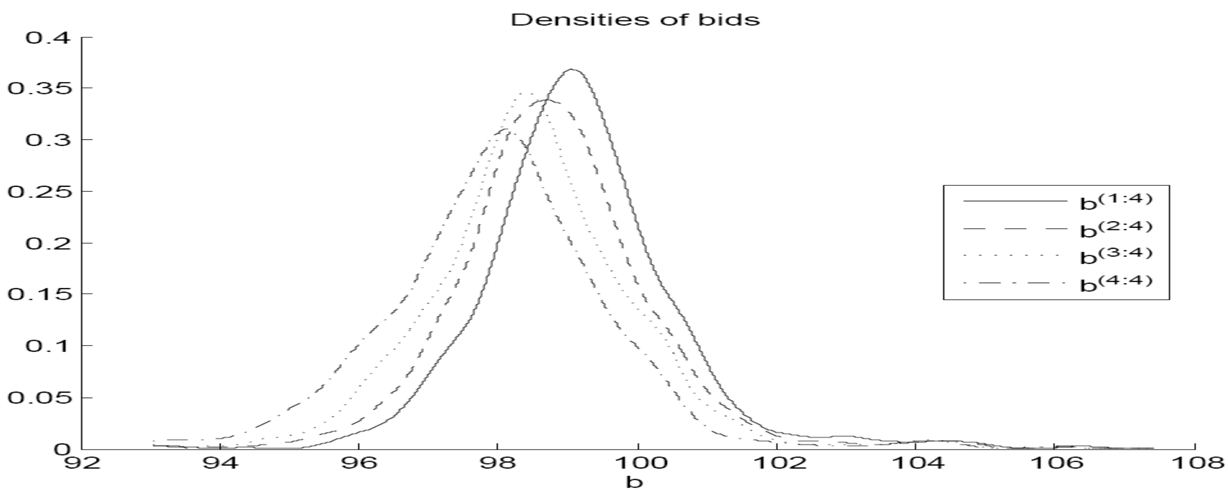


Figure 5(a)

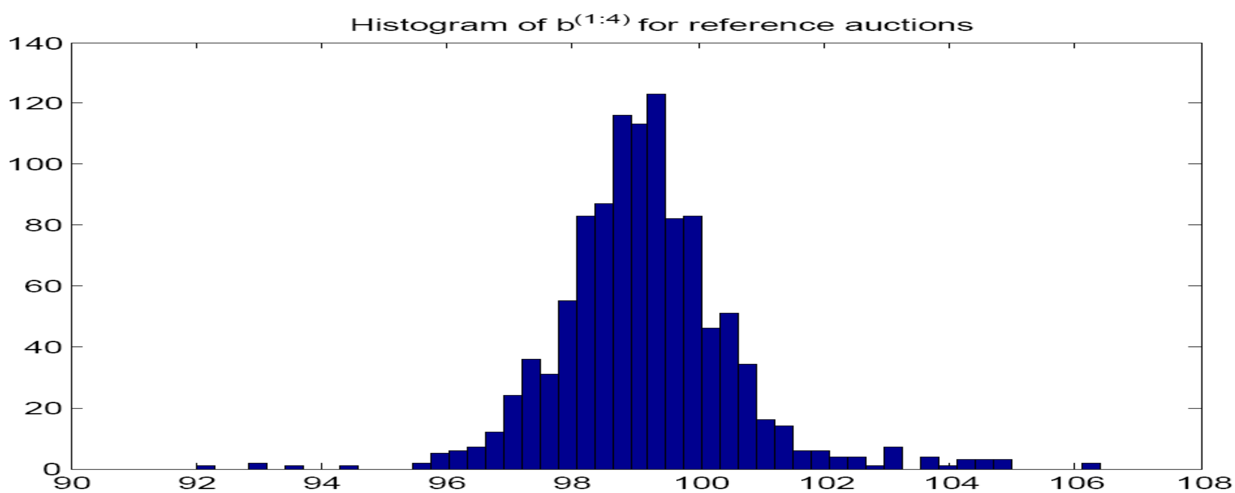


Figure 5(b)

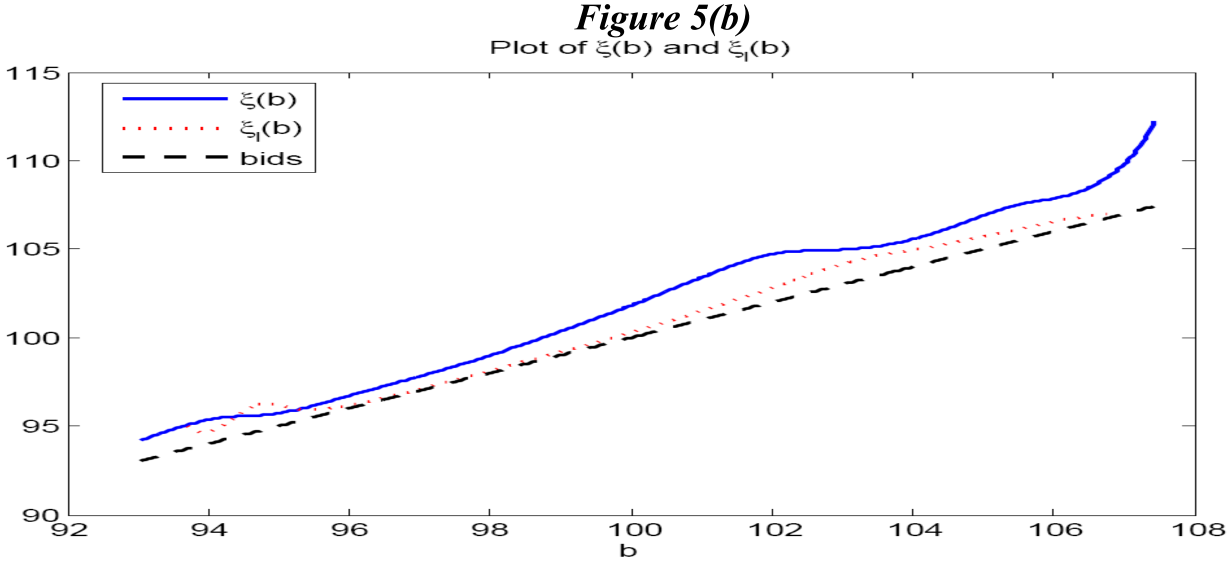


Figure 6

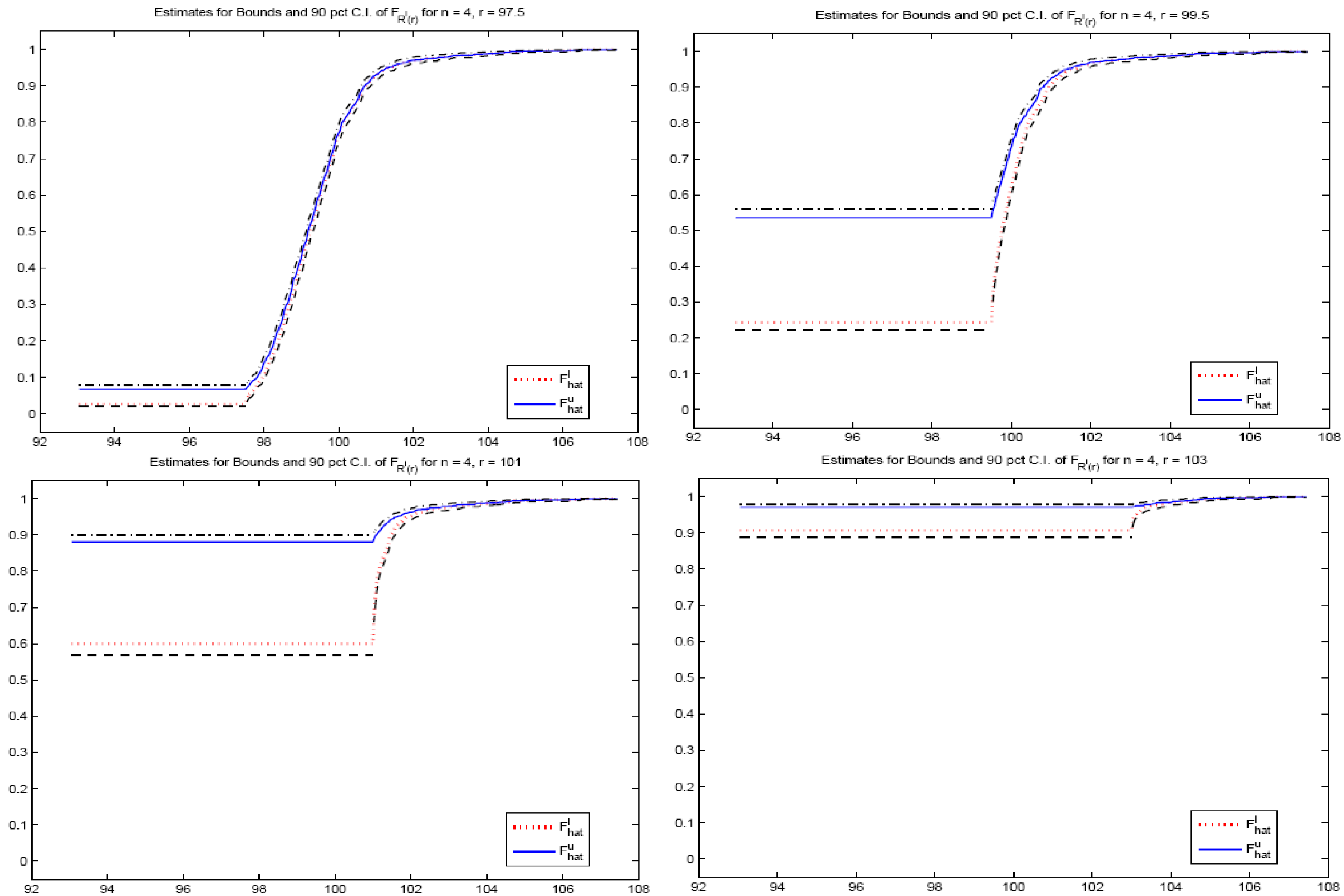

Estimates for Bounds and 90 pct C.I. of $F_{R'(r)}$ for n = 4, r = 97.5
Estimates for Bounds and 90 pct C.I. of $F_{R'(r)}$ for n = 4, r = 99.5
Estimates for Bounds and 90 pct C.I. of $F_{R'(r)}$ for n = 4, r = 101
Estimates for Bounds and 90 pct C.I. of $F_{R'(r)}$ for n = 4, r = 103
F^l_{hat}
F^u_{hat}
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
92
94
96
98
100
102
104
106
108

Figure 7

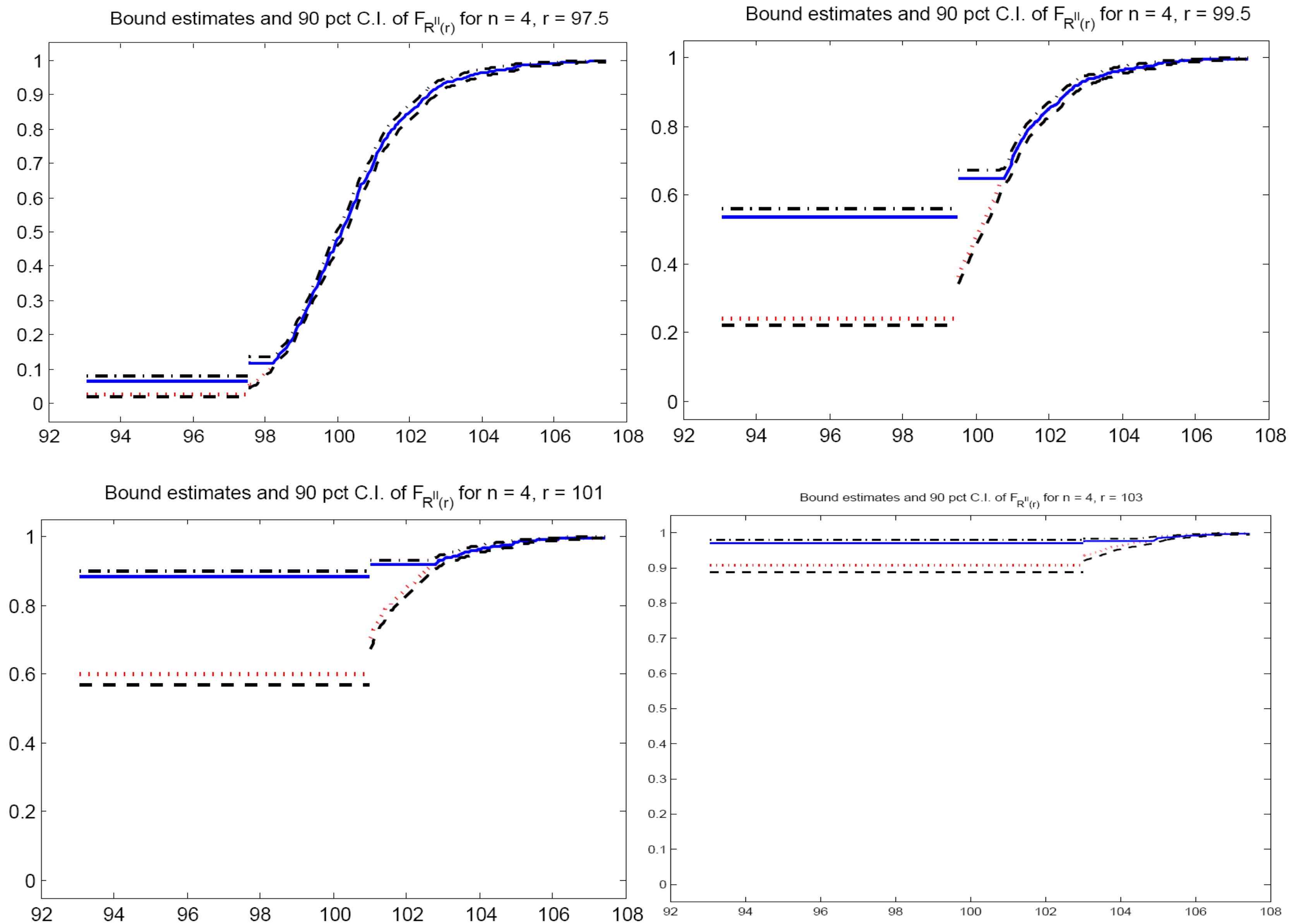

Bound estimates and 90 pct C.I. of $F_{R^{II}(r)}$ for n = 4, r = 97.5
Bound estimates and 90 pct C.I. of $F_{R^{II}(r)}$ for n = 4, r = 99.5
Bound estimates and 90 pct C.I. of $F_{R^{II}(r)}$ for n = 4, r = 101
Bound estimates and 90 pct C.I. of $F_{R^{II}(r)}$ for n = 4, r = 103

Figure 8(b)

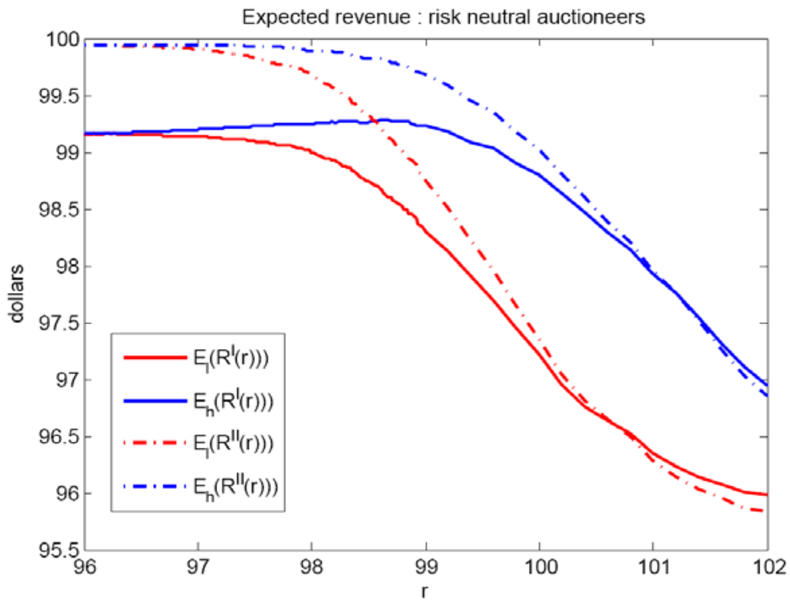


Figure 9(a)

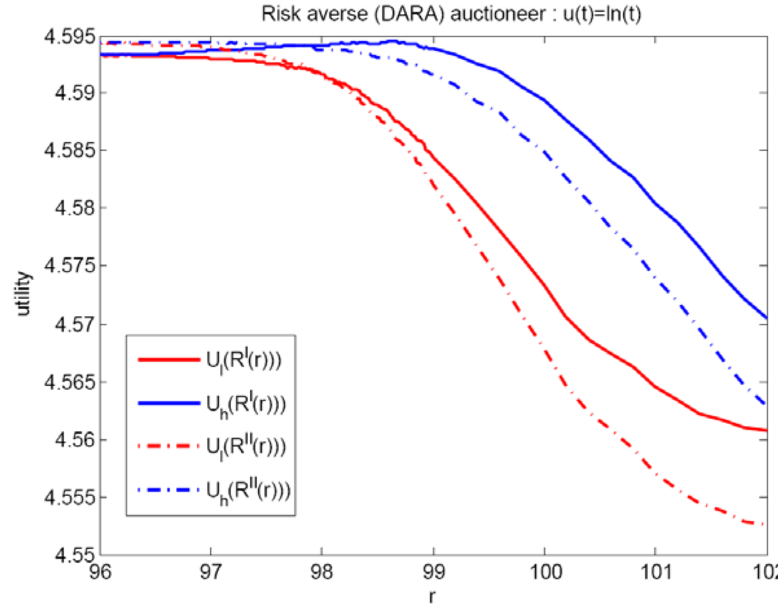


Figure 9(b)

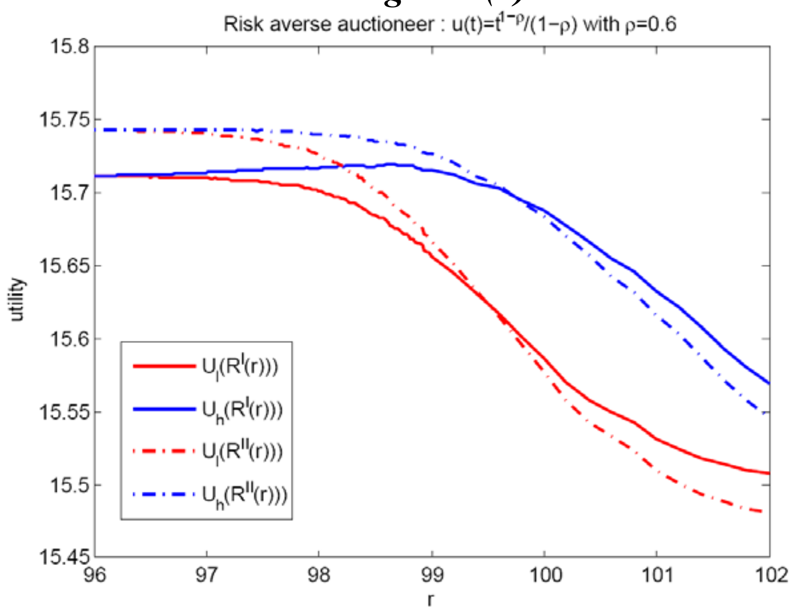


Figure 9 (c)

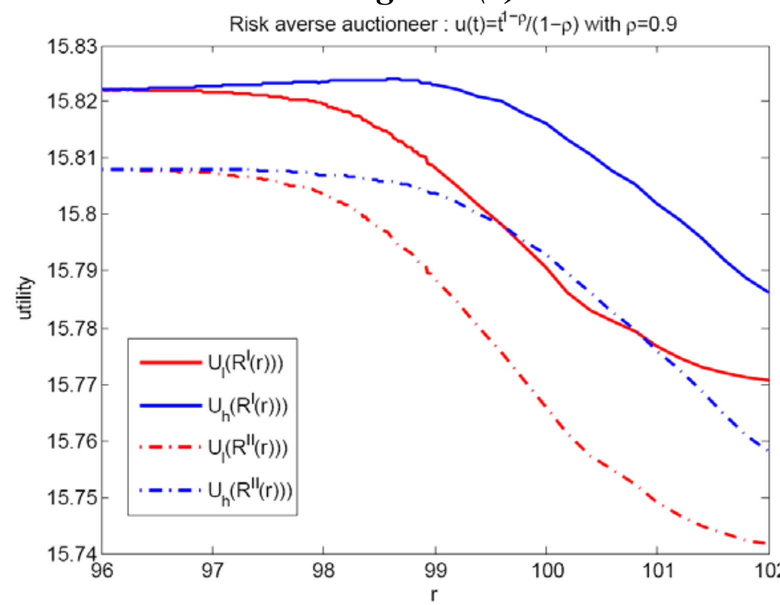


Figure 9(d)

Table 1(a) : Descriptive Statistics

# of bidding syndicates	2	3	4	5	6	7	8	9	10	11
# of auctions	608	971	1075	1007	852	687	531	406	258	158
Average par value ($mil)	4.90	9.22	17.33	23.13	27.02	27.27	30.20	32.68	26.54	33.57
Average price (/ $100 par)	99.17	99.31	99.51	99.97	100.32	100.44	101.00	101.06	101.19	101.64
Average spread	0.87	1.15	1.18	1.20	1.18	1.20	1.07	1.04	1.00	1.01
Average bid	98.73	98.77	98.95	99.42	99.79	99.94	100.53	100.59	100.76	101.22
Std. dev.	1.67	1.84	1.95	2.54	2.58	2.98	2.97	2.90	3.07	2.97
Minimal bid	90.28	85.20	86.47	87.03	86.68	13.26	93.44	91.67	93.94	93.87
Maximal bid	113.93	110.59	108.85	119.16	111.66	114.55	111.45	111.87	128.00	111.45

Table 1 (b) : Descriptive Statistics

Prices	
Min	91.17
Percentiles	
1	96.74
10	97.98
20	98.49
30	98.91
40	99.29
50	99.66
60	100.00
70	100.33
80	100.91
90	102.84
99	109.06
Max	128.00
# of auctions	6,721

All Bids	
Min	85.20
Percentiles	
1	95.32
10	97.37
20	98.07
30	98.56
40	98.99
50	99.40
60	99.81
70	100.28
80	101.14
90	103.54
99	109.30
Max	128.00
# of bids	37,547

# of bidders	# of auctions	
1	19	0.28
2	608	9.05
3	971	14.45
4	1075	15.99
5	1007	14.98
6	852	12.68
7	687	10.22
8	531	7.90
9	406	6.04
10	258	3.84
11	158	2.35
12+	149	2.22
Total	6721	100.00

WA Coupon Rate	
Min	0.0100
1	0.0214
10	0.0322
20	0.0355
30	0.0377
40	0.0392
50	0.0405
60	0.0419
70	0.0435
80	0.0454
90	0.0481
99	0.0549
Max	0.0671

Total Par Value	(in $million)
Min	0.105
1	0.385
10	1.275
20	2.160
30	3.200
40	4.485
50	6.000
60	8.581
70	12.000
80	20.415
90	45.000
99	297.831
Max	809.470

SecType		
Unlimited GO	4334	64.484
Limited GO	1061	15.786
Revenue	1326	19.729

Table 2 : Pooled Random Effect Estimates

	Est	Std Err	t-stat	p-value
wacr	1.520	0.122	12.49	0.00
wapn	-1.037	0.061	-17.11	0.00
sectype	2.476	0.458	5.41	0.00
BQ	-0.837	0.056	-15.00	0.00
totpar	1.764	0.108	16.32	0.00
type_cr	-0.680	0.121	-5.64	0.00
HR	0.221	0.175	1.26	0.21
HR_pn	-0.002	0.067	-0.03	0.98
NE	0.428	0.142	3.00	0.00
SW	-0.188	0.188	-1.00	0.32
South	0.226	0.121	1.87	0.06
West	-0.323	0.149	-2.17	0.03
NE_rating	0.017	0.177	0.10	0.92
SW_rating	-0.038	0.231	-0.16	0.87
South_rating	0.309	0.175	1.77	0.08
West_rating	0.389	0.215	1.81	0.07
d3	94.920	0.450	210.97	0.00
d4	94.968	0.445	213.26	0.00
d5	95.323	0.438	217.61	0.00
d6	95.579	0.443	215.91	0.00
d7	95.738	0.438	218.44	0.00
d8	96.128	0.443	216.92	0.00

Number of cluster	5123.00
F(21, 5122)	86.66
Prob > F	0.00
R-squared	0.43
Root MSE	1.98

Table 3(a) : GLS estimates for fixed number of potential bidders

number of bidders	4	5	6	7	8
Intercept	98.221	95.524	92.578	93.563	95.500
	169.540	91.620	76.000	79.570	64.370
WA coupon rate	0.611	1.495	2.205	2.021	1.797
	3.790	5.020	6.970	6.280	4.710
WA maturity	-0.896	-1.082	-1.025	-0.944	-1.278
	-8.690	-7.740	-8.430	-5.090	-5.730
Security Type	0.264	3.297	4.514	3.945	2.180
	0.390	3.110	3.960	3.370	1.400
Bank Qualified	-0.529	-0.723	-0.694	-0.848	-1.277
	-5.510	-6.580	-5.330	-5.820	-6.110
Ratings	0.061	-0.115	0.732	0.435	0.174
	0.170	-0.310	1.750	1.010	0.320
*Type*WACR*	-0.051	-0.935	-1.249	-1.036	-0.571
	-0.290	-3.330	-4.250	-3.380	-1.430
*Ratings*WAPN*	0.113	0.219	-0.158	-0.311	0.080
	0.900	1.500	-1.210	-1.530	0.330
Par amount	1.423	1.770	1.449	2.200	2.182
	6.220	8.390	8.750	8.570	6.430
N.E.	0.065	0.836	0.905	0.149	0.380
	0.280	2.810	2.670	0.280	0.860
South	-0.225	-0.052	0.747	0.382	0.369
	-1.130	-0.220	2.260	0.830	1.060
S.W.	-0.638	-0.443	0.116	-0.209	0.109
	-2.360	-1.290	0.290	-0.380	0.140
West	-0.807	-0.066	0.178	-0.600	-0.364
	-3.690	-0.260	0.460	-1.350	-0.590
*NE*ratings*	0.104	-0.572	-0.107	0.578	-0.120
	0.350	-1.590	-0.260	1.010	-0.210
*South*ratings*	0.506	0.794	0.048	0.349	-0.169
	1.570	2.290	0.110	0.680	-0.320
*West*ratings*	0.943	0.298	0.016	1.037	0.010
	2.700	0.660	0.030	1.760	0.010
*SW*ratings*	0.195	0.103	-0.181	0.308	-0.446
	0.520	0.240	-0.350	0.490	-0.520
number of auctions	1075	1007	852	687	531
number of bids	4300	5035	5112	4809	4248
'R-square'	0.29	0.48	0.44	0.43	0.35
'F-statistic'	17.24	30.24	26.26	25.47	15.73
'p-value'	0.00	0.00	0.00	0.00	0.00

Table 3(b) : Test of equal indices

WA Coupon Rate

	4	5	6	7
4	-			
5	-1.924	-		
6	-3.336	-1.157	-	
7	-2.917	-0.849	0.289	-
8	-2.184	-0.445	0.585	0.318

WA Maturity

	4	5	6	7
4	-			
5	0.767	-		
6	0.575	-0.219	-	
7	0.167	-0.424	-0.263	-
8	1.171	0.538	0.733	0.816

Type

	4	5	6	7
4	-			
5	-1.746	-		
6	-2.339	-0.553	-	
7	-1.992	-0.290	0.246	-
8	-0.858	0.427	0.866	0.647

Par amount

	4	5	6	7
4	-			
5	-0.788	-		
6	-0.064	0.853	-	
7	-1.600	-0.920	0.030	-
8	-1.336	-0.750	-1.453	0.030

Bank Qualified

	4	5	6	7
4	-			
5	0.945	-		
6	0.732	0.853	-	
7	-1.600	-0.920	-1.779	-
8	-1.336	-0.750	-1.453	0.030